INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

PROCEEDINGS

of the 33rd Conference of the International Group for the Psychology of Mathematics Education

In Search for Theories in Mathematics Education

EDITORS
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Volume 1

**Plenary Lectures, Plenary Panel,
Research Forums, Discussion Groups,
National Presentation, Short Oral Communications**

PME 33, Thessaloniki - Greece
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PREFACE

It is a great pleasure to welcome you to the 33rd Annual Conference of the International Group for the Psychology of Mathematics Education, which is held in Thessaloniki, at the Aristotle University and the University of Macedonia.

Thessaloniki is the second largest city in the country, in the district of central Macedonia with a continuous 3000-year history. With almost one million inhabitants, Thessaloniki is an important economic, commercial, intellectual and cultural centre for Greece and South-East Europe.

Its long multiethnic and multicultural history is documented in a wealth of monuments, from ancient ruins dating 23 centuries back to important churches dating from the 5th century and still in use. Conference attendees will have the opportunity to visit all monuments and museums in guided tours, organized by the conference secretariat during the conference.

The Aristotle University of Thessaloniki was founded in 1925. The structure of the University today, its range of activities and its size makes it the largest and most complex institution of higher education in the country. About 101,000 students are registered in the 42 Schools that cover a very broad scientific spectrum. The University of Macedonia is a newer and smaller institution, well organised and very potential one in human personnel and infrastructures.

Organizing a PME conference in Greece has been an endeavour long pursued by the regular Greek PME members, who have made efforts to turn the 2009 annual meeting of our society to a social and scientific success. The theme of the conference, "*In search for theories in Mathematics Education*", has been chosen in the hope that, as Ancient Greece provided the context within which Mathematics advanced theoretically, Modern Greece can become the threshold for enhancing the ongoing debate on this crucial issue.

After more than 30 years of high level research activity and the development of a variety of perspectives to analyze the complicated phenomena of learning and teaching mathematics, the community appears mature enough to invest on building up coherent and compound theoretical approaches, a task that will hopefully be promoted substantially by this conference's scientific activities.

The 33rd PME Annual Meeting seems to be one of the most highly attended in the history of International Group of PME. This fulfils us with the expectation that the many researchers from all over the world who will participate will contribute to the scientific programme of the conference in many interesting and scientific challenging ways. At the same time, we hope that the city and the country will provide an opportunity to everybody to experience some of the Greek culture and history.

The International Programme Committee and the Local Organizing Committee want to express their thanks to all PME members who came forward with contributing to the scientific programme of the conference as well as to the PME administrative structure for the invaluable help.

Organizing a PME conference is an extremely complicated and time consuming investment. PME33 owes its realization to the work and dedication of many people who I would like to thank for their continuous and effective support. I am certain that all this effort will lead to a scientifically fruitful and socially successful meeting.

Thessaloniki, May 2009

Marianna Tzekaki

SPONSORS

The conference received support from several sources to whom we are grateful:

Aristotle University of Thessaloniki

University of Macedonia

Democritus University of Thrace

University of Ioannina

Research Committee of AUTH

In-service Training of Preschool Teachers, AUTH

Greek Association of Researchers in Mathematics Education

The Prefecture of Thessaloniki

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THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

History and Aims of PME

The International Group for the Psychology of Mathematics Education, abbreviated to PME, is a group of researchers. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI). PME came into existence at the Third International Congress on Mathematics Education (ICME3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

Efraim Fischbein, Israel
Richard R. Skemp, United Kingdom
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Kevin F. Collis, Australia
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Nicolas Balacheff, France
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Carolyn Kieran, Canada
Stephen Lerman, United Kingdom
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Rina Hershkowitz, Israel
Chris Breen, South Africa

The present president is Fou-Lai Lin, Taiwan.

The Constitution of PME

The constitution of PME was adopted by the Annual General Meeting on the 17th of August, 1980 and changed by the Annual General Meetings on the 24th of July, 1987, on the 10th of August, 1992, on the 2nd of August, 1994, on the 18th of July, 1997 and on the 14th of July, 2005.

The **major goals** of the Group are:

- to promote international contact and exchange of scientific information in the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

PME Membership and Other Information

Membership is open to people involved in active research consistent with the aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other about their special and general interests. There are plenary lectures, research paper presentations, working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued twice a year, and can be found on the IGPME website.

Occasionally PME issues a scientific publication, for example the result of research done in group activities.

Website of PME

All information concerning PME and its constitution can be found at the PME Website: http://www.igpme.org

Honorary Members of PME

Efraim Fischbein (Deceased)
Hans Freudenthal (Deceased)
Joop Van Dormolen (Retired)

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33rd CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION PME (PME33)

Two committees are responsible for the organization of the PME33 Conference.

International Program Committee (IPC)

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LOC - National Component

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Despina Desli	Democritus University of Thrace
Ioanna Mamona	University of Patras
George Philipou	University of Cyprus

Conference Secretariats

Conference Scientific Secretariat

For matters related to the scientific issues of the conference (program, presentations, equipment, etc.): Erato Gazani, Department of Education, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece

Phone +30 2310 991294, Fax +30 2310 995098

E-mail: scient_pme33@auth.gr

Conference Administrative Secretariat

For matters related to the administrative issues of the conference (registration, payment, accommodation, excursions, travels, etc.): Symvoli Conference & Event Organizers, 20, I. Tsaluhidi str. GR- 54248, Thessaloniki

Phone +30 2310 433099, Fax +30 2310 433599

E-mail: admin_pme33@symvoli.gr

PME33 has a website at http://www.pme33.eu

PROCEEDINGS OF PREVIOUS PME CONFERENCES

The table includes the ERIC numbers and/or the e-address of the websites of the past conferences.

PME International

No.	Year	Place	ERIC number and/or URL
1	1977	Utrecht	The Netherlands Not available in ERIC
2	1978	Osnabrück	Germany ED226945
3	1979	Warwick	United Kingdom ED226956
4	1980	Berkeley	USA ED250186
5	1981	Grenoble	France ED225809
6	1982	Antwerp	Belgium ED226943
7	1983	Shoresh	Israel ED241295
8	1984	Sydney	Australia ED306127
9	1985	Noordwijkerhout	Netherlands ED411130 (vol.l), ED411131 (vol.2)
10	1986	London	United Kingdom ED287715
11	1987	Montréal	Canada ED383532
12	1988	Veszprém	Hungary ED411128 (vol.l), ED411129 (vol.2)
13	1989	Paris France	ED411140 (vol.1), ED411141(vol.2), ED411142 (vol.3)
14	1990	Oaxtepex	Mexico ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
15	1991	Assisi	Italy ED413162 (vol.1), ED413l63 (vol.2), ED413164 (vol.3)
16	1992	Durham	USA ED383538
17	1993	Tsukuba	Japan ED383536
18	1994	Lisbon	Portugal ED383537
19	1995	Recife	Brazil ED411134 (vo1.l), ED411135 (vol.2), ED411136 (vo1.3)
20	1996	Valencia	Spain ED453070 (vol.1), ED453071 (vol.2), ED453072 (vol.3), ED453073 (vol.4), ED453074 (addendum)
21	1997	Lahti	Finland ED416082 (vol.1), ED416083 (vol.2), ED4l6084 (vol.3), ED416085 (vol.4)
22	1998	Stellenbosch	South Africa ED427969 (vol.1), ED427970 (vol.2), ED427971 (vol.3), ED427972 (vol.4)
23	1999	Haifa	Israel ED436403
24	2000	Hiroshima	Japan ED452301 (vol.1), ED452302 (vol.2), ED452303 (vol.3), ED452304 (vol.4)
25	2001	Utrecht	The Netherlands ED466950
26	2002	Norwich	United Kingdom ED476065
27	2003	Hawai‘i	USA http://onlinedb.terc.edu
28	2004	Bergen	Norway http://emis.de/proceedings/PME28/
29	2005	Melbourne	Australia http://staff.edfac.unimelb.edu.au/~chick/PME29/

30	2006	Prague	Czech Republic http://class.pedf.cuni.cz/pme30
31	2007	Seoul	Korea
32	2008	Morelia	Mexico, http://www.pme32-na30.org.mx/

Copies of some previous PME Conference Proceedings are still available for sale. See the IGPME website at http://igpme.org/publications/procee.html or contact the proceedings manager Dr. Peter Gates, PME Proceedings, University of Nottingham, School of Education, Jubilee Campus, Wollaton Road, Nottingham NG8 1 BB, UNITED KINGDOM, Telephone work: +44-115-951-4432; fax: +44-115-846-6600;

e-mail: peter.gates@nottingham.ac.uk

PME-NA

No.	Year	Place	ERIC number and/or URL
1	1979	Evanston, Illinois	
2	1980	Berkeley, California	(with PME2) ED250186
3	1981	Minnesota	ED223449
4	1982	Georgia	ED226957
5	1983	Montreal, Canada	ED289688
6	1984	Wisconsin	ED253432
7	1985	Ohio	ED411127
8	1986	Michigan	ED301443
9	1987	Montreal, Canada	(with PME11) ED383532
10	1988	Illinois	ED411126
11	1989	New Jersey	ED411132 (vol.1), ED411133 (vol.2)
12	1990	Oaxtepec, Morelos, Mexico	(with PME14) ED411137 (vol.1), ED411138 (vol.2), ED411139 (vol.3)
13	1991	Virginia	ED352274
14	1992	Durham, New Hampshire	(with PME16) ED383538
15	1993	California	ED372917
16	1994	Louisiana	ED383533 (vol.1), ED383534 (vol.2)
17	1995	Ohio	ED389534
18	1996	Panama City, Florida	ED400178
19	1997	Illinois	ED420494 (vol.1), ED420495 (vol.2)
20	1998	Raleigh, North Carolina	ED430775 (vol.1), ED430776 (vol.2)
21	1999	Cuernavaca, Morelos, Mexico	ED433998
22	2000	Tucson, Arizona	ED446945
23	2001	Snowbird, Utah	SE065231 (vol.1), SE065232 (vol.2)
24	2002	Athens, Georgia	SE066887 (vol.1), SE066888 (vol.2), SE066889 (vol.3), SE066880 (vol.4)
25	2003	Hawai'i	(with PME27) ED500857 (vol.1), ED500859 (vol.2), ED500858 (vol.3),

			ED500860 (vol.4)
26	2004	Toronto, Notario	http://www.pmena.org/2004/
27	2005	Roanoke, Virginia	http://www.pmena.org/2005/
28	2006	Merida, Yucatan, Mexico	http://www.pmena.org/2006/
29	2007	Lake Tahoe, Nevada	http://www.pmena.org/2007/
30	2008	Morelia, Mexixo	(with PME32) http://www.pmena.org/2008/

Abstracts from some articles can be inspected on the ERIC website (http://www .eric.ed.gov/) and on the website of ZDM/MATHDI (http://www.emis.de/ MATH/ DI.html). Many proceedings are included in ERIC: type the ERIC number in the search field without spaces or enter other information (author, title, keyword). Some of the contents of the proceedings can be downloaded from this site. MATHDI is the web version of the Zentralblatt fur Didaktik der Mathematik (ZDM, English subtitle: International Reviews on Mathematical Education). For more information on ZDM/MATHDI and its prices or assistance regarding consortia contact Gerhard König, managing.

REVIEW PROCESS OF PME33

Research Forums. The International Programme Committee accepted 5 out of the 9 submitted RF proposals. Despite of the fact that all of them were interesting and relevant, the IPC decided to accept only five proposals, considering that this number allows the participants to really benefit from the presentations and discussions that would be otherwise attended by only a small part of them. Besides, the IPC had to choose the proposals that were most relevant to the PME33 theme and also to respect the need for a good geographic representation.

Working Sessions and Discussion Groups. There were 10 Working Session and 7 Discussion Group Proposals. The IPC accepted them all, recommending the change of three Working Sessions to Discussion Groups. This recommendation, which was accepted by the corresponding coordinators, was made on the basis that the proposed work did not fulfil the respective PME regulation requiring the participants of a WS to be engaged collaboratively in a joint activity.

Research Reports (RR). The IPC received 414 RR proposals. Each paper was blind-reviewed by three peer reviewers. The experienced reviewers contacted for this purpose were not, however, enough. Thus, more reviewers were invited and more reviews were asked from all the reviewers. The majority of the connected PME members responded to the request and contributed decisively to the successful completion of this crucial task.

Reviewers received proposals for review according to the research categories indicated in their Reviewer Information Form. The proposals were sent to reviewers according to the research categories marked by the author(s).

All papers with two or three acceptances were accepted. The members of the IPC re-examined all the proposals with one acceptance and two rejections. For the proposals that were finally accepted, a fourth review was added to the existing three ones. For the remaining papers, the IPC offered a Short Oral Communication (SO) or a Poster Presentation (PP) or agreed that the paper should be rejected.

Finally, 244 proposals were accepted, 126 were recommended as SOs, 24 as PPs and the remaining ones were rejected.

Short Oral Communications (SO) and Poster Presentations (PP). The IPC initially received and reviewed 151 SOs and 71 PPs proposals, 95 and 59 of which were accepted respectively. 16 SOs of the non accepted ones were recommended as PPs. In addition, 81 SOs proposals were re-submitted from RRs and 10 and 13 PPs were re-submitted from RRs and SOs respectively.

The reviewing process was completed during the 2nd Meeting of the International Programme Committee around the end of March 2009. Notification letters of acceptance/rejection were sent to the author(s) at the beginning of April 2009.

LIST OF PME33 REVIEWERS

The PME33 Program Committee thanks the following people for their help in the review process:

Abrahamson, Dor (USA)
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Akkoç, Hatice (Turkey)
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PLENARY LECTURE 1

The architecture and development of mathematical thought

THE ARCHITECTURE AND DEVELOPMENT OF MATHEMATICAL THOUGHT

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INTRODUCTION

For many, mathematics is a very special kind of activity of the human mind that enables some privileged persons to conceive of very abstract and bizarre concepts and solve very cumbersome, obscure, and impenetrable problems. For others, there is nothing special in mathematics. It is just one of many activities of the human mind which, when pulled to the extreme, requires a special talent to grasp the concepts involved and use them in original ways for the solution of important problems. We ally with this second interpretation. That is, we believe that mathematics does involve some special mechanisms of representation and mental processing which are appropriate for the representation and processing of quantitative relations. At the same time, we also believe that these mechanisms are constrained by the organization and the possibilities of one kind of brain, the human brain, and, at present, they are learned and practiced in one particular kind of culture, the human culture. Thus, any theory about the nature, the architecture, and the development of mathematics will have to specify the domain-specific processes and functions that it involves, the general potentials and processes of the human mind that sustain and frame its functioning, and their dynamic relations both in real time during problem solving and in ontogenetic time.

In this chapter we will attempt to satisfy these requirements in the context of a theory of cognitive organization and growth that is experimentally based and it integrates three traditions in psychology, namely the experimental, the differential, and the developmental tradition. Specifically, this theory models, first, the more dynamic, real-time, aspects of mental functioning to explain how information from the environment is recorded, represented, stored, and processed for the purpose of understanding and problem solving. Second, the theory also specifies the factors that are responsible for intra- and inter-individual differences in mental functioning and development. Finally, the theory models the development of mental functions in order to specify both their state and form at different phases of life and the causes and mechanisms underlying their transformation with growth. In the pages following we will first summarize the general postulates of this theory and then focus on mathematical thinking, in order to highlight its place in the architecture of the human mind, its inter-dependencies with the other components of the human mind, and its development. At the end of the paper we will summarize a series of studies that were designed to show how the various dimensions of the human mind specified by the

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 5-22. Thessaloniki, Greece: PME.

theory are related to school performance in mathematics and draw the implications of these findings for education.

THE ARCHITECTURE OF THE HUMAN MIND

The human mind is a hierarchical and multidimensional edifice that involves both general-purpose and specialized processes and abilities (Carroll, 1993; Gustafsson & Undheim, 1996). Figure 1 illustrates our general representation of the processes and functions involved in the developing mind (Demetriou, Mouyi, Spanoudis, in press; Demetriou, Christou, Spanoudis, & Platsidou, 2002; Demetriou & Kazi, 2001, 2006). Understanding, learning, or performance on any task, at a particular point in time, is a mixture of all of these processes.

General Processes

The general processes revolve around a strong directive-executive function (DEF) that is responsible for setting and pursuing mental and behavioral goals until they are attained. The constructs serving DEF may be specified from three different perspectives, that is, their (i) efficiency, (ii) their capacity, and (iii) the fundamental operations that they involve.

Processing efficiency refers to how well the person executes the processes activated in the service of DEF at a given moment. Technically speaking, processing efficiency refers to the ability to focus on, encode, and operate on goal-relevant information and inhibit or resist to goal-irrelevant information until the current mental or behavioral goal is attained. Thus, selective attention is the functional manifestation of processing efficiency. Ideally, processing is considered to be efficient when it is completed without mistakes and unnecessary mental operations that would result in excessive mental effort or waste of cognitive resources. A common measure of efficiency is *speed of processing*. Traditionally, the faster an individual can recognize a stimulus or execute a mental act, ignoring irrelevant information, if required, the more mentally efficient is thought to be.

Processing capacity is the maximum amount of information and mental acts that the mind can efficiently activate simultaneously under the direction of DEF. In the current psychological literature, working memory is regarded as the functional manifestation of processing capacity. The *directive-executive function* involves processes underlying setting mental goals, planning their attainment, monitoring action vis-à-vis both the goals and the plans, and regulating real or mental action requires a system that can remember and review and therefore know itself. (Demetriou, 2000; Demetriou & Kazi, 2001).

Figure 1, Note: Using brain as the background of this model simply implies that all systems are located somewhere in the brain but no specific location is associated to each system. Arrows indicate communication between systems.

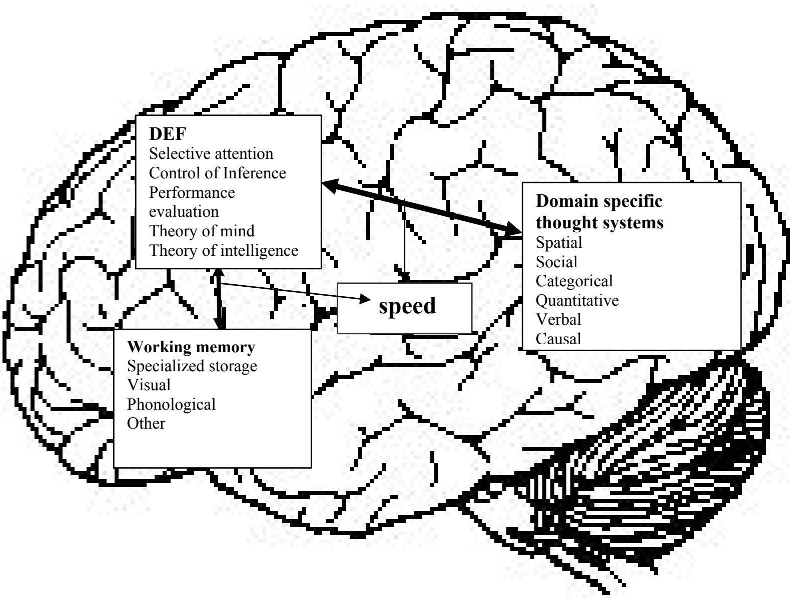

Figure 1. The general model of the domain-general and domain-specific systems of the human mind.

Domains of Thought and Specialized Processes

Specialized processes refer to mental operations and problem solving skills that are suitable for the handling (i.e., the comparison, combination, transformation) of different types of information, relations, and problems. Domain-specific mental operations serve special adaptational needs, such as orientation in space, communication with other members of the species, etc., represent and process different *type of relations* in the environment (e,g,, spatial, social, quantitative etc.,), and they are *biased to different symbol systems* that are better appropriate than other symbol systems to represent the type of relations concerned. Our research has uncovered the following six domains of thought that satisfy these criteria: categorical, quantitative, spatial, causal, verbal, and social thought (Demetriou et al., in press, 2002; Demetriou & Kazi, 2001). In the pages below we will focus on the domain of quantitative reasoning, which is the main object of this chapter.

Figure 1 illustrates our conception of the on-line relations between the representation of information in short-term storage, its control by DEF and more long-term knowledge and beliefs about it and about the person's relevant abilities, preferences, etc., and the domain(s) activated. In brain-relevant terms, this conception is consistent with the assumption that the brain involves circuits specializing in the representation of environment-relevant information (such as the parietal lobe for quantitative information or the occipital lobe for visuo-spatial information) and circuits (such as the frontal lobe) specializing in the surveillance, coordination, and regulation of these environment-relevant circuits (Dehaene, 1997). Efficiency as such refers to the communication between circuits as much as with the condition and functioning of particular circuits (Case, 1992b; Thatcher, 1992). Thus, development and individual differences in understanding and problem solving across domains may be caused by systematic variation in any of these systems or in their communication. We hope that this claim will become clearer in the pages following.

DEVELOPMENT

The Development of General Processes

Processing efficiency. There is ample evidence that processing speed changes uniformly with age, in an exponential fashion, across a wide variety of tasks, such as mental rotation, memory search, visual search, mental addition, and geometric analyses. That is, change of speed of processing is fast at the beginning (i.e., from early to middle childhood) and it decelerates systematically (from early adolescence onwards) until it attains its maximum in early adulthood (Demetriou et al. 2002). This pattern of change, which is illustrated in Figure 2, reflects the fact that, with age, the time taken by the brain to complete an operation becomes smaller due to improvements in the interconnectivity of the neural circuits in the brain and the improvements in the myelinization of neuronal axons that insulate the communication between neurons. As a result, the representation and manipulation of information in the brain becomes faster and more efficient (Case, 1992; Thatcher, 1992).

Working memory. There is general agreement that the capacity of all components of working memory (i.e., executive processes, phonological, and visual storage) increase systematically with age. In fact, the development of all three components seems to follow the same pattern of change and to be able to be described by a logistic curve which is very similar to the exponential curve that describes the change of processing efficiency (Demetriou et al., 2002). This pattern of changes in working memory is illustrated in Figure 2.

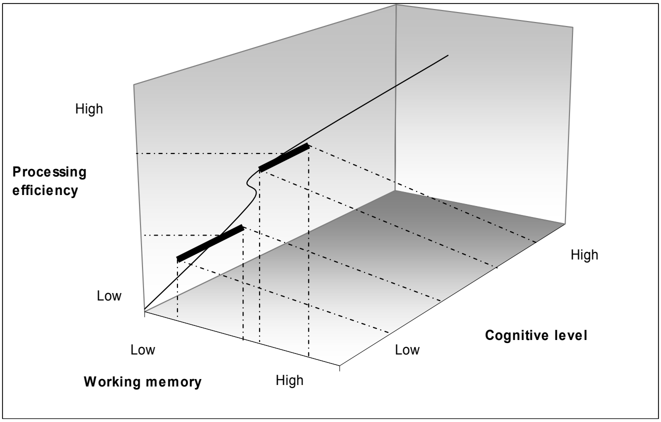

Figure 2. The idealized model of the relations between changes in processing efficiency, working memory, and thought.

Executive function, self-awareness, and self-representation. Zelazo and his colleagues (Frye, Zelazo, & Burack, 1998; Zelazo & Frye, 1998) have recently generated solid empirical evidence about the development of executive control. According to this evidence, children up to the age of three years can represent only a single goal and they cannot shift from one goal to another according to a certain rule. This becomes possible at about the age of five, when children are able to integrate the rules into a higher order rule that specifies when each of the two lower-order rules is to be used. In line with these findings, Band, van der Molen, Overtoon, and Verbaten (2000) showed that a global response control mechanism enabling children to inhibit all bending responses, if needed, is established by the age of 5 years. However, selective inhibition, that is, the ability to selectively inhibit different responses according to differentiated goals continues to develop until adolescence. Thus, during the primary school years executive control becomes differentiated and planful, thereby making action plans available to the thinker.

THE DOMAIN OF QUANTITATIVE THOUGHT

The Architecture of Quantitative Thought

All elements of reality can potentially undergo quantitative transformations. Things aggregate or separate so that they increase, decrease, split, or multiply in space or time for many different reasons. Obviously, perception, representation, and some kind of processing of quantitative information are important for adaptation for most living organisms. As a result, these processes are present in many other animals but humans (Dehaene, 1997). In humans, quantitative thought starts as a very simple perception of small quantities and ends up, under appropriate conditions, in understanding calculus and differential equations.

What are the core processes, the basic operations and rules, and the knowledge involved in quantitative thought? *Subitization* is an example of the core processes involved in this system. Subitization refers to our ability to specify the numerosity of small sets (smaller than 3 or 4 elements) by simply looking at them. The basics of a *mental number line,* which is used to represent, intuitively at the beginning, the quantitative relations between sets seems also to be a core process in this system. The mental number line may be conceived as the mental analogue of an actual line where each number occupies a particular place. This mental lining up of numbers enables the thinker to mentally inspect the line at particular regions of it so that a target number can be compared to other numbers which are smaller (on the left) or bigger (on the right) than it. Its neurological analogue involves several areas of the brain, such as the inferior parietal lobe, which are activated and orchestrated in the sake of representing the quantitative information abstracted by the eyes (or other senses as a matter of fact) from the environment (Dehaene, 1997). At the second layer of organization, quantitative reasoning involves actions enabling the thinker to deal mentally with the various quantitative transformations mentioned above. Prominent among these actions is counting, which enables one to specify the quantity of things that exceed the subitization limits. Piaget (Piaget & Scheminsca, 1952) was the first to clearly spell out how the internalization of actions on objects results in the construction of mental operations, their inter-coordination into structures, and their systematic development with experience. The third level of the organization involves all kinds of factual knowledge about the quantitative aspects of the world. Examples include knowledge about time reading, money values, and rules underlying everyday transactions, and numerical knowledge, such as, the multiplication tables.

The Development of Mathematical Thought

Obviously, there is both an evolutionary and a developmental relation between the three layers in the organization of a specialized domain of thought. Specifically, more basic layers are biologically more constrained, they appeared earlier in evolution, they are more important in functioning in the early stages of development and their early functioning generates the basic material for the construction of processes, skills, and concepts at the higher levels. This developmental linkage between functional levels in the organization of quantitative thought will become apparent in the outline of the development of mathematical thought to be given below (Demetriou, Platsidou, Efklides, Metallidou, & Shayer, 1991).

Infant core mathematics. A few weeks old infants can subitize the numerocity of sets including up to 3 elements and they are capable of distinguishing between different numbers of objects. Xu and Spelke (2000) argue that number representation in infants depends on a mechanism for representing approximate rather than exact numerosity. This mechanism represents information iconically (Wiese, 2003). Moreover, young infants are capable for performing simple arithmetic operations within the subitization limits. For instance, Wynn (1992) found that 4- to 5-month-old infants are capable of calculating the exact result of simple arithmetic operations, such as 1 +

1 = 2. These findings suggest that there is an unlearned core of numerical competence that precedes language or any kind of domain-specific formal training (Starkey, 1992). This core seems to lead to a global mental number line which involves very few numbers at the beginning which are iconically encoded, they are approximately related, and they form the foundations for the proto-quantitative schemes that emerge with the advent of language.

Proto-quantitative schemes. At the age of 2-3 years, children handle representations that are taken as single undifferentiated blocks that stand for familiar objects or concepts and have a transparent relation to them. As a result, relations at this early phase of development are not constructed as such but are intuitively “read out”, so to speak from the representational block. Thus, in regards to number, at the age of two years, children begin to use the sequence of number names. For example, in their daily experience they can imitate the number sequence in rhymes song (Fuson, Richards, & Moser, 1982). Moreover, they posses "proto-quantitative schemes" (Resnick, Bill, & Lesgold, 1992) that enable them to solve simple mathematical tasks that require judgment on the basis of absolute criteria (e.g., "few", "many", "a lot") or to make comparisons on the basis of a single dimension (e.g., “less”, “more”, etc.) (see Gelman & Gallistel, 1978).

Co-ordination of proto-quantitative schemes. At about the age of 3-4 children start to differentiate representations and thus to operate on two of them at the same time. Thus, at this level, co-ordination of proto-quantitative schemes becomes possible. For example, the “increase-decrease scheme”, which is geared on the representation of the number line, is co-ordinated with the basic principles of counting. This coordination enables children to specify quantities and numbers with certain accuracy and to grasp some aspects of cardinal and ordinal number. The steps in the development of cardinality are instructive in this regard. Specifically, according to Fuson and Hall (1983), at the beginning, when counting, children recite the last number with no clear idea that it relates to quantity, but because they realize it is a response an adult would expect. Latter on, they start to understand that the last number of the count relates to the quantity. That is, they start to realize that the last number in the series of numbers spelled out indicates a quantity. Finally, they come to realize that if they are stopped in the middle of a count they can say how many objects they have counted so far and then carry on, which indicates the beginning of integration of order with numerocity. Interestingly, at this stage, children can use pictorial representations of their counting. That is, they can translate their counts into accurate pictures.

Dimensionalization of quantitative schemes. At about the age of 5-6 years, representations or operations on representations are integrated with each other. The result is that proto-concepts evolve into dimensions and operations become ensembles that can be planed in advance. Thus, in the domain of quantitative thinking, the coordination of proto-quantitative schemes leads to actual quantitative judgments and estimations. For example they coordinate the “increase-decrease scheme" mentioned above with basic counting skills and the ensuing concepts of cardinality and ordinality, thereby acquiring a first understanding of number conservation. Moreover, by the end of this phase, cardinality and ordinality are well integrated with each other, so that children can translate order into numerocity and vice-versa. Numerical operations in action can also be applied and tagged to symbols (Griffin, 2004). As a result, children at this age show some understanding of the relations between operations, such as the inverse relationship between addition and subtraction or division and multiplication. Finally, children start to be able to use iconic representations, such as simple marks, to represent objects. This implies a grasp of the relations between multiple aspects of numbers and operations on them, as well as the relations between them and their representations. That is, children carry out procedures in the mind in just the same way as they would operate with tangible objects.

Integration of Quantitative Dimensions. In the next phase, at the age of 7 to 9 years, the representations and mental operations constructed above are integrated into systems that can be revised at will. As a result, thinking becomes analytical and fluid. In the domain of quantitative reasoning proper, mathematical concepts, such as cardinal and ordinal number, can be used as means for the representation and processing of different aspects of reality. This opens the way for the dimensionalization of reality. Thus, at this phase, the child can conceive of properties, such as, length, weight, or area, and operate on their relations. Moreover, simple formal mathematical relations can be processed (e.g., equations, such as 8 ? 3 = 5 and a + 5 = 8, can be solved) (Demetriou et al., 1991). This integration of concepts and operations into systems enables children to shift to more elaborate strategies in numerical problem solving. For example, they become able to abandon the "counting-all" strategy in favor of the "counting on". That is, in problems such as 8 + 5 = ?, they take 8 as their starting point and they go on from there counting another 5 units, instead of counting separately all of the objects to be counted (Krebs, Squire & Bryant, 2003). Moreover, there is evidence that 8-year olds have some implicit knowledge of negative numbers (Borba, & Nunes, 2000), implying the emergence of an abstract, potentially variable-like, conception of number.

Emergence of overarching mathematical constructs. Representations at the age of about 10-12 years are quite complex relative to the representations of the previous phase, because they can integrate multiple dimensions. That is, in all domains, two dimensions with at least two levels each can be represented and operated on. In the domain of quantitative reasoning, proportionality becomes possible initially as an ability to grasp proportional relations that appear obvious (e.g., problems involving numbers which are multiples of one another, such as 2/4 and 4/8). Simple symbolic representations can, moreover, be coordinated in order to specify a general quantity (e.g., equations, such as x=y+3, can be solved when y is specified) (Demetriou et al., 1991; Demetriou & Kyriakides, 2006). As a result, children start to be able to advance proofs of mathematical relations, such as "the sum of two odd numbers is an even number" by developing the proper arguments (Healy & Hoyles, 2000).

Bridging of overarching mathematical constructs. At the next level, at about 13-14 years of age, thinking starts to become emancipated from intuitive supports thereby becoming able to operate strategically on complex problems that require systematic differentiation of relevant from irrelevant information and integration of relevant information according to the current goal. This indicates an explicit understanding that the solution resides in the relations between the components of the problem. As such, this understanding provides a wholistic approach to problems, which enables the thinker to conceive of and explore alternative possible solutions and test them against each other until the best one is selected. This approach to problems enables the thinker both to reduce the problem load when complexity is the major obstacle to solution by appropriately partitioning the goal and operational complexity in manageable sub-goal stucks or fill in gaps in information through inter-relating other well defined information.

Thus, at this level, quantitative thinking can grasp counterintuitive proportional relations (e.g., the child can specify which of the two ratios, 4/5 or 7/8, is bigger) and solve algebraic problems where the unknown can be specified in reference to another, separately specified, construct (specify m given that $m = 3n + 1$ and $n = 4$) (Demetriou et al., 1991; Demetriou & Kyriakides, 2006). Moreover, at this age, visual proof becomes possible in geometry, indicating that adolescents become able to imagine how the triangles can be moved around from one configuration to another (Tall, 1995).

Grids of relational and generalized mathematical concepts. At the next level, at about the age of 15-16 years, the limitation that the components to be integrated are well defined is removed. As a result, adolescents become able to integrate implicitly related structures. For example, they can now specify the value of x when it is known that $x = y + z$ and $x + y + z = 20$ (i.e., 10) or when the equation $L + M + N = L + P + N$ is valid (i.e., when $M = P$). These problems require an abstract conception of number such that it leads to the understanding that any number can be expressed by alternative symbols and that symbols can be reciprocally defined in reference to each other, depending upon the particular relation that connects them. Thus, at this level, number is understood as a variable (Demetriou et al., 1991; Demetriou & Kyriakides, 2006).

Principled mathematics. At the next level, at the age of 16-17, adolescents start to be able to integrate relations at multiple levels and conceive of the underlying principles that interconnect them. Thus, systems of mathematics can be grasped at this level. The formal geometrical concept is constructed from the formal definition, and the properties of the formal object are only those which can be deduced from the definition (Tall, 1995). At this stage students are able to develop logical arguments by themselves and they appreciate the necessity for such arguments. They also understand the difference between definitions, axioms and theorems.

Relations between Mathematical Thought, Processing Efficiency, and Self-Awareness

The outline of development of the various processes given above suggests that there is development everywhere. Speed of processing increases, control of processing becomes more efficient and fast, working memory expands, self-awareness becomes more accurate, refined, and focused, and mathematical thought becomes increasingly more complex, versatile, abstract, and ingenious. How are all of these courses of development interrelated? A series of studies in our laboratory tried to answer this question (Demetriou et al., 1993, 2002)

One of these studies focused on the inter-relations between the three dimensions of processing efficiency, that is, speed and control of processing and working memory, and three domains of reasoning, namely, mathematical thought, which is of our concern here, and verbal and spatial reasoning (Demetriou et al., 2002) In this study, four groups of children and adolescents were involved in a longitudinal design. Specifically, 8-, 10-, 12-, and 14-year old participants at the first testing were examined in three consecutive years in all of these domains. To have indices of processing efficiency in these domains we measured the time needed to read a single familiar word, recognize a numerical digit, or a geometrical figure either under conditions of maximum facilitation, which represents speed of processing, or under conditions of interference, which represents control of processing.

Working memory was measured by tasks addressed to phonological and visuo/spatial short-term storage and the central executive. The phonological STS was addressed by verbal and numerical tasks. Participants were presented with series of words or numbers (two to seven) and they were asked to recall them in the order of their presentation. The visuo/spatial STS was addressed by a task requiring to store and recall shape, position, and orientation of geometric figures. The central executive was addressed by a set of tasks requiring one to combine either verbal with numerical or verbal with visual information at presentation and then recall the one or the other type, according to the instructions.

The reasoning tasks addressed quantitative, verbal, and spatial reasoning. Quantitative reasoning was addressed by two types of tasks. That is, numerical analogies of varying difficulty (e.g., 6 : 12 :: 8 : ?, 6 : 4 :: 9 : ?) and simple algebraic equations requiring to specify the arithmetic operations missing from them (e.g., (2 # 4) @ 2 = 6). Verbal reasoning was addressed by verbal analogies and syllogistic reasoning tasks. Spatial reasoning was addressed by mental rotation and coordination of perspectives tasks.

This is a very complex study that generated a wealth of data presented in detail in a long monograph (Demetriou et al., 2002). Here we will only focus on the relations between mathematical reasoning and the various dimensions of processing efficiency and working memory addressed by the study. Reference to verbal and spatial reasoning will only be made for comparative purposes. To specify these relations, a structural equation model was built where each of the three reasoning domains was regressed on all of these processes. The model was tested separately on the performance attained at each of the three measurement waves both before and after controlling for the effect of age. Figure 3 shows the part of the model that is concerned with mathematical reasoning. It can be seen that a very large part of the variance of performance on the mathematical reasoning tasks was accounted for by the condition of speed of processing (55%, 51%, and 37% at each of the three successive testing waves, respectively) and the central executive of working memory (17%, 49%, and 53% at the three testing waves, respectively). Therefore two aspects of processing and representational efficiency, speed of processing and executive control in working memory, enable one to predict with astonishing accuracy the condition of mathematical reasoning during a very crucial period of development, that is, from middle childhood to middle adolescence.

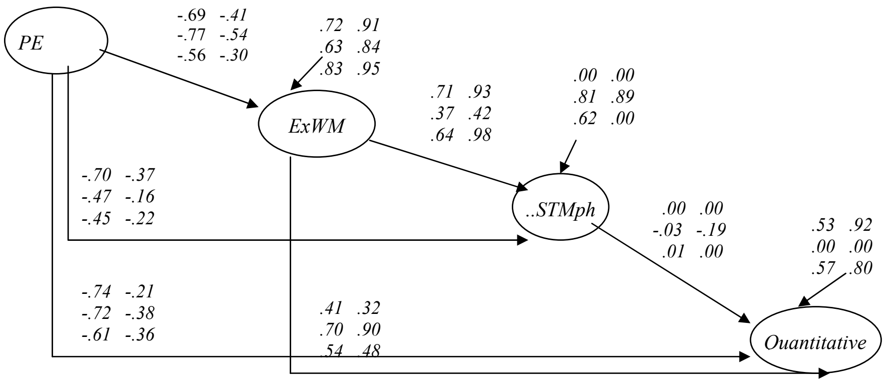

Figure 3. The structural equation model of the relations between processing efficiency, working memory and mathematical thought at three testing waves.

Note: The three coefficients in each domain stand for testing the model on each successive wave. Coefficients in roman letters come from testing the model on row correlations. Coefficients in italics come from testing the model after partialling out the effect of age.

Does the effect of these two dimensions vary with the development? It does in a very interesting way. It can be seen that the effect of speed of processing decreases from the one testing wave to the next (55%, 51%, and 37% of the variance at each of the three successive testing waves, respectively) whereas the effect of the central executive is not systematically related to age (17%, 49%, and 29% of the variance at the three testing waves, respectively). It is clear that these two factors are differentially associated with the development and functioning of mathematical thinking. This assumption is supported by some further findings concerning their effects on mathematical reasoning. Specifically, attention is drawn to the fact that the relation between speed of processing and mathematical reasoning decreased enormously at all three testing waves (4%, 14%, and 13% of the variance at each of the three successive testing waves, respectively) when this relation was purified from the effect of age. The relation of working memory with mathematical reasoning was much less affected and, in fact, it increased in the second testing wave (10%, 81%, and 23% of the variance at each of the three successive testing waves, respectively) as a result of this manipulation

This pattern of relations suggests that processing speed is the developmental factor in regard to the development of both working memory and mathematical thought. That is, changes in processing speed with age open the way for expansion in working memory capacity and the advance of mathematical reasoning to higher levels of functioning. Working memory is the individual differences factor. That is, for each level of processing efficiency, variations in the executive processes of working memory are associated with individual differences in the actual attainment of mathematical reasoning. In other words, the actual state of working memory conditions how much of the available processing potential, as specified by processing speed, is to be actualized into real skills and concepts in mathematical thinking. Moreover, the decrease of the role of speed of processing with age signifies that, with development, other factors come onto the stage, such as interests, motivation, special experiences, etc. We will focus on this part of the picture latter on.

Are the relations described above unique to mathematical thought or are they also present in other domains? Extending the model to include spatial and verbal reasoning suggested that some relations are unique to mathematical thought and some are general. Specifically, the very strong effect of processing efficiency and executive control on working memory is present in all three domains. However, dependence on specialized storage varied across domains. It was high in spatial reasoning and weak in verbal and mathematical reasoning. These differences suggest that the development and functioning of reasoning in different domains is differentially related to the condition of general processing functions and abilities.

Mathematical Thought, Self-awareness and Self-representation

Are people aware of cognitive processes when they do mathematics? Are they accurate in their self-representations of strengths and weaknesses in mathematics? How self-awareness and self-representation in this domain compare to self-representations in other domains of thought? A series of studies in our laboratory (Demetriou & Kazi, 2001, in press) were designed to answer these questions. The studies presented in Demetriou and Kazi (2006) show that self-awareness of cognitive processes appears very early in age, they are concerned with all types of cognitive processes, and that, by the age of 7 years, they are already refined enough to be able to compare specialized processes, such as counting and arithmetic operations in mathematics. Based on these findings, we suggested that self-awareness is actually one of the main constellations of processes that constitute general intelligence or *g*, the other two main constellations being general processing efficiency and general inferential processes. Moreover, these studies showed that at different phases of development, self-awareness reflects the state, form, and dynamics of the processes that are attainable within each phase, weakly and imprecisely at the beginning of the phase and strongly and precisely at the end. Moreover, within each developmental phase, awareness moves from the surface or content-based characteristics of the abilities to be attained in a phase, such as the objects or object characteristics involved, to their procedural and functional characteristics as such. In other words, the development of self-awareness seems to be a recycling process such that within each developmental phase is weak and imprecise and content-centered at the beginning and stronger, more precise, and process centered at the end.

This pattern of changes provides a developmental role to self-awareness. That is, the grasp of awareness at each cycle of development becomes part and parcel of the mental material that will be reorganized into the new inferential structures of the next cycle. That is, reasoning develops as a result of a formalization process that constantly maps onto each other inferential patterns and action schemes within and across domains, thereby generating new management, validation, and reasoning patterns. The grasp of awareness of the processes characteristic of each cycle is a sine qua non condition for the transition to the next cycle because it enables the thinker to redescribe the processes and schemes of the present level into a higher, more efficient and flexible level of representation (Karmiloff-Smith, 1992).

How does mathematical thought fare in this system of self-awareness and self-representation? Research strongly suggests (Demetriou & Kazi, 2001, 2006) that this is probably the domain that is transparent to awareness more than any other domain of thought. Specifically, structural equation modeling suggests that self-evaluation of the performance attained on mathematical tests closely covaries with the actual performance as evaluated by the researcher. In one such study, which involved participants from 11 though 16 years of age, 46% of the variance in self-evaluation of performance on the mathematical tasks addressed to our participants was accounted for by the condition of the actual performance in the domain. Moreover, another respectable 16% was accounted for by cognitive general ability. The corresponding values for the domain of causal, social, and spatial thought were very close very close to zero and 9%, respectively (Demetriou & Kazi, 2001). This privileged position of mathematics in the system of self-awareness reflects, on the one hand, that learning mathematics is an effortful, largely domain-free, enterprise that activates general cognitive ability, it allows self-observation, and it is guided by clear criteria for what is right and wrong. On the other hand, it may be the basis of the fact that people have rather clear attitudes to mathematics. They like them, when they know that they are or successful on them, and they dislike them when they know that they are not.

Processing Efficiency, Intelligence, and School Performance in Mathematics

A series of studies in our laboratory focused on the relations between actual school performance in mathematics and the various aspects of the architecture of the mind discussed here. In one of these studies, which involved participants from 12 to 18, we showed that as much as 71% of the variance of school performance in mathematics (72% in science and 27% in Greek) was accounted for by general cognitive ability. In another study we were able to decompose this effect into the components of general cognitive ability. Specifically, this study included classical measures of intelligence, (i.e., the WISC-R3) and measures of processing efficiency (i.e., speed and control of processing) in addition to cognitive development measures of the five domains of thought mentioned above. This study showed that a large part of the variance of school performance in mathematics was accounted for by these three components of general cognitive ability. That is 25%, 23%, and 6% of this variance was accounted for by processing efficiency, general IQ, and reasoning in the five domains (Demetriou, 2005). Figure 4 shows the model that decomposed performance in three school subjects into these three components.

CONCLUSIONS

This chapter outlined the general architecture of the human mind and specified the place of mathematical thought in this architecture. The general postulate is that the human mind involves both general processes and constraints and specialized domains of reasoning and understanding. Mathematics is one of these specialized domains. Each of theses domains is present from birth and its functioning begins with a limited set of core abilities and processes which enable the newborn to abstract simple domain-specific information quickly and accurately. Subitization and the mental number line are two of the core processes in the domain of mathematical thought. The domain-specific systems then take off and develop as a result of continuous differentiation, reorganization and recombination, and coupling and integration with domain-specific cultural productions and symbol systems. In the domain of mathematics, oral and written numerals, for instance, come to express and eventually mold the core number line. Algebra expresses the relations between multiple representations of the number line.

We have seen that development in the domain of mathematics from birth through the end of adolescence generates increasingly more complex, abstract, and rule-governed concepts, and more versatile, flexible, and planfull problem solving skills.

We have also seen that there is a very close relation between the development of mathematical thought and the development of processing efficiency and executive control processes in working memory. In fact, the relation was so strong that a very large part of the variance in mathematical thought is accounted for by the condition of these two indices of domain-free processes.

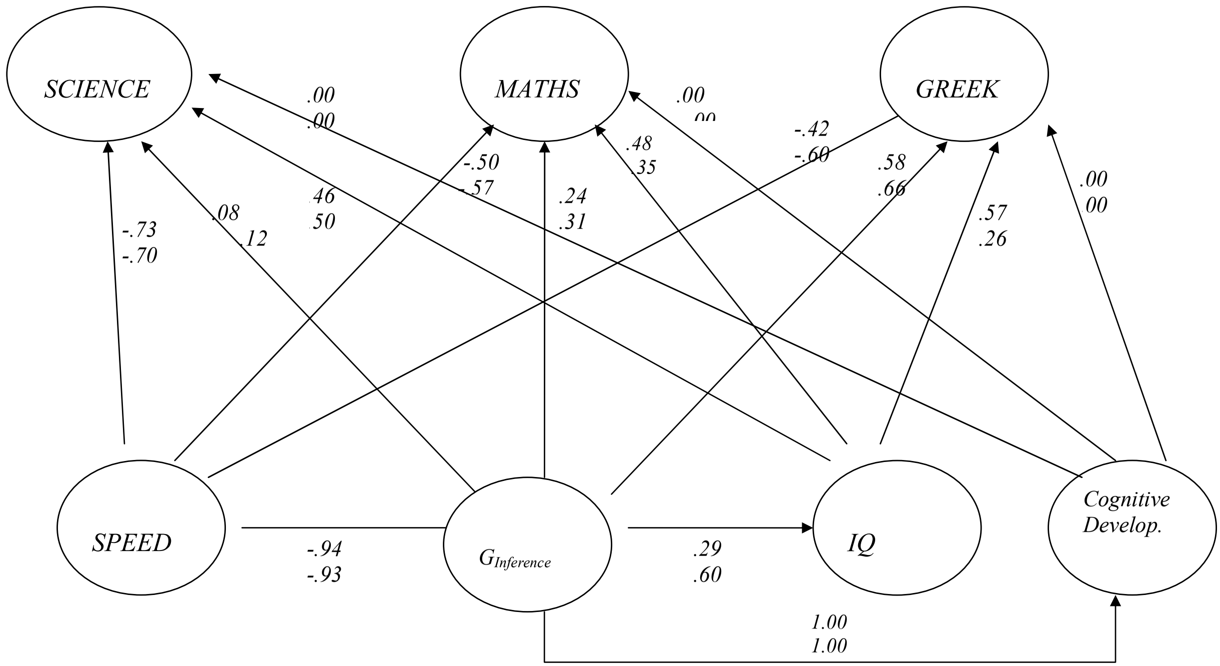

Figure 4: The structural model of the relations between the dimensions of mind and school performance in science, mathematics, and language

The implication of this finding is very clear: The constructions in mathematics at a particular age reflect the available processing and representational resources of the human mind to a large extend. The reader is also reminded that working memory explains individual differences in mathematical reasoning, for each level of processing efficiency at successive ages. In educationally relevant terms, this statement implies that having accurate information about these dimensions of domain free processes would greatly help the teacher decide what is learnable, at the ages concerned, of the various concepts and skills she wants to transmit and how individual children will respond to them. Practically speaking, the teacher must have access to measures of processing efficiency and representational capacity of her students. Modern technology makes these examinations a rather easy business, although special training is needed to address and interpret these examinations.

It is also notable that there is a close relation between self-awareness and self-evaluation, on the one hand, and the development of mathematical thought on the other. This set of domain-general processes complement processing efficiency and working memory as forces shaping the construction of domain-specific concepts and skills. That is, these processes actually enable the thinker to reflect on the relations between concepts and actions thereby reducing or projecting them into another more abstract and integrative system of actions and concepts. In other words, these processes provide the frame for shaping the potentialities afforded by processing and representational efficiency into real mathematical concepts. Therefore, processing efficiency sets the frame for what mental constructions are possible; domains, such as quantitative reasoning, provide the primary material for these potentialities to be materialized; and self-awareness abstracts general reasoning patterns by organizing and tagging domain-specific realizations into supra-domain inference patterns.

Finally, it is highly interesting that the measures of general cognitive processes and abilities were able to account for such a large part of variance in school mathematics. This suggests strongly that our measures of these processes (i.e., speed of processing and executive control in working memory) have captured the backbone of school mathematics, because they constrain the kind of constructions that are possible at successive age phases. Therefore, it is to be concluded that using the measures both for diagnostic and remedial purposes would enable the teacher of mathematics in the real classroom to plan her activities and interventions in ways that would maximize their efficiency.

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PLENARY LECTURE 2

What is first philosophy in mathematics education?

WHAT IS 'FIRST PHILOSOPHY' IN MATHEMATICS EDUCATION?

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INTRODUCTION

Emmanuel Levinas posed the question What is first philosophy? or What is the first philosophy? not as a quest for a fixed foundation for the rest of philosophy, but as a means of uncovering what is presupposed by our humanness in inquiring into philosophy (Levinas 1969). What presuppositions does any philosophizing make? In this talk I want to pose this question for our emergent discipline of mathematics education. What is first philosophy in mathematics education? What area of philosophical inquiry is the *sine qua non* of mathematics education research? Perhaps the deeper question is what philosophical assumptions are unavoidable for us in pursuing our inquiries?

Does it make any sense to ask this question? Are not all branches of philosophy equally near and equally distant from the concerns of mathematics education? I shall argue that this is not the case. Much is presupposed when we embark on research in mathematics education including philosophical assumptions, and it makes sense to ask if there is any priority or ordering among the latter.

Research in mathematics education depends on a number of presuppositions that are not usually articulated. Here is a preliminary listing.

1. We live in a material world, the Earth.[i]
2. The Earth is populated by various living creatures, including human beings, who are able to communicate with each other in a number of ways including, spoken and written language.
3. Human beings are organised into societies and have traditions and cultures.
4. These cultures include various forms of knowledge including mathematical knowledge.
5. Human societies have institutions for education and the passing on of knowledge, including schools and universities.
6. Mathematical knowledge is chosen in most societies and cultures as a central subject to be taught in school (and university).
7. The practices involved in the teaching and learning of mathematics in schools and elsewhere, together with the resultant area of knowledge and research is called mathematics education.

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8. These practices and this specialism are studied in universities and comparable institutions.

In deriving these claims I worked in the direction opposite to this list. Taking mathematics education as the apex of a pyramid – the given practice and field of study in which we work – I worked downwards to the more general and more common-sense assumptions that are tacitly implicated, arriving at last at the first (numbered) assumption. As in mathematical problem solving and proofs (Lakatos 1976), the results of this inquiry look more logical when inverted, starting with the general and leading to the particular. Hence the presentation as numbered above.

The outcome of this inquiry leads to the conclusion that, at the very least, mathematics education presupposes a number of objects:

- A world (the Earth) populated by variety of living creatures
- Human beings living on the Earth
- Communication between humans including spoken and written language
- Human societies with traditions and cultures
- Various forms of knowledge including mathematics
- Social institutions for passing on knowledge
- Mathematics as a central school subject
- Teaching mathematics as an organised social practice.

It can be objected that I am merely stating the obvious, and this is true. But not only are these background presuppositions obvious and true in some naïve sense, they are also relevant. Philosophical discourses sometimes dismiss as irrelevant basic facts such as these, like the existence of humanity, language and social practice and history! For example, to paraphrase an old joke in the philosophy of mathematics:

Platonism (or absolutism) does not require the existence of any persons,

Intuitionism requires the existence of at least one person,

Conventionalism (or fallibilism) requires the existence of at least two persons,

(Social Constructivism requires the existence of their ancestors, too!)

Such systematic blindness can be attributed to the positing of a disembodied individual rational self as the knower in Western philosophy, as the seat of reasoning and knowledge. Perhaps this is our inheritance from the Judeo-Christian intellectual tradition, with its individualistic emphasis. But it is surely important to acknowledge that our thinking being is both embodied and socially and historically situated, especially if our primary concern is the teaching and learning of mathematics and its theoretical basis.

Reflecting on the presuppositions in the list above opens up a number of different areas of philosophy as relevant to mathematics education, listed in Table 1.

Table 1: Objects of reflection and associated areas of philosophy

Philosophizing about	**Area of philosophy**

Mathematics	Philosophy of mathematics
Knowledge, knowing and the branches of knowledge	Epistemology
The world, human beings and other existents	Ontology
Human society and human interrelations	Ethics
The role of mathematics and knowledge in society, including their educational roles	Critical Theory (Frankfurt School)

The specialisms in Table 1 are not the sole areas of philosophy potentially raised by the preceding discussion. For example, teaching and schooling are the concerns of the philosophy of education; the use of language in communication raises the philosophy of language; human society and institutions introduces issues of political and social philosophy. However, to avoid an unnecessarily long list I shall focus on the primary areas of philosophy raised and already regarded as central to issues in mathematics education (epistemology, ontology, ethics) as well as those specifically concerning mathematics (philosophy of mathematics, Critical Theory). The philosophies of education; language and political/social philosophy all have an incidental role to play in mathematics education, but in my view are not prime candidates of themselves for a 'first philosophy'.

I shall consider the claims of each of the areas mentioned in Table 1 to be the 'first philosophy' of mathematics education in turn.

CRITICAL THEORY (FRANKFURT SCHOOL)

The Frankfurt School of social philosophers was founded 1923 during that crisis in Germany following World War 1 which led to the rise of Hitler and Nazism. There they developed Critical Theory drawing on the philosophy of Marx and Hegel, and on the psychoanalytic theory of Freud. Their theoretical standpoint was based on a commitment to egalitarian social justice values. It was a utopian perspective that presupposed the perfectibility of human society, and it viewed the prevailing functionalism as an ideology to be critiqued, as opposed to a rational, given 'truth', as some perceived it.

The Critical Theorists include Adorno, Fromm, Habermas, Horkheimer and Marcuse. One of the main targets of their critique is instrumental reason/rationality. Instrumental reason, of which mathematics is a central part, is the objective form of action or thought which treats its objects simply as a means and not as an end in itself. It focuses on the most efficient or most cost-effective means to achieve a specific end, without reflecting on the value of that end. It is seen as the dominant form of reason within modern capitalist society, leading to the destruction of nature, the rise of fascism and bureaucratic capitalism, and the reduction of human beings to objects of manipulation (Blunden n.d.).

Through its attack on instrumental reason Critical Theory offers a strong critique of the way mathematics is used in society, and this critique can be taken so far as to question the basic assumptions upon which mathematics education rests (Ernest 2009).

Many scholars have applied the insights of Critical Theory to critiquing education, perhaps the best known of whom is Paulo Freire (1972), who in later life included mathematics in his criticism (Freire 1996). But within mathematics education perhaps the best known scholars to develop the ideas of Critical Theory are Ubi D'Ambrosio,[1] Ole Skovsmose (1985, 1994) and Marilyn Frankenstein (1983), in the subfield that is now known as Critical Mathematics Education. D'Ambrosio (2007) offers a trenchant critique of the current state of the world and the roles of mathematics and mathematics education in it, in the following powerful statement.

> It is widely recognized that all the issues affecting society nowadays are universal, and it is common to blame, not without cause, the technological, industrial, military, economic and political complexes as responsible for the growing crises threatening humanity. Survival with dignity is the most universal problem facing mankind.
>
> Mathematics, mathematicians and mathematics educators are deeply involved with all the issues affecting society nowadays. But we learn, through History, that the technological, industrial, military, economic and political complexes have developed thanks to mathematical instruments. And also that mathematics has been relying on these complexes for the material bases for its continuing progress. It is also widely recognized that mathematics is the most universal mode of thought.
>
> Are these two universals conflicting or are they complementary? It is sure that mathematicians and math educators, are concerned with the advancement of the most universal mode of thought, that is, mathematics. But it is also sure that, as human beings, they are equally concerned with the most universal problem facing mankind, that is, survival with dignity.

D'Ambrosio points to the current critical state of society and the contradictory roles of mathematics and mathematics education as both complicit in the problems faced by all and contributors to the potential means of their solution. Thus from the perspective of Critical Theory, the most pressing issue for mathematics education is to contribute to the improvement of the human condition through addressing the universal problem facing humankind, namely survival with dignity. This diagnosis gains extra force through having been offered right before the present worldwide economic meltdown.

Critical Theory as a philosophy enters into mathematics education by insisting on its responsibility to offer values-based criticisms of society, mathematics and the social practices of mathematics education, most notably the teaching and learning of mathematics. This raises the following question. In analyzing the strengths and

[1] It may be the case that Freire and D'Ambrosio were directly influenced by Marx, as much as by the Critical Theory of the Frankfurt School, and Frankenstein is also a follower of Freire.

weaknesses of mathematics, society, mathematics education and their complex inter-relationships, is Critical Theory serving as a first philosophy? Important as the critiques it offers are, the answer to this question must be in the negative. For Critical Theory in its applications in mathematics education draws on a number of other philosophical areas and assumptions. These include ethics or values (as the base for the critique), epistemology and philosophy of mathematics (in the critique of mathematics and knowledge in general), and social philosophy and theory (for the social and educational critique). In some sense therefore Critical Theory serves rather as a last philosophy for mathematics education, because it brings together and combines many if not all other areas of relevant philosophy to analyze and underpin mathematics education as an area of knowledge, especially with regard to its role in society. It is a philosophical basis for the politics of mathematics education, combining social philosophy, ethical and epistemological dimensions of mathematics education.

THE PHILOSOPHY OF MATHEMATICS

Perhaps the most frequently considered area of philosophy in mathematics education is the philosophy of mathematics. After all, is not understanding the nature of the subject of mathematics that we teach and its philosophical underpinnings necessary both for teaching the subject thoughtfully and for research in mathematics education? Traditionally the philosophy of mathematics has been concerned with two main questions. How is mathematical knowledge justified, and what are the objects of mathematics? These are both relevant for mathematics education. However, as is well known, a recent 'maverick' tradition ((Kitcher and Aspray 1988) in philosophy of mathematics has not only challenged the traditional answers to these questions, but has also challenged the assumption that these questions are the sole or main concerns of the philosophy of mathematics. Elsewhere I have argued that the philosophy of mathematics should account for more issues than epistemology and ontology in mathematics (Ernest 1998). My claim is that an adequate philosophy of mathematics should account for:

1. **Epistemology**: Mathematical knowledge; its character, genesis and justification, with special attention to the role of proof,
2. **Theories**: Mathematical theories, both constructive and structural: their character and development, and issues of appraisal and evaluation,
3. **Ontology**: The objects of mathematics: their character, origins and relationship with the language of mathematics, the issue of Platonism,
4. **Methodology and History:** Mathematical practice: its character, and the mathematical activities of mathematicians, in the present and past,
5. **Applications and Values**: Applications of mathematics; its relationship with science, technology, other areas of knowledge and values,
6. **Individual Knowledge and Learning**: The learning of mathematics: its character and role in the onward transmission of mathematical knowledge, and in the creativity of individual mathematicians.

Reforming philosophy of mathematics to meet these broadened objectives, at the very least would provide an underpinning for the central focus of mathematics education, namely the teaching and learning of mathematics, especially through the last theme (no. 6).

As well as questioning its scope, the 'maverick' tradition in the philosophy of mathematics has challenged the traditional absolutist accounts of mathematical knowledge as certain, absolute, superhuman and incorrigible. The alternative fallibilist, humanist or social constructivist accounts of mathematical knowledge as fallible and humanly invented have underpinned or resonated with many of the most controversial theoretical developments in mathematics education. Over that past quarter century radical constructivism, social constructivism, enactivism, socio-cultural theory, postmodernism, poststructuralism, critical mathematics education, and even the problem solving and investigations movements in mathematical pedagogy have drawn on these newer philosophies of mathematics.

However it is not just these newer developments in philosophy of mathematics that are claimed to underpin mathematics education. The educational relevance of the philosophy of mathematics as a whole has been argued more widely.

As René Thom said:

> In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. (Thom 1971: 204)

As Reuben Hersh elaborated

> The issue, then, is not, What is the best way to teach? but, What is mathematics really all about? ... Controversies about...teaching cannot be resolved without confronting problems about the nature of mathematics. (Hersh 1979: 34)

In such discussions of the philosophies of mathematics embedded in the mathematics curriculum there is a shift from referring to formal academic philosophies of mathematics as discussed by professional philosophers to informal philosophies of mathematics, which are perhaps better described as images of mathematics.

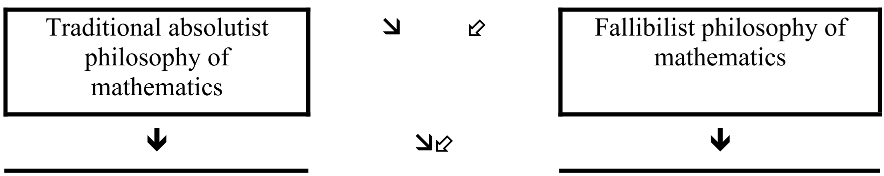
There is an analogy here with Tall and Vinner's (1981) distinction between concept definition and concept image. A concept definition is a formal, explicit and publicly justifiable description of the meaning of the concept, in written form (or its psychological correlate). A concept image is a cluster of connotations concerning the concept, which might include visual and other representations and linked ideas and properties as well as associated applications. It is a much more tenuous and subjective cluster of meanings in a variety of representational modes.

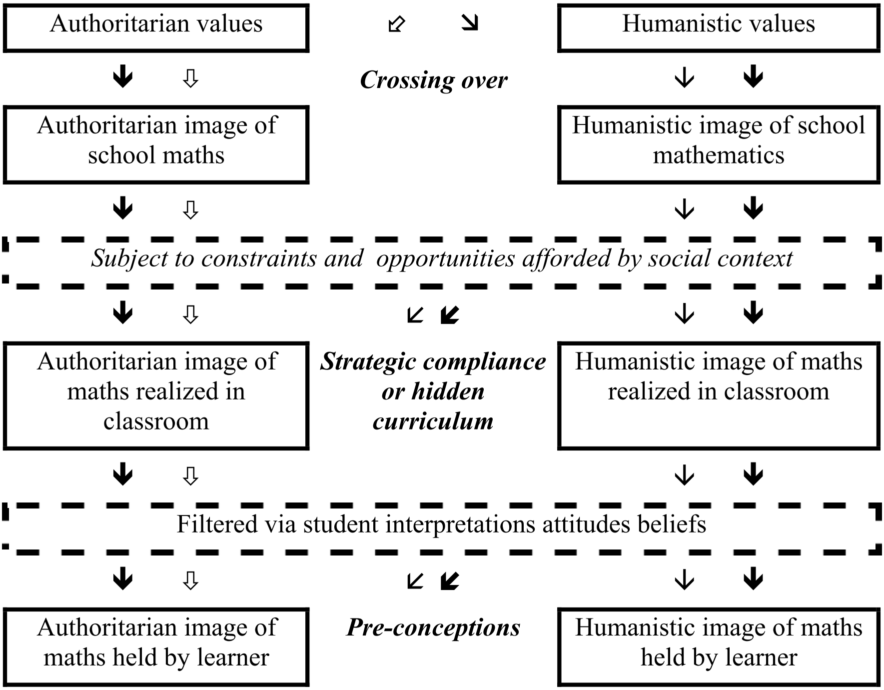
Likewise, an image of mathematics can include a wide range of representations and associations derived from sources including philosophy and written accounts of the nature of mathematics, but also including representations from the media, classroom presentations and parent, peer or other sourced narratives about mathematics. Personal images of maths can utilize mental pictures, including visual, verbal, and narrative representations, originating from past experiences, social talk, etc., and include cognitive, affective and behavioural dimensions, including beliefs and belief systems.

A great deal of research has been conducted into individuals' personal images, beliefs and attitudes about mathematics over the past twenty five years (Leder *et al.* 2003, McLeod 1992). This makes it a major growth area in research in mathematics education. But while images of mathematics in some loose sense are the counterparts of philosophies of mathematics on the personal or social level, there is a large philosophical gap between the two. The significance of images of mathematics in mathematics education research does not establish the primacy of the philosophy of mathematics in underpinnings mathematics education. I have argued elsewhere that there is no direct correspondence between philosophies of mathematics and the images of mathematics of school teachers, in the mathematics curriculum, in classroom representations of mathematics, or even student images of mathematics (Ernest 1995, 2008). To put it in simplified terms, whether a teacher has a traditional absolutist or fallibilist philosophy of mathematics may not correlate as well with their personal image of school maths so much as their values, whether authoritarian or humanistic. Furthermore, the enactment of this planned image of school maths in the classroom as a realized image is subject to the constraints and opportunities afforded by the social context of schooling. Lastly, the image of mathematics as held by the learner is only the result of this realized image after it has been filtered through the student's interpretations, pre-existing attitudes and beliefs. At each stage in this process there are swaying influences at work, such as a teacher's strategic compliance to social pressures in their realized classroom image of mathematics, the influence of the hidden curriculum of schooling on this realized image, or the impact of student preconceptions in their interpretation of classroom experiences in developing their own image of mathematics. These processes are all illustrated in Figure 1.

Figure 1 illustrates how an absolutist philosophy of maths combined with humanistic values can give rise to a teacher's humanistic image of school mathematics via 'crossing over'. Thus a deep commitment to the ideals of progressive mathematics education can, of course, co-exist with a belief in the objectivity and neutrality of mathematics. Likewise crossing over the opposite way would enable a fallibilist philosophy of mathematics to combine with authoritarian values to result in an authoritarian image of school mathematics.

Fig. 1: Philosophies of mathematics, values and images of mathematics





The overall point I am making here is that there is no logical implication leading directly from a teacher's philosophy of mathematics to their intended classroom image of mathematics, let alone to what images are realized in the classroom or are experienced by students. So although the philosophy of mathematics remains a vital issue for the teaching and learning of mathematics, this logical gap suggests that it is not the 'first philosophy' for mathematics education. There is no immediacy in the implications or influences of the philosophy of mathematics on the teaching and learning of mathematics.

I would argue that the influence of philosophies of mathematics on mathematics education are profound, but it is mediated by very complex processes that it is hard if not impossible to model, although it remains a fruitful problem with which to grapple for researchers.

EPISTEMOLOGY

What are the claims of epistemology to be the 'first philosophy' of mathematics education? There are three main planks to such a claim. First, epistemology is central because the teaching and learning of mathematics and mathematics education research are all about knowing and knowledge. Second, learning theories, i.e., theories of individual's learning are central both to the theory and practice of mathematics education. Third, the research methods and methodologies we use in our inquiries in mathematics education are all underpinned by epistemology. So this is potentially a real candidate for 'first philosophy'.

Epistemology is the theory of knowledge and concerns both the structure and character of knowledge, as well as the means for its justification. Mathematics education is an interdisciplinary field situated at the confluence of mathematics, the social sciences, and the humanities. Mathematical knowledge is justified at the highest levels by its casting into explicit and precise theories, with clear, step by step proofs of all of its claims and theorems. The social sciences draw upon a number of research traditions and paradigms for its knowledge claims, which I shall consider subsequently (point 3 above). Likewise, the humanities uses a range of methodologies including text based, conceptual and theoretical methods. All of these are used to a greater or lesser extent in mathematics education research, and are aspects of its methodology and epistemology.

The methodology of mathematics is a dimension of epistemology although it can also be classified as belonging to the philosophy of mathematics. It is the systematic study or theory of the methods of mathematics, and can be either prescriptive or descriptive. Prescriptive methodology offers a systematic account of the methods that can be used in making mathematics, including concept definition, conjecture and theory formulation, and proof construction. In contrast, descriptive methodology provides an account of the methods that have been used by mathematicians in mathematical practice.

Once there were high hopes that mathematical logic would fulfil the function of a prescriptive methodology of mathematics. However, mathematical logic does not capture the proof methods used by mathematicians in practice, not even the formulations of natural deduction (Kneebone 1963, Prawitz 1965) or tableaux systems (Bell and Machover 1977) which were intended to serve this purpose. Furthermore, the methodologies used in mathematical practice go well beyond proving, and include making definitions, making and testing conjectures, analogical reasoning, problem solving, theory construction, and so on (Corfield 2003).

In the history of mathematics, the distinction between prescriptive or descriptive methodology of mathematics has not been clear cut. Scholars from Pappus of Alexandria from around 320 CE (Boyer 1989) onwards have described methods used (Pappus distinguishes the methods of analysis and synthesis for solving geometrical problems), and used them to make generalized claims about potentially fruitful methodological approaches for mathematical research. One of the earliest known works in algebra, from about 825 CE, is al-Khowarizmi's *Al-jebr …*, which translates as [*the method of*] *the reunion of broken parts*. This and subsequent episodes in the history of algebra, as well as other areas of mathematics, illustrate how the heuristic mathematical methods addressed by the prescriptive methodology of mathematics become mechanized into algorithms or decision-procedures, which can be applied routinely. In both the history and the psychology of learning mathematics methods that are first encountered or invented as heuristic problem solving strategies subsequently become routinized and made specific, and thus turn into mechanical algorithms. This suggests a possible reason why mathematical methodology has never developed into a significant discipline in its own right, for if heuristics turn into algorithms, then what once were methodological rules lose this character as mathematics advances.

Of course not all heuristics suffer this fate. Following on from the landmark books *Rules for the direction of the mind* of Descartes (1628) to Polya's (1945) *How to solve it*, there is a myriad of publications which, although not helpful to research mathematicians, have become very prominent in mathematics education. Such treatments of heuristics for mathematical discovery have prompted the problem solving and investigations movements in mathematics education.

A philosopher who tried to develop the methodology of mathematics is Imre Lakatos. But his *Proofs and Refutations* (Lakatos 1976) has multiple functions: philosophical, historical, methodological, and pedagogical (Ernest 1998), and while it is of immense significance and impact, its diversity of approach does not serve to further the claims of epistemology (via methodology) to be the 'first philosophy' for mathematics education.

The second plank of the claim of epistemology concerns learning theories. This has been one of the most hotly debated areas within the mathematics education community for the past twenty five years. Various forms of constructivism from Piaget's theory of learning, through radical constructivism, to enactivism, social constructivism and socio-cultural theory have been hotly debated. As Ernst von Glasersfeld (1983: 41). said "To introduce epistemological considerations into a discussion of education has always been dynamite"

There is no doubt that the insights derived from this range of learning theories have deepened and enriched both research in mathematics education and the practices of the teaching and learning of mathematics. Some of the new areas of attention foregrounded include:

- importance of the identification of learner errors and alternative conceptions;
- attentiveness to the learner's previous learning, constructions and perceptions as a whole, including both cognitive and affective domains;
- a new focus on bodily movements and gestures in learning;
- attention to root metaphors as the basal grounds of learners' meanings and understanding;
- the importance of all aspects of the social context and of interpersonal relations;
- the value of an ethnographic view of the teaching and learning of mathematics as a social practice;
- the role of language, texts and semiosis in the teaching and learning of mathematics;

- the problematic nature of mathematical knowledge as a whole, as well as the learner's subjective knowledge,
- the fragility of all research methodologies.

This illustrates how the learning theories central to many of our research debates, have powerful implications for the practice of teaching and learning mathematics.

The third plank of the claims of epistemology as 'first philosophy' concerns research methodology. Since the means of validating knowledge is the central concern of epistemology, the research methods and methodologies we use in our inquiries are all part of epistemology. Following Habermas (1971) three main clusters of research methodologies, known as research paradigms, have been distinguished,. These are the scientific, interpretative and critical theoretic research paradigm. These are briefly summarised in Table 2 in terms of some of their salient theoretical features.

Table 2: Simplified summary and comparison of the three main research paradigms

PARADIGM	SCIENTIFIC	INTERPRETATIVE	CRITICAL
INTEREST	Prediction and control of the material world	To understand and make-sense of the human world	Social justice, emancipation
FOCUS	Validation of laws and theories	Exploration of meanings and actors understandings	Reform of social practices and institutions
WORLDVIEW	Scientific worldview	World of human meanings	Social world and its power relations
ONTOLOGY	Objects in physical space	Subjective or intersubjective reality	Persons in society within institutions
VIEW OF KNOWLEDGE	Objective knowledge	Personal or socially constructed knowledge	Objective multi-perspectival knowledge of all participants
METHODOLOGY	Experimental seeking general laws	Case studies of particular contexts	Critical action research in/on social institutions
METHODS	Mainly quantitative using predetermined instruments and categories	Mainly qualitative focusing on textual & spoken responses from individuals	Collection of all data types from overall context & all participants

INTENDED OUTCOME	Objective knowledge & truth in form of laws	Illuminative and illustrative case studies	Social reform, social justice intervention

Table 2 shows that research paradigms include many aspects of epistemology beyond the warranting of knowledge. They also include the interest behind the knowledge enquiry, the focus of the investigation, the assumed worldview and ontology, the view of knowledge, the methodology and methods used in the inquiry, and the overall intended outcome. Such a broad mission places research methodology at the abstract paradigm level in the social sciences rather than in pure epistemology as philosophers understand it.

Evidently while epistemology has a basis for its claim to be 'first philosophy'. However, it draws in many assumptions from other areas of knowledge and philosophy too, making such a claim far from clear cut.

ONTOLOGY

In initiating this inquiry I asked (and tentatively answered) the question, what are the fundamental objects whose existence is presupposed by any work whatsoever in our field? In effect, this is a foray into ontology, the science or metaphysics of being.

Echoing this focus, one of the dimensions presupposed in a research paradigm as illustrated in Table 2 is that of the underlying ontology. What kinds of objects do we take for granted as populating the universe we are working in, and what worldview is associated with them? In Table 2 three main perspectives are contrasted and each prioritizes a different aspect of our shared, lived-in reality. They are the:

1. Scientific world comprising material objects in physical space;
2. World of subjective and intersubjective reality comprising human meanings;
3. Social world and its power relations, comprising persons in interrelationships and within institutions.

Each constitutes a different ontology, a different theory of existents and existence. The underlying ontology constitutes a profound set of assumptions for mathematics education. More generally, in considering the claims of ontology as 'first philosophy' a number of fundamental questions are raised. What are mathematical objects? What are educational and mathematics educational objects? What overall theory of existents and existence are we assuming in our research, either overtly or covertly? Without presupposing essentialist answers to these questions, what is the nature or character of the objects?

The primary objects of study in mathematics education are human beings and their activities and relationships. Ontology poses the question: what is a human being? Philosophically this is a very fundamental question. Answering it with respect to our field of study brings up issues of identity, subjectivity, agency, and the learning career of students, our own development, and indeed that of all persons. What is a *human* being, and what is human *being*? Currently identity and its historical trajectory, in particular the learning career of students, is a growing research area in our field (Boaler 2002). This new focus attempts to get beyond the scrutiny of separate elements of learning and to consider the ‘nature’ of the learner as a whole.

Heidegger (1962) develops a complex metaphysics of being based on the idea that our understanding of ourselves and our world presupposes something that cannot be fully articulated, a kind of knowing-how rather than a knowing-that. At the deepest level such knowing is embodied in our social skills, how we interact with and share experiences and practices with others, rather than in our concepts, beliefs, and values. Heidegger argues that it is these cultural practices that make our lives meaningful and give us our identities. Although socio-cultural theory already emphasizes the import and tacit nature of the social practices in which all of our being, including the teaching and learning of mathematics, is enacted, there is potentially much more to be gained from the applications of Heidegger’s thinking to research in mathematics education.

Such considerations suggest that metaphysics, and in particular ontology, has potentially strong claims to be a first philosophy for mathematics education, but that such claims are not based on a fully worked out basis, and so are perhaps left in abeyance for the present.

ETHICS

The last candidate for a ‘first philosophy’ of mathematics education to be considered is ethics. This enters into mathematics education research in four ways. First of all, there is a vital need to be ethical in our research. As responsible and ethical professionals, it is incumbent on us at the very least to ensure that our research is based on the informed consent of any human participants, does not cause them any harm or detriment, and that we respect the confidentiality and non-identifiability of all individuals or institutions. Any research that does not fully conform to ethical standards is not only ethically flawed, but its claims to add to the sum of knowledge must be viewed as suspect. Unlike stolen money which is just as good in the shops as honest money, unethically derived knowledge is epistemologically as well as ethically tainted.

Second, as educational researchers we are participating in the great, age-old human conversation, which sustains and extends our common knowledge heritage. By sharing our thoughts, our findings both informally and formally, and through our publications, we are part of the public conversation from which others benefit. This great conversation, as Michael Oakshott called it, is not a means to an end, but an end in itself, and the conversation is inescapably moral and ethical. To participate you must value the contributions of others. You must listen with respect and humility, and when you have developed a voice, you contribute to the conversation, knowing it is much greater than you. The tacit values implied by participation are: valuing and respecting the voices of others, past and present; valuing the young who will get the chance to participate; not taking too seriously the trappings of power, earthly prizes, ego gratification, these will all be gone and forgotten as the great conversation rolls on; striving for excellence and high standards in oneself and others - both to be worthy of the great conversation, and to protect it; recognising that all human beings are part of this transcendent shared enterprise, and that all members of the human family deserve concern and respect. Mathematics education is one of the strands in the great conversation and we in its research community can be proud that our efforts and those of our predecessors have created and swelled one of the strands of this great shared enterprise.

Third, it is a self-evident truth that as human beings we are irreducibly social creatures. Humans as a species are essentially interdependent. We emerge into the world after our initial biological development within our mother's bodies. We must experience love and care from others in our early years to become fully functioning human beings. We must acquire language[2] and acceptable behaviour with others to participate in social life and practices. Without such skills we cannot survive and further the human race. Our species depends for its very survival on our ethical and cooperative behaviour with regard to our fellow humans.[3] In its highest form this dependency is expressed as the principle of reciprocity, embodied in all ethical belief systems and world religions as the Golden Rule: 'Do unto others as you would have them do unto you' (Wikipedia 2009). One source for this is the awareness that we are all the same but different (to paraphrase the title of Quadling's 1969 book on equivalence relations) and but for luck and contingency you and I as individuals could be in each other's situation.

Fourth and last, but far from least, prior to all such reflections, according to Levinas, we owe a debt to others that precedes and goes beyond reasons, decisions, and our thought processes. It even precedes any attempt to understand others. Levinas maintains that our subjectivity is formed in and through our subjected-ness to the other, arguing that subjectivity is primordially ethical and not theoretical. That is to say, our responsibility for the other is not a derivative feature of our subjectivity; instead, this obligation provides the foundation for our subjective being-in-the-world by giving it a meaningful direction and orientation (Levinas 1981). This leads to Levinas' thesis of 'ethics as first philosophy', meaning that the traditional philosophical pursuit of knowledge is but a secondary feature of a more basic ethical duty to the other (Levinas 1969).

Thus one can say that as social creatures our very nature presupposes the ethics of interpersonal encounters, even before they occur, and even before we form or reflect on our practices, let alone our philosophies. This is why Levinas asserts that ethics is the 'first philosophy' presupposed by any area of activity, experience or knowledge,

[2] By language I include all complex systems of human communication such as signing for the hearing impaired.

[3] I am not so idealistic or unrealistic so as to ignore the recurring presence of competition and contestation in human affairs at all levels. However, my claim is that human cooperation, mutual help and care must exceed competition, contestation and antagonism or else as a species we would have perished in the past or will perish in the future.

including mathematics education. If we accept his reasoning, then our quest is at an end. Ethics is the 'first philosophy' of mathematics education. It is the prerequisite for any human activity. It precedes Critical Theory, the philosophy of mathematics, epistemology, and ontology. It comes first because it is implicated and presupposed by our very human nature, and precedes any theorizing or philosophizing. Following Levinas' arguments, ethics trumps the other candidates for 'first philosophy' for mathematics education by preceding all other philosophies.

CONCLUSION

I have arrived at the endpoint of my quest, having identified ethics as the 'first philosophy' of mathematics education, having begun with an ethical philosophy (Critical Theory) and ended with ethics. This, on the one hand, is very satisfactory, because, as the argument runs, our social nature as animals precedes any reasoning and arguments, and we have an inbuilt necessity to respect, care for and love one another. On the other hand, this is very unsatisfactory, because ethics as the 'first philosophy' of mathematics education tells us nothing specific about mathematics, mathematics education or the teaching and learning of mathematics, other than to respect and value our peers, students and indeed all peoples. This conclusion is not quite empty of significance for mathematics education research, because it has immediate implications for learning theory. Acknowledging the primordially social character of human beings weakens the claims of theories like simple constructivism, radical constructivism, and enactivism that are expressed in individualistic terms. If such theories do not take into account our irreducibly social character, there is a strong case that can be made against them. Thus, for example, radical constructivism's account of the learner as a cognitive alien making sense of a world of experience, and constructing other persons as regularities in that world, in effect denies the social and ethical foundation of human being (Ernest 1994).

Although my conclusion that ethics is the 'first philosophy' of mathematics education has significance, there is a sense in which whole quest has been a conceit, based on a literary artifice. Like so many quests it is the journey rather than the destination that is important. This inquiry has enabled me to review the contributions of the most relevant branches of philosophy for mathematics education. I have illustrated a number of ways in which these branches are relevant for research in mathematics education. They do not always impact directly on the practicalities of our research, but they shine a light on it from an unexpected exterior space, enabling us to rethink and re-evaluate some of the taken-for-granted commonplaces of our practices.

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[i] I appreciate that the use of the pronoun 'we' presupposes that there is a community of speakers/listeners. In addition, the simple fact or act of language usage in this text presupposes that this is a community of linguistic speakers/listeners who are being addressed. Although these occur explicitly as assumption no. 2, their tacit implication from the start of my exposition in assumption no. 1 shows that any ordering is arbitrary. It is worth noting that these assumptions, whether acknowledged or not, are implicit in any human discussions or texts (and there are no other kinds), philosophical or otherwise, irrespective of whether they are explicitly admitted or not. Thus, for example, radical constructivism, which begins by positing a world and sentient beings acting in/on that world, and building up knowledge from such actions and reflection, already by its formulation presupposes that there is a community of human language users in the physical world (the Earth) that are in communication with each other. Of course a radical constructivist rejoinder is that their

theory of learning describes a prelinguistic activity, and it is only the after-the-fact theoretical description that makes the assumptions I have detailed.

MATHEMATICS EDUCATION AND VALUES

A reaction to Paul Ernest's plenary lecture

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I would like to start my reaction to Paul's paper by relating to the very end of his paper, namely, to his claim that ethics should be "the first philosophy of mathematics education." Personally, I am very grateful to Paul for that conclusion, even if I do not exactly understand it. I hope that, when expressing this view, I also speak in the name of other PME members who think alike. The title of the paper "what is the first philosophy in mathematics education?" is undoubtedly a call to discuss foundations. This has been a typical call about various domains of inquiry after these domains were practically established. A desire and also an urgent need rose to establish them also formally. One of the most famous foundational movements was the foundation of mathematics movement within which Paul worked on his Master thesis, as well as his Ph.D. thesis. In the beginning of his paper, Paul speaks about tacit assumptions of the domain of mathematics education, for instance, that "human beings live on Earth." Some of these assumptions are so obvious that no wonder they remain tacit. Why should we waste time on obvious claims? Other assumptions may seem obvious, but a deeper look at them reveals that they are not obvious at all. These assumptions, very often, relate to the meaning of notions involved in the discourse. Since Paul dragged us into a foundational discussion, I would like to spend some time discussing the meaning of two notions related to Paul's paper. The first one is "mathematics education" and the second is "the philosophy of mathematics education." In the domain of mathematics education what I am doing now can be considered as a part of meaning negotiation. In the domain of philosophy, it can be considered as explication. For the notion "mathematics education" I can suggest at least three meanings: 1. Mathematics education as an activity with which teachers and students are involved in mathematics classes. 2. Mathematics education as a domain which relates to mathematics education as an activity. Here we can distinguish at least two meanings: 2a. A research discipline which investigates all aspects involved in the learning and the teaching of mathematics (especially, mathematical aspects, cognitive aspects, emotional aspects, sociological aspects, political aspects etc.). 2b. A domain in which people recommend ways to improve and to enrich the activity of mathematics education where these recommendations are based on reflections or on personal experience of the people involved.

When Paul claims that ethics should be "the first philosophy of mathematics education," which of the above meaning is relevant to his claim? Before discussing an answer to this question I would like to clarify the meaning of the second central notion related to Paul's paper which is "the philosophy of mathematics education."

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 43-46. Thessaloniki, Greece: PME.

Here, I am a little bit confused. I am confused because in the title of the talk Paul speaks about "the first philosophy **in** mathematics education," whereas in the paper itself he uses the notions "first philosophy **for** mathematics education" and "first philosophy **of** mathematics education." Do all these three slightly different notions have the same meaning or do they have different meaning? (Sometimes, notions which are only slightly different from the vocabulary point of view are dramatically different in meaning). Since only Paul can answer this question and when I prepared my reaction, there was no time to discuss it with Paul, I would like to assume for the time being that the above three slightly different notions have the same meaning and I will try to explicate the notion "the philosophy of mathematics education."

So, what is philosophy of education? When I was a mathematics student I took some courses in philosophy. One of them was topics in philosophy. That course did not have any syllabus. The students were asked, in the beginning of the class, to raise any question in philosophy or about philosophy and the teacher, who was a famous philosopher, tried to answer it in short. One morning, one of the students asked: Sir, what is philosophy? The teacher's answer was the following: "Look," he said to the student, "if I give you an answer to this question, then you and all the other students in the audience won't have any reason to continue their studies in the department of philosophy. As a result of this I and my colleagues at the department will become unemployed. So, my answer to you is: continue your studies at the department of philosophy at least for three additional years and then you will find an answer to your question." A similar approach was leading Courant and Robbins (1948) in their famous book: What is Mathematics? They invited the reader to study some topics in mathematics which they presented in a relatively friendly way. Thus, my answer to the above question: "what is the philosophy of mathematics education?" can be similar to the answers given by my philosophy teacher and by Courant and Robbins: Read Paul Ernest's book (1995), the Philosophy of Mathematics Education, as well as several issues of the journal of the philosophy of mathematics education, and you will figure out what the philosophy of mathematics education is. By the way, the community of mathematics education should be grateful to Paul for both the book, a colossal product of work and thought, and the journal which gives to so many of us the opportunity to be reflective about our work in mathematics education. However, since I am not a professional philosopher, neither a professional mathematician, I will not follow the above approach to answer the question what is the philosophy of mathematics education. I will rather try to answer the following question: what can a person who has never heard the notion "the philosophy of mathematics education" understand by it when he hears it for the first time? Why do I replace the original question by another one? My reason for doing it is that the notion the philosophy of mathematics education is a relatively new notion. (If I am not mistaken, Paul was the first one who introduced it). Thus when a new notion is introduced to a certain intellectual community, it immediately evokes meaning (or, if you wish a concept image) which is formed by the circle of associations each one of us has in connection with the new notion. Since my time is limited I cannot elaborate on this in length. I will only say that my first association with the notion "philosophy of mathematics" was "philosophy of science." And what is the philosophy of science? The on-line Encyclopaedia Britannica claims that the philosophy of science is "a branch of philosophy that attempts to elucidate the nature of scientific inquiry." The free encyclopaedia, Wikipedia, says that "the philosophy of science is concerned with the assumptions, foundations, and implications of science." In my own words I would say that philosophy of science deals with all kinds of principles which lead scientists in their work. If we agree on that, and we also agree to form the meaning of the philosophy of mathematics education accordingly, then the philosophy of mathematics education should be concerned with the principles which lead the work done within the domain of mathematics education. But mathematics education in what sense? Paul has dragged us via brilliant philosophical analysis to the conclusion that ethics should be the first philosophy of mathematics education. Being a practical mathematics educator, I would like to undress Paul's analysis from its philosophical robe, although I do understand and sympathize with the importance of being philosophical, as well as with the importance of being Ernest (here I refer to Oscar Wilde's famous play). Paul borrowed the notion of "first philosophy" from Levinas, who was a French Jewish obscure philosopher. To be obscure is quite often an advantage if you are a philosopher. Another famous obscure philosopher is Wittgenstein. Surprisingly enough, the motto of Wittgenstein's most famous book (1922), Tractatus Logico-Philosophicus is (in a free translation from German): All that Man knows, which is not vague or foggy, can be said in three words. It is pity that the motto does not tell us these three words. However, we can consider it as a call for clarity. Coming back to the practicality which I want to represent, my request to authors of obscure texts is: Can you say it in simple words, please? The request here is to Levinas and not to Paul, who was quite clear through his entire presentation. In the simple words of Paul, Levinas' thesis of ethics means that traditional philosophical pursuit of knowledge is but secondary feature of more ethical duty to the other. Isn't this equivalent to the claim that acting ethically is more important than anything else? Isn't this equivalent to Levinas' claim, formulated by Paul, that "our responsibility for the other is not a derivative feature of our subjectivity; instead, this obligation provides the foundation for our subjective being-in -the-world by giving it meaningful direction and orientation"? And what does it mean to act ethically? Paul recommends us the Golden Rule as a key principle: "Do unto others as you would have them do unto you," and then sends us to the Wikipedia as a reference. The related article in Wikipedia mentions that many religions and cultures recommend ethical principles which are quite similar to the principle of the Golden Rule. The Jewish tradition, for instance, recommends the rule: That which is hateful to you do not do to your fellow. This rule is related in the Jewish tradition to a famous Talmudic sage, and no doubt that Levinas, being a Talmudic scholar, was strongly influenced by this sage. So, if the Golden Rule is the bottom line of Levinas' philosophy in simple words, why does he need philosophy in order to reach it, wheras it can be easily reached within other frameworks of thought? The answer is very simple. Levinas is a professional philosopher whose aim is to establish philosophically any ethical claim. However, the practical mathematics educator does not have to follow this line of thought. We can adopt the Golden Rule because we just like it as an educational principle. The crucial question is whether we can apply it to mathematics education in the first sense, namely, in mathematics classes? Paul does not give an answer to this question in his present paper and the "floor is open" to all kinds of suggestions. This is also true about mathematics education in the third sense (namely, a domain in which people recommend ways to improve and to enrich the activity of mathematics education). It is worthwhile to mention here that there are some attempts to combine mathematics teaching to values, for instance, that of Margaret Taplin (2009): Teaching Values through a Problem Solving Approach to Mathematics. On the other hand, it is quite clear that mathematics education in the second sense (namely, research discipline which investigates all aspects involved in the learning and teaching of mathematics) needs, in addition to the call to be ethical as researchers, a philosophy in the sense which I suggested above, namely, a philosophy that deals with all kinds of principles which lead mathematics education researchers in their work. In my opinion, and here, perhaps, I disagree with Paul, an appropriate candidate for this is Paul's third candidate for the first philosophy of mathematics education: Epistemology. It should deal with the mathematical aspects, the cognitive aspects, the emotional aspects, the sociological aspects and the political aspects of teaching and learning mathematics. Because of time limitations, I cannot elaborate on this. I just want to thank Paul again for his brilliant philosophical talk which, in addition to its being fascinating, it is also an important stimulus for foundational discussions in the community of mathematics educators in general, and among PME members in particular.

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PLENARY LECTURE 3

Understanding practices in mathematics education: structure and text

UNDERSTANDING PRACTICES IN MATHEMATICS EDUCATION: STRUCTURE AND TEXT

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The practices of mathematics education can be investigated at a wide variety of levels: from the actions of individual students or teachers through classroom interactions, school structures, curriculum specifications and materials, teacher development programmes, local, national or international systems of instruction and assessment. These levels are, however, inter-related. The study of a national curriculum gains significance as we see how it impacts upon and is interpreted by teachers and students. The study of an individual's actions makes more sense when these are interpreted in light of the broader context within which the individual is situated. In this paper I suggest some theoretical tools that can help to make such connections between the various levels of investigation, illustrating with a range of examples.

STARTING POINTS

I shall start by considering the theme of this conference 'In Search for Theories in Mathematics Education' from a linguistic perspective. Some points I find interesting:

- The nominal 'search' suggests a state rather than a process and thus has no temporal aspect – no end in sight – and no agent doing the searching.
- There is a nice grammatical ambiguity about the phrase 'in Mathematics Education': what is it qualifying? Does the search for Theories look within Mathematics Education? Or are Theories sought elsewhere to be used in Mathematics Education? Or is it the invisible agents of the search who are located in Mathematics Education?
- We, the participants in the conference, are not visible in this theme. How can we position ourselves in relation to it? Are we each individually 'in search'? Perhaps we may understand ourselves as in some way a part of a collective entity 'Mathematics Education' within which the search is taking place. Or are we outsiders, merely observing the search? Again there is a useful ambiguity that allows each of us to construct our own personal relationship to the theme.

I thank and congratulate the International Committee on their choice and wording of a theme that focuses us all clearly on the idea of a search for theories while allowing each of us to understand the nature of this search within our own personal and institutional context. Of course, in order to prepare this lecture I have had to make my own interpretation and position myself in relation to it. The interpretation I have

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 49-64. Thessaloniki, Greece: PME.

adopted leads me to tell a personal story of my own search for theories that might help me to address the problems within mathematics education that interest me. While this story is in one sense the story of an individual, it is simultaneously a collective story, influenced at every point by the contexts within which I have worked, both at a local level by people I have worked with and at a broader cultural level by the communities in which I have participated both within and beyond Mathematics Education. Communities that have themselves changed dynamically, influenced both by internal struggles and debates and by changes in the dominant discourses of national and international society.

In speaking of the inter-relationship between individual and collective, I have introduced one of the key characteristics of the perspective I adopt in the research I shall discuss here. In order to understand the practices of individuals it is necessary to understand how those practices and individuals relate to the social structures within which they are situated. At the same time, however, gaining insight into the way that social structures work demands attention to their operation at the level of individuals. In her plenary lecture at CERME6, Paola Valero drew attention to what she described as the network of mathematics education practices, including among others classroom practice, academic mathematics, teacher education, policy making, and advocated that research should seek to understand this network by focusing simultaneously at different nodes of the network. While my view of the network is perhaps more structured and hierarchical than hers, the implications are compatible: making sense of data arising from study of one node or level requires understanding of other connected nodes or levels.

Another key characteristic of my perspective is sensitivity to constraints and freedoms. As evident in my analysis of the conference theme, the authors' linguistic choices constrain but do not determine the interpretations that each of us may make. In turn, these choices were constrained but not determined by other factors, including the evolving practices of the PME conference and the wider mathematics education community and the general norms of academic practices and genres. This sensitivity orients me to ask the critical question "How might this be different?" – not in the ideal sense of "How might I prefer it to be different?" but in the practical sense of "What might be the differences if other choices had been or were to be made?"

Three fundamental themes have run through my professional career as a teacher and as a researcher: mathematics itself, language, and something that I shall call social issues. Initial interest in these themes can be linked to my personal history: family background, educational, professional and political experience, located in particular times and places. As a young teacher in a secondary school in a multiracial area of London, problems that concerned me (beyond the everyday concerns of survival in the classroom) included such questions as:

> Why do some students, who can do mathematics successfully in class and talk fluently about it, fail to reproduce this success in written tasks and examinations?

Why are the lower attainment groups in my school disproportionately full of students of African and Caribbean origin?

Why do so few students seem to appreciate mathematics in the way I do?

Why does such a lot of the teaching of my colleagues (and, lets be honest, myself) seem so boring and trivial?

Associated with these, as yet unsophisticated, questions, was the conviction that differences in students' (and teachers') achievements, interests and experiences in mathematics could not be explained solely at the level of the individual. This conviction was born of a political orientation rather than a sophisticated theoretical rationale. However, it is my belief that such orientations lie behind our choices between theoretical frameworks. My concern with social inequalities precludes adoption of a perspective that denies or ignores the influences of social relationships and structures on individual experience and achievement. My personal search for theory has thus been shaped by a need to understand how individual and social may be connected. There are various directions that I might take in this search and there are certainly theories that are compatible with this need that I will not be addressing. In particular, theories of learning and activity based in the Vygotskian tradition offer powerful ways of understanding such connections. (See (Hasan, 2002) for an account of the compatible and complementary contributions of Vygotsky, Halliday and Bernstein from psychological, linguistic and sociological perspectives respectively.) However, because many of my questions seek to address the uneven distribution of knowledge and educational success, I intend to focus here on the contributions of sociolinguistic, discursive and sociological theory to my way of understanding.

LANGUAGE, LINGUISTICS AND MATHEMATICS

As I have already indicated, one of my earliest research interests was in the nature and use of language in mathematics education, especially in the apparent mismatch between my judgement, through observation and oral discussion, of students' mathematical achievements and their performance on written tasks, in particular tasks involving reporting on the processes and outcomes of mathematical investigations. Attempting to research this interest entailed finding means of describing and judging students' mathematical writing. At that time (around 1990) most descriptions of mathematical language focused primarily on specialised mathematical notation (e.g., Ervinck, 1992; Woodrow, 1982) or vocabulary (e.g., Anghileri, 1991; Otterburn & Nicholson, 1976). However, my original observation suggested that specialised notation and vocabulary were not the main problems encountered by my students. Rather, the students failed to explain and justify their methods and results in ways that satisfied me or the assessment criteria they were attempting to meet.

There was beginning to develop a body of research looking more broadly at the use of language in classrooms and more extended spoken and written mathematical texts (e.g., Laborde, 1990; Pimm, 1987). In particular, the title of David Pimm's book

'*Speaking Mathematically: Communication in Mathematics Classrooms*' pointed to three important characteristics of the language in which I was interested: it is mathematical in some sense; it is for communication, so involves some form of social engagement; and it is situated within a particular context. It is thus not only the form of the language that is significant but also the role that it plays in interactions between individuals and in the broader social context. This understanding led me to see the Systemic Functional Linguistics of Michael Halliday as an appropriate means of approaching the problem of describing students' mathematical writing. Halliday made an early contribution to the characterisation of mathematical language (Halliday, 1974), also referred to by Pimm (1987), but, more significantly, his broader social semiotic orientation and his perception of language as essentially functional provide a strong theoretical basis for understanding the role of language within social practices, while his functional grammar provides powerful tools of description (Halliday, 1985). I have written elsewhere (Morgan, 2006) about social semiotics, the analytic tools it provides and their application to research in mathematics education.[i] My intention here is to discuss the interaction between linguistic analysis at the level of text and analysis at higher levels to enable fuller understanding of practices.

DISCOURSE, MATHEMATICS AND SOCIETY

To return to my original research problem, my task was not only to describe student writing but also to judge it or, rather, to understand how it was being judged. More specifically, how did the various characteristics of the texts produced by students relate to the ways in which their mathematics teachers and others involved in their assessment interpreted and valued the students' mathematical achievements? The explicit assessment criteria and teachers' overt statements about their assessment practices were not at a level of delicacy sufficient to understand how they related to the characteristics of individual student texts. For example, the teachers I interviewed claimed to value diagrams as a means of mathematical communication but, in practice, paid little attention to some types of diagrams and interpreted others (in particular, diagrams showing the detailed physical characteristics of practical apparatus used during the investigation) as evidence of lower level mathematical activity (see Morgan, 1998; Morgan & Watson, 2002). Moreover, the official assessment criteria made use of terms such as "appropriate means of communication", leaving open the question of how appropriateness might be judged. As Fairclough (1992) argues, use of *appropriate* disempowers those who are not already 'insiders', able to recognise and produce forms of text that will be highly valued by those in authority.

Teachers' evaluative discussion of student texts and the student texts themselves are elements of the same practice of school mathematics. They can only be understood in relation to one another. But there are also other texts (both written and spoken) that relate to these: curriculum documents, assessment criteria, teachers' guides, professional journals, school or mathematics department policies, classroom interactions, staffroom chat over a cup of coffee – among others. The constructs, values and relationships between the various participants that are present in these texts are components of the discourse of the practice, structuring what may be said or not said and by whom. The concept of discourse that I use here is within the tradition of Foucault (e.g., 1972), consistent also, in its concern with the distribution and exercise of power, with my initial concerns about social inequality. Unlike the broad sweep of Foucault's analytic approach, however, attention to the linguistic details of texts provides a means of identifying and understanding the operation of discourse at the day-to-day micro-level of classroom interaction.

Critical Discourse Theory provides a multi-layered way of thinking about language use in context that I find helps to address the issues in which I am interested. Texts (spoken or written, monologic or dialogic) are considered as the linguistic elements of social events. These events are shaped by more general social practices (of which 'orders of discourse' are the language-related elements) and by social structures (Fairclough, 2003). This theoretical perspective is reflected in the methodological approach to research. While data may take a variety of forms, it is always understood as arising within a communicative interaction – a lesson, a conversation, an interview, a written text – situated within a particular configuration of social practices and structures. Analysis from a critical discourse perspective focuses at three levels: the communicative interaction itself; the discursive resources used in the interaction and the orders of discourse from which they are drawn; the social structures and socio-cultural practices within which the interaction is situated (Chouliaraki & Fairclough, 1999, p.113). It sets out to establish how the linguistic and semiotic features of social interaction are systematically related to what is going on socially.

For example, in Evans, Morgan & Tsatsaroni (2006) we presented an analysis, drawing on critical discourse analysis among other theoretical tools, of part of a transcript of a group of lower secondary school students working on a mathematical problem. This analysis made use of semiotic features of the students' interaction but, in order to interpret these, drew on prior analysis of the structures provided by the national education system, traditional pedagogic practices and the local pedagogic practices of the particular classroom as well as an awareness of further resources available to the students from other practices, including 'everyday' discourses of the family or peer group. The structural analysis highlighted alternative interpretations – both for us as analysts and for the participants themselves - and identified possible tensions and contradictions that may be experienced by the participants within the interaction. An example of such potential for tension was the question "So how did you do it?" asked by one member of the group of another. We noted that this question could have different significance depending on the particular discursive resources drawn upon by each of the participants.

> Within the progressive pedagogy of this classroom it may call up the value placed on explaining mathematical activity and collaborating. Within a traditional pedagogy it may

represent a challenge by an evaluator (in a superior position) or a request for help from a student with lower status. (p.220)

The configuration of discursive resources brought together by the participants in the interaction and their pre-existing relationships shape the way in which the question functions in the practice of the group of students: perhaps opening out collaborative exploration of alternative methods of solution; alternatively leading to a defensive response, copying of answers or a refusal to share.

INTERPRETATION AND STRUCTURE

One of the problems with discourse analysis (or more generally any form of qualitative research) is that you are always asking: Where does this interpretation come from? How can this reading of the text be justified? The theoretical basis of social semiotics is that any text – the raw data – may only be understood through using knowledge of the immediate context of the practice of its production and consumption ("context of situation") and the broader cultural context shared by the participants in this practice (Halliday, 1978).[ii] It is through using this knowledge that the researcher may make claims about the ways in which the text functions for participants. But this raises further questions, including the practical and pragmatic question of how much knowledge of which parts of these contexts may be necessary to achieve a 'good enough' understanding. More fundamentally, how can the researcher (given sufficient information) arrive at a characterisation of these contexts? Fairclough argues that the only way this can be done is to draw upon the same 'insider' resources as the participants in the practice, but with self-conscious awareness of the common-sense assumptions of the practice (2001, p.139). An insider in a practice can interpret a text used within that practice in an authentic way, simply by virtue of being an insider. For the researcher, however, just being an insider is not enough as the assumptions, values and fundamental constructs of the practice are likely to be so naturalised that they are invisible and unquestionable. The researcher does not seek to understand the text as a participant does – to operate with it inside the practice – but to know the principles by which the text is understood and the functions it has within the practice. To achieve this it is necessary simultaneously to have insider knowledge and yet not to be in the practice at the point of analysis.

For researchers in education, insider knowledge may come from being or having been a student, teacher, teacher educator as well as a researcher, though it is important to recognise that changes over time and differences between contexts may affect the currency of such knowledge. This poses particular challenges for research across cultures, as educational practices – both at the level of the classroom and at the level of research – are not identical but have different sets of values and assumptions. Hence similar texts may not play the same role or be interpreted in similar ways. Recent years have seen a number of explorations of this issue through, for example, reanalyses of data from different theoretical perspectives (see, for example, some of the chapters in Clarke, 2001 or the 2006 *ESM* Special Issue 63(2) on Affect in

Mathematics Education), consideration of alternative approaches to didactical design (Miyakawa & Winsløw, 2009) etc. In the recent European ReMath project,[iii] the partners have used a methodology of 'cross-experimentation' with technological tools (Artigue et al., 2006) that, among other aims, seeks to study the use of software in classrooms in different countries, making visible the differences that arise within different national education systems and different research traditions. In this project we have seen differences in the ways teachers and students interpret and make use of the texts offered by the technological tools as well as differences in researchers' readings of the potential didactical functionalities of these tools and of student activity in the classrooms in which they are used.

We have to accept that no reading can be 'objective' as it must always be imbued with the values and assumptions of the reader. No interpretation of data is certain, as it must be constructed within a particular theoretical framework. The challenge is, on the one hand, to make the values and assumption of the framework explicit and, on the other, to find or develop a framework that matches one's fundamental philosophical and ethical beliefs and will proved insights into the issues of concern. As I indicated at the beginning, my own research concerns relate to the distribution of knowledge and academic success, to the role that language may play in this, and to the nature of pedagogy in mathematics classrooms, so it appears natural to make use of social theory that addresses the question of how the process of distribution operates, including linguistic aspects of this, and that provides ways of characterising pedagogy. There are a number of alternatives that match one or more of these criteria and offer useful possibilities for research in mathematics education, among which the work of Bourdieu and of Bernstein stands out and is being used by increasing numbers of researchers. The Research Forum at this conference on Sociological Frameworks in Mathematics Education Research offers some examples of such programmes of research and explores what such frameworks have to offer in rather more breadth and depth than I can in the space available. My intention here is to illustrate some of the ways in which Bernstein's sociological theory has informed my own research in interaction with the discourse analytic techniques already described.

THE FORMATION AND DESCRIPTION OF PEDAGOGIC DISCOURSE

The research into student writing and teachers' assessment practices discussed above took as its starting point the level of actions and interactions of individual students and teachers. Adopting a critical discourse approach necessitated drawing on other texts that provide a framework for teachers' practices, including official curriculum documents, practical guidance for teachers and teachers' professional journals, in order to know the resources available to teachers and to identify and understand they ways in which they draw on such resources to make sense of and account for their practices. It is also possible to take starting points at other levels. More recently I have been interested in considering the guidance provided for teachers in policy and curriculum documents and the ways in which the official discourse manifested in such texts provides resources for constructing the nature of teaching, learning and assessment, of teachers and students and relationships between them.

In the United Kingdom, both the curriculum and teaching methods have been increasingly specified by legislation and by non-statutory decree of official government agencies. These are regulated by inspection and high-stakes national examinations as well as by a plethora of documentation. Yet the practice in individual classrooms still varies. This observation matches the wealth of research internationally into curriculum reform that shows the lack of uniformity of its implementation and the persistence of 'traditional' forms of pedagogy. My study of teachers assessing students' mathematical writing identified a number of different positions that teachers adopted in relation to the task of assessing and to the regulatory framework. For example, some positioned themselves as 'official' examiners making explicit use the published assessment criteria, while others adopted a position as an advocate, imagining what the student might have been thinking and giving them the benefit of the doubt (Morgan, 1996). Working together with Anna Tsatsaroni and Steve Lerman, the origins of these positions in official and unofficial discourses of assessment were systematically identified and the resources provided by these discourses described in order to produce a model that provided a structure to classify possible positions and the relationships between them (Morgan, Tsatsaroni, & Lerman, 2002). This structure not only systematised the distinctive positions identified empirically but also predicted that such positions might occur.

One of the key concepts that informed this systematisation and further work that I have done, looking at teachers' and students' relationships to the curriculum, is Bernstein's notion of recontextualisation of discourses. As they move from one sphere of activity to another, discourses are transformed – recontextualised – according to a principle, consistent with the interests of relevant agents, that "selectively appropriates, relocates, refocuses and relates" to other discourses (Bernstein, 2000, p.33). In the context of mathematics education there are two moments of recontextualisation that concern me: that which happens in the construction by government agencies of official discourses of curriculum and practice and that which happens as the curriculum is deployed in the classroom.

I shall look first at an extract taken from the *National Framework for Teaching Mathematics* (DfES, 2001) a key document of the official discourse in England that sets out the forms of teaching expected of teachers in the lower secondary school.

> Good direct teaching is achieved by balancing different teaching strategies:
> - Directing and telling […]
> - Demonstrating and modelling […]
> - Explaining and illustrating […]
> - Questioning and discussing: questioning in ways which match the direction and pace of the lesson to ensure that all pupils take part (if needed, supported by apparatus, a calculator or a communication aid, or by an adult who translates, signs or uses symbols); using open and closed questions, skillfully framed, adjusted and targeted to make sure

that pupils of all abilities are involved and contribute to discussions; asking for explanations; giving time for pupils to think before inviting an answer; listening carefully to pupils' responses and responding constructively in order to take forward their learning; challenging their assumptions and making them think...

- Exploring and investigating […]
- Consolidating and embedding […]
- Reflecting and evaluating […]
- Summarising and reminding […]

How may we understand the description and prescription of teaching provided in this list? The headings of the list of teaching strategies succeed in incorporating aspects such as exploring, investigating, discussing, reflecting, which might be thought to belong to a student-centred discourse such as those associated with constructivist-based curriculum reform. On closer examination, however, it may be seen that there are also other discourses at play. Looking in more detail at the gloss on 'questioning and discussing', it becomes clear that it privileges a teacher-centred pedagogy with strong framing. It is primarily the teacher who is active (questioning, asking, challenging) and student activity is controlled; the teacher must: 'ensure that all pupils take part'; 'make sure that pupils of all abilities are involved'; 'make them think'. Questioning must match the direction and pace of the lesson rather than follow the direction of student activity and interests.

As well as these discourses about teaching, the extract draws on discourses about the nature of students and learning. There is an expectation that students will participate and, in particular, will think. At the same time, however, there is a strong assumption of differences between students, especially of deficit, as some students may need support to enable them to take part. The 'normal' student will take part as directed by the teacher; some others will need support to do so; those who refuse to take part or who participate in other ways (perhaps asking questions themselves rather than responding to teacher questions) are absent from the text.

The official discourse is thus a recontextualisation of several discourses of students, teaching and learning. These discourses originate in part in various academic theories – a precursor of this document was accompanied by a review of educational research that sought to justify its approach (Reynolds & Muijs, 1999)[iv] – as well as in everyday discourses about education, bringing these into conjunction and transforming them in the process. This raises the question of what the principles of this transformation may be – a question that I will not address here but which demands a wider consideration of whose interests may be served by the construction of such a form of pedagogy.

When considering the further process of recontextualisation of this official discourse as teachers make use of it in their classrooms, it is necessary to consider what other discourses teachers may draw on and what their interests may be as they selectively appropriate and transform the official guidance into classroom action. The discourses they draw on are likely to include a range of specialised academic and professional discourses encountered in the course of initial teacher education and further professional development activities as well as everyday discourses about education current in the community or encountered in the media. Analysing the structure of discourses that may contribute to the construction of pedagogy in the classroom provides a means of describing, understanding and, indeed, predicting the various ways that teachers may position themselves in relation to the official discourse. In the case of the *National Framework*, the conjunction of resources from several (not obviously compatible) discourses allows teachers to comply with the demands of the official curriculum while continuing to teach in a range of different ways. A fuller presentation of this argument is made in (Morgan, in press).

My final example takes a starting point at the level of the classroom, considering the recontextualisation that occurs as teachers implement a curriculum. In this case, two experienced teachers who were undertaking a Masters course in Mathematics Education collaborated as teacher-researchers in experimentation with a new technology.[v] Their participation involved planning and executing a set of lessons using a microworld constructed using E-Slate,[vi] a toolkit for building environments for educational exploration, as well as gathering data on students' use of the software and writing a reflective discussion of their experience.[vii] The microworld, named Fraction-Slider, provided two linked forms of representation of fraction: visually as a relationship between values shown by positions on two linked dynamic number lines (sliders) and, symbolically, as a rational number entered into a Logo procedure in either decimal (e.g. 0.25) or ratio (e.g. 1/4) form. The numbers entered in the Logo procedure determined the relationships between the values displayed on the sliders.

The teachers decided on two types of tasks making use of the microworld: first comparing and ordering fractions, then finding a fraction between two others. An outline plan was agreed for three lessons and tasks designed to use with students. However, detailed pedagogical issues were not discussed explicitly and it was left to the individual teacher-researchers to plan the conduct of their lessons. In practice, the two used similar lesson structures, starting each lesson with an interactive discussion with the whole class, followed by tasks that students engaged with individually or in pairs while the teacher provided support as needed, and finishing with further whole class interaction, reviewing what had been done during the lesson. Described in these terms, both teachers can be said to have adopted the voice of the official pedagogic discourse of the National Framework (DfES, 2001), which recommends such a 'three-part lesson' structure. On examining the detail, however, there were significant differences between the pedagogies of the two teachers.

One important area of difference was in the strength of the control maintained by the teacher over student participation in whole class interaction. Teacher 1 adopted a position of authority over all interactions and over what might be considered legitimate knowledge in the classroom, with a strong asymmetry between his role and that of his students. For example, as shown in the following extract from early in the first lesson, the teacher-student interaction followed a clear Initiation-Response-Evaluation pattern.

Teacher 1	…which is the largest fraction out of those two, D?
Student	3/6
Teacher 1	Why do you think it might be 3/6?
Student	Because three of the … three sixes …one…. [inaudible]
Teacher 1	Can anybody explain a little further, she's not wrong, I know she knows what she's talking about. E.
Student	Because three [inaudible]
Teacher 1	Right, excellent.

In contrast, Teacher 2 introduced each lesson with a discussion whose direction was not pre-determined. Initial questions were planned, but were more open in nature and concepts introduced by the students became part of the discussion. As may be seen in the following extract, Teacher 2 legitimated the students' contributions implicitly by echoing, revoicing or building on them rather than by making explicit evaluative comments.

Teacher 2	What do you think is happening here when you move the top slider? […] What do you think over here girls?
Student	I don't know. They just all seem to be moving when you move the top one along like that in a diagonal line.
Teacher 2	They're moving diagonally.
Student	They're moving proportionally, all three of them.
Teacher 2	Can you try and think about what those proportions might be? How would you try and work it out?
Student	If you move it like that
Teacher 2	Move it right over to the end

Bernstein orients us to distinguish pedagogies by considering the strengths of classification and framing. We can identify clear differences between the two teachers in the strength of framing – the location of control over pacing, sequencing of the material and evaluation. There were also differences in the classification – the way in which categories of knowledge were established and distinguished. From the start, Teacher 1 focused explicitly on fractions and calculations, while Teacher 2's introduction to the microworld did not even mention fractions until the word was introduced by one of the students. For a fuller discussion see (Morgan, 2007).

So how may we understand these differences, given that the two teachers worked together to plan the lessons and appeared to be engaged in the same enterprise? As one of the aims of the project was to explore the role of theoretical frameworks, this is where I shall start. While the relation between espoused and implicit theory and practice is not simple, it is possible to consider at least some of the theoretical resources available to the teachers and how they appeared to deploy these in order to make sense of their practice. Through their participation in the project the two teacher-researchers had come into contact with a number of theoretical ideas, in particular, socio-cultural theory and the notion of semiotic mediation. In addition, the design of the microworld itself and recent experience in a Masters course on the role of digital technologies in mathematics learning and teaching oriented them towards the ideas of constructionism. Both teachers described their planned student activity as "exploration", appearing to draw on the discourse arising in this theoretical field. The following extracts from their written reflections after the experiment, however, suggest that the two teachers appropriated these resources in different ways.

> Teacher 1: Pupils were able to get through far more questions … This exposed pupils to far more examples and hopefully enabled pupils to think more generally … A significant number of pupils began to be able to successfully predict outcomes by rounding fractions, '96/350 is about one quarter but 34/70 is just less than a half, so that must be bigger.' … It was interesting to see how the pupils felt free to just 'try any old fraction' … This … led to an arbitrary fraction being tied and then the denominator being 'stuck to' and the numerator being altered until successful, this technique was certainly only possible with this software … [one student] would set the main control slider to 100. 100 would then be their fractions denominator. They would then look at the two values on the two sub-sliders and chose for their numerator a number between those two numbers.

For Teacher 1, despite espousing exploration, it is "exposure" to examples which leads to generalisation. His main focus in reflecting on the outcomes is on student development of specific skills and strategies within the topic domain. The main role of the microworld is presented as facilitating acquisition of traditional forms of knowledge. This seems compatible with the discourse of the official curriculum discussed above.

> Teacher 2: Initially their talk mainly centred on the 'distance' of the sliders from one another, but some then started to talk about the movement of the sliders … What I thought was interesting about the replies was that those students who used a 'static' form of language ("the gap is bigger", "more space is taken up") tended to get the answer wrong, whereas those who used a more 'dynamic' language ("it moves faster", "it travels further") tended to get the answer right … Finding a fraction between 2/5 and 3/7 was hard as the fractions are so close together. This brought out a confusion about the meaning of 'in between' (does it have to be exactly in the middle?)

In contrast, Teacher 2's focus is on the language and meanings generated in interaction with the representations provided by the tool. This focus is compatible with the original framing of the aims and design of the study, presenting the microworld as a semiotic tool that may structure learning.

While the project in which they were involved privileged certain kinds of academic discourse about teaching and learning, the ways in which the two teacher-researchers planned, taught and talked about their work also drew upon the official curriculum discourse (with its recontextualisation of these and other academic discourses) and the more local discourses of teaching and learning current in their respective professional environments as well as everyday non-specialised discourses (cf. Morgan et al., 2002). Where they appropriated resources from discourses of research, including those of explicit theory, these acquired new types of meaning as they were put to new purposes within the practice of teaching, situated within particular institutional contexts. Considering their institutional contexts suggests one reason why the teachers selected differently from the available resources. The schools differed in the socio-economic backgrounds of the student population and in their dominant forms of pedagogy. Teacher 1's school served a mixed population, including many students from deprived backgrounds. It was a church school with a 'traditional' ethos, including strong control of student behaviour. In contrast, Teacher 2's school had a liberal 'progressive' ethos and a predominantly well-off middle class population. While beyond the scope of such a small study, this raises questions about how social structures may influence the forms of pedagogy and hence the access to knowledge available to different groups of students (cf. Atweh, Bleicher, & Cooper, 1998; Noyes, 2008; Zevenbergen, 1998). By describing and attempting to explain the paths taken by each of these teachers, it is also possible to ask how their practices might have been different. What choices were available to them and what might have enabled them to adopt different forms of pedagogy?

CONCLUSION

In narrating this partial account of my personal search for theory I have not had space to do full justice to the various sources that have influenced my thinking. In particular, the work of Bernstein offers a much more powerful and systematic set of concepts than the little I have touched on. Nevertheless, I hope I have provided a flavour of the resources I draw upon in order to address my research concerns. The multiplicity of theoretical tools, from linguistics and semiotics, discourse theory and sociology, matches the multi-layered conception of the field of study, incorporating individuals engaged in interactions within social practices and structures. My examples have taken as their starting points data arising in interactions occurring at different nodes within the educational enterprise: in classrooms, in assessment practices, in curriculum policy. There are of course many other possible starting points that could contribute to understanding the complexity of practices within mathematics education. My contention is that, whatever the starting point, it is necessary both to look within – at the level of the text itself – to understand how the interaction operates locally and to look beyond – at the discourses it draws on and the structures within which it is situated – in order to understand how it arose and how various participants may use it in different ways.

NOTES

[i] The linguistic tools provided by Halliday have been extended to address the analysis of multimodal texts (e.g., Kress & van Leeuwen, 2001). There have also been some

efforts to develop specialised tools for mathematical texts of various kinds (e.g., Alshwaikh, 2008; O'Halloran, 2005) within a social semiotic perspective.

[ii] It is no surprise to find that Halliday's use of context of situation and context of culture draws on ideas originating in ethnography as ethnographic researchers struggle with similar insider-outsider dilemmas. However, unlike some ethnographic approaches, the critical discourse analyst does not attempt to withhold preconceptions, inducing theory from the data, but is likely to make explicit use of social theory to structure the analytic approach (Chouliaraki & Fairclough, 1999).

[iii] 'Representing Mathematics with Digital Media', funded by the European Community, Framework 6 Programme, IST–4–26751–STP. See http://remath.cti.gr. The participating research teams were: University Paris 7 Denis Diderot, Paris, France; National Kapodistrian University of Athens, Educational Technology Lab, Athens, Greece; Consiglio Nazionale delle Ricerche, Instituto Tecnologie Didattiche, Genova, Italy; MeTAH and Leibniz, IMAG, Grenoble, France; University of London Institute of Education, London Knowledge Lab, London, UK; University of Siena, Department of Mathematics, Siena, Italy.

[iv] Though see (Brown, Askew, Millett, & Rhodes, 2003) for a critique of the way this research was used.

[v] This work was undertaken by the TELMA (Technology Enhanced Learning in Mathematics) European Research Team, part of the Kaleidoscope Network of Excellence, funded by the European Community (IST–507838) under the Framework 6 Programme. See http://www.noe-kaleidoscope.org.

[vi] E-Slate was devised by the TELMA Athens partners (NKUA-ETL). See http://e-slate.cti.gr/

[vii] The analysis offered here uses the data collected by the teachers and also uses their analyses and reflections as data. Both teachers have seen this analysis and agreed to its publication.

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PLENARY LECTURE 4

Instruments for Learning and Teaching Mathematics - an attempt to theorize about the role of textbooks, computers and other artefacts to teach and learn mathematics

INSTRUMENTS FOR LEARNING AND TEACHING MATHEMATICS

AN ATTEMPT TO THEORISE ABOUT THE ROLE OF TEXTBOOKS, COMPUTERS AND OTHER ARTEFACTS TO TEACH AND LEARN MATHEMATICS

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Looking into teaching and learning of mathematics, it is obvious that various embodiments and processes (materialised or written in sometimes everyday language) play a decisive role in doing, teaching and learning mathematics. Regardless of the epistemological approach used, this can be traced back to the importance of patterns and structures within mathematics, which have to be (re)presented in artefacts. The artefacts are hopefully turned into instruments by the learner/user of mathematics. The paper will try to interpret this situation from a theoretical point of view - mainly "instrumental genesis". Empirical research on the most important traditional learning/teaching instrument, the textbook, and the well discussed new technology (computers and -sometimes educational- software) will be taken as prototypes to illustrate the theoretical statements.

“The Computer forces us to think about topics, which should have been analysed long before.“

„Der Computer zwingt uns zum Nachdenken über Dinge, über die wir auch ohne Computer längst hätten nachdenken müssen.”

(Hans Schupp, Didactician of Mathematics, 1993)

My starting point is an attempt to better understand certain aspects of mathematics and its teaching & learning. I begin with an epistemological perspective on mathematics, imbedded in a particular way to look into human cognitive practice. A short excursion hopes to show that this perspective is not too particular, because I continue with a general reflection on artefacts as necessary ingredients for the teaching and learning of mathematics. Details of a particular understanding of the interplay between artefacts and mathematics are presented – condensed in the concept of ‘semiotic mediation’. This concept is complemented by an elaboration of the concept of ‘instrument’, which is a mixed entity of artefact and ‘schemes’. A more or less individual perspective is followed by an analysis of the institutional, social role of artefacts and schemes, in all: of instruments. These ideas are summed up in a tetrahedron model of instruments for teaching and learning mathematics. A

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brief overview on research about Dynamical Geometry Software (DGS) tries to illustrate these theoretical deliberations, also showing that the theories of semiotic mediation and instrumental genesis do not completely cover the research on DGS. Methodological and consequences for research in general are formulated in order to end with a look back on mathematics and its teaching and learning in general.

ARTEFACTS ARE UNAVOIDABLE IN RELATION TO MATHEMATICS

As I assume that Didactics of Mathematics is the 'sum' of scientific activities to describe, analyse and better understand peoples' joy, tinkering and struggle for/with mathematics, the relation between mathematics and its human users has to be analysed. My entry into this Research on Mathematics Education is a description of human cognitive practice related to mathematics, which is inspired by a definition of human cognitive activity given by Wartofsky: "... The crucial feature of human cognitive practice ... (is) the ability to make representations ... This I traced to the ... production of artifacts ... (defined as) Anything which human beings create by the transformation of nature and of themselves: thus also language, forms of social organisation and interaction, techniques of production, skills." (Wartofsky 1979, p. xiii). In order to better structure this vast range of artefacts, Wartofsky offers a classification of artefacts:

- '**primary'** artefacts: "directly used in the production"

- '**secondary**' artefacts: "used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out"

- '**tertiary**' artefacts: "constitute a domain in which there is a free construction in the imagination of rules and operations different from those adopted for ordinary 'this-wordly' praxis" (Wartofsky, p. 202/209)

Even if we could have a discussion if certain pieces of mathematics are primary or tertiary artefacts, it is obvious – and easily illustrated by examples – that teaching and learning of mathematics uses a wide variety of secondary artefacts like textbooks, traditional calculators like abaci, Cuisinaire rods and software more or less specifically designed for the use in the teaching and learning process.

In addition to this factual illustration of the importance of artefacts for teaching and learning mathematics, I start from the assumption that artefacts play a particular role for the discipline of Mathematics – compared to other disciplines. The 17th ICMI-study on "Mathematics and Technology – Rethinking the Terrain" (for the proceedings see Hoyles & Lagrange 2009) nicely condenses this assumption. It bluntly states: "Because of its epistemological nature any immediate relationship with mathematics is impossible; any relation passes through a mediation process. Ideal, immaterial, non-perceivable entities such as numbers or figures acquire existence, can be thought of and shared, only through their materialization in a concrete perceivable entity, generally referred to as representation." (Drijvers et al. 2009, p.

133). Because of its epistemology, mathematics cannot do without man-made representations, of artefacts in order to be accessible for women and men.

Excursion on epistemology: constructivism – Platonism - formalism

For an epistemology of mathematics, there are basically two options: Either one starts from the assumption that mathematics is already in the world and has to be re-discovered when learning/teaching/making progress in mathematics. In order to make mathematics visible, some sort of representation has to be developed (be it a primary or tertiary artefact to give life for the disciplinary evolution or be it a secondary artefact for education in relation to mathematics). This platonistic or realistic epistemology of mathematics stands in contrast to a (nowadays more common) constructivist approach, starting from the assumption that mathematics has to be (individually and/or socially) constructed to make it a tool for societal use and/or individual play. The consequence for representations of mathematics is the same as with platonistic/realistic perspectives: At least for a societal use, mathematics needs to be (re)presented somehow with the help of man-made representations, with artefacts.

Even a formalist position trying to avoid the dichotomy of Platonism and constructivism - like "Mathematics is the science of formal systems" (see Curry 1970, p. 56) as a way to cope with the ontological crisis of mathematics in the first half of the 20th century - has to create some record of the systems in order to make the systems accessible for research and communication on the formal systems.

If scientific research has something to do with cumulative knowledge, replication of results and inter-subjectivity, some sort of record, some inscription (see Dörfler 2006) is needed to create the possibility of scientific development without re-inventing the wheel in every scientific activity. Some sort of record is needed to enable mathematics to be handed over to a successor and/or the next generation of mathematicians, to make progress based on what has been achieved already.

ARTEFACTS: NECESSARY FOR LEARNING / TEACHING MATHS

Having seen the importance of artefacts for mathematics itself, for the discipline of Mathematics, we now turn to teaching and learning of mathematics. If one looks closer into the processes of teaching and learning (e.g. of mathematics), Wartofsky's second category of artefacts is an additional hint that (wo)men can profit from the use of artefacts. I want to strengthen this: I start from the assumption that one cannot do without artefacts when teaching and learning whatever necessary. This is especially true for mathematics – as we have seen from an epistemological standpoint in the section above. Taking into account the recent history of Didactics of Mathematics (or Mathematics Education Research), the statement can be further illustrated by its own history and a continuing interest in teaching/learning material (for a rather global proof see the proceedings of the 100th anniversary of ICMI, Menghini et al. 2009). In addition to this, the theories, which came into being when technology use in mathematics was analysed, are also strongly focussed different artefacts. One may be neglecting the subtle, but existing differences of the various theoretical developments to research information technology and its use in relation to mathematics. Because of the fast developing technology and its use in education, research had to theorise the role of artefacts, namely computers, software and communication technology like the internet, to better understand technology enhanced teaching and learning of mathematics.

Only a minority of researchers realised that the necessity of taking into account the artefacts was not restricted to research on new technology, but is unavoidable for analysing mathematics education in general. Artefacts like textbooks, embodiments like Cuisinaire rods for numbers and Geoboards, rulers and compasses for points, circles and segments have always played a major role in mathematics education. To cite from a preparatory study for the TIMMS studies: "But despite the obvious powers of the new technology it must be accepted that its role in the vast majority of the world's classrooms pales into insignificance when compared with that of textbooks and other written materials" (Howson 1995, 21). Consequently, theory based research on artefacts can learn from research on new technology, but should not be restricted to this recently developed teaching aid for mathematics education. As will be shown in the following, theory can learn from theories about educational use of information and communication technology, but one should have in mind that more fundamental and further-reaching topics are at stake within such theorising.

The widening of the focus of research is not only motivated by theoretical arguments, but can also be illustrated by empirical research: some colleagues analysed the analysis of the actual documentation work of French teachers of mathematics, their "looking for resources, selecting/designing mathematical tasks, planning their succession, managing available artefacts". The research by Gueudet&Trouche followed an instrumental genesis approach (see below) and suggests to analyse the wider range of work on different resources as "the core of teachers' professional activity and professional development". In doing so, one may be able to better understand the actual processes of change in teachers' professional practice (Gueudet&Trouche 2009, all quotes from the abstract). Researching the use of artefacts in general is a focus of attention not restricted to information and communication technology.

FROM ARTEFACTS TO INSTRUMENTS: INSTRUMENTAL GENESIS

Mathematics and artefact: semiotic mediation

An artefact, which is somehow important for mathematics (be it use, teaching or learning of mathematics), has to "speak to" mathematics in some way. There should be some hints to mathematics, which makes the artefact significant for mathematics, the artefact should contain signs of mathematics. Semiotics (as the science of signs) tries to understand this relevance by analysing the signs used and the mathematics it stands for. (I obviously take Semiotics as the science of signs that is something, which "stands for", points to something else). Insofar, Semiotics only analyses the relation between artefacts and something else – like for instance mathematics (for a recent account of the relation between Mathematics and Semiotics see the thematic issue of Educational Studies in Mathematics, vol. 61(1); for the editorial Saenz-Ludlow&Presmeg 2006).

The basic idea of semiotic mediation (at least as I understand it!) is, that the same sign stands for, points to different "things" when teaching / learning of mathematics is analysed. The same sign, the same "signifier" has different "things" signified, has different meanings (!) for different persons involved in the scene. The very same sign mediates something totally different. For instance: While the expert (e.g.: a teacher) sees a mathematical structure and a statement on angle bisectors in triangles in the drawing produced with the help of a Dynamical Geometry Software ("DGS"), the novice (the learner) may only see three segments forming a triangle and some three straight lines in the middle of the respective angles. While the expert is thinking (maybe: talking) of three angle bisectors "surprisingly and always" meeting in one and only one point in a triangle, the novice does not recognise any surprising fact (for an early empirical illustration of this difference see Hölzl 1995). The different messages from the "same" sign have to be brought together in order to let learning take place, at best / hopefully according to the expert view. This can be done by diligently orchestrated classroom discussions, which make the learners aware of the scientific messages mediated by the artefacts and discovered with the help of the expert, the teacher (for details and examples from the teaching and learning of the function concept see Falcade et al. 2007).

The dynamics in this description comes from the difference of two actors' perspectives in the scene: The learner is confronted with a system of signs s/he may (not) be able to interpret. On the other hand, the second actor, the teacher – in her/his role as being responsible for the continuity of the socially desired knowledge – wants to offer, if not inculcate (geometrical) knowledge to the novice. This theoretical reconstruction (of semiotic mediation) leads to two important aspects of teaching and learning with the help of artefacts: It introduces two major players of the game (the teacher/expert and the learner/novice) and it shows the importance of the teacher's role for learning in a highly conventionalised field like mathematics – using theoretical concepts from social constructivism in line with Vygotsky. Consequently, semiotic mediation approaches tend to stress the role of the teacher in learning processes and – in this way - introduces human beings into the interplay of artefacts and mathematics. This is what we will be looking into in the following section.

MATHEMATICS AND INSTRUMENTS

Above, I deliberately avoided the word "instrument", using "artefact" as the fundamental theoretical concept. In doing so, I followed a caveat linked to one of the most often used theoretical ideas when analysing teaching and learning with the help of artefacts, namely "instrument" and "instrumental genesis". Before going into details, I just want to cite from the ICMI-17-study proceedings: Apart from certain differences within different 'instrumental genesis' approaches, "language issues play a role. … the word instrument … is used in a different sense from its meaning in natural language: in the case of somebody playing the piano – the instrument in daily-life language – the instrument from the theoretical perspective is more than the piano alone and includes the piece that will be played, as well as the schemes the player uses while doing so." (Drijvers et al. 2009, 124).

Instrument: artefact + utilization schemes

Within semiotic mediation, two users of artefact mediated learning / teaching came into sight: the teacher/expert and the learner/novice. The theoretical concept of "instrument" exactly puts these actors into focus – somehow changing from Semiotics to Psychology. We now look into the relation between one or more artefact(s) and a user, trying to theoretically describe this interaction by means of the development of 'utilization schemes' the user develops (for details of the following see the initial description by Rabardel 1995/2002, especially chapters 6-8). Within this approach, the instrument is a "mixed entity" made up of an artefact component and 'utilization schemes'. "We define the instrument as a whole incorporating an artifact (or a fraction of an artifact) and one or more utilization schemes" (Rabardel 2002, p. 65). This definition borrows the concept of 'scheme' from Piaget, using the work of G. Vergnaud. Rabardel (following Vergnaud) takes four ingredients of a scheme, namely

"- anticipations of the goal to be reached, expected effects and possible intermediary stages;
- rules of action along the lines of "if-then" which allow the sequencing of subjects' actions to be generated;
- inferences (reasoning) that allow the subject to calculate rules and anticipations based on information and the operational invariants system he/she disposes of;
-- operational invariants that pilot the subject's recognition of elements pertinent to the situation and information gathering on the situation to be dealt with" (Rabardel 2002, p. 79).

From this description of utilization schemes, we learn that schemes are closely linked with goals of an activity. In using a scheme, the user looks forward to use certain means for identified ends, using a scheme is a purposeful activity to reach a certain goal. As with other aspects of this theory, the importance of goals in utilization schemes makes this theoretical approach part of activity theory, inspired by theorists like Leontiev and Vygotsky (explicitly referred to by Rabardel). For Rabardel, Vygotsky marks the transition from material instruments (analysed by Leontiev) to psychological instruments (see Rabardel 2002, p. 57), a move which is in line with the wide definition of artefacts suggested by Wartofsky.

For mathematics education research and as indicated by the double "-" in the quotation from Rabardel, 'operational invariants' are most important for the identification of utilization schemes. If a researcher wants to identify utilization schemes, s/he has to look for operational invariants in the activity of his subjects of study. From a didactical point of view, the 'rules of action' and the inferences are also most important in order to better understand the second component of an instrument, namely the utilization schemes. Rules of action and inferences may somehow link the utilization scheme to the contents of the scheme, this is where mathematics can be detected in utilization schemes.

If one goes back to the Vergnaud concepts of 'scheme', one could find the ideas of "concepts-in-action" and "theorems-in-action" as major components of the schemes he analysed (see for example Vergnaud 1983). Vergnaud has deliberately chosen words, which are deeply linked to mathematical ideas like 'concepts' and 'theorems'. As a consequence of the non-mathematical nature of utilization schemes to be analysed, Rezat (in his analysis of mathematics textbook use) suggests to generalise these concepts to 'beliefs-in-action' in order to adequately mirror the purpose and contents of utilization schemes developed with the use of different artefacts, which may be more or less linked to mathematics. I quote an example from his work about the use of mathematics textbooks by learners in grade 6 and 12 (Rezat 2008, p. 4-181):

> Utilization schemes "… were found in conjunction with five different activities: solving tasks and problems, to look up something, practising, performing follow-up coursework, anticipation. In order to solve tasks and problems students rely heavily on worked examples. This scheme can be characterized by the rule-of-action: Whenever I have to solve tasks and problems from the book I study the worked examples. This scheme is supported by the belief-in-action: Worked examples help to solve tasks and problems".

This quote nicely illustrates the importance of the rules of action and the beliefs in action linked with different utilisation schemes of the artefact mathematics textbook. The instrumental genesis approach was most operational and effective in an attempt to describe and analyse the use of mathematics textbooks by student learners (for the study as a whole see Rezat 2009).

Individual and institutional instrumental genesis: schemes and 'technology'

Looking back to what has been developed so far, it looks as if instruments (the mixed entity of artefact and utilization scheme) are an individual affair. Within instrumental genesis, the user learns to pursue certain goals with the help of artefacts – and if s/he develops operational invariants and rules of action to follow because her/his inferences promise to arrive at the goals, an instrument in the theoretical sense is born. But Rabardel already opened up a more general meaning to instrumental genesis by stating: "Finally, it should be stressed that utilization schemes have both private and social dimensions. The private dimension is specific to each individual. The social dimension comes from the fact that schemes develop in the course of a process in which the subject is not isolated. Other users, as well as artefact designers, contribute to this emergence of schemes" (Rabardel 2002, p.84). To put it in a more general statement: Instruments are not restricted to individual use, but there is room for instruments used by groups of individuals, and maybe even: whole institutions.

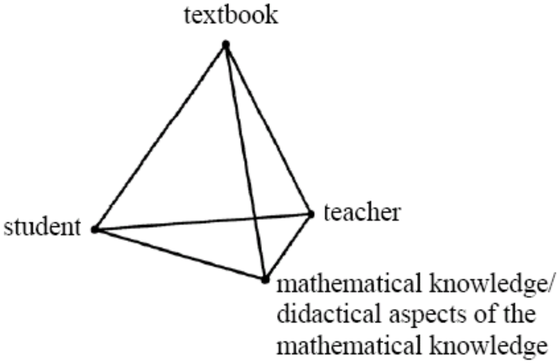
As a sort of empirical illustration of this idea, I again cite from Rezat's study about textbook use: "The analysis of the data revealed that students did not only develop individual utilization schemes of the mathematics textbook, but that some schemes emerged repeatedly. It is argued that these schemes that could be found in different groups of students of different ages seem to reflect typical schemes of utilizing a mathematics textbook within a given culture. It is suggested to call these schemes culturalhistorical-utilization-schemes (CHUS)" (Rezat 2008, p. 4-181).

Another illustration could be the research and conceptual development by Trouche (2004/2005), who put forward the concept of "orchestration" to describe the way, computer technology is used within the classroom. The specific classroom situation he describes as a viable one (with individual handheld calculators for every student, a "sherpa" student managing a data projector for the screen of one of the calculators and a teacher) can be either seen as an institutional utilization scheme or a complicated, but individual utilization scheme of the teacher in charge of the class. More generally speaking: "orchestration" can be a concept to group institutional utilization schemes of (not necessarily) technical artefacts or a concept to describe the utilization schemes of the teacher. Anyway, the concept is a valuable reminder of the variability of possible classroom organisations against too deterministic ideas of (possible) uses of artefacts by teachers.

Besides these empirical illustrations, Monaghan (2007) has put forward convincing arguments that it is exactly the difference between individual and institutional utilization schemes which makes a difference in some instrumental genesis approaches. The original instrumental genesis approach - somehow linked to the Brousseau theory of situations "théorie de situations (TSD)" - is basically interested in individual instrumental genesis (i.e. in psychological questions), whereas the later analysis of CAS use is more based on the "anthrological approach" of Didactics of Mathematics (inspired by Chevallard). From its very foundation, the latter is focused on the difference between the individual, private versus institutional role of tasks, techniques and theory – and, as a consequence, is more appropriate to take into account a perspective somehow transcending the individual instrumental genesis, analysing inter-individual, if not institutional, social phenomena.

Summary: From triangles to a tetrahedron

Looking back to what I have elaborated so far, one can see a red thread through the whole development. I started from the relation between (mathematical) knowledge and artefacts, used semiotic mediation as a reason to introduce two human players (the expert/teacher and the novice/learner) to concentrate on the interaction between these human players, mathematics and different artefacts when learning/teaching mathematics. If I want a simple model of these interplays, the best I can imagine is a tetrahedron with mathematics, artefacts and two human players as endpoints. In 2006, analysing the role of textbooks within the activity of teaching and learning mathematics, Rezat (2006, p. 4-413) suggested the following model:



After changing “textbook” to “(secondary) artefact”, we have a good model for the use of instruments in the teaching and learning of mathematics. The segments between student or teacher and the artefact allow for instrumental genesis, the segment between artefact and mathematical knowledge models semiotic mediation, while the bottom of the tetrahedron is the well-known didactical triangle. In all, this tetrahedron models the basic relations important for a scientific analysis of instrumented teaching and learning mathematics. As a consequence of this being a model, it is obvious that it may be worthwhile to think of something surrounding this tetrahedron, e.g. all those persons and institutions interested in the teaching and learning of mathematics, the “noosphere” (cf. Chevallard 1985/91). For a full account of institutional and societal influences on teaching and learning mathematics with the help of instruments, it may be appropriate to have even more spheres surrounding this tetrahedron.

DYNAMICAL GEOMETRY ENVIRONMENTS (DGE): AN INSTRUMENT

In order to show the validity and helpfulness of the theoretical concepts described above, I will now give an account of research on Dynamical Geometry Environments (DGE), one of the best researched instruments for teaching and learning mathematics worldwide. In this section, I basically follow the line of arguments of Laborde et al. 2006, somehow re-writing the section on DGEs and that on research trends of this paper (loc.cit., pp. 284-295). This will show the potential of the theoretical concepts described above, but it will also point to restrictions and important research questions, which are not in reach of semiotic mediation and/or instrumental genesis approaches.

Laborde et al. start from a broad description of DGE, from a description of the artefact Dynamical Geometry Environments. Drag mode, locus of points and macros are identified as the basic characteristics of this type of artefacts, which are said to be helpful in teaching / learning of mathematics, especially geometry, because of certain affordances of these artefacts. Here we have an analysis of constraints and affordances of DGEs, mentioning such features like the "move from the spatial to the theoretical" and the importance of "dependency and functional relationships" within DGEs (quotes from section headlines, loc. cit p. 285). An important part of research on DGEs is about the "use of drag mode" (described loc. cit. from p. 286 onwards) with an interesting classification of drag mode uses developed by Italian colleagues (for a comprehensive presentation of this research see Arzarello et al. 2002). With respect to this paper, the most interesting result is a classification of drag modes, which should be seen as utilization schemes of the drag mode. Some of the descriptions of these modes explicitly give the goal of the utilization scheme and the rule of action ("Wandering dragging is moving the points on the screen randomly in order to discover configurations", Laborde et al. 2006, p. 287). It remains unclear if the Italian colleagues describe individual and/or institutionalised utilization schemes – presumably because the main focus of this research was exploratory and descriptive in order to better understand the use of the drag mode in DGEs.

Upcoming research on DGEs for truly spatial, three-dimensional geometry can additionally illustrate a phenomenon, which can easily described from an instrumental genesis approach, but could be a surprise for other research approaches: An analysis of the artefact 3D-DGEs immediately shows the necessity of innovative approaches to the question of how to design the input management of 3D-DGEs. Being of a planar, two-dimensional nature, the classical solutions for plane geometry DGEs (i.e.: mouse on a mouse-pad to allow for direct manipulation, display on a flat screen) do not work with three-dimensional geometry. In fact, two 3D-DGEs (ArchimedesGeo3D and Cabri-3D) present different solutions for this unavoidable problem. As a consequence, 3D-DGE-software already exists in the form of two fundamentally different artefacts. Research into the use of these artefacts shows consequences of these design decisions in terms of use of the artefacts (see Hattermann 2008). Hattermann succeeded in finding utilization schemes for 3D-DGEs, which could not be found in 2D-DGE research (like the "degree of freedom-test" for 3D-DGEs). One can also learn from this research that only data gathering of a certain duration offers a chance to identify utilization schemes.

In addition to utilization schemes for the drag mode, schemes of use for the locus of point feature of DGEs have also been studied (see Jahn 2002) – even if the paper by Jahn is not explicitly referring to schemes. Consequently, Jahn does not look into the characteristics of utilization schemes, but offers a description of learners' activities, which can be read in terms of instrumental genesis. To my best knowledge, macros – the third characteristic feature of DGEs - have not been intensively researched up to now (for some ideas on macros see Kadunz 2002).

Research on DGEs was always closely linked to a better understanding of Geometry and its teaching and learning, which could be seen in relation to the semiotic mediation potential of this artefact. The distinction of "robust versus soft constructions" (Laborde et al. 2006, p. 288) is a nice example of an artefact mediated concept with a high geometrical potential, namely the distinction between ephemeral and geometrically necessary features of a drawing / a figure (for the introduction of this distinction see Parzysz 1988).

In contrast to this evidence for the pertinence of the theoretical concepts, the discussion of the role of DGEs in the "proving and justifying processes" (loc. cit., p. 289) shows that semiotic mediation and instrumental genesis do not completely cover the diversity of the research into DGEs. The general research trends mentioned in the end of Laborde et al. confirm this ambivalent result on the coverage of the theories presented here: The "nature of geometry mediated by technology" (headline describing a research trend, loc. cit., p. 291) can be taken as an additional, general indication of the pertinence of the theories presented here. Research on task design and the role of the teacher is less closely linked to semiotic mediation and instrumental genesis – even if they can be linked to these concepts.

(METHODOLOGICAL) CONSEQUENCES FOR RESEARCH

If research in Didactics of Mathematics is theorising on teaching and learning mathematics with the help of instruments in the way described above, certain consequences for an adequate methodology of research are evident: The first and most obvious consequence is the insufficiency of a mere analysis of the affordances and constraints of the artefact "as such". In the course of the instrumental genesis, users tend to be more creative than the designers / engineers who develop the artefact. The concept of 'catachreses' (Rabardel 2002 in the beginning of chapter 7) is a reminder that human beings do not always restrict their use of artefacts to the way the developers or designers have thought of, but make unintended use of artefacts, sometimes even create utilization schemes, which are far away from those imagined by the creators of the artefact. "User studies" are of utmost importance if Didactics of Mathematics is interested in teaching and learning of mathematics mediated by instruments (and this seems to be meant for the majority of teaching/learning situations).

With the difficulty of identifying the theoretically most important 'operative invariants' of utilization schemes, research has to look into process-data of the use of artefacts and cannot do with mere product-data. An immediate consequence of this is that studies have to collect data on at least a certain time span in order to have a chance of capturing invariants within the data gathered. The option of finding the invariants with cross-individual comparisons within the product-data of a short moment, a "snapshot" type of data should be rejected because of the theoretical approach called "instrumental genesis" (mind the word 'genesis'!). Only a sort of "long" term research seems to be compatible with an instrumental genesis approach – even if this does not easily fit into the difficult budget situation of a research institution or the time constraints of a dissertation project. From the thesis of Rezat, one can also learn that innovative data gathering methods have to be invented in order to capture as much as possible of the use of an artefact without violating the ecological validity of the data.

As can be illustrated by the didactical tetrahedron, user studies (not only for textbooks and/or new information technology) can be defined by different types of users using the same artefact. An obvious illustration of this statement would be a study on textbooks as teaching and learning instruments of teachers (instead of learners) and would be as worthwhile as the one on instrumental genesis of student learners. The dissertation by Johansson (2006) on textbook use of Swedish mathematics teachers can be seen as a first attempt in this direction.

From studies in the "anthropological approach" inspired by Chevallard, one can learn about the difficulties of research into institutional instruments, i.e. artefacts and its institutional utilization schemes. The document analysis widely used within the anthropological approach may give an indication of the utilization schemes intended by those in power within educational institutions. Document analysis may be particularly appropriate within a nation with a centrally prescribed intended curriculum for all schools. Nevertheless, certain questions remain open: How to research institutional, societal instrumental genesis on the various levels of the educational system? How to research the differences between 'intended' and 'realised' utilization schemes, which proved so important in large-scale studies like TIMSS or PISA? What about the differences between the 'intended', the 'realised' and the 'achieved' curriculum and the respective social utilization schemes? And what about an analysis of institutional utilization schemes in comparison to individual utilization schemes for the same artefact?

In addition to this caveat and from the section on research into DGEs, it is also evident that more theoretical concepts (than semiotic mediation and instrumental genesis) are needed and should be used when analysing teaching and learning with the help of artefacts. The European project on comparing different theoretical approaches to technology use in mathematics education (for a short description cf. Artigue 2009, section 23.2.5) can be read as a hint to problems and potentials when different theoretical approaches are brought together, maybe even mixed or contrasted.

DIDACTICS IS A HUMAN SCIENCE! WHAT ABOUT MATHS?

As a sort of conclusion, I come back to my introductory statement about Didactics of Mathematics. The instrumental genesis approach – as a major way to theorise about teaching and learning mathematics with the help of artefacts or instruments – is an excellent illustration for the role played by human beings within Didactics of Mathematics. Using an instrumental genesis approach, Research in Mathematics Education proves to look into the way, people build up a relation to mathematics. Didactics of Mathematics looks into the relation between mathematics and its human users. It is a human science (in German: 'Humanwissenschaft'). To me, and from discussions with mathematicians, it is an open question how to classify Mathematics as a scientific discipline. Because of its (immaterial?) objects, Mathematics cannot be classified as natural science (any more, this may have been different some centuries ago). Some colleagues argue, Mathematics is the most material science because of the importance of inscriptions within Mathematics, which can hardly be overestimated. Calling Mathematics a structural science (because it is sometimes said to be the science of structures) is only sort of a way to avoid a classification. Or is Mathematics a special sort of human science – with its reliance on consensual finding of truth (see the empirical study of Heintz 2000) as a very special research methodology?

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PLENARY PANEL

- David Clarke (Panel Convenor)
- Deborah Loewenberg Ball
- João Pedro da Ponte
- Minoru Ohtani
- Barbara Jaworski

THEORETICAL PERSPECTIVES IN MATHEMATICS TEACHER EDUCATION

David Clarke

University of Melbourne

"Mathematics Teacher Education" identifies its primary concern as the learning of mathematics teachers, while the focus of that learning is the mathematics education of others. Conceiving of mathematics teacher education as a process in which both relevant knowledge and sophisticated practices are progressively constructed and refined, we become heirs to the wealth of theory developed to explain either knowledge construction or the incremental refinement of practice or both. The tasks of this plenary panel include the exploration of the possible role(s) of theory in the context of mathematics teacher education, the description of some relevant theories, together with illustrative examples of the use of such theories in illuminating or informing mathematics teacher education, and the identification of key issues, concerns or considerations in the selection and utilisation of theory in the study or implementation of mathematics teacher education.

INTRODUCTION

"Mathematics Teacher Education" identifies its primary concern as the learning of mathematics teachers, while the focus of that learning is the mathematics education of others. Naively, one can ask, "how (and why) should the theories of mathematics teacher education differ from the theories relevant to mathematics education?" One approach to this question is to consider the nature of the knowledge that mathematics teacher education seeks to nurture and promote. Only slightly different, is to consider the sophisticated practices that might represent the consequences of participation in effective mathematics teacher education. Although both these approaches take as their point of entry the goals, purposes and intended outcomes of mathematics teacher education, a theory of mathematics teacher education (if such a theory were to exist) must concern itself with the educative process whereby the knowledge and the practices are developed during the mathematics teacher's on-going progression from novice to increasingly accomplished teacher. In conceiving of mathematics teacher education as a process in which both relevant knowledge and sophisticated practices are progressively constructed and refined, we become heirs to the wealth of theory developed to explain either knowledge construction or the incremental refinement of practice or both. The tasks of this plenary panel include the exploration of the possible role(s) of theory in the context of mathematics teacher education, the description of some relevant theories, together with illustrative examples of the use of such theories in illuminating or informing mathematics teacher education, and the

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identification of key issues, concerns or considerations in the selection and utilisation of theory in the study or implementation of mathematics teacher education.

THE SITUATED NATURE OF TEACHER KNOWLEDGE

The connection of theory with practice is prioritised in all four plenary panel presentations. The presentation by Deborah Ball addresses the issue of what a practice-based theory of mathematical knowledge for teaching might look like and how it might contribute to mathematics teacher education. The grounding of theory in practice also constitutes a recognition of the culturally-situated nature of both the theory and of any recommendations it might generate regarding accomplished teaching practice. This possible cultural specificity is acknowledged by Ball and the co-authors of her paper, and does not necessarily detract from the potential value of their model of mathematical knowledge for teaching as an analytical tool for the study of mathematics teacher education. It is, however, worth giving thought to the situatedness of teacher knowledge; for example, with regard to the language in which the practices of mathematics teachers are described.

It might be expected that the internationalisation of the mathematics education community would afford an expansive reconception of the practice of mathematics teaching reflective of the wide diversity of classroom practices found in mathematics classrooms around the world. Ironically, this same internationalisation has strengthened the establishment of English as the lingua franca of the international mathematics education community and thereby silenced the expression of some of the subtle and sophisticated constructs by which mathematics teachers and teacher educators in non-English speaking countries would describe and evaluate the practices occurring in their mathematics classrooms.

This silencing extends to the articulation of those theories by which we might model, study, inform and optimise mathematics teaching practice. The most widely recognised example is the frequent mistranslation of the Russian term “obuchenie” (as employed by Vygotsky, 1962) as either “teaching” or “learning” when the term actually signifies the co-incident occurrence of both activities in the one practice collaboratively undertaken by teacher and students in their mutual pursuit of student mathematical learning (see Clarke, 2001, 2006, for a more complete discussion).

Our capacity to identify the occurrence of key elements of mathematics classroom practice (particularly those actions undertaken by mathematics teachers) is critically dependent on the pedagogical vocabulary at our disposal. The extent and sophistication of this vocabulary varies greatly from language to language. In my experience, English appears to be among the most impoverished languages in this regard. This has profound implications for any discussion of teacher quality conducted in English. If a teacher action can be named, then its occurrence can be identified and it becomes possible to consider whether the action is being carried out well or less well. If the action cannot be named then it will be absent from any catalogue of desirable teacher actions and consequently denied specific promotion in any program of mathematics teacher education. Actions considered as essential components of the mathematics teacher's repertoire in one country: for example, mise en commun[1] (France), pudian[2] (China) or matome[3] (Japan), will be entirely absent from any catalogue of accomplished teaching practices in English. The paper by Jaworski in this plenary panel highlights the absence in Norwegian of any term equivalent in meaning to "inquiry." Such absences have profound consequences, not just for our capacity to describe classroom practice or distinguish pedagogical alternatives but in the form in which we conceptualise and articulate our theories.

THE PROLIFERATION OF THEORIES

Mathematics teacher education as a field embraces knowledge and practices related to mathematics learning, mathematics teaching, and the mathematics curriculum, and each of these domains brings with it a suite of relevant theories. As a community concerned with mathematics teacher education and the role of theory, one approach is to organize the plethora of available theories in order to make clear the domain of relevance, the epistemological basis, the privileged constructs, and the aspirational intentions of each theory. Prediger, Bilkner-Ahsbahs, and Arzarello (2008) usefully distinguish static conceptions of theory, for example the definition offered by Niss (2007, below), from dynamic conceptions, in which a theory is seen as "a tool in use" (p. 166).

> "an organized network of concepts (including ideas, notions, distinctions, terms, etc.) and claims about some extensive domain, or a class of domains, consisting of objects, processes, situations and phenomena" (Niss, 2007, p. 1308).

However, even the most well-developed theories (for example, the Theory of Didactical Situations (Brousssseau, 1997), Variation Theory (Marton & Tsui, 2003), Activity Theory (Engeström, 1999) and Positioning Theory (Harre & v. Langenhove, 1999)) are under continual revision and refinement as they are adapted to meet the needs of new conditions and to address new research questions or to accommodate new data. It seems reasonable to require that a theory have evidence of the sort of situated structure characterised by Niss' definition, while also recognising the need

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[1] Mise en commun – a whole-class activity in which the teacher elicits student solutions for the purpose of drawing on the contrasting approaches to synthesise and highlight targeted key concepts.

[2] Pudian – typically, an introductory activity in which the teacher elicits student prior knowledge and experience for the purpose of constructing connections to the content to be covered in the lesson.

[3] Matome – a teacher-orchestrated discussion, drawing together the major conceptual threads of a lesson or extended activity – most commonly a summative activity at the end of the lesson.

for theories to be fluid and evolving according to the demands of new situations and the purposes of researchers.

The mathematics classroom is the nexus of a limitless variety of purposes, histories, and overlapping contexts and the application of any theory to such settings must be towards purposefully partial insights. Equally, the identification of relevant theory must include consideration of the nature of mathematics teacher education and of the contribution that theory/ies might make. In the second of the plenary panel papers, João Pedro da Ponte examines the nature of teacher knowledge and the elements that influence mathematics teacher education programs. Central for da Ponte in the selection or development of relevant theory are questions of purpose, position and role of theory in relation to programs and individuals, and of individuals in relation to theory and research.

THE SOCIO-CULTURAL CONTEXT OF MATHEMATICS TEACHER EDUCATION

Contemporary interest in the socio-cultural context of education (learning, teaching, and teaching-learning) is amplified when mathematics teacher education is viewed from an international perspective. At this point, the situatedness of all educative processes pervades not just considerations of the context in which mathematics teacher education is undertaken, but fundamental considerations regarding what constitutes knowledge for teaching and what is considered to be accomplished practice. Teaching as an activity is culturally situated to a greater extent than almost any other body of practice. It is possible that even medical and legal practice are not more culturally grounded.

The review of research into the teacher's role in classroom discourse by Walshaw and Anthony (2008) draws on an extensive literature that omits any reference to research into the practices of classrooms situated in Asian countries or to the writings of researchers situated in Asian countries. Given the attention afforded to the success of school systems in countries such as Japan, Korea and Singapore in international tests of mathematics achievement, this omission is all the more remarkable. The distinctive character of classrooms situated in different cultural traditions has been the subject of several publications (Clarke, Keitel & Shimizu, 2006; Fan, Wong, Cai & Li, 2004; and Leung, Graf & Lopez-Real, 2004). There is an internal coherence and consistency of message in the literature about classroom discourse arising from what might be called the Western canon[4] in educational research. The review by Walshaw and Anthony (2008) is particularly assertive regarding the role of student conversation in mathematics classrooms.

[4] Use of the term 'Western canon' draws on the literary connotations of the term and is intended to invoke associations both of claimed authority and of contested legitimacy.

> There is now a large body of empirical and theoretical evidence that demonstrates the beneficial effects of participating in mathematical dialogue in the classroom (Walshaw & Anthony, 2008, p. 523).

This statement is supported by a long list of references drawn entirely from Western sources.

By way of contrast, Li (2004) in discussing "A Chinese Cultural Model of Learning" made the following observation.

> As Kim (2002) has shown, Asian students not only do not believe that speaking promotes thinking as do Western students; they believe that speaking interferes with thinking (Li, 2004, p. 132).

The instructional practices of teachers in mathematics classrooms in different cultures are predicated on pedagogies that privilege different forms of student action in the mathematics classrooms and these pedagogies are, in turn, based on implicit theories of learning that assign spoken mathematics a very different function in the learning process (Clarke & Xu, 2008). Our theorizing about the connection between classroom practice and learning and, in particular, our research into the practices of mathematics classrooms must draw on (and contribute to) theories that accommodate culture as one essential aspect of the situated nature of learning, rather than ignoring the pervasive role of culture in framing our attempts at theorizing. This cultural situatedness is particularly evident in the varied writings on the role of language in the mathematics classroom. In the third panel presentation, Minoru Ohtani uses a culturally situated account of Japanese "Lesson Study" as the focus of a search for a fruitful theoretical perspective on mathematics teacher education and "Lesson Study" in particular.

THEORETICAL SELECTIVITY

Theories are among the tools by which we seek to "understand" the practice of mathematics teaching for the purpose of optimising its effectiveness. Just as particular theories are differently situated both epistemologically and with regard to those aspects of mathematics teaching to which they attend, so we, as researchers, are inclined to select our theoretical tools because the actions and outcomes they privilege resonate with educational values that we already hold. This theoretical reproductive process can only amplify our pre-existing assumptions regarding what is to be valued and what is to be discarded.

This theoretical selectivity can take at least two forms - the advocacy of certain practices based on the findings of research reflective of (i) a particular theory of learning or (ii) particular cultural norms. The concern with both lies in the possibility that the resultant recommendations ignore other, potentially effective alternatives, through the theoretically-enforced blindness of the research to specific constructs, processes, or outcomes. The recent synthesis of western research on teacher discourse by Walshaw and Anthony (2008), cited earlier, provides a powerful illustration of such "evidence-based" but culturally selective advocacy. As argued above, I would suggest that such cultural selectivity almost inevitably results in a form of theoretical selectivity embodied (but also possibly concealed) in the language in which the theory is framed and the actions to which it attends.

Of course, theoretical selectivity can be much more explicit. In their critique of "minimal guidance" instruction, Kirschner, Sweller and Clark (2006) make the insightful observation that "it may be an error to assume that the pedagogic content of the learning experience is [should be] identical to the methods and processes (i.e., the epistemology) of the discipline being studied" (p. 84). In particular, their assertion that "The practice of a profession is not the same as learning to practice the profession" (p. 83) highlights a critical issue in the design of instruction in mathematics, and identifies this issue explicitly with subscription to particular theories of learning.

The paper by Kirschner, Sweller and Clark (2006) questions the effectiveness of problem-based learning (PBL), increasingly employed in university settings in Medicine, Law, Engineering and Education. The characteristic that such courses share is their vocational specificity, which confers an intuitive appeal on any pedagogy involving problems rooted in settings and situations peculiar to the targeted profession. The other point of appeal of inquiry-based education in the professions is the plausible claim that such an approach simultaneously addresses content goals while also promoting the development of the clinical, forensic, or problem solving skills that represent a form of vocationally-specific super-curriculum. The socio-cultural learning theories typically invoked by the advocates of inquiry-based educational programs are epistemologically disjoint from cognitivist approaches. This is particularly evident in the use made of Activity Theory by both Jaworski and Ohtani, where collaboration and inquiry are prioritised as social practices by which teacher learning is advanced.

Nonetheless, it is possible to argue that mathematics teacher education is immune to the major criticisms that cognitivism would raise against inquiry-based approaches. Kirschner et al. (2006) suggest that the class of pedagogies they characterise as minimal guidance instruction are only successful "when students have prerequisite knowledge and undergo some prior structured experiences" (p. 82). Individuals being "taught" to be teachers of mathematics have an extensive experiential base, albeit one positioned differently in relation to the practices of the setting into which they are now being initiated as teacher. Of course, practising teachers should meet the prior knowledge and experiential criteria with ease. Even from a cognitivist perspective, mathematics teacher education may be one domain in which the learner is best supported through participation in the legitimate practices of the field. The four plenary panel papers are differently positioned in relation to the range of theories encompassed in the terms "cognitivist" and "socio-cultural." The value evident in the combination of perspectives reinforces the importance of relinquishing any aspirations to a single grand theory of mathematics teacher education in favour of a more pragmatic selective inclusivity that sees alternative theories as potentially complementary rather than necessarily opposed.

THEORETICAL INCLUSIVITY

To illustrate the operation of such theoretical inclusivity, consider the Interconnected Model of Teacher Professional Growth (Fig. 1 - Clarke & Hollingsworth, 2002). The Interconnected Model is an empirically grounded description of possible learning pathways for teachers of mathematics. There are similarities with the Ball et al. model of Mathematical Knowledge for Teaching, in that a specific theory of learning is not prescribed.

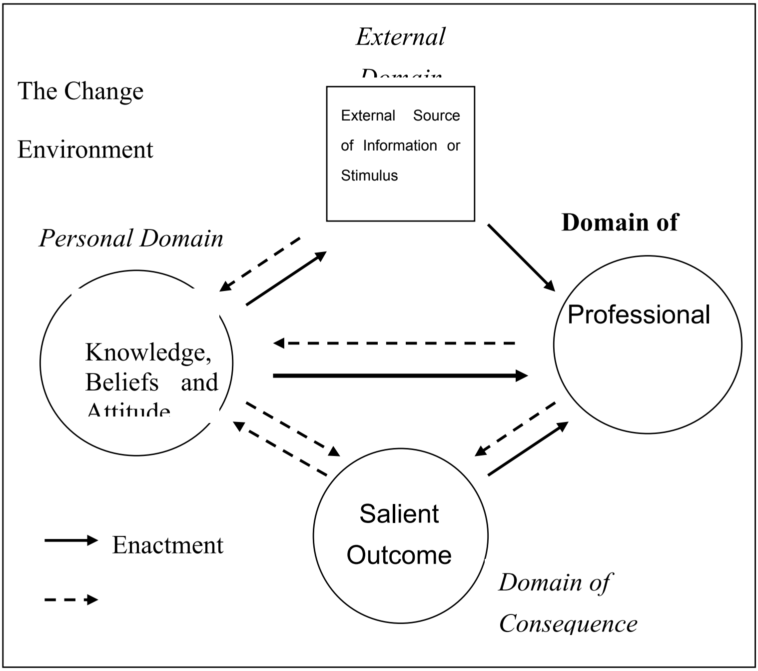

Figure 1. The interconnected model of teacher professional growth

Since the Interconnected Model is fundamentally a change model, realised through teacher/learner reflection and enactment, it accords priority to (i) experimentation/inquiry, (ii) values, (iii) knowledge construction as the internalisation of valued practice, and (iv) the catalytic role of the informed other. Essential to the Interconnected Model is the role of the "Change Environment" in facilitating or restricting the processes of enactment and reflection by which teacher professional growth is advanced. In this sense, the Interconnected Model aligns itself with socio-cultural theories according a situated character to learning. But these are many, and it can reasonably be asked, "With what theories of mathematics teacher education to mathematics teacher education is the Interconnected Model consistent?" The framing of such a question assumes a position of pragmatic inclusivity, that sees theories as tools to be employed where they are most likely to provide fruitful insights. The question further distinguishes empirically-derived models from the theories that might be invoked for their explication.

The Interconnected Model is consistent with any mathematics teacher education process that locates the responsibility for enactment and reflection with the teacher. Given its emphasis on classroom experimentation and consequent reflection, the Interconnected Model is easily aligned with teacher education programs adopting an inquiry-based approach, including both Japanese Lesson Study and the Norwegian inquiry-based program described by Jaworski. In the fourth panel presentation by Barbara Jaworski, the developmental process that results from critical alignment in an inquiry community is analysed through an activity theory perspective. Critical alignment offers a dynamic conception of community of practice (Wenger, 1998) in which practice evolves through a persistent process of reflective perturbation.

KEY QUESTIONS AND CONSIDERATIONS

Point One: By what criteria do we select those theories that might (i) accommodate (and possibly explain) the results of existing research into mathematics teacher practice and learning, and (ii) inform the future development of mathematics teacher education? Since any set of criteria reflect a system of values, this requires that we identify our aspirations regarding mathematics teacher education. But these aspirations presume a conception of "the effective mathematics teacher" and such a conception is, in turn, predicated on a set of valued student outcomes that would constitute the goals of mathematics teaching, the achievement of which would signify that effective mathematics teaching had occurred.

Point Two: Differences in something as fundamental as the prioritising of spoken mathematics suggests that the cultural situatedness of mathematics teaching practice is profound. Does it follow that mathematics teacher education is similarly culturally situated and that the content, the methods and the intended outcomes of mathematics teacher education programs will inevitably differ between cultures? One aspect of this cultural situatedness will be the roles accorded to mathematics teachers, teacher educators and researchers in the processes and practices of mathematics teacher education.

Point Three: What are the implications for mathematics teacher education, if we treat mathematics teaching "quality" or "competence" as a cultural artifact rather than a culturally-neutral absolute? Evidence and experience suggest that mathematics teacher education programs in different cultures appeal to different learning theories in promoting accomplished practice, and accord different roles to teachers and teacher educators in its achievement.

Point Four: The significance of synthesis. The criteria and the methods for methodologically inclusive research synthesis are under development (Suri & Clarke, 2009). There is also a need for strategies by which we might connect research results pertaining to different domains (eg. Calculator use and teacher questioning) within the broad field of mathematics education and interpret these findings in our mathematics teacher education programs. There remains the challenge of synthesising across research programs undertaken from distinct theoretical perspectives. For example, can our teacher education programs be informed by the findings and recommendations of research projects conducted from cognitivist and from socio-cultural perspectives?

It becomes incumbent upon the research community to: (i) acknowledge the influence of theoretical selectivity on research; (ii) report research in such a way as to make explicit both the theoretical positioning of the research and its implications for data construction and analysis; and, (iii) engage in the strategic synthesis of research undertaken from different theoretical perspectives.

In an academic environment in which theories proliferate, the papers in this plenary panel have been designed to raise some of the most important concerns about the selection, development, and utilisation of theories in the field of mathematics teacher education. Albert Einstein's aphorism is frequently quoted: "There is nothing so practical as a good theory." But which theory? And whose practice might it inform? These are questions that the mathematics education community needs to address specifically and regularly at the inception of each research project and each mathematics teacher education program.

A PRACTICE-BASED THEORY OF MATHEMATICAL KNOWLEDGE FOR TEACHING

Deborah Loewenberg Ball, Mark Hoover Thames, Hyman Bass, Laurie Sleep, Jennifer Lewis, and Geoffrey Phelps

University of Michigan

A central challenge of professional education is to prepare novices for skilful beginning practice. Doing this depends on robust theory about the relationship between teaching practice and teacher education. Theories of instruction can help provide a foundation for a professional curriculum centred on learning the practice of teaching. Teachers need to know and be able to use mathematics in the work of teaching pupils and they need to be able to carry off specific instructional practices that enable pupils to learn. This paper reports from a set of coordinated studies that have developed a practice-based theory of mathematical knowledge for teaching and theories about ways to teach it, designed assessments keyed to this practice-based theory of teachers' usable knowledge of mathematics, and developed a coding scheme for appraising the mathematical quality of instruction.

INTRODUCTION

A central challenge of professional education is to prepare novices for skilful beginning practice. Doing this depends on robust theory about the relationship between teaching practice and teacher education. Theories of instruction provide a foundation for a professional curriculum centred on learning the practice of teaching.

One particularly vexing problem has been to determine the mathematical knowledge needed for teaching. Teachers need to do more than simply *know mathematics*. They need to be able to *use* mathematics in the work of teaching pupils. Research on teachers' mathematical knowledge has investigated what practicing teachers *do* know, and our literature is replete with studies that show the lacks in teachers' content knowledge. Others have created lists of what teachers *should* know, based usually on the curriculum they are responsible for teaching or on expert opinion about what would be good for teachers to know. These lists are not empirically connected to the work of teaching - that is, we do not know whether teachers who know these things actually teach better than those who do not. Although important, neither of these approaches to identifying the mathematical knowledge needed for teaching is tied to the work of teaching, and hence, neither is warranted by the demands of the work.

The introduction of *pedagogical content knowledge* (Shulman, 1986) as a special domain of teacher knowledge was important for distinguishing the personal knowledge of content (knowing content for oneself) from the special amalgam of content and pedagogy needed to teach the subject. Important here was the naming of

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 95-98. Thessaloniki, Greece: PME.

a type of content knowledge uniquely needed by teachers—a subject-matter-based form of professional knowledge. The notion of pedagogical content knowledge quickly caught the imagination of researchers, not only in mathematics but also in science education. Still, the term was underdefined, and scholars and teacher educators used the notion in different ways.

MATHEMATICAL KNOWLEDGE FOR TEACHING

Our research group decided to investigate the question more directly by asking, "What do teachers *do* in teaching mathematics, and how does what they do demand mathematical reasoning, insight, understanding, and skill?" We oriented our investigation of the mathematical knowledge needed for teaching in studies of the practice of teaching. We sought to uncover the ways in which mathematics is involved in contending with the regular day-to-day, moment-to-moment demands of teaching. Our analyses lay the foundation for a *practice-based theory of mathematical knowledge for teaching* (Ball & Bass, 2003). This approach can be seen as a kind of job analysis, similar to analyses done of other mathematically intensive occupations that range from nursing and engineering physics (Hoyles, Noss, & Pozzi, 2001; Noss, Healy, & Hoyles, 1997) to carpentry and waiting tables.

By "mathematical knowledge for teaching" (MKT), we mean the mathematical knowledge *needed to carry out the work of teaching mathematics.* We focus on the tasks involved in teaching and analyse the mathematical demands of these tasks. Obviously, because teaching involves showing pupils how to solve problems, answering learners' questions, and checking their work, mathematical knowledge for teaching requires understanding the school curriculum. However, it also requires mathematical understanding beyond what can be seen on the tables of contents of school textbooks or in curriculum frameworks.

The fundamental questions that orient this theoretical approach are:

1. What are the recurrent tasks and problems of teaching mathematics? What do teachers *do* as they teach mathematics?
2. What mathematical knowledge, skills, and sensibilities are required to manage these tasks?

By "teaching," we mean everything that teachers do to support the learning of their pupils. Clearly we mean the interactive work of teaching lessons in classrooms, and all the tasks that arise in the course of that work. But we also mean planning for those lessons, evaluating pupils' work, writing and grading assessments, explaining learners' progress to parents, making and managing homework, attending to issues of equity, and justifying one's decisions to the school head.

Central to the progress of this work has been a large longitudinal database, documenting an entire year of the mathematics teaching in a grade 3 public school classroom. The records collected across that year include videotapes and audiotapes of the classroom lessons, transcripts, copies of pupils' written class work, homework, and quizzes, as well as the teacher's plans, notes, and reflections. Records of practice from a range of other U.S. classrooms have complemented this database. A second major resource has been the fact that our research group comprises individuals from a wide range of different disciplines and experience. By analyzing these detailed records of practice, with different perspectives and knowledge, we seek to develop a theory of mathematical knowledge as it is entailed by and used in teaching (Ball, 1999). An important weakness of this work is the gap left by building this theory on teaching practice in a single country (the United States). Given that teaching is a cultural practice (Stigler & Hiebert, 1999), the theory we are developing may be limited for explaining the mathematical demands of teaching in other cultural settings. Scholars have begun to test the validity of the theory in other countries (see Delaney, 2008, 2009; Delaney, et al., 2008), with promising results for the robustness and revisability and extension of a more culturally broad version of a practice-based theory of mathematical knowledge for teaching.

A MULTI-DIMENSIONAL MODEL

We define the mathematical knowledge we are studying as mathematical knowledge "entailed by teaching"—in other words, *mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to pupils*. To avoid a narrow perspective, we use a generous conception of "need" that allows for the perspective, habits of mind, and appreciation that matter for effective teaching of the discipline.

This program of research, based on studies of practice, and tested empirically (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2004, Hill, Rowan, & Ball, 2005) has yielded a multi-dimensional model of the mathematical knowledge needed for the work of teaching, depicted in Figure 1. We use the generic label ("content knowledge") to highlight the conceptual structure of the theory, and to encourage scholars in other disciplines to test its broader applicability.

Note the labels, "subject matter knowledge" and "pedagogical content knowledge" that head the model. On the left hand side of the diagram is the mathematical knowing in teaching that is purely *mathematics*; the right hand side comprises *mixtures* of knowing about pupils or knowing ways of teaching in the context of particular mathematical topics. By "common content knowledge" (CCK), we mean the knowledge of mathematics needed *in common* with others who also know and use mathematics. This is distinct from "specialized content knowledge" (SCK), which is a form of knowing mathematics not needed by those who do not teach. This specialized kind of knowing includes being able to figure out whether an unusual learner solution works in general or is just a fluke, for example, or considering the differences and equivalences among several different representations for a division problem.

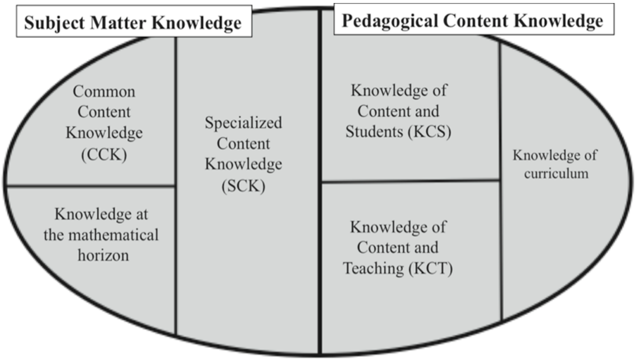

Figure 1. Mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008)

"Knowledge at the mathematical horizon" is the understanding of the broader set of mathematical ideas to which a particular idea connects. It is the sort of understanding that gives teachers peripheral vision for where they are and where their pupils are heading, to be conscious of the consequences of how ideas are represented, or the later development that is enabled — or possibly impeded — by decisions within the current work. The domains represented on the right of the diagram ("knowledge of content and students" [KCS], "knowledge of teaching and students" [KCT]; and "knowledge of curriculum) comprise the special amalgams that are deeply embedded in the work of teaching—knowing, for example, what makes a topic difficult for pupils, or the ways in which learners tend to develop understanding of a particular idea, or ways to sequence and structure the development of a mathematical topic, including representations likely to help pupils learn.

CONCLUDING SUMMARY

In summary, our approach to developing theory about the mathematical knowledge needed for teaching is framed in relation to practice. It is centered on the work and its demands rather than on what teachers do know or might need to know. Working from practice, we ask about the situations that arise in teaching that require teachers to *use* mathematics. Still, despite our expressed intention to focus on knowledge *use*, our categories remain rather static. Needed is to understand how teachers reason about and deploy mathematical ideas in their work. This includes skills, habits, sensibilities, and ways of reasoning as well as "knowledge." The questions we pose are intended to measure teacher knowledge in the context of its use, but how knowledge is actually used and what features of teacher thinking shape its use remain tacit and unexamined. How to capture the common and specialized aspects of the mathematical *reasoning* of teaching practice, as well as how different categories of knowledge come into play in the course of teaching, is a next important step in this program of work.

EXTERNAL, INTERNAL AND COLLABORATIVE THEORIES OF MATHEMATICS TEACHER EDUCATION

João Pedro da Ponte

University of Lisbon

Theories about mathematics teacher education depend critically on how one responds to questions such as: What is the nature of teacher knowledge? And how is such knowledge to be appraised? By the external views of the researcher, by the internal views of the teacher group, or by any other way? The external perspective of the researcher is able to mobilize important theoretical and empirical tools, however, this external perspective is often unable to grasp essential elements of the complexity of teacher knowledge, practice, and identity. Just capturing the views of teachers brings with it the concern that these views are often contradictory, assuming different values, orientations, and agendas from one teacher group to another, depending on grade level, school system, country, world region, and so forth. Another approach is to combine the experiences and perspectives of teachers and researchers. This paper explores the question of relevant theories in mathematics teacher education from all three perspectives.

In this paper I sketch some elements that I regard as important to understand teacher education and to develop theories about it. These include how we regard (i) teachers and their activity; (ii) teacher education processes, including settings and contexts; and (iii) the researcher's perspective.

THEORETICAL PERSPECTIVES ABOUT THE TEACHER

Mathematics teacher education concerns developing teacher knowledge, teacher practice, and teacher identity. Therefore, theories about mathematics teacher education depend critically on how one responds to questions such as:

1. What is the nature of teacher knowledge? Is it declarative knowledge that we can assess by asking the teacher orally (e.g., interviewing) or on a written way (e.g., testing), or is it practical knowledge that must be noticed through the observation of the actual activity of the teacher?
2. How is such knowledge to be appraised? Is it by the external views of the researcher, by the internal views of the teacher group, or by any other way?
3. What is the relation between teacher knowledge and teacher practices? Are practices just framed by teacher knowledge, or are there other factors that have an essential role in framing teachers' practices?
4. How does teacher identity develop, how does it relate to teacher knowledge and practice?

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Several responses are possible to these questions and the companion plenary paper by Ball et al. provides one model of teacher knowledge. Cognitive theory assumes (explicitly or implicitly) that knowledge and cognitive processes (such as knowing in action and problem solving) are the central elements that must be observed and understood to explain and to frame learning processes, including teacher education processes. For a long time, this was the dominant view about the teacher. Since the 1990s, however, we have witnessed the emergence of theories that emphasize social processes, such as social interactions between participants, communities of practice, and activity structures involving participants.

Research in mathematics teacher education has been strongly influenced by cognitive perspectives about the teacher, focusing, for example, on teachers' knowledge of mathematics, teachers' pedagogical content knowledge and teachers' beliefs and conceptions; more recently, social perspectives about cognition, including situated cognition, communities of practice, etc. are increasingly used as a lens to look at teachers (see, e.g., Ponte & Chapman, 2006). Open issues in this matter, include: a) Are social perspectives, just by themselves, a sound base to frame teacher education, or it is necessary to combine cognitive and social theories? b) If we need to combine theories, how might this be done? Is teacher development well described by general principles about learning (either cognitive or social) or do we need specific theories about it?

THEORETICAL PERSPECTIVES ABOUT TEACHER EDUCATION PROCESSES, SETTINGS, AND CONTEXTS

Mathematics teacher education is a complex process that takes place in formal teacher education settings and in informal situations that involve the teacher but also other actors (such as teachers from the same and other subjects, students, parents, and school administrators), and that is related to the social and institutional features of the society and school system in which the teacher works.

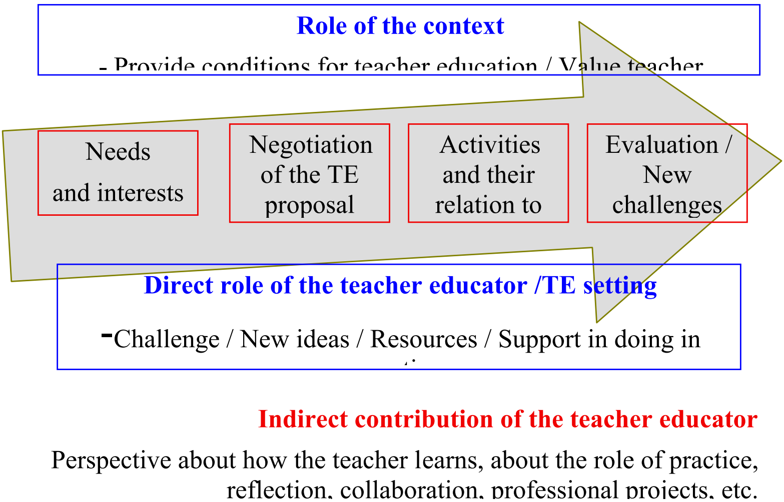
In fact, there are many elements that influence the teacher education process, either at pre- or in-service levels (see Ponte & Chapman, 2008). We must note that teachers' knowledge, teachers' practices and teachers' identity are at the centre of this process, but their development must be understood in reference to the collective identity of the professional community of teachers, including the established values and norms of the profession and the processes of professional interactions.

Elements that influence the nature of mathematics teacher education programs include, for example:

1. Teachers' characteristics, including their motives, interests, knowledge, beliefs, and conceptions.
2. The characteristics of teacher educators and of the other social actors related to the activity of the teacher.

3. Program features such as teaching approaches, purposes and objectives, curriculum and materials, assessment instruments and procedures, and their overall organization and pedagogical approach.
4. Sociocultural features of the society, including the roles and values promoted by government authorities, the media and the general public.
5. Organization of the educational system, including processes related to teachers' careers (such as recruiting and inducting into the profession, certification, contracts, promotions) and to the curriculum and evaluation system.
6. Research, including its emphases, values, priorities, and ways of disseminating results.

Teacher education, either carried out as formal processes, or just induced by informal settings, therefore, requires careful design (Loucks-Horsley, Hewson, Love & Stiles, 1998). In designing such a teacher education process, we assume an underlying theory about the needs and interests of the teachers and about the activities that they will perform to promote such development. Such activity is framed by an initial negotiation and a final evaluation that may involve the identification of new challenges. Also critical are the roles of the context and the direct and indirect contributions of teacher educators.



At the core of the design of the teacher education process is a view about the relationship between theory and practice, concerning how teachers learn or develop professionally: a) Do they learn directly from practice, whether it is the practice of a mentor or of a more experienced teacher, or their own practice, with no explicit role for theory? This is the "pedagogical tradition" that informed teacher normal schools in the XIX century, and well as some views of teacher education as apprenticeship; b) Do teachers first learn theory, to apply later in practice? This is the "modern" or "technical rational" view, still prevalent today in most pre-service and in-service teacher education courses; or, c) Do they learn theory and practice, in some combination, using theory to question practice and using practice to identify and understand empowering theory? This is an emerging "inquiry tradition", which receives currently wide support among mathematics teacher educators (Lampert & Ball, 1998; Llinares & Krainer, 2006) and is the subject of the accompanying plenary panel paper by Jaworski.

Questions to consider on this matter include, for example: a) Is there a role for each perspective, related to some kind of objectives, in some specific contexts? b) What hinders a wider dissemination of the inquiry perspective? And, c) What may be specific to designing *mathematics* teacher education processes?

THE RESEARCH PERSPECTIVE

Another dimension in which theories of teacher education differ concerns the relationship that the researcher has with the teacher education process. Is the researcher an agent external to the teacher education process that just analyses it from the outside? Is the researcher someone that tries to study the perspectives of the participating teachers, from their point of view? Is the researcher someone that studies the process essentially from the point of view of the teacher educator?

1. The researcher that adopts an external perspective is able to describe many important aspects of teaching that may be of interest to different social groups (and politicians). However, this external perspective is often unable to grasp essential elements of the complexity of teacher knowledge, practice, and identity. In some cases, it produces results that are unhelpful to support teacher development, reinforcing the view of the "deficient" teacher, so common in the research literature.

2. Capturing the views of teachers may reveal the reason why participants do what they do, showing that they are rational actors. However, teachers' views are often contradictory, assuming different values, orientations, and agendas from one teacher group to another, depending on grade level, school system, country, world region, and so forth. Inside a single school there are often teachers with opposite views and practices in relation to essential issues of curriculum, teaching and learning. As a consequence, knowing teachers' perspectives is essential to frame teacher education, but it is not enough to design it.

3. Another approach is combining the experiences and perspectives of teachers and researchers. This can be achieved in a variety of ways – educating teachers to act as researchers, letting researchers teach in elementary and secondary schools, or promoting collaborative projects involving teachers and researchers (Jaworski, 2004). To study the intricacies of teacher knowledge, practice, teachers by themselves, as it enables each one to bring their own expertise to shed light in very complex issues. Such an approach has its own problems: Who sets the agenda? How are activities negotiated? What happens to the knowledge generated? Does it have a real effect on

teacher practice? And on teacher education practice? Ohtani's paper in this plenary panel addresses these issues explicitly in the context of Japanese "Lesson Study."

CONCLUSION

I finish with some provisional thesis about our topic:

1. Theories about teacher education are needed to design, evaluate, and research teacher development and teacher education processes.

2. Such theories need to make explicit a view about the teacher, teacher learning, and teacher professional development as well as about teacher education processes, contexts, and designs.

3. Such theories need to take into account the role of the researcher *vis a vis* the teacher education process, as external, internal, and collaborative perspectives will always provide different (complementary) pictures of the process.

IN SEARCH OF THEORETICAL PERSPECTIVE ON THE "LESSON STUDY" IN MATHEMATICS

Minoru Ohtani

Kanazawa University, JAPAN

In this paper, Japanese "lesson study" is used as a situative vehicle to support discussion of one theoretical approach to mathematics teacher education. In this presentation, three aspects of lesson study will be addressed: (i) what is meant by "lesson study"; (ii) how is "lesson study" organized; and (iii) why is "lesson study" important for teachers' professional development. Based on these aspects, the presenter proposes that "Activity Theory", especially "Developmental Research" (Engeström, 1990), provides a promising scheme for understanding communal cultural practice in mathematics teacher education.

"LESSON STUDY" IS CULTURALLY-HISTORICALLY SITUATED

Recent international comparative research (eg., Clarke et.al, 2006) shows persuasive vision of mathematics classroom practice as a socio-culturally mediated milieu. Detailed analysis on mathematics lessons taught by "competent" teachers revealed complex variations both across classrooms and among countries with respect to task structure, lesson events, language use and so forth.

I believe that this is also true for mathematics teachers' professional development. Relevant evidence can be found in the recent enthusiasm in the US and elsewhere for Japanese "Lesson Study" which is a good mixture of grass-rooted practice and governmental support, tracing its origins back almost one hundred years. In his PhD dissertation, Yoshida gave a detailed ethnographical description of how "Lesson Study" is organized in one Japanese elementary school (Fernandez & Yoshida, 2004). "Lesson Study" follows several steps, which consist of discussing lessons teachers planned and observed collaboratively, called "study lessons" (Fernandez & Yoshida, 2004: 7-9):

- Collaboratively planning the study lesson;
- Seeing the study lesson in action;
- Discussing the study lesson;
- Revising the lesson (optional);
- Teaching the new version of the lesson (optional);
- Sharing reflection about the new version of the lesson.

Basically, almost all elementary and the majority of middle schools conduct "Lesson Study" within the school on a regular basis in which entire teaching staff of the school engage in a sustained manner in order to attain school educational objectives. Pre-service and novice teachers also engage in "Lesson Study" in collaboration with

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university mentors, expert teachers and inspectors from the local education board. In some case, groups of teachers from different schools organize voluntary circles and conduct "Lesson Study" in accordance with common interests. Thus, Japanese "Lesson Study" is seen as a typical form of Japanese teachers' professional development.

THE COLLABORATIVE AND SYSTEMIC NATURE OF "LESSON STUDY"

Among important lessons drawn from Japanese "Lesson Study", Yoshida puts it:

> [Japanese practice of lesson study] provides us with a vision for how to create a process whereby teachers can engage in collaborative, sustained, and grounded reflection about their practice from which they stand to learn great deal. …. In the United States there is currently much talk about the need to professionalize teaching. In the context of these conversations one often hears calls to break the pervasive isolation of teachers that makes it difficult for them to learn from each other and develop a joint body of professional knowledge (Fernandez & Yoshida, 2004: 229-230).

Such pervasive isolation of teachers does not happen in Japanese schools. A simple example is the

configuration of the teachers' staffroom (Fig. 1).

Fig. 1: A typical configuration of a Japanese teachers' staffroom

The figure shows a typical Japanese school teachers' staffroom, where every teacher has their own working desk with laptop PC, teaching materials, and several kinds of documents. Teachers from the same grade level are seated beside each other for collaboration (a kind of division of labour). They prepare and discuss their lesson plans when they have time to spare. In the picture (Fig. 1), a female teacher is checking her lesson plan for later discussion with her colleagues in order to improve the study lesson. In the upper right hand corner, we can see "Mission Statements" of the school. Thus, the configuration of the staffroom sets a base for collaborative professional development of teachers in Japan.

Another important factor that facilitates collaborative practice is the unstable staff recruitment process. At the end of the school year, the local education board announces personnel changes of every school's staff. Part of school staff, including principal and vice-principal, goes out and comes in each year. This means that Japanese teachers have no freedom to choose at which school they will teach. Such personnel relocation or reshuffling makes further collaboration among school staff essential. In order to attain educational objectives and to cover the school-based curriculum with new members, it is crucial for the school to establish a community of practice. Such socio-institutional conditions require "Lesson Study". "Lesson Study" functions as not only school-based professional development but also to ensure the maintenance of school missions and the covering of the school-based curriculum. School as a communal activity system is not stable but evolves through a systemic "Lesson Study" approach.

Recent educational reform and movements such as the revision of the national mathematics curriculum and the introduction of national assessment of mathematics literacy have significant impact on school practice. Such reform movements generate new forms of expansive learning opportunities and distributed agency on the part of the teachers. Essentially, "Lesson Study" functions as the mechanism for adapting to such changes.

THEORETICAL PERSPECTIVE ON "LESSON STUDY"

As exemplified at various points in the above two sections, a promising theoretical perspective on "Lesson Study" seems to be "Cultural-Historical Activity Theory" (Engeström, 1990). I believe it offers a powerful theoretical lens for describing and analyzing school communal practice. Starting from Vygotskii's theory of mediated action (Vygotskii, 1984) and Leont'ev's concept of collective and systemic formation with multiple mediation (Leont'ev, 1975), Engeström elaborated a triangular structure (Fig. 2), which can be applied to various social activity systems.

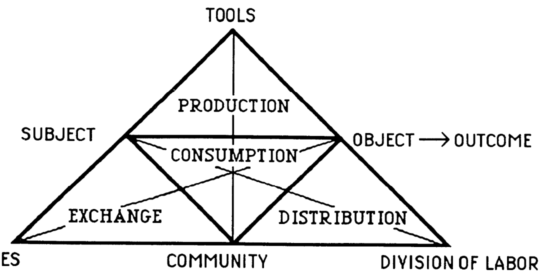

Figure 2: The meditational structure of an activity system (Engeström, 1990: 79)

The top section of the figure comes from Vygotskii's individual tool-mediated action in cultural-historical settings. However, individual action has meaning only in relation to the components at the bottom of the triangle, which includes other people who share the same object (community), social norms and conventions (rules), and the division of object-oriented actions among members of the community (division of labor). In view of the upper part of the triangle, "Lesson Study" is considered as a tool, which mediates between teacher (subject) and students (object). However, "Lesson Study" exists as such only in relation to the components at the bottom. As a rule, the teacher has to prepare lesson plans in accordance with a certain format, which is agreed upon in the school community. The teacher makes lesson plans in collaboration with others. Teachers are divided into small sub-groups, which are in charge of certain grades. Components at the bottom of the diagram play an essential role in Japanese "Lesson Study".

"KNOT WORKING" AMONG DIFFERENT ACTIVITY SYSTEMS

The activity-theoretic approach also aims at the construction of a new model of activity jointly with a different activity system. In essence, the activity-theoretic approach is interventional and transformational teamwork. In that sense it is called "Knotworking" (Engeström, 2008). Engeström puts it:

> The notion of *knot* refers to rapidly pulsating, distributed, and partially improvised orchestration of collaborative performance between otherwise loosely connected actors and activity system. Knotworking is characterized by movement of tying, untying, and retying together seemingly separate threads of activity (Engeström, 2008: 194).

Although as yet only limited research directly uses this approach, it seems to offer a powerful perspective for future research.

An example of such "Knotworking" is the collaborative design of a teaching unit of direct proportion with teachers from elementary, secondary, and tertiary education (Ohtani, 2007). Elementary school focuses on teaching general characteristics of proportion in the range of the positive rational numbers. Proportion is formulated as an internal ratio using a numerical table. Then, a graph is introduced. The expression "y= fixed number×x" is not taught in elementary level. On the other hand, in junior secondary school, the importance of numerical tables is reduced and algebraic expressions of the form $y=ax$ become predominant instead. Thus, in the teaching of proportion, many differences are found between teaching in elementary school and in junior secondary school. Teachers from different activity systems have tried to resolve the problem of curriculum articulation. This kind of temporal collaboration is also becoming visible in the Japanese "Lesson Study" tradition.

CRITICAL ALIGNMENT IN AN INQUIRY COMMUNITY

Barbara Jaworski

Loughborough University

This paper takes as its entry point developmental research being undertaken between mathematics teachers and teacher educators (as both researchers and practitioners) in a European setting to promote better understandings in theory and in practice of mathematics teaching-learning at all levels of education. Developmental research is research with both insider and outsider dimensions in which practitioners inquire into their own practices and/or research the practices of others. Such research both charts the developmental process and contributes to development. The paper draws on research projects in which teacher educators and teachers have formed inquiry communities to develop mathematics teaching and learning. The paper discusses how these perspectives cohere or alternatively raise issues through which complexity in mathematics teaching-learning can be addressed.

INTRODUCTION

I start from a question raised in the introductory paper by David Clarke, and the two responses that follow this question:

> How (and why) should the theories of mathematics teacher education differ from the theories relevant to mathematics education?"
>
> One approach to this question is to consider the nature of the knowledge that mathematics teacher education seeks to nurture and promote. Only slightly different, is to consider the sophisticated practices that might represent the consequences of participation in effective mathematics teacher education

I should like to add a short, but important rider to the first response, that is to consider also the *knowers* involved and *how their knowing relates to the impact of knowledge on or in practice*. The idea of impact on or in practice then relates strongly to the second of the two responses. The *sophisticated* practices are sophisticated, or we might say, also, *complex* because they are rooted in the different worlds of those who engage – the students, teachers, academics, researchers, mathematicians, school leaders, parents and all other stake holders in the mathematics education process. The ways in which we configure these worlds and their participants in relation to mathematics teacher education knowledge and practice is theory-based. It is my view that theory in mathematics teacher education differs from that in mathematics education because of the necessary complexity of inter-relationships between the knowers involved. I slipped in the word "we" in the above sentence, and the nature of this "we" is fundamental to conceptualisation of mathematics teacher education practice. Who are "we"? For example, Joao Pedro da Ponte draws attention to a propensity (evident in the literature) of researchers in mathematics education to treat teachers as deficient; that is as a group of people who lack knowledge and have to

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 109-116. Thessaloniki, Greece: PME.

somehow be enabled to gain the knowledge they lack. Such a position is theoretically rooted in a conception of knowledge as an external entity and the possibility of conveying knowledge from those who have it to those who do not. If, on the other hand, all those engaged in mathematics teacher education practice (the "we") are seen as co-learners in an educational enterprise with multiple layers of goals, each bringing essential knowledge related to their own communities and practices, then the theoretical groundings on which mathematics teacher education is based look very different. It is from this latter position that I offer a perspective.

AN INQUIRY COMMUNITY

To simplify somewhat, I consider mainly *two* groups of participants, *teachers* and *teacher educators* – the latter group is also referred to variously in the literature as educators, academics, didacticians or professional development leaders. I reserve the word *researchers* to refer to those practitioners who *also* engage in research, and this can include participants from either of the two groups, thus *teacher-researchers* or *teacher-educator-researchers*. In mathematics teacher education, the two groups might be seen to have a common goal – to improve opportunity for students to engage with mathematics in the best possible ways to support and build their mathematical concepts and fluency. Because I talk here about complex practices, it seems clear to me that the *best possible ways* are what we are all striving to know. Neither of the two groups has this knowledge in any absolute sense, but both bring important elements of it; through a co-learning process, both can increase their own knowing and importantly the community knowing on which practices can be well founded. A collaborative enterprise with mutually understood goals is fundamental; so is a mutually expressed desire to look critically at current practices to inform understandings and make productive development possible.

Thus, I am proposing a process of critical, collaborative co-learning and the construct that is constantly reaffirmed for me as central to this process is that of *inquiry*. Inquiry is about asking questions and seeking answers, recognising problems and seeking solutions, exploring and investigating to find out more about what we do that can help us do it better. The overt use of inquiry in practice has the aim of *disturbing* practice on the inside, of challenging the status quo, of questioning accepted ways of being and doing. Such use of inquiry starts off as a mediating tool in the practice, and shifts over time to become an *inquiry stance* or an *inquiry way of being in practice* (Cochran-Smith & Lytle, 1999; Jaworski, 2006).

I find that the theory of community of practice (CoP), as introduced by Lave and Wenger and particularly as developed by Wenger (1998), provides a grounding position from which to conceptualise inquiry activity. Wenger speaks of people *belonging* to a community of practice, having *identity* with regard to a community of practice, in terms of three dimensions: *engagement, imagination* and *alignment*. So, for example, in practices of mathematics learning and teaching, participants *engage* in their practice alongside their peers, use *imagination* in interpreting their own roles in the practice and *align* themselves with established norms and values of teaching within school and educational system. Holland, Lachicotte, Skinner and Cain (1998, p. 5) write, "Identity is a concept that figuratively combines the intimate or personal world with the collective space of cultural forms and social relations". Identity refers to *ways of being* (Holland, et al. 1998) and we can talk about *ways of being* in teaching-learning situations, which assume *alignment* with what is normal and expected in those situations. For example, the mathematics teachers within a particular school have identity and alignment related to their school as a social system and group of people. Any individual teacher or teacher educator has identity related to their direct involvement in day to day practice, but constituted through the many other communities with which the individual aligns to some degree. Individual identity and group identity are complexly related.

CRITICAL ALIGNMENT

A community of practice becomes a community of inquiry when participants take on an inquiry identity. That is, they start overtly to ask questions about their practice, while still, necessarily, aligning with its norms. In the beginning, inquiry might be seen as a tool enabling investigation into or exploration of aspects of practice – a critical scrutiny of practice. Thus, we see an inquiry identity growing within a CoP and the people involved becoming inquirers in their practice; individuals, and the community as a whole, develop an inquiry way of being in practice, so that inquiry becomes a norm of practice with which to align. We might see the use of inquiry as a tool to be a form of *critical alignment*; that is engagement in and alignment with the practices of the community, while at the same time asking questions and reflecting critically. Critical alignment, through inquiry, is seen to be at the roots of an overt developmental process in which knowledge grows in practice (Jaworski, 2006).

Such a theoretical position accords well with research in a developmental paradigm: that is where the research itself is seen both as a tool to promote development and as a means of studying the developmental processes involved (Jaworski, 2003). Developmental research typically involves collaboration between groups of practitioners with goals to develop the practices involved. Thus, there is explicit intention to look critically at practice, to innovate and to explain current and new practices. Such research might involve action or design cycles in which practitioners plan their activity, observe it critically during action, reflect analytically on their observations and feed back to further cycles of activity. Reflection and analysis lead to new knowledge in practice and, with further analysis, to new knowledge in the academy.

With regard to practical settings, there are now many documented projects in which academics and teachers work together as researchers to learn more about provision of mathematical opportunity for students and students' associated achievement in mathematics. In one such project in Norway, teachers and didacticians worked together to create a community of inquiry in which both had their own roles and goals associated with their respective knowledge in practice. For didacticians particularly, this related to the design of mathematical tasks for workshop settings with teachers, and to an associated promotion of pedagogic and didactical issues relating to classrooms. For teachers particularly, it related to their participation in workshops leading to design of activity for students in their own schools together with teacher colleagues and (if invited) didacticians. Teachers and didacticians each held specialist knowledge, which together allowed exploration of complex situations linking workshops and school classrooms with goals for students' learning. Overt use of inquiry led to each group looking critically at their own engagement, while aligning with the norms and expectations of that engagement.

Unsurprisingly, the flow of activity and learning was neither simple nor smooth. The groups of practitioners had much to learn about each other's ways of seeing, and ways of engaging in practice. A mutual discourse had to develop in order for genuine sharing to take place. For example, there is no Norwegian word to translate "inquiry", so the search for an understanding of what "inquiry" might mean had to include finding words in the language that might express meanings suggested by theory and in the literature. Activity had to allow for differences to be recognised and debated; such debate, even though not always comfortable at the time, led to an increased depth of awareness which fed back to a more knowledgeable practice. For example, didacticians' visions of planning for inquiry in classrooms had to be modified significantly before they could appreciate what inquiry in classrooms might realistically entail. Teachers had to recognise their own power to influence development, drawing on knowledge that went beyond what didacticians could offer. We used activity theory, particularly the expanded mediational triangle of Engeström (1999) to trace such issues and tensions and explain outcomes (Jaworski & Goodchild, 2006).

SUMMATIVE COMMENTS

The theoretical ideas which I have no more than sketched here provide both a foundational underpinning in which all people and groups are seen as knowers and learners and a working paradigm in which practices are acknowledged and respected, people's involvement in those practices valued, mutual goals and activity conceptualised, and issues and tensions recognised and traced. Respect and value are of course closely related to the identities, practices and ways of seeing of those involved. This is where inquiry and critical alignment are extremely powerful. Inquiry as a tool starts to open up possibilities, to challenge existing assumptions and to create opportunity for innovative practice. Development of an inquiry way of being allows community awareness to grow and particular sensitivities to be recognised and tackled. An important outcome for any participant group is to become more aware of the knowledge and needs of those in other groups. From such awareness, particular developmental activities can be planned jointly to draw on the recognised expertise and to maximise collaborative potential. Such outcomes take considerable time. The Norwegian projects are into their sixth year, and it is only now that some of the more powerful potentials are becoming evident and realisable.

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RESEARCH FORA

Teacher knowledge and teaching: ***considering a complex relationship through three different perspectives***
Co-ordinators: Deborah L. Ball, Charalambos Charalambous, Mark Thames, Jennifer M. Lewis

Critical perspectives on communities of mathematical inquiry
Co-ordinator: Susie Groves

Mathematical gift and promise: ***exploring and developing***
Co-ordinators: Alexander Karp, Roza Leikin

Sociological frameworks in mathematics education research
Coordinator: Stephen Lerman

The enactivist theory of cognition and mathematics education research: ***Issues of the past, current questions and future directions***
Coordinators: Jérôme Proulx, Elaine Simmt, Jo Towers

RESEARCH FORUM 1

Teacher Knowledge and Teaching:
considering a complex relationship through three different perspectives

RF1: TEACHER KNOWLEDGE AND TEACHING: VIEWING A COMPLEX RELATIONSHIP FROM THREE PERSPECTIVES

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This paper introduces our research forum, planned and conducted with colleagues from Canada, England, Germany, Israel, and the United States. The first section frames the problem and describes our approach, which includes the analysis of two video segments from three different perspectives. We then describe how we selected the video segments and provide an overview of the two segments.

FRAMING THE RESEARCH FORUM

Attention to teacher knowledge is not new. Indeed, more than two thousand years ago, Aristotle argued that, "teaching is the highest form of understanding." Shulman's (1986) call for increased attention to teacher knowledge restored scholarly interest in teachers' intellectual resources and their role in teaching. Yet, despite significant progress, we are still at the beginning of untangling the "complicated relationship [that] exists among teachers' knowledge, their teaching practices, and student learning" (Mewborn, 2003, p. 47).

Seeking to increase our collective understanding of this relationship, this research forum brings together three different perspectives in conceptualizing teacher knowledge and its role in teaching. Briefly, these perspectives are *Mathematics for Teaching* (Davis & Simmt, 2006), the *Knowledge Quartet* (Rowland, Huckstep, & Thwaites, 2005), and *Mathematical Knowledge for Teaching* (Ball, Thames, & Phelps, 2008).

Brent Davis and his research group view the notion of mathematics for teaching through the interpretive framework of complexity science, a theory that examines how systems learn. Defining *mathematics for teaching* as a "distinct branch of mathematics," these researchers organize their inquiries around "concept studies"—a collective learning structure through which mathematics educators identify, interpret, interrogate, invent, and elaborate images, metaphors, analogies, examples, exemplars, exercises, gestures, and applications that teachers invoke (explicitly or implicitly) in efforts to support student understandings.

The *Knowledge Quartet* of Tim Rowland and his colleagues represents a practice-based framework developed inductively from analysis of videotaped lessons taught by novice teachers. This framework comprises four categories of knowledge: *foundations* include teachers' knowledge, beliefs, and understanding acquired before and during teacher preparation; *transformations* concern aspects of knowledge-in-

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action as demonstrated during planning and in teaching; *connections* pertain to knowledge displayed when teachers draw connections between and among mathematical ideas; *contingency* is manifested by how a teacher responds to unanticipated events that emerge during instruction.

Also studying records of practice, Deborah Ball and her colleagues define *Mathematical Knowledge for Teaching*, functionally, as the knowledge demanded by the work of teaching. This knowledge supports a teacher in routine tasks of teaching, such as providing and evaluating explanations of why certain algorithms work, selecting and using appropriate representations, and understanding student errors. They distinguish *common content knowledge* needed in other mathematically intensive professions from *specialized content knowledge,* unique to the work of teaching, and they distinguish these two domains, not requiring knowledge of students or pedagogy, from pedagogical content knowledge as characterized by Shulman (1986), which they refine into several sub-domains.

These three perspectives—neither representative nor exhaustive—conceptualize the relationship between teacher knowledge and practice in highly developed but contrasting ways. The purpose in bringing them together is to reveal connections and potential synergies that would contribute to the field's understanding of assumptions, options, and concerns in addressing the mathematics that matters for teaching. To this end, each research group sought to address the following research questions using its perspective to analyze two segments drawn from different mathematics lessons:

> RQ1. What is the nature of the mathematical knowledge needed for the work of teaching mathematics? How is this determined?
>
> RQ2. How is this knowledge actualized in teaching? How do we know?

Because we consider the selection and the nature of the two segments consequential for our analysis and findings, the remainder of this paper documents the process and the challenges encountered in selecting these segments and then it briefly describes the two segments. The three papers that follow this introductory paper address the two research questions, each drawing on one of the perspectives. Commentaries then explore potential synergies among these perspectives.

SAMPLING TEACHING: AN OFT-UNSPOKEN PART OF ATTEMPTS TO EXPLORE THE FOCAL RELATIONSHIP

At the outset of our work, it became clear that selecting samples of teaching for analysis was actually the first step toward addressing our research questions: the samples *frame* the arguments a researcher can make about the knowledge needed for the work of teaching as well as how this knowledge is manifested in teaching.

Take, for example, a lesson segment in which students rarely speak in the lesson. Analyzing such a segment cannot tell us much about the mathematical demands of interacting with students (of interpreting and responding to their thinking), simply because such interactions are not immediately evident. Or consider another extreme case, in which students work on an investigation and the teacher observes them without intervening. Again, with little access to what is "going on" for the teacher, we might be limited in what we could say about the insides of the work of teaching and the knowledge needed for that work.

Excluding these extreme cases, however, does not simplify the matter: Should the chosen segment represent "exemplary" or "typical" teaching? What types of practices should the lesson segment feature and why? Should it illustrate mathematical "affordances" that enable what teachers can do or mathematical "limitations" that impinge on the work? Does grade level, mathematical strand, or topic matter? Would it matter if we analyzed a lesson on the meaning of fractions rather than on operations with fractions? Also, which parts of a lesson (e.g., introduction, group discussion, or conclusion) should one pick? Are these parts meaningful and can they be identified in lessons from different countries? Although not exhaustive, these questions suggested a need for explicit selection criteria.

Once we ventured into setting such criteria we were confronted with further challenges. Take, for example, the question of sampling between "exemplary" and "typical" teaching. Beyond the research questions for this forum, the choice between "exemplary" and "typical" teaching depends fundamentally on researchers' questions and practical agendas. For instance, are we interested in mathematics for modal practice, for expert teaching, or for specific kinds of teaching? And, what about the ambiguity of terms such as "exemplary," even among members of the same research group? Later, after a first pool of teaching samples was identified, it became clear that these samples were illuminating certain aspects of teaching but obscuring others. Furthermore, in these samples, what was "visible" to some of us was not readily accessible to others.

Our decision to document some of these challenges is meant to encourage other researchers to do so because we believe that this important part of the process of exploring the connection between teacher knowledge and practice is often left unspoken. To select segments, we adapted three criteria from Thames (2009). The segments provide: (i) access to *interactions among the teacher and students around content*, as these are central to teaching; (ii) access to *teacher thinking and reasoning during instruction* (perhaps implicitly), as these are key to making sense of the teaching and gaining insight into the practice; and (iii) *enough of the context* to make the actions of the teacher and students interpretable (such as access to the curriculum materials being used).

THE LESSON SEGMENTS AND SOME CONTEXTUAL INFORMATION

The analyses presented in the next three papers are based on segments drawn from two different lessons: one given to students (age 6-7) by a preservice teacher (Chloe) in the U.K. and another delivered to upper elementary students (age 10-11) by a seasoned teacher (Karen) in the U.S.A. (Teacher names are pseudonyms.) The two clips have points in common and points of contrast. Both segments are introductory parts of lessons and address number and operations, a prominent curricular strand in both countries. However, they represent lessons given by teachers differing in years of experience to students of different grades in different countries. Furthermore, although both pertain to numbers and operations, Chloe's segment focuses on figuring out strategies for adding and subtracting numbers —thus mainly dealing with *operations*— and Karen's segment revolves around remainders and their meaning in whole-number division —thus largely focusing on number *meaning*. In our analyses, we do not pit one segment against the other but use both to illuminate different aspects of the work of teaching and the relationship being explored. The reader is cautioned, though, that what is reported in the next three papers is inescapably constrained by the aspects of teaching these two lesson segments make more or less visible. (The complete transcripts of the two analyzed lesson segments appear on http://www-personal.umich.edu/~dball/presentations/index.html.)

Chloe: Addition and Subtraction with 9, 11, 19, and 21

Chloe is a preservice teacher with an average background in mathematics, as suggested by the grade she received on the national mathematics examination (taken at age 16). The segment was drawn from a lesson she gave toward the end of her 18-week practicum experience. The lesson objective stated in Chloe's plan was that "Children should be able to subtract 9, 11, 19, and 21, using appropriate strategies." This objective was taken directly from the English National Numeracy Strategy Framework (DfEE, 1999) for Year-2 mathematics; this Framework also gives examples about the strategy Chloe is seen using (e.g., 70 – 11=59, because it is the same as 70–10–1, ibid., p.4/35). The students in Chloe's class were predominantly white and their socioeconomic backgrounds were diverse, but without extremes of affluence or poverty.

The lesson begins with a warm-up, mental-math activity in which students are asked to figure out the number they need to add to a given number to first get ten, and then twenty. This is followed by the segment considered in our analysis. Chloe puts a ten-by-ten number grid on the whiteboard and, using two counters to show the starting number (the first addend) and the end number (the sum), she asks students to recall from the previous lesson how they were adding 9, 11, 19, and 21 to a given number by moving up, down, right, and left. She then shifts to the lesson of the day: again using the number grid and the counters, she asks students to subtract first 19 from 70 and then 9 from 70. Next, she summarizes the "rules" for subtracting 9, 11, 19, and 21 on the whiteboard. After this segment, students work individually on a worksheet of additions and subtractions with 9, 11, 19, and 21.

Karen: Remainders in Whole-Number Division

Karen is an experienced teacher with relatively high levels of mathematical knowledge for teaching, as measured by a *Learning Mathematics for Teaching* test in numbers and operations, geometry, and algebra. According to the scale developed from a sample of 640 teachers, Karen scored in the 93rd percentile. At the time she taught this lesson, Karen had taught in Grades 3 to 6 for 37 years. Like the students in the other classes at her school, Karen's students represented a diverse group in terms of race, native language, and socioeconomic background.

The segment comes from a fifth-grade lesson on interpreting remainders in whole-number division problems. As Karen explained during a post-lesson interview, her goal for the lesson was to focus on the meaning of numbers, because her students "*ha[d] not been trained in really thinking about math a lot,"* and they were more inclined to simply *"take the numbers and just do something with them."* After a brief warm-up activity, in this segment, Karen asks her students to figure out $n \div 3 = 273$ using the idea of inverse operations discussed in previous lessons. After asking a student to present and explain his thinking and after Karen illustrates how the idea of inverse operations applies to a simpler example ($7 \div 1 = 7$), she asks her students to identify all possible remainders for division by six. Students give different answers and Karen encourages them to recall the conditions about remainders they discussed previously. The students are then asked to apply these conditions to figure out the remainder to division by two other numbers (4 and 8) and to define the meaning of *remainder*. The segment ends with students solving $74 \div 3$ mentally and discussing it. After this segment, Karen and her students explore three different situations for interpreting the quotient (and the remainder) in whole-number division "story" problems (ignoring the remainder; rounding the remainder to the next greater whole number; and reporting the exact quotient). The students are then asked to solve several division problems and interpret the quotients and remainders they get.

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CONCEPT STUDY AS A RESPONSE TO ALGORITHMETIC

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We offer brief conceptual analyses of the principal images and metaphors that frame Chloe's enacted understandings of NUMBER *and* SUBTRACTION*, and Karen's enacted understandings of* NUMBER *and* REMAINDER*. We suggest that while their figurative anchors for these concepts are rich, powerful, and personally coherent, they are not necessarily conscious. Their use of these anchors, therefore, may not be explicit or coherent enough for novice learners. Based on these analyses we look to 'concept study'—a structure to interrogate and elaborate the images, metaphors, and other figurative aspects of mathematical concepts—as one means to address apparent disconnects between teachers' enacted mathematics knowledge and the sense that their students make of mathematics.*

THE PULL TOWARD RULES

Rules and predetermined procedures seem to hold a strong appeal in the mathematics classroom. Even when teachers are intent on promoting conceptual understanding by, for example, using manipulatives, engaging in group discussions of mathematical terms, and/or encouraging the use of mental arithmetic, the pull towards enacting algorithms of arithmetic in a rigid fashion—a phenomenon that we call "algorithmetic*"*—is often irresistible. Algorithmetic limits the range of mathematical interactions in the classroom by closing off interpretive possibilities.

While it is tempting to explain the tendency toward algorithmetic by attributing it to teachers' poor understanding of particular mathematical concepts, this explanation is partial at best. As we attempt to demonstrate in our analyses of the video clips of Chloe and Karen, the mathematical understandings that teachers enact in their classrooms are often quite rich. The critical issue is not the teachers' knowledge of explicit procedures, but rather their lack of awareness of the implicit associations that they use in their practices. We argue that many teachers might have only limited conscious access to their own enacted mathematical understandings. Mathematical structures and metaphors that are transparent to teachers tend to be used implicitly in pedagogical situations. The lack of explication of enacted metaphors and the relationships among them obfuscates the study of mathematics for young learners and can make mathematics seem disjointed and arbitrary. Teachers who do not have an adequate awareness of the images they invoke (and, in particular, their possible contradictions and tensions) often over-emphasize the underlying mathematical rules or procedure when they sense confusion on the part of learners.

From the perspective of formal mathematics, a math teacher can't go wrong by repeating pre-established mathematical facts. But from our perspective, conscious excavation by teachers of the implicit structures of their own mathematical understanding presents a worthy alternative to algorithmetic. In the last part of this paper, we outline just such an alternative method in the application of 'concept study.' But first, we analyse the examples of Chloe and Karen, with a focus on the implicit associations that these teachers employ.

CHLOE: DEAD (OBVIOUS) METAPHORS

Chloe demonstrates one of the most significant barriers to effective pedagogy: adults are sometimes unable to separate representations of concepts that young learners are unable to assemble into anything coherent. What Chloe can't see as different, some of her students can't see as the same.

Chloe's understanding of subtraction of quantities that are near multiples of 10 is varied and draws on multiple metaphors of SUBTRACTION. Specifically, we observe that three principal and distinct interpretations of NUMBER are woven into Chloe's teaching. They are communicated to the students by means of vocabulary, gestures, and artefacts that are not always compatible with each other. The dominant oral interpretation is NUMBER as *count-of-objects*, which is reinforced in each of the 54 times that Chloe uses 'take away' to refer to SUBTRACTION. Implicit in this phrasing is a conception of SUBTRACTION as the decrease of the cardinality of a set that results when objects are removed from the set.

The two other interpretations are visual, and are drawn from the main visual artefact of the lesson – the numbers chart. The first interpretation is NUMBER as *position-in-a-linear-sequence*. In this interpretation, SUBTRACTION is the operation of counting back along the number line. Chloe uses this interpretation when she checks the calculation 70 – 19 = 51 by counting down from 70 to 51. The second interpretation is NUMBER as *position-in-a-2-dimensional-grid*. In this interpretation, SUBTRACTION is the operation of moving a marker horizontally, vertically (and even diagonally) along the grid. This interpretation is used throughout the lesson and appears to be a source of much confusion for the students.

Arguably, reconciling Chloe's three interpretations of SUBTRACTION (i.e., take away, count back, and move marker up/down) should not be difficult, as it only requires the relatively minor leap of mapping objects onto linear positions. In reality, the numbers chart introduces a rather sophisticated positional interpretation of NUMBER, one that invokes the power of place value in calculation. Organized into a 10×10 grid, the numerals 1–100 become something quite different from a linear list. The new entity has emergent properties that are as confusing as they are powerful.

Chloe has clearly reconciled the interpretations in her mind and is using them interchangeably. Her knowledge of aspects of SUBTRACTION is not the source of the students' confusion. In fact, Chloe comprehends SUBTRACTION so well that her various interpretations of the concept have become transparent to her. She cannot help but comment:

Chloe: No, 70 take away 20, it's dead easy 'cos we can all count in 10s, and then add one more.

We hear the phrase "dead easy" as meaning "I am no longer able to distinguish among possible concrete realizations of NUMBER and SUBTRACTION at play here." As Rorty (1989) explained, the figurative devices that infuse her understandings have become literalized: they are dead metaphors, or in this case, *dead obvious* metaphors.

Not only have the figurative devices for referring to SUBTRACTION become transparent to Chloe, but also the relationships among them are taken for granted. On several occasions students begin to answer a subtraction question by describing movement along the grid, only to be stopped by Chloe who insists on an alternative interpretation.

Student: We go up 1.

Chloe: Don't tell me what we'd go up, tell me what we'd take away. Yeah.

For neophyte students her response is probably puzzling. Even though they were explicitly instructed to use the numbers chart to find the answer, the teacher rejects their method when they do so.

Another potential source of confusion for students is the use of similar language to refer to different metaphors that signify SUBTRACTION. For example, in the previous example, the phrase "go up one" was used in the sense of "subtract 10," rather than the more commonly encountered sense of "add 1." This type of confusion can be mediated only if the teacher is explicitly aware of the metaphors at play, and if she negotiates among them by being attentive to nuances of language.

As Chloe's lesson proceeds, it is not clear that anything meaningful is being learned about SUBTRACTION. Rather than grappling with possible interpretations of the operation or interrogating how it is simultaneously enabled and obscured by the place value system, the students become increasingly confused about specific artefact-related operations. When the students continue to respond with wrong answers, Chloe resorts to algorithmetic:

Chloe: ... when you subtract 19 or nine you're always adding one right at the end. Subtracting 11 or 21, you're always taking away one more at the end.

"Always" indicates that the students are now bound by a rule. The confusion about "what is SUBTRACTION?" is eliminated at the expense of closing off interpretive possibilities. Students who memorize the rule are sure to do well by following the prescribed method and discounting other possibilities.

KAREN: CLOSING DEFINITIONS

Karen demonstrates another barrier to effective mathematical pedagogy: the use of techniques normally associated with promoting mathematical thinking can mask adherence to single-minded interpretations and an orthodoxy of algorithmetic.

Karen is a very experienced teacher who has taught for 37 years. Her lesson on remainders in whole-number division has been perfected through many iterations, as shown by its easy flow and Karen's quick responsiveness to students' comments. In fact, it appears that Karen has had to deal with students' questions and misconceptions about remainders so many times that her interpretations of DIVISION and REMAINDER had become as unambiguous as possible. Herein lies the main problem of Karen's lesson. Karen enacts, and limits her students by teaching them to enact, a very limited set of metaphors of mathematical concepts. It is likely that these metaphors are what Karen deems to be "best practices"—the metaphors that are likely to cause least confusion for the students. However, Karen's commitment to singular meanings also diminishes the depth of mathematical interactions with her students.

Throughout Karen's lesson, she operates from a conception of NUMBER as *count-of-objects*, which compels particular interpretations of DIVISION and REMAINDER. Karen uses a strictly quotative interpretation of DIVISION, in which the number of objects in each group is known, and the unknown quotient is the number of groups. Here, then, a REMAINDER is what is "left over" after one has made all the possible groups.

Karen's interpretations lead her to emphasize several properties of DIVISION and REMAINDERS with which teachers of higher grades would likely take issue. For example, when talking about DIVISION, Karen refers to the dividend as the "bigger" number and the divisor as the "smaller" number. These notions will likely be challenged as soon as students encounter division of fractions or decimals. In discussing possible remainders, Karen maintains that they cannot be zero – an assertion that is clearly at odds with, for example, the role of REMAINDER in modular arithmetic. However, since Karen operates from a conception of REMAINDER as something tangible that is "left over," it is easy to see why she would disallow a remainder of zero.

One of the ways in which Karen tries to promote mathematical thinking is by instructing her students to perform mental arithmetic. Indeed, it is impressive to see Grade 5 students perform calculations such as 3×273 and $74 \div 3$ in their heads. But closer inspection reveals that the students are applying the standard multiplication algorithm without actually using the intermediary devices of pencil and paper. Mental multiplication in Karen's class is performed in exactly one way: from right to left, units to hundreds. When Nick wants to "carry," Karen stops him at once.

Karen: You didn't carry because you're doing this in your head.

Karen's rejection of the possibility of carrying when performing mental multiplication appears to be in tension with her insistence on a "left over" interpretation of REMAINDER, since both are rooted in physical, set-based grouping metaphors. But this tension is revealed only once teachers are explicitly aware of the metaphors and associations that underlie their mathematical understanding. Only then can teachers proceed to use their awareness of such tension for creative pedagogical purposes.

Like mental multiplication, mental division in Karen's class is essentially the long-division algorithm in disguise, as evidenced by the exchange

Karen:	Seventy-four divided by three. Do this in your head, please. Then raise your hand. How many tens groups can I make?
SS:	Two.

The "tens groups" to which Karen is referring are not the groups-of-3 that we would expect in quotative division. Karen is referring to the tens digit in the long division algorithm. Here she is operating with a conception of NUMBER as *digit-in-standard-algorithm.*

Karen's lesson is an example of how arithmetic lends itself to unambiguous rules, unequivocal definitions, and orderly algorithms. An experienced teacher can run tight lessons with many opportunities for individual practice of the algorithms and group repetition of the rules. Unfortunately, however, this instructional coherence comes at the expense of rich interpretive possibility. In her post-lesson reflection interview Karen complains that "these children have not been trained in really thinking about math a lot." She goes beyond the textbook because she believes that "they need it in order to start thinking about the mathematics." Unfortunately, what Karen calls "thinking about mathematics" appears to be faithful recitation of previously-learned rules, justifications, and mental algorithms. In thinking about mathematics-for-teaching, we are far less concerned about Karen's misconception that zero is not a remainder, than we are about the restrictive worldview of arithmetic that she enacts. And, ironically, the textbook reveals a considerably richer set of interpretations for notions of DIVISION and REMAINDER.

For us, then, the principal issue around implicit associations in Karen's lesson is quite different from the issue in Chloe's lesson. For Chloe, the wealth of interpretations presented great possibility, but that possibility was closed off by a failure to notice and connect the interpretations that had been invoked. In Karen's case, the assumptions and associations are more explicit but are confined to unambiguous packages that close off mathematical thinking and imagination.

CONCEPT STUDY: PARTICIPATING WITH MATHEMATICS

As mathematics educators, we are centrally interested in how recent research into the embodied bases, figurative aspects, semantic qualities, and distributed nature of mathematical concepts could be applied to improve teachers' mathematical knowing. As Chloe and Karen demonstrate, most teachers have a wealth of embodied and enacted mathematical knowledge. However, left on the level of implicit association, this knowledge can be as constrictive as it is enabling. Chloe and Karen provide clear demonstrations of this fact in their lessons. In both cases, there is a wealth of uninterrogated tacit knowledge that actually gets in the way of effective teaching.

The lack of explicit awareness, we believe, is a major contributor to these teachers' emphases on procedures and process.

Our response to this sort of situation has been to engage in *concept study* (Davis, 2008) with practicing teachers, through which we represent, interrogate, and elaborate in-class enactments of mathematics. The phrase "concept study" borrows from two prominent strands in contemporary mathematics education research: the mathematical emphases of "concept analysis" (e.g., Leinhardt, Putnam, & Hattrup, 1992) and the collaborative structures of "lesson study" (e.g., Fernandez & Yoshida, 2004). Concept analysis, which was particularly prominent in mathematics education research from the 1960s to the 1980s, has been focused on explicating logical structures and associations of mathematical concepts. The more recent lesson study movement is a collective-oriented structure for the articulation, critique, and further development of mathematics teaching strategies. To these we add a few other important ingredients in the concept study mix, including (as mentioned) emergent research into the figurative dimensions of mathematical concepts (e.g., Lakoff & Núñez, 2000), the embodied nature of knowing (e.g., Varela, Thompson, & Rosch, 1991) and the situated and distributed character of knowledge (e.g., Lave & Wenger, 1991).

The aim of concept study is not to home in on better procedures or best practices. Rather, the purpose is to investigate the tendency toward prescriptions and algorithmetic by engaging practicing teachers as co-producers, rather than consumers, of research and mathematical insights. To that end, some emphases within concept study have included:

- enabling open definitions – excavating the images, metaphors, applications, and exemplars that are used to frame a concept, and structuring those aspects in a way that anticipates and enables elaborations as learners move through levels of mathematics instruction;
- exploring implications – getting a sense of the local network of associations around the concept by examining nearby nodes, hubs, links, and entailments;
- mapping out a landscape – identifying horizontal and vertical curriculum connections while looking for similarities and differences among interpretations;
- inventing meta-interpretations (blending interpretations into more encompassing models, and subsequently examining those models for limitations and possible entailments).

This list is neither complete nor a prescriptive recommendation for action. Rather, it is an indication of the sorts of activities that, under the right circumstances, should prove to be compelling to practicing teachers.

Concept study seeks to involve teachers with emergent insights into mathematical knowing not by *introducing* them to new explicit ideas, but by *including* them in the elaboration of extant implicit knowledge. Such elaboration is not merely a matter of identifying associations and implications. Rather, it is nothing less than a reconfiguration of teachers' relationship with mathematics. Since children and their teachers contribute to cultural mathematics through the selection and development of images, metaphors, analogies, gestures, and exemplars that come to be woven into conceptual understandings, teachers have to be active participants in this work as well.

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KAREN AND CHLOE: THE KNOWLEDGE QUARTET

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THE KNOWLEDGE QUARTET

Research shows that feedback on observation of lessons taught by student ‘trainee’ teachers typically focuses heavily on organisational features of the lesson, with very little attention to the subject-matter being taught (e.g. Borko & Mayfield, 1995). The Knowledge Quartet (KQ) is an empirically-based conceptual framework for the discussion of the role of trainees’ mathematics SMK and PCK, in the context of lessons taught on their school-based *practicum* placements. From the KQ-perspective, as will become apparent later, the distinction between different kinds of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching.

The KQ is the outcome of grounded theory research in which 24 lessons taught by elementary school trainee teachers were videotaped and scrutinised. We identified aspects of trainees’ actions in the classroom that seemed to be significant in the limited sense that they could be construed to be informed by their mathematics SMK or PCK. This inductive process generated a set of 18 codes, subsequently grouped into four broad, super-ordinate categories, or ‘dimensions’ - the Knowledge Quartet. For example, the third dimension, *connection*, is a synthesis of four of the original 18 codes, namely: making connections; decisions about sequencing; anticipation of complexity, and recognition of conceptual appropriateness. A detailed methodological account is given in Rowland (2008a).

Our conceptualisation of the KQ here is necessarily brief. The first dimension, *foundation*, consists of trainees’ mathematics-related knowledge, beliefs and understanding acquired ‘in the academy’, in preparation (intentionally or otherwise) for their role in the classroom. The second dimension, *transformation*, concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself. A central focus is on the representation of ideas to learners in form of analogies, examples, explanations and demonstrations. The third dimension, *connection*, binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content. Our conception of this coherence, as indicated above, includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks. Our final dimension, *contingency*, is witnessed in classroom events that were not planned for. In commonplace language it is the ability to ‘think on one’s feet’. A more detailed conceptual account is given in Rowland, Huckstep and Thwaites (2005). In effect, the genesis of the KQ in observations of teaching, and this conceptualisation of the KQ itself, constitute our answers to the two research questions underpinning this Research Forum. The warrants for this claim become clearer in the application of the

KQ framework to 'real' lessons: therefore we now proceed to the analysis of Karen's and Chloe's lessons, through the lens of the Knowledge Quartet.

KAREN'S LESSON

The lesson segment under consideration begins with the class solving problems of the kind $a \div b = c$, where c and one other variable are given, and the third has to be found. In particular, when b, c are known, a useful strategy for finding a is to recognise that division by c and multiplication by c are inverse functions. This is the focus of attention in the first three minutes of the video clip under consideration. The lesson then shifts to whole number division-with-remainder, and consideration of what the possible remainders could be, for a given divisor. In the identified segment, Karen does not address the textbook lesson objective, which is on three ways of interpreting the remainder (and the whole-number quotient in fact) in 'real' problem-solving situations. Our KQ-lens homes in on *mathematics content knowledge* in teaching, and the more generic aspects of Karen's management of learning are outside its scope. That is not, of course, to say that such matters are unimportant, or that the KQ-analyst would have no opinions about them, but the KQ is not designed as a tool to take them into account.

Foundation. This dimension of the KQ includes teachers' content-related beliefs and attitudes, as well as knowledge of content and pedagogy. The number operations and calculations that we observe in the lesson are straightforward, and would not be expected to be problematic from the teacher's point of view. Below the surface of the arithmetic lie two big ideas with significant implications beyond the immediate lesson objectives. The first of these, the notion of inverse operations (or functions), is strongly emphasised in the first portion of the video clip. Karen elicits and then uses the related vocabulary fluently and flexibly.

3	Karen:	You're going to use a check problem right. The inverse, aren't you? You're going to go the opposite direction. You know what it's related to.

The second big idea, central to the second phase of the lesson, is a result in the Theory of Numbers, the curiously-named 'division algorithm', which states that given integers a, d (with d positive), there exist unique integers q, r such that $b=qd+r$ and $0 \leq r < a$. The theorem assures that any of the usual whole-number division algorithms will yield a unique remainder between zero and one less than the divisor (inclusive). Karen tackles this important result head-on with the class, but her notion of what is to be understood is restricted, in one detail.

25	Karen:	Okay, boys and girls if I'm dividing by six […] what remainders could I have? […] What do you think, Daniel?
26	Daniel:	Zero.
27	Karen:	I could have zero, but it's not a remainder.

It seems that Karen's notion – her concept-image in fact (Tall & Vinner, 1981) - of 'remainder' is of something, as opposed to nothing, 'left over', when the process of removing divisor-sized groups has been exhausted. Daniel recants:

28 Daniel: I mean you could have no remainders.

29 Karen: I could have no remainder. What other remainders could I have?

Karen reinforces the same point later:

71 Karen: What does a remainder tell us? What does a remainder tell us, Kim?

72 Kim: There's some leftover.

73 Karen: There's some leftover, meaning ...? There is some leftover, I agree with that.

The distinction between 'no remainder' and 'zero remainder' is psychologically significant. The belief that zero cannot be a remainder resonates with human phylogeny, and the late acceptance of zero *as a number* (Barrow, 2001). It can be argued that Karen's rejection of zero as a possible remainder is of no great pedagogical significance, even though it could be regarded as relevant 'horizon knowledge' here (Ball et al., 2008). Maybe she didn't study Number Theory herself, and maybe few of her students will go on to study it. On the other hand, the section 'Math Background' in the notes in the Teachers' Handbook (Harcourt Math 5th Grade, Chapter 11, lesson 6) explain that "Remainders can have any whole-number value from zero up to, but not including, the value of the divisor".

Before moving on, we touch on Karen's knowledge and beliefs in connection with student understanding and calculation methods. Mental calculations are performed by the students on several occasions in this segment of the lesson. The complaint that school mathematics is in the thrall of meaningless written algorithms goes back at least three decades (e.g. Plunkett, 1979) and Karen seems to have 'turned the tide' in her classroom. Nick's evaluation of 3x273 in the opening moments seems to have been a mental calculation. Karen leaves nothing to chance when she probes:

7 Karen: All right Nick, tell he how you thought about this? If I want to multiply three times two hundred seventy-three in my head, what do I do?

Nick says that he multiplied "three times three" and got 9, then "three times seven".

13 Karen: Three times seven? Where did three times seven come in?

14 Nick: Three times seventy.

15 Karen: Oh, thank you. Three times seventy.

The joint exposition concludes with the addition of 9, 210 and 600, articulated in that order by Nick, and listed in a column on Karen's overhead projector (OHT) display with units and tens 'lined up' vertically with the relevant digits of 273 above them. We infer several things from this episode. First, that the culture of mental calculation as a "first resort" (DES, 1982) is embedded in Karen's classroom (and there is further evidence of this later). Secondly, that Karen promotes a meaningful, holistic approach to the numbers through emphasising their *value* (seventy, not seven) rather than mere manipulation of digits, and that this is also engrained in and understood by the students [turns 13-15]. Thirdly – and this is more speculative – that this approach to mental multi-digit calculations seems to be grafted onto the earlier acquisition of algorithmic skill in carrying out the same calculations by right-to-left column digit manipulations. The Dutch Realistic Mathematics Education (RME) approach turns this sequence on its head, so that written calculation methods evolve from prolonged use of holistic mental methods (e.g. Treffers, 1987). This, of course, is not an issue for Karen in isolation, but for the whole school, and maybe for the State.

From a KQ-perspective the application of subject knowledge in the classroom always rests on foundational knowledge. We take this as justification for the extent of this section of our analysis, and the brevity of the three which now follow.

Transformation. Attention to this dimension of the KQ always leads us to consider, especially, the choice and use of representations and examples in the lesson. With regard to the lesson objective on interpreting remainders (and quotients), Karen's lesson departs from that proposed in the textbook in the significant sense that she poses 'bare' division situations for consideration rather than problems set in context. In that regard, the only sense in which the mathematics is 'represented' is symbolic, as sounds and inscriptions on the OHT. Given the objective of interpretation in real problem-solving situations, it is relevant to note that no 'real' problems have been posed or solved in the 9-minute segment under scrutiny. An interesting alternative approach, in keeping with the stated lesson objective, and Karen's initial focus on 6 as divisor, could be to adopt a realistic quotative division representation, such as putting eggs into cartons, each holding 6 eggs.

Regarding the choice and use of examples, it seems that Karen has not prepared hers in advance. Sometimes she rethinks and rejects her first choice:

> 77 Karen: So if I have a problem that says seventy-four divided by two (writes 74÷2 on the overhead, but then erases it). No, that's too easy. How about three? Seventy-four divided by three. Do this in your head, please.

In some ways even this example is still not well-matched to the learning objective (Rowland, 2008b), since the effort of performing the calculation seems to get in the way of the target concept that she wants them to focus on, i.e. the remainder. Finally, we note Karen's choice of example to clarify the notion of inverse operations:

> 25 Karen: […] if we have seven divided by one equals seven (writes 7 ÷1=7, as she talks) that seven times one is going to equal seven, right?

The choice of 1 as divisor is unfortunate because (a) in many ways this is an exceptional case, and (b) this is a classic case of 'confusing the role of variables' (Rowland, 2008b), since 7 appears as both dividend and quotient. This same comment about variables applies to the first example visible on her OHT i.e. $81 \div n = 9$.

Connection. In the early portion of the lesson, Karen seems to be emphasising the important connection between multiplication and division. The connection dimension of the KQ includes the sequencing of instruction and the related tasks and exercises.

Looking at Karen's OHT, we see the sequence $81 \div n = 9$, $248 \div n = 62$, $385 \div n = 55$, $n \div 3 = 273$. It would be interesting to discuss the rationale for this sequence with Karen, although, as we remarked earlier, it may have been spontaneous. Incidentally, this illustrates the fact that one incident or episode in a lesson might relate to two or more dimensions of the KQ. It is apparent, however, that the first three problems are likely to be solved by division, not by multiplication, so the inverse-operation theme is less apparent. Anghileri (2006, pp. 77-78) stresses the importance of discussing number *triples* in multiplicative relationship, and bringing together all four links between the three elements. (Her example is 3x8=24, 8x3=24, 24÷3=8, 24÷8=3.)

Contingency. Contingent moments arise one after the other when teachers interact with a class, as Karen does, and their responses to these opportunities draw in various ways on their mathematical content knowledge. We pick out just a couple of them for attention and illustration. The occasion when Daniel proposed "Zero" as a possible remainder, which we raised earlier, is one of them. Another is when Nick volunteers 813 for the product of 3 and 273.

5	Karen:	Nick says eight hundred thirteen. What do you say class? […]
7	Karen:	All right Nick, tell me how you thought about this? If I want to multiply three times two hundred seventy-three in my head, what do I do?

Karen's response to Nick's error draws on some generic, Socratic pedagogical strategies, but we argue that they have a particular relevance in the mathematics classroom, where the authority of the teacher is tempered by the rational democracy of the subject. She successfully avoids evaluating Nick's answer, and makes it available for public scrutiny and resolution [turn 5].

In conclusion, our analysis confirms Karen's relaxed and confident management of the mathematics classroom, and also identifies a few matters that it would be interesting to discuss with her, to support her reflections on the lesson.

CHLOE'S LESSON

The learning objective stated in Chloe's lesson plan is: "Children should be able to subtract 9, 11, 19 and 21 using the appropriate strategies". Chloe reminds the class (student age 6-7 years) that in their previous lesson they added 9, 11, 19 and 21 to various 1-digit and 2-digit whole numbers. Chloe demonstrates how to subtract these same numbers by subtracting 10 or 20 first, then adding or subtracting 1. We emphasise that the process of *selection* in the commentary which follows has been extreme, but this mirrors selection made necessary by time constraints in post-observation discussion with trainees.

Foundation. It becomes clear that Chloe will emphasise mental, sequential strategies, perhaps with some use of informal jottings (DfEE, 1999, p. 2/4). This is in keeping with the *National Numeracy Strategy* (NNS) in England, which, following the Dutch RME approach, emphasises mental calculation methods in the early grades. Sequential (or cumulative) strategies for two-digit addition and subtraction begin with one number (for subtraction, the minuend) and typically move up or down the sequence of integers in tens or ones (e.g. Anghileri, 2006). We limit ourselves here to the observation that her efforts to teach these particular mental strategies – especially the 'compensation' strategies with 9 and 19 – in a meaningful way, with the 100 square, meet with very limited success. In the end she falls back on:

57 Chloe: Let's write these rules on the white board so we remember. Then we'll have something to follow when we're doing our sheets.

– and Chloe lists a menu of 'what to do' rules.

Transformation. Chloe uses the 100 square as a model of the sequence of two-digit positive integers. It is useful for representing ordinal aspects of the sequence, and for representing the place-value aspects. The discontinuities at the 'ends' of the rows, however, come into focus when Chloe demonstrates subtracting 19 from 70. We wonder whether Chloe had considered using the *empty number line* (e.g. Beishuizen , 1999) as an alternative way of representing the numbers and their difference, to clarify when compensation is necessary, and why. Chloe makes full use of the 100 square in her exposition, but is frequently dismissive of children's use of the spatial language that it invites. For example, at one point she places the counter on 70:

49 Chloe: From seventy I want to take away nine. What will I do? Rebecca?

50 Rebecca: Go up one.

51 Chloe: No, don't tell me what I'm gonna go up or move, tell me what I actually do. Rebecca?

This would seem to relate to the format of the examples in the NNS *Framework*, which adhere to *symbolic* representation of the compensation strategy, e.g. "24-9=15 because it is the same as 24-10+1; 35+19=54 because it is the same as 35+20-1". Indeed, the four 'model' solutions that Chloe writes for reference on the board use the same representation.

Connection. Given her use of the 100 square to demonstrate the strategies, there was scope for some discussion of the links between vertical and horizontal spatial movements on the board and the tens-ones structure of the numbers under consideration. As we have remarked, she actively discouraged children's reference to the spatial analogue.

Contingency. There are no compelling distractions from Chloe's planned agenda for the lesson in this episode, although the child's question about using the number squares for the exercises might be a case in point. Various children's use of up/down language on the 100 square, to which we have already referred, might have been usefully explored rather than dismissed. A similar opportunity presented itself when, in the review of adding 9 at the beginning of the lesson, Christopher comments that this involves a diagonal move on the 100 square. Chloe simply dismisses the idea at the time, although she does return to it later, appreciative of his feelings perhaps but not of his spatial versatility.

19 Chloe: Christopher was right in saying you move diagonally on a number square, but because we're focusing on showing all our working out, and writing out our sums underneath the boxes today, you need to say that you add ten and take away one.

We reiterate that a single event or episode can frequently be considered from the perspective of two or more dimensions of the KQ, as demonstrated in our commentary.

Acknowledgement

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MATHEMATICAL KNOWLEDGE FOR TEACHING: FOCUSING ON THE WORK OF TEACHING AND ITS DEMANDS

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In this paper, we use Chloe's and Karen's clips to illustrate how we study the work of teaching and how we then infer the knowledge needed for this work. To this end, after outlining our notion of mathematical knowledge for teaching*, we elaborate on four tasks that can be identified from analyzing Chloe's and Karen's clips: using representations to support students' thinking, recording the mathematical work of the lesson, selecting and sequencing examples, and attending to and using mathematical language to clarify key ideas. We use this analysis to illustrate how we study teaching in order to understand the nature of mathematical knowledge needed for teaching mathematics and how this knowledge is actualized in this work.*

STUDYING THE WORK OF TEACHING AND ITS DEMANDS

Few would question that teacher knowledge matters for the work of teaching. Yet research on how teachers' knowledge affects teaching, and consequently student learning, has been equivocal (Fennema & Franke, 1992; Wayne & Young, 2003). To understand the knowledge needed for the work of teaching and how this knowledge is actualized in this work, over the last 15 years we have pursued a practice-based and disciplinary-grounded approach to study the work of teaching. Broadly speaking, this approach unfolds in three stages: first, we observe and analyze records of actual practice to identify the work of teaching as implied by these records; second, we analyze the mathematical issues and demands that arise in this work; finally, we consider the knowledge implied by those issues and demands —what we call *mathematical knowledge for teaching* (MKT). Our conceptualization of MKT, however, goes beyond the strict boundaries and limits of knowledge, to also include skills, reasoning, habits of mind, and sensitivities needed to teach mathematics effectively (cf. Ball, Thames, & Phelps, 2008).

In analyzing actual records of teaching, our focus is not on what teachers (should) accomplish, but rather on the teaching tasks featured in these records and on the demands that these tasks present for teachers, especially with respect to developing a classroom that honors both the integrity of the mathematics and that also takes students' ideas and contributions seriously into account (Ball, 1993). Due to space limitations, a full exposition of our approach is not possible here. However, we use Chloe's and Karen's clips to give a glimpse of this work. To this end, we first list several teaching tasks that could be identified when studying these clips; we then focus on four such tasks, and use these tasks as the platform to discuss how we attempt to understand the knowledge needed to teach mathematics effectively.

IDENTIFYING THE WORK OF TEACHING AND ITS DEMANDS IN CHLOE'S AND KAREN'S CLIPS

Identifying Tasks in the Work of Teaching

In the thirteen-minute excerpt of Chloe's lesson, we see Chloe engaging in several teaching tasks. These include: explaining mathematical ideas (e.g., the idea of adding or subtracting by compensating by one), using representations to make these ideas explicit, using and sequencing numeric choices and examples that keep the mathematics workable and coherent, recording the mathematical work of the lesson so that ideas can be followed, and managing multiple solutions strategies shared by students in discussion. Similarly, in the eight-minute segment of Karen's lesson, Karen is shown engaging in a multitude of tasks; to name a few: selecting and using examples to exemplify the notion of remainders, providing explanations while building on students' ideas, directing students' attention to the meaning underlying the division algorithm, and attending to and using mathematical language for remainders. The careful reader will notice that all these tasks share a common denominator: they involve important mathematical work —these are the tasks on which we focus our attention. Other tasks involved in the work of teaching (e.g., bringing an ill-behaved student back to order, encouraging and rewording students for their participation in the lesson), although unquestionably critical for the work of teaching, are not the focus of our analysis.

Analyzing the Mathematical Issues and Demands Arising in Teaching

From the gamut of teaching tasks in Chloe's and Karen's lessons which have a *mathematical* component, for the purposes of this paper we focus on four: using representations to illuminate key mathematical ideas and thus support student thinking; recording the mathematical work of the lesson; selecting and sequencing examples; and attending to and using mathematical language to clarify key ideas. Following our custom in analyzing records of practice, we do not to evaluate the teacher's work but attempt to understand the mathematical issues and demands arising in this work *as featured in these clips*.

Using representations to support student thinking. Chloe uses a visual representation to scaffold children's understanding of the *near-tens* strategy, when adding or subtracting 9, 19, 11, and 21: a *Hundred Numbers Chart*, that is, a 10 x 10 grid including numbers 1 through 100. She also supplements this chart with two counters: one to demarcate the initial quantity (the first addend in the case of addition, and the minuend in the case of subtraction); and another, which she moves horizontally or vertically, to illustrate the quantity being added or subtracted. Throughout this clip, Chloe is deliberate in having students consider these two movements (the horizontal and vertical) as distinct. This is evident from her interactions with Cameron. When Chloe asks students how they can add nine on the *Hundred Numbers Chart* and Cameron responds that they can go diagonally, Chloe does not accept his description and scaffolds him to decompose this composite movement into its two main components: first you go down and then you move one number to the left:

Chloe: What's the easy way to add ten on a number square? Cameron.

Cameron: Go diagonally.

Chloe: Not diagonally. To add ten you just go…

Cameron: Down.

Chloe: Down. … So move your red counter down, one of the red counters down. … So that adding ten.... And then what do we do, Rhiannon, for adding nine?

Rhiannon: We take away one.

Such a decomposition is important for supporting students' thinking, because it allows the teacher to build on the affordances of the representation she selected. Because of the way numbers are arranged in this 10x10 grid, the *Hundred Numbers Chart* conveys not only the place-value notion of our contemporary number system, but could potentially represent the *near-tens* strategy which is the main focus in Chloe's lesson. To clarify, consider two of the examples discussed in Chloe's clip: 9 + 11 and 70 – 9. Following the *near-tens* strategy that Chloe is seen teaching in this clip, the addition can be decomposed as 9 + (10+1) and the subtraction as 70 –(10–1). This algebraic decomposition aligns very well with the geometric moves on the *Hundred Numbers Chart*. To add or subtract ten, one needs to move the counter vertically, because the numbers in each column are increasing (when moving down) or decreasing (when moving up) by ten. To add or subtract one, one needs to move the counter horizontally, because the numbers in each row are increasing (when moving to the right) or decreasing (when moving to the left) by one. This close correspondence between the algebraic and the geometric decomposition of numbers and moves suggests that the representations that teachers choose for their teaching, but foremost how these representations are used in teaching, can help illuminate key mathematical ideas and processes.

Recording the mathematical work of the lesson. Chloe uses magnetic markers on the *Hundred Numbers Chart* to signal attention to the numbers used in the *near-tens* strategy rehearsed in this lesson, and to highlight the placement of these numbers on the chart that signifies the intervals of tens and ones. At the close of the clip, Chloe also moves to the white board to record the number sentences that have been discussed. For 70 –19, she writes 70 – 20 + 1 on the board and mentions:

Chloe: So, how do we write that out? It's seventy, take away twenty, and we add one. And that gives us the answer.

She then encourages the students to use exactly the same approach to figure out the difference of 70 – 9:

Chloe: Remember with nineteen it was take away twenty add one, so nine is going to be seventy take away ten... Cameron?

Cameron gives the correct answer (i.e., adding one), which Chloe records on the board as 70 – 10 + 1. Prompted to identify the final answer, Cameron incorrectly proposes 61 instead of 59, probably because, consistent with his articulation, he was adding (instead of subtracting) one to the difference 70 – 10. In the remainder of the clip, the students are also shown encountering difficulties as to whether when subtracting 11 one needs to add or subtract one. Although *prima facie*, one could consider the work of recording on the board an easy task, the mathematical intricacies involved in the *near-tens* strategy, as exemplified in Chloe's clip, suggest that this not the case. To illustrate these inherent difficulties, it is instructive to first consider the use of this strategy when *adding* 9 and 11 —which was the lesson of the previous day. Take for example the addition of 70 + 9, which, following the *near-tens* strategy can be decomposed into 70 + (10–1). In the case of this addition problem, moving from 70 + (10–1) to 70 + 10 – 1 is relatively straightforward. However, this is not entirely analogous in subtraction. How do you move from 70 – (10–1), and more critically, how do you record this step on the board to explain to *second graders* that you need to add instead of subtracting one?

Selecting and Sequencing Examples. Both Chloe's and Karen's clips lend themselves to discussing teachers' selection and sequencing of examples.

In Chloe's clip, we see the teacher using the four number choices suggested by her curriculum (9, 11, 19, and 21) to illustrate the *near-tens* strategy. To illustrate the critical role that teachers' selection and sequencing of examples might have in supporting student learning, it is helpful to return to Chloe's recording on the board just discussed and consider whether a different sequencing of examples might have helped address the complexity of adding or subtracting one when adding or subtracting 9 and 11. Although when adding it might not make much difference if one considers 70+9 first and them moves to 70+11, in subtraction, the sequencing of examples might be consequential for student understanding, because subtraction 70–11 [i.e., 70 – (10+1)] does not involve the additional complexity of subtracting a negative inherent in subtracting nine [i.e., 70–9 =70 – (10–1)].

In considering Chloe's clip, the question also arises as to which numbers better lend themselves to this *near-tens* strategy. Could Chloe, for instance, use this same strategy to teach addition or subtraction by 8, 12, 18, and 22 —where compensating by two (instead of one) is needed? Would it make sense to use this same strategy to teach addition and subtraction by 6, 14, 16, and 24? Or to put it in broader terms, if our intention in teaching is not simply to help students become facile in using certain strategies, but also to help them understand the applicability and usability of these strategies, which numbers should the teacher use to reach this goal?

Karen is also featured selecting several examples in her work. At the outset of the segment under consideration, Karen writes the equation n ÷ 3 =273 on the overhead, apparently expecting students to solve it by using the "inverse operation." To then reinforce the idea of using an inverse operation, she employs a simpler example (7 ÷ 1 = 7) and uses two arrows to show the "inverse path," as illustrated in Figure 1.

$$7 \div 1 = 7$$

Figure 1: Illustration of the "inverse path."

Later on, Karen asks her students to identify all possible remainders of 6 – a task suggested by her curriculum materials. She also asks students to consider all the possible remainders for another two examples, not identified in her curriculum (i.e., 4, and 8). Finally, at the culmination of this clip, Karen selects the division 74 ÷ 2, but immediately changes it to division 74 ÷ 3, on the grounds that the first one was "too easy."

In considering the significance of teachers' selection and sequencing of examples for structuring student learning opportunities, one could contemplate the affordances and potential limitations of Karen's pertinent work in this segment. The selection of example n ÷ 3 =273 does not only afford considering the "inverse path," for a simpler example (e.g., n ÷ 3 = 80) could suffice in this respect; in fact, such an example, could, perhaps, more clearly elucidate the idea of multiplying the divisor by the quotient. However, the work on the example n ÷ 3 =273 in Karen's segment submits that the selection of this example created a space for not only engaging students in some mental math, but also for discussing a place-value idea that the example n ÷ 3 = 80 might have not facilitated: when mentally carrying out the multiplication 273 x 3, Nick, one of Karen's students, is pressed to articulate that one actually multiplies "three times *seventy*," and not "three times seven," as the numbers involved in this multiplication might imply. Karen's selection of the example 7 ÷ 1 = 7 is also worth some consideration. From one respect, this example is numerically simpler than the original example Karen used; from another respect, however, one could legitimately point out that the appearance of seven as both the dividend and the quotient could have obscured the distinct role that these numbers need to assume to more clearly illustrate the idea of using an inverse operation. Along those lines, one could argue that Karen's selection of three *even* numbers when discussing the remainders (i.e., 6, 4, and 8) might inadvertently have reinforced a misconception that only even numbers might qualify when discussing divisors and remainders. Finally, this excerpt leads us to consider the affordances and limitations of choosing division 74 ÷ 2 instead of division 74 ÷ 3 at the culmination of the lesson segment. What mathematical ideas and processes does each example help clarify and what mathematical ideas and processes might it obscure?

Attending to and using mathematical language to clarify key mathematical ideas. This task of teaching is clearly illustrated in Karen's segment when she is shown working with her students on defining the meaning of the remainder.

In everyday language, remainder is often conceived of as "that which is left over." In contrast, from a mathematical viewpoint, remainder in *whole-number* division is the whole number *r* which satisfies the condition r = D–d x q (where D=dividend, d=divisor, and q=quotient) and which can take any value in the set [0, …, q}. The everyday and the mathematical meanings of the remainder are, however, not totally compatible; indeed, Karen's segment shows how working at the interface of the everyday and the mathematical meaning of certain terms requires a close attention to the mathematical language used to clarify the meaning of such terms.

The everyday meaning of remainder is brought up at several points of the lesson, but more explicitly when Karen asks students to define "what the remainder tells us." Kim defines the remainder as "there is some leftover." By definition, this everyday meaning of the remainder excludes zero as a remainder, for zero actually denotes the *lack* of any quantity. The challenge then becomes: How should the teacher deal with this discrepancy? Karen's work on this segment —whether deliberate or not— suggests that a plausible way to do so could be to pay close attention to the mathematical language used to define the term in play. This is particularly evident from her interactions with Daniel. To initiate the discussion about remainders, Karen asks students to identify all the possible remainders of six. Interestingly, the first remainder that is proposed (by Daniel) is zero, to which Karen responds that zero is *not* a remainder. Daniel then suggests that "you could have no remainders," probably implying that when the remainder in a whole-number division is zero, there are actually no remainders. This short sequence of exchanges could be considered to include an implicit definition about the remainder that attends to both its everyday and its mathematical meaning: from a mathematical viewpoint, it is accepted that when carrying out a division, one can get zero in the remainder *spot*. However, this result would imply —in Karen's words— that "there is no remainder," given that a remainder actually suggests that *something* must be leftover.

We emphasize that we look to these excerpts to trigger for us what mathematical demands arise in the course of teaching, without evaluating the teachers' performance. In this instance, we build on this episode to pinpoint the care that needs to be given by the teacher in defining mathematical terms (alone or while interacting with students) so that these terms honor the ideas that students bring to school but also are compatible with the mathematical meaning of these terms.

KNOWLEDGE NEEDED FOR TEACHING

The foregoing discussion provides just a glimpse of how we work on analyzing actual records of practice to understand the work of teaching and its demands, and consequently the knowledge needed for this work. Building on our analysis of several such records, over the last 15 years we have developed a framework that provides a heuristic for different types of knowledge needed for teaching. So far, we have identified six such domains. We illustrated each with an example from Chloe's and Karen's lessons, with reference to the two clips just discussed. To teach the lesson on adding and subtracting using *near-tens*, Chloe needed to have understood this strategy and its constituent parts —we identify this basic knowledge as the *common content knowledge*, the knowledge used by professionals not necessarily in teaching. However, possessing such knowledge does not suffice for the work of teaching. Teachers need an understanding of the mathematics they are teaching that allows for them, among other things, to use mathematical language with care and precision when defining mathematical ideas —we refer to this type of knowledge as *specialized content knowledge*. Selecting and using appropriate representations and examples also appears to be pivotal to the work of teaching, for these devices can support or hamper teachers' attempts to help their students build meaning; we use the term *knowledge of content and teaching* to capture the knowledge that is needed to make educated decisions for selecting and using such means in teaching. The teacher, additionally, needs to be able to build on and unpack students' ideas —take, for example, the episode in which Chloe is shown interacting with Cameron to clarify that "moving diagonally" actually means "moving down and then moving to the left." We use the term *knowledge of content and students* to refer to this domain of teacher knowledge. Strong *knowledge of the content and the curriculum* could also help the teacher consider the curriculum in terms of the key mathematical ideas that certain lessons are designed to help students understand (e.g., the *near-tens* strategy in Chloe's lesson). Teachers' awareness of the content that comes next in the same grade or is taught in upper grades could also help them present the content in ways that support students' thinking in the future, or at least do not hamper it —we refer to this knowledge domain as *horizon content knowledge*. To illustrate this type of knowledge, consider, for example, the implications that the idea of zero not being a remainder (in Karen's segment) could have for studying real-number division and modules in upper grades.

CONLCUSION

In this paper, we used Chloe's and Karen's clips to provide a snapshot of how we work on analyzing and understanding teaching and the knowledge needed for teaching. We do so by analyzing actual records of practice, by identifying mathematical tasks for teaching, and by identifying the *mathematical issues and demands entailed in teaching*. To then determine the knowledge needed for this work or how knowledge is actualized in this work, we consider what it would take on the part of the teacher to respond to the aforementioned issues and demands in ways that honor the integrity of the mathematics, but also respect and build on students' ideas and contributions. As our work in analyzing Chloe's and Karen's videoclips suggests, when studying records of practice our intention is not to determine what the teacher should have done, but rather to understand what knowledge teachers draw upon to undertake the multiple demands required to teach mathematics effectively.

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TEACHER KNOWLEDGE AND TEACHING: CONSIDERING THE CONNECTIONS BETWEEN PERSPECTIVES AND FINDINGS

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Teacher knowledge for teaching mathematics has received immense international attention in recent years. Yet, theoretical/conceptual perspectives to serve as the basis for analysing teacher knowledge and its relationship to the work of teaching are sparse. The work of the three groups participating in this Research Forum address, each in its own way, this shortcoming of current research. All three perspectives acknowledge the complexity of knowing mathematics for teaching. All also attend to both the mathematics and the work of teaching. But they do it in different ways.

Brent Davis and Moshe Renert examine carefully the mathematical concepts, which are the focus of the two clips. They show, for example, how in both Chloe's and Karen's cases "NUMBER [conceived] as *count-of-objects"*, is linked, in Chloe's class with "SUBTRACTION as the decrease of the cardinality of a set", and in Karen's class with "REMAINDER [a]s…'left over'". Similarly, later in Chloe's class "NUMBER [conceived] as *position-in-a-linear-sequence*" is linked with "SUBTRACTION [a]s the operation of counting back along the number line", and so on. Davis and Renert emphasize that lack of awareness of such implicit associations that teachers use in their teaching is a critical issue, attributing the pull towards rules in mathematics classrooms to this kind of lack of awareness. Davis and Renert suggest that teachers need explicit rich knowledge of logical structures, figurative dimensions, and associations of mathematical concepts, combined with knowledge of mathematics teaching strategies.

Tim Rowland and Fay Turner approach the issue of teacher knowledge using the Knowledge Quartet (KQ), an empirically-based conceptual framework, developed originally for pre-service teachers during their practicum training. Rowland and Turner focus on classifying situations in which mathematical knowledge surfaces in teaching. Using four dimensions: foundation, transformation, connection and contingency, Rowland and Turner identify several problematic aspects in Chloe's and Karen's knowledge exhibited in the clips, and their relationship to the work of teaching. For example, the psychological significance of the distinction between "no remainder" and "zero remainder" in Karen's lesson, as opposed to its "no great pedagogical significance".

Deborah L. Ball and her colleagues attest that they approach the notion of knowledge rather broadly, "Our conceptualization of MKT, however, goes beyond the strict boundaries and limits of knowledge, to also include skills, reasoning, habits of mind, and sensitivities needed to teach mathematics effectively". They start from practice, analyzing first the work of teaching by identifying teaching tasks, such as, using representations to support student thinking. Then they attend to the mathematics that appears in this work. Finally, they analyze the knowledge needed for teaching. To do the latter they use a framework they developed that provides a heuristic for considering different types of knowledge needed for this work: common content knowledge, specialized content knowledge, knowledge of content and teaching, knowledge of content and students, knowledge of the content and the curriculum, and horizon content knowledge.

In their chapter on 'Competing paradigms in qualitative research', Guba and Lincoln (1994) discuss the interdependency of theory and research findings. They point to an interesting, sometimes unforeseen, connection between the theoretical framework used by the researcher and the findings of the research: "theories and facts are quite *interdependent*—that is, that facts are facts only within some theoretical framework" (p. 107). Moreover, they assert, 'Not only are facts determined by the theory window through which one looks for them, but different theory windows might be equally well supported by the same set of "facts."' (p. 107). The three analyses of Chloe's and Karen's clips show some similarities and some differences. This raises interesting questions:

- How similar are the interpretations and understandings of Chloe's and Karen's clips made by the three groups?
- Are the differences in the interpretations and understandings of Chloe's and Karen's clips made by the three groups related to the different perspectives used by each group for the analysis?
- Are the different perspectives compatible? Do they complement each other?

The answers to these questions are not trivial. In Even and Schwarz (2003) we exemplified how analyses of a piece of practice—a lesson—by using two different theoretical perspectives led to different interpretations and understandings of the same lesson. Both analyses showed that students did not behave mathematically as desired by the teacher. However, the two approaches suggested different interpretations of the situation and of the sources to the problems observed.

Often, understanding of the complicated practice of teaching and learning mathematics requires the use of different perspectives. This Research Forum will help us understand the potential contribution of each of the three perspectives suggested by the participating groups to advance understanding of teacher knowledge and its relationship to the work of teaching.

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TWO LESSONS - THREE VIEWS - SOME COMMENTS

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The purpose of this Research Forum is to consider two lessons from three different theoretical perspectives. In my reaction, I comment on the three groups' analyses without attempting a further interpretation of the two lessons.

THE TWO LESSONS - WHERE ARE THE CRITICAL ISSUES?

Two lessons serve as the essential substrate of the Research Forum work: one lesson given by a novice teacher (Chloe) and another one given by an experienced teacher (Karen). The goals of both lessons are only lightly shining through: Chloe seems to intend to teach an alternative strategy for adding and subtracting; Karen seems to clarify the role of the remainder in dividing two natural numbers. In my opinion, both lessons fall short on these goals.

The critical point for me is the lack of any effort on the teachers' part to create some kind of systematic coherence in the lesson and beyond. Creating coherence obviously requires that the teacher possess a lot of professional knowledge. Thus, the question arises: Which of the three theoretical views, offered to us by the three approaches — Mathematics for Teaching (MfT), the Knowledge Quartet (KQ), and Mathematical Knowledge for Teaching (MKT) — is (more or less) able to detect the facets of teachers' knowledge necessary in these lessons?

To be more explicit, I would have expected any of the three perspectives to engage in the following critical issues: In *Chloe*'s lesson: Is the *near-tens* strategy encountered as such? Is the 10x10 number grid representation used appropriate? Are students forced to generalize? In *Karen*'s lesson: How mathematically sound is the concept of "remainder" as taught? Is continuation possible? Are the chosen examples appropriate? And *generally*: Is the issue of "coherence" a target of the analyses? Do the analyses provide the teachers with some constructive hints to rethink the lessons?

THE THREE VIEWS - ARE CRITICAL ISSUES DETECTED?

The KQ seems an easy-to-apply framework, as it gives four basic dimensions one can check. However, things remain unanswered as it goes about the longer range perspective. The *connection* dimension seems to me too locally bounded, and does not reflect how things develop further in upper grades. The KQ therefore immediately detects the (fatal) error of Karen that zero isn't a remainder, but seems weak in identifying the rupture in Karen's lesson from "inverse operation" to "division with remainder"; the KQ characterizes this incoherence only as a "shift." The KQ detects, as MfT does, the problematic choices of examples in both lessons. However, the KQ characterizes the example 7÷1=7 in Karen's lesson as the "classic case of confusing the variables", while the MfT critique about the choice of 70 as a starting number in Chloe's lesson concentrates on the teacher's comment, "It's dead easy."

The MKT approach and the MfT approach seem to me, in a sense, complementary. While MKT focuses on the mathematics itself, sometimes even on the "mathematical horizon", MfT emphasizes that the critical point of teachers' knowledge is the "lack of awareness of the implicit associations that they use in their practices". So to say, MfT cares about the "horizon of mutual understanding." These are really two sides of the same coin.

Thus, it is easily understood that MKT detects that Chloe does not make explicit the strategy she is going to teach: she does not generalize the strategy of adding and subtracting say 18 or 23, nor does her class use brackets to indicate this strategy (as shown in the MKT paper). On the other hand, the MfT recovers that certain mental images of the concept of natural number are not as explicit in Chloe's knowledge, and this apparently influences the students' insight into subtraction.

COHERENCE - COGNITIVE ACTIVATION - CONSTRUCTIVENESS

The stage remains open: All three views give insight into the complexity of teacher knowledge. Exhibiting different perspectives onto one and the same practical event is therefore the method of choice to proceed to a conceptualization of teachers' professional knowledge in mathematics. To me, there are three orientations:

"*Coherence*" is the key issue of lesson construction. This coherence must be visible to the students in the classroom, and — at the other end — the teacher should be committed to care for coherence over longer periods. This plea comes from different sources — from recent developments (Lin, 2005), but also from the European "Didaktic" tradition (cf. Wolfgang Klafkis's contribution in Westbury et al., 2000). "*Cognitive activation*" seems to be a good orientation, when we point to what we want to accomplish within the social setting of the class. It is the target of teachers' knowledge (Neubrand & Seago, 2008). "*Constructiveness*" seems to be crucial for being practical. Each framework for characterizing a teacher's knowledge realized in a lesson should show a potential to assist the teacher constructively.

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RESEARCH FORUM 2

Critical perspectives on communities of mathematical inquiry

RF2: CRITICAL PERSPECTIVES ON COMMUNITIES OF MATHEMATICAL INQUIRY

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While the notion of Communities of Inquiry (CoI) has its origins in philosophy, there has been widespread interest in mathematics classrooms as Communities of Mathematical Inquiry (CoMI). This paper outlines the structure and content of Research Forum 2: Critical Perspectives on Communities of Mathematical Inquiry, gives a brief review of research in the area, and highlights some key issues.

ABOUT THIS FORUM

The goal of this Research Forum is to present critical perspectives on *Communities of Mathematical Inquiry* (CoMI) and to engage participants in a (possibly continuing) international research community of inquiry addressing the questions that will be raised in the forum.

This forum will include five brief presentations based on the presenters' papers in this volume. The first session will set the scene in terms of the theoretical frameworks to be adopted, a summary of the field, and the main issues and findings. In the second session, presentations will relate to the underlying assumptions that shape teaching as a cultural activity, or what Stigler and Hiebert (1999) refer to as the *cultural scripts* in different countries, and the affordances and hindrances that these imply for classroom functioning as communities of mathematical inquiry.

Both sessions will allow substantial time for discussion, focused on the questions:

- What are the benefits, if any, of applying CoMI in a classroom?
- What current practices constitute versions of CoMI in your country?
- How might CoMI be developed in a classroom?
- Which are the affordances and hindrances for CoMI in a classroom?
- How can international collaborative research support the development of CoMI?
- How might this "group" establish a collaborative research program?

However, as the aim is to create a community of inquiry, participants will be encouraged to frame their own questions for discussion.

A continuing discussion group will be formed for those participants who wish to participate in an extended community of inquiry with a view to possibly carrying out collaborative research in the area.

Space restrictions dictate that this paper can only provide a very brief introduction to *Communities of Mathematical Inquiry*, research in the area, and some key issues.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 153-182. Thessaloniki, Greece: PME.

COMMUNITIES OF MATHEMTICAL INQUIRY

The notion of *Communities of Inquiry* (CoI) originated with the work of the philosopher C. S. Peirce (1839–1914), who argued that we "come to know the world via a communal and pluralistic community of inquirers engaged in a scientific method of inquiry" (Pardales & Girod, 2006, p. 301). In the classroom context, the notion of *Communities of Inquiry* has underpinned the *Philosophy for Children* movement (for example, Lipman, Sharp & Oscanyan, 1980; Splitter & Sharp, 1995).

Key features of classrooms functioning as communities of *philosophical* inquiry are the development of skills and dispositions associated with good thinking, reasoning and dialogue; the use of subject matter that is conceptually complex and intriguing, but accessible; and a classroom environment characterised by a sense of common purpose, mutual trust and risk-taking.

Inherent in *Communities of Inquiry* is the belief that social interaction is a requisite for learning and that students and teachers can engage together in scientific discourse where progress is made through building on one another's ideas which are publicly displayed and evaluated (see Lipman, Sharp & Oscanyan, 1980; Wells & Mejia-Arauz, 2006; and also Greeno, 1992; Bereiter, 1994, who refer to progressive or scientific discourse rather than *Communities of Inquiry*). Thus, in a community of inquiry, participants are engaged in confronting problematic situations and participating in dialogue and argumentation (in the sense of Krummheuer, 1995).

Within the PME and the wider mathematics education community, there has been widespread interest in mathematics classrooms functioning as communities of inquiry (or *inquiry classrooms*). Research into *Communities of Mathematical Inquiry* (CoMI) has addressed all of the three aspects of CoI, although it is often impossible to divorce one aspect from another as they are so closely intertwined. Much of the research, has focused on progressive discourse, collective argumentation and the socio-mathematical norms associated with these (for example, Cobb, Wood & Yackel, 1991; Yackel, 2001; Yackel & Cobb, 1996; Williams & Ryan, 2001; Brown & Renshaw, 2004; Groves & Doig, 2004; Groves & Fujii, 2008). This research spans the development of skills of thinking, reasoning and dialogue and the development of a classroom climate that encourages students to see themselves as mathematical thinkers, and models the teacher as being *in* authority rather than as *the* authority (Splitter & Sharp, 1995).

Other research foci include: the importance of appropriate and conceptually rich tasks and questions that provoke mathematically rich activity (c.f. the Japanese *hatsumon*), the need for *conceptual press*, and the need to interweave the concrete and the abstract (for example, Kazemi, 1998; Groves & Doig, 2002; Groves, Doig & Splitter, 2000); the assumptions underlying mathematical thinking and the process of inquiry in CoMI and the teacher and student actions that indicate these (Goos, 2004); and the role of CoMI in teacher education and teacher professional development (for example, Jaworski, 2004; Williams, Corbin, & McNamara, 2002).

CULTURE AND IDENTITY

In the tradition of Hans Freudenthal and the *Realistic Mathematics Education* (RME) movement in the Netherlands, the late Leen Streefland attempted to establish CoMIs in primary classrooms by giving students responsibility for their learning, using realistic problems, addressing students as "researchers", and using their solutions as the source of mathematics for the lesson (in much the same vein as seen in the Japanese structured problem-solving lesson – see for example, Fujii, 2009). Streefland also believed in creating a classroom climate of mutual trust where all students could participate in constructing mathematical meaning (c.f. Splitter & Sharp, 1995). According to Elbers (2003), students in Streefland's classes

> began to view themselves in different roles and speak about themselves in different ways. They talked about their new identities and responsibilities as members of a community of inquiry. They criticized each other when somebody fell back into the habits of conventional lessons. (p. 81)

The term *normative identity* is used by Cobb, Gresalfi and Hodge (2009) to refer to the interactively constituted classroom obligations with which students need to identify in order to be regarded as a successful "doer" of mathematics, and use *personal identity* to refer to the extent to which a student identifies with or merely complies with, or even rejects, these obligations – that is, the extent to which students turn "obligations-to-others" into "obligations-to-oneself" (p. 47). They also distinguish between *conceptual agency* – involving choosing methods and developing meanings for themselves – and *disciplinary agency* – using established methods. They argue that these distinctions provide a useful tool of analysis and apply it to contrast the mathematical learning in two classrooms. In the "design experiment class", the students, who had developed sophisticated reasoning and argumentation skills, also considered themselves to be succeeding mathematically due to their ability to make substantial contributions to classroom discussions.

These two descriptions not only resonate with CoI, but also describe the way in which Japanese mathematics classrooms operate, at least at the elementary school level, where the goal of learning is not for individual benefit but for whole-class progress (Fujii, 2009). Lewis (1998) characterises Japanese elementary schools by "the three C's: connection, character, and content" (p. 32), with teachers seeing as their first priority helping children develop a strong emotional connection to school, while fostering friendship, co-operation and responsibility, and giving children time to convince themselves of the concepts contained in the "frugal, shared curriculum" (p. 35). Learning is seen as a co-operative activity, with success being "judged by whether one sets and meets rigorous personal goals and does one's best for the group" (p. 36). Inagaki, Hatano and Moritas (1998) see classroom as a community of learners rather than individuals, with even the silent students learning. Thus, in Japan, notions such as *distributed authority* are supported by deep-seated cultural factors, which may account for the difficulties often associated in Western countries with creating CoMIs beyond research projects specifically targeting such practices.

PROGRESSING MATHEMATICS WHILE PROGRESSING DISCOURSE

In earlier work, Cobb and his colleagues (for example, Yackel & Cobb, 1996: Yackel, 2001) focus on the *socio-mathematical norms* associated with learning to communicate in ways that progress mathematical understanding (see also Kazemi, 1998). Sherin (2002), however, highlights the difficulties teachers have in moving from a traditional classroom to what she calls a *discourse community* by drawing attention to the "balancing act" required of teachers to use students' ideas as the basis of discussion while progressing the mathematical content.

Sawyer (2004) contrasts the strongly framed vertical discourses usually associated with teacher-directed mathematics teaching with the more loosely structured, context based horizontal discourses that are more closely related to everyday life. She observes that the former is usually regarded as being more conservative, but also more visible in terms of its pedagogy, than the latter, with "invisible" pedagogies often being seen as more progressive, but disadvantaging groups who are unaware of the rules of participation. In her explanatory case study, she describes the way in which a first grade teacher enacts a "radical pedagogy" which focuses on

> collective access to valued forms of knowledge … as students work together to learn the rules of these discourses … [and which has the] potential to enable very young students from diverse backgrounds to tackle the complex interrelationships of vertical and horizontal discourses in real life problems, while ensuring that mathematics remains at their heart. (Sawyer, 2004, pp. 457; 462)

It is no accident that this class and their teacher were involved in *Philosophy for Children*, and that it was the "culture" from the philosophy lessons that was explicitly continued in mathematics lessons.

CONCLUSION

Research into *Communities of Mathematical Inquiry* has increasingly looked at inquiry as a collective activity which is supported and constrained by cultural assumptions and practices at the local school and broader community levels. As Stigler and Hiebert (1999) point out, our efforts at improving teaching often ignore the fact that teaching is a cultural activity and overlook the insights we can gain into our own cultural scripts through comparative research.

One purpose of this Research Forum is to bring together an international community of researchers with a common interest in furthering research into *Communities of Mathematical Inquiry*, with a particular focus on the ways in which cultural factors impact on such practice. While cultural beliefs that underpin teaching in Japan and the West have frequently been highlighted (for example, Stigler & Hiebert, 1999; Groves & Fujii, 2008), there are many nuances that still need to be explored in order to paint a more complete picture of *Communities of Mathematical Inquiry* across countries and cultures. It is our hope that this Research Forum will stimulate an ongoing program of collaborative research.

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COMMUNITIES OF MATHEMATICAL INQUIRY: A PRIMARY PEDAGOGY IN PERIL

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This paper outlines some examples from an Australian education system, and its classrooms, that provide evidence of practices that are considered as antithetical to establishing and maintaining Communities of Mathematical Inquiry (CoMI). Although some possible solutions are posed, implementation is left open for readers to consider, as contexts vary widely from jurisdiction to jurisdiction.

INTRODUCTION

Belonging to, or being part of, a community is highly valued in today's society, if the rise of so-called "social software", like *Facebook*, is to be taken as evidence. However, this paper argues that "community" as a facet of education or learning, is being threatened and may well become extinct in a large number of the world's schools. More disturbingly, the spread of the "un-community" is not confined to school education.

For example, in a recent article, Rochford (2008) decries the alienation of university students from the university community, and that "the higher education system in Australia is based on presumptions of human interaction that are antithetical to the traditional idea of a university community" (p. 41). Further she quotes Hager (2005) who suggests that students have become passive receivers of education, not part of a learning community. This is at the tertiary education level, but what is happening in the compulsory years of schooling, and more particularly the primary years?

In any discussion about community in mathematics learning, ideas that consistently emerge are those resonant with the notions suggested by Lipman and his colleagues (Lipman, Sharp, & Oscanyan, 1980) and Splitter (Splitter & Sharp, 1995) for *Philosophy for Children.* For example, Splitter and Sharp list the following marks of a philosophical discussion: reasoning and inquiry; concept formation, and meaning making (pp. 127-130). All of these marks form the reason that it is from this philosophical perspective that the present paper views effective mathematics pedagogy.

The first point that must be made is that many educators consider that *Communities of Inquiry* (CoI), and related ideas, such as *Communities of Mathematical Inquiry* (CoMI), are productive forms of pedagogy, and particularly so by members of the mathematics education research communities (Groves & Doig, 2004; Groves, Doig, & Splitter, 2000). Anthony and Walshaw (2008) claim that "the classroom community is the cornerstone for developing a sense of belonging in mathematics classrooms" a view supported by Pegg and his colleagues (2007) among others.

Practitioners too may hold these opinions, as shown by Doig, Groves and Splitter's (2001) report of the project *Mathematics classrooms functioning as communities of inquiry: Models of primary practice*[1]. In this project, video-data of mathematics lessons were collected from ten Year 3 and 4 primary classrooms in Victorian government schools, and a lesson from each of the Year 3 and 4 classrooms at the Japanese School of Melbourne. Primary teachers, primary school principals, and mathematics educators who participated in a series of focus groups were unanimously supportive of CoMI as representing effective mathematics pedagogy and, further, that the Japanese primary mathematics lessons represented examples of CoMI. The effectiveness of the pedagogy in the so-called Confucian Heritage Cultures (CHC) (see, for example, Watkins & Biggs, 1996) is supported by evidence from numerous international comparative research studies. Thus, it is argued, that CoMI is an effective pedagogy for mathematics teachers to employ.

If this is the case, why then claim that CoMI is a pedagogy in peril? The answer is, that, if we accept that Asian primary lesson patterns are good examples of CoMI, these represent only a part of school practice, albeit a large part in terms of student numbers. In other jurisdictions, it can be argued that CoMI is not supported as a pedagogy, but is threatened, or replaced, by pedagogies antithetical to CoMI. Threats from two quarters are considered in this paper. The first is from the education system, and the second is from classroom support materials. The examples are taken from current documents and Internet support materials for teachers in Victoria, Australia.

EFFECTS OF THE EDUCATION SYSTEM

In Victoria the Department of Education and Early Childhood Development (DEECD) is responsible for curriculum, curriculum support, and also supporting classroom pedagogy. To this end, DEECD maintains a large and complex web-site that allows children, parents and teachers to access every aspect of school education under DEECD's purview. Part of this enterprise entails suggesting what are considered effective pedagogies. In our case, while the suggested pedagogy is not mandatory, it is the most prevalent to be found in classrooms. The pedagogy, it is suggested, should be differentiated to maximize children's learning – in other words, differentiated to attend to individual children's needs, a laudable objective.

However, given a class of twenty or more children, individualized instructional practices are impractical for many teachers, and so a modified pedagogy is employed. This is a class management strategy that has a whole class period at the beginning of a lesson (orientation), followed by small group work (teaching), and finishing with a whole class "share time" (plenary). This arrangement is popularly known as the "hamburger model". Each of the parts of this hamburger, unfortunately, as recommended, or in practice, do have different foci. For example, the orientating first

[1] *Mathematics classrooms functioning as communities of inquiry: Models of primary practice* was funded by the Australian Research Council. The Chief Investigators were Susie Groves (Deakin University), Brian Doig and Laurance Splitter (The Australian Council for Educational Research).

part may deal with yesterday's lesson on length, the small group work may have several foci, and the plenary at the end may be a sharing of all the disparate parts.

An example lesson observed in the previously mentioned *Mathematics classrooms functioning as communities of inquiry: Models of primary practice* had three groups in the group work part: one group was using sale catalogues to try and purchase $1000 000 of property, a second group of two children was asking classmates to guess the total value of a mixed collection of coins in a jar, and the children in the third group were tossing coins to observe sequences of heads and tails tossed. At the end of the lesson in the plenary, the class teacher had children from each group report their "findings". Obviously, each group had some difficulty in understanding other groups' findings. When asked about this, the class teacher pointed out that each day the groups would change activity and by the end of the week, when everyone had tried all activities, there would be a whole class discussion. It is easily seen that this arrangement is antithetical to forming a CoMI.

Thus, starting with a laudable suggestion, the end result is antithetical to effective pedagogy: the devil, as is said, is in the detail.

There is a further issue with the notion of classes working in small groups. The figures below show the physical and dialogical arrangements of a hamburger-style lesson. In the figures T represents the teacher, S the students, and the arrows show the main direction of information flow.

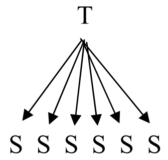

Figure 1: The orientation part of the lesson.

Figure 1 shows the orientation part of the lesson. The instruction and questions flow from teacher to students. It is typically a rehearsal of known work, and is usually in a simple question and answer format. Clearly this is not intended to be part of a CoMI and it would be churlish to deny the usefulness for many children of this part of the pedagogical model.

Figure 2 shows the group work part of the lesson. Note that the bottom group is the focus or teaching .group. This group receives special attention due to their mathematical needs, while the other (two) groups work independently of the teacher. The teacher independent groups may not be studying the same topic as each other, as outlined in the sample lesson from the *Mathematics classrooms functioning as communities of inquiry: Models of primary practice* project described above.

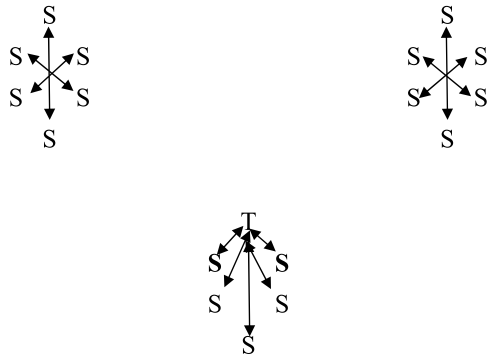

Figure 2: The group work part of the lesson.

Figure 3 shows one of the children describing their experiences and learning to the teacher and the remainder of the class. Each child would be expected to take this rôle in turn. The class and the teacher would also ask questions, and at times the plenary could become dialogue, but observation suggests this rarely happens.

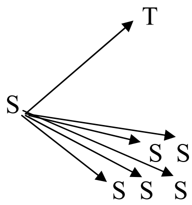

Figure 3: The plenary part of the lesson.

It is clear that the hamburger model of pedagogy is antithetical to CoMI due to its structural elements. It would appear that the dilemma for CoMI is how to address individual student needs while enabling a form of dialogue and community.

IN THE PRIMARY CLASSROOM

Support at the primary classroom level of the most obvious examples of CoMI-antithetical practices are to be found in text books. In Victoria the common practice of text-book writers and publishers is to take the curriculum content, which is expressed in terms of achievement outcomes (objectives), and fragment this further in the following manner. Each major topic in mathematics (for example, number, geometry, measurement) is subdivided into smaller topics that can be taught in a single school term. In Australia there are four ten-week terms in a school year (although some terms may vary slightly from this general pattern in some years). In this way, each major topic is "visited" every term, and is usually seen as an example of (Bruner's) spiral curriculum. Next, the term sub-topics are re-divided into smaller parts that can be taught in single lessons. But these are presented in the text-book on non-consecutive pages, for example, pages 14, 26, 41, 55, 68, 80, …. A consequence of this process is that these myriad parts of the curriculum are interleaved, and no coherent or sustained learning can take place. To complete the disarray, each page of the text-book may have two or three different aspects of these parts per page! For example, a measurement page may have tasks that require measuring the length of real objects in centimetres, working out how many centimetres there are in a metre (!) and writing a given number of centimetres as whole metres. While this example is based on Dean and Nightingale (2003) the format of their book is not unique but is rather ubiquitous.

The Victorian primary teacher is supported then by text-books, and other materials, that accept that an "atomized" curriculum is both efficient and effective. And yet, research into the practices of effective teachers of mathematics does not support the use of such "atoms", but rather suggests that effective teachers more often implement, *inter alia*, sequences of lessons rather than single one-off activities, and make explicit connections between ideas (Doig, 2003). Further, CHC classrooms demonstrate that a sustained focus is enabling of a CoMI and thus it can be claimed that the use of a text-book of the nature above is antithetical to a CoMI.

CONCLUSION

In the preceding paragraphs we have described examples of pedagogical practices and their supporting materials that are, intentionally or not, antithetical to CoMI as a basis for effective mathematics classroom practice. This is despite evidence from CHC classrooms that CoMI is an effective pedagogy. The question that remains is how to change the present situation in the face of current practice.

Clearly, pilot classrooms need to be established that can demonstrate the effectiveness of CoMI as a pedagogical practice. Such classrooms would need to be open to visits from other teachers, parents, and other interested parties. Data of CoMI effectiveness for learning that is patent and understandable to the intended audience needs to be collected and widely disseminated. Convincing an education system to trial approaches that they see as antithetical to their own prescribed, or suggested, models would be a large and lengthy task. Perhaps mathematics education researchers, such as members of the PME community, can form research networks, with individuals contributing their findings to form a critical mass. Perils may yet be prevented.

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A JAPANESE PERSPECTIVE ON COMMUNITIES OF INQUIRY: THE MEANING OF LEARNING IN WHOLE-CLASS LESSONS

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This paper aims to characterize Communities of Inquiry from a Japanese perspective through clarifying the meaning of learning in whole-class lessons in Japan. In order to accomplish this aim, classroom discourse, students' and teachers' reflective comments in post-lesson video-stimulated interviews are analysed. This analysis reveals an actual process of what and how students' conceptions are presented and accepted through classroom discourse. Some intended teacher behaviours that make whole-class discussion progressive are also identified.

INTRODUCTION

There is widespread agreement in many countries about the critical role played by progressive classroom discourse in whole-class lessons in the generation of new mathematical understandings for students. Japanese whole-class lessons can provide examples of such discourse. However, as results from the TIMSS 1999 Video Study (Hiebert et al., 2003) reveal, Japanese lessons seem to be significantly different from those in other countries, so that there may be a different philosophy concerning the meaning of community or learning in Japanese whole-class lessons. Although it is not always the case, Japanese lessons have a typical flow or structure, particularly when problem-solving oriented mathematics lessons are conducted. Such lessons consist of four components: 1) Understanding the problem for the day; 2) Problem solving by students; 3) Comparison and discussion (students presenting solutions); and 4) Summing up by the teacher. It could be said that Japanese teachers have just one ideal lesson script for Japanese lessons (Jacobs & Morita, 2002). Foreign people have noticed these components through observing Japanese lessons (Becker, Silver, Kantowski, Traversand & Wilson, 1990, Stigler & Hiebert, 1999). The reason why Japanese lessons are structured as described above may reveal a cultural assumption behind these lessons. The aim of this paper is to characterize *Communities of Inquiry* from a Japanese perspective by clarifying the meaning of learning in whole-class lessons in Japan.

For this purpose, classroom discourse, students' and their teacher's reflective comments in post-lesson video-stimulated interview data from the *Learner's Perspective Study* (LPS) (see, for example, Clarke, Keitel & Shimizu, 2006) are analysed. In particular, this paper focuses on the question in the interview asking the teacher to identify important moments of the lesson. Such moments seem to have something valuable to contribute in terms of the meaning of learning in whole-class lessons.

THE LESSON

After repeated viewing of the LPS lessons, Lesson 5 of J1 was chosen. It exemplifies the Japanese lesson structure and the type of discussions that occur in such lessons. The J1 unit of lessons deals with the concept and representation of linear functions.

The problem context for Lesson 5 is folding a 12 cm x 15 cm rectangular piece of coloured paper and considering the perimeter of the figure (see Figure 1). Students were asked to decide independent and dependant variables and make a table and graph of the relationship. Two kinds of solutions emerged: the independent variable as the length of the folded part, which is white part in Figure 2, as student NO reported, and the dependent variable as the perimeter of the whole rectangle. The alternative response was, as students NI and TA reported, the independent variable being the length of the coloured part and the dependent variable being the perimeter of only the coloured part of the rectangle (see Figure 3).

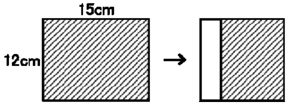

Figure 1: The Task

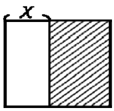

Figure 2: Student NO

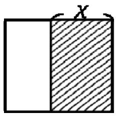

Figure 3: Students NI &TA

IMPORTANT MOMENTS IDENTIFIED BY THE TEACHER AND STUDENTS

Table 1 shows the lesson structure and important moments identified by teacher K and students N1 and TA at the post-lesson video-stimulated interview.

According to Table 1, the teacher identified nine important moments, while student NI identified eight such moments and student TA seven. It is interesting to see that the times and moments identified are fairly consistent between the teacher and the students, although the precise times are slightly different.

Concerning the lesson structure, Understanding the problem took about 12 minutes of class time, the Problem solving phase about 13 minutes, the first Comparison and discussion about 17 minutes, and the second Comparison and discussion about 4 minutes. There was no Summing up phase in this lesson.

In the Understanding the problem phase, one or two moments were identified by both the teacher and the students (Teacher 3:50, 9:29; Student NI 9:23; Student TA 6:00). An appropriate understanding of the task for the day is surely a necessary condition for whole-class discussion later. In the interview, the teacher clearly stated that she needed to make sure that everyone understood what to do. In fact, the teacher let two students demonstrate their idea. The students interviewed said these students' ideas were different from their own, and therefore they felt this moment was important.

Process of Problem Solving		Teacher K	Student NI	Student TA
Understanding of Problem	01:33:22	3:50		6:00
		9:29	9:23	
Plobrem Sloving	12:53:28	14:00 16:50	14:25 16:00	14:22 17:30
Comparison and Discussion 1 Student NO's Presentation	25:42:20	28:00 29:30	27:09	24:44
Student UM's Presentation		32:26		
Student TA's Presentation		33:30	34:30 37:45	37:26 40:00
Comparison and Discussion 2	42:39:28	42:50	42:55 45:20	43:02

Table 1: The important moments identified by teacher and students

In the Problem solving phase, while students work on the problem, Japanese teachers walk between desks monitoring student activity. In this lesson, the teacher monitored the whole class and came twice to student NI and TA, who were both interviewed. These times were identified as important occasions by both the teacher and these students (Teacher 14:00, 16:50; Student NI 14:25, 16:00; Student TA 14:22, 17:30). At these moments, students NI and TA were struggling with identifying an independent variable, so they asked the teacher for help when she came to them the first time, and at the second time, they reported briefly on their work.

In the Comparison and discussion phase, the teacher asked students to raise their hands and the first student to report was selected. The first student (NO)'s solution (see Figure 2) is probably the most common or typical one. Both the teacher and the students identified this as an important moment (Teacher 28:00, 29:30; Student NI 27:09; Student TA 24:44). In the interview, the students said that the solution presented by NO was different from their own and therefore important.

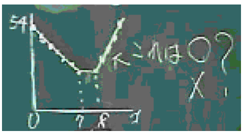

Figure 4: Student UM's graph

In NO's solution, the graph became like a V-shape, which was sketched on the blackboard. The second solution by student UM was also almost a V-shape graph, but without an apex (see Figure 4). Therefore the teacher deliberately selected him even though he did not raise his hand. However the student UM had already corrected it, so there was not much to discuss. The teacher identified this moment as important (32:26), but the two students did not. So the teacher represented UM's graph for whole class to discuss at the second Comparison and discussion phase. This time was identified as an important moment (Teacher 42:50; Student NI 42:55; Student TA 43:02).

DISCUSSION AND CONCLUSION

The important moments and reasons why they were chosen reveal teachers' and students' ways of thinking about or attitude towards learning in whole-class lessons in Japan. First of all, the class seems to appreciate and value as important knowing different ideas. The different ideas are represented not only by students spontaneously, but also through careful selection by the teacher in a manner such as:

> Also, I saw that a number of you were having trouble with your graphs when I was going around …. Wasn't it your group, TA? Could you maybe explain to the class? Yes, come up to the front. (00:32:36)

The evidence for valuing other students' ideas is also found in the teacher's recommendation to write down their friends' ideas by saying things such as:

> Well, as we did yesterday, write down someone's opinion in colours. (J1, Lesson 7, 19:14:03)

Valuing other students' ideas in whole-class lessons indicates a community of inquiry, supported by a prerequisite for whole-class lessons in Japan described by Lewis (1995) as: "all children, regardless of their academic achievement, are genuinely valued members" (p.177).

The goal of learning in whole-class lessons in Japan is not for individual benefit but for whole-class progress. A Community of Inquiry needs to have progressive discourse to raise the level of whole-class thinking. More generally, to deepen understanding, we need to see "not A" together with "A", where "A" is your own idea. The J1 teacher gave an appropriate example of "not A" to the class with student UM's graph, in order to create "progressiveness" in the whole-class discussion.

Although the teacher valued all the students' ideas, she chose which ones to present to the class. Student TA's idea was highlighted at the end of the lesson. Student UM's graph was also highlighted, although it was incorrect (see Figure 4). On the other hand, some students' idea were not presented, for example the idea that the independent variable is the width of rectangle, as shown in Figure 5, and the table written by student UN in Figure 6 which was based on this idea. Students TA and NI also had the same idea in the beginning of the lesson. In fact student NI made a table like student UN, but only up to the value of 8 in the independent variable. Later, she changed the meaning of the independent variable to the length of the coloured part.

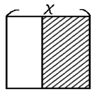

Figure 5: An idea not presented

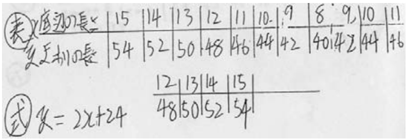

Figure 6: Student UN's table

This idea of NI and TA (see Figure 3) was considered in detail during the sixth and seventh lessons. This is a typical example of the way in which students' ideas are treated with care: the teacher identified the moment when student TA explained her idea (33:30) as important, while student NI identified the same moment (34:30). However the teacher did not understand what student TA meant, therefore the time when student NI explained it again was identified as important by both students NI and TA (37:45, 37:26). These moments seem to illustrate a shared thinking process between the teacher and students, which could be regarded as evidence of the existence of a community of inquiry.

Students NI and TA were not afraid to come to the front to explain their idea. They seemed to want their teacher to understand what they were thinking. Eventually, their idea affected the entire content of the following two lessons. In other words, the sources of mathematical ideas, and responsibility for learning were left to the students (Hufferd-Ackles, Fuson, & Sherin, 2004; Goos, 2004).

The teaching and learning of mathematics in whole-class lessons in Japan engages students in collaborative forms of inquiry through the structured lesson process. Among the four components of the lesson, the Comparison and discussion phase is the most difficult part for teachers to deal with, simply because, the teacher may have to orchestra the whole-class discussion in order to create a reflective discourse and to develop students' mathematical understanding. This process is called *neriage* in Japanese, meaning carefully, meticulously polishing one's thought or creating a progressive discourse in a whole class. Unsuccessful *neriage* becomes only "show and tell". The process and conclusion of *neriage* usually emerges in the blackboard writing. For further research, it is interesting to analyse blackboard writing, focusing on how individual and/or shared ideas appear on the blackboard.

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THE CLASSROOM AS A COMMUNITY OF MATHEMATICAL INQUIRY

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This paper defines community of inquiry in terms of three dimensions, namely: the nature of the classroom as an environment in which all participants see themselves as "one among others", the dynamics or "moves" driving classroom activity, and the conceptual level of the content which gives substance to the inquiry process. Dialogue is identified as the dynamic interplay between the external conversation of the classroom community and the thinking of its participants, including the "meta-cognitive scaffolding" which occurs when students shift from first-order thinking to "thinking about thinking". All three dimensions play a part in helping students think more deeply, which suggests that the community of inquiry has much to offer in the context of students' understanding and appreciation of mathematics.

The concept of *Community of Inquiry* (CoI) has its origins in the work of the American Pragmatist philosophers C. S. Peirce and John Dewey, with considerable influence from Vygotsky's theories of language, mind and society (Peirce, 1955; Dewey, 1916/1966; Vygotsky, 1978). This concept also forms the pedagogic and epistemological foundation on which *Philosophy for Children* (P4C) has been constructed (Lipman, 2003, Splitter & Sharp, 1995; Lipman, Sharp, & Oscanyan, 1980; Splitter, 2000). More recently, researchers and teachers have sought to adapt and apply CoI across a range of school-based disciplines, including mathematics (Cobb, Gresalfi, & Hodge, 2009; Cobb, Stephan, McClain, & Gravemeijer, 2001; Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, 1999; Yackel & Cobb, 1996; Lampert, 1990; von Glasersfeld, 1995; Bishop, 1985); technology (Garrison & Anderson, 2003), and history (Seixas, 1993; also Pardales & Girod, 2006).

"CoI" refers to a particular kind of classroom environment or culture, in which students *engage together* in various forms of *inquiry*, where the latter is understood to be any mode of thinking that is motivated by, and directed toward, clarifying, solving or resolving something which is regarded as both problematic and worthy of attention. Inquiry is both cognitive and affective: inquirers – of whatever age and in whatever subject area – are curious (asking lots of questions), persistent and focused (following the inquiry where it leads), and open-minded (being prepared to consider new perspectives, to "own" mistakes, and to self-correct).

Inquiry eschews the kind of vacuous relativism which declares all points of view to be equally valid, but also the rigid dogmatism of an imposed authority; it encourages a healthy sense of scepticism but steers clear of the paralysis of universal doubt; and it guides thinkers to be clear about their beliefs, values and commitments, while avoiding claims to absolute certainty. The collaboration characteristic of a CoI is not incidental. For multiple thinkers to be jointly engaged in an inquiry, there must be a form of public communication which *represents the thinking of the community*. In a CoI, this kind of communication is called “dialogue”.

This brief characterization contains elements of what I call the *Three Dimensions of Inquiry*. These dimensions will now be explored in more detail.

DIMENSION 1: THE NATURE OF THE CLASSROOM ENVIRONMENT – SEEING ONESELF AS “ONE AMONG OTHERS”

The community of inquiry strives to build and maintain a harmonious balance among its members. It is grounded on, and committed to, principles of fairness and equality. Every individual feels valued and cared for, by and within the community; it is a “safe place” in which all contributions are respected (albeit not necessarily agreed with). However, this sense of personal empowerment also requires each person to see herself as part of something larger – the community itself, which is simply the network of relationships which links each member to each other member. This idea is not new: to regard myself as a person is to regard myself as being related to others who also regard themselves as persons. I capture this interdependence by saying that all members of the community sees themselves as *“one among others”* (Splitter, 2007). Cobb and his associates refer to the “possible relationships between *individual* students’ mathematical development and the classroom *social* processes in which they participate” (Cobb, 1997, p. 259; also Cobb, 2006; Yackel & Cobb, 1996).

DIMENSION 2: CLASSROOM DYNAMICS AND PROCESSES – WHO DOES WHAT, WHO THINKS, WHO TALKS, WHO CONTROLS

The dynamics in a CoI reflect or reveal the types of thinking that take place. They are the “moves” made by those engaged in the inquiry. One such move is the *meta-cognitive scaffolding* that occurs when students shift from engaging in the (first-order) mathematical task to making that task the object of investigation and discussion (“thinking about thinking”). Such moves will typically involve the kinds of analysis, synthesis, creative thinking and evaluation that are the hallmarks of higher level thinking.

Cobb and his associates have documented many classroom conversations involving children as young as first grade engaged in making this shift. It requires a measure of reflection and abstraction, moving from a computational activity (in which children are working something out) to a conceptual activity in which they are, for example, comparing and discussing the different explanations that were posited as solutions, realizing that it is “OK” to make mistakes, etc. (Cobb, 2006; Yackel & Cobb, 1996; Cobb et al*.,* 1997). Lampert (1990) makes a similar point:

> When a student is in charge of revising his or her own thinking, and expected to do so publicly, the authority for determining what is valid knowledge is shifted from the teacher to the student and the community in which the revision is asserted. (p. 52)

Dialogue and inquiry: Two sides of a coin

We must also consider the dynamic interplay between the external conversation or dialogue – which reveals the thinking of the community – and the thinking of participants – which, in turn, can be regarded as internalized dialogue (Vygotsky, 1978, Garrison & Anderson, 2003). Given the role of the community in nurturing the growth of each participant as someone who is learning to think well, we cannot accept the traditional dynamic in which the teacher claims sole responsibility and control. On the other hand, the CoI is not reducible to such familiar alternatives as individualised instruction and group work, unless those involved have internalised the pedagogy of inquiry into their own behaviour. It is unrealistic to expect children to become inquirers – that is, reflective, responsible, caring thinkers who build upon one another's ideas – if they lack the tools and dispositions needed for the task.

DIMENSION 3: CONTENT AND CURRICULUM – SPECIFICALLY KNOWLEDGE, CONCEPTS, JUDGEMENTS, UNDERSTANDINGS

A CoI has a commitment to content, as has the "traditional" classroom, but here the concern is with content that emerges from the interpersonal construction that we call "inquiry". Further, while one goal of inquiry is to find answers to questions posed by the community (else why bother to ask them?), satisfaction is gained as much by the process itself as by any answers arrived at. What matters most is the extent to which the "answers" that do emerge mark an increase in *understanding* of the questions, concepts and issues involved (von Glasersfeld, 1985; Bishop, 1985; Smith, 1983; Yackel & Cobb, 1996; Cobb & McClain, 2002).

CoI as a "meaning-making" environment

The yearning for things in our experience to make sense is as strong as our desire to know the truth of things. To say that something – a word, a story, an event, a topic, even a life – makes sense to us is to say that we can establish lines of *connection* between it and other items in our experience *which are already meaningful to us* (Splitter & Sharp, 1995). Similar characterizations can be found in the mathematics education literature, for example:

> one gets the feeling of understanding when a new idea can be fitted into a larger framework of previously-assembled ideas. A metaphor that fits this quite well is the notion that one assembles ideas in one's mind much as one assembles a jig-saw puzzle. Each new candidate piece, like each new idea, can be used only if it fits into the aggregate of pieces that have previously been assembled. (Maher, 1999, p. 88)

Regarding meaning in such recursive terms brings into sharp focus an epistemological problem facing many students of mathematics: they are expected to build new understandings on top of those which they have previously constructed; but if the latter are shaky and insecure, whatever is added will also be shaky.

Concepts and "meaning-making"

Inquiry is a process of *digging deep*, rather than staying on the surface of a topic. Thinking tools such as probing questioning and reasoning are important here, but they are not enough. There must be something which is judged by students (not just the teacher) as *worth "digging into"*. An important component of inquiry is to move among different levels of complexity: between specific, concrete situations ("real world" examples), and more thought-provoking, abstract and challenging contexts where consequences, reasons and criteria are formulated, examined and evaluated. This sense of movement is well captured by Lampert (1990, p. 31), who refers to the "zig-zag path from conjecture to proof (or refutation) and back to axioms", and the need to adopt "the inductive attitude which requires a ready ascent from observations to generalisations, and a ready descent from the highest generalizations to the most concrete observations" (see also Splitter, 2000).

Such contexts are *conceptually rich:* at the heart of any subject area which has the potential to generate inquiry is a framework of concepts which carry the "seeds" of understanding for the discipline itself. These concepts (including *number, function, identity, set/class, operation, fraction/ratio, order,* in mathematics) are *contestable* (not clear-cut, hence ripe for further inquiry) and *central* to any deeper understanding of the subject; they also operate at a level of generality not captured by memorizing formulae and solving individual problems. This is a far cry from the traditional mathematics classroom which is based on – and limited to – the final authority of teacher and text book.

Cultivating a dialogue in which the "zig-zag" between lower and higher conceptual levels occurs requires setting the scene in a number of ways. These include: beginning the lesson with the presentation of a rich and relatively complex problem which "would have the capacity to engage all of the students in the class in making and testing mathematical hypotheses" (Lampert, 1990, p. 39). Lampert's own illustration requires students to figure out the last digits in 5^4 , 6^4 and 7^4 . Within the lessons themselves, we can identify the zig-zag pattern, as the children move from concrete illustrations and examples (as Lampert puts it: talking about what *I* did personally), to conjectures, hypotheses and attempted proofs (talking about what *you* – i.e., *one* – do).

DIALOGUE AND MATHEMATICS IN THE CLASSROOM: A CHALLENGING COMBINATION

The stereotype of mathematics as a subject with "hard" answers to questions/problems and fixed methods for solving them works against creating a dialogical environment in the classroom, particularly when the teacher is seen as the expert who always knows the answers in advance (which is very likely the case for all but the most senior grade levels). Philosophy, by contrast, presents a different stereotype: a subject with *no* definitive content; abundant in questions but with no answers or, at least, no mutually agreed-upon answers. Notwithstanding the threat of *relativism* ("any answer is as good as any other, so who cares?"), which is based on a mistaken view of philosophical thinking, the barrier to dialogue described above simply does not exist: even trained philosophers cannot claim to *know* the answers to philosophical questions.

Moreover, the meta-cognitive scaffolding described above, while exemplified in philosophical dialogue, is less easily accommodated in the mathematics classroom. Notwithstanding the work of Cobb et al – which amply demonstrates that such meta-cognitive discourse both can and should take place, even at the most junior levels – mathematics curricula are usually highly structured and constrained (Cobb & McClain, 2002; Cobb, 1997). Those teachers who want to encourage reflective discourse may be tempted to bring the focus back to the first level subject matter, which they inevitably need to do in order to "move on".

The way forward here requires the kind of shift in the culture of mathematics teaching that writers have emphasized in recent years (McTighe et al, 2004, Cobb & McClain, 2002; Cobb, 1997, 2006; Cobb et al., 1997; Lampert, 1990; von Glasersfeld, 1995; Splitter, 2009). This shift calls on teachers and curriculum designers to focus on developing students' deep understandings of key concepts, paying particular attention to forms of thinking and dialogue that embrace both first order and higher order thinking, including meta-cognition. The framework of the classroom CoI has much to offer here.

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CHAT PERSPECTIVES ON COMMUNITIES OF MATHEMATICAL INQUIRY

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Cultural-Historical Activity Theory (CHAT) offers a critical analysis of inquiry as an object of collective activity in (a) schooling (b) mathematics and mathematics learning. I take it that a community of inquiry might grow from a fairly stable group that has made the collective activity of inquiry together into a habitual practice, and discuss the constraints that schooling in an audit culture impose that make this a highly rare and even dangerous aim.

INTRODUCTION

The concept of a Community of Mathematical Inquiry (CoMI) needs some clarification: I will take it from the *Philosophy for Children* tradition that a Community of Inquiry (CoI) is an "ideal" form of a community in search of truth (Splitter & Sharpe, 1995). It then becomes an empirical and strategic question as to whether something like a CoI or CoMI exists, can be brought into existence or is even "possible" in certain social, cultural or historical circumstances. These are the issues I address here in relation to mathematics education research.

The use of the notion of a classroom of learners, or a group of reflective practitioners becoming a community of practice (hereafter CoP) or a community of inquiry (hereafter CoI) has been quite widely used across the field of education (e.g. Wells, 1999). These studies usually refer to communities of practice as offered by Lave and Wenger (1991) and Wenger (1998): sometimes as well as to the Philosophy for Children literature, and sometimes also to the CHAT literature (typically by Vygotsky, Leontiev, Engestrom and Cole).

However, for Wenger the conceptual framework of CoP is an analytical tool – thus it is always an open question whether the concept is useful in a given social context. Not all communities or social groups share a "practice", a common enterprise, and a repertoire of discourse and competences which are essential elements of a CoP; sometimes they share a practice which is by no means characterised as inquiry.

I argue, indeed, that only rarely is it useful to think of a mathematics classroom as being a CoP, and even when a classroom community does truly engage together in a mathematical enterprise, it is rarely truly an inquiry. The same can be said of many a staff room and many a teachers' in-service course or professional development group. While a teacher might aim to develop a classroom community into a CoP/CoI/CoMI, this cannot be done by fiat/declaration – and we might need to understand the constraints and problems involved. Activity Theory was developed specifically to help us understand learning and development in schools, inter alia, and can help us here.

CHAT PERSPECTIVE ON THE "SUBJECT" AND "OBJECT" OF ACTIVITY

The CHAT perspective I draw on is mainly the legacy of Vygotsky and his colleagues, especially Leontiev and lately Engestrom and Cole among others: for a summary see Roth and Lee, 2007, and in the mathematics context, see Williams and Wake, 2007.

Leontiev (1981) says that for him (and therefore for what we call Activity Theory) "activity" makes no sense without a collective social, culturally-historically formed "object". The object of an activity is both the material object of joint labour (the animal being hunted; the history text being read) and at the same time its "ideal" motive and need (the food and clothing envisaged to be consumed; the need for understanding history, perhaps?) But what is the motive for reading history in school? Leontiev asks us to imagine a student reading a history book, who is told by a colleague that he has the wrong book, as it is no longer on the syllabus for the course. We may find the student puts the book down in disgust – in such a case we might infer that the motive was the upcoming exam, the winning of grades for university entrance say (what our project has come to call the "exchange value" of education within an audit culture, and a performance economy of schooling: see Williams, 2009). The activity might be said to be schooling-going, whose motive is something like grade-producing, in a competitive economy of educational status.

On the other hand, if the student were instead to ignore the exam syllabus, and carry on reading (or maybe reluctantly turn to other work) then we might infer that there was some interest in history itself: now it becomes a matter of social and cultural analysis what this activity of historical inquiry is about, but let us hypothesise for the moment the motive is to understand history and so humanity's present condition and possible futures as a result (in contrast to "exchange value" one might describe this as the "use value" of education, in Marx's terms for commodity value).

Maybe there is a community of practice or inquiry that this student is becoming inducted into without even knowing it, but their action was just to read the history book and become interested, possibly without any direct interaction with a CoI. But CHAT suggests that this motive is social, and connects the student by a cultural-historical thread with the activity we chose to call historical inquiry.

The psychological disposition of the learner, (i.e. the "personality" of the subject of this activity, in Leontiev's technical terms) then is intimately bound up with the nature of the social motives of their various activities: of grade producing as a good student in the activity of schooling, or of understanding the human condition, as in the activity of inquiry. A crucial question then is: to what degree is it possible to develop such inquiry activity in school?

"SCHOOLING" OR "INQUIRY" IN CLASSROOMS AND SCHOOLS

Engestrom (1987) offers us some insights into the nature of schooling (what he calls school-going) as an activity: his approach is interactional and structural, and seems to work well for activities that are institutionalised, such as commodity production in a factory, health promotion in a health clinic or hospital, or grade producing in school. There are instruments of teaching and learning (curricula, texts, syllabuses). There is a community with a division of labour (e.g. between teachers, and learners, inter alia) and norms or rules of behaviour (the teacher/head-teacher controls, etc).

The cultural-historical weight of hundreds, if not thousands, of years of schooling can be felt in each of these elements of the school system. Many of these elements are of course not necessarily conducive to inquiry. But more than this is involved: the school system (including its instruments, rules and divisions of labour) is continually being reproduced by the work of other systems, too: the publishers of the texts, the political system that governs (and funds) through the audit system, the adjacent educational and industrial systems that feed or produce schools, and so on. All these adjacent systems also bear a great weight of cultural-history, which serves to reproduce themselves and schooling, which in turn constrains activity in schools or classrooms.

Many (including Engestrom, 1991) have sought to break free from the "exchange values" of schooling by appealing to the "use values" of mathematics: directing attention to mathematics in scientific or social inquiry, to modelling and problem-solving, or to the interesting nature of mathematics itself. Many of us perhaps have struggled hard to develop work in school and classrooms with this object; I argue this is consistent with the CoI ideal.

Some small examples: I recall a lesson that was inspired by the school secretary asking me for the third time how to work out the price of goods before the 15% sales tax (VAT) was added (the school did not pay the tax, but the brochures advertised prices included it). I told the following class I'd like them to help me make some kind of table or ready-reckoner or other aid to help with this task. It was a "real" task – it had use value – and was not obviously mediated by its value in the curriculum, to their grades etc. The result was that the class seemed to come together for the half hour and collectively produced some solutions: the activity transcended schooling for a moment, for a lesson, before returning again to the routines of schooling.

Entire projects were subsequently built (by me and others, in the UK and elsewhere) around the aim of making mathematics useful in modelling and applications; and we have some evidence in recent studies that these can achieve real benefits for students (see Davis et al., 2009). However, these initiatives often do not last: parents and students may not understand these inquiry initiatives or share the motive. Innovative teachers/administrators move on and find their legacy disappears very quickly when they depart the institution. Some excellent texts are produced only to disappear when the publishers find a cheap and less useful re-write more profitable. The institutions that inspect teaching may come to undermine inquiry if it is deemed not to make for the most efficient scores on exams. The professionalism of teachers in regard to assessment and coursework may be undermined by political systems that hold them accountable for their funding.

I suggest this is due to a conflict between the ideals of the "mathematics culture" (the CoMI and the subject culture) and the "audit culture" which rewards (i) students' grade producing (with access to scarce resources) and (ii) schools' aggregate performances (with approval, funding, promotions etc). Thus, we examine audit a little more closely.

USE AND EXCHANGE VALUES IN THE AUDIT CULTURE

I claim the contradiction between use value and exchange value of education is played out in many systems that surround and impinge on the classroom and the school. What is the use and exchange value of mathematics education (see Williams, 2009)? We see a high exchange value for mathematics in students', teachers' and administrators' and policy makers' accounts, where test performance and exam grades "count" beyond everything else. Mathematics is a shortage subject, provides access to highly valued and scarce resources such as elite university courses, and jobs; there is even some evidence that the sole difference of having advanced level maths and not has been worth a significant amount of money over a career. Particularly we have seen very high value attached to a quantitatively literate education in banking, insurance, and the city.

On the other hand the use value of mathematics is more controversial. It is functional in many a practical way through its applications, in work, leisure and citizenship, critical or otherwise. But it may be considered useful in another sense in providing enjoyment, interest and pleasure. Indeed it seems that for many of the students we have studied, this aspect may signify a high degree of competence (also associated with a significantly high exchange value). Thus, we must take care not to under-value use, nor to think that exchange value is entirely dissociated from or unrelated to its use. A key problem may be that the two forms of value tend to become alienated through the audit culture, as implemented in the performance cultures in schools.

Following Power (1999), my argument proceeds as in Williams (2008). The audit culture is driven by an accountability system of targets, measures and rewards that allows politicians and managers to move between sectors such as health, policing and education. Auditing performance is required by accountability: professionals must be held accountable for the resources they use, and as such audit is therefore driven by a certain – sometimes healthy – distrust of local practices. But in its extreme form, a "colonisation" of local practice by an imposition of proxy measures is widely reported in every public sector in the UK, e.g. the impact of 3-hour targets on medical staff in Accident and Emergency units in hospitals has proved dangerous when it begins to drive (or colonise) practice, as sick patients may be sent away or readmitted rather than add to the unit's failure to meet target statistics (one hospital recently is reported to have killed several hundred patients as a result).

But arguably in mathematics education colonisation produces another kind of extreme form, where many teachers complain they now simply teach to the test, even if this means teaching bad mathematics, and sidelining interest and enjoyment. This is because the use value of mathematics education to a customer is not quite so life and death as the value of one's health. On the other hand the exchange value of mathematics to the body politic and to students alike can be compelling: only the top grade is acceptable for entry to an elite university, only a good degree result will do for the CV to enter into a highly paid job.

However, this very colonisation can be dangerous for policy, since the public exposure of the malign influence involved can undermine policy and hence the credibility of political intervention itself. Therefore the audit measures are politically more powerful if allied credibly to local knowledge, for example using locally developed evaluation instruments. Thus the devolution of audit to local management can acquire space in some areas of the public services. Supporters of inquiry might then exploit the contradictions inherent here in two ways: (i) by enhancing the (appearance of) use value of inquiry mathematics to wider political forces in society, and (ii) by ensuring that these are measured and integrated into the audit system that ensures inquiry mathematics has exchange value for students.

But this will always be double edged: the more nearly the audit instruments reflect practice on the ground the more credible the audit system is itself, and the more obedient to these systems becomes all the social forces involved. Today, some schools are able to ignore some grade producing pressures; some schools administrators may even genuinely believe that teaching for inquiry will enhance the value of the education they offer their students and may persuade parents to support them (for example, we see some freedom in certain independent sectors, and in some schools that have sufficient capital to ensure their students do very well anyway). These exceptions prove the rule.

Finally, we confront the problem of the use and exchange value of mathematics, mathematics education, and mathematical inquiry – a problem not tackled by CHAT hitherto. I have argued elsewhere that a strictly Marxian/ist analysis of education suggests that learning (as opposed to teaching) is not technically labour, but may result in enhanced labour-power, a uniquely useful commodity in a capitalist society that the state is bound to nurture for its use value- while at the same time controlling its cost of production. This contradiction manifests itself in the contradiction in classrooms between spending time (time = cost) and understanding through inquiry (enhancing potential for mathematics to be useful and do work).

My sleight of hand in equating inquiry with understanding here hides the many facets of the use of mathematics: in leisure, in work, and in labour. Enhancing the critical powers of future labour may have the unintended consequence of enhancing the workers' criticality generally – dangerous for the employer, the state, and capitalism, and possibly the intellectual worker too (even university management these days may not tolerate academics' insubordination).

Especially interesting is the spectacular usefulness of much mathematics several generations if not hundreds of years after it is produced. The aesthetic of mathematics is evidently mathematics' quintessential use, and provides its utility – eventually. In general the time for education to pay off plays to a generational time-scale, and many contradictions arise because of this time factor: policy is always mediated by history and ideology and is deeply political. So we cannot avoid the politics of inquiry.

In **conclusion**, I have highlighted some of the elements of schooling and educational institutions (indeed public sector institutions) generally that serve to constrain and the potential of communities to develop inquiry mathematics. On the other hand, I have suggested some contradictions in the system that we might exploit, though this is always a risky, double-edged business.

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RESEARCH FORUM 3

Mathematical gift and promise:
exploring and developing

RF3: MATHEMATICAL GIFT AND PROMISE: EXPLORING AND DEVELOPING

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SETTING THE AGENDA

This research forum is aimed at designing a research agenda in the field of education of the mathematically gifted by encouraging authentic discussion focusing on the research performed during the past decade. The analysis of the research literature in the field demonstrates that the research issue of giftedness and the education of the gifted was neglected to some extent during the last three decades of research in mathematics education (see for detailed analysis Leikin, forthcoming). Thus, we will make an attempt to understand what has been done and design a research agenda for the field and hope for the continuation of international research collaboration that will start at this forum. We will observe the historical development of the education of the gifted and then will focus our attention on several ongoing studies that take place in different countries all over the world. The questions that will be addressed at the forum will include (but not be limited to) the following:

- How was mathematical giftedness addressed at different historical periods, in different cultural contexts, by different nations, and what interconnections and mutual influences have existed between general and gifted education and education in different countries?
- How may mathematical giftedness be identified in students of different ages?
- How is mathematical giftedness associated with mathematical creativity, critical thinking, advanced mathematical thinking?
- What evidence does point to psychological, social, and pedagogical difficulties that arise in the course of such education; and what evidence does point to the successful forms and techniques of such education?
- What knowledge and skills are required from the teacher in order to support the realization of the mathematical potential of the mathematically gifted? How can the teachers be prepared for this kind of teaching? What might be done during teacher education to help the future teachers meet effectively the challenges of teaching gifted students?

We will take a broad view of mathematical giftedness, so that the discussion will also have to address the problem of developing potentially mathematically talented students (Usiskin, 2000), that is, the problem of a general education that makes it possible to identify and to nourish mathematical talent.

In preparing the Forum, we are operating with the understanding that mathematical talent is not something that is given once and forever, but is rather an opportunity, or a kind of promise (Sheffield, 1999), which might be realized, but might also be lost

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under unfavorable circumstances (Milgram, 2008). Following the conclusions of Russian psychologists (e.g. Leontiev, 1978), we believe that specific aspects of a personality (including mathematical giftedness) are formed and discovered through human activity. Therefore, we believe that it is vital to involve students in such stimulating forms of activity that make it possible to develop their talents and to realize their promise (Usiskin, 2000).

Following Krutetskii (1976), we believe that it is important to study and observe the features and characteristics that are specific to mathematical giftedness. At the same time, we believe that it is necessary to take into account the social side of teaching mathematically promising students, noting various factors that might stand in the way of the realization of their talents or, conversely, be conducive to it—this is particularly important for parts of the population that have been historically educationally underserved (Walker, 2006). Consequently, we will address the *psychological aspects of identifying the mathematically gifted* and the *experiences of mathematically promising students situated in different socio-cultural contexts*.

We believe that what can lead students to discover and realize their mathematical talents are activities that are challenging, fundamentally free of any elements of routine, inquiry-based, and rich in authentic mathematical problem solving. The principles for organizing such forms of activity, which have been formulated in classic works (Polya, 1973; Schoenfeld, 1985), can be enunciated anew thanks to technology, which role in the activation of students' thinking has been repeatedly noted (e.g., Yerushalmy, forthcoming) and which application in the teaching of the mathematically gifted deserves continued study. We will address the *organization of problem solving in working with the mathematically promising and the role of technology* in organizing problem solving.

Our position is that the teaching of mathematically promising students is impossible without creative and well-prepared teachers (Evered & Karp, 2000; Thornton & Peel, 1999, Leikin, 2003). A teacher’s role in the mathematics classroom is to stimulate mathematical reasoning of high-aptitude students in a special way in order to get them to be engaged in learning activities. The "devolution of a good problem" by the teacher is central in a pedagogical situation designed explicitly to encourage students to learn a particular content (Brousseau, 1997). We are interested in discussing with the audience the *kinds of pedagogical contract that can be suitable in teaching gifted students and the issues of the mathematical curriculum for the gifted and promising*.

Teachers' knowledge (e.g., Shulman, 1986), and beliefs (e.g., Thompson, 1992) determine their decision making when planning, performing, and reflecting. Teachers' knowledge and beliefs are interrelated and both have very complex structures. We will discuss *the knowledge of mathematics for teaching associated with education of the gifted and ways in which teachers' knowledge, beliefs, and skills can be developed to prepare them for teaching mathematically gifted learners.*

HISTORICAL PERSPECTIVE: SEVERAL TOPICS THAT MERIT STUDY

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INTRODUCTION

The education of the mathematically gifted has a many-sided history. In an earlier article (Karp, forthcoming), I made an attempt briefly to trace this history as a whole, using the USSR and the United States as my main examples. Here, I will attempt to outline several separate, although related, topics that I believe deserve a historical analysis. I should note that there are many topics which do not appear on the list below, but which are not in my opinion therefore any less important.

RECOGNIZED MATHEMATICAL TALENTS: THEIR FORMATION AND THEIR MANIFESTATION

Cultural history is full of stories about the triumphal appearance of talented individuals before an unexpecting public. Russians, for example, will recall how the young Pushkin read a poem at an open school exam, thus instantly becoming the focus of society's attention, or how a novel submitted by the young Dostoevsky instantly placed him among the ranks of the greatest Russian writers. Such stories about mathematicians are many fewer in number (probably the only one that readily comes to mind, in fact, is the story about the young Gauss finding the sum of an arithmetic progression, but even this is only a story about a student being recognized in front of a single class).

Specialized studies (Karp, 2007; Choi, forthcoming) of the lives of mathematically gifted individuals and observations of their teachers show that, although talented students are often quickly noticed by their no less extraordinary teachers in mathematics classes, only their successes in specially organized competitions and similar events bring them wider attention.

Nor is this surprising. The very nature of the workings of mathematical talent (and of talent in many other scientific fields, as well) implies that it can be noticed only by someone who has special preparation in mathematics, and even more importantly, that such talent can manifest itself only under the right conditions. What kinds of conditions, then, have turned out to be favorable for the manifestation, ripening, and recognition of mathematical talent?

There have been times when a whole number of outstanding mathematicians have appeared in a small country almost simultaneously—such was the case in Hungary during the first half of the twentieth century, and it is natural to connect this phenomenon with Hungary's system of specialized education (Vogeli, 1997). We know how many outstanding mathematicians were educated (Bell, 1937). It is important, in our view, to carry out an extensive and systematic analysis of surviving accounts of the mathematical education of highly gifted students.

GOVERNMENT, SOCIETY, AND THE MATHEMATICALLY GIFTED: THE POLITICAL HISTORY OF THE EDUCATION OF THE MATHEMATICALLY GIFTED

Above, we noted that special efforts on the part of society were necessary in order for mathematical talent to be brought to light and to be brought to maturity. But this fact automatically connects the formation and development of the mathematically gifted with the policies that are carried out in the field of education (and thus, not just in this field) by the government of the country.

Russian (Soviet) schools with an advanced course of mathematics, which produced tens if not hundreds of highly qualified mathematicians, appeared in the late 1950s and early 1960s on the crest of a general process of liberalization in the country. A broad recognition of the differences in the inclinations and ambitions of young people would have been impossible in the absence of such a liberalization. It would have been impossible, too, to create conditions that broke up the uniformity of curricula and requirements. And there could have been no broad social movement to create and maintain such conditions — a movement in which hundreds of mathematicians and physicists became involved. Similarly, the so-called period of stagnation in the Soviet Union during the 1970s-80s brought about a substantial deterioration in the position of these schools, which were sometimes simply shut down, and sometimes transformed under pressure from the government, which feared these hotbeds of free thought, even if this thought was mathematical, not political (Donoghue et al, 2000). The government's policies, however, were inconsistent. The government recognized the usefulness of such schools, which in essence served to prepare personnel for the military-industrial complex, but at the same time, was unable to go against its nature.

The complicated history of Russian schools for the mathematically gifted is only one example of the still unwritten political history of the education of the mathematically gifted. Today, when such education exists in one form or another everywhere in the world, from the United States to North Korea, it is interesting to ask: how exactly has such education interacted (and how does it continue to interact) with the government in different countries? What are the mechanisms and patterns of this interaction? What are the most important forces and the most important channels of influence in such interaction? What are the contradictions and difficulties inherent in such interaction? All of these questions merit research and analysis.

THE MATHEMATICALLY GIFTED AND GENERAL EDUCATION: A HISTORY OF CONNECTIONS AND INTERACTIONS

Tannenbaum (2000) has ironically pointed out that American society is never interested in teaching both the most and the least successful achievers at the same time. Indeed, historically, the peaks of public interest in these two objectives have not coincided. This fact is interesting for the study of the aforementioned interaction between prevailing social trends, on the one hand, and the various forms assumed by mathematically gifted education, on the other. The connection between the development of general education and the education of the mathematically gifted, however, appears obvious.

On the wall of virtually every American classroom, there once were (and sometimes still are) quotations from Polya about how to solve a problem. It is clear, however, that Polya's work and methodology evolved under the influence of the aforementioned Hungarian system of teaching the mathematically gifted in which he himself was brought up. History offers other examples of the influence (direct and indirect) of the education of the mathematically gifted on general education; influence in the opposite direction is even easier to trace, if only because the teaching of the mathematically gifted student begins precisely within the context of general education.

And yet, general education and mathematically gifted education are even now often explicitly or implicitly contrasted with one another. We believe that it is important systematically to collect historical data illustrating how they interact. This involves examining the most varied aspects of such interactions—from studying how one program influences another or how methodological techniques are transferred from one context to another, to analyzing the work of teachers who are educated within one framework and go on to teach in another.

THE HISTORY OF INTERNATIONAL INFLUENCES IN THE EDUCATION OF THE MATHEMATICALLY GIFTED

But it is not only general education and specialized gifted education that exert a mutual influence on one another. Influences between different countries are no less significant. It is almost always naive to ask in what country some methodological technique or approach was invented: arriving in a new country, even outright borrowings assume a somewhat new shape. And yet it is useful to start thinking about channels of international influence in education in general and in mathematically gifted education in particular, if only because this can facilitate the recognition of general patterns and national-cultural peculiarities.

Russian schools for the mathematically gifted were influenced by the already mentioned Hungarian schools of the early twentieth century, while these Hungarian schools themselves arose within the framework of a system that was based on French models (Vogeli, 1997). On the other hand, when schools that were officially called "schools with an advanced course of mathematics" appeared in socialist Hungary, the models that they were based on were Soviet.

What exactly was borrowed? How were the borrowings transformed? What were the channels through which such influences were transmitted? The answers to such questions are by no means completely clear even in the case of the aforementioned exchange of ideas that led to the creation of schools for the mathematically gifted. We know even less about other, no less important examples of the international exchange of ideas concerning the teaching of mathematically gifted students.

PROBLEMS FOR GIFTED STUDENTS: WHERE DID THEY COME FROM?

Finally, let us turn to the history of concrete materials for mathematically gifted students, first and foremost to the problems that are used in their education. To trace the origins of a single problem is usually impossible. But if we are dealing with a group of problems or a general topic, we are often able to trace where they came from and how they have been used, and thus also to discern the specific philosophy underpinning their employment.

Turning once again to the Russian experience, one can, for example, take note of the great prominence that contemporary mathematically gifted education in Russia accords to the study of the quadratic trinomial with parameters. One can trace how and when such problems began to appear more and more frequently on college entrance exams; and one can identify the period (the 1940s) when such problems began to appear — in relatively easy versions — on school exams in ordinary schools. Finally, one can at least in part connect the popularity of such problems in the 1940s with translated (French) problem books that became widely used in the Soviet Union.

Why did old problems turn out to be appropriate and necessary in this or other cases? What about them attracted mathematics educators? How did the problems that were used and the approaches to teaching students to solve them change over time? Such a history of problems can shed light both on the processes discussed above and on the development of particular teaching styles.

CONCLUSION

Historians—historians of mathematics education included—find an interest in the bare reconstruction of historical facts. But such a reconstruction of historical facts is also useful for mathematics educators, whose interests lie far from history, and for the development of mathematics education as a whole. The history of mathematics education is a kind of memory and that preserves successes, mistakes, and failures. It is impossible to understand what is taking place in today's classrooms without understanding how today’s practices developed. It is naive to expect that history will give direct answers to today's questions, but without history it is unlikely that such questions will ever be answered. We believe, therefore, that research in the history of mathematically gifted education along the lines proposed above will ultimately turn out to be useful for those who engage in such education in practice as well.

PSYCHOLOGICAL ASPECT: IDENTIFICATION OF GIFTEDNESS IN EARLIER AGES

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INTRODUCTION

The issue of identifying mathematically gifted students has long been debated and documented (Davis & Rimm, 2004). The main reasons for which this issue is still unresolved are: the absence of a clear definition for mathematical giftedness, the lack of appropriate mathematically oriented instruments for students' identification, and the heterogeneous nature of mathematically gifted students, who vary in the range of abilities they demonstrate. Furthermore, gifted students exhibit different behaviours and are not always those with the highest achievement records, most disciplined, attentive, tidy or organised in the classroom (Johnsen, 2004).

The challenge of identifying mathematically gifted students has served as a motivation for a research program funded by the Research Promotion Foundation of Cyprus. Specifically, the aims of the project were (a) the clarification of the definition of mathematically gifted students with different profiles; (b) the supply of corresponding valid and reliable identification tools; and finally (c) the provision of curricular materials for mathematically gifted students and their teachers. In this article we only focus on the identification process planned to pursue in the project. We present the principles and the theoretical model that guide this project as well as the methodology we will employ for the identification of mathematically gifted students in elementary school.

PRINCIPLES FOR THE IDENTIFICATION OF MATHEMATICAL GIFTEDNESS

Adopting Renzulli's definition (1978), mathematically gifted students are those who have above-average ability, creativity in mathematics and task commitment in their pursuit of a solution to a mathematical problem. It is obvious from this definition that we need tools for measuring ability, creativity and commitment and as far as we know there has not been a single instrument that can capture these three very diverse dimensions in mathematics. However, it may be possible to use a combination of valid, reliable, sensitive and objective tools and approaches which will allow the collection of such information. For instance, one may include achievement tests, aptitude and intelligence tests, creativity tests, portfolios, interviews, observations, check-lists, nominations, and self-reports. Several researches argued in favour of this multiple-criteria approach, which constitutes the first principle for constructing identification tools (Hoeflinger, 1998).

The context and the purpose of identification tools is considered as another important principle for choosing and/or developing appropriate tools. For instance, in our research program the purpose is to identify *mathematically* gifted students and not *generally* gifted students. Thus, special attention should be given to the context of these instruments in order to ensure that they highlight mathematical strengths and weaknesses, cognitive abilities and the strategies and patterns of behaviour of mathematically gifted students.

Inclusiveness is the third principle, which prevents what Birch (1984) called a "narrow identification". This principle ensures that identification tools are not biased in any way by gender, race, colour, socioeconomic status, and geographical location.

The last principle for developing such tools refers to the flexibility and continuity of the process for identifying gifted students whose abilities may not be immediately apparent (Davis & Rimm, 2004).

MODELS FOR THE IDENTIFICATION OF MATHEMATICAL GIFTEDNESS

The above mentioned principles for the identification of gifted students are encompassed in a large extent by Gagné's (1991) differentiated model of giftedness and talent, which we adopted in our research project. Gagne's model provided us with a clear guidance of the way to organise the identification and enrichment program so as to serve the needs of the mathematically gifted students and especially those of young ages and those of high ability that may not be achieving to their full potential.

Gagné's model makes a clear distinction between giftedness and talent. According to Gagné (2003), giftedness designates the possession and use of untrained and spontaneously expressed natural abilities in five aptitude domains: intellectual, creative, socioaffective, perceptual/motor, and others. The degree of these abilities needs to place the individual in the top 10% of its age peers. The development and level of expression of these natural abilities are partially inherent and can be observed in many of the tasks that a child may be occupied with. They can also be more easily and directly observed in young children because the environmental influences and schooling would have not yet had a large impact on them. Furthermore, they are still apparent in older children that may have not been given the opportunities or appropriate schooling that other children may had been fortunate to have.

Talent designates the superior performance of the individual in one or more fields of human activity, in our case mathematics. Talent emerges from natural abilities and is a consequence of the students' learning experience. Again the individual needs to be in the top 10% of age-peers that have been receiving training in mathematics.

It becomes apparent that in Gagné's model natural abilities may be transformed via maturation, formal and informal training, into well-trained abilities and talent in mathematics. Gagné's model acknowledges the potential of natural abilities and also the influence that may be exerted by other factors on mathematical talent.

Unlike other models, Gagné's model accounts and suggests reasons for which some gifted students may not be amongst the high achievers in their class. It allows teachers and researchers to make a clear distinction between potential and achievement. It clearly suggests that identification should not be based only on traditional instruments which measure mathematical abilities, but to consider this as one of a range of aptitudes and abilities that characterise gifted students.

THE PROPOSED MODEL FOR THE IDENTIFICATION OF MATHEMATICAL GIFTEDNESS

In our study the proposed identification process will be based on Gagné's (2003) distinction between natural abilities and talent. Thus, we reviewed several instruments which measure natural abilities such as general intelligence, cognitive abilities, visualization, nonverbal and verbal reasoning, attention, memory, general problem solving skills and creative abilities. The reason for this extensive review was to identify common factors that significantly contribute to natural abilities that designate giftedness and develop a test that will measure these abilities.

To measure mathematical talent we reviewed different achievement tests in mathematics, the TOMAGS Primary and Intermediate tests (1998) as well as a number of other national and international mathematics tests such as SATs, TIMSS and PISA. The reason for this review was the identification of major aspects of mathematical abilities that these tests measure and the development of a mathematical test that will support the identification of students with mathematical talent. Our aim is to develop a mathematical test that will sufficiently capture what we may call mathematical talent in a coherent and comprehensive manner. The tasks in this new mathematical test will measure aspects such as: Logical thinking; qualitative and spatial relationships; ability to perceive and generalize; ability to reason analytically, deductively, inductively; ability to find rational, economical solutions; flexibility and reversibility of mental processes; ability to remember mathematical symbols, relationships, methods of solution; mathematical perception of the world and mathematical creativity (Bicknell, 2008).

Special attention for the design of both the natural and mathematics ability tests will be given to their administration, scoring and time needed to be completed. Identification tools which are designed to be administered by trained researchers in a laboratory are by default very difficult or even impossible to be used in the regular classroom by teachers. At the same time, their duration and sometimes the necessity to examine each student individually, makes them impractical to use in everyday classrooms. It is important to have instruments that can be easily administered, scored and completed in a reasonable period of time.

Furthermore, our purpose is to go beyond the top 5% of the population that may traditionally be identified as gifted and select the top 15%. This will provide a wider threshold that will allow the selection of gifted students that may not be supported by various environmental factors. All these students will participate in a mathematics enrichment program and their progress and mathematical talent will be measured for a second time in a later phase.

All the tests for natural and mathematical abilities will form an objective basis for our assessment. However, they will not be the sole measure of giftedness and mathematical talent. It is possible that some talented student may not be achieving to their potential due to various environmental reasons and added measures will provide significant information. After accepting the necessity of an identification process using multiple criteria, the instrument to be developed will be combined with students' observations, as well as with teacher, parents and peer interviews and checklists as it is shown in the following section.

IDENTIFICATION PROCEDURE

The specific steps that we will follow in our research program regarding the identification procedure are presented below:

Phase One – Information Gathering in Schools: (a) Students' results in natural abilities test, (b) Students' results in mathematics test, (c) Observations of students in mathematics learning environments, (d) Teacher checklist and interviews.

Phase Two – Collaboration with Parents and Peers: (a) Parents' checklist and interviews, (b) Peer checklist.

Phase Three – Analysis of Results: Analysis of test results, interviews and checklists.

Phase Four – Mathematics Enrichment course.

Phase Five – Analysis of Results and Reports: **(a)** Students' results in mathematics test, (b) Writing of students' reports.

CLOSING REMARKS

The identification process described in this article takes into considerations many of the factors and principles mentioned in the literature and may also help in the development of curricular materials for mathematically gifted students and their teachers. In addition, the proposed model for identification allows the distinction between natural abilities and mathematical talent. It also appears to be promising for the identification of students in the early years and those that for various external reasons may be failing to reach their full potential in mathematics.

WHAT IS SO SPECIAL ABOUT PROBLEM SOLVING BY MATHEMATICALLY GIFTED STUDENTS?

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BACKGROUND

Reviews of the literature on mathematical giftedness frequently result in the lists of traits by which "gifted" students positively deviate from the "normal" ones. For instance, a synthetic of literature on mathematical giftedness and mathematical thinking by Sriraman (2005) revealed that the construct of mathematical giftedness has been defined in terms of superiority in mathematical processes such as: abstracting, generalizing and discerning mathematical structures; independently discovering mathematical principles; managing data; thinking analogously and heuristically; decision-making in problem solving situations; reversing mathematical operations; visualizing problems or relations; distinguishing between empirical and theoretical principles; appreciating mathematical proofs.

The comprehensiveness of such lists creates an impression that the gifted are superior in virtually all components of mathematical thinking and problem solving. This impression frequently transforms into a public belief that the gifted employ qualitatively different kinds of cognition (e.g., Rogers, 1986). In turn, this belief creates a severe pedagogical issue expressed best in the words of an elementary school gifted child: "We need teachers who help us, not haunt us" (from Lewis, 1982, reprinted in Davis & Rimm, 2004, p. 53).

To which extent does the belief that mathematically gifted individuals possess "different cognition" rely on evidence? How can the apparent exceptionality of problem solving processes of the gifted be captured and explained? The article at hand addresses these questions by discussing some sources of invalidity in research on cognitive characteristics of high achieving school children.

THREATS TO RESEARCH ON PROBLEM SOLVING OF THE GIFTED

The threats to validity of mathematics education research at large are well known in the research community. This section discusses three additional threats that I deem quite specific for research on exceptional mathematical problem solving.

The threat of overestimating creativity of high achievers due to inappropriate instruments

It is widely accepted that winners of high level mathematical Olympiads are mathematically gifted individuals (e.g., Andreesku et al. 2008). Indeed, this stance complies with any after-the-fact definition of giftedness (Stankowski, 1978) once we agree that winning a medal at, say, International Mathematical Olympiad (IMO) is an outstanding mathematical achievement. Are IMO winners necessarily creative mathematical thinkers? The question has merit, in particular, because creativity is considered an indispensible part of multi-dimensional definitions of giftedness from the late 1970th (e.g., Renzulli, 1978; Milgram, 1989). In school mathematics, problems requiring creativity are frequently defined as those ones that entail constructing (subjectively) new knowledge or finding (subjectively) new connections within the existing body of knowledge (Silver, 1997).

The last definition implies that success in solving any problem may be an indicator of the solver's mathematical creativity only if the problem is novel for the solver. In a way, this simple logical conclusion challenges some past attempts to create universal questionnaires for mathematical creativity measurement (e.g., Livne, Livne and Milgram, 1998) as well as the attempts to establish the extents of creativity of different solutions to the same problems with no reference to the knowledge of the potential solvers (e.g., Ervynck, 1991). A good example of dealing with this issue is given in a study by Leikin and Lev (2007) that included a measure of conventionality of the solutions to the utilized in their research problems. The measure took into account if the particular solutions were included in the mathematics curricula of different groups of their study participants.

As to Olympiad problems, their use in research on mathematical creativity is not always defensible. For instance, Andžāns (2008) pointed out that three out of six problems proposed at IMO-2007 in Vietnam can be traced back to previous IMOs or to competitions in the former USSR. As a rule, the IMO participants use the problems from the past Olympiads in their training. Thus, one's success in solving Olympiad problems can be attributed to the quality of his or her memory, aptitude or the transfer ability, and not necessarily to creativity. In sum, mathematical creativity of the winners of mathematical Olympiads cannot be taken for granted and should be further investigated. The investigations of problem solving creativity should utilize tasks that are feasible but novel even for the top achievers.

The threat of circularity due to (hidden) similarities in a way of sampling and a way of probing

I present this threat with an example of an imaginary study. Consider a study aimed at exploring the differences between gifted and not gifted students related to the ability to decompose complex mathematical problems into manageable sets of sub-problems. Assume that gifted students are defined in the study as those with high scores in Raven Progressive Matrix Test (RPMT). The sample consists of gifted and not gifted students of the same mathematical background. Suppose that the main finding of the study is that the gifted students over performed the not gifted in decomposing the given mathematical problems. How informative such a finding would be? In order to address this question one can turn to the literature about cognitive abilities involved in processing RPMT. Carpenter, Just and Shell (1990) argued that coping with RPMT items requires "the common ability to decompose problems into manageable segments and iterate through them, the differential ability to manage the hierarchy of goals and subgoals generated by this problem decomposition and the differential ability to form high level abstractions" (p.429). In light of this quotation, the main finding of our (imaginary!) study looks unsurprising at best. In fact, the superiority of the gifted students has been implicitly embedded in the study design by the chosen way of sampling. The study might be much more interesting if it were utilize another, not so directly connected to the decomposition ability, operational definition of giftedness.

Contemporary literature suggests a variety of approaches to defining mathematical giftedness (e.g., Davis & Rimm, 2004; Leikin, Berman & Koichu, in press). All the approaches – from traditional psychometric to modern multi-dimensional ones – vary with respect to: (i) cognitive and co-cognitive traits included in conceptualizations of mathematical giftedness, (ii) types of tasks chosen for identifying the gifted, and (iii) levels of achievements needed in order to treat a student as "gifted" or "promising". The common feature of all these approaches is that they define as the gifted the most successful solvers of particular types of problems within a particular cohort or population. The above example shows that a cognitive trait under investigation should not belong to the set of traits chosen for building a sample in accordance with a particular definition of giftedness.

The threat of misinterpretation of the role of a particular cognitive process due to the complex nature of mathematical problem solving

Schoenfeld (1992) suggested that *knowledge base*, *heuristics*, *monitoring and control*, *beliefs and affect*, and *practices* are the major attributes of mathematical problem solving. Contemporary mathematics education research relies on this suggestion and attempts to understand the role of each attribute as well as how they come to cohere (Schoenfeld, 1992; Mamona-Downs & Downs, 2005).

An investigation of a particular problem solving attribute requires systematic observations of students' performances in problem solving situation that can be resolved by the effective use of the chosen attribute, and not by all the attributes. Such situations are especially difficult to find for exceptionally gifted students. For example, research on heuristics requires the use of problem solving situations that cannot be resolved just by retrieving relevant elements of the knowledge base from the memory. However, we know from research on intellectual giftedness (which is not alien to mathematical giftedness) that gifted students have larger and more efficiently organized memory, have larger and more elaborative organized knowledge bases (e.g., Robinson & Clinkenbeard, 1998). Another example: a study on monitoring can result in meaningful comparison between gifted and not gifted students only on condition that all the students are given a chance to look back at their solutions and see what they have done wrong. Practically this means that all the students, including the gifted ones, should be observed when they encounter a difficulty or an obstacle in problem solving. However, since the giftedness is associated with *successful* problem solving, such studies are rare. A fragment of such a study will be presented in my conference talk.

CONCLUDING REMARKS

This paper explicitly calls for: (i) awareness when adhering to a particular definition of giftedness; (ii) a-priory analysis of the relationships between the chosen research variables and the chosen way of sampling; and (iii) attention to appropriateness of research tools to the research goals. I believe that following these recommendations may benefit research aimed at characterizing the unique cognitive traits of the mathematically gifted. The recommendations are also relevant for testing the apparent discrepancies between research on general intellectual giftedness and research on mathematical giftedness. Such discrepancies exist, for example, regarding the role of metacognition in problem solving. Mathematics education researchers frequently connect high metacognitive abilities with high problem solving competences (e.g., Schoenfeld, 1992). Cognitive psychologists are less conclusive about the role of metacognition. For instance, Robinson and Clinkenbeard (1998) pointed out in their review that intellectually gifted students are found to know more about metacognition, but they may not use a wider variety of metacognitive strategies than other students, and they do not apparently monitor their strategies more than other students.

It should be noted here that research on general intellectual giftedness also does not dispel the mystery of exceptional problem solving (e.g., Dai & Renzulli, 2008). Thus, the question put in the title of this paper still provides us with fascinating research agenda in the fields of mathematical giftedness and mathematical problem solving.

(UN)LIMITED OPPORTUNITY: THE EXPERIENCES OF MATHEMATICALLY TALENTED BLACK AMERICAN STUDENTS

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INTRODUCTION

This paper describes findings from a study of the formative and professional experiences of US-born Black mathematicians who have earned the PhD in a mathematical science. This study is informed by previous work, completed at a public high school in New York City, focused on high achieving students of color, their academic communities, and their mathematics performance. The purpose of this paper is to explore the experiences of the mathematicians and the high school students, with the goal of informing the research community about psychological, social, and cultural factors that contribute to their success, as well as to describe some of the institutional mechanisms that thwart and facilitate opportunities for Black mathematicians and high achieving high school mathematics students to pursue, and excel in, the field.

RELEVANT LITERATURE AND THEORETICAL FRAMEWORKS

There are pervasive and well-established discourses (of difficulty, isolation, and exceptionality) about the mathematics achievement, potential, and opportunity of Black Americans. Martin (2009) and others note that the overwhelming focus in much of mathematics education research has been to narrowly conceive US Black achievement in mathematics as a problem of underachievement. The much discussed "achievement gap" in US school performance delineates higher average achievement in mathematics as the purview of White and Asian American students on the one hand, and lower average achievement on the part of Black and Latino/a students on the other. Very rarely is the concomitant "opportunity gap" that contributes to these patterns of achievement discussed in meaningful and critical ways. Further, there is no room in the structured discourse for the possibility—and presence—of Black American high achievers in mathematics. Research that documents the ways in which Black Americans are successful in mathematics is noticeably less prominent.

In a previous study of high achieving urban Black and Latino/a high school mathematics students, academic communities (largely informal networks comprising family members, peers, and teachers) were integral to those students' success (Walker, 2006). Relatives encouraged students, explained concepts, and assisted with homework and test preparation; peers served as both tutors and tutees and discussed math with respondents in and out of school; and teachers provided enrichment and discussed mathematics in a larger context than "school" mathematics. In this paper, I describe some of the academic communities that Black mathematicians belong to and create to foster success for themselves and others.

It would be ahistorical and atheoretical to omit a discussion of race and how it affects the lives and professional experiences of Black mathematicians in the US. Despite the centrality of race to the experiences of many Black mathematicians, in general, most studies of mathematicians (with the exception of Kenschaft (2005) and a few others) ignore the powerful contexts of race and ethnicity. The complicated history of slavery, discrimination, and racism in the US has contributed to the perceptions of Blacks as intellectuals that continue to have a profound impact on discussions of education and learning. The degree to which one's mathematics identity might also encompass ethnic and academic identities (Boaler, 1999; Hilliard, 1995; Martin, 2009) has emerged as a prominent theme in this work. The development of one's identity (ethnic, academic, and/or mathematical) is situative and fluid, and exploring the relationships across these types of identities—particularly those between ethnic (e.g. Cross, 1991) and mathematics identity—I argue, is integral to understanding mathematics success.

METHODOLOGY

The methods used for the high school students' study are described in detail in Walker (2006). For the Black mathematicians' study, 25 Black mathematicians to date have been interviewed. An interview protocol was developed and semi-structured interviews were conducted in person or via telephone. All interviews (on average, interviews were about an hour in length) were audiotaped and transcribed. A research assistant and I read each interview several times to generate a list of master "codes". We coded each interview for major themes and in this paper, two of the major themes that have emerged to date are discussed below. All mathematicians quoted in this paper have been identified using pseudonyms.

FINDINGS

The Importance of Academic Communities

As was the case for high school students, Black mathematicians' families have played a critical role in their mathematics development. Often, this is through exposure to mathematics that they may not be exposed to in school. For example, Craig Thomas (PhD 1990s) remembers:

> I remember being on the porch [when I was about 8 or 9] and [my grandfather] asked me if he walked halfway to the end of the porch, and then halfway again, and then halfway again, how many steps would it take him to reach the end of the porch? I may have guessed five or something, I don't know. So then he actually proceeded to do it,…the idea was that he was converging—he didn't use the term convergence of course—he never actually reached it but he got closer and closer, and of course he didn't say "within epsilon". But anyway, I have fun when I'm teaching about convergence to really tap into it at this early level. [Craig Thomas]

In addition to families, Black mathematicians' teachers have a profound influence on their development and access to rigorous mathematics. Eleanor Gladwell (PhD 1980s) reports:

> I think it's because of him [a mathematics teacher] that I really excelled in math in high school. For example, when my class got to trigonometry, the county would not allow them to teach trigonometry because there were not enough students; you had to have enough for a big class. Well, he only had five, six, or eight. So he decided that we needed trigonometry to go to college. So he agreed that if our parents would bring us back in the evenings, he would teach us trigonometry. And they did. That's how we learned, that's how we got our trig. [Eleanor Gladwell]

The role of academic communities (comprising family members, teachers, and peers) is paramount in the experiences of many of the mathematicians interviewed to date. Perhaps the best summary statement about these communities is expressed by Lorraine Murphy (PhD 2000s):

> I think that a lot [of my success in becoming a mathematician] is about encouragement. I was fortunate enough to be in a family that was supportive and stressed education…I had teachers who encouraged me and entered me into contests and had a strong belief in me. When I got to the point [in her graduate program] where there were people that didn't believe in me, I had had this whole long history of people supporting and believing in me, so that didn't just totally destroy it for me. I think a lot of people get to that destruction point much earlier than I did. So I really think between my family and my teachers and my mentors, that is what really kept me going and sticking with it. [Lorraine Murphy]

Race, ethnicity, and opportunity

Some mathematicians grew up in the segregated South, attending high school in the 1940s and 50s. Many of these participants describe the high quality of mathematics instruction that they received in legally segregated schools that had fewer resources than white schools, which challenges the dominant view of the "inferior" education provided by these all Black institutions. Further, family members were committed to their education, even if these family members had not themselves been able to complete primary or secondary school. This echoes Walker (2006)'s findings that high achieving mathematics students received encouragement and support from family members who may not had the same school opportunities themselves. In short, academic support comes in many forms, and the tradition that many schools have of ignoring or denigrating family members who are not considered to be well educated is a mistake.

Despite the challenges of racism and discrimination, particularly for mathematicians coming of age during the pre-Civil Rights era, Black mathematicians broke through barriers and made great contributions to science. However, they are often seen as novelties and exceptions (this has been true since the time of Benjamin Banneker, who is widely known as the first Black "mathematical person" in the US). It is clear from all of their stories, regardless of age, that themes of race, novelty, opportunity, and intellect continue to figure prominently in both discourse *about* Black mathematicians and in Black mathematicians' *own* discourse.

Linus Heubert (PhD 2000s) reports an experience in graduate school:

> I remember [fellow white students] having a conversation about some model. So we're talking about that, and they were talking about black people and how they poll, they knew everything. I remember sitting there the whole time. And then they went on to talk about how black people should take a different SAT because the SAT didn't measure them correctly. So I suppose they felt they were being enlightened and things like that, and, you know, it was upsetting. And I'm sitting there thinking, "I probably did better on the SAT than everybody in this room." [Linus Heubert]

Wayne Leverett (PhD 1960s) describes his perceptions of opportunity as a Black American who came of age in the 1950s and 1960s:

> When I was coming up, it's highly unlikely that you would know a PhD. In fact, when I got to [this private industry job], my good [White] buddy would say, "Wayne, when I was growing up I had an uncle who was a mathematician so I kind of knew what they did and I knew that I wanted to be one. But in your case, how did you know you wanted to be one?" Now, in my case it was different: you could not see all the way to a Ph.D.
>
> At the high school level, I could look around and say, "Whew boy, if I could get to be a [math] teacher in this high school, that would be a good lifestyle." And that's as far as I can see. And then you get to college and you meet [a Black mathematician] and he points you a little bit further. And you say, "Maybe I could get to be a professor [here]." And then you get lucky and get a chance to go to [other] places. But it happens in stages…Today's kids coming up, I'm thinking of these kids here [at the conference],…If I had been able to see that when I was coming up, oh my goodness, I think I would have said, "Oh, I ought to be a dean at MIT." You know, it's different these days. But I'm sure any old person you talk to will say that things are much different today then when I grew up and that's all I'm saying. One has to be envious of the opportunities that young kids have these days. [Wayne Leverett]

Many of the Black mathematicians interviewed to date—regardless of their ages—in some way feel responsible for bringing others like them into the field:

> Definitely the first important thing for me is this continuing to add more African American PhDs and women to mathematics. Probably people will be saying that I am too young for this to be my most important thing, but because I was so influenced, I feel obligated to pass that on. So to get students excited about math and about deep math, that is my main focus. [Elizabeth Greene, PhD 2000s]

Many Black mathematicians have created or belong to informal and formal networks that support their own work and induction into the field. They have formed such communities to facilitate access for others and their own socialization into a profession that has not always supported their talent and development. (See Newell, Gipson, Rich, & Stubblefield (1980) for a discussion of the discriminatory practices of national mathematics organizations in the mid-20th century). I theorize that the notion of an academic community that supports their mathematics engagement may have first developed for Black mathematicians during their formative years. Further, in addition to fostering Black mathematicians' interest and achievement in mathematics, these experiences also inform the types of academic communities Black mathematicians form as professionals. (There are several formal organizations and programs formed by Black mathematicians, including the National Association of Mathematicians (NAM), founded in 1969, the Cooperative Research Fellowship Program founded at Bell Laboratories in 1972, and the Conference for African American Researchers in the Mathematical Sciences (CAARMS) founded in 1995).

INVOLVING STUDENTS IN EXTRA-CURRICULAR SCHOOL MATHEMATCAL ACTIVITY: VIRTUAL MATHEMATICAL MARATHON CASE STUDY

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INTRODUCTION

CASMI (www.umoncton.ca/casmi) - the Interactive Science and Mathematics Community - has an almost 10 years of history of proposing rich mathematical problems on-line for all school levels trying to attract more students with challenging mathematics and science (Freiman & Lirette-Pitre, 2009). Some of the site activities we directed to gifted population.

Based on the observed literature, we assumed that participation in the online competition may potentially contribute to the development of mathematical giftedness through (1) solving non-routine problems and (2) persisting when facing challenges. We designed a Virtual Mathematical Marathon (VMM) that we conducted on the CASMI web site during summer 2008. The total of 80 problems had been posted twice a week, 4 problems each time. One hundred and ninety four students submitted their answers to some of these problems on a voluntary basis. In this paper we outline the participation of schoolchildren in the marathon.

BACKGROUND

Relationship between success and persistence in gifted

According to Renzulli's model, there are three basic traits of successful outstanding individuals: above average general ability, high level of task commitment, and creativity (Renzulli, 1978). Task commitment seems to be less explored in research, although Renzulli insisted that "energy brought to bear on a particular problem (task) or specific performance area" (Renzulli, 1978, p. 182) is a very important aspect of giftedness.

We studied the relationship between the success of the participants in problem solving and their decision to continue participation in the marathon. Our analysis unfolds some interesting participation patterns that may be researched further.

The role of competitions in the education of the gifted

Our work was based on the conception that mathematical challenge is critical characteristic in promotion of mathematical talent (Barbeau & Taylor, 2009). The role of mathematical competitions has been analyzed by Taylor, Gourdeau and Kenderov (2004) as a way to promote students' creativity through enrichment and challenge. Bicknell (2008) mentioned competitions as an important part of provision for gifted and talented students that gives them satisfaction and enhances self-directed learning skills.

The idea to use a virtual form of mathematical competition came from a view of online problem solving environments as potentially challenging and motivating educational resource according to the work of Renninger & Shumar (2004), Jones & Simons (2000) as well as our experience with the Problem-of-the-Week model on the CASMI site (Freiman & Lirette-Pitre, 2009, Sullenger & Freiman, 2008). Based on the observations on the role of competitions in gifted education mentioned above, we decided to comprise some competitive components in our CASMI model. We choose virtual mathematical marathon as an activity potentially attractive for our students.

We aimed to provide students with an opportunity to discover their talent which they can not normally demonstrate in regular classroom (Taylor, Gourdeau and Kenderov, 2004) thus we considered marathon as a stimulus for improving students' informal learning. Fomin, Genkin and Itenberg (2000) described that during the marathon that they conducted on face-to-face basis, their students managed to increase the number of problems they solved, relatively to other non-competing frameworks in which the same students participated. Additionally these students found that marathons were more interesting and attractive than other, better known forms of Olympiads (Fomin, et al.).

The role of virtual learning environments

Internet can be a suitable challenging environment for organizing mathematical competitions and problem solving enrichment activities, potentially contributing to the development of mathematical ability and giftedness (Johnson, 2000). Experiences of virtual communities such as MathForum (mathforum.org) and NRICH (nrich.math.org), and our CASMI site (www.umoncton.ca/casmi) confirm in certain ways these expectations (Renninger & Shumar, 2004, Piggott, 2005, Freiman, Lirette-Pitre and Manuel, 2007).

PARTICIPATION IN THE VIRTUAL MATHEMATICAL MARATHON

According to our model of the VMM, cycles of 4 non-routine challenging problems have been posted twice a week on the CASMI website (total of 8 per week). We conducted the marathon during 10 weeks from the beginning of June till the middle of August 2008, with overall 80 problems. Every registered member could login on the web-site, choose a problem, solve it, and submit an answer by selecting it from a multiple-choice menu. The automatic scoring system immediately evaluated students' success in solving each marathon problem as well as his/her competitive result as compared to other students' results.

Four problems in each cycle were of 4 levels of difficulty. The tasks were developed and ranked collaboratively by a group of experts in mathematics and mathematics education. The scoring was provided as follows: level 1(the easiest problem) was scored with 3 points, level 2 with 5 points, level 3 with 7 points, and level 4 (hardest) with 10 points. To support students' participation in the marathon, unsuccessful attempts were also awarded with 1, 2, 3, and 4 points according to the difficulty of the problem. Each participant could join the marathon, solve as many problems as he/she wishes, withdraw, and come back at any time, continuing solving ongoing problems.

The most of 194 children tried at least one problem were aged from 10 to 14 years old. Usually, after submitting their answer to one problem of the cycle, children were trying to solve all 3 remaining problems. 133 of 194 participants dropped out after only one cycle. 13 of them left the marathon despite having solved all 4 problems correctly, and 33 have left after getting 4 incorrect answers. After the 5^{th} cycle, 18 participants continued the competition (less than 10% of the total number); half of them (9) were trying to solve the tasks after the 10^{th} cycle, five (5) of them stayed till the end of the marathon.

Further, 11 other participants stopped after having all 4 answers of one cycle wrong, whereas only 3 stopped their participation after having all 4 correct answers. We found that 22 of 194 children have participated in at least 3 of 5 consecutive cycles. It is interesting to notice that no one of this group of 22 has left the marathon after correctly solving all 4 problems of the ongoing cycle. Four of them have left the marathon after receiving 3 of 4 correct answers: three of them did not come back later; one of them came back for one more cycle. Ten (10) of them have left the marathon at some moment after an unsuccessful attempt: 0 correct answers of 4 (6 participants) or 1 correct answer of 4 (4 participants).

Discussion and conclusions

Our data may suggest few interesting observations. First, the most of children who left the marathon after participating in more than one cycle did it after five cycles, exactly when the summer vacation began. Further analysis is needed to get a better understanding of this finding because our data are insufficient to conclude whether the lack of success may also influence some students’ decision to withdraw from the marathon while others, more persistent participants, continued the competition disregarding intermediate results.

Another promising path to explore further is to learn more about participants who showed a ‘leave-and-back’ behaviour trying to make some isolated attempts, which may point at the existence of links between the persistence and the interest. While those attempts were not always successful, the fact of returning to the competition from time to time may indicate some additional level of interest. Finally, we found that the difficulty level doesn’t seem to be a factor that influences participation behaviour since most participants who solved the level-1 problems (easy) were also trying to solve other, more difficult problems. However, we need to look deeper at the problems themselves in order to understand how different type of problems and may affect children’s’ participation.

That was our first experience in doing virtual mathematical marathon. We are planning to repeat it again the next summer, and also try to pursue it during the school year. We will then be able to compare different sets of data. We need to collect some data of participants’ perception of the activity and its impact on their interest in doing more challenging mathematics. Surveys and interviews would give more insight into the question of how the persistence in solving mathematical problems can be developed and nurtured.

TEACHING THE MATHEMATICALLY PROMISING: FOCUSING ON ONE TEACHER

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INTRODUCTION

Rather than using the term *mathematically gifted* I use the notion of *mathematically promising* students – those who can achieve a high level of mathematical performance when their potential is realized to the greatest extent (Sheffield, forthcoming; Leikin, forthcoming). Realisation of students’ potential depends on the extent to which learning opportunities suit students' individual abilities, their affective characteristics and personality. The role of the education system is to provide appropriate learning opportunities, and teachers are agents of the education system in achieving this goal.

Who can teach mathematically promising students? In this short article I outline some components that seem to be crucial in teaching the mathematically promising, and devise some recommendations for teacher education.

I attempt to identify critical characteristics of teaching the mathematically promising through careful reading of the book edited by Karp (2007) in memory of his colleague - the brilliant mathematics teacher A. R. Maizelis (referred to in this article as A.R.). A.R. taught mathematics in a special school for mathematically promising students (now Mathematical Gymnasium 30, Saint Petersburg, Russia). The book includes biographical information about A.R. and memories of his graduates and his teacher colleague, as well as reflective notes about his 50 years of teaching experience. My own memories of A.R., as his former student, guide my choices of excerpts from the book.

Before presenting the teacher's views on his practice, I present a short outline analysis of A.R.'s graduates' memories.

THE TEACHER THROUGH THE EYES OF HIS MATHEMATICALLY GIFTED STUDENTS

Several themes that characterise critical features of teaching the mathematically gifted arise from the analysis of these texts. All the following themes are repeated in the memories of different people, and the list of themes presented below is far from complete. (Page numbers are from Karp, 2007)

☒ Learning is magic:

The time when we were his students seems to be magical (Paramonov, p. 114)

☒ Learning in a system, ☒ Learning to learn, ☒ Having one's own opinion, ☒ Seeing beyond mathematics:

A.R. provided us with a system of knowledge, taught us to learn, not to be afraid of disagreeing with authoritative opinion, not to be closed within mathematics, to be curious about many areas of science, arts, music. … (Bychkova, p. 63)

☒ Learning to think and understand what you do and why ☒ Seeing mathematics as an art, ☒ Admiring mathematics:

A.R. never showed us how to solve a particular type of problem, he did a great thing: he taught us to think, to understand what we are doing and why. The most important for me is that I learned mathematics as an art and not just as science. Love and admiration for mathematics as an art stay with me all my life. (Gobaraeva, pp. 66-67)

☒ Happiness from communication with the teacher, ☒ Happiness from meeting difficulties, ☒ Extraordinary personality, ☒ Atmosphere of understanding and mutual respect:

I remember the feeling of happiness. First, happiness from communication [with the teacher]. … Second, happiness that we experienced from solving difficult problems… Now …we understand how important this [overcoming difficulties] was for formation of our character…

I think that everything was determined by the high-quality personality of A.R., by his professionalism, thanks to which an atmosphere of understanding and mutual respect was created. This was extremely important at that age. (Kurnikova, pp. 96-97)

☒ The lesson "ends quickly", ☒ Learning through self-esteem, ☒ Hard work and endeavour even for the highly able, ☒ Telling stories:

> … the lesson ended very quickly – as if it lasted 15 minutes.
> Homework … included problems that you might not solve, but if you solve – you are rewarded. Here our self-esteem started working … We endeavoured.
> During the first meetings he succeeded to show us that we are not wunderkinds, we are just regular students, and if we want to achieve our goals, we have to work hard.
> He had a brilliant sense of humour. For each case [mathematical topic, or concept] he had cases from scientists' biographies, idioms, metaphors, and jokes…
> It seems that mathematics is an abstract subject. But A. R. explained in such a way that we clearly saw that mathematics is connected to all the fields of life. (Grigorov, pp. 68-74)

☒ Being happy for the students:

> One of his main talents was an ability to challenge each student, from high achievers to low achievers. He saw perfectly the abilities of each student… A. R. raised the bar to the height that he believed his students could overcome.
> He was candidly happy for the success of his students (Petrov, pp. 115-119)

☒ Finally:

> Each of us owes him not less not more than the future. .. (Grigorov, p. 68)

TEACHING THE MATHEMATICALLY PROMISING THROUGH THE EYES OF THE TEACHER

A.R. in his article *Notes of an old teacher* (Maizelis, 2007) claimed that he did not generalise his experience that "has been accumulated during half century", rather he retrospectively described episodes meaningful for him as a teacher and shared with others some of his "techniques". (All the excerpts in this part of the paper are translated from Maizelis, 2007).

Being proud for students' mathematical discoveries

His students' discoveries provided A.R. with the most meaningful memories from his teaching. He admitted that discoveries did not happen in every lesson and chose three episodes in which his students discovered mathematical proofs or rules. A.R. remembered the names of the students, lesson details and provided exact mathematical examples from those lessons.

Mathematical norms in the former Soviet Union (Russia) included learning proofs of theorems from textbooks and presenting them to the class. The first episode "panic discovery" happened when a student, who had not learned a proof for a theorem , presented the following proof for the theorem If line AB is parallel to line CD in plane P then AB is parallel to P:

> If line AB is not parallel to plane P, then it intersects the plane in a point that does not belong to CD. Then AB crosses BC. This contradicts the given condition.

A. R. continues:

> Till now I had not noticed such an easy proof. … I congratulated Misha Altshuler and he told me after the lesson that he invented the proof on the spot since he had not opened the textbook.

In the second episode, A. R. described "the first discovery" of Zenya Shumilkina who –he stresses– earned her PhD in mathematics and physics and, "what is more impressive", become a very successful inventor.

In the third episode, he told a story in which a theorem was called by the class "Eruh's theorem" for the name of the student who discovered it. This was a

> game ... that develops understanding… that knowledge is acquired through mental attempts and not by means of memory.

What A.R. learned from these and other stories:

> Do not be too careful, as many teachers are; homework should include an approachable mathematical task even it was not studied yet; a teacher should trust in his students; and the ideal lesson is one in which a teacher does not take part.

Some additional insights and recommendations: "techniques"

Due to the constraint of space, in this article I only outline the themes that A.R. described as important components of the teaching process. For each of the themes in the text there is an elaborated explanation and a concrete mathematical example.

☒ Different topics should be connected

☒ Students need to have a feeling of mental development as well as to see the future development of a topic

> Students' intellectual activity is activated when they feel that their knowledge is growing, and, moreover, comprehend perspective of knowledge implementation, development and their productiveness in life and in science.

☒ Different solutions to one problem should be an integral part of teaching mathematics

☒ Algorithmic knowledge is a tool for solving "interesting problems"

> Equivalent transformations, factorisation, solving equations, and calculations stop being a purpose in themselves, and become tools for solving interesting problems.

☒ Extra-curricular mathematics integrated into the lessons and homework is a tool to raise students' interest in the subject.

☒ Developing intuition and visualisation should precede formal and rigorous proof.

☒ The teacher has to be sensitive to ability level; he has to give more support to those who need many attempts.

> Difficulty of proof depends on the class. Trivial theorems do not exist. One student may find a proof easy, yet for another the same proof will stretch his abilities. The teacher's purpose is to identify the latter, and award him for his success.

☒ Students should "be infected with creativity"

CONCLUDING REMARKS

A.R. Maizelis developed a system of original teaching strategies and techniques that he used with his students in a flexible manner. He was an expert in heuristic implementation of these strategies and techniques. His work matched to a great extent contemporary theories of education: e.g., Brousseau's (1997) theory of didactical situation and Mason's (2002) attentiveness and noticing. He succeeded in proving that these theories can be applied in practice with mathematically promising students while his former students highly appreciated this practice. His erudition beyond mathematics, combined with deep and broad mathematical knowledge at an advanced level, allowed him to be flexible in teaching and to constantly "raise the bar" for his students. His intuition and attentiveness enabled him to match "the height of the bar" to each and every student he taught, taking into account affective characteristics of his students. But there was more than this – his humanity. A.R.'s teaching was similar to individual mentoring for at least 100 students each year (over 50 years). Probably the strongest characteristic of his teaching the mathematically promising, and one that allowed realisation of their potential, was his ability to respect his students and be proud of their success.

Should the teachers of gifted be gifted? I assume that some readers will answer "yes" while other will say "this is impossible". This implies two directions for education of teachers of mathematically promising students. First, the educational system needs to attract more gifted mathematicians with pedagogical talents to teach mathematically promising students (this is not the case nowadays). Second, educational programs for future teachers of the mathematically promising should include the "techniques and strategies" outlined above and they should connect theory and practice. I suggest that these programs must allow future teachers to work with gifted students, and they should include analysis of multiple cases of teaching the gifted, probably through personal communications with the teachers.

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RESEARCH FORUM 4

Sociological frameworks in mathematics education research

RF4: SOCIOLOGICAL FRAMEWORKS IN MATHEMATICS EDUCATION RESEARCH

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The last two decades have seen a shift towards what has been called the social turn (Lerman, 2000), in which researchers have come to see the inseparability of culture and context in cognition. Indeed 2005 saw PME's constitution change to encompass perspectives other than psychology, reflecting this shift and also foregrounding issues of social justice. Socio-economic factors remain the major influence on who fails in school mathematics, though social recognition and participation are equally important dimensions of social justice. Therefore no research group in mathematics education, least of all the leading international group, can ignore the social disadvantages reproduced in mathematics classrooms in most countries of the world.

Thus, in this forum, we will explore the ways in which theoretical frameworks drawn from the field of sociology may and do inform research in mathematics education and point towards ways in which more equitable outcomes may be achieved. We will consider the various ways in which current research makes use of theory associated with, for example, the work of Bernstein, Bourdieu or Foucault, or from sociolinguistics, and other related fields, and will discuss how such frameworks can shape our research questions and methodologies and form a basis for change in mathematics education. The symposium will also consider the importance and influence of the various ideological stances of researchers in identifying research questions, designing research projects and interpreting research data. The emphasis will be on orientations and what these open up and obscure and on the ideologies at work in all research design, rather than on research results in a manner of speaking. The forum will be reflective, in that we will ask, "What do we 'know' or understand as a result of such research?

In session 1, which will be chaired by Candia Morgan, there will be 3 short presentations, each followed by audience discussion. Session 2, chaired by Steve Lerman, will consist of 2 presentations and a workshop, in which participants will be invited to raise research questions informed by sociological theory, in relation to the PISA and/or TIMMS studies.

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THE RELATION BETWEEN THE OFFICIAL AND THE PEDAGOGIC RECONTEXTUALISING FIELD IN THE EDUCATION OF TEACHERS OF MATHEMATICS

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SETTING THE CONTEXT

In setting out how sociological work enables one to pose and research particular questions we have chosen, in this paper, to focus on mathematics teacher education, though we could have chosen many other topics, highlighting the contribution of Basil Bernstein and those who have extended and developed that body of theory.

The way that initial teacher education has been restructured under the influence of national and global policies in increasingly many countries raises issues about whether teachers are enabled to engage with the complex issues involved in how school reproduces social class and other inequalities. Further, in order to be able to resist attempts to narrow down the knowledge base on which teachers' judgements can be exercised we need to have a better understanding of what sociologists and others perceive to be the current problem in the education of teachers: the increasing regulation of the content and standards in the service of policy goals set by governments often with little or no consultation with teacher educators.

Within universities mathematics has relatively high status whilst mathematics education, together with other sub-fields of education, generally has low status. We understand, from Bernstein, that the stronger and the more established according to traditional academic standards the discipline the more able it is to resist external pressures and to satisfy external demands without making big sacrifices concerning its own goals about the kinds of knowledge it values. It is also the case that mathematics education in its scope is practice orientated and therefore more easily drawn into discourses of what is 'useful', 'practical' and 'relevant' knowledge. Mathematics educators might find it difficult to resist, and sometimes even find it appealing and challenging to focus on how to satisfy external demands on how our student teachers are to meet ever changing standards or curricula. It could be said that often we are training teachers for what changes in school, rather than providing them with an education that gives them a perspective on changes that will come.

Sociologists of education can provide us with tools to examine the origins and also the effects of powerful discourses such as current policy ones. It is important to note that Bernstein's theory, in particular, is not polemical; it is intended to enable the analysis of relations between policy and practice (Beck, 2009) so that one is better able to engage and resist changes to which one does not subscribe.

Changes are currently multi-faceted and far reaching as to their implications, and transformations are far too great to ignore. As Apple writes: "There clearly are now major shifts in many things: in what counts as legitimate knowledge and legitimate teaching; in how such knowledge and teaching are to be evaluated; in the dominance of new managerial emphases; in the role that economic realities, as defined by the powerful, play in all this; and what these changes mean in the actual experiences of people working in schools, health care, social services and other institutions." (Apple, 2009, p. xiv).

We need also to question the state of our own enterprise and whether the space for researchers to do autonomous work, under this change, is shrinking. Here the scene across the world differs. In Australia, for example, the development of a new national curriculum for mathematics is being led by a well-known researcher. However in the UK the picture is the opposite. The voices of researchers are still there but they are weaker and weaker and the official is becoming stronger and stronger.

THE SOCIOLOGICAL TOOLS – RELEVANT CONCEPTS

Whether we talk about schooling or about professional education, Bernstein's (2000) theory recommends to focus on the way *official knowledge* (curricula) is selected, organized, transmitted, and on how criteria of 'legitimate' knowledge are established through practices of evaluation. The concept of *recontextualisation* (Bernstein, 1990, 183-184), that expresses this basic idea, is central in this theory, directing us to the processes through which knowledge, drawn from different disciplines and established areas of knowledge production, such as mathematics, is selectively appropriated, relocated and refocused 'for the purposes of their selective transmission and acquisition'; or else, to how pedagogic discourse 'constitute its own order and orderings' by selective appropriation of other discourses.

The idea of different fields, such as of knowledge production, recontextualisation and reproduction, presupposed in the concept of recontextualisation, implies also the concept of *boundary*, the most fundamental concept in Bernstein's theorization (1990). Boundaries (boundary maintenance) relate to power. They are social conventions and practices which create and keep apart categories, be they people, agencies, types and fields of knowledge, or stages in processes. The function of a boundary, material or non-material, is always symbolic: the strength of the boundary is inscribed in the consciousness of subjects, regulating their actions, behaviour, manner and pedagogic identity. As such, the boundaries define what is legitimate and illegitimate, e.g., who has the power to educate or to research, to be assessor or learner. For instance, in current policy arrangements teachers are cast more as perpetual learners rather than as professionals (Tsatsaroni & Sarakinioti, 2008).

In the current conjuncture, we can appreciate better the importance of the concept of boundary, for it is the basis for making important distinctions both within the field of education and between this field and other fields of social action. Thus in studying processes of knowledge construction for pedagogic purposes, *within* education, an important distinction is that between the *official pedagogic recontextualising field* and the *pedagogic recontextualising field*, the former dominated by the state and its agencies, the latter consisting of pedagogues, specialized journals, researchers' associations etc. (Bernstein, 1990; Morgan, Tsatsaroni & Lerman, 2002). As this relation changes over time in different national systems operating within different societies, it is always important to study where the boundary between these two subfields and their respective agencies and agents is drawn: the stronger the boundary between them, the less the space and the power for pedagogogic agencies outside the official recontextualising field to influence and have control over processes of knowledge construction, the pedagogies for its transmission and the criteria of evaluation of the legitimate knowledge. An apposite example in the context of teacher education in England is the Training and Development Agency for Schools which has specified 'standards' for both initial teacher training and teachers' subsequent career progression; representing efforts by government agencies to prescribe and control the knowledge base of the teaching profession and to subject it to 'modernisation' by promoting 'generic' models of training (see below) (Beck, 2009).

Using again the notion of boundary to study the relations *between* the field of education and other fields of action, with reference to processes of knowledge recontextualisation, an important distinction is between *horizontal* and *vertical discourse* (2000). There are at least three important ideas that can be drawn from the distinction Bernstein makes between these two different kinds of discourse. First, with some help from Bourdieu (1998, 31-34) and Habermas (1987, 367-373) we can see this pair of concepts as pointing to the functional differentiation of different fields of activity in modern societies, education being one sphere where vertical discourse dominates, having the space to specialize and develop into distinct structures and forms. Second, given the boundary, institutionalized in the course of history, that creates insulation and keep apart the different categories of discourse, vertical discourses are distinguished by the formal and systematic organization of their knowledge forms, as against the fields of practical activity which, having as their main value the functional use of knowledge, are organized in segmental ways. Hence, third, the differences observed in the transmission processes between the two kinds of discourse, expressed in their differing *orientations to meaning*. In particular, vertical discourse historically aims to initiate individuals and groups of learners into systems of meaning (symbolic systems), while horizontal discourses orientate people to context dependent meanings, and to typical ways of responding to specific situations (See Hassan, 2004, for finer distinctions).

Therefore, in researching processes of knowledge construction for pedagogic purposes, like the initial education of teachers and teacher professional development courses, we need to ask how boundaries establish those agencies and agents who are legitimized to control the process and the criteria, i.e. what orientation to meaning is of value in specific pedagogic contexts.

Genericism described by Bernstein (2000, 41-63) and Bernstein scholars as being the pedagogic form whose most distinctive feature is "the particular relationship they claim to have with 'everyday knowledge', and the skills and competencies supposedly required in a widening range of occupational and other spheres" (Beck, 2009, p. 5), is currently promoted by governments at all levels of education. We therefore, need to ask whether such model of pedagogy and the kind of discourse thus privileged is an adequate base for teachers as professionals, operating as they do in a field that is becoming ever more complex and demanding and that increasingly plays a major role in the reproduction or challenging of social inequalities.

We can only stress the importance of Bernstein's analytical concepts for mathematics education research by referring to Young's (2006, 25) point that, currently, there is "a shift from a reliance on *generalizing principles* associated with specialist professionals to a reliance on *procedural principles* associated with regulatory agencies"; and by inviting reflection on Beck's (2009, 12) conclusion: "We have not, as yet, reached the point Bernstein imagined…where there remains *only* the voice of the ORF [Official Recontextualising Field] – but at least in the field of the professional formation of teachers in England, we do seem to be moving steadily further in that direction".

LANGUAGES OF DESCRIPTION

In this section we describe briefly just some of the work in mathematics education in the sociological tradition. For mathematics education to be less vulnerable it has to *honour* its achievements and to try to establish a tradition of sociological work to think about important issues in mathematics education. Most productive studies are those that manage to add to or extend the language of description and create new languages that are appropriate for a problem. In recent studies of mathematics teacher education programmes in universities in South Africa within the QUANTUM project (e.g. Davis, Adler &, Parker, 2007) the researchers developed a sophisticated language for the description of different images of the teacher and of mathematical knowledge for teaching and its acquisition by inservice teachers. Ensor's (2001) study was an examination of what beginning mathematics teachers drew upon from their methods course as they moved into their classroom experience. Morgan, *et al* (2002) developed a language to describe how teachers are positioned when evaluating students' extended work in mathematics when they are caught between official discourses and their own unofficial practices.

Teacher education is an important area but sociologists have done work in other areas of mathematics education. Within this context we can refer to Dowling's (1998) study of school mathematics texts, Sethole's (2005) work on the role of everyday contexts in the learning of mathematics (see also Cooper & Dunne, 2000), and Brown's (2006) work on school/family relations and the teaching and learning of mathematics, amongst many others.

SOME RESEARCH QUESTIONS

To what extent do forum participants find that mathematics teacher education is focused on preparing new teachers to perform according to the latest pedagogic demands?

What are the relations between research (unofficial field) and policy (official field), in participants' own countries?

Do the concepts above help to conceive of one's local, national situation in different ways?

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SOCIOLOGICAL FRAMEWORKS IN MATHEMATICS EDUCATION RESEARCH: DISCURSIVE APPROACHES

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INTRODUCTION

Over the last forty to fifty years, in social science and educational research generally, the importance of sociological approaches has been that sociology has been seen to be that social science which most clearly challenges the assumptions of psychology. Thus it is important for an organisation like PME to examine the possibility of sociological challenges to 'the psychology of mathematics education' research programme(s).

The approach in this contribution is 'broad-brush' and personal, in that it relates to Jeff's academic experience, though the aim is to weave this discussion here with the issues raised in Candia's plenary.

Jeff is going to introduce the discussion by recalling major themes as he perceived them in contact with sociology and sociologists over his academic career since the late 1960s, in particular at the London School of Economics as a student of statistics, and as a beginning lecturer in an unusually fertile inter-disciplinary social sciences setting, the Enfield College of Technology, later part of Middlesex University.

In the late 1960s and early 1970s in the UK, sociological theory was focussed on the works of the three 'greats': Marx, Weber, and Durkheim. Other courses, called for example 'Social Problems', might induct students into issues like 'deviancy' and ethnographic approaches. Often taught explicitly were two or three classic dilemmas, the use of which was thought to help distinguish different approaches, and even the work of the three 'greats'. They can be summarised as:

(1) structure vs. action

(2) order vs. conflict

(3) official vs. deviant perspectives.

I shall discuss each briefly in turn, before showing how the key features of discursive approaches, as they have developed up to today, can be seen to cope with these classic dilemmas in sociology and social theory.

(1) *structure vs. action*: This dilemma addresses the extent to which the human being is strongly influenced or largely determined by the person's position within a social structure - or, alternatively, is relatively unconstrained, and free to act in ways that make sense according to their meanings and understandings of the social world (see e.g. Giddens, 1979). At that time in the UK, the social structure at issue was still usually considered to be that of social class stratification - rather than gender, or ethnicity, say. Such approaches were used to explain differences in achievement at school.

This dilemma is closely related to others, such as: *determinism vs. freedom*; and *society (or collective) vs. individual* (Henriques et al., 1984) - both of which have resonances with long traditions of philosophical thought; indeed, this searching for answers in philosophy was characteristic of many sociologists at the time. We can also note the fundamental role that accommodating these issues somehow plays in current discursive approaches, as shown in Candia's plenary - as well as in the longstanding concern in mathematics education community with the relationship *context vs. task (or behaviour)* (see e.g. early PME plenaries from Alan Bishop in 1985 and Christine Keitel in 1986).

(2) *order vs. conflict*: This relates to a long-standing anxiety (from the 17th century in Europe) in social and political philosophy, as to whether the 'natural state' of human beings is peaceful or 'a war of all against all' (Hobbes), and how 'the state of society' might manage people's natural tendencies and appetites (Horton, 1966). 'Structural-functionalist' sociologists focussed on how order, balance and consensus might be maintained - and conflict theorists wondered how the 'hegemony' of the ruling class, and society's ideology (e.g. Gates, 1997) and the 'false consciousness' of the dominated classes, might be countered with 'true' class consciousness.

(3) *official vs. deviant perspectives*: The possible existence of *multiple perspectives* is raised by adherents to various versions of conflict theory, and this idea was used in ethnographic studies of community and deviancy from the early 20th century. In time (mostly from the late 1970s in the UK), sociological approaches were embraced by gender, ethnicity and 'cultural studies' researchers, since sociology seemed supportive of their aims to do innovative work on the situations of their subjects. A version of this dilemma, *outsider (OR researcher) vs. insider (OR participant)* is

discussed by Candia. Of course, the admission of multiple perspectives raises the spectre of relativism.

DISCURSIVE APPROACHES

Let us see how the key ideas of discursive approaches address the dilemmas of sociological theorising proposed above. *Discourses* can be seen as systems of 'constructs values, and relationships between the various participants […], structuring what may be said or not said, and by whom' (Morgan, 2009), and thereby motivating actions within the related *practice(s)* (or '*discursive practices'*), and allowing participants to account for their own actions, and to evaluate those of others. Discourses find expression in the *texts* produced by participants (*ibid*.).

Critical Discourse Analysis (CDA), as outlined by Candia, focuses at three levels:

(i) the communicative interaction itself;

(ii) the discursive resources and 'language-related elements' used in the interaction;

(iii) the social structures and socio-cultural practices within which the interaction is situated (Chouliaraki & Fairclough, 1999, p.113).

Thus the use of discourse as a central concept allows the consideration of both (inter)action and the meanings of the participants, as well as the social structures and practices which provide the context.

Second, the use of discourse as a central concept allows an escape from the horns of the 'truth' / 'ideology' dilemma. It is realised that it is not only *true* ideas that have the power to motivate (and also that not only *false* beliefs need explanation!). Discourse is seen not merely to *describe* the world, a world, but crucially to *construct* our world. Thus Foucault talks about 'regimes of truth'.

Third, Alfred Schutz's insistence on the importance of sociologists grounding their understanding of social world on participants' commonsense understandings of the world of everyday life, while also developing their own 'second-level understandings' (e.g. Berger & Luckmann, 1967) has resonance in Norman Fairclough's claim that, to have a 'good enough' understanding of the contexts of participants' practices, one must 'draw upon the same "insider" resources as the participants in the practice, but with self-conscious awareness of the common-sense assumptions of the practice' (Morgan, 2009). This latter distancing or 'reflexivity' (Bourdieu, 2004) of course helps to provide a protection against relativism.

In addition, Candia's emphasis of the researcher's responsibility for developing and arguing for their own ('second-level') interpretation reverberates with Giddens' (1976) emphasis of the need for a 'double hermeneutic'. (For recent discussions of this issue, see Bourdieu, 2004).

The other key concept in the discursive approaches that we have used is that of *positioning* (Evans, 2000; Morgan, 1998). This concept is an attempt to avoid both horns of the determinism vs. freedom dilemma, and to understand the context of a participant's actions as 'result[ing] both from the general social availability of positions in discourse, and from the investment for the particular person to take up a specific position. It is not at all "freely" chosen, nor is it fully determined, but there are "reasons" (Hollway, 1989) for it' (Evans, 2000, 133). This leads to a two-stage description of the context of an activity like mathematical problem-solving, which we have described in recent work as the *structural phase* and the *textual phase* (Evans, Morgan, & Tsatsaroni, 2006). In the classroom described, the structural phase involves 'a prior analysis of the structures provided by the national education system, traditional pedagogic practices and the local pedagogic practices of the particular classroom as well as an awareness of further resources available to the students from other practices, including 'everyday' discourses of the family or peer group' (Morgan, 2009), that is, an analysis of the 'available' discourse in the setting. The textual phase focuses on 'semiotic features of the students' interaction' and 'the particular discursive resources drawn upon by each of the participants' (*ibid*.).

Candia has given examples of how the structural phase works, and also of the use of sociolinguistic tools in the textual phase. In the textual phase, much interest focuses on the way that particular discursive resources are 'drawn upon' - and indeed *unexpected* resources are 'called up' - by participants. Is this intentional, or at least conscious ('… on reflection, I can see why I did that …') - or might it be unconscious?

In our earlier work, we have found that trying to understand the play of affect and emotions is often crucial to understanding which discursive resources are drawn on in the textual phase. That is, meanings are not only cognitive - and participants have emotional 'investments' in their use of particular terms or signifiers (Hollway, 1989; Denzin, 1990). We have suggested that emotion can usefully be understood as a 'charge' of 'energy' (cf. Collins, 1990) attached to signifiers in a semiotic chain (Evans, Morgan, & Tsatsaroni, 2006). In our analysis of classroom interaction, we found that drawing on psychoanalytic concepts as researchers, allowed us to suggest a richer interpretation of pupils' reactions to problem solving - in terms of anxiety, resistance and isolation. (Of course, we were not able to do this by using only sociological concepts.) Other work has used psychoanalytic concepts such as *defense, repression* and *rationalisation* (e.g. Henriques et al., 1984; Walkerdine, 1988; Evans, 2000; Brown, 2008; and many others).

CONCLUSION

In this short and 'broad-brush' discussion, we have tried to show how discursive approaches may be used to resolve some of what Jeff called the dilemmas of 1960s and 1970s sociology, and of social theory more generally. However, these approaches are not in themselves purely sociological. Candia's plenary shows that they draw on sociolinguistic and semiotic perspectives and approaches. Further, we have argued for the importance of psychoanalytic concepts.

Despite the opposition in previous years referred to above between sociology and psychology, some of the approaches referred to here may be styled as 'discursive psychology' (e.g. Henriques et al., 1984; Hollway, 1989; Walkerdine, 1988 - and, in a somewhat different vein, see e.g. Edwards, 1997). Indeed, approaches using 'discourse' as a central concept are helping to produce a theoretical basis for studies in education, including mathematics education, which is full of inter-disciplinary promise (e.g. Evans & Zan, 2006, and other papers mentioned in Candia's Plenary).

Certainly, using discursive approaches makes us examine our own involvement in the production of research 'truths' - for example, about 'what works' in classrooms and in adults' use of mathematics.

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GOVERNMENTALITY AS A FOURTH PURPOSE OF MATHEMATICS ASSESSMENT

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Like all educational assessments, the purposes driving the assessment of mathematics learning go to the issues of generating mathematical credentials, educational accountability and mathematics learning (Broadwood & Black, 2004; Black & Wiliam, 2007). For instance, looking at the case of PISA (see my rationale for doing this below) we find the "main aim" of assessment is to "determine the extent to which young people have acquired the wider knowledge and skills in reading, mathematical and scientific literacy that they will need in adult life" (OECD, 2006, p. 9) and that, *inter alia*, the results will "allow national policy makers to compare the performance of their education systems with those of other countries" (p. 9). The unstated but implied purpose here thus relates directly to accountability and has implications for learning and certification. Moreover, in the context of most formal educational situations, it is generally assumed that these key purposes are also served by and impact on pedagogy, and this leads to linking the fate of assessment to issues related to the mathematics curriculum (Pinar, 2004). It follows from this that critique of assessment, like critique of the curriculum it is part of, must have a broader social contextual framework than simply the curriculum itself. In particular, the validity of assessment strategies, methods and items need to be looked at from perspectives that go to both the cultural ends of education as well as the purely technical. Messick (1989) and others (Gipps, 1999), for instance, attempt something like this in their concept of "consequential validity". Here, because assessment is often seen as part of the experience of formal education, it seems right to treat the educational consequences of assessment processes as relevant to the educational trajectories of students. Crucial to this is both the knowledge acquired and also the potential for knowledge to be acquired.

For me, this analysis suggests that a fourth purpose for assessment needs to be identified – and this must address the conditions that are required and implied by the production and reproduction of social order. I would tentatively suggest that these constitute a fourth purpose for assessment, and one that might be theorised in Foucauldian terms via his concept of governmentality. Several lines of analysis could be opened up here. One would be at the level of educational systems: How do international comparisons and league tables fit in with and advance the regimes of government (Meadmore, 1995)? How is the discourse of accountability co-opted by government to register the mandates of those in powerful positions? By government here, I of course refer to both jurisdictional government as well as institutions such as departments of national government, schools, etc. Another alternative would be to investigate the many detailed ways test items enact "strategic games", and thereby both limit and release behaviours of teachers and pupils in order to satisfy policy ends. A key question here could be: Seen in what light are power relationships made to present as the natural and rational choices of those governed? A third kind of analysis would be to explore how these two "levels" of assessment interrelate. Clearly other alternatives also exist.

In this paper, I want to focus on the second of these. In order to do this I first discuss aspects of the Foucauldian take on government as relations of power. I then take these theoretical resources to a preliminary analysis of a PISA test item; discussion and conclusions follow.

THEORETICAL RESOURCES

For Foucault, the modern state governs not by exerting condign power or imposing sheer discipline, but by shaping and aligning the identities of its subjects to a prefigured end. These constitutive processes are managed by orders of rationality, they are the implementation of 'regimes of truth' and order. Because these power relations go to the prefigured ends of government, Foucault refers to them as technologies of government; and this form of power he calls governmentality (Foucault, 1979). As Klein (1996) notes, governmentality

> "applies itself to immediate everyday life which categorises the individual, marks him (sic) by his own individuality, attaches him to his own identity, imposes a law of truth on him which he must recognise and which others have to recognise in him. It is a form of power which makes individuals subjects." (Foucault, 1982, p. 781, cited in Klein, 1996, p. 376)

Here we see that the formation of identity (the way a subject knows herself and others know her) is one aspect of governmentality. Other kinds of work are also performed here, however. In the *History of Sexuality, Volume 2* (1997) for example, external norms, imperatives, codes, systems and schemas get to be expressed as a commentary on the self by the self with respect to behaviours, desires and dispositions – ethics; and thoughts, communications and ways of doing things against conceptual and procedural knowledge schemas – knowledge. Thus, kinds of self-regulation become what Foucault (1988, 1997) refers to as a 'techniques of self' and these go to building both what Foucault refers to as the 'ethical subject', and also to what I provisionally refer to as the 'knowing subject'. If the work of governmentality is done well, then the governed subject assumes identities and occupies ethical spaces and knowledge frameworks that align with the ends of government at the level of the political, the state and society as a whole. Foucault calls the end of such processes 'subjectivation'. Because governmentality, working through the processes of subjectivation, is both the condition and consequence of the modern Western state, I believe distinctive issues about the broader functions of assessment as a component of the curriculum are raised. Can we discern the power relations of governmentality at work in mathematics assessment?

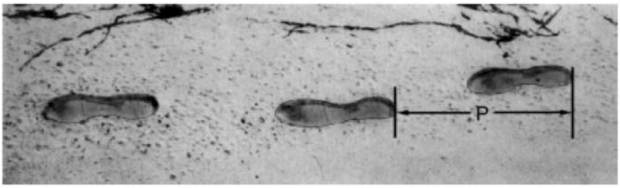
As Ball (1990) and Meadmore (1995) and others have noted, the institutional experience of schooling can be construed as a form of governmentality in which the 'scholastic identity' (Ball, 1990) and the professional identity of the teacher (Evetts, 2003) use the curriculum as both a regulative and self-regulative device. Previously, drawing on the work of Black, Wiliam and Broadfoot, above, I identified certification, accountability and learning as three key purposes of assessment. Now, based on the Foucauldian analyses I have presented here, I suggest that assessment can also be seen as a technology of government. This means that governmentality can be construed as a fourth key purpose of assessment. In the empirical analysis below, I wish to take this speculation further.

ANALYSIS OF A PISA ITEM

Being suggestive only, my empirical analysis focuses on a single item drawn from a set of PISA 2006 released items mathematics (OECD, 2006a). I have chosen PISA items for a number of strategic reasons: I mention only two. First, because the "primary aim of the PISA assessment is to determine the extent to which young people have acquired the wider knowledge and skills in reading, mathematical and scientific literacy that they will need in adult life" (OECD, 2006b, p. 9), I considered that PISA test items could be less impacted by individual national systemic priorities and more influenced by thinking that has standing across numerous international jurisdictions. Thus if evidence of governmentality was obtained, this would reflect orientations that are deeply embedded in shared cultural forms, and thus be of potentially more general significance. Second, subsequent studies might then explore cultural hegemonies of international significance.

The item I consider is set out below.

WALKING



The picture shows the footprints of a man walking. The pacelength P is the distance between the rear of two consecutive footprints.

For men, the formula, $\frac{n}{P} = 140$, gives an approximate relationship between n and P where,

n = number of steps per minute, and

P = pacelength in metres.

Related to this scenario, a number of questions are proposed. I consider the first of these (see below), together with PISA mandated scoring codes.

Question 1: WALKING *M124Q01- 0 1 2 9*

If the formula applies to Heiko's walking and Heiko takes 70 steps per minute, what is Heiko's pacelength? Show your work.

WALKING SCORING 1

Full Credit

Code 2: 0.5 m or 50 cm, $\frac{1}{2}$ (unit not required).

- $70/p = 140$
  $70 = 140p$
  $p = 0.5$.
- 70/140.

Partial Credit

Code 1: Correct substitution of numbers in the formula, but incorrect answer, or no answer.

- $\frac{70}{p} = 140$ [substitute numbers in the formula only].
- $\frac{70}{p} = 140$
  $70 = 140p$

  $p = 2$ [correct substitution, but working out is incorrect].

OR

Correctly manipulated the formula into $P=n/140$, but no further correct working.

No Credit

Code 0: Other responses.
- 70 cm.

Code 9: Missing.

In Question 1, students are asked to apply the given formulae with the value $n = 70$, and show that $P = 0.5$ m. In this the mathematical knowledge targeted involved two items (i) substitution, and (ii) algebraic manipulation. In order to test these, however, the crucial step required is (iii), being able to relate the question to the scenario. This requires the student to correctly assign the value 70 to the variable n. In order to do so, the student must recognize that "number of steps per minute" and "70 steps per minute" at this stage of the response process are interchangeable. However, complicating this literally formulaic performance is the formula itself. A paradoxical situation arises. On the one hand a purely symbolic performance suffices to obtain a Full Credit Code 2 response – yet on the other hand the question offers an apparently unneeded visual modality. Nevertheless, in mathematical discourses, visual modes typically intensify semantic developments, and this, in the context of an assessment item, would tend to suggest that 'meaning' is central to full credit in this item. However this is not so - in fact, in this instance the visual mode operates in the *opposite* direction: it points *away* from, rather than towards, meaningful responses to this item.

Two reasons support this view. Firstly, note that in the accompanying picture the P together with arrows are superimposed over the image, and thus the visual mode is explicitly and purposively linked to the symbolic mode. Indeed, the free floating symbols used to do this double as both visual and symbolic cues signifying how the visual mode is to be experienced as rational, and the how the symbolic mode is to be "read". However, seeing these symbols as a formula presents further problems: for though a formula, the expression is not presented *as* a formula. Expressions in an implicit form are known to present a challenge to students who routinely encounter independent and dependent variables separated explicitly. Secondly, triggered by the literalism of the framing visual modality, a student who seeks to make "literal" sense of the formula may be stumped. The formula implies, for instance, that the slower a man walks the shorter his paces (and conversely with increased speed). But because this can only be intuitively sensible within a certain (but in the question, unstated) range, the formula as it stands in the scenario *cannot* be taken literally. Nevertheless, the range of modalities offered invite, actually insist, on a form of literalism that if maintained, would actually block a Full Credit response to this item. Yet, on the other hand, entry to the Question 1 requires a form of semantic purchase. This paradox intrigues me, and I think it worth exploring in order to better see and understand how various responses to this item are possible, and how they might be associated with various regimes of truth and different subject positions. Certainly a student who values mathematics as a meaningful engagement, could be waylaid; whereas a similar student who preferred to search for formal symbolic relationships would might well be favoured.

DISCUSSION

But how does the Foucauldian notion of governmentality help us understand this potentially troubling situation? For me it appears that knowledge of mathematics cannot be sufficient for a successful and timely response to this item. Certainly the student must *know* mathematics, but more is required - the student must be the *knowing subject* that gives facility and, possibly, has awareness of the strategic game of laying aside semantically pregnant cues and, indeed, seeing these as actual obstacles, not pathways, to progress. In addition, the student must be the ethical subject that, in these moments, will not succumb to the provocations of semantic cues however implicit, and suspends the official process announced in the framing scenario of the question - in the name of 'the answer', and in the name of 'the assessment'. And the student must have an identity by which the student knows herself to be the person who cannily sniffs out the real goal, locates the official problem inscribed in unofficial and other-directing cues and, with this in view, knows she will exercise a disposition and confidence to both reproduce and verify the manner of official sense making.

So far I have outlined why I think this mathematics assessment item confuses rather than elucidates inter-modal interactions and how this has implications for subjectivation. This is the first part of my argument. What I now want to consider is how the subjectivation obtained serves political ends. Here I need to be especially brief. Firstly, Foucault viewed the state as a "structure in which individuals can be integrated, under one condition: … this individuality [is] shaped in a new form, and [submits] to a set of very specific patterns" (Foucault, 1982) - thus the state does not operate "above individuals, ignoring what they are and even their very existence", but through them as specifically subjectivated beings. Borrowing from this, the outline of an alliance of this mathematics assessment item with a neo-liberal state begins to emerge. According to Lemke, individuals in a neo-liberal state regulate themselves in accordance with the "imperatives of flexibility, mobility and risk taking" (Lemke, 2002, p. 6); and here the "process of individualization … endangering collective bonds" is paradigmatic. But previously I showed that successful negotiation of this assessment item required an economy of effort that could only be obtained by purposively disregarding certain semantic cues. In this way that the recuperation of the merely formal, symbolically mediated, mathematical problem could be achieved. However, this strategy could also be perceived as hazardous – as subsequent questions could be conditional on a semantic (not merely formal) purchase[1] Certainly readiness to be flexible, to be mobile within the multimodal micro setting of the scenario, would be a good hedge against this risk; this could lend rationality to a "just in time" approach. Interestingly, however, these possibilities are not without irony: because of neo-liberal subjectification, the pristine valorised subject – by virtue of which neo-liberalism acquires certain ethical attributes – is immediately nullified. What emerges is neither a more distinctive nor more nuanced subject – but precisely the opposite. Though the processes of individuality are vigorously championed at one level, they are simultaneously effaced at another.

SOME RESEARCH QUESTIONS

Recently, Kenway *et al* (2006), quoting Kirzner (1984) described the entrepreneur as an "'agent that spurs society to take advantage of existing scattered and dispersed knowledge' and who 'generates and harnesses new technological knowledge, and discovers entirely new bodies of resources that had been hitherto overlooked'" (p. 41). In its neo-liberal form, however, entrepreneurial work is not simply economic. On the one hand it combines both "techno-scientific knowledge and business acumen", and on the other hand, it is often organized within bureaucratic settings in which various kinds of institutional entities are brought into broad coalition and mundane interdependence. Kenway *et al* call such agents of production, 'technopreneurs'. I suspect there is much to explore here that could have interesting implications both for the theory of assessment, as well as the pragmatics of mathematical assessment. For example, (albeit) in one assessment item we found multimodal connectives were strategically ignored and meaning subordinated to formalism. In terms of neo-liberal rationality of the technopreneur this corresponded to the actions of certain techniques of self (the dispositions to mobility, flexibility and risk taking). But at the level of educational discourse, these choices can also be interpreted as a tactical challenge to the "rightness" of the priorities of the educational avant-garde. Choosing a formal approach, it would seem, might enhance a formal result – yet in this event meaning is decoupled from form *both* at the level of the task *and* at the level of the result.

My concluding questions are thus:

To what extent does the idea of governmentality help us understand the subjectivity privileged by mathematics assessment?

Indeed, more broadly, to what extent do these ideas help us explore the constitution of institutionally framed mathematics?

Can these ideas thus be useful in bringing together analytic and evaluative criteria relating to mathematics curriculum issues?

Notes

1. As far as I could ascertain, in this example, they did not.

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SOCIOLOGICAL FRAMEWORKS IN MATHEMATICS EDUCATION RESEARCH: GLOBAL POLICY TRENDS AND ADULTS' MATHEMATICS EDUCATION

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INTRODUCTION

Global players in the area of education and economic development are promoting the launching of additional international comparative studies of adults' performance in mathematics: in the last few years, this has included PIAAC for adults generally, and TEDS-M for trainee teachers. They join several series of studies:

- those started in the mid-1960s by the International Educational Association, currently run under the TIMSS projects
- PISA, the series of assessments commissioned by the OECD for 15 year olds (Lingard & Grek, n.d.)
- International Adult Literacy Survey (IALS) and Adult Literacy and Life skills Survey (ALL), dating from the mid-1990s and from the early years of this decade respectively, the latter commissioned and managed by Statistics Canada, NCES of the USA, OECD and UNESCO.

This paper will focus on the development of PIAAC, with which I have been involved. This is currently taking place under the auspices of the OECD, and the project is being managed by a consortium of organisations in the US and Europe. The design of items for the three scales - literacy, numeracy, and problem solving in technologically-rich environments (PS/TREs) - under the leadership of 'expert groups' of academics and educational developers, has just been completed in spring

2009; there will be a field trial in 2010 and the full run in 2011, with results to be available in 2013 / 2014. PIAAC aims to follow earlier versions of such surveys, e.g. IALS and ALL, but with some crucial developments. The questions have been designed to allow comparisons within countries over time with results from ALL, and it is also hoped that the results can be related to those of PISA, the OECD's assessments for 15 year olds.

In this presentation, I aim to provide the basis for discussing several key issues:

- policy issues that are aimed to be addressed using the results
- developments from earlier studies, IALS and ALL
- the methods of conceptualising and measuring adult competencies, in particular numeracy, for this study.

PROGRAMME FOR THE INTERNATIONAL ASSESSMENT OF ADULT COMPETENCIES (PIAAC)

OECD policy-makers present the PIAAC strategy as one that will 'help [the participating countries] to:

- Identify and measure differences between individuals and across countries in key competencies and other economic and social outcomes believed to underlie both personal and societal success.
- Assess the impact of competencies on economic and social outcomes, including individual outcomes such as integration into the labour market, employment status and earnings, participation in further learning and education throughout the life cycle, as well as aggregate outcomes such as fostering economic growth or creating social equity in labour market outcomes and social participation.
- Assess the performance of education and training systems in generating the required competencies at the levels required by social and economic demands.
- Clarify the policy levers that, once ''deficiencies'' in key competencies have been identified, lead to enhancing competencies through the formal educational system, in the work-place, through incentives addressed at the general population, etc.' (Schleicher, 2008, 2-3).

PIAAC involves rather more countries that either of the earlier studies of adults. It will involve 28 countries, 22 in Europe, and a slightly larger number of languages (or versions of languages). The cross-national nature of the project is justified on the ground of economies of scale, providing a comparative perspective for policy-makers, displaying greater variance in scores and situations, and allowing monitoring of progress towards international (e.g. EU Lisbon declaration from 2000) targets (Schleicher, 2008).

Numeracy has been defined for the purposes of designing the items for PIAAC as:

> the ability to access, use, interpret, and communicate mathematical information and ideas, in order to engage in and manage the mathematical demands of a range of situations in adult life. (Gal & Numeracy Expert Group for PIAAC, 2009)

This is an attempt to conceptualise mathematical thinking in context. An expanded definition of 'numerate behaviour' is given, which specifies a number of dimensions of numerate behaviour:

- the context: everyday life, work, societal, further learning
- the type of response: see the verbs in the above definition, plus 'evaluate / solve'
- the mathematical content: quantity & number, dimension & shape, pattern / relationships / change, data & chance
- multiple representations (ibid.).

Numerate behavior, in turn, is understood as 'founded on the activation of several enabling factors and processes':

- mathematical knowledge and conceptual understanding
- adaptive reasoning and mathematical problem-solving skills
- literacy skills
- beliefs & attitudes
- numeracy-related practices and experience
- context/world knowledge.

Thus, a feature of PIAAC is the production of demographic and attitudinal information in a Background Questionnaire.

The crucial innovation in the fieldwork methods of PIAAC is the designation of the 'default' assessment method as via computer (laptop in the respondent's home). Other developments include an 'expanding range of skills', including problem solving in technologically-rich environments (PS/TREs), and a set of self-report indicators of the use of job-related skills at work.

PIAAC is being designed to be repeated, in order to build up time series of information for participating countries. It will also aim to study the correlation of skills levels with the 'success' of individual and countries, how well education and training systems generate the competencies surveyed, and how policy might improve the effectiveness of these systems.

DISCUSSION

I hope the short account above will provide sufficient background for several issues to be discussed:

- the advantages and dangers of international comparisons on such a global scale, including if and how research from a social justice framework might be organised to exploit such opportunities;

- the advantages and disadvantages of using such a specification of adult numeracy as the basis for the research;
- the advantages and disadvantages of aiming to use computer assessment as the 'default' method, as is beginning to happen in other areas of mathematics education;
- the effects of the availability of such test results on educational policy-making in the participating countries (cf. Lingard & Grek, n.d.).

Studies such as PIAAC aim to produce, on an international comparative basis, information to feed into a central theme of educational policy discussions at the current time - those concerned with 'what works' in classrooms and in adults' use of mathematics.

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SOCIOLOGICAL TOOLS IN THE STUDY OF KNOWLEDGE AND PRACTICE IN MATHEMATICS TEACHER EDUCATION.

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In this short paper we outline how we[1] have put Basil Bernstein's theory of pedagogic discourse to work to interrogate knowledge and practice in mathematics teacher education. A central problem that we have pursued is answering the question: *what* is constituted as *mathematics for teaching* (MfT[2]) in teacher education and *how* it is so constituted? Our discussion here draws from Parker's study of pre-service teacher education (Parker, 2009), which in turn develops from and builds on earlier work reported in Adler & Davis (2006), Davis (2005), Davis, Adler & Parker (2007), and Adler & Huillet (2008). Embedded in the question is an understanding that, in practice, selections of content into mathematics teacher education are varyingly drawn from the domains of both mathematics and teacher education. We present here how we have worked to adequately describe pedagogic modalities in the field of mathematics teacher education and the ways in which practice (in this instance, mathematics for teaching) might be specialised. We illustrate the methodology through analysis of an instance of teacher education pedagogic practice. While the methodology itself is our focus, the particular example provides a compelling story at the heart of which is the problem of integration of knowledge(s) within a pedagogic practice. Here, a constructivist pedagogy is at work, but differentially with respect to teaching/learning *mathematics* and teaching/learning *mathematics teaching*. The example illuminates mathematics and teaching, and their co-constitution in a particular pedagogic context, together with implications for equity.

Our methodology

For Bernstein, an interactional practice (IP) (Bernstein 1996, p 31) constitutes the recognition and realisation rules that enable the 'what and how' for constructing an expected legitimate text. Within an IP a text is anything within the context that attracts evaluation. Our methodology hinges on describing the full duration of a selected IP by breaking it up into *evaluative events*. We unpack an evaluative event through four moments of pedagogic judgement recognisable over a temporal segment of classroom interaction: existence (E), reflection (R), necessity (N), and contingent notion (C). These moments of judgement are theoretically[3] necessary in order to fix the meaning (if only temporarily) of the concept/ notion/ idea/ behaviour in focus within the pedagogic context. The operation of evaluation across these moments enables us to identify the way in which meaning is (re)produced: we identify the grounds on which meaning is communicated through the movement of judgements from existence though reflection and contingent necessity, if they exist. It is accepted that all judgement, hence all evaluation, necessarily appeals to some or other locus of legitimation to ground itself, even if only implicitly. *Legitimating appeals* can be thought of as qualifying reflection in attempts to fix meaning. We therefore examine *what* is appealed to and *how* appeals are made during the movements through reflection into necessity in order to deliver up insights into the constitution of MfT. The empirical field of mathematics teacher education, through the sites we have studied, suggest at least four different domains through which appeals operate: mathematics itself, curriculum knowledge, experiential knowledge and the authority of an adept (Davis et al, 2007).

Identifying the object of acquisition

The first step for observing evaluative judgement in any IP is to identify the object(s) of acquisition. In teacher education practices we have studied, we found there was always a tension between at least two such objects: notions of mathematics (M) and notions of teaching (T). It is expected that this is likely in any teacher education context where there are a number of different knowledge discourses to be acquired[4]. The analytic space for identifying the objects of acquisition of any particular 'evaluative event' is constituted by recognising aspects of M and T, and then considering which is *primary* in the particular event, marking it M or T, and which object is secondary (in the background/ assumed as known), marking it m or t. The possibilities constituted by the tension between M and T are indicated in Fig 1[5].

	M	**m**
T	MT	mT
t	Mt	mt

Fig 1: Analytic space for recognising objects of acquisition in MTE (mathematics teacher education).

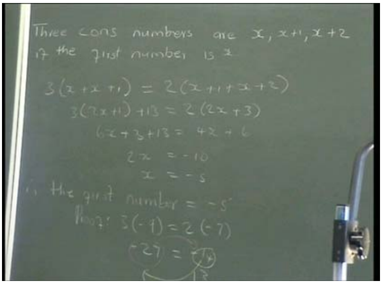
Recognising an event and its sub-events

The beginning of an evaluative event can be recognised in terms of the announcement of the existence of an object in its immediacy, and the end (or sometimes a pause in the judgement of evaluation and a move into a new evaluative event) by the announcement of a *new* object to be acquired. In general the evaluative event may proceed over a long period of time and can be considered as made up of a number of *sub-events,* each one contributing to the primary object and moving through the moments of judgement.

Our example comes from a curriculum or mathematics methods class focused on solving and evaluating word problems. Over the duration of 1 ½ hours, five main events were recognised. The first event was a continuation from the previous lecture, and in itself represents a sub-event. Since there was no access to that interaction, it is named here as Event 1. Event 1 moves through five sub-events, Event 1.1, 1.2 … and so on. In each case, while the overarching object of acquisition remains the same (in this specific instance, an orientation to solving and evaluating 'word problems' algebraically through 'unpacking' a specific example), the interaction moves through a number of different specific objects which together assist with the move towards necessity in relation to the main object. A brief account follows:

Event 1.1: There is an announcement of existence through projecting a specific word problem from the OHP and requesting students' solutions for it (the problem had clearly been given to students in the previous lecture and they were expected to come to the class with the solution). A number of solutions are produced and students must consider which, if any are correct and why this is so.

Event 1.2 The lecturer selects one student to explain their full working of the problem. This is written on the chalk board and considered by the whole class. In the first focus the class is asked to evaluate the *algebraic correctness* of the first part of the solution (the representation of the consecutive numbers).



Event 1.3 Attention is still on the solution written on the board, but the focus moves to the process of *translation from a 'word problem' to a symbolic representation* of the problem. Focus is on the meaning of the first line of the solution and the *translation from words to algebraic expressions* that will later be used to formulate an equation.

Event 1.4 The focus is still on the solution written on the board and the process of *translation from a 'word problem' to a symbolic representation* of the problem, but in this sub-event the focus moves to the *formulation of an equation* that can be used to solve the problem, and on evaluating *whether this represents the problem accurately or not*. The student's formulation is discussed and negated and replaced by other formulations which are judged as correct.

Event 1.5 The focus moves to *different (equally correct) representations* of the problem and at this point the problem is solved and the meaning of the translation and meaning is contingently fixed for this particular problem.

The 'notion' conveyed includes: doing such word problems is a process that involves translation from words to symbols and doing this successfully depends on carrying mathematical meaning from the words into the symbolic representation; there are different (equally correct) ways in which the meaning can be expressed; all correct ways of expressing this meaning will result in the same correct solution; the grounds for making decisions and legitimating a text (in this case a particular expression) as correct are to be found in the mathematical meaning itself. All legitimating appeals are to the domain of mathematics.

This is accomplished within a discussion based pedagogic practice, where discussion moves between lecturer directed whole class discussion (where students can and do punctuate the flow), as well as small group discussion amongst students. Space restrictions here require that we simply assert this general pattern of interaction, and point to the weak internal framing with respect to social relations and pedagogy. The internal framing with respect to sequencing is also weakened, for example, after E1.4 the lecturer signalled the move to E2, however a student interjected and took the focus back to E1, which enabled the movement to E1.5 and the fixing of an instance of contingent necessity. In contrast, the lecturer clearly controls the selection of tasks and student responses that are used. Selection of contents framing is relatively strong.

Generating and presenting data

The methodology enables a number of analyses which are presented here in summary tables. Firstly it enables recognition of the movement of events across the duration of the lecture, in particular the changing patterns in the primary object of acquisition through different sub-events. In Table 1 we see the movement in this case is from Mt → Tm → Mt → Tm → Tm →Tm → Mt. The table reflects further, recognition of the pedagogic resources used in the interaction and the movement in evaluative judgements across time.

Primary/ secondary object	Sub-events	Duration of sub-events	Main resource used	Movement in evaluative judgements	Comment
Mt	E 1.1 - E 1.4	30 min	Student solution to word problem	R	all legitimating appeals made to M
Tm	E 2.1	1 min	L exposition – reflecting on E 1	E	appeal made to authority of L
Mt	E1.5	2 min	L exposition; L pulls together discussion (E1.1 to E.1.5)	N	meaning contingently fixed; criteria summarised by lecturer
Tm	E 2.2	2 min	L exposition – reflecting on E 1	E	major appeal made to authority of L
Tm	E 3.1	4 min	Student's method for checking word problem	R begins	L sets up independent homework
Tm	E 4.1 - E4.3	22 min	students formulation of word problem	R	major appeals to experience or authority of L
Mt	E 5.1	30 min	L presents set of word problems to be solved	R begins	appeals to M. L sets up independent homework

Table 1: Movements in the object of acquisition across the IP

Secondly it enables an analysis of the movement through different forms of interaction within the pedagogic context, as summarised in Table 2. Across the duration of events there is a spread of different forms of interactions, so opening up different discursive spaces for pedagogizing knowledge.

	Whole class discussio	Small group	Individua l work	Lecturer expositio	Lecturer question and	Student presentati	Lecturer questioni	Student questioni
Mathematics	4	2	2	1	0	3	5	0
Proportion sub-events (N=6)	66.7%	33.3 %	33.3 %	16.7 %	0%	50%	83%	0%
Teaching	3	0	0	3	0	4	4	0
Proportion sub-events (N=7)	42.8%	0%	0%	42.8 %	0%	57%	57%	0%

Table 2: Forms of pedagogic interaction across events in our example

Thirdly it enabled an analysis of the distribution of the legitimating appeals (grounds through which a specific text is legitimated or negated within the IP) across all events/sub-events of the lecture. The summary is presented in Table 3.

	maths	Maths education	Metaphorical/everyday knowledge	Experience of lecturer/student teacher	curriculum	Authority of the lecturer
Mathematics	6	0	0	0	0	0
Proportion of appeals (N=6)	100 %	0%	0%	0%	0%	0%
Teaching	1	0	0	3	1	3
Proportion of appeals (N=7)	14.3 %	0%	0%	42.8%	14.3 %	42.8%

Table 3: Summary of legitimating appeals across the IP example

The production of the three summary tables presented here will be discussed in more detail in the presentation. Here we reflect on this data and discuss some of the insights made possible through our methodology.

DISCUSSION

An analysis we have summarised enables a deeper understanding of how evaluation operated within this pedagogic context to open up access to the (re)production of privileged texts, in this instance *MfT*. The analysis illustrates how a selection of school mathematics content is used as a means for developing a discussion based-approach approach to teaching school mathematics. This practice provides for access to multiple texts: for *relearning selections from school mathematics*, to *model an approach to teaching* these selections, and *to provide a pedagogic space for developing mathematics education/teaching knowledge to enable practical realisations of this teaching approach*. A pedagogic space, primarily grounded in experience, is created for reflecting on school mathematics learning and developing an approach to teaching and learning mathematics. A key feature of this practice is seen in the *variation* of types of classroom interaction, and the possibility for different voices to be heard and evaluated. This is specifically enabled through the questioning that the lecturer uses to guide whole class discussions, small group work, and to interrogate the thinking involved in students' solutions.

What is most illuminating within this context is the clear difference between pedagogic judgements in evaluating acquisition of mathematical texts in contrast to teaching texts. In this particular slice into IP, we see that when M is the primary object/ text/ notion, it is rooted within the field of mathematics itself. The authorising field for judging mathematical products is the discursive field of mathematics. Students were to voice their ideas, but they were expected to justify their positions with reference to convincing mathematical arguments. A *selection of* students' (incorrect) work was a major resource for enabling reflection on what was to be acquired and for movement towards necessity through negation. The movement through the various evaluative events suggests a wider pattern, where lecturer sets up work for students to do, and then in the next lecture uses their productions to enable access to the evaluative criteria for judging legitimate productions.

When T is the primary object of acquisition, the practice *looks* similar, in that it follows the same patterns of interaction (putting up students' productions to examine possibilities of legitimate (re)production). The major grounds for evaluating any particular possibility is, in contrast, based in *experience*; the experiences of the student teachers themselves and of the lecturer. While particular realisations were put up for evaluation, the grounds for pedagogic judgement were diffuse and few discursive resources were available from the field of mathematics education or education more generally to ground judgement. The recognition and realisation rules for the T objects that were the focus of acquisition remained implicit. Indeed, while implicit, it is the learning/teaching practice itself that serves as a *model* for 'best' practice. Aspects of this model (of how to teach) do appear to surface as primary objects on occasions (for example in E2 above), however the possibility of its acquisition is structured in the form of reflection on the practice that has been modelled, and the evaluative rules for the practice remain implicit. Discursive resources from the field of ME, while appearing to structure the practice itself, are not made explicit or used to reflexively interrogate the model.

What then of ‘how’? As briefly mentioned, where M is the primary object, framing is relatively weak in terms of sequence and pacing – but overall selection and evaluative criteria are controlled quite firmly by lecturer. For example, while what gets to be put up on the board is generated by the class discussion and the productions that students come up with, the lecturer selects this from what has been generated (she does not for example, ask for a volunteer). The lecturer steers the conversation through the way in which evaluation takes place in the classroom, and in such a way that there is no question about what is a legitimate. What is incorrect is clearly negated and replaced by a correct production. Criteria are fairly strongly framed; grounding is clear, and to be authorised from within mathematics itself. However, the pacing and the sequencing is relatively weak with students input being a critical factor in how the focus moves from event to event. Social relations are flattened and students and the lecturer interact within the context as knowledgeable participants. An invisible pedagogy operates in which it appears that the students’ have considerable control, however, the context is closely managed by the lecturer. When T is in focus, a similar pedagogy is implemented. However the framing of evaluative rule now appears to be weak: the grounding is not firm; things may or may not be accepted and the grounds upon which this is to be decided are not clear. The knowledge base for acquisition is opaque, but at the same time is taken for granted. Experience belongs to all and all experience is valid.

The practice fits the description of a competence-based pedagogic modality described by Bernstein. With respect to the M objects of acquisition it appears to be informed by a constructivist pedagogy that focuses on students building knowledge of school mathematics through negation of their sensible ideas and thus creating the possibility for the acquisition of a specialised mathematical voice. On the other hand with respect to T, acquisition is structured through a form of constructivism in which the evaluative rules are implicit and knowledge is localised and based on experience. Thus the possibility is that knowledge and practices related to mathematics education/mathematics teaching will be differentially distributed across different groups. The substantial discursive resources that appear to be structuring the pedagogic interaction, and so the ‘modelling’ of teaching, remain implicit and so not available for developing reflexive competence in mathematics teaching practice.

Other cases of mathematics teacher education (in Davis et al, 2007) yield similar insights: of a complex interaction between mathematics and teaching as objects of acquisition, how teaching is modelled in each case, and the discursive resources made available for reflection. Our analysis and so methodology suggests that competence models might well be ubiquitous in form across ranging pedagogic contexts; but differ substantively with respect to how consciousness might be specialised.

RESEARCH FORUM 5

The enactivist theory of cognition and mathematics education research: Issues of the past, current questions and future directions

RF05: THE ENACTIVIST THEORY OF COGNITION AND MATHEMATICS EDUCATION RESEARCH: ISSUES OF THE PAST, CURRENT QUESTIONS AND FUTURE DIRECTIONS

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INTRODUCTION TO THE FORUM

A number of intentions triggered this research forum on enactivism and mathematics education research, and those are significant to highlight as they have in return structured the content and form that this forum takes. First, there has been and continues to be a substantial amount of research and writing on issues of enactivism undertaken by mathematics education researchers; thus we wanted to highlight and synthesize this body of research. At the same time, although much research has been conducted within the enactivist perspective, many of those contributions, and their authors, are not always well known and have often been seen merely as "interesting" orientations or "alternative" perspectives – but clearly not mainstream. Because we believe enactivism offers an insightful orientation which shows promise for enhancing our understanding of mathematics teaching and learning, we wanted to bring forth the nature and wide spectrum of enactivist contributions in order to share and create dialogue with the PME community about significant issues raised through this orientation. A third intention is in reaction to what might be thought of as a hegemony of constructivism in the mathematics education literature. We believe that enactivism, as a theory of cognition, offers a more encompassing and enlightening perspective on learning, teaching, and epistemology.

Therefore, the following concerns will orient and be continuously present in the research forum unfoldings: retrospectives (as well as perspectives and prospectives) on research studies and writing done on enactivism in mathematics education will be shared; contributors will focus on insightful features that enactivism offers us; particularities of enactivism as a theory of cognition will permeate all discussions and presentations; and finally, but not least, interactions and discussions will take place about the ideas put forward.

Recurring theme through the research forum: Natural drift

Implicitly or explicitly, Maturana and Varela's (1992) concept of *natural drift*, including notions of *structural coupling* and *structural determinism*, represents an overarching theme discussed in the forum. For a number of us, these notions represent some of the most significant contributions of enactivism as a theory of cognition – we summarize these below, slightly adapted from Proulx's (2008a) PME contribution.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 249-278. Thessaloniki, Greece: PME.

Maturana and Varela's theory is grounded in biological and evolutionary perspectives on human knowledge and processes of meaning making, closely rooted in Darwin's theory of evolution, albeit extending from it on some issues (see Maturana & Varela, 1992, pp. 93-117; Maturana & Mpodozis, 1999). Darwin used the concept of "fitting" to make sense of the process of survival of species. Hence, for species to survive, it must continuously adapt to its environment, to *fit* within it. If not, it would perish. The concept of fitting is, however, not a static one in which the environment stays the same and only the species evolve and continue to adapt. Darwin explained that species and environment co-evolve, and Maturana and Varela added that they *co-adapt* to each other, meaning that each influences the other in the course of evolution. This idea of co-evolution/co-adaptation is key in regard to the origin of changes or adaptations of the species to its environment. Maturana and Varela call this *structural coupling*, as both environment and organism interact with one another and experience a mutual history of evolutionary changes and transformations. Both undergo changes in their structure in the process of evolution, which makes them "adapted" and compatible with each other.

> Every ontogeny occurs within an environment […] it will become clear to us that the interactions (as long as they are recurrent) between [organism] and environment will consist of reciprocal perturbations. […] The results will be a history of mutual congruent structural changes as long as the [organism] and its containing environment do not disintegrate: there will be a *structural coupling*. (1992, p. 75, emphasis in the original)

Here, the environment does not act as a selector, but mainly as a "trigger" for the species to evolve, as much as species act as "triggers" for the environment to evolve in return. Maturana and Varela explain that events and changes are occasioned by the environment, but they are determined by the species' structure.

> Therefore, we have used the expression "to trigger" an effect. In this way we refer to the fact that the changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system*. The same holds true for the environment: the living being is a source of perturbations and not of instructions. (1992, p. 96, emphasis in the original)

Maturana and Varela call this phenomenon *structural determinism*, meaning that it is the structure of the organism that allows for changes to occur. These changes are "triggered" by the interaction of the organism with its environment. They give this example: A car that hits a tree will be destroyed, whereas this would not happen to an army tank. The changes do not reside inside of the "trigger" (inside the tree), rather they come about from the organism interacting with the "trigger." The "triggers" from the environment are essential but do not determine the changes. In short: changes in the organism are dependent on, but not determined by, the environment.

Thus, if one uses structural determinism to make sense of the learning process, one understands that the response of the learner is dependent on the environment he/she is put in, *but* is determined by the learner's own way of making sense and interpreting. Thus, the response to a stimulus is not in the stimulus *per se* but is in the person that responds to it. And in return, the teaching/learning situation is seen in a structural coupling sense, where a history of adapted meanings is being developed and to which both teacher and student are adapting and transforming, while keeping adapted.

The various contributions to the forum

In addition to addressing these issues, implicitly or explicitly, each contribution from groups of researchers will tackle issues that are at the heart of enactivism in mathematics education. To do this we intend to offer brief historical accounts of issues tackled by the earliest researchers to have used this framework in mathematics education, followed by discussions of current questions and themes addressed by contemporary researchers using enactivism, and ending with new directions and orientations for mathematics education research. In particular, four themes are addressed: issues of perceptually guided action and embodied cognition, a familiar theme at PME, will be tackled through its enactivism interpretation by Ibrahim-Didi, McGarvey, Namukasa and Thom; Glanfield, Martin, Murphy and Towers will address issues of emergence, co-emergence and identity; issues of ethics will be discussed by Begg, Dawson, Mgombelo and Simmt; and Brown, Coles, Lozano and Reid will tackle issues of methodology. In addition, there will be a retrospective discussion about enactivism and mathematics education by Sawada and Kieren and a prospective presentation about new orientations by Proulx.

The form of the forum: A truthful theoretical positioning

This outline represents what we intend to address through the forum. But, in order to stay true to enactivist principles, we see these possibilities as starting points and we believe the forum, through the presentations and various discussions, will be laying down its path while walking (Varela, 1987). In that sense, our orientation toward structural coupling and structural determinism are not only overarching themes for the forum, but also structure the way in which the forum unfolds. Our hope is that through this forum we are able to expand the space of the possible (Davis, 2004).

Enactivism: A community's story

Tom Kieren and Daiyo Sawada are the two key scholars who brought enactivist theory to the mathematics education community at the University of Alberta – from which many of the contributors to this Research Forum have their roots. They trace their interest back to 1982 when they were introduced to the work of biologists Maturana and Varela. At that time they were struck by the coherence of this theory of how cognition comes about without reference to the representation of external reality. Inspired by the systems approach for understanding cognition, Sawada and Kieren rethought their work in the context of "living systems as a basis of educational research." For Sawada, the Maturana and Varela theory of living systems implied that enactivism in (mathematics) education was just another name for the process of "bringing forth" in the praxis of education. For him, it meant that in any pedagogical situation, the ontological (the "where" and the "who") was as salient as the epistemological (the "what" and the "how"). For Kieren, this means studying mathematics knowing of persons through the lens of the inter-action of and the co-emergence of a person(s) and the otherness which includes others.

Through the display of excerpts from a video interview with both Kieren and Sawada, and with the other themes addressed, we attempt with this research forum to pay tribute to this genesis and look forward to gathering an international community of scholars who share an interest in enactivism as a theory of mathematics knowing.

PERCEPTUALLY GUIDED ACTION: INVOKING KNOWING AS ENACTION

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Khadeeja Ibrahim-Didi, Emirates College for Advanced Education, Emirates
Lynn M. McGarvey, University of Alberta, Canada

Contemporary studies in cognitive science and mathematics education reveal knowing to be a dynamic, contextually contingent, and body-centered phenomenon. This view contrasts with other perspectives that regard the relation between knowing and the material and bodily world as one of abstraction. Grounded in our embodied experiences, knowing arises from recurrent patterns of perceptually guided activity.

INTRODUCTION

In this presentation, we explore four exemplars to reveal how the relation between knowing and bodily experiences is redefined when we take actions to be perceptually guided. Section one examines a four year old boy's reasoning as it emerges through action and suggests how teachers might attend to and support reasoning of young children. Section two draws from a grade two student's physical exploration of two material objects to question the idea of knowledge as an abstract entity independent of the bodily experiences in which it arises and to reveal knowing as well as the evolution of it as necessarily bodily in nature. Section three involves a small group of secondary school students interacting with each other and with concrete objects to understand what would otherwise be an abstract concept—repulsion. Section four is about a pair of students engaging in talking and listening, using materials and writing when working on a mathematical task. This section illustrates how regularities that emerge from preceding actions in turn guide further actions.

Young Children Reasoning in Action

"Most of our everyday mathematical understanding takes place without our being able to explain exactly what we understood and how we understood it (Lakoff & Núñez, 2000, p. 28). Yet, descriptions of mathematical reasoning tend to emphasize discursive aspects such as conjectures, explanations, justification and argumentation. While verbal and written accounts provide audible clues as to how a person reasons, it necessarily underestimates young children's capabilities and inappropriately separates reasoning and action. The following example demonstrates that listening for abstractions neglects reasoning that emerges through action.

Logan (4 years old) built a tower (Figure 1) after being introduced to soft blocks in his preschool class. Over the next six weeks he had three additional opportunities to build with the blocks. Logan created a tall tower (Figure 2) in the last session.

Figure 1: Session one

Figure 2: Session four

Throughout the four sessions the teachers repeatedly asked Logan and the other children to explain how to make stable towers. Logan's oral and gestural responses included, "don't make them so big … make them smaller"; "put [the blocks] on slowly"; "put them flat"; and "hold it [the tower while placing a block on top]". The teachers appeared dissatisfied with Logan's responses and unsuccessfully prompted him to add to his explanations (e.g., "Put them on slowly? Anything else?"). Yet, the evolution of Logan's structures and his explanations that describe stability principles in physical terms are particularly strong demonstrations of his growing understanding of the properties of the blocks and principles of stability (e.g., wider base of support and vertical alignment of blocks). Supporting Logan's mathematical reasoning does not require that he rise above his sensorimotor actions toward abstractions, but to recognize the necessary connection between his reasoning and action.

Geometric Ideas and Conceptions Within Bodily Transactions[1]

If it is our physical bodies that enable the very possibility for us to sense and make sense of the world as we live within it, then it is untenable to conceive knowing as ever being an abstract entity independent of the bodily experiences in which it emerges and evolves (Deleuze & Guattari, 1994; Merleau-Ponty, 1968, 1986). This point is illustrated in the following excerpt from a lesson in which the first author co-taught mathematics to first and second grade children. Here, the emergence of particular geometric ideas and conceptualizations are observed as Owen (grade 2), counted and explored the faces, edges, and vertices of two different sized rectangular prisms with his hands. Importantly, it must be borne in mind that focusing on Owen's hands in no way suggests this to be the only location at which he sensed or made

[1] This research was supported by the Social Sciences and Humanities Research Council of Canada.

sense about the objects. Rather, his hands as one possible *and* observable site offers a glimpse into the sensuous nature of his physical experiences and from which interpretations about his conceptions of the two prisms can be made (see Figure 3).

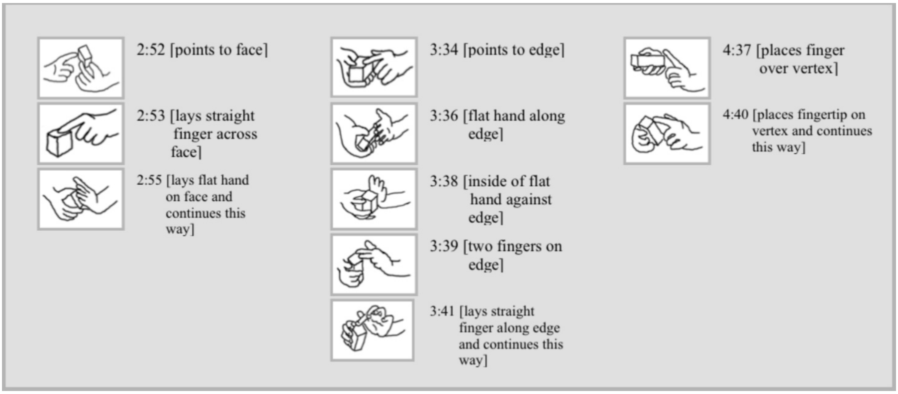


Figure 3: Owen's hand positions as he counted the two prisms' faces, edges, and vertices

On one level, Owen uses his hands to locate and enumerate each of the objects' 6 faces, 12 edges, and 8 vertices. Yet compellingly on another level, it is Owen's recurrent actions of positioning and repositioning his finger(s) or entire hand(s) on the prisms that suggest his re-cognition of the objects. The eventual geometric ideas and conceptualizations brought forth by Owen are identified transactionally[2] as: (a) his flat hand pressing against the prisms' flat, two-dimensional faces, (b) his straight index finger laying along the prisms' straight, one dimensional edges and, (c) his fingertip touching the dimensionless vertices of the prisms. The geometric sense he demonstrates about the rectangular prisms is not knowledge drawn or abstracted from his bodily experiences but knowing that emerges and evolves *as* it is continuously grounded in and defined by the transactions of his physical body and the material objects.

Reasoning: Beyond the individual

When knowing is perceptually embodied and rational (Lakoff & Johnson, 1999) how might the rationality of a complexly configured learner be understood?

Mariya, Layla, Muneera and Aisha were considering *why* a looped strand of hair placed on a frictionally charged Styrofoam plate moves in response to a finger that is placed near it. In the midst of the conversation that focussed primarily on the attraction of electrons in the thread toward the finger, Mariya covered the thread with her fingers and pushed it down against the plate and released the thread as if to

[2] This term is used to highlight the irreducibility of Owen's physical actions and the rectangular prisms he manipulates.

observe the effect. She paused and repeated the action while Layla, Muneera and Aisha looked on.

59. Laila: [Ey! Remember there is something that *repel*s. Hmmm?]

60. Muneera: [Hmm ... And you think you can offer some exceptionally great ideas?]

61. Layla: [That ... No. Now *look*. When I do this (moving finger near the thread) *maybe* there might be negative charges here. It might be that it is also negative ... *so it might even be repelling because of that*].

62 Mariya: [Yes (tentative)]

Figure 4: Reasoning about charge

How do actions that direct trajectories of interactions influence the ability of a group to imagine *cause* and *effect*? Can this interaction be considered *rational*? The collectively entraining interactions—while not rational in the psychological sense—may present an organic embodied sense of rationality that goes beyond that which is considered individual and mental (Gigerenzer, 2006). The acknowledgement of Mariya's action and developing significance of Layla's use of the term saw a collective, *heuristic* reaching out (Varela, 1992) into the space of possible reason. Understanding the cause for the movement of the thread developed in a particular direction, a *leaning-toward* by the emerging learning system, promoted by the collectively embodied significance of perceptually available actions as *plausible* reasons.

Actions and Interactions Guiding Attention: A Succession of Episodes

To Bateson (1980), from our actions and interactions emerge regularities. This was certainly the case with a pair of middle school students, Irene and Lillian, during a number theory activity. They were finding numbers that are formed by adding consecutive numbers (i.e., 9 = 2 + 3 + 4). The girls' initial work involved listing numbers to 10, checking each of the numbers using domino materials and crossing out numbers such as 2, 4 and 7 that could not be arranged using consecutive addends (Episode A, Figure 5). They examined the sequence of numbers not crossed out. In their words, "there was not much of a pattern". It is then that Irene interjected:

80 Irene: Why don't we list down the numbers in a pattern?

81 Lillian: Yes [Lillian replies after looking at Irene for a while. Irene turns to write—work in Episode B as Lillian watches. When it comes to writing 11, Irene

pauses and without saying anything she and Lillian look in quest at 11.]

82 Irene: **No it can't, so let's just go on with the list**" [*Irene writes down 12 as she says*] **12** [*They loudly count together*] 1 plus 2 plus 3…

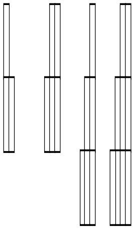
USING DOMINOES	EPISODE A (Lillian writes)	EPISODE B (Irene writes)
	1- 2-- 1 + 1 3-- 1 + 2 4-- 2 + 2 5-- 2 + 3 6-- 1 + 2 + 3 ~~7-~~ 8-- 2 + 3 9-- 2 + 3 + 4 10- 1 + 2 + 3 + 4 3^{+3}, 5^{+1}, 6^{+1}, 9^{+1}, 10^{+1}, 11 …	3 = 1 + 2 5 = 2 + 3 6 = 1 + 2 + 3 9 = 2 + 3 + 4 10 = 1 + 2 + 3 + 4 11 = 5 + 6 12 = 3 + 4 + 5 ~~14~~15 = 1 + 2 + 3 + 4 + 5 14 = 2 + 3 + 4 + 5 16

Figure 5: Irene and Lillian's talking, use of material and written work

"Bring the dominoes," Lillian requested as they were examining the table in Episode B (turn 94 of the transcript). The girls *folded back* to work with the dominoes, repeating the same arrangements done earlier. Pirie and Kieren (1994) observe that students usually *fold back* to inner, earlier levels of understanding when faced with difficulty at an outer, latter level. While examining the geometry of the sums, they noticed a regularity: "Wait. This one the one is there, this one the one is gone, the next one the …" (Irene, turn 102). Beyond attending to the four arrangements as a group and to the shapes of the arrangements, they began attending to the initial number in each of the strings of addends—some begin with one, others with two and so on. It appears the talking and writing done during Episode B enabled the girls to attend to a key aspect in Episode C and D—selecting an initial term generates a sequence of numbers that have the consecutive terms property. Thus, "almost every [number] … apart from one… two …four" (turn 288-290) satisfied the property. Talking, using materials and writing had guided and ordered how they perceived in successive episodes.

CONCLUSION

Through four exemplars we revealed how embodied cognition is demonstrated through an integration of many modalities including sensing, perceiving, acting and observing. The abstract in a mathematical domain is an aggregate of readiness-for action. Higher cognitive structures are not separate from or superior to embodied experiences but instead, emerge from recurrent patterns of perceptually guided actions.

CO-EMERGENCE AND COLLECTIVE MATHEMATICAL KNOWING

Florence Glanfield, University of Alberta, Canada
Lyndon Martin, University of East Anglia, UK
M. Shaun Murphy, University of Saskatchewas, Canada
Jo Towers, University of Calgary, Canada

This sub-theme gathers together the expertise of two research teams – Towers & Martin, and Glanfield & Murphy. In our respective dyads we have been investigating the processes of learners (both teachers and students of mathematics) as they interact to generate mathematical and/or pedagogical knowing. Our analytical frameworks differ, as we will show in this piece of writing, and yet each is rooted in, and continues to draw sustenance from, enactivist thought. In this piece we explore, in particular, the way in which enactivism considers the relationship between individual and collective cognition and how this relationship prompts co-emergent phenomena.

To begin, it might be helpful to explore enactivist perspectives on thinking. Cartesian thinking has emphasised the ideal of a modern self as solitary, coherent and independent of context. Such thinking has then positioned this radical subject as the reference point for what is known (and worth knowing). The ideal knower, in this frame, has been the autonomous individual. Conversely, enactivist thought prompts a reorientation to the collective body, both in terms of what is known and of who is doing the knowing. Starting from Darwin's evolutionary metaphors, enactivist thought focuses on the "dynamic interdependence of individual and environment, of knowledge and identity, and of self and other, rather than on their autonomous constitution" (Davis, 1996, p. 8). It prompts attention to the structural dynamics of knowing and the co-emergence of knower/known. It is the notion of co-emergence, and how in our work we seek to characterise it, that we elaborate in this paper.

The starting point for our two research programmes has been a severing of this attachment to the individual. In the case of Martin and Towers, this severing, while initially being simply philosophically attractive, became theoretically compelling as we struggled to trace individual threads of mathematical thought and expression in particular data examples. At the time, we were using the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994) to trace the mathematical thinking and actions of individual students working together on a piece of mathematics. However, one or two troublesome pieces of data began to gnaw away at us. In these extracts, try as we might we could not tease apart the actions and dialogue to create separate, coherent, individual traces of understanding. It was as though the mathematical understanding in these episodes was emerging not at (or not only at) the level of the individual but, significantly, at the level of the group. In taking seriously the enactivist prompt to loosen our hold on the assumption of individual streams of thought and action, we were drawn to consider the troublesome interactions not as interacting individual monologues but as a single voice. This emergence of the possibility of group mind was a liberating (though not instantaneous) recognition, and one that would not have been possible for us without the orientation engendered by an immersion in enactivist thinking and literature and in particular by Maturana and Varela's work on what has become known as the Santiago theory of cognition. A key idea here is that cognition—mind—is a process, not an object. Hence, the possibility of group mind as a process occasions new ways to think about interacting bodies. Interactions (between students, let's say) consist of *reciprocal perturbations* (Maturana & Varela, 1992, p. 75). Such perturbations trigger mutual changes in the system and a history of recurrent interaction leads to structural congruence or coupling between the participants. This system is emergent—its properties emerge at a certain level of complexity that may not be present previously. Much of our work is concerned with investigating this complexity—trying to understand the characteristics of, and conditions for, collective (rather than individual) mathematical understanding.

In a recent paper (Martin, Towers, & Pirie, 2006) we offered the beginnings of a theoretical framework that focused on the collective mathematical activity of learners collaborating in small groups. We suggested that by using the lens of improvisational theory it was possible to observe and account for acts of mathematical understanding that could not simply be located in the minds or actions of any one individual, but instead emerged from the *interplay* of the ideas of individuals, as these became woven together in shared action, as in an improvisational performance. Improvisation is broadly defined as a process "of spontaneous action, interaction and communication" (Gordon-Calvert, 2001, p. 87). It is a collaborative practice of acting, interacting and reacting, of making and creating, in the moment, without script or prescription, and in response to the stimulus of one's context and environment. Improvisational theory has an established history as a mechanism for interpreting group process and we have found it particularly valuable in interpreting the growth of collective mathematical understanding as a process. Its constructs are consistent with enactivist principles, as we will endeavour to show here and in our presentation, however it enables a more fine-grained analysis and interpretation of group processes than the current literature on enactivism has offered. In that sense, we see our invoking, and adaptation, of improvisational theory within the domain of mathematics education as a coherent extension of the enactivist literature.

Enactivism, then, has prompted us to pay attention to the relationship *between* things in a mathematical environment (ideas, fragments of dialogue, gestures, silences, diagrams, etc.) rather than to what each of those things might mean or represent in their own right and for the individual generating them. This idea has manifested itself in our own work through the developing and elaboration of the theoretical notion of improvisational coactions, a way of viewing collective action that focuses on and describes the process through which structural coupling might occur. Sawyer (2003) talks of improvisational activity as being conceived of "as a jointly accomplished co-actional process" (p. 38), and for us the use of the term coaction emphasises, in a powerful manner, the notion of acting with the ideas and actions of others in a mutual, joint way. More precisely:

> we use the term coaction as a means to describe a particular kind of mathematical action, one that whilst obviously in execution is still being carried out by an individual, is also dependent and contingent upon the actions of the others in the group. Thus, a coaction is a mathematical action that can only be meaningfully interpreted in light of, and with careful reference to, the interdependent actions of the others in the group. (Martin, Towers, & Pirie 2006, p. 156)

We propose that improvisational coaction is a particular way of working with the ideas offered by others, with a specific set of identifiable characteristics (drawn from improvisational theory), through which the co-construction of a shared conceptual structure (Teasley & Roschelle, 1993) can occur, and from which collective mathematical understanding can flow. Significantly, the notion of improvisational coaction places as much responsibility on those who are positioned to respond to an action as on the originator. Indeed, it is what others make of the offered action that determines whether it facilitates a continued flow of mathematical actions or merely exists as some kind of isolated, static offering. Improvisational coacting is a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built upon, developed, reworked and elaborated by others, and thus emerge as shared understandings for and across the group, rather than remaining located within any one individual. We therefore see improvisational coaction as being a specific kind of interaction, but whereas interaction allows for reciprocal, complementary collaboration, without the requirement to be mutually building on the just-offered action, improvisational coaction goes beyond this and requires mutual, joint action. The notion of joint action and shared knowing is our starting place to explore how this set of ideas has emerged simultaneously in our colleagues' work with practicing teachers.

In the research of Glanfield and Murphy alongside teachers, we came to understand how teachers described their identities in relationship to the children and colleagues with whom they worked. Our contribution to this larger work, is in how we narratively inquired into teachers' mathematical knowing as enacted in their narratives of practice. As teachers talked and wrote, their self knowledge referred repeatedly to the knowledge they constructed and held in community. Furthermore, this knowledge was then enacted in mathematical curriculum making (Clandinin & Connelly, 1992) in classrooms with children.

An important aspect of the teachers' unfolding mathematical knowledge and practice was how their histories shaped present practice. Jane, a teacher who participated in our research, often talked about her experiences as a child as she described her current work with kindergarten children. Jane's historical descriptions foregrounded memories of isolation, not knowing, and tension in mathematics as she recalled her childhood in school. Outside of school she had different memories of mathematics, memories that showed relationship, embodied experiences, and a sense of knowing.

When she began to see her self as a competent mathematics teacher and knower after twenty years of practice she referred to the people whose ideas and support helped her take up this identity.

Kieren posed enactivism as a view of mathematical knowing which "observes knowing as occurring in the *inter-action* of the individual and the world which (s)he is shaping and in which (s)he is acting" (1995, p. 16) and therefore "how each [teacher] thinks or acts with respect to a mathematical situation is fully determined by his/her structure and his/her lived history of mathematical actions in relevant situations" (p. 17). Teachers are both individuals and a part of the environment in which they find themselves acting and inter-acting, in present contexts shaped by past experience. Through the occasions of discussing and thinking about mathematical knowing with us and colleagues in the research group, and by referencing experiences with their students, and in their own reflections the teachers continually brought forth an embodied mathematics educator identity.

Developing understanding then, from an enactivist perspective, is "an ongoing interpretation which cannot be adequately captured as a set of rules and assumptions since it is a matter of action and history, an understanding picked up by imitation and by becoming a member of an understanding which is already there" (Varela, 1992, p. 252). The actions of the teachers were shaped by the environment, people, and also "determined by the [teachers'] own lived histories of actions" (Kieren, 1995, p. 10). In other words, the teachers' knowing could be understood in relation to people, places, and time, narrative commonplaces (Connelly & Clandinin, 2006) that helped us as researchers shape an understanding of the teachers' experiences. As we worked with these teachers over time, and this work occurred away from their classrooms, a research community was developed and sustained by all members. This community contributed a further layer of people knowing in relationship over time. For example, Jane became more certain in her knowing, and willingness to express it, as she began to see herself as a mathematical knower in relationship with the other teachers and us as researchers. Her mathematical identity hence co-emerged with that of others.

The research group, or mathematical community, brought forth a world of knowing at the same time that each individual was bringing forth a world of knowing. Pirie and Kieren (1994) described individual understanding as a dynamical process, one that can be observed in action and inter-action. Kieren and Simmt (2002) suggested that collective understanding is also dynamic and can be observed in the patterns of actions and interactions in the group. Individual narratives find meaning in their expression, but also in the larger narrative being shaped by the community.

For Jane, and the other teachers in the research group, their individual narratives often arose in response to the narratives of others. As we began to encourage a more reflective stance the teachers began to talk about how their mathematical teaching practice was shaped in a relational space. Teaching and learning are complex phenomena because they involve human beings. Individually and in community, teachers are dynamic, unpredictable and inter-related (Davis & Sumara, 1997). Connelly and Clandinin (2006) wrote of narrative, the study of ways in which humans experience the world, as both phenomenon and method; that is, experience is a narrative phenomenon that can best be understood by inquiring narratively into it. Bruner (1986), who wrote of narrative as a mode of knowing separate from paradigmatic knowing, suggested that it is our "sensitivity to narrative" that provides the "major link between our own sense of self and our sense of others in the social world around us" (p. 69).

Enactivist thought, then, has shaped our two research programs by providing not only conceptual structures with which to weave our understandings but also a community of knowing, doing, and being within which to practice, one that is now being extended by our participation in this Research Forum. Our knowing continues to co-emerge as we collaborate on this endeavour, determined by our individual conceptual structures but dependent on our interactions.

STORIES OF MATHEMATICS EDUCATION AND ETHICS: PATHS WE HAVE LAID IN WALKING

Andy Begg, Auckland University of Technology, New Zealand
Sandy Dawson, University of Hawaii, United States
Joyce Mgombelo, Brock University, Canada
Elaine Simmt, University of Alberta, Canada

We explore the ethical implications of an enactivist perspective. We acknowledge that in teaching and research situations we are part of complex living systems in which knowers, knowing and the knowable co-emerge; of particular importance are the relationships between these parts. We assert that the essence of these relationships is ethics. By ethics we do not mean ethical knowledge or principles as described in codes of conduct, but ethics at a deeper personal and social level that might be termed ethical action. Our paper presents a set of stories that illustrate questions of ethics and implications for ethics that emerge for us as we engage with others in the context of our work as mathematics educators and educational researchers.

LAYING DOWN A PATH WHILE WALKING IN THE DARK: BEGG

My path relates to *thinking in schools*. Why? Because thinking is a key competency (or aim) in curriculum documents, yet little is done about it. I'm travelling this path with an enactivist perspective in which I assume that body/brain/mind are inseparable and that humans are complex emerging living systems nested within other similar systems. When considering thinking I began with the cornerstones of philosophy—*being* (ontology), *knowing* (epistemology), and *doing* (ethical action), and wondered, are these inseparable aspects of the process of *living*? And, I wondered, is *knowing* (the process, as distinct from knowledge or what is known) the same as *thinking*?

For me being, knowing and doing are inseparable complex emerging processes, knowing is the process of thinking that occurs at many levels from conscious to subconscious, and thinking is all brain processing whether it occurs consciously or not. Fox (2008) cited the work of a number of researchers and reported that the brain is actually busier when not focussing on specific tasks that require concentration. This subconscious activity explains how children learn words and language rules, and numerous aspects of their worldviews through imitation before they start in formal education; and for me this activity provides a possible explanation of how ethical decisions are be made without conscious awareness of the process or the product. One can appreciate this from the following example:

> … you see an elderly woman, and she drops a coin on the ground, your own attitude makes you bend down and pick it up for her. But what do you do when it is a young man who drops a coin? (von Foerster & Poerksen, 2002, p. 148).

For me, the inseparability of being, knowing and doing, and the notion that much of our thinking is not conscious, has implications related to ethical action. Whether I see decision making as 'the totality of the conditions for deciding the bestowal of esteem or disdain within the system' (Luhmann, 1995, p. 236), or believe that 'ethics relates … to … the rights of living beings' (Badiou, 2001, p. 4), or simply, as doing the right thing, I want to be more consciousness of such decisions and my thinking that underpins my actions. I see conscious ethical thinking occurring when as researchers we anticipate events when getting ethical approval, and with children when possibilities are discussed before some venture. But generally I see a need to develop a greater awareness of less-than conscious thinking so that, at least with hindsight, I may take more informed ethical actions in the future. I see this occurring in schools that emphasise reflection, and use strategies such as class meetings and thinking circles.

GAINING TRUST AND RESPECT WHILE LAYING THE PATH: DAWSON

> Darkness was falling as the Director of Education led me through the jungle to a secluded open space, lite with coconut husk torches, where the men of his village would practice the standing dance for the upcoming Yap Day festival. Women were forbidden to watch and outsiders were not usually invited to such rehearsals. I was surprised when the Director, after a meeting earlier that day, had invited me to drive out to his village to witness what I found to be a raucous, fun filled, and at times bawdy rendition of a spirited dance replete with sexual innuendo. (Event occurred on Yap, 2001)
>
> I was nervously driving up the mountainside deeper and deeper into the jungle. The mentors and novice teachers had said to me, "Come join us for a drink at the Sakau[3] bar after school." I knew that issuing such an invitation to an outsider was rare and signalled a degree of acceptance. This was the first time I had been asked to join the project's local research colleagues for the Sakau ceremony. (Event occurred on Pohnpei, 2002)

[3] Sakau, which is extracted from the pounded roots of a pepper shrub, contains up to 14 naturally occurring painkillers that, unlike alcohol, sedate the drinker making the ritual of consumption a quietly social act.

My co-presenter (Simmt) notes below that she came "…to understand mathematics knowing in action as the co-emergence of embodied thought, social relationships, and cultural forms and practices." The emerging trust and respect, evident in the vignettes about my experiences on Yap and Pohnpei during the early stages of the Mentor Project (Dawson, 2008b), was the consequence of immersing myself within indigenous cultural forms and practices, benefiting from the social interactions with local informants who invited me into their villages, and the inquisitive, non-judgmental and respectful manner with which project participants were approached.

The Mentor Project story (Dawson, 2008a) describes how collaborating with teachers of mathematics, mathematics specialists, and college level instructors of mathematics in ways that impact them in a positive fashion was a zigzag path highlighted by the gradual development of trust and respect among project participants and staff. Honouring and partaking in local customs and practices was a pre-requisite for entry to and the maintenance of an ever-deepening conversation that took place over the eight-year life span of the project. The Mentor story is about the manifestation of an ethics of caring (Noddings, 1992/2005). Noddings argues that caring is the foundation of ethical decision-making: caring is "…a state of being in relation, characterized by receptivity, relatedness and engrossment". The trust and respect that emerged during the Mentor project was fostered by an ethics of caring that was fundamental to the relationship among the project's various participants, cutting across languages and cultures, and the vast expanse of the western Pacific Ocean.

THE ETHICS OF PEDAGOGICAL RELATION: MGOMBELO

My story revolves around the question of ethics in *pedagogical relation.* In my experience as a teacher and a researcher in mathematics teacher education, many of my insightful moments come from listening to student-teachers' questions right after their experience of teaching during their practicum. How do you deal with students who refuse to engage in meaningful tasks and insist on asking you to provide them with rules and procedures? How do you deal with students who do not do their homework? How do you deal with students who lack basic skills in mathematics? These are just some of the questions that student teachers ask. For me the lesson to be learnt from these questions is less about learning student teachers' struggles or concerns during their practicum than that of implications for my own teaching. Attentive, thus to the questions, I see the questions as pointing to a fundamental question in a pedagogical relation: How do we act towards our students? It seems to me that there is something in a pedagogical relation that presents itself as a call of our students and compels us as teachers (educators) to respond or simply to act. It is this "something" that I address as ethics that exists in space between teacher and student in a pedagogical relation.

In practice, the call of our students which demands our action produces double bind situations where the teacher, student and knowledge are concerned. Herbst (2003, p. 207) expresses this double bind in mathematics education in terms of tensions that teachers have to face when managing task enactment and knowledge development. For example in case of how to represent mathematics objects: On one hand, "a teacher may be compelled to identify precisely which features of the representations used in the task are relevant to the mathematical ideas targeted by the task". And on the other hand, "the teacher may be compelled to maintain a certain degree of vagueness regarding what in those representations is relevant so that students are reserved the opportunity to mathematize, to make deliberate choices instrumental to their inquiry"

How we respond, interpret, act in these double bind situations, depends on our perspective of who we are as teachers or students in pedagogical relation. As von Foerster (1992, p. 10) states succinctly, if I consider myself as "an independent observer who watches the world go by" then "because of my independence I can tell others how to think and to act: 'Thou shalt. . . .,' 'Thou shalt not. . . .' However, if I consider myself "as a participant actor in the drama of mutual interaction, of the give and take in the circularity of human relations" then "because of my interdependence, I can only tell to myself how to think and to act: 'I shall...', 'I shall not...' (von Foerster, 1994). For him, these two perspectives distinguish between moral codes and ethics: the former is the origin of moral codes and the later is the origin of ethics.

THE ETHICS OF MATHEMATICS KNOWING IN ACTION: SIMMT

One of the most profound lessons of my journey as a researcher was coming to understand that when I say, "I know" it is much more than knowing some "thing." It is the profound understanding that we human beings live with the implications of our knowing, and because we live with others, our knowing has implications for them.

"This is Dr Simmt," my collaborating teacher introduced me to the new grade seven class. "She is from the university. As part of her research Dr. Simmt is going to teach you mathematics this year." Looking back I wonder what the youths were thinking about that morning when we first met. Did they really understand what it meant to be part of a research study? Did I understand the implications of researching mathematics knowing in a classroom context?

I approached the study with enactivist assumptions about knowing: our day-to-day knowing is not centred on problem solving but living; it is in living/knowing that we bring forth our lives and the world we live in (Varela, Thompson & Rosch, 1991); the immediacy of perception and action in our moment by moment activity demands immediate coping (Varela, 1999). My intent was to explore the implications of mathematics knowing in a highly inter-active classroom which engaged some of the common practices teachers are being encouraged to use: group activity, concrete manipulative use, multiple representations, student explanations, public display of student "thinking," and so forth—the many practices that are encouraged by "constructivist" views of learning without question about the implications of such practices beyond the student's mathematics. From earlier research (Simmt, 2000) I had come to understand mathematics knowing in action as the co-emergence of embodied thought, social relationships, and cultural forms and practices. Hence I would observe for implications in these domains.

The lessons we learn from our doing/knowing marks us— I thought it was a relatively benign moment. The grade seven students had worked in pairs and triplets to figure out the product 3 x –4. They created posters to publically "show how they knew the product was –12. In a large group discussion we went over the many different explanations and the students' distinct ways of thinking about the problem. With only a minute or two before the dismissal bell was to ring, I quickly scanned the room trying to determine if any of the posters we had not yet got to revealed a different way of thinking than had already been represented. I was ready to ask the students to pack up their books when one of the girls blurted out, "You don't have ours yet." She pointed to work that looked very much like many of the other posters. Three sets of four circles with minus signs in them. "I think this is like the others," I said conscious of the time. "Ours is different though. Three times four negative is negative twelve because the four is bigger." There was a collective gasp in the class. I was confused and asked, "what would 4 x –3 equal." "Positive twelve," she responded confidently. There were a number of her peers quick to correct her. The bell rang and the students rushed to clean up. It was 6 weeks later when I realized she hadn't offered any class contributions since our introduction to integer multiplication.

RESEARCHING THE UNSENSEABLE? - IMPLICATIONS FOR RESEARCH METHODOLOGY IN ENACTIVISM

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Educational research is characterised by involving research on the hidden psychological processes of beings embedded in a complex social process too multi-faceted to grasp. How then do we learn anything about it? We answer from an enactivist perspective that we learn about education in the same way that we learn about everything else, through limited but good-enough iterative processes.

We are involved in different ways in longitudinal research and have worked together in different combinations over more than 10 years. What are the methodological issues that we face as David researches the need to prove, Laurinda works with teacher development, Maria (Lolis) develops the idea of algebraic learning and Alf focuses on teachers working together within a mathematics department in a secondary school? In each case, what we are interested in is not possible to know about through direct perception, so what is it possible to perceive?

The methodological issues, for us, are related to the observer, often ourselves or, when working collaboratively, a range of observers and to accounting for how we see more over time than we did when we began. How do we learn about our interests?

What do we see learning as? A key methodological starting point is that we observe ourselves observing. We have found the works of Maturana and Varela support us in being able to articulate both the centrality of the observer (Maturana, 1988) alongside the way different observers with different experiences see different things; and these multiple views support us in working with complexity rather than feeling the need to come to a single view, a truth. A second shared assumption is the inseparability of perception and action. Varela describes perception as perceptually guided action within his definition of enactivism (Varela, 1999, p. 12). We can see actions and these give some experiences with which to track patterns over time, both within the actions that others make and in our own 'seeings'. The making and refining of distinctions related to our practices as teachers and as researchers is at the heart of what we name learning. Enactivism thus gives us an epistemology that has important implications for our methodological stance (Reid, 1996). These ideas will be illustrated briefly through some descriptions of our work, together and apart.

LOLIS – ALGEBRAIC LEARNING – ACCOUNTING FOR

At the beginning of my PhD I wanted to learn about children's learning of algebra. I started with some questions that made me address my research in particular ways and that arose from my previous experiences with the teaching and learning of mathematics. As I immersed myself in the process of research, the choices I made selected particular features of the context I was exploring. My ideas on algebraic learning were modified as I engaged with the research literature on the learning of algebra, in conversations with people, and in interactions with the participants of my project. I went through a process of continual development and change and, because of my presence and my actions the people I related to also changed. An enactivist approach meant that I needed to be aware of my assumptions while I carried out my investigation on algebraic learning. I tried to do this by being specific about my questions and the criteria that allowed me to distinguish certain events from others.

Following the enactivist approach to cognition, I considered learning as the ongoing structural change that allows individuals or groups to act effectively in a changing environment (Maturana, 1987) and therefore I approached the learning of algebra through the exploration of effective actions in different learning contexts. This allowed me to investigate learning through the observation of students' activities. In the process of research, I had to develop specific ways of characterising effective behaviours in the classrooms I observed. When the criteria for distinction are clearly specified, it is possible for readers to follow discussions about distinctions with little difficulty. I distinguished effective behaviour when I observed conduct which allowed the individual to continue participating in a certain environment. For example, in a given classroom, I found that asking questions was effective for the students, giving rise to dialogue and further interactions. In another classroom, a question asked by a student was considered to be an interruption and therefore was not encouraged by the participants. Behaviour that is effective in a certain environment will normally be promoted, and hence occurs frequently. I characterised actions as effective whenever they occurred frequently in the classroom, and when their presence did not disturb or interrupt what seemed to be the 'normal' flow of events in a lesson.

Different perspectives were taken in the exploration of effective behaviours in the classrooms. I accessed students' activities through lesson observations, interviews and a test. I did not try to write a theory of children's learning of algebra. I tried to explain what I observed in a coherent manner, one that a community of researchers in mathematics education might consider useful and acceptable. Throughout the process of research I talked about my project with other researchers and, ultimately, my work had to meet certain criteria for it to be acceptable, that is, for it to be "good-enough" in a particular context.

ALF – ROLE OF WHOLE CLASS DISCUSSION – ITERATIVE PROCESS

My research interests are in how mathematical classroom cultures develop and the role of the teacher within this, more specifically the role of whole class discussion (particularly the teacher's role) in the development of these cultures and individual learning. In my current project, I take video recordings of lessons and am looking at sections of classroom dialogue where there are significant student contributions ("significant" being an ill-defined notion - students perhaps doing more than giving a short answer to a question).

The limited but good-enough iterative process I go through personally when first looking at classroom data (having selected a shortish 'significant' section and transcribed it) is to ask myself the following questions: why is this being said now? what pattern does it follow? what pattern does it break?

I have adapted these questions from micro-analysis techniques within linguistic ethnography (for example, see Rampton, 2006). "Good enough" feels like an accurate description of why I keep using them. So far, these questions have helped me see more in the data in a way that does not feel like I am bringing a lot of theoretical baggage (in terms of pre-categorisations) to my viewing. The enactivist influence is there in the importance of pattern.

In my own analysis of data, the iteration comes from a repeated cycling between interpretation and more detailed viewing. This analysis itself is part of a larger iterative process, in which I work on video or transcript data with a group of teachers (whose lessons the video recordings have come from). At some point I will share my analyses and get feedback. I also audio record some of these conversations, which in turn become part of my data for analysis (my interest in developing mathematical classroom culture, extends to mathematical department culture - especially, what is the same/different between classroom discussion and these teacher discussions).

I recently went back to reading the article "What the Frog's Eye Tells The Frog's Brain" (Lettvin, Maturana, McCulloch, Pitts, 1988) (Maturana later described this article as marking a watershed in his thinking), and had a new lighting on an old awareness, which I attempt to capture below:

> Lettvin et al. (1988) studied signals sent from the frog's retina to its brain under a variety of laboratory conditions. They found *no* correlation between neural activity from the eye and anything we might naively assume to be the case, such as certain colours, certain intensities of light, certain shapes. The only way they found consistent correlations to regular and predictable patterns of neuron firing was with items that could only be described in a more abstract way, e.g., *a small object (of any shape) entering the visual field from the right with a fast and changing movement against a stable background* (say, 'bug to the right'). So, it is not the case that the frog's eye provides a representation of the world which is processed in its brain, by de-coding the representation, to conclude (not in awareness presumably) 'bug to the right', which then triggers activity. It is as though, instead, the processing has already been done - the patterns of activity coming directly from the frog's eye can only be correlated with something as high level (to an outside observer) as 'bug to the right'. This then triggers a response directly.

There is a directness between perception and action, that human self-consciousness perhaps masks; at least this is the case during periods of effective behaviour. I follow Varela (1999) in believing that self-consciousness only kicks in, for humans, when our behaviour suddenly stops being effective for some reason (for example, I am in the flow of this writing and suddenly become aware I am cold – my self-consciousness is now in play as I decide whether to do anything about this before carrying on writing). I want students to do mathematics in a smooth and effective manner, but, when doing research, it is partly to disrupt the smoothness of the link between perception and action that I engage in the iterative behaviours described above; in order to help me slow down my sense-making of situations, see more detail and hear from other perspectives.

LAURINDA – TEACHER DEVELOPMENT – MIDDLE WAY

How do student teachers learn? When I started working as a teacher educator it became clear that neither discussions about what their images of mathematics were in the group, nor giving advice of the 'try this' kind, something to do in response to a perceived problem supported shifts in practice. Over time, I became aware that what did seem to lead to a range of possible strategies being used were statements in a middle position to the other two extremes. These 'purposes' are different each year, although a common one at the start of the year, with a motivation of needing to prepare a first lesson for a small group of students, is something like 'How will I know what they know?' In researching my own practice as a teacher educator and acting in the role, it is working with these emerging patterns over time that is crucial. The student teachers see the mathematics teaching, whilst I see more in relation to the patterns of their energies in learning.

DAVID – GOOD-ENOUGH UNDERSTANDING

In my work with Vicki Zack I explore grade 5 students' reasoning in relation to proving, and one aspect of our work has been looking at the way that "good enough" understandings allow students to remain engaged in a proving process in spite of not having the whole picture the teacher has (see Zack and Reid 2003, 2004). Students use good-enough understandings as placeholders, to be replaced when or if more complete understandings become available. When confronted by many complex ideas (as when presented with a proof for the first time) they make many tentative, temporary decisions, and keep diverse and sometimes contradictory possibilities 'in the air' waiting at times to the end to make sense of what has transpired. They behave 'as if' their understandings are sufficient, as long as they do not fail in some way (In enactivist terms their understandings "satisfice". Good-enough understanding means 'making do' on occasion, and moving on. Being good enough is not a weakness to be overcome, however. In fact, we contend that being good enough is all we can do.

DAVID AND LAURINDA – AFFECT – THE UNSENSEABLE

Our current collaborative research looks at the ways in which somatic markers (Damasio, 1996) influence teacher decision-making and students' reasoning, and the degree to which those markers can be observed by us, by colleagues, and perhaps by the teachers and students involved. Because somatic markers are a part of unconscious mental activity they cannot be observed by introspective reflection. In fact, the stories we tell after the fact about our decision making are likely to include inventions to account for the influence of somatic markers of which we are not aware. *How then can we research something we cannot observe?* The method of examining decision points in a person's actions seems to hold promise. We can observe changes in behaviour, indicative of unconscious decision making, and consider what markers based on past experience might account for those decisions. Our experiences sharing data at conferences suggests that colleagues see similar events as suggesting the sort of unconscious decision making accounted for by Damasio's hypothesis of somatic markers. This leaves us optimistic that it will be possible in our work to observe the effects of somatic markers in a range of contexts, to distinguish positive and negative somatic markers, and to suggest ways in which they form and evolve in mathematics classrooms.

REFLECTIONS

Our thoughts, methodologically at this point, are related to what we can say about learning that can help us say something about learning about learning; what makes enactivist research enactivist is a continuing iteration that we have illustrated through ideas of "middle", "good-enough", effective behaviour and "the observer". A second distinguishing feature of enactivist research is captured in Varela's distinction between deliberateness and deliberate analysis (1999, p. 32) – the former is a self-consciousness, inexpert way of being; the latter describes how, even as beginners in any field we can learn like experts if, "after acting spontaneously, [we] reconstruct the intelligent awareness" (*ibid*.) that led to the action. In our work with others, and in how we use ourselves, we strive for such spontaneous, effective action, whilst also bringing watchful attention to how we observe. One thing we can all do is categorise and, as researchers, we make certain distinctions; we engage in an iterative process of working with these distinctions to refine them; to learn.

SOME DIRECTIONS AND POSSIBILITIES FOR ENACTIVISM AND MATHEMATICS EDUCATION RESEARCH

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I use this closing space in order to offer paths and possibilities for enactivism in mathematics education research. In that sense it is an opening space. (This said, it has to be seen as portraying my interpretation, stemming from a young scholar's understanding of these future and possible paths.) Four themes were addressed by the different expert groups. These are important issues that have been written about in mathematics education research, and the ongoing projects described in those pages that build on this history of research demonstrates how vivid the perspectives and possibilities are for future research. I will not, thus, go down the route of trying to summarize those discussions in this closing note. The route I undertake concerns two significant aspects that enactivism can continue contributing significantly to in mathematics education research: issues of *learning*, and of *teaching*. Addressing these issues is also a way of pointing to differences between enactivism and constructivism – one of the Research Forum's initial intentions[4].

ISSUES OF LEARNING

Within the concepts of structural determinism and structural coupling, learning is not seen as a causal event determined by an external stimulus. Rather, learning arises from the learner's own structure as it interacts with its environment. Thus, learning is dependent on, but not determined by, the environment. In mathematics education, if one accepts the concept of structural determinism, then one accepts that anything offered as a mathematical situation, a task, an intervention, an explanation, etc., namely the environment offered to students, is at most a "trigger." The students' explorations will be oriented by their own understandings of this environment, and what constitutes issues to explore *for them*. Varela refers to this as problem posing.

Problem posing and problem solving

Varela (1996; Varela et al., 1991) explains structural determinism in terms of the difference between problem solving and problem posing. Problem solving implies already present problems situated in the world and lying "out there" waiting to be

[4] I also discuss specific distinctions between both theories in Proulx (2008b); ones that I do not address here.

solved, independent of us as knowers. For Varela, because of our co-determination with the environment in which we live, because we have a structure and because we are coupled with that environment, problematic situations emerge for us in the sense that we specify the meaning that these situations have and how we deal with them – we bring them forth. These problems do not lay "out there," objective and independent of our actions. We specify the problems we encounter because of our structure that enables us to act and recognize things in specific ways.

> The most important ability of all living cognition is precisely, to a large extent, to pose the relevant questions that emerge at each moment of our life. They are not predefined but *enacted*, we *bring them forth* against a background. (Varela, 1996, p. 91, emphasis in the original, my translation)

The problems we encounter and the questions we undertake are as much a part of us as they are part of the environment – they emerge from this coupling. We interpret events as issues to address: we see them as problems to solve. We are not acting on pre-existing situations, our co-determination and interaction with the environment creates, enables and specifies the possible situations to act towards. The problems we solve are then implicitly relevant for us and are part of our structure. Our structural determinism allows these *to be* problems for us, as the environment "triggers" them in us. Some "issues" of the environment that would "trigger" elements in some persons do not "trigger" the same elements in others. The effects of the environment are not *in* the environment, they are made possible by the organism's structure in interaction with its environment, in the evolving coupling with it.

This is important because this view of problems offers a frame that explains that the issues addressed and explored in classrooms are the ones that resonate with and emerge from the students' structures or ways of knowing. Simply put, regardless of the environment (tasks, interventions, etc.) offered to students, the issues that will be addressed or the orientation of the explorations taken cannot be pre-decided. Although these will be "triggered" by this environment, they are explicitly determined by the students' knowledge/structure. Thus, there is no linear path or trajectory that can be pre-traced and, therefore, despite of the planning and intentions of the teacher, there is no guarantee that specific issues will be dealt with, and many emergent issues can pop up continuously (see Remillard & Kaye Geist, 2002, concept of "openings in the curriculum," as well as the theoretical construct of "objectives to work on" I have developed in Proulx, 2005). The "environment" is there to trigger: what students learn is determined by who they are and what they know.

Implications for learning

In my 2008 PME paper, I discussed two outcomes of this position in regard to the learning of teachers in teacher education situations. I documented, using empirical evidence, how one's structure can limit one's possibilities for learning, restricting that person's capacity to make sense and develop meaning from a situation; something that Simon (2007) had refer to as "we see what we understand," or simply that we learn what we *can* learn. As well, I illustrated how one's structure can orient one to "only see" some aspects in a situation, leading one to focus extensively on some aspects, leaving others aside. But, mainly, I discussed how structural determinism enriched the potential interpretations of these events and helped to understand more deeply the learning mechanisms at play in them.

This said, these elaborations were only a start. We need to know more about the use of structural determinism (and structural coupling) as a framework of analysis for students' learning of mathematics. We need to better understand how this perspective can influence teaching and learning in mathematics, as well as research in mathematics education. What implications can this view have for our understanding of students' mathematical learning and understanding? Can structural determinism, as a frame of analysis, help to better understand or make richer sense of the dynamic at play in learning mathematics? And if so, how? What does it permit? What does it afford? How can, and in what ways does, this perspective affect research in mathematics education? There is a need to probe these issues at a theoretical and empirical level. We need to develop studies around this frame to better understand mathematical learning, as well as better understand the frame of analysis itself.

ISSUES OF TEACHING

This second aspect builds on the aforementioned questions and research perspectives in relation to the teaching dynamic. If there is an aspect in mathematics education research that enactivism has not paid enough attention to, it is issues of teaching and how enactivism opens the way for alternative understandings of it.

Constructivism has, implicitly or explicitly, brought a reframing of the teacher as a "facilitator" or "guide on the side" in contrast to the teacher as someone who "tells/give" or "broadcasts" information. As Kieren (1995) explains in his paper, enactivist thought appears to dissociate from both perspectives. As I began exploring in my 2008 PME paper, enactivism seems to offer an alternative view of the teacher and of teaching, an aspect that needs to be more developed and thought through in (theoretical and empirical) research studies. I explore here some of these issues, in regard to alternative paths that enactivism offers for teaching.

Structural determinism and teaching

The concept of structural determinism implies that the knower has agency over the type of effect that can be produced on him or her: learning is dependant on, but not detemined by, teaching. Put differently, the way the learner responds and learns from a situation is determined by the learner him/herself, not the situation. The effect on students of what a teacher says or does is not pre-determined: it is determined by the students' own structure (their knowledge). Obviously, the "inputs" of the teacher provoke something, influence the process, and play a role. But the type of role and how this will be taken up is determined by the structure of the student and not by the actions or words of the teacher.

What implications does this view have for the teaching dynamic? If students can only see what they already understand, it could mean that nothing new can be worked on with them since it needs to be already known by them. If this is so, this means that our structure not only determines what we learn, but also restricts and stops us from learning. This is clearly not so, and Maturana offers some explanation of this process, which places an importance on the outside environment to provoke reactions/learning from the learner.

Maturana (e.g., 1987; Maturana & Mpodozis, 1999) makes a distinction between our structure and our actions, which he terms "conducts." The conducts demonstrate, and are permitted by, one's structure/knowledge. These conducts are determined by the structure, but are also dependant on (couples with) the environment in which they are enacted. It is in the interaction with the environment that one's conducts arise, in its structural coupling with it. So, these conducts are coupled with, and embedded in, the environment within which they are made possible.

By emerging from the coupling of one's structure and environment, the triggered conduct is "new" or created from this coupling. Thus, this conduct can trigger back in return. It triggers back one's structure, by having emerged from a world of possibilities *in the coupling of structure and environment*. This emergent possibility, this conduct, influences (read: triggers) the structure itself and offers it new possibilities; possibilities for change, for learning. Thus, this generated conduct in return affects one's own structure. [As well, in this structural coupling, the conduct triggers changes in the environment. I address this issue below.] This is how our structure evolves, through a continuous interplay between our structure (that determines possible conducts) and the conduct that emerges from the interaction/coupling with the environment. Our structure triggers conduct, and this conduct, by carrying aspects of the environment with it and therefore being un-thought of and possessing a new character, triggers our structure in return. It is a circular, never-ending, loop of structural change. One can infer that a new environment has the potential to trigger new conducts which in return can trigger changes in one's structure. Therefore, what is present in the environment is of fundamental value to generate these new conducts and in return to generate the structural transformations. The potential for learning lays here, where the environment in which the learner is placed triggers new conducts as the learner is in constant interaction/coupling with this environment.

In the same sense that the environment acted as a trigger on the organism or species, the teacher's interactions and interventions act as triggers in the learning process of students. The environment in which the students is placed is of fundamental importance to trigger learning. The teacher acts as a trigger for students and has the opportunity to open new possibilities for them, new ways of making sense and of understanding. It is by bringing or throwing "something" into the learning environment that the teacher can create something and potentially trigger students' learning. This, therefore, calls for the teacher to be active in the learning process/teaching dynamic, where his or her actions are to be seen as triggers for students' learning. This view challenges the view of teacher as mere facilitator or guide (Kieren, 1995); it calls for a teacher that puts oneself within the action and acts vigorously in this learning space to trigger and provoke something in learners.

Structural coupling and teaching

The notion of structural coupling adds to this theorization of the teaching dynamic. Structural coupling brings the idea that both students and teacher evolve and co-adapt to each other in the learning process/teaching dynamic. There are two major outcomes from this. The first one is that the teacher becomes *complicit* (Sumara & Davis, 1997) in the students' knowledge. The teacher influences what is learned by interacting and coupling with students. Hence, the teacher is "within" the students' knowledge and cognitive acts. By being structurally coupled with students, the teacher influences and orients the learning that happens, hence is seen as complicit in this knowledge production. Cooney (1988) also alludes to these ideas:

> Both mathematicians and mathematics educators cannot escape their responsibility for shaping their students' philosophies of mathematics no matter how implicitly or subtly those philosophies may be communicated by their instructional methods, the means by which they encourage students to learn mathematics, and the means by which they assess their students' learning of mathematics. (p. 359)

Secondly, with this structural coupling, just as the teacher's actions act as triggers for the learning of students, the students' actions, comments, interactions, and so on, *reciprocally* act as triggers for the learning of the teacher.

> […] teachers can observe and learn from and with their students, helping them bring forth a world of mathematical significance and in fact, bringing it forth with them […] learning is a reciprocal activity […] in which the students and the teacher learn from one another and the situation in which they exist. (Kieren, 1995, p. 2)

This means that students and teacher develop a history together and become structurally coupled: they both learn and co-evolve in that history of relationship. Thus, there is not an issue of "facilitating" or "guiding" here, nor of directing/imposing what to do and think: there is an issue of participation, of genuine interaction for promoting (mutual) learning.

Implications for research

This being said, what does it mean concretely in practice? For mathematical teaching and learning? One orientation that has begun to be developed is the one of the teacher as an authentic and genuine participant in students' learning (see Davis, 2004).

We have however not much evidence of what it means in the classroom and we need to know more about it. In what ways does the fact that "learning is dependent on, but not determined by, teaching" affect the teaching dynamic? What liberties and constraints does this afford? In what ways does the fact that the teacher needs to be pro-active and participative in the learning environment enable him or her new actions that the facilitator would restrain him or herself from doing? In what ways does (or does not) Mgombelo's above argument of the teacher's double-bind fit with this participative and genuine perspective? In what ways does the consideration that the teacher learns in the teaching dynamic affect this dynamic or the actions that can be posed? In what ways is the fact that the teacher is complicit in, and part of, students' learning afford or constrain actions, as well as provoke ethical issues? In what ways does the requirement for the teacher to be active in the learning process affect his or her actions? What does being an authentic and genuinely participative teacher mean?

I believe these are fundamental questions to address for two reasons. The first one is that answering or considering these questions *reframes* importantly what it means to be a teacher of mathematics. The second is that we need to know more of the *impact* of this perspective on the role of the teacher of mathematics and how this gets put into action in practice. There is an important future for enactivism in research in mathematics education concerning the answering of these questions about teaching.

CONCLUDING THOUGHTS

I have not insisted on commenting the previous four themes, since the various expert teams have worked extensively on these issues. In so doing, these expert teams have laid down the past and the present issues in enactivism and mathematics education research, thus opening paths for the future. My attempt in this closing space has been to raise issues still underdeveloped in enactivism and mathematics education research, in order to illustrate the potential of addressing these vibrant issues.

The two routes I have attempted to explore were the ones of learning and of teaching. These routes were not chosen randomly. I have specifically chosen them because they have been, to some extent, what could be called the bread and butter of constructivist theory in mathematics education – and I deeply believe, as a young scholar, that enactivism can offer alternative and fruitful perspectives on these vibrant issues for our mathematics education community. However, a lot still needs to be done along those lines…

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DISCUSSION GROUPS

ONLINE MATHEMATICS TEACHER EDUCATION

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Salvador Llinares — University of Alicante, Spain

In recent years, there has been an increase in research on teacher learning and development in online contexts. Research in mathematics education has been engaged in this debate. On PME 32 we had a discussion group with a title similar to the one above in which we discussed different models of online mathematics education and as well as it means to develop research in online environments. Research questions and we began to share some issues about the nature of this new research agenda and the theoretical frameworks.

The aims of this discussion group for PME 33 are to continue the debate about specific research issues and approaches available and to propose an additional discussion on examples of research carried out. Some questions come from the discussion group in PME32, and other issues have arisen regarding characteristics of mathematics teacher education programs using online methodology (e-learning or b-learning) and aspects that provide potential advantages or complement traditional classrooms. How can some good features of face-to-face teaching and learning be reproduced in the on-line environment? How can prospective teachers and teachers be encouraged to use an on-line environment? A second group of issues is related to how we approach studying teacher learning and teacher development in this context. Can teachers collaborate to solve mathematical problems? What theoretical approaches are suitable for addressing the study of teacher learning and development in these new contexts? To what extent do the new ways of interaction generated in the online learning context determine what is learned and/or the ways in which the teacher learning and development is understood?

To address some of these questions considering specific cases, we have planned the following activities. In the first session, a video excerpt from online research developed by the first author of this DG will be presented. Small groups will analyse the video and propose issues to be discussed in a large group at the end of the first session. Participants will be invited to focus on differences between the data presented and data from research developed in other settings. In the second session, participants who have participated both on the first session of this group **and** in last year's group session will have the opportunity to present their research in five minutes. If there are too many who fulfil the requirements and would like to present, we will divide the group. At the end, we will discuss the possibility of forming a WG to work on a publication.

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LOGICAL AND PROBABILISTIC GAMES. DIDACTICAL USE AND DIDACTICAL ANALYSIS

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Logic, probability and games have a long tradition in education and are related due to historical, epistemological, psychological and educational reasons. Although the decline of the movement of "modern mathematics" had as consequence the elimination of logic from the curriculum in many countries, the necessity of its teaching is widely recognised (A.S.L., 1995; N.C.T.M., 2000) and the related didactical problems remain open (ICMI/IASE Joint Study, 2006; ICMI Study 19, 2007; Hatzikiriakou & Metallidou, 2009).

One way to address the above problems is the utilisation of logical and probabilistic games for the assessment, the investigation and the improvement of logical and probabilistic reasoning, and the introduction of the related logical and probability theory. The proposed games (Chasiotis, 1996), require only a set of ordinary playing cards, which are manipulated according to predetermined rules, formulated in everyday language, free of logical and mathematical terminology.

The proposed Working Session is part of an action research project about the study of the logical and probabilistic reasoning, based on the didactical use and the didactical analysis of the above games, as well as on the comparative study of the teaching of logic and probability in all grades (Chasiotis, 2007). In the frame of this project, four versions of the above games, in the form of worksheets, and a questionnaire, are addressed to different groups of students, teachers and researchers.

During the first part of the Working Session we will present some versions of the above games and we will invite the participants to play and analyse them logico-mathematicaly. In the second part we will focus on the didactical use and the didactical analysis of the presented games.

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PROSPECTIVE PRIMARY MATHEMATICS TEACHERS IN ACTION: EXPLORING CAPABILITIES AND CONSTRAINTS

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AIMS

These discussion groups are designed to develop our understanding of:

1) the potential of prospective primary mathematics teachers
2) the constraints under which they work and how these might be ameliorated.

BACKGROUND

There is a growing understanding of the characteristics of successful teachers (Shulman, 1986; Ball, Bass and Hill, 2004). Our knowledge as to how teachers become experts is less well developed although Rowland, Huckstep and Thwaites' (2005) 'Knowledge Quartet' provides a detailed account of the key attributes we might expect of elementary teachers towards the end of their teacher education courses. Here we focus on the evolution of prospective teachers into classroom practitioners.

SESSION 1: POTENTIAL AND IMPLICATIONS

Using participants' diverse experiences and expertise we will consider the potential of prospective mathematics teachers in action using observational and interview data collected from primary classrooms half way through their education course. Implications for teacher education will be exchanged and discussed.

SESSION 2: CONSTRAINTS AND POSSIBILITIES

Building on session 1, and using supplementary commentaries from intending primary teachers, we will broaden the focus of discussion to explore both the perceived and the real constraints with which prospective mathematics teachers typically contend. Strategies to ameliorate these will be exchanged and reviewed.

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COMMUNICATION ISSUES IN THE MATH CLASSROOM: VIRTUAL MONOLOGUE AS A REFLECTION TOOL

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This DG is concerned with mathematical classroom communication in general, and with ways of enhancing teachers' communication and reflection skills in particular. Specifically, we will be looking at data from a workshop for teachers, where the tool of *virtual monologue* (VM) was used for practicing these skills. VM was introduced by Leron and Hazzan (1997) as a tool for reflection and communication by researchers, and has been transposed into an activity in a teacher's workshop by Ejersbo (2007). Originally, the VM was conceived as a way to convey the researcher's view of the student's state of mind, reproduced in a monologue in the student's own voice. In the teachers' workshop, the VM was transposed into a tool for teachers to help them reflect on their own communication skills.

The data for this DG consists of two parts. The first part is a short "taxicab scenario", brought in by one of the participants in the teachers' workshop, where the teacher and one of his students appear to experience a crisis in communication. The second part is the various VMs composed by the participating teachers to interpret the taxicab scenario, in both the student's and the teacher's voices. In the workshop, it was hoped that by composing these monologues, the teachers would be indirectly reflecting on their own practices and communication skills. Our experience and our data support the conclusion that such reflection has indeed taken place.

In the PME DG, after a short introduction, the participants will first create their own VMs on the taxicab scenario, and then reflect on their experience and on the resulting texts from five perspectives: One, the communication issues raised by the taxicab scenario itself; two, the various interpretations embodied in the VMs composed by the DG participants; three, comparing and analyzing the VMs produced in the teachers' workshop; four, reflecting on the relevant classroom communication issues in general and on the use of VM as a tool for enhancing teachers' communication and reflection skills in particular; five, sharing the DG participants' own experience in using other methods for working with teachers on reflection and communications skills in the mathematics classroom

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EQUITY AND DISCOURSE IN MATHEMATICS CLASSROOMS: A FOCUS ON STUDENTS

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David Wagner, University of New Brunswick
Judit Moschkovich, University of California, Santa Cruz

In mathematics education in the last 20 years, there has been an increased focus on discourse in mathematics classrooms. In many classrooms students are now asked to read, write and talk about mathematics much more than ever before. Along with this renewed focus on discourse in mathematics classrooms has come a focus on issues of equity, in terms of access, achievement, identity, and power (Gutierrez, 2008).

Studies have tended to focus on teacher 'discourse moves' or pedagogical strategies to promote equitable mathematical discourse. More rare is a focus on student activity, contributions, and perspectives: how students from diverse backgrounds participate in mathematical discussions, perceive classroom norms around mathematical discourse, learn new norms, think about the choices available to them, or think about associated equity issues. Therefore, this discussion group will focus on students' participation, discourse, and perspectives as they relate to issues of equity in mathematics classrooms.

We will devote one session to discussions of data from the organizers' classroom research in discourse-intensive mathematics classrooms. These data include video, audio and transcription excerpts from classroom interactions, and ethnographic interviews. The discussion group will address the following questions: What are the multiple ways students participate in small group and whole class discussions? How do students respond to teachers' discourse moves? What issues of equity do students identify in mathematics classrooms? How do students navigate issues of identity in various contexts of mathematics discussions? How do students' moves create more (or less) equitable learning environments for themselves and their peers?

For the second session, participants will be invited to bring samples from their own data, and we will work in small groups to develop strategies for analysing student activity, participation, and perspectives as these relate to issues of equity in mathematics classrooms. The topic for this discussion group forms the basis for an upcoming special issue of the *Canadian Journal of Science, Mathematics, and Technology Education*, to be edited by Indigo Esmonde and Judit Moschkovich. All participants of this discussion group will be invited to submit papers to this special issue.

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PRE-SCHOOL COMMUNITIES AND MATHEMATICS EDUCATION RESEARCH ISSUES AND FINDINGS

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Dina Hassidov, Western Galilee & Zefat, Academic College , Israel
John Oberman, Shaanan Academic Teachers College, Israel

In recent years the interest in the pre-school children's knowledge of mathematics has increased immensely. Different research strands raise the importance of this domain. Young children come to kindergarten with a lot of informal mathematical concepts and variable skills and abilities. Several recent studies have shown that pre-school children can solve a variety of problems before formal instruction on mathematical operations.

The aim of this Working Group is to build a community of PME members who have researched mathematics education issues with pre-school children (age 3-6).
Members of the group from different countries will present ongoing research and conclusions from previous studies. We hope that members will bring to the group a variety of dilemmas and issues relevant to the learning of mathematics with pre-school children as seen in their countries. Sharing of this information will enable the group to prepare a plan of action in order to advance further research in the study of pre-school mathematics education.

At present the material available has shown that mathematical activity formal and directed can utilize the natural intuitivism of children in pre-school.
In the first stages of the working group we will spend the time in the formation of the group, collecting and evaluating previous research together with making the first draft of issues to be related to in new and continuing research.
We are suggesting that the group will relate to the following issues.

1. Pre-school mathematics and the move to school.
2. The involvement of parents in pre-school mathematics.
3. A curriculum for pre-school mathematics.
4. Cultural and gender differences in pre-school mathematics.
5. Which mathematical domains should be related to-number concepts, geometry and real situation problem solving?
6. Mapping of the development of mathematical thinking with pre-school children.

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STUDYING TEACHER LEARNING FROM VIDEO

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Researchers in mathematics education are increasingly relying on video for use in teacher education programs, professional development, and as a data source to study teacher learning. However, relatively few presentations at recent PME meetings have explored important issues related to using video in these ways, and in particular, how we can most effectively study teacher learning from video.

Using video in mathematics professional development and teacher education has the potential to support collaborative learning by providing a shared common experience. As teachers watch purposefully selected clips from mathematics classrooms (focused either on the teacher, a group of students, or both), teachers have the unique opportunity to develop an important type of knowledge – that of how to interpret and reflect on classroom practices.

At the present time, there are many unknowns about what teachers actually learn from watching video of classroom instruction in professional development settings. Closely connected to this question of *what* teachers learn is how we can *study* teacher learning. Our Discussion Group will provide the opportunity to explore these issues. Possible theoretical and methodological challenges of studying what teachers learn from video and how to study it might include: 1) What do teachers learn about student thinking from watching video? 2) How can we study what teachers learn about student thinking from watching video? 3) How do clips focus teacher thinking on strategic instructional practices? 4) How do clips provide opportunities for teachers to make shifts in their thinking about content and pedagogy? 5) How can we study shifts in thinking? 6) How do teachers transfer new understandings about content, student thinking or pedagogy to their everyday practice? and, 7) What are the best ways to study changes in teachers' knowledge and practice related to their experiences watching video in professional development?

This session will create an opportunity for participants and session leaders to collectively examine similarities and differences in how they use video to study teacher learning. To begin, we will survey the discussion group participants to understand the different interests in using video to study teacher learning. We will then briefly introduce two mathematics professional development projects, the Problem-Solving Cycle and Learning and Teaching Geometry project, and show video clips as a launch into discussions. The video clips will be subtitled and transcripts will be provided for increased language access for participants to video content. Discussions will center around: 1) the learning opportunities provided by the videos, and 2) the various methods we can use to study teacher learning from video. Small groups will be created based on participant interest around theoretical or methodological issues.

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INVESTIGATING TRANSITION

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There has been a widespread concern over the lack of preparedness of students making the transition from pre to post compulsory, and from upper secondary to university mathematics. In this session we aim to better understand how students transit from (1) lower to upper secondary education (age 15/16); and (2) from school to university in mathematics (age 18/19). It appears that students experience different difficulties at different stages, and develop different strategies to make these transitions successful. At the same time institutional practices afford, or hinder, students developing a mathematical disposition and an identity that supports their engagement with mathematically oriented subjects in upper secondary and tertiary education. We will discuss, and investigate on the basis of existing research, how student experiences of mathematics education practices may interact with background social factors to shape students' self-identity, dispositions and decision-making in the two transition phases.

In the first part of the session the organisers will provide short synopses of research work being conducted in the UK (e.g. TLRP, Transmaths projects[1]). We know from previous research that particular factors have been shown to be influential in terms of students developing identity as learners of mathematics, amongst them the following: socio-cultural factors (e.g. class, gender, ethnicity, parental culture, etc); institutional practices (e.g streaming, test-oriented); distinct classroom practices (e.g. discussion-based pedagogies); students' affective dispositions (e.g. self-efficacy); student learning outcomes. For each of these, we will summarise our results and present them for discussion, which will form the second part of the session. We will also show examples of survey data, and video clips of pedagogic practice in the different phases of secondary and tertiary education, in order to elicit engagement of participants. Our particular concern relates to the following questions:

- How do we theorise mathematics in transition?
- What are the commonalities and differences between these two transitions?

At previous conferences we have identified 'pockets of research' in the field, colleagues who are researching these transitions in mathematics, and we aim to provide a forum where these and other questions can be discussed internationally.

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[1] http://www.lta.education.manchester.ac.uk/TLRP/,
http://www.lta.education.manchester.ac.uk/TLRP/summary2.htm,
http://www.education.manchester.ac.uk/research/centres/lta/LTAResearch/transmaths/

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MATHEMATICS AND KINDERGARTEN TEACHERS – A CHALLENGE TO THE RESEARCH COMMUNITY

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This discussion group will provide an opportunity for raising the pressing issue of preparing and supporting preschool teachers in their endeavours to engage young children in mathematical activities. A growing number of countries now mention mathematics as part of their preschool curriculum. This trend calls for supporting preschool teachers in creating a mathematically rich environment, which will facilitate mathematical thinking in their kindergartens. We will discuss the roles of the kindergarten teacher in promoting mathematics in the kindergarten. These roles include choosing and implementing activities that will explicitly promote the acquisition of mathematical concepts in the kindergarten. A major aim of the discussion group is to discuss ways in which the research community can support the kindergarten teachers, cognitively, emotionally and didactically in these efforts.

In the first session we will focus on the types of knowledge a kindergarten teacher needs in order to promote mathematical thinking among young students. We raise several questions for discussion: (1)What mathematical content and mathematical processes does a kindergarten teacher need to know? (2)What pedagogical content knowledge is necessary for the kindergarten teacher? (3)Are current theories of teacher knowledge suitable for investigating kindergarten teachers' knowledge for teaching mathematics? (4)What are appropriate tools for investigating kindergarten teachers' knowledge?

In the second session we focus on how kindergarten teachers' knowledge for teaching mathematics may be promoted. We raise several issues such as: (1)How may professional development programs for kindergarten teachers resemble and differ from programs aimed at other levels of teaching? (2)Which, if any, theories of mathematics education should be presented to kindergarten teachers? (3)How can professional development for kindergarten teachers support a change in practice?

In both two sessions we will draw on related publications (e.g., Clements & Sarama, 2007) and on official documents (e.g., National curricula).

References

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COORDINATING PSYCHOLOGICAL AND SOCIAL ASPECTS OF CLASSROOM LEARNING

Michelle Zandieh, Chris Rasmussen, Andrew Izsák

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We propose to continue the 2008 Discussion Group regarding recent complementary lines of research that examines interactions between psychological and social aspects of classroom learning. For over 15 years, mathematics educators have recognized that both individual and social aspects are central to mathematical thinking and learning. A main challenge has been to respond to two theoretical positions on learning that can appear to be in opposition. One position, often traced back to Piaget, gives priority to individual psychological processes. A second, often traced back to Vygotsky, gives priority to social and cultural processes.

The *emergent perspective* (e.g., Cobb & Yackel, 1996) has been one of the most visible theoretical perspectives that seeks to transcend past divisions between individual and social accounts of classroom learning. This perspective emphasizes reflexive relationships between the learning of classroom communities—characterized in terms of social norms, sociomathematical norms, and classroom mathematical practices—and the learning of individuals—characterized in terms of beliefs and understandings that are psychological correlates of norms and practices.

We will discuss the theoretical and methodological challenges of conducting research that investigates such questions as (a) how is the learning trajectory for a class related to the learning trajectory of various individuals in the class; (b) how can one determine the emergence of a particular norm or a taken-as-shared mathematical practice in classrooms where there is little student debate; (c) how do the teacher and students in a given classroom interpret lessons in which they participate together; (d) what is the relationship between classroom mathematical practices and larger disciplinary practices such as defining, symbolizing, and proving; (e) how is an individual's participation in particular mathematical practices related to his or her acquisition of knowledge; (f) how are notions of classroom mathematical practices related to other notions of practice used both within the math education research community and within the larger social science research community? The presenters will offer brief examples from their own research to initiate discussions of the challenges one encounters when investigating relationships between psychological and social aspects of classroom learning.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175-190.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 290-290. Thessaloniki, Greece: PME.

WORKING SESSIONS

TEACHING AND LEARNING MATHEMATICS IN MULTILINGUAL CLASSROOMS

Richard Barwell, University of Ottawa, Canada
Philip Clarkson, Australian Catholic University, Australia
Núria Planas, Universitat Autònoma de Barcelona, Spain
Mamokgethi Setati, UNISA, South Africa

Multilingualism is a widespread feature of mathematics classrooms around the world, whether it concerns multiple language use, bilingual education, second language education, immersion education or other situations. Teaching and learning mathematics in multilingual settings can present challenges for teachers, learners and policy makers. Research has increasingly addressed some of these challenges and sought to understand the complexity of mathematics education in multilingual settings from a variety of theoretical perspectives.

In 2008, ICMI initiated its 21st study with a focus on mathematics education in multilingual settings (co-chairs, Mamokgethi Setati and Maria do Carmo Domite). This working session, which is organised by members of the International Program Committee for the study, will be devoted to examining the aims and focus of the study.

The working session will focus on the following key questions:

- What does multilingualism look like in contexts of mathematics learning in different parts of the world?
- How do these different multilingualisms interact with learning, teaching and understanding mathematics?
- What methodological issues arise in these different contexts?
- What role does policy play in mediating multilingualism in mathematics education in different contexts?

ACTIVITIES

Over the two working sessions, our work will draw on the discussion document for the ICMI study, as well as empirical data from a variety of settings. Participants will discuss aspects of the discussion document. They will also analyse the samples of empirical data. Discussion of these analyses will inform and enrich the discussion of the document. Points arising from this working session will be fed back into the work of the study.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 293-293. Thessaloniki, Greece: PME.

TEACHERS RESEARCHING WITH UNIVERSITY ACADEMICS

Laurinda Brown, Jarmila Novotnα and Merrilyn Goos

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The University of Queensland, Australia

This Working Session (WS) will be the culmination of a process that started in 2007, as a follow-up to a Research Forum at PME 30. At PME 31, a framework was developed for analysing ways in which university academics and teachers might conduct research together. Several participants from this session decided to prepare and start projects which formed the basis of the WS at PME 32, where examples of research collaborations between teachers and university academics were presented and discussed with feedback being given to the researchers in the style of a reflecting team (the focus being on the articulation by the presenter in response to probing questioning from the participants).

For PME 33, we will use the two WSs to work on reviewing draft papers that are being developed from proposals that were received for a Special Issue of the Journal of Mathematics Teacher Education (JMTE SI) on *Teacher Change* (First Guest Editor, Laurinda Brown). Some participants in the previous WSs have been encouraged to develop their proposals for submission to JMTE by August 15th, 2009.

In each Working Session, we aim to take at least three papers and, after initially considering the aims of the SI and a brief introduction to the theoretical frame and ways of working illustrated by the papers, set up reflecting teams for each paper from participants in the sessions. In the final session, we will reflect on the process of developing a project related to teachers researching with university academics and turning such projects into journal articles. We aim to circulate draft articles to participants in the WS where these are known in advance.

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Goos, M. (2008). Critique and transformation in researcher-teacher relationships in mathematics education. *Symposium on the Occasion of the 100th Anniversary of ICMI*. http://www.unige.ch/math/EnsMath/Rome2008/partWG3.html (09/02/2008)

Novotnα, J., Zack, V., Rosen, G., Lebethe, A., Brown, L. & Breen, Ch. (2006). RF01: Teachers researching with university academics. In: Novotnα, J. Moraovα, H., Krαtkα, M. & Stehlνkovα, N. (Eds.). *Proceedings PME 30*. Praha: PME. Vol. 1, pp. 95-124.

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Novotnα, J., Brown, L. & Goos, M. (2008). Teachers researching with university academics. In: Figueras, O. et al. (Eds.) *Proceedings of PME 32*, Vol. 1, p. 201. Morelia: PME.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 294-294. Thessaloniki, Greece: PME.

GESTURE, MULTIMODALITY, AND EMBODIMENT IN MATHEMATICS

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St. Mary's College of California, USA

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Luis Radford
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The central purpose of the Working Session is look at mathematical thinking, learning and communication from a perspective that encompasses multiple modalities of expression and different semiotic systems (Radford, Edwards & Arzarello, 2009). These modalities include gesture, speech, written inscriptions, and physical and electronic artefacts. In addition to considering how mathematics is communicated, the Working Session will address the notion that, in our acts of knowing, different sensorial modalities (tactile, perceptual, kinesthetic, etc.) are integral parts of our cognitive processes. Thus, the basis for mathematical thinking, as well as language and thought in general, is found in our embodiment as physical beings (Lakoff & Núñez, 2000). In the Working Session, we will continue the process of investigating the entailments of, and evidence for, this stance.

The session will be participatory, and attendees will be asked to share videotaped data, interpretations, theories and ideas. We will meet as a whole group as well as in small groups, within which participants will be able to sign up to give brief presentations. Themes and topics addressed in previous years include:

- Gesture and semiotics
- Conceptual integration and conceptual metaphor
- Gesture and embodiment in young children and blind students
- Dynamic geometry and other computer-based tools
- Graphing and other visual modalities
- Language, culture and the body in mathematics

The session will not be limited to these topics, but will be based on the interests of the participants.

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Lakoff, G. & R. Nunez (2000). *Where mathematics comes from*. NJ: Basic Books.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 295-295. Thessaloniki, Greece: PME.

INTERNATIONAL PERSPECTIVES: GENDER AND MATHEMATICS EDUCATION

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Joanne Rossi Becker — San Jose State University
Helen Forgasz — Monash University
Olof Bjorg Steinthorsdottir — University of North Carolina

In 2005 and 2006 lively discussion group sessions, focusing on several areas of interest related to gender and mathematics, addressed issues associated with countries showing large gender gaps in achievement and those that did not. In the working group sessions in 2007 and 2008, possible factors contributing to large gender differences in achievement and efforts to narrow the gaps, in countries such as Korea and Iceland, were discussed. Colleagues from Australia, where research related to gender and mathematics has been actively undertaken, presented findings and implications for future research agendas. Results obtained from a longitudinal study on New Zealander girls' experiences in mathematics learning were also presented.

In 2008, draft papers for a book to be published in 2009 entitled, "International Perspectives on Gender and Mathematics Education", were shared, and there was a call for more papers from various countries. Twenty papers from eleven differing countries are now being considered for inclusion in the book.

ACTIVITIES

After the organisers complete the reviewing process for the book "International Perspectives on Gender and Mathematics Education", authors will be identified and invited to present at the Working Group. In particular, chapters for which there is the potential for in-depth discussion and future research opportunities will be selected.

Day 1: a brief introduction, participants will be divided into smaller groups according to their areas of interest; the group topics will be predetermined by the organizers. In these groups, participants will discuss, critique, and offer their ideas and suggestions about future research directions to participant authors whose work is being examined.

Day 2: the main ideas gathered from the previous day's small group discussions will be shared. Issues identified in the book chapters invite further research as well as methodological approaches for addressing these issues will be discussed. Participants' email addresses will be collected and subsequently distributed to enable all involved to stay in contact. This action will also facilitate further research opportunities and the sharing of information after the conference.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 296-296. Thessaloniki, Greece: PME.

GENERIC PROVING: UNPACKING THE MAIN IDEAS OF A PROOF

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Learning proofs is known to be hard for students of all ages. Generic proving (GP) is an interactive didactical process for addressing this difficulty. GP is situated in the context of exemplification in general, and generic examples in particular. When we want to convey vividly the essence of a mathematical concept, we use a *generic example* – an example complex enough to "see the general in the particular" (to quote Mason and Pimm). Thus, 36 is a good generic example of *perfect square*, but 4 is not (being too special). When we want to convey the essence of a theorem, e.g., *a perfect square has an odd number of factors*, we can still use a generic example: 36 has 9 factors (namely, 1, 2, 3, 4, 6, 9, 12, 18, 36). This example, however, does not give us a clue as to *why* the theorem is true. For this we use a *generic proof* – a proof carried out on a generic example. Thus, we list all the different ways 36 can be written as a product of two factors: $36 = 1 \cdot 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9 = 6 \cdot 6$, from which it is evident that the factors come in pairs plus one stand-alone, thus totalling an odd number (specifically, $9 = 2 \cdot 4 + 1$). In general, a generic proof is intended to convey vividly all the *main ideas* of the proof, separating out the creative parts from the technicalities of notation and formalism of the complete proof. We have now seen an example of a generic proof, from which some of its advantages could already be gleaned, but this example is too simple to be considered a *generic example of a generic proof*. Many subtle and important issues would only surface in the context of more complicated proofs.

In the first part of the Working Session, participants will analyse in small groups some richer examples of GP, where the main ideas of the proof emerge gradually in the context of one or more *partial generic proofs*. In the second part, we will use the examples of GP generated in the first part, to reflect from several perspectives (mathematical, educational and philosophical) on the following more general issues.

- What is a good generic example in the context of a generic proof?
- What are the strengths and weaknesses of a generic proof?
- How big is the gap between a generic proof and the complete proof of the same theorem, and how can it be bridged?
- Not all proofs are equally amenable to a generic version. Can we characterize the proofs (or parts thereof) that are so amenable?

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 297-297. Thessaloniki, Greece: PME.

THE ROLE OF THEORY IN UNIVERSITY-LEVEL MATHEMATICS EDUCATION RESEARCH

Co-ordinators: Elena Nardi[1] and Paola Iannone[1]

Also: Irene Biza[1], Victor Giraldo[3], Alejandro S. González-Martin[2], Marcia Pinto[4]

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University-level mathematics education research is a relatively young research area that over the last twenty or so years has started to embrace an increasingly wider range of: **theoretical frameworks (1)**, e.g. cognitive/developmental, sociocultural/anthropological and discursive; and, **methods/methodologies (2)** e.g. quantitative/experimental, qualitative/ethnographic and narrative (Artigue et al, 2007). Variation also characterises research in this area with regard to at least two further issues: **the role of the participants (3)**, students and mathematicians, in the research – from 'just' subjects of the research to fully-fledged co-researchers; and, **the degree of pedagogical action (4)** involved in the research – from external, non-interventionist research in which researchers collect and analyse data, but proceed to no action for reform beyond mere recommendation, to developmental/action research in which researchers identify problems and devise, implement and evaluate reform of practice.

In this Working Session we will examine **the role of theory in university-level mathematics education research** and we will do so in two ways: **substantive** and **methodological** (**1** and **2** above, in Meetings 1 and 2 respectively of the WS). At the heart of our exploration will be the **potential insight -** into, for example, how cognitive, emotional, social and cultural aspects of a student's mathematical experience shape their attitudes towards and understanding of mathematics – **that can be gained from a coherent fusion of theories.** We envisage the two 90′ meetings including brief exposition followed by group-work and discussion:

Meeting 1. Welcome and session outline (10′); Examples of theoretical frameworks: brief reminders (20′); Work in groups: analysis of a sample of data from different theoretical perspectives (30′); Discussion (30′).

Meeting 2. Examples of methodologies: brief reminders (20′); Work in groups: given a hypothetical university-level mathematics education research topic, question and context, design a study that employs a range of methodological perspectives and methods (30′); Discussion (30′); Closing (10′).

Particularly in Meeting 2 participants will be asked to also consider issues **3** and **4**.

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2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 298-298. Thessaloniki, Greece: PME.

PROBLEM POSING IN MATHEMATICS LEARNING: ESTABLISHING A THEORETICAL BASE FOR RESEARCH

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University of Ploiesti	Illinois State University	University of Michigan

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University of Delaware	National Autonomous University of Mexico	Hiroshima University	University of Bucharest

In recent years, problem posing has received more attention in the mathematics education community – both as a means of instruction (to engage students in learning important concepts and skills and to enhance their problem-solving competence) and as an object of instruction (to develop students' proficiency in posing mathematics problems). Experience with mathematical problem posing can promote students' engagement in authentic mathematical activities. For example, problem-posing activities can allow students to encounter many problems, methods and solutions, and promote students' creativity – a disposition to look for new problems, alternate methods, and novel solutions. Many studies have detected positive effects on students' problem-solving achievement and/or their attitudes towards mathematics when problem posing has been systematically incorporated into mathematics instruction. Some scholars have recognized that problem posing is an important part of mathematical activity, yet research on problem posing has not yet become a major focus in mainstream mathematics education research. Thus, the field would benefit from an effort to systematize the theoretical curriculum and pedagogical foundations for and the empirical findings accumulated to date in problem-posing research.

Our working session will engage participants in discussion on the following topics:

1. Problem posing as an integral component of school mathematics.
2. Contrasting the cognitive components of problem posing and problem solving in mathematical thinking.
3. Problem posing and discourse in mathematics classrooms.
4. Problem posing processes and how these relate to creativity.

Coordinators of the working session will also engage participants in making specific plans for proposing a special issue of *Educational Studies in Mathematics* and for planning the structure and chapters of a book on problem posing.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 299-299. Thessaloniki, Greece: PME.

NATIONAL PRESENTATION

RESEARCH IN MATHEMATICS EDUCATION IN GREECE AND CYPRUS

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In this study we present a brief summary of the research in mathematics education carried out in Greece and Cyprus. In both countries this research is quite a recent advancement. In the paper we first elaborate on the development of the current situation with respect to mathematics education in the two countries, we highlight concerns and challenges faced by the scientific community, we summarise the main areas of research activity and we finally discuss the impact of the relevant research on the mathematics education reality of the two countries as well as on the development of the respective research community.

THE CONTEXT OF MATHEMATICS EDUCATION IN GREECE AND CYPRUS

Some characteristics of the Greek Mathematics Education. The Greek education system has traditionally maintained a centralized character. Decisions are made in practice at a Ministry of Education organization called the Pedagogical Institute which operates as a consultant to the Ministry. There is a national curriculum accompanied by a single textbook determined by this Institute for each school subject, including mathematics and reform is mainly perceived as changes to the curriculum occasionally accompanied by short in-service courses to teachers. Mathematics education practice is embedded in the wider educational paradigm in Greece which has a revelatory character accompanied by an agenda for all students to be exposed to knowledge rather than to construct knowledge for themselves or engage in knowledge building activity with some degree of autonomy. This is coupled by sensitivity to the preservation of a cultural identity often resulting in emphasis on ancient Greek history and achievements including those of the Greek mathematicians. At the end of the secondary education, there are competitive national exams for entry into University which traditionally maintain a high status and are something most pupils are expected to strive for. This is ingrained in social values to such an extent that the phenomenon of coaching or private auxiliary schooling is widespread, especially but not exclusively in the final two years of secondary education. With respect to the preparation of secondary teachers, there is no teacher education certificate. A Bachelor's degree in mathematics together with a "pass" in competitive national written exams is sufficient for employment as a mathematics teacher. With respect to in-service education, there is no systemic mechanism or infrastructure for teacher development apart from an initial short taught course provided to newly appointed teachers. The teaching profession in the state system is tenured and poorly paid and no in-service assessment mechanism is in place.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 303-322. Thessaloniki, Greece: PME.

Teaching is inevitably traditional and the teaching profession is perceived as relatively static. In the last 15 years however, the Ministry has been allocating funds for the establishment of Master's courses in Education. Furthermore, in the wake of funding for the use of digital technologies, there have been projects funding middle scale teacher education, production of educational software and teaching material.

Some concerns and challenges for the Greek mathematics education community. From the point of view of the research community, one of the main problems created by the situation described above is the rigidity in values and beliefs within education but also in the wider community with respect to change and alternative educational practices. The implementation of innovative educational practices even at small scale is thus not very welcome. This is enhanced by the exam-cramming nature of the final years of schooling and the resulting inflexibility of the curriculum. In turn, this situation restricts avenues for research-led reform in mathematics education and poses obstacles to the implementation of teaching practices informed by research. Mathematics education knowledge per ce can hardly reach teachers, especially at the pre-service level for secondary education and in-service level for both primary and secondary education. Finally, there is only gradual acceptance and respect for research in mathematics education as a means to generate reform in the education system. One of the challenges is thus to gain a clearer and more comprehensive picture of the impact of these contextual constraints and resistance to teaching and learning practices. Another challenge is the design and implementation of interventions in educational practice including teachers as designers of educational scenarios and plans so that their personal pedagogy is recognized and becomes an object for reflection and community building. A third challenge is to work with pre-service and in-service teachers so that they may move from perceiving their profession as static to a developing one and adopt a more reflective and open stance to reform initiatives.

Some characteristics of the Cypriot Mathematics Education. The Cypriot Educational system developed after 1960, when Cyprus became an independent state, borrowing elements from the British and primarily from the Greek educational system. The size of the country facilitates quick communication and spread of ideas and "directives" to all parts of the island. Not surprisingly, it is highly centralized; all decisions are taken at the Ministry of Education, there is a uniform national curriculum and one textbook for each subject, which for years used to come from Greece. The influence from Greece was always predominant as a sign of the will to preserve the historic and cultural identity. After the establishment of the Pedagogical Institute (1976) some text books, including mathematics books were developed in Cyprus, though the effort to follow international trends had only a limited success. Young Cypriots had to pursue higher studies abroad, mostly in Greece and England. In the former case the entrance exams were organized by the Cypriot Ministry of Education. Another drawback with regard to mathematics education has been the ever-increasing expansion of the para-educational system; a coaching system undertaken by small-sized registered institutions but also by in-service teachers of mathematics at their own homes. Apart from the economic burden on the parents and time load on the students, this afternoon system resulted in the sacrifice of real learning to the benefit of short term outcomes; since the ultimate goal was "to pass the exams", motivation became strictly utilitarian and most students developed negative attitudes towards mathematics. After the establishment of the University of Cyprus (1992) the situation began to improve, despite the resistance by several societal groups.

Concerns and challenges for the Cypriot mathematics education community. Though from 1992 the Department of Education took over the preparation of primary and pre-primary teachers, the pedagogical education of the secondary teachers became its responsibility as from 2007. Until then, most secondary teachers used to view the role of the Department of Education as the successor of the Pedagogical Academy. Teaching was mostly traditional and the educational community had a low respect and limited interest in participating in research programmes run by mathematics educators. Gradually, however, the situation is moving forward; research in mathematics education is reaching more interested people. Mathematics education research attracts the interest of teachers and of the administration, due to the impact of local and international conferences organized on the Island and also due to the MA programme in mathematics education established in 1997 among other reasons. For instance, the first two mathematics educators were consulted in developing the new series of mathematics texts which was completed by 2000. A recent sign of this recognition was the official assignment of the redesign of the whole mathematics curricula for K-12 to a Committee of mathematics educators. The education community aspires to play a major role towards the effort to materialize the Government proclaimed project to reform the whole educational system, including the administrative structure, evaluation, pre-service and in-service training, the curricula and the educational material. The main concern for mathematics education is to let teachers and administrators move from mere recognition to actual participation in research oriented reform.

THE DEVELOPMENT OF THE TWO RESEARCH COMMUNITIES

The Greek research community. Research activity in mathematics education in Greece was initiated by the end of 1980s by a small number of faculty members working in the Mathematics Departments and by a number of mathematics education researchers in the newly established departments of primary and early childhood education who had received their doctoral degrees from European Universities. This first group of researchers tried to establish mathematics education in their departments, despite the prevailing adverse climate due to several reasons such as: existing beliefs and attitudes among teachers, mathematicians and educators towards mathematics education; increased teaching load and lack of other colleagues to cooperate as in most cases there were the only ones working in this field in every department; furthermore, there were no funded research activities in the country at that time. Nonetheless, this small group of people started to build some cooperation in the Greek or International conferences.

Initially, the researchers started to build their own research in the field by cooperating with teachers, PhD students and other colleagues from Greece and abroad. A small number of research projects were also funded by the Centre for Educational Research at the Ministry of Education, the Secretary of Research and Technology at the Ministry of Development and the European Community. Since 1994, most university departments of Education and of Mathematics started to organise graduate programmes that led to Master's Degrees in Education or in Mathematics. Mathematics Education was usually part of the Mathematics or the Education Master's Degree while in the Mathematics Department of Athens University the first Master's Programme in Mathematics Education was established in 1994. Initially, the focus of this programme was mostly on Mathematics but since 2003 the programme became cross-departmental including the Education Department at the School of Philosophy and the Faculty of Philosophy & History of Science and a joint venture between the University of Athens and the University of Cyprus. This cross-fertilization has a notable benefit for the studies while giving the opportunity for more communication among the researchers. These organised graduate programmes had an impact on the production of new researchers and on the professional development of teachers. Nowadays, the Greek community of the mathematics education researchers consists of faculty members as well as of researchers working abroad, school advisors, teachers and newly graduated PhD students. These scientists have started to make their presence in the international research scene by contributing to conferences like PME, CERME, ICME and journal publications.

Apart from the production of new researchers through postgraduate programmes, Greek researchers have been exerting some impact on the mathematics education system. They have been involved, for instance, in in-service teacher education initiatives by the Ministry and in the production of educational material for the curriculum. Moreover, Greek researchers took advantage of the interest and impetus of the Ministry of Education to include the use of digital technologies mainly in secondary education in the mid-nineties (Odysseia Project, 1996-2000). Through the Computer Technology Institute, they participated in the shaping of policy to introduce pedagogical innovation based on the use of digital media, supported the translation of a group of well-known pieces of exploratory software for mathematics education and the proliferation of a small number developed in Greece[i]. They took part in the design and implementation of seminars to prepare teacher educators within a two-phased project to provide in-service courses to mathematics teachers aiming to affect about 4000 schools. These seminars addressed issues of mathematics education, pedagogical innovation and uses of exploratory software for experiential mathematical activity. This teacher education initiative involved the preparation of documentation, the availability of software and the development and use of community building distance education infrastructure. This is seen as infrastructure for scaling up the project of in-service teacher education. Furthermore, ways of describing educational activity plans developed at the academic level[ii] were modified with the collaboration of teacher educators from Odysseia. They were then used both to work with teachers at the wide scale, but also as the central means of communicating ways of using digital media in mathematics education. They were also used as a means for teacher empowerment in the sense of recognition and reification of teachers' personal pedagogies in the sense that all teachers were asked to develop scenarios and then use them to discuss issues of epistemology and of teaching and learning mathematics.

The Cypriot research community. The Department of Education of the University of Cyprus filled the first position in mathematics education in 1992, in the same year when it accepted its first students. Two more positions were soon filled and by the end of the decade, the group of four math educators was completed. The first Cypriot participation in PME Conference appeared in 1995 in Lisboa. The first Master's programme in Mathematics Education was launched by the Department of Education in September 1997 and the first PhD Dissertations were completed by 2003. Every year since then an average of three new doctoral dissertations are accepted by the Department. In the meantime, this small group of researchers supported by graduate students had an increasing research activity and production as witnessed by their contribution to PME, CERME, ICMI and other international conferences as well as by publications in international journals. For instance in the proceedings of the PME 28 (Norway, 2004) the Cypriot group has contributed 12 research reports (and a participation in a research forum); the same number of research reports were also published in the proceedings of the PME 30 (Prague, 2006). Among other activities, the Cypriot group organized CERME 5 (2007). As from 2008 three private institutions have been accredited by the State as Universities; all three of them have departments of education, and mathematics education is among their priorities.

As regards the areas on which Cypriot researchers have focused their interest, not surprisingly they vary. In any case, one could mention the research on the affective domain, in the wider sense of the concept (in connection to several central parameters in education), problem solving, representations, the application of technology particularly as regards dynamic geometry, etc. Some of the projects undertaken were funded by the University, by the Cyprus Research Promotion Foundation and by European programmes.

Cooperation between the Greek and the Cypriot Research Communities. Though historically Greeks and Cypriots share several common characteristics in terms of culture, language, etc., there are also many differences in terms of the context in which the two mathematics education communities operate. For example, the issue of the impact of research on practice has a different meaning in the Cypriot educational system than in the Greek Educational system. The structure of the university, the schools and the society in general is also different. Moreover, the researchers in Greece are spread out in different geographic areas, which makes cooperation on a regular basis difficult while in Cyprus they are mainly around the same city and a large number of them work in the same university. Relations between the two communities were always very close and cooperation has also been developed between some Greek and Cypriot researchers both at the research and teaching level. This cooperation became even closer since 2004 when the Greek Association of Researchers in Mathematics Education (GARME) was established. This association aims to promote research in Mathematics Education both in Greece and in Cyprus, through the regular means and methods applied by similar associations worldwide. Among other activities, GARME organizes a conference biannually, each time at a different university and has already published the first three issues of its journal “Research in Mathematics Education” (in Greek).

The Greek Mathematical Association (EME) is another organization through which Greek and Cypriot mathematicians have traditionally pursued their academic and professional development. EME was established in 1918 and it also organizes a Conference each year while publishes a few journals, mostly catering for the needs of students and teachers in practice.

THE APPROACH FOLLOWED IN PREPARING THIS REPORT

Since no comprehensive data base was available, the Council of GARME, in order to prepare this report, asked each of our colleagues to send us a short summary of their research with main results. After receiving this summary from all interested people (about 40 researchers), the next step was to look for a possible categorization of this volume of summaries. Even though we are aware that any grouping of the published research would unavoidably seem superficial due to the obvious fact that people frequently combine variable parameters and issues in their work, in this review we could not avoid categorization - for organizational purposes. In what follows, this review is organized in five sections: Mathematics teaching and learning; Affective domain in learning and teaching mathematics; History of mathematics and its connection to teaching and learning mathematics; Technology in learning and teaching mathematics; and mathematics teacher development.

MATHEMATICS TEACHING AND LEARNING

The learning of mathematics constitutes a major research stream for the Cypriot - Greek mathematics education community. There are shifts that can be identified over time with respect to the themes studied, the theoretical frameworks and the data collection tools adopted. The early years of the relevant research activity are characterized by a fairly narrow focus on mathematics concepts and processes (dominantly arithmetic and algebraic) and pupils’ difficulties in dealing with them, within a theoretical framework rarely going beyond constructivism. A noticeable broadening of themes can be observed during the 1990s, encompassing new content areas, the exploitation of the use of alternative representation modes and technological tools and the employment of theoretical perspectives, which allow the viewing of mathematics learning not as an exclusively psychological process but also as a socio-cultural one. Each phase of this course of research development was influenced by the international dominant research tradition. Nowadays, this scene, trying to capture the complexity of the mathematics learning process, tends to focus more and more on the sources and paths of meaning making which students are engaged in when dealing with mathematics in formal or informal contexts. In this section, the research activity on mathematics learning is first discussed along two dimensions, namely, to the cognitive and to the socio-cultural factors involved. Then, the focus is turned on the research concerned with aspects of teaching mathematics.

With respect to the cognitive factors influencing the learning of mathematics, children's early mathematical development represents a major area of interest. There is a variation of research themes in this area, some of which sustained attention over the years, such as (mental) representations (Kaldrimidou, 2002), reasoning (Tzekaki, 1996; Gagatsis & Patronis, 1990), cognitive styles (Pitta-Pantazi & Christou, 2009) and concept acquisition and development (Nunes, et al., 2003; Zacharos, 2006), mostly in relation to elementary arithmetic, and others that became part of the research agenda fairly recently, e.g., stochastic thinking and statistical concepts (e.g., Kafousi, 2004, Skoumpourdi, et al., 2009). The research paradigms utilized in these contexts tended to view young children's learning predominately from a constructivist perspective and exploited mainly qualitative research tools.

Other areas of cognitively oriented research that can be identified include mostly mathematical transversal issues, such as advanced mathematical thinking, mathematical modelling, problem solving, the role of (external) representations in learning mathematics, inductive/deductive/proportional reasoning, the development of cognitive and meta-cognitive abilities and proving. The majority of the respective studies were carried out within the context of primary education, some concern secondary students and very few university students. In particular, there has been considerable research output on the learning of advanced mathematical ideas, mainly of functions (Kaldrimidou & Ikonomou, 1998; Elia & Spyrou, 2006), but also of Calculus and Real Analysis concepts (Biza & Zachariades, 2007; Mamona- Downs & Downs, 2008). This research has been shaped by a series of related theoretical considerations, e.g., concept image, students' meta-cognitive conceptions, conceptual change, etc.

Problem solving has attracted the interest of a number of Greek and Cypriot researchers predominately as a meaning construction process but also in relation to mathematical modelling, which offers a unique venue for developing students' problem solving abilities (Mousoulides, et al., 2008; Mamona - Downs & Downs, 2005). In addition, the role of external representations in learning mathematics has been the subject of extensive research, which concentrated on the exploitation of multiple models/ representations (mostly geometric) for this purpose, revealing aspects of the strategies employed by pupils to access the targeted ideas via the models offered (Gagatsis, et al., 2004, 2006). Finally, reasoning, cognitive and meta-cognitive abilities as well as proving are of concern in a number of studies, which attempt to investigate the place, the power and the development of these processes and abilities within the context of learning mathematics (Modestou & Gagatsis, 2007; Pitta-Pantazi & Christou, 2009).

Of course, the above presentation of the cognitively oriented research cannot be exhaustive. For example, there are studies which do not strictly fall in the categorization adopted above, like those focusing on pupils' difficulties and performance in specific mathematical content areas (e.g., probability and statistical thinking, Lambrianou & Afantiti-Lambrianou, 2003; algorithms, Kourkoulos & Keyling, 2004) or on the relationship between students' mathematics performance and their attitude towards learning mathematics (e.g., Barkatsas, et al. 1998). Also, there are studies concerned with the development of tasks and the introduction of innovative approaches to facilitate the learning of mathematics in different content areas (e.g., geometry, Markopoulos and Potari, 2000; Triantafillidis, 1995; Elia & Gagatsis, 2003; Kynigos, 1993; probability, Tatsis, et al., 2008; algebra, Marmara & Hatzikiriakou, 2007).

With reference to the socio-cultural factors influencing mathematics learning, the research developed is much less compared to the one related to the cognitive factors but does not lack quality and significance. The relevant studies address learning as a mathematics meaning construction process which is shaped by contextual parameters. Some of these parameters are of social nature (e.g., gender, Chionidou- Moskofoglou, 2003, social interactions and practices, Patronis, 1997; Kynigos & Theodossopoulou, 2001, Tatsis & Koleza, 2006, 2008; Chassapis, 1999; Chaviaris, et al., 2007) and others are related to cultural issues (e.g., learning mathematics in culturally diverse contexts, Stathopoulou & Kalabasis, 2007; Sakonidis, 2005; Chronaki, 2005). The empirical work under consideration employed a variety of theoretical lenses for its purposes, such as Vygotsky's theory, symbolic interactionism, the situated learning approach, Bishop's enculturation/acculturation perspective and ethnomathematics. The results of the relevant studies shed light on the functioning of a number of practices and tools of social as well as cultural nature, which play a decisive role in the establishment of shared meaning, such as the socio-mathematical norms and the technical and symbolic tools respectively.

Mathematics teaching and indirectly learning are the subject of a number of classroom studies which focus on how teachers' decisions and actions shape the mathematical meaning constructed by the pupils. In particular, a series of studies attempted to analyze complex mathematics classroom phenomena from an epistemological point of view, i.e., with respect to the nature of the mathematical meaning shaped. Few of these studies examined how teachers' management of the epistemological features of the mathematical knowledge influence this meaning in a variety of contexts, e.g., in primary / secondary classes, in algebra / geometry lessons and in traditional and innovative settings (e.g., Kaldrimidou, et al., 2000). Another part of the studies under consideration looked at how teachers' instructional practices determined the nature of the mathematics knowledge emerging in a number of occasions, e.g., when teachers intervene in pupils' working on a task (e.g., Tzekaki, et al., 2002). One of the basic findings of these studies is that teachers manage the epistemological features of mathematics in a homogeneous manner, which distorts the mathematical meaning emerging in the classroom. Considering the complexity of classroom teaching, Potari and Jaworski (2002) focused on the relationship between mathematical challenge and students' sensitivity in order to explore the meaning of effective teaching. The results indicate that, when there is a balance between these two elements, there is some evidence that students construct the targeted mathematical meaning. Regarding effective classroom practices, Cobb et al. (1997) contend that participation in reflective discourse supports pupils' mathematical development. Finally, teaching practices and their impact on students' learning have also been of concern in a study, which looked at differences between primary and secondary instructional characteristics and, through a designed intervention, attempted to make students' respective mathematical experiences compatible (Sdrolias & Triadafillidis, 2008).

AFFECT AND MATHEMATICS TEACHING AND LEARNING

We note that researchers unanimously agree that mathematics learning encompasses cognitive as well as affective variables. Learning could be considered as a function of multiple variables including cognitive, affective and socio-cultural ones. Humans are cognitive, affective and social beings, in the sense that all human mental processes occur in context and are inseparably combined with and strongly affected by affects. The affective domain involves emotions, attitudes, beliefs and values, and it is highly connected to motivation, metacognition and self regulation. Affects are also a social construction within a social group and group values have a deep influence on learning. In this section we focus on affect as related mainly to cognitive variables.

Greek and particularly Cypriot researchers have examined several dimensions of the affective domain (attitudes, beliefs, motivation) in connection to one or more cognitive or social factors related to mathematics teaching and learning. Attitudes and beliefs have been examined in several contexts in many papers; Philippou and Christou (1998) have examined the impact of a teacher preparation programme, based on the history of mathematics, on preservice teachers' attitudes about mathematics. The primary finding was that a considerable proportion of students enter the Department with very poor attitudes towards mathematics, a subject they will be called to teach after graduation; they found that these attitudes had improved after the students have been exposed to the programme of study. The same programme was also evaluated in terms of students' efficacy beliefs to teach mathematics; it was found to have a positive impact on students' efficacy beliefs, as measured soon after they were employed as school teachers (Philippou, & Christou, 2002).

The development of pre-service teachers' efficacy beliefs and epistemological beliefs were also examined in connection with their mathematics knowledge, the other pivotal factor contributing to teachers' performance in teaching mathematics. In this regard, Charalambous et al. (2008) focused on the nature and development of preservice teachers´ efficacy beliefs during both coursework and fieldwork, as well as of in-service teachers' epistemological beliefs. In a similar vain, Nicolaou and Philippou (2007) examined students' efficacy beliefs in connection to their ability in problem posing, as well as to their general performance in mathematics.

The teacher is by far the key factor in any effort to reform the mathematics curriculum. In this regard, teachers' concerns arising from the introduction of a new mathematics curriculum have attracted the interest of Cypriot researchers. Christou et al. (2004) have found deficiencies that maintain and even increase teachers' concerns, arising primarily from limited teacher preparation. In this respect, Charalambous et al. (2002) have concentrated on a specific unit of the new curriculum to further examine teachers′ concerns in connection to their efficacy beliefs.

Other researchers studying affective variables include Markou and Philippou (2005), who examined students' motivational beliefs, self-regulated learning and problem solving ability, and Panaoura and Philippou (2007), who focused on metacognition.

HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION

The integration of history and epistemology of mathematics in teaching has attracted the interest of many Greek researchers. The idea of using the history of mathematics in contemporary pedagogy, as a means to promote learning, springs from the adoption of a "genetic" approach, applied to the teaching of mathematics. The naive "recapitulation principle" has gradually been replaced by the genetic approach, a more balanced consideration of history, which in simple terms examines the pros and cons, the "whys" and "hows" of using history in mathematics education. The evolutionary approach of teaching mathematics has recently gone a step forward to examine ways in which history can illuminate the present either in the form of an epistemological laboratory, or in the sense of developing general schemes in the form of guidelines, which could be applied in special cases (Tzanakis & Arcavi, 2000).

Research in this area could be summarized on the basis of the two major non-strictly separated questions, namely the "whys" and "hows" of using history in mathematics education. All research in this field has someway or other focused on using history either as a tool or as a goal (why) mainly through three different approaches: the illumination, the module (or unit), and the history-based approach. Greek and Cypriot researchers have contributed in this field in various ways. Tzanakis had a notable contribution as a researcher and as a leading person in various international organizations and activities, e.g, he served as the Chair of The HPM Study Group (2004-2008), the Organizer of the *ASG Meeting of the HPM Group* at the ICME 11 etc. Among his publications we note a major contribution in the ICMI study (Fauvel & van Maanen, 2000), in which Philippou had also a contribution in chapters 2 and 4.

We have already mentioned the development and application of teacher mathematics preparation programme based on the history of mathematics. This programme was organized along the evolutionary line following the development of mathematical ideas; it might be considered as an example of the history-based approach, organized along the development of mathematical concepts and methods from the pre-Hellenic period to our modern times, aiming at motivating students' learning and enhancing their affective relation with mathematics, by humanising mathematics.

Another strand of research concerns the study of possible parallelisms between the historical development and the cognitive development of mathematical ideas (Thomaidis & Tzanakis , 2007). In this respect, several aspects have been empirically examined, both with students and teachers, taking into account the differences between the modern classroom and the social milieu of the scientific community in the past.

A third strand of research concerns the ways to integrate original sources in classrooms and their educational effects. In this context, some researchers have implemented the use of original texts of ancient Greek Mathematics in teaching high school students that resulted in the development of relevant educational material.

Some scholars examined the role of the history of Geometry in learning and teaching of Mathematics, in view of the Constructivist paradigm and the socio-cultural approach. Geometrical constructions that “solve” the three famous problems of antiquity have also been studied, while other researchers tried to trace the development of complex and non-regular shapes in published books of mathematical education during the past two centuries.

TECHNOLOGY IN TEACHING AND LEARNING MATHEMATICS

So far, there has been a range of theoretical frameworks and constructs employed to study uses of digital media for teaching and learning mathematics, mainly within a constructivist or a socio-constructivist approach to learning focusing on the generation of meanings, on problem solving, on the process of students’ mathematizations, on the use of representations and on the creation of socio-mathematical norms in the classroom. The main methodological stance has been twofold. That of design research embedded in a wider culture in both countries for inclusion of academic researchers in searching for ways to generate educational reform and that of case-study research focusing on meaning making per se. This research ranged from studies including the design and development of exploratory digital media to those involving the development of tasks and microworlds with existing tools and to those only addressing learning practices in case studies or at the classroom level. The learning of a variety of mathematical concepts has been addressed, such as the tangent line for upper secondary and tertiary level, ratio and proportion, the tangent function, periodicity, vectors and a variety of geometrical concepts for secondary and primary levels. With respect to teacher education, the challenges of teachers’ epistemologies and beliefs have been studied as they used digital media to design student tasks and as they are engaged in communities of argumentation and practice in blended in-service courses. With respect to the design and development of digital media, E-slate Turtleworlds and MaLT integrate turtle geometry and dynamic manipulation of variable procedure values (in 2d and 3d geometry respectively), Euclidraw combines a dynamic geometry book with a computer algebra system, Cruislet applies a variety of mathematical concepts to navigation in 3d geographical space, E-slate data-handling uses the ‘Tabletop’ software idea on a powerful relational database and Iris integrates mathematics and art on an Imagine-Logo platform.

For example, Christou et al. (2004) investigated students' emergence of proof as they engaged in activity with dynamic geometry software. Biza & Zachariades (2007) described a method to introduce late high school and early university students to the concept of derivative with the use of Euclidraw while Kynigos and Gavrilis (2006) focused on students' meanings with respect to the concept of periodic co-variation using a specially designed Turtleworlds microworld in a classroom case study. Kordaki and Potari (2002) carried out case study research to investigate high school students' understanding of preservation of area using a specially designed microworld and adopting a Vygotskian tool-artefact perspective. This study was followed by a study focusing on students measurements of area in the same context (Kordaki, 2003). Papadopoulos and Dagdilelis (2006) also focus on students' understanding of area in irregular shapes with Cabri, using a framework from distance education. Markopoulos and Potari (1996) adopted a constructivist perspective as they studied primary level students' meanings as they were formed in a longitudinal case study with the use of a specially designed piece of software. With respect to research on stereometric concepts, Christou et al. (2006) discuss the design principles of a 3D math dynamic geometry software while Kynigos and Latsi (2006) study students' conceptions of the concept of vector with a MaLT microworld simulating a computer game. Much earlier, Kynigos et al. (1997) elaborated on the design and longitudinal development of a component based authoring system for mathematical microworlds. Within the framework of classroom research, Kynigos and Theodossopoulou (2001) studied the kinds of socio-mathematical norms developed in student triads engaged in collaborative investigations with Turtle Geometry. Extended classroom research was also carried out by Psycharis and Kynigos (2004) on students' generation of meanings around ratio and proportion with the use of a Turtleworlds' microworld.. Finally, Veriki and Chronaki (2008) adopted a social perspective in addressing issues of gender and uses of technology outside the school.

Little research work has been carried out on teacher education in the uses of technology in teaching and learning mathematics. Kynigos (2007) studied how teachers' beliefs were challenged as they collectively designed and developed microworlds for their students to engage in experiential mathematical learning while Makri and Kynigos (2007) focused on teachers' reflections through on-line discussions on teaching and learning mathematics.

MATHEMATICS TEACHER DEVELOPMENT

This area of research has been developed around two dimensions: one concerns mathematics teacher knowledge and beliefs and the other mathematics teacher professional development. Most research in this area focuses on pre-service teacher development at primary level while few studies refer to secondary mathematics teachers and in particular at the in-service level.

Concerning teachers' beliefs, the Cypriot team has focused on epistemological and self efficacy beliefs of both pre-service and in-service primary school teachers as it has been discussed in the section "Affect and Mathematics Teaching and Learning". Another area of study considered by both communities is teachers' conceptions about students' mathematical assessment. Two examples are the studies of Philippou and Christou (1997) and Sakonidis & Klothou (2007). The former study adopts a cognitive perspective while the latter a social perspective and in particular Bernstein's theoretical framework to examine the pedagogical discourse of teachers with respect to assessing. The main methodological tools of both studies are questionnaires and the mostly quantitative data analysis. The findings indicate micro and macro factors that influence teachers' conceptualization of assessment. Another example of research on beliefs considering a rather situational view of beliefs is a longitudinal case study of a primary school teacher from the last year of her university studies until the third year of her teaching in school (Potari & Georgiadou-Kabouridis, 2009). Two specific beliefs related to mathematics learning and teaching were investigated through the teacher's actual practice and reflections. The methodological tools included interviews and classroom observations and the data analysis was qualitative. The findings indicated that the period of the university was crucial to the formation of these beliefs and that the relation between beliefs and practice was dialectical. The impact of pre-service education on student teachers' beliefs was also identified in a study by Koleza and Iatridou (2006), which concluded that perspective teachers' participation in problem solving affects their self efficacy beliefs. Furthermore, Kynigos and Argyris (2004) also conceived a dialectic relationship between beliefs and practice in the context of computer-based exploratory mathematics in the classroom. In this study elementary teachers' beliefs and practices were investigated after eight years of innovative practice involving one-hour-per week computer-based mathematics classroom activity with small cooperating groups of pupils. Through quantitative and qualitative analysis of video recordings and semi-structured interviews, the study indicates that teachers' actions may be influenced by their belief systems as well as by wider cultural perspectives. At the secondary level, the study of Biza et al. (2008) examined secondary mathematics teachers' pedagogical and epistemological beliefs about the role of visualization in proving. These beliefs were investigated through qualitative analysis of teachers' written responses on a number of hypothetical classroom scenarios and of focused interviews with some of the teachers. The findings indicate that the teachers lacked awareness of the limitation of the visual representations in proving or disproving claims.

In terms of mathematics teacher knowledge, there are a number of studies that report prospective teachers' difficulties in mathematics in general or in specific mathematical areas. For example, the results of the study of Lemonidis et al. (2006) indicated that prospective primary school teachers' mathematical performance varied in different content areas and was related mainly to prospective teachers' mathematical background in high school and not to their mathematical experiences at the university. On the other hand, the work of Stylianides et al. (2007) focused on specific strategies that the prospective teachers developed while facing written tasks related to proof by induction. Concerning prospective secondary school teachers, the work of Giannakoulias et al. (2007) investigated prospective secondary school teachers' difficulties concerning the properties of the set of Real Numbers and reported certain misconceptions while the work of Stylianides et al. (2004) investigated undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. The methodological tools of most research studies in this area were written tests and questionnaires. These tests include mathematical tasks or hypothetical classroom scenarios that are designed by taking into account specific students' difficulties reported in the literature. The scenarios have been used mainly for exploring in-service teachers' knowledge (e.g., Biza et al., 2007; Mastorides & Zachariades, 2004). Recently, there are few research examples that link mathematics teacher knowledge to the actual practice (e.g., the work of Charalambous (2008) at the pre-service level and the work of Zachariades et al. (2008) at the in-service level) and others that focus on the way in which teaching practices shape the mathematical meaning constructed in the classroom (e.g., Kaldrimidou et al., 2008) The methodological tools of these studies included classroom observation and interviews and their results indicate different dimensions of mathematics knowledge for teaching.

The last area of research is about teachers' professional development. This theme has been studied both at the pre-service and in-service level and especially in primary mathematics teacher education. Although one issue on teacher development is the challenging of teacher beliefs, something which has been discussed previously in this section, the focus of most studies that are reported here is the context that supports teacher's development. Paparistodemou et al. (2007) analysed prospective teachers' lesson planning, actual teaching and reflections and indicated their difficulties to identify the central mathematical issues in their designed tasks and to link theoretical knowledge about children's mathematical learning to their lesson planning and actual practice. In the context of in-service education professional development has been studied through longitudinal studies in the context of collaboration between teachers and researchers (e.g., Georgiadou- Kabouridis, 2001; Georgiadou-Kabouridis et al., 2004; Sakonidis et al., 2007). The studies adopt a community of inquiry perspective and consider teacher professional development through teachers' involvement in action research. The findings indicate the crucial role of the teacher educators – researchers in encouraging teacher reflection and the development of teachers' beliefs concerning specific areas of mathematics teaching and learning.

CONCLUDING REMARKS

We found this synthesis of Greek and Cypriot research in mathematics education a challenging task which can only portray what is going on in broad terms and cannot easily capture the dynamic and evolving nature of this research. What we feel is important however, is the added value of Greek and Cypriot researchers who feel more and more as part of the GARME community that appears to have an impact on the nature and the orientation of the research. It is through this community that they can gain by directly sharing ideas in their native language, coming into contact with mathematics teachers all over Greece and Cyprus, incrementally contributing to systemic reform in both countries and finally contributing perhaps in a more coherent and potent way to the growing knowledge about mathematics education worldwide. With respect to the impact of the research community in educational practice, we still have a long way to go given the challenges posed by the Greek and Cypriot context. The cultural and systemic obstacles described at the beginning of the paper can only be gradually addressed and researchers' crossing over to decision –making mechanisms is inevitably taking its time. As in each country, there are specific and idiosyncratic contextual issues bearing on attempts by researchers to give time to contribute to systemic reform initiatives. In Greece for example, the educational paradigm, the exam system, the centralized curriculum and the lack of access to teacher support are particularly pertinent. In Cyprus, however, despite the long prevailing adverse situation, in recent years there are encouraging signs of improvement. Overall, the research carried out so far provided some insights into the challenges posed by the specific educational contexts and had an impact on the research development as well as on the implementation of the mathematics education policies.

If we may be so bold as to make a first attempt to synthesize across this research in order to pinpoint some particular characteristics we can say that Greeks and Cypriots have made a contribution to research on mathematical meaning making and affect and on forging connections between the history of mathematics which is engrained in our culture and mathematics education. Furthermore, particular strands are emerging on challenging teacher beliefs and on teacher education methods within a context of centralized education and minimal in-service (and pre-service at secondary level) education. That is, this research is grappling with the issue of how to initiate teacher education so as to constructively challenge teachers to perceive their profession as a developing rather than a static one and to develop sensitivities to issues of teaching and learning mathematics. It is within this framework that research on teacher epistemology, pedagogical and content knowledge is emphasized by the two communities. Finally, through participation in European research projects mainly in uses of technology in mathematics education and in middle scale systemic teacher education projects, Greek and Cypriot research is contributing to knowledge regarding the integration of theoretical frameworks, contexts and practices and to the quest for finding new conceptual tools to understand design perspectives at large scale.

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[i] Cabri Geometre, Geometer's Sketchpad, SimCalc, Tabletop and Function Probe, E-slate Turtleworlds and DataHandler, MachineLab Turtleworlds (MaLT), Cruislet <http://etl.ppp.uoa.gr/_content/download/index_download_en.htm>, < http://etl.ppp.uoa.gr> <http://etl.ppp.uoa.gr/_content/download/index_download_en.htm> and Euclidraw, http://www.euclidraw.com/Gr_fls/EUC_greek.html

[ii] See for example the European Project 'ReMath', http://remath.cti.gr.

SHORT ORAL COMMUNICATIONS

THE ROLE OF CONTEXT: USING THE WORLD TO UNDERSTAND MATHEMATICS OR MATHEMATICS TO UNDERSTAND THE WORLD?

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Exploring a sample of curriculum documents reveals a shared aim to equip learners to use mathematics effectively in the real world, but different purposes for which contextualised problems might be used. I argue that there currently exists an important but disguised confusion which poses significant challenges for curriculum designers.

I shall report a study of curriculum documents from the UK, USA, Singapore, New Zealand and South Africa which reveals that whilst there is a common recognition *in their aims* of the importance of children being able to make use of mathematical ideas in everyday life and to make sense of the world, the advice on curriculum content offered to teachers often has a narrow focus on the use of contextualised word problems as the way to enable children to accumulate the understanding and skills to apply mathematics beyond the classroom. Amongst the considerable volume of research concerning the use of such problems there is substantial evidence of their failure to provide a bridge between the classroom and the real world. Indeed it appears that many children answer word problems 'correctly' precisely because they attend primarily to the form of the question, and largely ignore the context, leading them to 'Perceive school word problems as artificial, routine-based tasks which are unrelated to the real world'(Verschaffel, Greer & Torbeyns, 2006, p.60).

I suggest that the problem with word problems is symptomatic of an inherent confusion about the role of context in mathematics education: do we use the real world to understand mathematics, or mathematics to understand the real world? Contextualised word problems appear to be about the real world, but I argue that often they are actually about mathematics, and that we need to look elsewhere to find school activities which fulfil the aims in our curricula by offering children opportunities to see the world through a mathematical lens.

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CONCEPTIONS OF BETWEEN-RATIOS AND WITHIN-RATIOS

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In this study, with two prospective elementary mathematics teachers, Rita and Mark, I studied the nature of conceptions of between-ratios and within-ratios and made distinctions with respect to those conceptions. In two one and one-half hour written sessions followed by one-hour long clinical interviews, these two prospective teachers' use of informal and formal strategies and justifications behind those strategies in the context of ratio were examined. Their justifications suggest that an understanding of between-ratios does not require an understanding of within-ratios; and an understanding of within-ratios might develop to some degree independently of an understanding of between-ratios.

Several researchers have studied the conceptions of ratio, mainly within-ratios, such as speed (Thompson, 1994), ratio-as-measure (Simon & Blume, 1994), identical groups conception and ratio-as-quantity (Heinz, 2000). In this study the focus was more on between-ratios and the relationship between conceptions of within-ratios and between-ratios.

Results of the study showed that Rita had abstracted the result of division of the quantities from the same measure space, between-ratios, as the multiplicative change from one situation to the other. On the other hand, she did not have any meaning for the result of division of quantities coming from different measure spaces, within-ratios, neither as an intensive quantity nor as a per-one interpretation. Similarly, data showed that Mark, at the beginning of the interview, abstracted the quotient of the quantities in within-ratios as per-one interpretation. At the same time, he did not have any meaning for between- ratios. That is, he had not abstracted that between- ratios represent the change factor from one ratio situation to the other.

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DEVELOPMENT OF A COURSE FOR PRE-SERVICE MATHEMATICS TEACHERS ON FORMATIVE ASSESSMENT: THE CASE OF QUESTIONING[1]

Hatice Akkoç, Erhan Bingolbali, Fatih Ozmantar

As Heritage (2007) points out, assessment and teaching have been traditionally seen as reciprocal activities as a result of measurement concerns such as high-stakes accountability of testing. Despite its importance for learning and teaching, assessment has not been a main focus of teacher training courses. In this study, we designed a course for pre-service mathematics teachers, which aims to develop their assessment skills. This paper focuses on how a pre-service teacher developed the knowledge and skills of questioning during a lesson for the purpose of formative assessment.

A three-hour workshop was conducted which aims to develop the knowledge of assessment for different purposes (summative and formative) in the process of teaching and skills for applying classroom assessment for formative purposes to achieve the objectives of a lesson. A case study was conducted and a female pre-service mathematics teacher's (Rana's) development was investigated. Rana was asked to prepare a lesson plan which introduces the concept of derivative before and after the workshop. Using this lesson plan, she taught a derivative lesson in the school placement after the workshop. She was also interviewed on her lesson preparations.

Results indicated that Rana's knowledge of questioning aiming at formative assessment has dramatically improved. In her first lesson plan, she did not use any questions in the process of teaching to provoke student thinking. She considered assessment as a separate activity which occurs after a teaching episode rather than as an integral part of the lesson. In that sense, she considered assessment for summative purposes. On the other hand, in her second lesson plan which she prepared after the workshop, she planned questions aiming at provoking student thinking. In practice, she used the planned questions and interpreted answers from students on the spot and asked follow-up questions accordingly. With regards to the course design, it can be comfortably claimed that the course contributed to Rana's development a new perspective for formative assessment.

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Heritage, M. (2007). What Do Teachers Need to Know and Do? *Phi Delta Kappan*, 89 (2), 140 – 145.

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THE NATURE OF RISK

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The aim of this short oral presentation is to discuss the nature of risk and to expose the issues on peoples' understanding of risk. The intended study will be based on design research, to explore tools to analyze students' 'thinking-in-change' on risk. Theoretical elaboration will be supplemented by my analysis of data to be presented.

ISSUES IN THE NATURE OF RISK:

In this short oral paper, I will summarize some significant issues in the study of risk. Risk is an important socio-scientific concept impacting deeply on decision-making in everyday life yet there is confusion about the meaning of the notion.

Risk is often presented as a probabilistic idea, or alternatively the emphasis is placed on impact. A common proposal is that risk involves the coordination of probability and impact. Campbell (2007) seeks to clarify this confusion through an example of two bridges crossing over a ravine. One bridge is about to collapse whereas the other looks better. Which bridge should one choose to cross the ravine? Perhaps the better choice of bridge is clearly the latter because there is less chance of the bridge collapsing. But what if the latter bridge crossed a very deep part of the ravine where as former one crossed a shallow part of the ravine? Now the impact of crossing the first bridge is greater. Surely, the decision will be based on a trade off between probability and impact. If we imagine the judgement of risk to be underpinning such a decision, thus, risk is in this case, a coordination of impact and likelihood of event.

Firstly, many people would constrain their thinking about risk to a single dimension perhaps because making such a coordination of two dimensions is difficult for many reasons. Some argue that probability is subjective but there can be little dispute that in most situations the judgement of impact is subjective. Risk perception is affected by the media, which can magnify the judgement of both likelihood and impact. People's judgements of risk are affected by personal experience or a judgement about a population, the degree of control that can be exerted over the situation by the individual and many other factors that will be explored in the presentation.

How then can we learn to make better judgements of risk? I aim to design a microworld to perturb thinking about risk by enabling the student to embrace the complexity hinted at above. The rationale for software design will be shown at PME.

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A BALANCE IN CHINESE MATHEMATICS INSTRUCTION AND ITS CULTURAL INFLUENCES

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The goal of this study was to investigate the features of a balanced mathematics classroom teaching in China under the influence of Chinese culture from analysing videotapes of classroom teaching. The research questions are: How does Chinese culture influence mathematics classroom teaching? Is there any pattern in conceptual understanding, procedural fluency, and real word application?

METHODOLOGY

Subjects. A total of 15 lessons from three 3rd-8th grade teachers from three cities in China were observed and videotaped. Of the three teachers, all have been teaching for more than three years.

Data Analysis. Used Analytical Model and NVivo 8 software for data analysis. Analytical Model included a sequence of eight interacting, non-linear phases: 1) Videotaping classroom teaching attentively, 2) Discussing and describing the video data, 3) Identifying critical events (moment-by-moment), 4) Transcribing, 5) Coding, 6) Constructing storyline, 7) Review teachers' reflection and lessons, and 8) Composing narrative. Two researchers coded the video lessons, and coding was based on analyzing the major components for balanced teaching in the MSA approach (Wu & An, 2006): 1) Models for conceptual understanding (M); 2) Strategies for procedural fluency (S); 3) Applications for real world connection (A).

RESULTS

The results show that the three teachers had their own unique teaching patterns in three indicators of the MSA. Although each teacher differed slightly in three indicators, their teaching was well balanced with focusing on modelling and strategy development. Teacher reflection data confirmed why Chinese teachers have such teaching patterns: they are influenced by the Confucian philosophy of learning: 熟能生巧(practice makes fluency),知其然，知其所以然(Knowing answers and knowing why such answers), and 学以致用(Apply what you have learned).

SIGNIFICANCE OF THE RESEARCH

Well-balanced teaching combining three components of the MSA has a great impact on students' learning. In order for teachers to develop a balance in teaching, culture plays a vita role. This study suggests that researchers and educators need to take cultural consideration and develop teachers' disposition toward effectiveness of teaching mathematics, thereby developing their unique patterns in a balanced way of teaching mathematics.

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IMPACT OF MATHEMATICS EDUCATION RESEARCH IN THE CLASSROOM AND WHAT THE MATHEMATICS CLASSROOM COULD *SAY* TO US RESEARCHERS

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This study is part of our doctorate research (ANDRADE, 2008) that investigates the relationship between research and classroom in Mathematics Education. In this presentation, we will discuss about the impact of Mathematics Education research in the classroom and what the classroom could show to us researchers. The methodology used is orientated mainly in Analysis of discourse. The survey of data/facts and their analysis include discourses of 71 Mathematics Education researchers (44 Brazilians and 27 from other countries: Australia, Canada, Denmark, France, Israel, New Zealand, Portugal, Southern Africa, United Kingdom, and United States) that was obtained through open and discursive research questionnaire. We will approach results regarding to the questions 02, 04 and 05 it. Question number 02 concerns the evidence of the Mathematics Education research impact in the classroom. It was observed that the idea of the research that cause impacts involve well-structured projects, as well as teacher education one. However, there is the observation that the real impact occurs when one modifies the non-standard model classrooms. There is also a strong defense for the research of the collaborative type, action research, participatory or similar, in the belief that they can have a better impact in the classroom than other methods of research. Question number 04 - how research can contribute more effectively with the change in the Mathematics classroom - showed that the teacher is seen as the main vector of relationship between research, researcher and classroom. Question number 05 - what the mathematics classroom could say to us researchers – showed that is especially through whom/how is the student that it can "tell" something both to us researchers and the research. It can also tell something about the teacher, how is his/her practice. It may point everything or almost everything that a researcher needs to conduct research in Mathematics Education.

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TOWARDS A RESEARCH AGENDA ON CRITICAL MATHEMATICS EDUCATION AND ACTIVITY THEORY: A VIEW ON GEOGRAPHY STUDENTS' PRACTICES IN MATHEMATICS

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Our main goal is to open a research agenda founded on a tentative harmonic dialogue between critical mathematics education and activity theory, from both theoretical and methodological point of view. The possibility of dialogue arose from the analysis of a mathematics education practice of undergraduate students of the geography program at the Federal University of Minas Gerais (UFMG), Brazil.

The main concern of critical mathematics education is the development of *mathemacy* (Skovsmose, 1994), whose objective is not to merely develop the ability to carry out mathematical calculations, but also to promote the critical participation of students/citizens in society, analyzing and discussing political, economic, and environmental issues where mathematics plays a major role as a technological support. Cultural historical activity theory, by its turn, approaches cognition and behaviour as embedded in collectively organized activity systems which are mediated by artefacts (Engeström, 1987). Within this framework, activity is conceptualized as social practices oriented at objects. Here, people motives play a central role given that objects are conceptualized assuming that an entity becomes an object when it meets a human need.

Putting in dialogue these two theoretical perspectives raises a number of research questions that we think may constitute the beginning of a research agenda:

1) How do forms of conceptualizing mathematics education shape mathematical thinking in people?

2) What are the implications for mathematics education of considering mathematics as a mediating artifact?

3) How can we adopt a transformative approach to curricula in order o integrate critical mathematics education in the *traditional* educational activity system?

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PRESERVICE ELEMENTARY MATHEMATICS TEACHERS' PROOF PERFORMANCE

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The purpose of this study is to determine preservice elementary mathematics teachers' proof performance on task-based interview questions. Individual interviews were conducted with 16 participants. It can be claimed that their performance was changed in terms of proof types, the most difficult one was proof by contradiction and the easiest type was found as direct proof.

INTRODUCTION

In mathematics education proof process is important for knowing and doing mathematics, constructing the base of mathematical perception, and using mathematical knowledge in different situations (Stylianides, 2007). Knuth (2002) noted that the process of proof should be dealt in early grades mathematics. Therefore elementary mathematics classrooms need teachers to provide such learning environment to the students (Almeida, 2003). The purpose of this study is to determine preservice elementary mathematics teachers' proof performance. After Almeida's (2001) proof scale was administered to 104 preservice teachers, 16 of them were selected with regard to the points from the scale. Five task-based interview questions were directed each interviewees individually. According to the findings, it can be said that the interviewees taken from lower group performed better that those from higher group. Furthermore it was observed that, interviewees did the pictorial or schematic part of the proofs but when it came to algebraic part, most of them failed to conduct it. The easiest proof type was found as direct proof and the most difficult proof type was found as proof by contradiction. Almost all of the students did not perform well on those statements. Induction and proof by counter example were easily applied to the statements by interviewees. Finally, the interviewees made some beneficial suggestions related to making proofs as a part of mathematics instruction.

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CAN FORMATIVE ASSESSMENT REPLACE SUMMATIVE ASSESSMENT?

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This study set out to find ways in which the formative and summative goals of assessment could be achieved using the same instruments. The motivation for this integration of goals was to achieve time efficiencies, as suggested in (Wiliam, 2000), while optimising the quality of assessment information available to the teachers. The study took place in a secondary school for girls. Three teachers, including the principal author, took part and helped collect data. The topic that provided the mathematical focus for this study was Measurement. A program of formative assessment was employed during the entire topic. Two of the participating teachers used student assessment worksheets designed by the researcher throughout the instructional sequence. At the end of the topic, the participating teachers predicted the marks that would be achieved by their students on the end-of-unit test.

The researcher then interviewed the other two teachers to ascertain the sources of their predictions. The quantitative analysis of the teachers' predictions has shown that they were capable of predicting the performance of the majority of their students within an accuracy margin of 10%. Another finding of the study was that all participating teachers were most accurate in predicting the performance of the students who achieved the lowest 25% of the test marks. The latter finding contradicted other published research, namely (Biuhagiar & Murphy, 2008; Coladarci, 1986). The study suggests that a teacher's reflection on formative assessment instruments complements, but does not replace, an end-of-unit summative test for the purpose of generating an accurate report of a student's performance.

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INVESTIGATING THE SOCIALIZATION OF SECOND LANGUAGE LEARNERS OF MATHEMATICS

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There is a growing body of work on the learning and teaching of mathematics in various multilingual settings. In one such setting, students learn mathematics through a second or additional language in situations in which their home language(s) are not widely used. Previous research has explored second language learners' participation in mathematics: e.g. Moschkovich (2002) on how language functions as a resource for mathematical thinking for bilingual students; Clarkson (2007) on how students may privately switch between languages in solving mathematics problems in Australia. Other researchers, meanwhile, have looked at the broader patterns of second language classroom interaction: e.g. Khisty (1995) on how a focus on mathematical meaning supports bilingual learners of mathematics.

This paper arises from the initial stages of an ethnographic study of second language learners in mathematics classrooms (funded by SSHRC, no. 410-2008-0544). Theoretically, the study is framed by notions of socialization, both into the discursive practices of mathematics (Lerman, 2001) and into the language practices of English (Duff, 1996). This perspective entails a concern with the interplay between individual students' participation and the patterns of interaction at whole-class level. In this paper, I discuss a methodological approach to investigating socialization in mathematics that uses ideas from linguistic ethnography and in particular, Hymes' notion of *speech events*, defined as activities that are 'directly governed by the rules or norms for the use of speech' (Hymes, 1972, p. 56). Using data from the study, I look at how a focus on speech events in the observation and analysis of mathematics classroom interaction allows the individual and the whole class, as well as both mathematics and language, to remain in view.

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AN EXPLORATION OF THE RELATIONSHIP BETWEEN FACTORS AFFECTING STUDENTS' VIEWS ABOUT PROOF AND THEIR PERFORMANCE IN PROOF

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The aim of this study is twofold; to determine effective factors in students' views about proof and to confirm proposed structural model including interrelationships of these factors and proof performance. This study not only helps instructors to gain useful insights on students' views about proof but also provides evidence on how to foster students' proof performance with respect to the identified factors.

The corpus of literature indicates that students' views about proof comprise of numerous factors. The prominent factors are; self-confidence, beliefs, perception about proof, anxiety, metacognitive processes, motivation, role and importance of proof. Little has been done so far to develop a survey in a broad sense including all these aspects. Furthermore, the factors outlining students' views about proof are believed to have significant impact on their performance in proof. Yet to date, no research has proposed a model including these factors and their possible relation to proof performance.

In this respect, this study primarily sought for developing and validating a survey on students' views about proof. Secondly, a structural model is proposed to analyze and to explore factors that are influential in students' performance in proof namely in direct, indirect proofs and refutations.

This study is administered to prospective mathematics teachers and undergraduate mathematics students at public universities in Turkey. Data is collected through a survey on students' views about proof which was developed by researchers and the proof questionnaire that was developed by Riley (2003). Items corresponding to aforementioned factors were determined from the review of literature and appropriate items were included in the survey. Factor analysis technique is used to confirm the components of the survey. Hypothesized model examining the interrelationships between these factors and students' performance in proof is proposed and tested by using structural equation modeling (SEM) techniques.

The outcomes of this study draw attention on important factors that have potential impact on students' successful engagement with proof and highlight consideration of these identified aspects when teaching proof at mathematics classrooms.

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MATHEMATICAL COMPETENCES IN STEERING DOCUMENTS

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INTRODUCTION

This study is an ongoing project where we only have very preliminary results, and the main aim of this paper is not to present results, but to discuss the methodological issues when using content analysis on the course syllabus for the Swedish compulsory school. Content analysis (Krippendorff, 2004) is in most cases in the literature used on large materials, for example all news articles from a specific year that concern a certain political issue. We will discuss what happens when we try to use the method on a comparatively small document of a very different kind.

CONTENT ANALYSIS

Content analysis is performed through the application of a coding tool, or a *codebook*, that specifies the *recording units* that are of interest in the analysis. Since the focus of the study is on mathematical competences in the course syllabus, the codebook consists of the six competences defined in the project: Problem solving, Procedural Fluency, Reasoning, Representations, Connections, and Communication. The recording units are supposed to be the smallest unit of information in a text, so we have divided the material in 'chunks' and then, when possible, connected each chunk to a specific competence.

The course syllabus for the Swedish compulsory school consists of 5 pages of text (9000 characters in 64 paragraphs). The splitting of the text resulted in 205 units of description. 120 of these were coded to the six competences:

Problem solving	Reasoning and proof	Applying procedures	Representations	Connections	Communication
31	13	28	5	9	34

Table 1: Result of the coding of the course syllabus in mathematics

In the work with the course syllabus we have experienced many situations where the meaning of the text has been uncertain. The use of double coding has given us a possibility to clarify many situation, but there are still interpretations that might differ from how teachers see the text. It is an interesting observation that a document that is supposed to ensure equal opportunity for all students contains so many words and sentences that are difficult to understand and interpret.

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OBJECT RELATIONS AND EPISTEMIC ACTIONS OF LOW-ACHIEVING STUDENTS

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Starting from a "theory of action" point of view, the theoretical framework of an empirical study is presented. Aim of the study will be developing a theory for fostering low-achieving students, focussing on the students' constructions of mathematical knowledge, their specific relationships to mathematical objects including their motivational status and the interplay between both.

The current state of an ongoing research project is presented aiming at establishing a theory as a basis for fostering low-achieving students from the ages of 13 to 17. We have finished a pilot study exploring the field and establishing a theoretical frame for an empirical study deepening our insight into motivational and epistemic conditions of the students' processes of constructing mathematical knowledge. Low-achieving students attend our project with biographies full of experiences of incompetence in the subject of mathematics. These experiences have led to subtle kinds of disengagement in mathematics that cause additional experiences of incompetence stabilising forms of disengagement. The outcomes are broken relationships to mathematical things and unsuitable mathematical knowledge. In our pilot study the following four research questions emerged as most relevant for our aim: How can disengagement of low-achieving students be overcome? How do these students create a relationship to mathematical things? How do they construct mathematical knowledge? What is the epistemic role of context for low-achieving students?

In order to answer these questions, we use a network of three theories: The theory of abstraction in context (Hershkowitz et al., 2001) provides a lens to analyse the construction of knowledge. Oerter's action theory (1982) is the background theory for investigating the students' relationships to mathematical things and the Self-Determination Theory (Deci & Ryan, 2000) will help us finding a way out of the self enforcing spiral of disengagement and experiences of incompetence.

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PEDAGOGICAL CONTENT KNOWLEDGE FRAMEWORK: USING IT AS A TOOL FOR COURSE DESIGN[1]

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Shulman's (1986) notion of pedagogical content knowledge (PCK) has mainly been used as a tool to diagnose or follow the development in pre-service teachers' PCK or/and pedagogy or/and content knowledge. The notion, we think, also has the merit to guide the design of mathematics teaching courses and their delivery as part of pre-service teacher education programs. This conviction has directed us to design courses using the notion of PCK as a framework. In this paper we show how this notion is used as a guide to design and deliver mathematics teaching courses and also show how the courses shape a pre-service teacher's (Rana's) PCK regarding derivative.

Based on and inspired by the prior works of Grossman (1990), we determined the following five components of PCK for the design of our courses: (i) knowledge of students' difficulties and misconceptions for a particular concept, (ii) knowledge of multiple representations for a particular concept, (iii) knowledge of instructional strategies and methods for a particular concept, (iv) knowledge of curricula with regard to a particular concept, (v) knowledge of assessment of a particular concept. These five components of PCK were used in developing courses contents. The courses were applied to a group of pre-service mathematics teachers and the notion of the derivative made the content dimension of PCK.

A participant of the courses (Rana) prepared a lesson plan which introduces the concept of derivative before and after the courses and she taught a derivative lesson based on the lesson plans in the school placement after the courses. The results revealed significant changes in Rana's plans in terms of the way derivative was introduced, role teacher was given, instructional strategies adopted, curricula materials used, employment of assessment and integration students' difficulties into the instructions. The courses designed in the light of the PCK, therefore, improved Rana's PCK dramatically. Our experience, as a result, allows us to state that PCK framework not only enabled us to design the courses and develop their contents but also provided us with a roadmap to deliver the contents in a comprehensible manner.

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RECOGNIZING MATHEMATICS LEARNING IN INQUIRY DIALOGUE

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Sometimes, in a mathematics classroom, the sound of an engaged student emerges. She proposes a solution, asks a question, or wonders about something that she thinks might be true. She *inquires* (Lindfors, 1999). What should the teacher do with her planned lesson, when the student imposes with an inquiry?

As researchers in four classrooms, we work together with the teachers to identify students' engagement and inquiry in mathematical dialogues (Alrø & Skovsmose, 2004), and to develop room for mathematical inquiry in mathematics classrooms. Our motivation is threefold: (1) in inquiry dialogue, students' learning is "on display" and it is possible to learn something about students' knowledge that is of importance for the teacher(Lindfors, 1999; Alrø & Skovsmose, 2004). (2) If we want the students to develop an inquiry stance towards the subject, the inquiries of students should be valued in the mathematics classroom. (Borasi, 1992; Lindfors, 1999) (3) Students' inquiries into mathematical questions can be good starting points for dialogue where mathematics may be taught and learned. (Alrø & Skovsmose, 2004)

In this report, we try to identify the mathematical ideas that students contribute to classroom dialogue, and how they are followed up on by the teacher and other students. Other research issues will be reported on later.

Our analysis finds many instances of students' mathematical inquiries in class- room discourse, even in settings where the classroom episode from the teacher's perspective is not an "inquiry episode" (Lindfors, 1999). These instances is found in the classroom video material, but also in teacher-didactician discussions after classes. In these discussions, such themes are sometimes initiated by didacticians (observing and participating as teacher's aide), and sometimes by the teacher, the initiator finding the student contribution challenging and worthy of discussion. Several of the teacher-initiated discussions are examples where the student contribution was not discussed further in the classroom.

The project is part of the research consortium "Teaching better mathematics».

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ARGUMENTATION IN MATHEMATICS: THE ROLE OF THE TEACHER

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The interest in argumentation in Mathematics Education is very recent. It was only in the eighties that the discussions about this issue stood out as an effort to attack the problem of the specificity of mathematical proof in relation to argumentation in mathematics and to establish links between epistemological, cognitive and educational perspectives (Douek, 1999). The importance of argumentation in mathematics classrooms is deeply connected to an emphasis on learning mathematics with understanding and to the need of developing students' mathematical reasoning (Yackel & Hanna, 2003). In this context, several curricular documents and studies underline the need for the involvement of students in learning activities where they must share, explain and justify their statements and results, where disagreements are overcome by presenting mathematically meaningful and valid arguments and where they are challenged to formulate, evaluate and prove mathematical conjectures.

The creation of learning environments that facilitate the students' involvement in argumentation is a very complex process, which poses serious difficulties and several problems to teachers (Lampert, 2001). Little is known about this process, about teachers' work related to it or about associated facilitating or hindering factors.

This presentation will focus on selected results of a research study, framed on the interpretative-collaborative paradigm, in which one of the goals was to describe and to analyse the work of two middle school teachers who wish to involve their students in mathematical argumentation activities (Boavida, 2005). Particularly, it is intended to reveal aspects and contexts that stand out as particularly relevant for the emergence and development of mathematical argumentation, as well as the challenges faced by the teachers when planning and creating a classroom environment which facilitates the fostering and the development of those activities.

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TEACHERS' PERCEPTIONS ABOUT THEIR UNDERSTANDING OF CHILDREN'S NUMBER KNOWLEDGE

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This study explored teachers' perceptions and understanding of a research-based framework describing children's cognitive development in early number. Survey and interview data from twenty-eight teachers were collected to determine teachers' perceptions of their understanding of the framework, their ability to use it to assess students' mathematical development and to plan appropriate instruction for individual children.

The Learning Framework In Number (LFIN) is used by teachers participating in a state-wide professional development numeracy program operating in over 1700 primary schools across the state of New South Wales, Australia. The LFIN is used by teachers to not only identify the level of development each child has attained but provides instructional guidance as to what each student needs to work towards. A stimulus for the current study, was the need to know what teachers understand about the LFIN and how it impacts on their teaching and assessment practices.

A survey was distributed to teachers of three purposively selected primary schools that had been identified as implementing the numeracy program, Count Me In Too. Survey questions related specifically to individual teachers' perceived understanding of the LFIN, their confidence using it to assess students' development in number and the impact this knowledge has on their instructional decision-making. Another section of the survey required an open-ended response to a scenario involving a description of a student's reaction to a mathematical task. Teachers were asked to use the available evidence to approximate the child's performance as described by the LFIN and to make suggestions about the types of activities/learning experiences that would most suit the child's level of understanding. Responses to this section of the survey were analysed according to a pre-determined rubric—the various levels of the rubric indicative of a respondents' ability to interpret the child's cognitive development in terms of that described by the LFIN.

Interestingly, the majority of teachers rated their understanding of the LFIN and their confidence using the LFIN as only Level 3 or lower; yet a quarter of all respondents achieved the highest possible rating when given a realistic case scenario to assess and plan instruction and fifty percent of respondents provided responses that were rated in the top two levels of the rubric. This raises questions as to the impact teachers' lack of confidence in their own abilities to determine student's mathematical development may have on their willingness to follow-through with appropriate instruction for their students.

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A GENDER PERSPECTIVE IN TIMSS 2007: THE CASE OF SINGAPORE

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In this study we look at the gender-related data from the Trends in International Mathematics and Science Study in 2007 (TIMSS 2007) for Singapore. Girls in Singapore have performed significantly better than boys in TIMSS 2007 at both grades 4 and 8. Similar results were obtained in the TIMSS 2003 whereby, in addition, girls from Singapore performed better than boys in all reported content domains at grades 4 and 8. In this presentation, we report the secondary analysis of data for Singapore from the TIMSS International Data Base and the International TIMSS Report (Mullis, Martin, & Foy, 2008), with a focus on gender.

In our analysis, we looked into the overall performance of the students and their performance by gender within content domains and cognitive domains at both of tested levels, grades 4 and 8. Furthermore, we looked into how the teacher's gender impacted on the students' performance. Students' self-confidence and attitude were also examined from a gender perspective. In the data for Singapore, it was noted that larger percentages of students were taught by female teachers in mathematics. Girls taught by female teachers outperformed students from all other gender combinations of students and teachers. In each of the three cognitive domains, girls had higher achievement scores in mathematics than boys, both in Singapore and internationally. Also, it is worth noting that at grade 8, boys had higher self-confidence in mathematics than girls.

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AS TIME GOES BY: TECHNOLOGY, EMBODIMENT AND CARTESIAN GRAPHICS

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This paper addresses and discusses the role of technology in the emergency of students' understanding time as component axe in Cartesian graphs distance x time. The role of the embodiment theory will be discussed; not only as our framework but also as a possibility in helping teachers to better understand about students' thinking.

In this research, our interest was on the symbols used by the students to register changes over time, specifically how they used the axe time in the Cartesian graphic, and how they change their thoughts when using the sensor and calculator.

This study took place in Brazil, in a junior high school class with twenty-eight junior high school students. They worked in an activity of sketching a Cartesian graph for a movement situation in three steps; step one- reading a story and using pencil and paper; step two- working with a graphic calculator and a CBR sensor; step three revisiting the graph done on step one. A theoretical framework based on embodiment theory was used for analysing students' meaning production and how they change their thinking about time representation during the activity. The results pointed out to a change in perception of time, from an abstract entity to a material one; and to an understanding of time-axe.

For these students, the source domain was enriched with the possibility of thinking in a two-dimensional way. Before using technology all they could rely on was a one-dimensional way of thinking, the footprint on the ground, now they think in terms of the game "match the graph". They were relating the abscissa and the ordinate in order to read and to produce a dxt graph.

We can say that the embodiment theory opened, in this study, a new window of understanding students' meaning production for us researchers and teachers. Technology allowed the students not only to solve the first task but also to turn palpable something abstract.

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EXPLORING PATTERNS AND ALGEBRAIC THINKING

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Mathematics isn't only symbolic manipulation according to a set of archaic rules, but the understanding of patterns (Devlin, 1998). The passage from arithmetic to algebra is one of the major difficulties that students face and teachers should diversify strategies in order to allow their students to develop algebraic thinking and the sense of symbol (Arcavi, 2006). According to Orton and Orton (1999) patterns are one of the possible paths when thinking of presenting algebra and, consequently, improving algebraic thinking. The definable goal of this research lead to the understanding of the use of patterns in class, in a context of investigation tasks, in order to develop algebraic thought. One of the attempts of dealing with this set of problems has been done within four research prompts: 1) the image of Mathematics; 2) mathematical connections; 3) the understanding of Algebra; 4) mathematical communication.

The present study was done taking as a starting point a 8th grade class, using a qualitative and interpretive methodology, based on case studies. The researcher is both instrument and participant-observer. Questionnaires, interviews, direct class observation and written reports provided the necessary data.

The final results show that the use of patterns as a base and stimulus, in a context of investigation tasks, may contribute to the ultimate understanding of Algebra, granting the improvement of algebraic thinking or, specifically, the sense of the symbol by defying students to use different representations, to identify and generalize relations and to analyse its meanings. Furthermore, it also lays mathematical connections, enhances mathematical communication by means of developing their ability to use non-ambiguous and adequate language, written or spoken, and sets up a revised image of Mathematics for students.

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LEARNING MATHEMATICS: A LOOK AT CONCEPTIONS OF FUTURE PRIMARY TEACHERS

Maria Elisabette Brisola Brito Prado and Nielce Meneguelo Lobo da Costa

UNIBAN BRASIL UNIBAN BRASIL

This study is part of an ongoing research aiming to understand how future primary teachers – who will teach math to children – think about their own learning process. As a premise we understand that thinking through their math learning process, even though it is different from the child process, will interfere when they plan and create their future classes. This premise is based on an education concept in which learning and teaching process are understood as connected and interdependent. (Charlier, 2001).

How do children learn? How do I know they learn? How do they use the developed knowledge to solve math problems? Several national assessments promoted by government agencies have been reveled children insufficient math comprehension at the end of primary school (SARESP, 2007; SAEB, 2005), so that this matter is currently on Brazilian educators debate. They believe one of the causes could be a mistaken pedagogical practice, so that as a consequence, the central problematic might be related to teacher education.

According to Adler and Jaworski (2004) is essential to investigate believes and conceptions that support math teachers practice and the relation between teacher actions and teacher/students learning. This investigation used a qualitative methodology and was applied with 64 students in the last quarter of a three years Pre-Service University Course to Primary Teachers. An inquiry intended to emerge conceptions and believes about future teachers' math learning.

The analysis revealed some conceptions categories and, using CHIC software (Classification Hiérarchique Implicative et Cohésitive) in the analysis of textual registers, some ambiguous conceptions were identified and will be discussed in our presentation in details.

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IN THE INTERFACE: MATHEMATICS TEACHERS FINDING THE CURRICULUM

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Recent research has identified misalignment between the demands of the curriculum, particularly reform mathematics curricula and classroom teaching. In South Africa, classroom research suggests that many teachers simply ignore the new curriculum and continue to teach how they taught before; others support the ideals of the curriculum but struggle to enact it in practice while others create hybrid practices, drawing on the new curriculum as well as their traditional practices (Brodie, 2007, Jansen, 1999).

The central question that this paper addresses is how do teachers come to make meaning from curriculum statements in relation to their teaching practices. We report on a teacher development activity in which teachers engaged in conversations as they attempted to map test items taken from an international test against the South African mathematics curriculum statement. We argue that this activity opened up for teachers key important aspects of the intended curriculum: content coverage, cognitive challenge, and sequence and progression in the curriculum. The repeated forwards and backwards analysis of the test items in relation to the various assessment standards in the curriculum statement helped teachers to gain a focused understanding of their professional knowledge and experience in relation to the intended curriculum.

From the perspective of teacher development we argue that teachers need an artefact, external to the intended curriculum, which they can use to mediate between the intended curriculum and their professional knowledge and experience, which shapes the enacted curriculum. An internationally set test is ideal for this purpose. We argue for a structured process of interface in which teachers are actively engaged in curriculum translation of the mathematical material that is embedded in the test in relation to the curriculum statement. The paper presents a model of this interface.

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TARGETS AND CONCEPT OF A MODEL TEST FOR THE INTERNALLY DIFFERENTIATED DEVELOPMENT OF MATHEMATICAL COMPETENCIES

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A theory-supported concept for an efficient individual enhancement of competencies in heterogeneous learning groups was developed in cooperation with 24 maths teachers. At present this internally differentiating teaching concept is tested with 50 school classes in a four-year school test.

Many teachers are frequently asking the following question, complaining that there are no theoretical or empirical studies and practicable answers in the teaching methodology: how can heterogeneous learning requirements in secondary school maths lessons be treated in a way that a maximum number of students in a class is adressed both cognitively and motivationally and effective learning results are achieved for all?

With model test MABIKOM (development of internally differentiating mathematical competencies in maths lessons, with the support of newest technologies) – fostered by the Lower Saxony and Texas Instruments – efforts are being made since 2008 in quarterly several-day project meetings to find answers to this question. As a first step and in view of planned competence-oriented teaching lessons the teachers described the relevant differences of students. Keeping the problem in mind the provided teaching concept (cf. Komorek et al, 2007) was revised and prepared for a first test in the classes 5, 7 and 9. In addition to the teaching and learning methods developed in cooperation with the participants of the project, profiles were elaborated (cf. www.math-learning.com) as well as prototypical teaching examples which have already been tested individually in the project and improved on the basis of field reports. The presentation highlights the cooperatively developed teaching concept.

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FRACTION SKILLS AT THE START OF SECONDARY EDUCATION IN THE NETHERLANDS

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Discussions on the mathematics curriculum among stakeholders such as educators, mathematicians, policy makers and employers in the Netherlands parallel those that occur in many Western countries. Such discussions express a widely felt concern about the basic mathematical skills of students at the beginning of both secondary and tertiary education. However, the discussion on curricular reform is frustrated by a lack of knowledge of students' actual skill level in pivotal mathematical domains.

To overcome this lack of knowledge, we conducted a study on students' skill level in the fraction domain at the start of secondary education in the Netherlands. Drawing on a framework in which skill levels correspond with the notion of emergent modelling (Gravemeijer, 1999), we asked the question whether students meet the expectations of the written curriculum.

We designed a fraction skill test of 78 items on the basis of core fraction concepts from research literature and bases on common textbooks. To this test we subjected 218 students in their first year of secondary education.

The outcomes of this test show an unexpected variance in skill level that does not correspond with the expectations on the basis of the written curriculum. In our dataset these differences in test scores can not be ascribed to the primary school that students attended. We discuss the implications of this finding for both curricular reform and further research in the fraction domain.

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[i] Eindhoven School of Education is a joined institute of Eindhoven University of Technology and Fontys University of Applied Sciences

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EXPERIENCES OF MATHEMATICS I THE TEACHER EDUCATION. A PHENOMENOLOGICAL ORIENTED NARRATIVE ANALYSIS OF STUDENTS TEXTS.

Tone Bulien

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I will present the theoretical approach and the analytical choices of my PhD thesis (Bulien, 2008). The thesis is a study of texts from and interviews with six Norwegian teacher students enrolled in a compulsory course in mathematics. It is a critical constructive descriptive investigation where the aim has been to listen to the students sharing their experiences studying mathematics.

The thesis proceeds from a phenomenological perspective, using narratives as an important feature in both the analysis itself and the presentation of the results. Using phenomenologically oriented knowledge sociology (Schütz, 1962,1964; Zahavi, 2001) and theories of narrative analysis (Polkinghorne, 1998) a description of the students' perceptions of teaching and learning mathematics, both prior to and in the course of the compulsory course, is made visible through narratives. The methodology employed is narrative analysis. The students' experiences are divided into four main areas of beliefs (Leder, Pehkonen, & Törner, 2002): beliefs about mathematics in general, beliefs about themselves as practitioners of mathematics, beliefs about teaching mathematics, and beliefs about how mathematics are learnt.

One of the results indicated that the students' experience of the compulsory course in mathematics did not depend on their previously held beliefs on mathematics education or their attitudes towards mathematics in general.

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EIGHTH GRADERS' INTERPRETATION OF BINOMIAL SUMS AND PRODUCTS THROUGH ALGEBRA TILES

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This study examines 8th grade students' interpretation and sense-making of binomial sums and products modeled with algebra tiles. This study is part of Project CoSTAR (Coordinating Students' and Teachers' Algebraic Reasoning)[1] that has as its main purpose the coordination of research on students' understandings and teachers' practices and interpretations of students' actions relative to algebraic reasoning. Algebra tiles, serving to model binomial sums and products, stand out as one of the most commonly used manipulative materials in the US by middles school students. Multiplicative structures can to some extent be modeled by additive structures; however, they have their own characteristics inherent in their nature, which cannot be explained solely by referring to additive aspects. We base this present study within a framework of *operational invariants*, which are preconceptual or conceptual knowledge necessary for proficiency in a conceptual field (Vergnaud, 1996). The most important operational invariants are *concepts-in-action* (which provide ways of picking up, selecting, and categorizing relevant information) and *theorems-in-action* (which serve to make possible inferences and calculations on the ground of the information available) (p. 237). Grounded in this theory, we developed a series of terminology in an attempt to reveal 8th graders' operational invariants arising from their verbal descriptions, actions, hand gestures, drawings in the context of binomial sums and products modeled with algebra tiles.

This study took place in a middle school's 8th-grade classroom in the southeastern US. Our data consist of four videotaped classroom lessons, six student interviews, three teacher interviews. We undertook a retrospective analysis using constant comparison methodology in order to generate a thematic analysis. Our main result is that *concepts-in-action* in the process of representing sums and products with algebra tiles were available to most students, however, constructing *theorems-in-action* required mastery in connecting tile representations to algebraic expressions. Using the same tiles to represent both lengths and areas was a cause of confusion for many students. Students generating true theorems-in-action were the same ones who suggested the use of the negative sign combined with the addition operation rather than subtraction operation, when negative quantities were involved in representing binomial sums and products via algebra tiles (e.g., in the multiplication example (x+5)(x-4), those students preferred to work with (x+5)(x+(-4)) as a product of binomial sums, each representing lengths; and consequently obtained x^2+(1x)+(-20) as a sum of areas of rectangular tiles instead of x^2+x-20).

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IDENTIFYING EPISTEMIC AND COGNITIVE CONFIGURATIONS IN THE ELEMENTARY ALGEBRAIC REASONING

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Juan D. Godino — University of Granada. Spain

A methodology of analysis where the configuration of mathematic objects and meanings, introduced by the Onto-Semiotic Approach of mathematics cognition and instruction (OSA) is applied (Godino, Batanero and Font, 2007; Font, Godino and Contreras, 2008) to characterize the mathematics objects and meanings put into effect during the resolution of mathematic problems.

In the onto-semiotic approach the "epistemic analysis" is the identification of the objects and meanings that intervene in the expert solutions of a task, which will be used as a reference to interpret the solutions given by students. OSA proposes five types of objects or entities that intervene and emerge during the mathematic practices carried out to solve a problem: linguistic objects, concepts, properties, procedures and arguments. The epistemic analysis has a twofold purpose for the teacher trainer: The first is to explore objects and meanings that come into play during a problem resolution process; the second is to identify possible meaning conflicts and to predict difficulties and mistakes that could emerge on the solutions that students give to the problem. A finding of this research is that the problem structure poses a number of interpretative challenges and semiotic conflicts in correspondence to concepts that are interpreted in terms of linguistic elements. We show how the particular configurations of primary entities facilitate or hinder the arithmetic or algebraic problem representation.

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TRENDS IN NUMBER RESPRESENTATIONS AMONGST FIVE- TO EIGHT-YEAR-OLD STUDENTS

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This presentation focuses on the kinds of responses produced by children to form both conventional and unconventional number representations. The study aimed to answer the question: Are there progressions in the development of number representations (a) in writing; (b) orally; and (c) through base-ten valued tokens from Kindergarten through Grade Two?

Seventy-one interviews were conducted with children in Kindergarten through Grade Two in which the children were asked to represent numbers of two- through five-digits in writing, orally, and through base-ten valued tokens. Representations were then classified by the type of response produced. We found that most types of responses were significantly associated with a particular grade level. That is, for these categories, the chance of producing that type of response was much higher for certain grade levels than for others. For instance, responses that resemble the "sounding out" of numbers are more common amongst first graders. We wonder if this is due to the emphasis on reading and phonics in the first grade in the USA. Children are taught to sound out words and spellings and therefore, it appears that they are attempting to do this with the tokens and numerals but with numbers instead of phonemes.

Surprisingly, while the probability of a conventional written or oral representation was lowest in Kindergarten, it remained stable from Grade One to Two. Since the magnitude of the numbers presented increased across grade levels, this result indicates that once a given number range is mastered, a higher range presents the same difficulties that had once been presented in the lower range. Therefore, by the end of the second grade, many students have still not appropriated more rules underlying the number system than in the first grade; they have simply mastered more number ranges while sidestepping the rules of the number system as a whole.

Unlike written numbers, conventional tokens were most likely to be used amongst second graders. Thus, it appears that the use of conventional base-ten token composition does increase with grade-level. This also indicates that the rate of conventional token use is not related to the rate of conventional written numbers since this did not increase from Grade One to Grade Two.

These results indicated that there are identifiable progressions in the representations that children produce when representing external numbers in these systems. While these progressions may be due to age-related factors or to school-related factors, we can still note that it is imperative to pay attention to not just the rates of conventional representations amongst students but also the unconventional responses as well.

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6TH GRADE MATHEMATICS TEACHERS' CONCERNS ABOUT A REFORMED MATHEMATICS CURRICULUM[1]

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Research studies that examine teachers' implementation of new educational programs reveal that teachers' concerns about new practices impact their change. Investigation of teachers' concerns provides information not only about the degree to which teachers implementation of approaches proposed in new curricula (Christou, Eliophotou-Menon, & Philippou, 2004) but it also portrays teachers' reactions toward innovation and their needs emerged during implementation of it. Based on these assumptions, the main objective of this study is twofold. The first objective is to investigate the concerns that Turkish mathematics teachers expressed about recent innovation in mathematics curriculum. The second objective is to explore the role of personal characteristics (age, gender, experience) on the development of teachers' concerns about new mathematics curriculum. The participants of this study were 316 (194 female and 122 male) elementary school mathematics teachers selected using a stratified sampling method. Teachers' concerns regarding new 6th grade mathematics curriculum are identified by administering the Stages of Concern Questionnaire based on the Concerns-Based Adoption Model (Hord, Rutherford, Huling-Austin, & Hall, 1998) and through an open-ended question.

The analysis of the qualitative and quantitative data indicates that the teachers mainly have personal and management concerns. The results of the Kruskal-Wallis and follow-up Mann–Whitney tests show that (1) males have more awareness concern than females; (2) females have more personal and impact concerns than males; (3) teachers at the age level of 20-29 have more personal and impact concerns than teachers at the age level of 50-59; (4) teachers who have 3-5 years of teaching experience have more personal concerns than teachers who have 20 and above years of teaching experience. These findings direct our attention to the importance of attending to the teachers' concerns and needs about new mathematics curriculum and its implementation. The differences in teachers' concerns identified in this study in terms of age, gender, and experience can contribute to the design of more effective professional development programs for in-service teachers.

[1] Funding for this project was provided by the Scientific and Technological Research Council of Turkey.

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MODEL-ELICITING ACTIVITIES CAN CHANGE TEACHER'S MATHEMATICS-RELATED BELIEFS

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Mathematics teacher's beliefs have become an important issue for teacher education since 1980. Research (Philipp, 2007; Forgasz & Leder, 2008) show that teacher's instructional decisions and teaching practices are influenced by the beliefs they hold. The trend of studies about teacher's beliefs gradually emphasize on how to change teacher's beliefs.

This study reports the change of sixteen in-service secondary mathematics teachers' mathematics-related belief after a nine-week course which links to a master's degree program for in-service teachers. The process of this study included two phases. Phase 1, these teachers as the role of students engaged in three model-eliciting activities (MEAs, Lesh & Doerr, 2003) such as Big Foot, Parking Lot and Volleyball problems. These teachers were divided into small groups for solving one MEA cooperatively in every two weeks. They also wrote reflection journals to compare these MEAs to show their understanding of modelling pedagogy and their beliefs. Phase 2, every group designed one MEA in the last three weeks of the course. They used the Six Principles (Lesh & Doerr, 2003) to design MEA and to evaluate these MEAs by themselves and with one another. This evaluating process also showed their mathematics-related beliefs. Data collections included pre-test and post-test by the Teachers' Mathematics-related Beliefs Questionnaire (TMBQ, six-point Likert scale, 39 items, developed by us, α=0.84, three subscales included Object (11 items), Self (15 items), Context (13 items)), reflection journals, interview reports, video tapes of the course and researchers' observation journals.

According to the reflection journals and the data of TMBQ, 12 out of 16 teachers got higher scores on the post-test. They especially have much more improvement on the Object subscales than the other two subscales, the Self and the Context. These teacher's beliefs tended to be reformed approach. Why these teacher's beliefs on the Self and the Context subscales did not change as much as the Object subscale will also be discussed.

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EXPLORING PERCIEVED BELIEFS OF TEACHERS PARTICIPATED IN LESSON STUDY AND OPEN APPROACH

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This study is a part of three-year professional development project implementing lesson study and open approach conducted by the Center for Research in Mathematics Education. Unlike Japanese lesson study, this project modified Japanese lesson study by incorporating open approach and emphasizing “collaboration” in every phase of lesson study cycle. Thus, teachers, researchers and outside experts participated in *collaboratively* designing research lesson, *collaboratively* observing their friend teaching the research lesson, and *collaboratively* doing post-discussion or reflection on teaching practice (Inprasitha and Loipha, 2007). Every week, 78 teachers from 4 schools participated in each phase of lesson study cycle through 3 academic years.

The purpose of this paper was to present teachers’ perceived beliefs about students’ changes and teachers’ changes. The data was collected by a questionnaire consisted of open-ended questions. The questionnaire was distributed to 78 teachers and got 43 respondents. Besides from quantitative analysis of questionnaires, the qualitative analysis was used for analysis teaching practices of one teacher whom the researcher had observed her teaching practice at one school for the entire 2007-2008 academic years.

It was found that this modified Japanese lesson study provides a chance, which has never has before, for participating teachers to reflect upon their teaching practices and their existing beliefs. Comparing before entering the project, teachers viewed their students must listen to their lectures and viewed their roles were either as lecturers or explainers. After participating in the project, most of the teachers had perceived beliefs about students’ changes were as following; students could expressed their ways of thinking and the reasons underpinning their solutions, recognized various ideas from other people, and had variety ways of thinking and were enjoyable in learning mathematics. Teachers’ perceived beliefs about teachers’ changes were as following; teachers changed their roles to act as facilitators and observed their students’ ways of thinking, provided more chance to prepare their research lessons than they were used to be, recognized their critiqued friends and worked with other teachers as team working. These perceived beliefs of teachers suggested that those teachers are aware of their own existing beliefs about teaching practices. Those perceived beliefs should be considered the critical stage before they can change their beliefs and associated beliefs about their teaching practices.

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INVESTIGATION OF THE KINDERGARTEN TEACHERS OF CRAFT KNOWLEDGE IN TEACHING MATHEMATICS

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This article reports on the craft knowledge of kindergarten teacher in teaching mathematics. Using a study of 18 teacher's mathematical lessons, and interviews regarding theses lessons to examine the role of craft knowledge appeared in play in the teaching practice. The subjects came from kindergarten teachers in a public elementary school, they have taught over 15 years. The teachers were classified with expert teachers and non-expert teachers for comparing with their craft knowledge which different teachers behaved in mathematical teaching practice. This research focuses on mathematical lessons of "temporal concepts" for analysis of craft knowledge.

The findings revealed that the expert teachers looked at mathematics as an interesting subject, so they liked to teach mathematics in the class and set time to emphasize key concepts of time with daily life of children for their instructions, but non-expert teachers just focused on read-time for teaching. Moreover, the expert teachers made hand–on materials for children to understand abstract time-concepts, on contrary, non-expert teachers liked to draw on sit-tasks for children to memorize the instruction of read-time. Addition, the expert teachers applied games to know children's learning but the non-expert teachers neglected to know children understanding from her instruction. In the aspect of content knowledge, the expert teachers of articulate in abstract concepts made children understand, however, the non-expert teachers made confused particularly in read-time. In conclusion, the craft knowledge consists of mathematical knowledge which could be taught understandably, teachers' instruction referring to their view on mathematics, and making mathematical materials to assist teaching. Impressively, the consensus of craft knowledge was to teach mathematics for fun that is priority for all teachers. The research submits suggestions that providing mathematical programs for kindergarten teachers in in-service education.

Key words: craft knowledge, children mathematics, temporal concept

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MIDDLE SCHOOL STUDENTS' REASONING ABOUT A MISSING VALUE LINEAR STEEPNESS PROBLEM

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By the end of middle school, students should have a strong grounding in proportional reasoning and its applications (National Council of Teachers of Mathematics, 2000). Understanding proportions is essential for the study of slope and linearity. We present middle school students' correct and incorrect solutions to a missing value proportionality problem involving the steepness of two lines. Our research questions are: 1) What strategies do grade 5 and 7 students use to solve a missing value problem regarding steepness? 2) How do students justify their solution strategies?

This mixed method study including pencil-and-paper assessments of students in grades 5 and 7, as well as group interviews of grade 7 students. A greater percentage of grade 7 students than grade 5 students answered the experimental task correctly. Still, 69% of grade 7 students answered the experimental task incorrectly. A greater percentage of grade 7 students than grade 5 students incorrectly used additive reasoning. Small group discussion was able to convince some grade 7 students to switch to visual or proportional reasoning and to correctly solve the experimental task.

This research shows that within the context of steepness, providing students with opportunities to solve proportionality problems in small group settings may help additive reasoners begin to move towards proportional reasoning. Further research should explore additive reasoning in other steepness contexts as well as other common errors exhibited by the students.

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MENTORING ELEMENTARY SCHOOL MATHEMATICS TEACHERS

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This report investigates mentoring from the perspectives of 7 elementary school mathematics teachers and their head of department for mathematics in Singapore. This is a qualitative study situated within an interpretive theoretical frame. Transcripts of weekly group discussions, interviews, and teachers' process of constructing story sums were the main data sources that were used to investigate the teachers' views on mentoring during the professional development. Preliminary findings indicate that through greater collaboration between university and teachers, the professional development model of this study fostered new platform for resource sharing and preparing for changes in the national curriculum and increased opportunities for mentorship for teachers.

With the revised 2007 Singapore mathematics curriculum, new mathematics topics were introduced to the elementary school syllabus. 7 teachers in one elementary school in Singapore formed a study group with their head of department in mathematics and a university mathematics educator to prepare themselves teach some of those new topics. The participants in the study included the 7 teachers and their head of department for mathematics. The group decided to work on fractions for 6 months and they met weekly to discuss the concept of fraction, plan, observe and critique mathematics lessons on fractions. In this paper, we report the group's participation in one of the subtopics in fraction - fraction divided by fraction. How the teachers found the mentoring processes beneficial to their pedagogical content knowledge through the study group was explored. The main data sources that were used to investigate the teachers' views on mentoring during the professional development were transcripts of weekly group discussions, interviews, and the teachers' process of constructing story sums involving fractions divided by fraction. The data were read, re-read and coded to the research questions. They were analysed using constant comparison method (Lincoln & Guba, 1985). The teachers and their head of department responses were separated, in order to compare response patterns. The following themes emerged from the preliminary data analysis: perceptions of professional development of the participants and expert in subject-matter knowledge. Based on observations and data feedback, our findings indicate that through greater collaboration between university and teachers, the professional development model of this study fostered new platform for resource sharing and preparing for changes in the national curriculum and increased opportunities for mentorship for teachers.

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A STUDY OF IMPROVING 7TH GRADERS' MATHEMATICAL PROBLEM SOLVING AND COMMUNICATION ABILITIES THROUGH INQUIRY-BASED MATHEMATICS TEACHING

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National Chunghua University of Education, Taiwan (R.O.C.)

INTRODUCTION

Studies on students' abilities, thinking, beliefs, attitudes, and cognition development are forming a growing body of literature about learning mathematics (e.g. Tall, 2004). One of the underlying themes of these studies is to gain an understanding of how learning mathematics comes about. It is in the context of this theme this article is framed, with discussing students' growth of problem solving and communication abilities in the context of inquiry-based mathematics teaching.

METHOD

The study was conducted with twenty-seven 7th graders who were provided opportunities to learn mathematics through inquiry, such as, observation of patterns, making conjectures, testing of conjectures, estimation of results, reflection on the significance of their inquiry, and exchanging ideas with peers and the teacher. Data collection consisted of video-taped classroom observations, students' interviews, students' worksheets and researcher's field notes. Data analysis adhered to the procedures similar to Strauss and Corbin (1990), which were analyzed separately but simultaneously and then compared in order to examine the relationship between inquiry-based mathematics teaching and the changes in students' problem solving and communication abilities.

RESULTS

The results indicated that the inquiry-based mathematics teaching had a positive effect on the subjects' problem solving and communication abilities. Students who engaged in this study were able to construct deeper understanding of the contexts of problems, use a variety of appropriate strategies to solve problems, monitor and reflect on their solving processes, interact with peers and the teacher, and communicate their thinking coherently and clearly to others. Furthermore, the results also revealed that students with low mathematics achievements were more significantly influenced by inquiry-based teaching than the students with high achievements.

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PROSPECTIVE TEACHERS' STRATEGIES TO SOLVING A QUADRATIC GENERALISING PROBLEM

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Celia Hoyles — Institute of Education, University of London

Research on pattern generalisation has largely focused on students' strategies, reasoning and more recently, justification of their generalisations. But teachers' generalisation strategies and practices seem rather less examined in the literature. To draw attention to what prospective teachers bring to the classroom when they teach pattern generalisation, this study aims to shed light on teacher strategies when constructing multiple expressions for a generalising problem. Unlike a typical textbook task, the problem involved a quadratic rule in one variable and was relatively unstructured to allow teachers to explore the pattern more freely.

27 prospective secondary school teachers in Singapore solved the generalising problem in written format; individually first and then in groups so they could discuss different ideas to produce multiple solutions. After marking the scripts to check that each solution led to a correct expression of the rule, the solutions were then analysed to examine the teachers' strategies for establishing the general rule in each solution.

The findings of the study indicate that the teachers produced two to five solutions per group, resulting in 13 different equivalent expressions of the general rule, comprising numerical or figural approaches or a combination of these. A detailed analysis of those solutions that drew on the figural approach further reveals three different categories of generalisation strategies: *constructive*, *deconstructive* and *reconstructive*. The first two categories were originally developed by Becker and Rivera (2006) for linear generalising problems and were found to be applicable in this quadratic generalising problem as well. The third category is, however, a completely new category that we introduce to the classification scheme. Subsequently, four teachers' solutions representing each of the three figural categories and the numerical approach were chosen and given to the teachers, asking them to choose the method that (1) was closest to what they would adopt in class to teach students to derive the rule, and (2) they believe would best help students to construct the rule. The analysis of data showed that both the constructive and reconstructive methods (41% each) were the closest to the teachers' choices for teaching. The teachers (70%) also believed the latter was most helpful to students in rule construction. Samples of some teachers' solutions will be shown.

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LIKES AND LEARNING: STUDENT PERCEPTIONS OF THE VALUE OF DIFFERENT MATHEMATICAL TASK TYPES

Barbara Clarke — Monash University
Doug Clarke — Australian Catholic University
Helen O'Shea — Monash University
Peter Sullivan — Monash University
Anne Roche — Australian Catholic University

The Task Type and Mathematics Learning (TTML) project is investigating the best ways to use different types of mathematics tasks, particularly in Grades 5 to 8. The project focuses on three types of mathematical tasks:

Type 1: Teacher uses a model, example, or explanation that elaborates or exemplifies the mathematics.

Type 2: Teacher situates mathematics within a contextualised practical problem to engage the students, but the motive is explicitly mathematical.

Type 3: Teacher poses open-ended tasks that allow students to investigate specific mathematical content.

These task types are not claimed to include all possible tasks and the categories are not mutually exclusive, with many tasks difficult to classify uniquely.

A Grade 5 and a Grade 6 teacher developed a three-week unit of work on Statistics, consisting of 14 tasks chosen from the three task types above. At the completion of the unit, all students in both classes were given a list of all tasks, marking with "1" the one they "liked most", "2" the one they "liked second most", "A" the one they "learned the most maths from", and "B" the one they "learned the second most maths from." They were also asked to provide a written justification of their first choice in each category.

The most popular task (totalling first and second preferences) involved developing a way or recording outcomes of a common children's game (*Rock, Paper, Scissors*). The second most popular task was the *Average Height of the School*, where the opportunity to work together to measure appealed.

In relation to tasks from which students believed they learned most mathematics, the data were probably more surprising. The open task, *Seven People Went Fishing,* gained the highest ratings. The task was as follows: A group of 7 people went fishing. The mean number of fish caught was 7, the median was 6 and the mode was 5. How many fish might each of the people have caught?

All 14 tasks were rated highly by at least some students. Although it will be important to replicate this study in different contexts, there is evidence that students are thoughtful about their preferences, and are a useful resource as teachers plan the use of different task types in the mathematics classroom.

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ASSESSING TEACHER PEDAGOGICAL CONTENT KNOWLEDGE: CHALLENGES AND INSIGHTS

Doug Clarke Anne Roche Ann Downton

Australian Catholic University (Melbourne)

As part of a two-year teacher professional learning project, a major aim of which is to enhance teacher pedagogical content knowledge (PCK), the authors developed questionnaire items which have the potential to assess and show improvement over time in teachers' PCK. The research project involves 32 primary schools in Melbourne (Australia) which are participating in a two-year program with our university, consisting of 12 full days of professional development for teachers (including workshops, professional reading, and between-session activities) and in-classroom support from the research team. The project aims to enhance teacher pedagogical content knowledge and student learning.

Our current framework has the following components: Pathways; Selecting; Interpreting; Demand and Adapting. Frameworks developed by Ball and Bass (2000) and Chick, Baker, Pham, and Cheng (2006) were helpful when constructing our own.

Each questionnaire (for Grades K-2, 3-4, 5-6 respectively) involved four to six items set in the context of classroom scenarios, where, for example, teachers indicate a particular student difficulty with content, propose suitable learning activities, or order tasks according to their relative cognitive demand.

We have learned that it is important that teachers are asked to justify their choices in a multiple-choice questionnaire. There were many examples of correct choices accompanied by faulty justification. Of course, these written explanations can be difficult to code, and every item was coded twice, with moderation and refinement of codes following any initial disagreements.

While there was pleasing growth evident in teacher's PCK, we also learnt that lack of improvement may reflect the clarity of particular items or the quality of the professional development in relation to the content and/or framework element. Items have been altered accordingly in the second year of the project.

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EQUAL IN DIVERSITY: REFLECTIONS ON PRIMARY TEACHERS' LEARNING[1]

Anne D. Cockburn, Carlo Marchini and Paola Vighi

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This session is intended as a catalyst for future discussion and research by members of the PME community. It arises from an unexpected observation made during an international study exploring mathematical misconceptions in primary schools (Cockburn and Littler, 2008). The research team anticipated finding significant variations in the type of mathematical misconceptions exhibited by pupils in each of the 4 project countries (Italy, Israel, Czech Republic and England). This was not the case and, indeed, that there was a large commonality in what was observed even though, in discussion, we noted that there remarkable differences in the way the teachers taught and had been trained. For example the equals sign, together with arithmetic algorithms and structural properties, gave rise to misconceptions (Parslow-Williams and Cockburn, 2008) which, in some cases, extended well beyond the primary years of schooling (Marchini and Cockburn, 2008). Closer examination of the data revealed that, although many of the misconceptions may have originated in the earliest years of schooling, frequently they did not become manifest for several years. As we will demonstrate, despite the many differences in culture, language and experience, it was striking that the teachers in Italy and England discussed similar implications for their future practice.

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FUTURE MATHEMATICS TEACHERS' IDEAS ABOUT PROOF: AN EXPLORATIVE STUDY USING THE ISIS PROBLEM

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[2] Centre for Instructonal Psychology and Technology, University of Leuven, Belgium

[3] Portland State University, USA

[4] Postdoctoral Fellow of the Research Fund K.U.Leuven, Belgium

The Isis problem, which has a link with the Isis cult of ancient Egypt (Davis & Hersh, 1981), asks: "Find which rectangles with sides of integral length (in some unit) have area and perimeter (numerically) equal, and prove the result." The problem can be initially approached using routine expertise but then requires (for almost all school and college students) adaptive expertise, yet relies on the most rudimentary technical mathematics. It can be extended in numerous ways, for example by asking which triangles with integral sides have the corresponding property (a significantly more difficult problem) or by shifting up dimensionally to ask which cuboids with integer sides have volume and surface area numerically equal. Interesting questions then arise as to which proofs for the original problem are extendible. The problem is notable for the multiplicity and variety of proofs (empirically grounded, algebraic, geometrical) and associated representations. A selection of such proofs provides an instrument for probing students' ideas about proof.

A group of 39 Flemish pre-service mathematics teachers was confronted with the Isis problem. More specifically, we first asked them to solve the problem and to look for more than one solution. Second, we invited them to study five given proofs (factorization, tiles, unit fractions, graph, table) and to rank these proofs from best to worst. We will present different self-found proofs in this group of Flemish pre-service mathematics teachers, as well as their rankings of and comments on the five given proofs. The results highlight a preference of many students for algebraic proofs (factorization and unit fractions) as well as their rejection of experimentation.

Because the Isis problem relates two quantities of different dimensionality, it also connects with the considerable body of research showing that students do not understand the basic principle that linear enlargements by factor k result in 2-dimensional quantities, such as area, being enlarged by a factor of k^2, and 3-dimensional quantities, such as volume, by a factor of k^3 (De Bock, Van Dooren, Janssens, & Verschaffel, 2007), a principle that explains many phenomena in biology and engineering.

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DECODING THE DISCOURSE OF DIFFICULTY IN HIGH SCHOOL MATHEMATICS

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This paper focuses on critical discourse analysis as a theoretical framework for examining mathematics classroom discourse. Critical discourse analysis assumes that utterances in the classroom occur in relation to various orders of discourse. Orders of discourse are entrenched linguistic norms that correspond to habits of social interaction and are central to processes of cultural reproduction (Fairclough, 2003). This paper uses transcript data from a high school calculus course to argue that teacher utterances about the difficulty of mathematical content function as an "order of discourse" that positions speakers within power relations and structures and regulates specific kinds of authority.

This paper emerges from a three-year research project on mathematics discourse and teacher identity. I discuss the effectiveness of critical discourse analysis for making sense of classroom interaction, and I address a methodological issue that haunts all studies of classroom discourse, that being the challenge of decoding discourse so as to understand the way that *subjectivity* is constituted and enacted (sometimes in contradictory ways) in brief and seemingly spontaneous utterances that combine everyday language with the unique lexico-grammatical features found in the mathematical register. In other words, this paper explores the mapping between socio-cultural features of subjectivity and lexico-grammatical features of particular utterances commonly heard in the mathematics classroom.

I examine transcripts from a high school mathematics classroom, and focus on the "discourse of difficulty" that functions to regulate student participation and whole-class negotiation of texts. Repeated attention to the difficulty of the course content operates as a means of managing authority in the classroom discourse. References to difficulty explicitly situate the teacher as the gatekeeper whose benevolent role is to facilitate the decoding of texts. In decoding the texts in terms of difficulty, the teacher reduces the "meaning" of each new text to its relation to prior school texts, focusing on skills of recognition and procedural mastery. This "intertextual" structuring of classroom interaction directs student attention to the textual features of the mathematics register, while diminishing their authority to produce new meanings from these texts.

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DEVELOPING NUMBER SENSE: THE TEACHER'S ROLE

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IPS-ESE Setúbal IPS-ESE Setúbal

In the context of a project centred in the development of number sense we studied the role of the teacher in promoting number sense. In this presentation we will characterise the teaching practice model associated with number sense development.

Number sense has been considered one of the most important components of elementary mathematics curriculum. The development of personal strategies of calculation and its implications to solve problems in real situations are recommended by both international literature (Fuson, 2003) and Portuguese curricular documents.

The project *Number sense development: curricular demands and perspectives* aims to study the development of number sense in elementary school (5-12 years old). In the context of this project the first author of this presentation developed a case study centered in a primary teacher with the objective of investigating: (1) roles that the teacher assumes which will help to develop number sense; (2) difficulties the teacher experiences when preparing and using mathematical tasks which will potentially develop number sense.

In this presentation we will characterize the teaching model associated with the exploration and discussion of tasks that intend to promote number sense development. Classroom practices will illustrate this model and its development from the process-oriented teaching model proposed by Yang (2003).

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GRAPH PAPER: A MEDIATING TOOL IN SOLVING HISTORICAL GEOMETRICAL PROBLEMS

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This study investigates the role of tools in solving historical geometrical problems. In history, social environments of classrooms are full of material objects, tools and artefacts. These are the result of human intentional action. They are constructed for a purpose and validated within the educational community. Tools are mediators of human thought and behaviour, there-for its use and internalization relates to mental functions (Vygotsky, 1978). An aspect of the mediation involved is that tools and artefacts may help scaffold our activities in geometry problem solving. Historical problems are a rich cultural heritage in mathematic education. Tzanakis and Arcavi (2000, p. 208) mention three different main approaches to the integration of history in mathematics education: (1) Learning history, by the provision of direct historical information", (2) "Learning mathematical topics, by following a teaching and learning approach inspired by history", and (3) "Developing deeper awareness, both of mathematics itself and of the social and cultural contexts in which mathematics has been done".

With the above considerations in mind students of 10^{th} grade were asked to solve the well known geometrical problem presented by Socrates in Meno by Plato: "A square of side two feet has area four square feet. Doubling the area, we draw another square of eight square feet. How long is the side of the new square? "

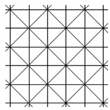

Figure 1

The above problem was posed to 41 students of the Music School of Serres. The students used a graph paper (Brock and Price 1980), as can be seen in the figure 1. They gave different and ingenious solutions of the problem. Graph paper use as a mediating tool and the non algebraic solution which were objected will be discussed in the presentation.

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FACTORS THAT PREDICTS GEOMETRY SELF EFFICACY OF PRESERVICE ELEMANTARY TEACHERS

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The aim of this study is to examine those factors which contribute to the geometry self efficacy of a sample of 312 preservice elementary teachers. Data were collected by using self efficacy scale toward geometry, Van Hiele geometric thinking level test, geometry attitude scale, and geometry achievement test. A correlation analysis revealed that the relationship between geometry self efficacy and all other variables are significant. Moreover, the findings from a regression showed that a combination of these variables was able to predicts geometry self efficacy significantly.

Self-efficacy, the conviction in one's ability to successfully organize and execute courses of action to meet desired outcomes (Bandura, 1986), has been claimed as a reliable predictor of mathematical performance (Pajares, 1996). Furthermore many researches showed that teachers or teacher candidates self efficacy is related many factors like achievement, attitude or some other affective or cognitive characteristics. However the factors contributing geometry self efficacy has not yet been searched. In order to fill this gap in the literature, the current study aimed to examine those factors which contribute to the geometry self efficacy.

Data of the study was collected from 312 (176 female, 136 male) senior preservice elementary school teachers from three universities. Four instruments used in the study are self efficacy scale toward geometry, Van Hiele geometric thinking level test, geometry attitude scale, and geometry achievement test.

A correlation analysis revealed that the relationship between geometry self efficacy and each of the variables of geometric thinking level, geometry attitude, and geometry achievement. Moreover the findings from a regression showed that the combination of geometric thinking level, geometry attitude, and geometry achievement was able to predicts geometry self efficacy significantly.

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COLLABORATIVE GROUP TALK AS A MEANS OF CONSTRUCTING A SHARED MATHEMATICAL LEARNING SPACE

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Collaborative group work is regarded as a successful means of engaging students in mathematical activity, citing learning outcomes or affective aspects as measures of this success. In a review of small group talk, Good, Mulryan and McCaslin (1992) describe “clear and compelling evidence that small group work can facilitate student achievement as well as more favourable attitudes towards peers and subject matter”. The research studies described in this review rarely explore how groups maintain a shared learning space through which this learning occurs.

The study presented here examined the peer talk of students working together in collaborative groups learning mathematics in a socio-constructivist classroom environment (Edwards 2003). It was undertaken in naturalistic classroom conditions in which students between the ages of 11 years and 15 years were audio-recorded as they worked on open-ended mathematical problems. Transcripts of this peer talk were analysed using Mercer’s (1995) three-level linguistic, psychological, and cultural model to analyse peer discussion.

Mercer’s psychological level identifies to what extent reasoning is visible in the student talk. This involves the communication structures between learners, the extent to which learners control the content and direction of the talk, and the ‘ground rules’ established for what constitutes valid talk within the group.

Using the analysis at the psychological level, transcripts of student talk are presented which provide evidence of how all members of the group gain access to the mathematics being constructed. The analyses provided here offer evidence of how the dynamics and interactions of groups support the construction of a continuous (or shared) learning space to support their mutual mathematical learning.

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EMERGING MATHEMATICAL EXPERTS AND PROOF

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The purpose of the research was to investigate ways of thinking (Harel, 2007) utilized by emerging experts in mathematics; that is, students pursuing a doctoral degree in mathematics. One research question was: How did these students work together to generate a proof for the following unfamiliar conjecture? **Conjecture:** *Let **f** be a strictly increasing function from [0, 1] to [0, 1]. Prove that there exists a number **a** in the interval [0, 1] such that **f(a)**=**a**.*

RESULTS

All six pairs of students were able to create a proof for the conjecture in the 40 minutes they were provided. In contrast to novices, who tend to work backward from the desired solution (Larkin, McDermott, Simon & Simon, 1980), the emerging experts in this situation instead worked forward from the "knowns," broadening the search space rather than trying to narrow it. Another "expert-like" behavior observed was their attempts to reduce or re-configure the problem to a state that they were familiar with and could solve, for example, by asking, "What if it was continuous?" The students continually tried to re-use results and strategies they already knew, working within a search space that was expansive and recursive, rather than the more linear search space of the novice. They also needed to coordinate different "pieces" of knowledge necessary for crafting a solution, in particular, the fact that it was an increasing function (and the entailments of that fact), and the fact that there exists a set consisting of the infimum of all x such that $f(x)<x$.

DISCUSSION

This study explores a currently under-investigated pool of mathematics learners, doctoral students with emerging expertise in the discipline. A deeper understanding of this level of expertise can, in the long run, contribute to more effective instruction, guidance and curriculum to assist students in learning mathematics and in carrying out its defining practice, mathematical proof.

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PUTTING THEORY INTO PRACTISE: THE EVOLUTION OF AN IN-SERVICE COURSE FOR MATHEMATICS TEACHERS

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This presentation concerns ways of effectively carrying out an in-service education course. Previous implementations of the course revealed discrepancies between the expectations of the participating teachers and those of the teacher educator, and the main focus of the presentation is on how to deal more effectively with this discrepancy. My research involved several years of developing and running various iterations of a particular in-service course (Ejersbo, 2007). Analyses of the course and classroom observations resulted in gradual changes in the course. In the presentation, I will explain why and how these changes took place. The changes include the idea of a *meta-didactical transposition* – a variation on the well-known notion of 'didactical transposition' (Chevallard, 1985). Unlike the original notion, which referred to the transposition of mathematical ideas, here we are transposing *didactical* ideas from the research literature into activities for the teachers in the inservice course.

Using a design research methodology (Cobb et al., 2003), I ran the redesigns for several cycles, and will report on three main points. First, how theoretical ideas were transposed from the research literature into the practice of the in-service course. Second, how it was possible during the course to challenge the teachers' normal routines of daily teaching and preparation. Third, ways of giving the teachers alternative ideas for their teaching practice.

One of the results of the design research, which will be reported in the presentation, is the emergence of three guiding principle for the meta-didactical transposition. They concern how to find useful theories and transpose them into practical activities. One of the guiding principles, for example, states that in the preparation phase, theory precedes activities but in the implementation phase, activities precede theory.

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CODE-SWITCHING IN MATHEMATICAL FIGURED WORLDS

Indigo Esmonde

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In this short oral presentation, I will present some video clips from a recent analysis of ‘figured worlds’ in heterogeneous mathematics classrooms. Figured worlds are “a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others” (Holland, Lachiotte Jr., Skinner, & Cain, 2001, p. 52). In the figured worlds of classrooms, these characters typically include teachers and students, and specific expectations for how each will behave.

Every person participates in a number of different figured worlds in their lives, such as the figured world of schools or ‘the street’ (Hatt, 2007). In fact, within a single location, like a mathematics classroom, several figured worlds may come into play at various times (Jurow, 2005). In a study of classroom figured worlds in one mathematics class in California, U.S.A., several mathematical figured worlds were identified: one that was more aligned with the teacher’s style of speaking, and one more aligned with peer-to-peer talk. In such a classroom, students must make sense of the hybrid discourse practices available to them, knowing that their choices influence how they are positioned within that figured world.

A particularly interesting phenomenon was the process of ‘code-switching.’ All people are proficient to some degree at code-switching as they pass through the various contexts in their lives. In our data corpus, we observed lower SES students’ discomfort with code-switching into these official mathematical discourses. This discomfort was displayed through stylised ‘acting’ of the roles required for the teacher-aligned mathematical figured world. In the short oral presentation, I will show several video clips, and invite discussion from the audience about research techniques to capture this phenomenon, and implications of this phenomenon for equitable mathematics instruction.

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TEACHER PLANNING MATHEMATICAL ACTIVITIES

Patricia Flores Universidad Pedagógica Nacional, Mexico

Olimpia Figueras and François Pluvinage Cinvestav, Mexico

The main purpose of the study is to investigate the effects of teachers' class planning on the mathematical activity they can foster in their classroom and the ways of thinking about the achievement of their students.

Four teachers of primary school participating in a professional development program were asked to plan activities for solving additive word problems using a slide rule in either one of two versions: an applet created with Cabri Java (Adjiage & Pluvinage, 2007) or a physical model built with cardboard. Teachers carried out the activities in order to focus on the relationships among the structure of the mathematical activity and the hypothesis they brought into play regarding students' reactions and actions for mathematics knowledge building. Researchers knew teachers were not acquainted with the slide rule but it was an important issue for planning activities due to the fact that they had to innovate, not only repeat activities already mastered (Guin & Trouche, 1999). A researcher made a demonstration of possible uses of the slide rule for solving additive word problems with natural numbers. A discussion was carried on about possibilities of working with decimal numbers or fractions. For that purpose teachers had to manufactured a new model. Classes were videotaped for analyses.

Discussion and results

A need to have the learning situation controlled, to know the possible results and answers to questions students might pose was identified in the classes of three participants in the study. Teachers lost sight of the students' invention of word problems and their own possibility of gaining knowledge concerning student's understanding of concepts underlying the usage of the slide rule. No emphasis was placed on relationships between data and the function of the slide rule components. However, children adapted their activity to the instrument's usage, it fostered the representation of problems posed, the awareness of how information is organized and how it can be transformed; aspects that teachers considered difficult to bring up while working with pencil and paper or even using the number line.

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DELIMITATION OF THE REIFICATION PROCESS WITH RESPECT TO OTHER COGNITIVE PROCESSES

Vicenç Font[(1)] and Juan D. Godino[(2)]

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In this short oral communication we establish a delimitation of the reification and decomposition processes with respect to the particularization and generalization processes and to the materialization and idealization processes, using the holistic viewpoint that provides the onto-semiotic approach to mathematics cognition and instruction.

In this short oral communication, first we look at the problem of the delimitation of the process of reification with respect to other processes in teaching and learning mathematics from the holistic viewpoint that underlies the onto-semiotic approach to cognition and mathematical instruction (Godino, Batanero & Font, 2007; Font & Contreras, 2008). This delimitation permits a more detailed analysis, and consequently a better comprehension of each of these processes as well as of their combined presence in mathematical activity.

We show also how the use of objectual metaphor in the mathematics classroom discourse leads the students to interpret the mathematical entities like "objects with existence". On the other hand, the discourse about ostensive objects representing non ostensive object that do not exist and about the identification (differentiation) of the mathematical object with one of its representations leads the students to interpret the mathematical objects as being different from its ostensive representations. As a consequence, the classroom discourse helps to develop the students' comprehension of the non ostensive mathematical objects as objects that have "existence".

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SINGLE-SEX GROUPINGS FOR MATHEMATICS TEACHING IN GRADES 7-10

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Single-sex classes have been viewed as one way of addressing gender differences in mathematics participation and achievement favouring males. While girls have been found to enjoy single-sex classes more than boys do (e.g., Jackson, 2002), findings on achievement gains from these settings are mixed. It has also been recognised that other factors, both within and beyond the single-sex settings, are implicated in students' mathematics learning outcomes (e.g, Streitmatter, 1997), both cognitive and affective. The aims of this study were to determine i. the extent to which single-sex classes are adopted for the teaching of mathematics in co-educational schools in Victoria (Australia) at grades 7-10, and ii. whether teachers supported the practice.

Using an online survey, Victorian (Australia) secondary mathematics teachers were asked: i, Are single-sex settings used in your school for mathematics learning at grades 7-10, and ii. Do you agree with the school's policy about single-sex classes (Yes/No). Why? Over 40 teachers responded to the survey, but only 28 taught in co-educational schools. The responses of the 28 teachers are shown in Table 1.

Sample size: N = 28	**Yes**		**No**		
Are single-sex classes used for mathematics at grades 7-10 in your school?	5 (18%)		23 (82%)		
	Yes	***No***	***Yes***	***No***	***No answer***
Do you agree with the school policy?	5	-	18	3	2

Table 1. Summary of responses from the 28 teachers about single-sex mathematics classes at grades 7-10.

As seen in Table 1, teachers in schools with single-sex classes were supportive of them, generally believing that girls benefitted. In schools without single-sex classes, most agreed, but some disagreed, with the school policy. Among those disagreeing, girls were thought to be disadvantaged by mixed-sex classes. Interestingly, no-one mentioned positive or negative effects of single-sex mathematics classes on boys. Girls, however, were viewed as 'civilising' influences on boys in mixed-sex settings. The small sample and mixed findings precluded the generalisability of findings.

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RESEARCHING IDENTITY AND AFFECT IN MATHEMATICS EDUCATION

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Inés Mª Gómez-Chacón — Univ. Complutense de Madrid, Spain

In researching identity in mathematics education we are focusing on four general aspects: 1) a consistent conceptualization of identity, in particular of mathematical identity; 2) the exploration of possible analytical tools resulting from this conceptualization to identify one's mathematical identity in a given institutional mathematical context; 3) identification of how these identities emerge in terms of one's mathematical experiences; 4) reflection about teaching strategies to foster the development of a healthy mathematical identity by students. In what concerns aspect number 1, we follow Gee (2000) when he says that identity means being a certain 'kind of person'. This being a 'kind of person' is shaped by our experiences in the several social practices we participate in and can change across contexts and time; it depends on our different performances in multiple contexts. From this perspective, one's 'integral' identity is constituted by multiple context-based identities. In the context of mathematics education, being a certain 'kind of person' can be understood in terms of the relationship the subject (student, student-teacher, teacher) develops with the discipline. Gómez-Chacón (2000) says that this relationship involves social information and knowledge, and interaction with that knowledge. This interaction, in its turn, can be characterised by the subject's personal experiences in mathematical practices and how this subject positions her/himself in relation to mathematical knowledge. We argue that these are infused with affect: beliefs, values, emotions, motivations, attitudes and success and failure attributions. In our presentation we provide data from some of our studies, one with secondary students (Frade & Machado 2008), the other with primary student teachers (Gómez-Chacón 2006), to illustrate aspects 2, 3 and 4 and interrelationships between these aspects and affect.

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PROMOTING TEACHERS' UNDERSTANING OF STUDENTS MATHEMATICAL REASONING

John Francisco

University of Massachusetts

This paper reports on the experiences of a group of mathematics teachers in a 3-Year after-school NSF-funded research project involving urban, low-income, middle grade students from a minority community in the United States. For one year, the teachers observed students in mathematical investigations, providing a context for the development of mathematical ideas. The paper describes instances of teachers attending to a variety of aspects of students' mathematical reasoning. The purpose is to show that research sessions on concept development can provide much-needed opportunities for teachers to learn about students' mathematical reasoning

THEORETICAL FRAMEWORK

Effective teaching requires knowledge of students' mathematical reasoning. There is a strong positive correlation between teachers' understanding of how students build knowledge and the students' performance (Rowan, Chinag, and Miller, 1997). The concept of Pedagogical Content Knowledge (PCK) includes "[pre]conceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (Schulman, 1986; p.9). Teaching also requires the ability to interpret and evaluate students' non-standard mathematical ideas and explanations (Ball and Bass, 2003). Yet, middle grade mathematics teachers continue to fail to teach by the NCTM Process Standards, one of which is on "reasoning and proof" (Jacobs, Hiebert, Givvin, Hollingsworth, Garnier, & Wearne, 2006). The paper provides evidence that research settings on concept development can provide teachers with learning opportunities on students' reasoning.

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DEAF ADOLESCENTS SOLVING SUBTRACTION PROBLEMS BETWEEN PEERS

Mariana Fuentes & María del Pilar Fernández-Viader

Autonomous University of Barcelona - University of Barcelona (Sapin)

As part of an ongoing research we present two examples on cooperative learning in subtraction operations and problems with signing deaf students (aged 13:06 -15:11 and 12: 07-13:04). We observed that the two students who explained the procedure to solve a problem found an adapted way to help his/her companion. We think this is due to the fact they share the Zone of Proximal Development (ZPD), and also because of the common use of a visual language and efficient visual strategies in which they incorporate the digits with different uses and functions.

COOPERATIVE LEARNING

Philosophy of cooperative learning establishes that in this relation of giving and receiving help in a reciprocal way, both students benefit of this process. Results on addition and multiplication were presented in PME32 ((Fernández-Viader & Fuentes, 2008a; Fernández-Viader & Fuentes, 2008b).

Method

Students solved operations in symmetrical and asymmetrical couples. Each student posed the operations and problem to the other student, this one solved the operation and then the first one corrected the exercise. Then they interchanged roles.

Results and conclusions

Confirming previous results, the students who explained found an adapted way to help his/her companion because of being in the same Zone of Proximal Development (ZPD) and also because they use the same code, so they feel comfortable to justify and explain the procedures. To have a shared language is to enabling curriculum access too. Studying these kinds of strategies is worth for designing teaching strategies and teacher training.

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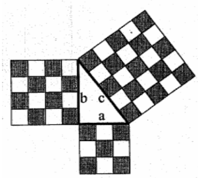
I THOUGHT IT WAS A PROOF

Anne Berit Fuglestad and Simon Goodchild

University of Agder

The research reported here is situated in a binary project in which teachers in local schools collaborate with a group of didacticians at the University of Agder within the projects Teaching Better Mathematics and Learn Better Mathematics (Fuglestad & Goodchild, 2008). The theoretical framework takes a socio cultural perspective of learning and development, with an aim to establish communities of inquiry, working on the common enterprise (Wenger, 1998) of developing the teaching and learning of mathematics. 'Inquiry' is taken as fundamental for both development and research. Inquiry is intended to stimulate reflection upon what might be improved and approaches to achieve the desired improvement. Developmental-research methodology is adopted in which interacting cycles of development and research inform each other (Gravemeier, 1994).

Activities within the project include workshops, school visits and meetings with teachers and didacticians. In workshops mathematical topics are presented followed by work in small groups. During the second year of the project a workshop focused on reasoning and proof. In this report we examine the discussion in a small group activity. Our research questions are: What does the event tell us about teachers' knowledge of mathematics? What is revealed about the social dynamics of the group?



In a group discussion the teachers worked on the Pythagorean theorem, and whether the example drawn in the figure can be taken as a proof. There were doubts, perhaps it was? A further question was to complete a valid proof of the theorem. The discussion revealed that some teachers initially were not sure if it was a proof and what was necessary for proving. However, their discussion led to a joint conclusion that one example is not sufficient, although some pupils might be satisfied with it. Furthermore, by joint efforts working from participants' partial recall about the related geometry and algebraic expressions they were able to reconstruct a proof. The evidence recorded on video reveals lively engagement and collaboration building on each others wondering, questions and comments. Further details will be given in the presentation.

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CHANGING TEACHERS' PERSPECTIVES OF MATHEMATICALLY TALENTED STUDENTS: THE CASE OF RONA

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In many elementary schools, mathematics is taught in mixed-abilities classes where the needs of mathematically talented students (MTS) need to be met as along with others. Yet, many teachers are not aware of the MTS in their classroom and are often ill prepared to meet their special needs. "From One End to the Other"[1] is a mathematics teachers' program aimed at instruction of MTS in heterogeneous classes.

In this paper we report on the changes of one program participant, Rona, focusing on three categories: *awareness, practice,* and *self-efficacy*. The results are based on opening, mid, and ending questionnaires as well as interviews.

Changes in awareness of MTS. In the beginning of the year Rona characterized MTS as being bored in heterogeneous classes, hardly ever making arithmetic mistakes, and rarely having learning disabilities. As Rona participated in the program, she became aware of different characteristics, e.g. how they approach challenging problems, their ability to explain their solutions, the quality of the solution method, motivation etc.

Changes in practice. In the past, Rona allowed talented students to accelerate their advancement in the current subject or to help slower students, claiming she had "no time to look" for appropriate activities. By the end of the year she reported giving these students activities aimed at deepening their mathematical knowledge, using differential teaching techniques, and changing the way in which she implemented these activities in the class. Rona began to take a more active role in the learning process, listening and discussing the activities with her students.

Changes in teacher's self-efficacy. Rona claimed that she now had the courage to work with MTS in a group, she was self-confident to give her MTS assignments that she would have ordinarily skipped in the past, feeling that "Although I myself am not gifted I can definitely help those students develop their talent."

Going through a meaningful change in *awareness* and *practice* concerning MTS may not be enough. If Rona had believed that only gifted teachers were able to meet the needs of talented students, she may not have implemented the more challenging activities in her class. It is the combination of *awareness, practice* and *self-efficacy* which is necessary if the needs of the mathematically talented students are to be met.

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EXPLORING THE AFFECT/COGNITION RELATION IN PROBLEM POSING SITUATIONS

Barbara Georgiadou-Kabouridis — Greek Ministry of Education

Marianna Bartzakli — Greek Ministry of Education

Researchers have defined problem posing in different ways (Mamona-Downs, 1993). In this study we adopt Stoyanova's (1998, p. 165) approach to problem posing as "the process by which students construct personal interpretations of specific situations and formulate them as meaningful well-structured mathematical problems".

This study reports findings from research conducted in one Year 1 (children aged 6 to 7 years) mathematics classroom in an urban school in Greece. Research was undertaken throughout the school year, during which students were engaged in problem solving and problem posing situations. Data were collected by using ethnographic methods, such as participant observation, students' artifacts, interview and the teacher's journal (Eisenhart, 1988). In the data analysis we view the process of problem posing as a canvas on which we explore and identify incidents that reveal the interweaving of affect and cognition. Affect is studied through the students' enthusiasm for mathematics when they are highly motivated and encouraged by the teacher to participate in problem posing activities. Cognition is investigated through the students' anticipated reactions to the teacher's fostering when they create well-structured word problems.

Results showed that the teacher created such a classroom environment, which encouraged students to build their knowledge and develop their thinking and learning skills while constructing their own word problems. The different kinds of problem posing situations that the students were engaged in stimulated the variety of their reflective responses. Students' different creativity levels of problem posing related to their ability of developing their classroom experiences and to their desire to challenge their own and their peers' problem solving abilities. Moreover, there was evidence of students' metacognitive processes while reviewing and reformulating their problems.

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HOLISTIC ANALYSIS AND EVALUATION OF MATHEMATICS TEACHING AND LEARNING PROCESSES

Juan D. Godino[1], Vicenç Font[2] and Carmen Batanero[1]

[1] University of Granada; [2] University of Barcelona

In this sort oral communication we first synthesise a theoretical model about mathematical cognition and instruction, which provides conceptual and methodological tools to pose and deal with research problems in mathematics education. We then describe some applications of this framework to carry out a systematic and global analysis of a mathematics teaching and learning processes.

The *onto-semiotic approach* (OSA) to mathematical knowledge and instruction (Godino, Batanero and Font, 2007) is growing as a theoretical framework for Mathematics Education impelled by issues related to teaching and learning mathematics and by the aspiration of achieving the articulation of the diverse dimensions and perspectives involved in them. This work of articulation cannot be made through the superimposition of tools taken from different and heterogeneous theories. Steiner (1990) conceived Mathematics Education as a scientific discipline in the centre of a complex, heterogeneous, social system and proposed, beside Mathematics, other referential sciences for our discipline, such as: Epistemology, Psychology Pedagogy Sociology and Linguistics. Each of these disciplines focuses its attention on partial aspects of the issues involved in teaching and learning mathematics, using their specific conceptual tools and methodologies. The OSA theoretical system is based on elements taken from these diverse disciplines and intends to develop a unified approach to didactic phenomena that takes into account their epistemological, cognitive, socio cultural and instructional dimensions.

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HOW STUDENTS' UNDERSTANDING OF MEASUREMENT IS FORMED?

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To investigate the status of measurement in school mathematics in Iran, we designed a research study in three parts. At the first part, we developed a framework for investigating the place of measurement in school mathematics curriculum by reviewing the related literature and observing a number of people from different walks of life doing different measurement activities supported by the literature. At this stage, our purpose was to see that how ordinary people without formal instruction on measurement, engage with these activities. We carefully documented our observations and analyzed them to develop our framework. As a result, we identified ten components for our framework including mathematical knowledge related to measurement, real measurement activities, standard measurement tools, non-standard units of measurement, standard units of measurement, individual frames of reference, estimation, understanding proportion, technology, and culture in which, people were used their individual frames of reference more confidently and more naturally.

At the second part of the research, we applied the framework to analyze the content of mathematics textbooks of grades 1 to 8 regarding measurement. The results of the content analysis showed that the main emphasis of these textbooks is on mathematical knowledge related to measurement, standard measurement tools and standard units of measure. In addition, some of the texts have paid attention to real measurement activities, non-standard units of measure and estimation at the surface level. However, the components of individual frames of reference, understanding proportion, technology, and culture have been ignored completely.

Finally, at the third part of the study, we observed 10 secondary school students doing measurement activities, those who received formal instruction on measurement through the primary and lower secondary school mathematics textbooks that have little or no room for students to use their individual frame of reference. Through the data analysis, we found out that these students as well, use their own frames of reference when acting in informal settings. Based on these data, we developed a new model that suggests a spectrum that at one end, there are physical individual frames of reference and at the other, there are mental individual frames of reference. The findings of this part of the research indicated that students mostly use the physical individual frames of reference over the mental individual frames of reference. They also use their individual frames of reference that are not accurate enough and are not learnt through school mathematics. These findings revealed that students' understanding of measurement in Iran is mostly formed by their individual experiences rather than formal instruction.

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DO DEFINITIONS INFLUENCE MATHEMATICS TEACHERS?

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The basis for this research is a belief that the ways teachers perceive definitions in mathematics should affect their teaching. In particular, it is expected that teachers' understanding of definitions in mathematics will influence the classroom discourse.

Data was collected during a visit to a group of four student teachers, as their teacher at the university college, during their teacher practice. The data consists of notes from the classroom and the following discussion with the student teachers together with recordings of a subsequent conversation with them. The lesson observed was a 2nd grade lesson on subtraction. The student teachers introduced the term minus by the statement "minus is the same as taking away". There occurred some episodes, related to different models and methods used for subtraction, about which the student teachers themselves initiated discussion. Still it was problematic to engage them in connecting the threads and discussing the effects of the statement "minus is the same as taking away" on the pupils understanding of the ensuing classroom activity.

The data indicate that the student teachers are uncertain about some of the purpose of mathematical vocabulary, as illustrated by the following inquiry.

> How important is it really to learn [words such as addition and subtraction]? If [the pupils] know that plus is to put together and minus is to take away, what is then the point of imprinting [the words addition and subtraction]? (Student teacher.)

My analysis indicate that the student teachers perceive definitions as primarily an economical tool for communication, i.e. it enables them to use fewer words to express the same content, and not as a tool for structuring and developing mathematics.

Research seems to be lacking on how teachers' knowledge of and about definitions influence their teaching practice (cf. Edwards & Ward, 2005). This is also relevant for teachers in the primary grades, as to how teachers handle the gap that exists between school mathematics and formal mathematics. The reported research will be extended to a further study of these topics. This research is part of the project *Learning Conversation in Mathematics Practice* (LCMP) which is financed by the Norwegian Research Council (NFR) and Bergen University College.

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EXAMINING PROSPECTIVE TEACHERS' REASONING OF FUNCTIONS IN SCHOOL MATHEMATICS

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This research examines prospective teachers' reasoning as they reflect upon the notion of function in school mathematics. These conceptions of prospective teachers are considered of significance in the process of transforming subject matter for the purpose of teaching. That is, in the prospective teachers' development of pedagogical content knowledge.

When two groups of prospective teachers during interviews were given opportunities to reflect upon the presence of functions in school mathematics, it regularly had the effect that prospective teachers came to consider the concept of function as present in school mathematics, but "hidden" in their experience and not made visible until the end of compulsory school, causing them to wonder why this is the case. Reflection upon the significance and presence of different concepts in school mathematics seems to be unfamiliar to prospective teachers. The findings of this study imply that these kinds of reflections may possibly raise a series of questions to serve as a starting point for valuable didactical discussions with prospective teachers, which are related to subject-matter knowledge and other forms of knowledge in the tradition of Shulman and his colleagues (e.g., Shulman, 1986; Grossman, Wilson & Shulman, 1989).

The study indicates that prospective teachers' reflect upon the presence of functions in school mathematics impacts their reasoning. They diverge from a line of thought in which the concept of function has low presence and relevance in compulsory school, and begin to notice situations where the concept of function is present, though not explicitly stated in school mathematics. This process of change in prospective teachers' conceptions of the presence of functions in compulsory school may possibly contribute to emphasize the significance of the process of transformation of subject matter knowledge to pedagogical content knowledge. Broadly speaking, this refers to the specialized knowledge that teachers have of how to represent content knowledge in multiple ways to learners, and even though a variety of concepts in mathematics are not explicitly stated, it represents powerful lines of thought which are issues for attention in the preparation of mathematics teachers.

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VISUAL PROOFS: HIGH SCHOOL STUDENTS' POINT OF VIEW

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The potential of visual proofs has been pointed out previously (Arcavi, 2003). However, students avoid using visual considerations (Eisenberg & Dreyfus, 1991). In order to examine this discrepancy we investigated high school students' views on visual proofs. We selected nine statements with suitable visual and algebraic proofs from Nelson (1993). We designed a questionnaire in two parts. Part I examined to which extent high school students accept visuals proofs as legitimate. Part II examined to which extent and for what purposes high school students prefer using visual proofs over algebraic proofs. During the questionnaire design, we conducted two sets of interviews in order to (i) select the most suitable from among the nine statements and (ii) adapt the formulations to high school students' thinking habits.

The questionnaire was administered to 142 high school students. The findings of part I show that 64% accept visual proofs as legitimate at least 6 out of 7 times. However, 77% of the students believed that their teacher would not accept visual proofs as legitimate. Many wrote explanations like "It is unacceptable proving in that way. My teacher is not proving like this and he expects us to do the same". Some characterized visual proofs as lacking mathematical language. The findings of part II show that 56% of the students would use a visual proof alone or in combination with an algebraic one for explaining a statement to a friend. On the other hand, only 13% prefer using a visual proof rather than an algebraic one for proving a statement (In class or in an exam). This gap indicates that the lack of using visual proofs may be due to the low esteem that high school teachers have, in their students' eyes, for visual proofs. These findings lead us to the conclusion that high school students are receptive for visual proofs. This research also indicates that the students have their own opinions and thoughts about this subject. We hope these results will encourage teachers to increase the use of visual proofs in their classrooms.

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IN-SERVICE TEACHERS' UNDERSTANDING OF DIVISION

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Teachers', prospective teachers' and students' knowledge of division as well as mathematical problem solving as an important didactic tool have been studied extensively (Ball 1990; Schoenfeld 1992).

We study the conceptual and procedural in-service teachers' understanding of division in a realistic problem solving setting. The following problem was given to 230 in-service teachers. "We want to make cards the dimensions of which are 1.5cm times 3.5cm, using a sheet of cardboard the dimensions of which are 1m times 5cm. How many cards can we make leaving unused as little amount of cardboard as possible?" After their papers were collected, they were presented with three different solution strategies supposedly adopted by three different students George, Jim and Helen.

Only 45 teachers solved the problem correctly and recognized Helen's strategy as the correct one. 40 teachers followed George's or Jim's strategy, but they recognized Helen's strategy as the correct one. 140 teachers followed George's or Jim's strategy and they did not recognize Helen's strategy as the correct one. Finally 5 teachers followed Helen's strategy but their poor understanding of division led them to consider Jim's and Helen's approach as equivalent.

We have identified two different sources for wrong answers: a) unsuccessful modelling (i.e. the geometrical aspects of the problem are not considered or, when they are considered, the cutting is done in a way that leaves too much paper unused.) b) Incomplete understanding of the long division algorithm and especially inability to correctly interpret the remainder (when the division performed is Euclidean) and the quotient (when the division is performed in the context of the decimal number system.)

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TEACHING STATISTICS: CONCEPTS VS METHODS

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Should introductory statistics courses focus on concepts or on methods? Recently some educators have advocated sharply reducing methods in order to thoroughly teach concepts. We believe that students learn concepts just as well when they are learned in the context of statistical methods. We report here on the results of a survey of medical students four to five years after taking introductory statistics. Although many of the details of the methods were forgotten, the concepts were learned and retained. The "methods" format provided a much richer applied format for learning the concepts.

THE MAIN SECTION HEADING STYLE IS PME HEADING 2

For many university students, statistics is a required course. It is also a dead end course, with no sequel in the curriculum. Therefore, it is important for students to leave the course with a grasp of basic concepts: what is variation and why do we need to worry about it, what is a confidence interval, what are the elements of a statistical test of hypotheses. It is also helpful for students to get exposure to the statistical methods most often used in their field of study.

Some writers have argued that methods should be de-emphasized to allow for more time to teach concepts. Two drawbacks: limited scope of methods and limited focus on applications.

To assess how well students retain knowledge of statistics, we conducted a survey of 4th and 6th year medical students at Tel Aviv University. All of them took a methods-based introductory statistics course in the first year of studies. This course included 42 lecture hours + 20 hours of exercise sections, covering basic statistical concepts, types of medical research, methods for descriptive statistics, introduction to probability, concepts in statistical inference and methods for analysing data.

Completed questionnaires were obtained from 96 students, a response rate of 51%. Most of the students reported that they had difficulty understanding the "methods" section of research articles but had much better ability to understand the "results" section. Most had already begun work on a research project and only 37% reported strong understanding of the methods in their own proposal.

We conclude that the course does provide ability to read professional literature but is not sufficient as a base for conducting research. We are convinced that the students need a second course focused on study design later in their program.

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MATHEMATIZING, SUBJECTIFYING, AND IDENTIFYING IN MATHEMATICAL DISCOURSE - PRELIMINARY IDEAS ON A METHOD OF ANALYSIS

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Leaning on the commognitive framework (Sfard, 2008), I suggest a method for analysing processes of identity building which are part of the discourse of learning mathematics. The mutual influence of student's mathematical learning and identity has been studied extensively (e.g. Boaler & Greeno, 2000). However, the actual *process* of this mutual shaping has gained less attention. Studying these processes is the aim of my research project, in which I am following 3 groups of 7th grade students, who studied algebra with me as the teacher. In this talk I intend to present the main elements of my conceptual framework and of its analytic tools.

I conceptualize the activity of learning mathematics as interplay between acts of *mathematizing* - communicating about mathematical objects, and acts of *subjectifying* - communicating about those involved in the process.

Subjectifying actions can be categorized according to how general their message about a person is. Utterances in which an interlocutor evaluates what she just did can be conceptualized as *specific* participation-evaluation utterances. The next level is the *general* evaluation of one's participation. The last category is *membership-evaluation*, which takes place when the speaker evaluates who she *is*. I define identifying utterances as subjectifying acts that *signal that the identifier considers a given feature of the identified person as permanent and significant.*

Subjectifying can be done in different ways: it can be verbal or non-verbal, direct or indirect. I propose that non-verbal and verbal indirect subjectifying actions may often be categorized as 'identifying' provided they are *repetitive,* and *consistent*. Additionally, *emotional* declarations and gestures are key signs of *significance*, and thus will be categorized as 'identifying'.

Based on these categorizations, I have built an analysis system of the students' discourse, that will hopefully enable me to better understand how identifying is intertwined with mathematics learning, and more specifically, which identifying acts are connected with learning difficulties in math.

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[1] This report is written as part of my dissertation for Ph.D, with Prof. Anna Sfard as an advisor.

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BECOMING A CONFIDENT MATHEMATICAL LITERACY TEACHER: THE ROLE OF THEORISING AND REFLECTING ON CLASSROOM PRACTICE

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THE STUDY

In response to the need for teachers in the new high school subject of Mathematical Literacy in South Africa, tertiary institutions have offered large scale teacher development programmes aimed at re-skilling non-mathematics teachers to become qualified Mathematical Literacy teachers. This paper reports on research undertaken with 275 teachers in the final semester of their re-skilling programme.

Drawing on the framework for learning from teaching proposed by Hiebert *et al* (2007), on literature on the mathematics teacher as inquirer/researcher (Adler, 1997) and teacher confidence (Graven, 2004), this study investigates the extent to which the exposure to mathematics educational theory and personal experience of action research affected these teachers' confidence and their perceptions of personal competence and efficacy in teaching Mathematical Literacy.

Teachers completed questionnaires, including quantitative and qualitative items, at the beginning and at the end of the semester. The questionnaires, adapted from standard confidence and efficacy instruments, probed teachers' perceptions of their preparedness, confidence and self-efficacy beliefs in taking on their role as Mathematical Literacy teachers. These questionnaire items were subjected to factor analysis to establish themes in the responses. The resultant subscales were analysed for evidence of change over the course of the semester.

The findings of this study suggest that personal and teaching efficacy beliefs became more positive, and all five confidence subscales showed statistically significant improvement. Qualitative analysis of teachers' reflective writing indicates that exposure to some of the theories of mathematics education, and their reflective research projects were considered helpful as they became Mathematical Literacy teachers.

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ANALYSING THE APPLICATION OF REPRESENTATIONS IN PROBLEM-SOLVING : A CASE STUDY OF QUADRATIC INEQUALITIES

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Quadratic inequalities (simplified as QI) play an important role in mathematics. The study of QI is the foundation of various mathematics topics such as algebra, linear planning and the investigation of functions. In school mathematics, the emphasis of QI is on "the link between algebra and geometry". And the topic is a turning point of learning analytic geometry, because in learning equations students study point-wise or piecewise properties, while in QI learning they study global properties.

The purpose of this study was to investigate the cognitive characteristics of QI concepts, and to analyze the application of representations in problem-solving. The study used diagnostic evaluation, consisting of 21 problems during a 60-minute math lesson. The sample consisted of 111 11^{th}-grade students, and they studied IQ about 4 months before the test. They had also studied how to draw graphs of functions.

The cognitive characteristics of QI concepts: (a) In handling algebraic inequalities, students tend to have only procedural knowledge. However, they often have weak understanding of conceptual knowledge. (b) In handling graph of inequalities, they can translation equation to graph, but some of them also have misunderstandings. For example, when some of them see the statement of $f(x) > 0$, they misunderstood it as the range of $x > 0$ (on the right-half of the x-axis).

The application of representations in problem solving: (a) Students are usually confined in the representations that the questions gave them. They do not link the problems to other representations. (b) The linking between different representations is weak. They rarely transform graphic representations to algebraic ones, and vice versa.

To develop mathematical concepts students need to do it through understanding their representations. We found that it is not so easy for students to understand some representations of QI, and suggest more emphasis be put on this respect in teaching.

Goldin & Kaput (1996) suggested that student should use more diagrams to create strong links between representations in order to solve problems in creative and meaningful ways. Today, the learning of QI in Taiwan, students tend to operate on a single representation. We suggest teachers use computer technology to provide a multiple-representation environment to supply students with more experience of linking representations.

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TEACHER'S PROBLEM-POSING BEHAVIOUR CHANGES IN MATHEMATICS CLASSROOMS

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This study reports the problem-posing behaviour changes of two Taiwan mathematics teachers, Mr. Jiang, and Miss Yu. They used the MiT(Mathematics in Taiwan), a reform-oriented mathematics textbook in Taiwan, to teach their junior high school students learn mathematics in classrooms. The MiT emphasizes that students studying in Taiwan should learn mathematics by their living culture. However, the teacher's work is to some extent the opposite of the student's. The teachers must recontextualize their knowledge and the students must redecontextualize their knowledge (Freudenthal, 1991; Brousseau, 1997). The MiT as the Mathematics in Context (Romberg & de Lange, 1998) is a non-traditional textbook. We suppose that teachers can conduct reform-oriented teaching easily if they had the reform-oriented textbook, MiT. Yet, we soon found out we were wrong. Most teachers in Taiwan, at the beginning, do not know how to teach mathematics based on the MiT. This study is designed to investigate the changes in teacher's teaching, especially their problem-posing behaviours when they used using the MIT as textbooks in their classrooms.

We observed these teachers' teaching inside classrooms, took the teacher's interviews, made discussions with them before and after classes, observed their working in instructional module workshops, and encouraged them to change their teaching based on the MiT.

The findings show that Jiang and Yu have changed their problem-posing behaviours gradually during the school year and emerged into three phases. Although Jiang and Yu have different feature and strategies in problem-posing, they have the same model of change. Phase one, they tend to automatically and unconsciously remove the contexts and the situations of the problems in the MiT. Phase two, they tend to keep the contexts on purpose. Phase three, they tend to keep the contexts and the situations of the problems. The differences in teaching goals and strategies between Jiang and Yu will be also discussed.

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EXAMPLES AS MEDIATING ARTIFACTS IN CONJECTURING

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In this paper we report a study concerning examples as mediating artifacts in conjecturing. We used activity theory as an analytical framework to analyze students' behaviours and the process of conjecturing. Our points of view are that of Mason & Johnstone-Wilder (2004, p.142) that claim that "Participating in a conjecturing atmosphere in which everyone is encouraged to construct extreme and paradigmatic examples, and to try to find counter-examples involves learners in thinking and constructing actively". Activity theory also emphasizes social interaction within an activity context and the processes of conjecturing that take place through interaction and mediation.

The aim of this paper presented here was to describe how students generating and classify all kinds of examples to formalize mathematical conjectures. Concerning the use of examples in conjecturing, we observed how the exploration on examples might be fruitful, if it is carried out within a Proceduralized Refutation Model and not done "at random". Students reflected on results, and on the internal structures of examples, in unfamiliar mathematical statement. The results of the case study showed that

- After taking the examples as the mediating artifacts, students could initially understand the statements, and thus examples became the important artifacts to connect the students with the statement.
- Students can distinguish between supportive and rejected examples by constructing their own example spaces.
- Students can develop the ability of generalization by looking for the common properties and then make conjectures.

To summarize, according to the Vygotskian and activity theory perspectives, students should be enculturated with psychological ''toolkits''. In essence, examples are rich psychological and cognitive artifacts for co-construction, and when internalized, these examples serve as conceptual toolkits.

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GOAL ORIENTATION CHANGE AS A RTI INDICATOR FOR TIER 2 MATHEMATICS INTERVENTION

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Currently, limited availability of technically adequate measures for the Tier 2 intervention in the primary grades is a major issue. Assessing student's RTI (response to intervention), mathematics teachers need more assessable assessments that are sensitive to progress. Non-cognitive factors are important for predicting students' performance (Nicholls, 1984). A lot of researchers have tended to postulate motivational achievement goal theory for its role in providing more appropriate reasons for an individual to engage in a learning situation (Covinton, 2000; Elliot, 1999; Pintrich, 2000). Dweck suggested that one's implicit theory of ability creates a meaning system that influences the types of goals that are salient to the individual (Dweck, 1986; Dweck, 2000). The author follows Dweck's definition and views goal orientation as domain specific. Goal orientation is also conceptualized as the result of student's disposition and learning history in this study. So, a student's goal orientation can be modified by appropriate or supportive learning experiences. The purpose of this study is to investigate the predictive validity issue of a computerized mathematics learning approach scale (CMLAS) change measure. The change of CMLAS is applied as a RTI indicator for the tier 2 mathematics intervention. A dynamic assessment model which combines mathematics computerized adaptive testing (MCAT) and a two-phase ICT infused intervention is implemented in this study. The first intervention phase uses a spatial game to modify student's mathematics learning approach. The second intervention focuses on fostering the slow learner's automaticity of numerical operation. Thirty-four at risk or learning disable students are included in this study. After first intervention phase, the proportion of mastery orientation students increases significantly (form 38% to 68%). All subjects are classified into 3 different groups (remain mastery, remain neutral and change toward mastery) according to their CMLAS change after the first intervention phase. When the second intervention phase is over, the differences on mathematics growth slope between groups are examined. The growth slope mean of students who change toward mastery is significantly higher than the mean of students who remain neutral. The preliminary results suggest that the assessment of goal orientation change could be a promising predictor for mathematics growth, unconditional to students' prior ability. Due to the limit of sample size, further cross validation studies are needed before any screening procedure could be proposed.

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USING ODD AND EVEN NUMBERS TO DEVELOP ALGEBRAIC REASONING

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For those students who complete their schooling with inadequate algebraic understandings, access to further education and employment opportunities is limited. The ongoing concern, both in New Zealand and internationally, with the number of students in this position has resulted in increased research and curricula attention of the teaching and learning of algebraic reasoning. One response to addressing the problem has been to integrate the teaching and learning of arithmetic and algebra as a unified curriculum strand in policy documents (e.g., Ministry of Education, 2007; National Council of Teachers of Mathematics, 2000). Within the unification of arithmetic and algebra, students' intuitive knowledge of patterns and numerical reasoning are used to provide a foundation for transition to early algebraic thinking (Carpenter, Franke, & Levi, 2003).

The research reported in this paper demonstrates how an examination of odd and even numbers provided young students with a valuable context in which to learn how to make conjectures and construct generalisations. Episodes in the paper are drawn from a larger 3-month teaching experiment (Cobb, 2000) focused on developing early algebraic reasoning. The study was conducted at a New Zealand urban primary school and involved 25 students aged 9-11 years.

Results suggest that the use of argumentation, concrete materials, and specific teacher interventions were important factors which supported students to begin using arithmetic understandings as a basis for early algebraic reasoning. Findings of this study affirm that the context of odd and even numbers can provide students with effective opportunities to make conjectures, justify and generalise. Argumentation and teacher intervention supported students to model their conjectures on material and use increasingly sophisicated justification strategies.

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COMING TO 'KNOW' MATHEMATICS THROUGH 'TALKING AND DOING' MATHEMATICS

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In a recent PME paper Mike Askew (Askew, 2007) drew our attention back to re-consider the viability of Vygotskian notions of scaffolding, within current mathematics classrooms. In his paper, Askew challenged the usefulness of the metaphorical view of scaffolding as a tool for results in mathematics education. Askew linked this view of scaffolding—considered as a tool for results—to recent reform measures introduced in Britain. In these reform measures the policy makers specify detailed learning outcomes for mathematics education. New Zealand has followed a similar route with the introduction in recent years of a 'Numeracy Development Project' (Ministry of Education, 2004a). Similar to the British model described by Askew, this project has predetermined learning outcomes and a detailed script for teachers to use. Implicitly suggested within this model, is the idea of scaffolding as a tool which will give results—student acquisition of mathematical knowledge and strategies through teacher led instruction. Askew, drawing on Vygotskian theories interpreted in the work of Newman and Holzman (1997), proposes that rather than considering scaffolding as a tool for result, it should be considered as a tool-and-result. In this frame, scaffolding as the mediating tool is as much a part of the learning, as is what is learnt.

In this paper I illustrate what happens to 'talking and doing mathematics' when scaffolding is used in what resembles scaffolding as a tool for results and in contrast as a tool-and-result. I show through two classroom episodes the two forms of scaffolding used to support student learning and the different forms of student learning in relationship to 'talking and doing mathematics' which emerges as a result.

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VISUALISATION AS A MEANING-BESTOWING PROCESS IN THE LEARNING AND TEACHING OF ABSTRACT ALGEBRA

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Abstract Algebra is considered by students as one of the most difficult topics of their university studies. They often feel discouraged after the mandatory course typically taught – in the UK – in Year 2 and their majority does not choose to study any more Abstract Algebra courses thereafter. Previous research (e.g. Nardi, 2000) attributes student difficulty with the topic to its multi-level abstraction and the less-than-obvious, to students, *raison d'être* of concepts such as cosets, quotient group etc. Bestowing meaning to these new concepts is therefore a fundamental part of the students' encounter with the topic. Constructing appropriate visual imagery can provide crucial support to this meaning-bestowing process (Zazkis et al, 1996).

The study we draw on here is a close examination of the cognitive, social and emotional aspects of mathematics undergraduates' learning experience in Abstract Algebra. Our data consists of: observation notes and audio-recordings of 20 lectures and 12 group seminars of a 10-week Year 2 Abstract Algebra course attended by 84 students in a well-regarded mathematics department in the UK; regular interviews with 13 of the 84 students, the course lecturer, two group seminar leaders and two group seminar assistants; the 84 students' coursework and exam papers.

Our preliminary scrutiny of the data suggests that students' typically problematic relationship with visualisation (e.g. Presmeg, 2006) is even more so in the case of Abstract Algebra. While students (e.g. as evident in the interviews) repeatedly express – often with intense emotion – a need for 'pictures' that will illuminate the nature of the novel to them Abstract Algebraic objects, they are at the same time reluctant to attempt a construction of such images (e.g. as evident in the visual scarcity of their written work) or engage with the images on offer by their lecturers (e.g. as evident in the lecture observations). Our analyses currently explore whether issues such as student lack of experience/practice with visualisation and student uncertainty about the mathematical status of visualisation lie behind this reluctance.

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INNOVATION COMPETENCE AND PRACTICE TEACHING

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The 'Learning Conversation in Mathematics Practice' (LCMP) research project has engaged student teachers, schoolteachers and teacher educators in a collaborative study of the communicative conditions for learning. The project reported here is, as part of LCMP, emphasising on professional development and empowerment of student teachers. These student teachers are studying mathematics / mathematics education and take part in the developmental initiative entitled 'Real-life Education' in schools. As part of their practice teaching, they are challenged to develop new and alternative designs for teaching, in collaboration with teachers, teacher educators and employees from the designated workplaces. The planning, implementation, reflection and evaluation of teaching and learning processes are to take place in an experimental mode, which facilitates the 'trying out' of new ideas, for learning mathematics at work-places and in classroom. The research focus is on the communication that develops in the context of the student teachers' professional development as mathematics teachers. In collaboration between student teachers, teachers and didacticians the role of communication in the critical process of teaching and learning mathematics is questioned. Data is drawn from such conversations, as tape recordings and videos. The analysis is based on a dialogical perspective as developed by Alrø and Skovsmose (2002) and from a text theoretical and bakhtinian perspective as developed in Johnsen-Høines (2004; 2007). When the data are analysed and discussed in the context of the student teachers learning, the notion of innovation competence, which is regarded as the competence of being able to challenge already existing practices and critically to think, plan and act in alternative ways, is distinguished as an important feature. One theme for inquiry is on identifying how innovation competence can be stimulated in a conversation that focuses on possibilities rather than on limits. Such meta-discussions are found to be important in the context of empowerment for the professional development of the student teachers.

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DEVELOPMENTAL CHANGES IN THE INFLUENCE OF LANGUAGE ON YOUNG CHILDREN'S MATHEMATICAL PERFORMANCE

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Our longitudinal study explored the role of language in young children's mathematical development between ages 5 and 7 years by comparing the growth rates of four achievement subtypes. Subtype comparisons indicated that language plays a minor role in young children's mathematics; however, further analysis suggested that compensatory strategies adopted by those with learning difficulties may mask the importance of language in young children's mathematics.

Jordan and colleagues (2001, 2003) previously explored the role of language in mathematics by testing children aged 7-9 years longitudinally on a battery of mathematical tasks from the following four achievement subtypes: mathematical difficulty only (MD), reading difficulty only (RD), comorbid mathematics and reading difficulties (MDRD) and typically achieving (TA). Their subtype comparisons indicated that language plays a greater role in some mathematical tasks than in others.

The aims of the present research were twofold: Firstly, to use two key subtype growth rate comparisons (MD v MDRD and RD v TA) to explore the role of language in performance on seven mathematical tasks in children aged 5-7 years. Secondly, given Hanich et al.'s (2001) suggestion that RD children may be able to compensate for their poor reading skills by utilising other intact skills, the role of language was further explored by correlating measures of verbal and non-verbal ability with performance on the seven mathematical tasks for each subtype. Overall, results revealed little difference between the performance of MD and MDRD, or between RD and TA, on the seven tasks, suggesting that language plays only a minor role in the performance of young children on the seven mathematical tasks. On the other hand, the correlational analysis indicated that language ability is related to the performance of typically achieving children on most mathematical tasks, and that those with learning difficulties may adopt alternative routes to processing.

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STUDENTS' LEARNING OF MATHEMATICS IN CLASSROOMS WHERE COMPUTERS ARE USED

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Computers are generally seen to have the potential to make a significant contribution to the learning of mathematics, but limited non-interventionist research exists about student learning of mathematics in authentic classroom situations where computers are used. This presentation concerns a study set within this research context, reporting on the mathematical learning of two students as they completed a task set by their teacher.

The data was taken from a mathematics lesson for 11 to 12 year olds. The topic for the lesson was linear functions of the form $y = mx + c$ and the teacher stated that the intended learning for the lesson was 'making a connection between a straight line graph and its equation'. The lesson took place in a computer room and the teacher began by showing the class how to produce graphs using a computer graphing package. She gave worksheets to all the students with instructions to work in pairs to complete the questions on the sheet. This involved students producing sets of graphs using the software and writing down what they noticed about each set of graphs.

The learning of the two students was analysed was analysed using a theoretical perspective which suggests that 'dialectics' of action, formulation and validation take place in the mathematics classroom, and that all three types of dialectic are required for mathematical learning to take place (Brousseau, 1997). Further, transitions between the pragmatic/empirical field and the mathematical/systematic field are also required for student learning in mathematics (Balacheff, 1991; Noss, Healy & Hoyles, 1997).

The findings of the study reveal disappointing levels of mathematical learning in terms of the theoretical perspectives adopted. These findings, informed by the non-interventionist paradigm adopted, in which naturalistic classroom situations are studied, point to the need to support teachers in the engineering of classroom tasks to take into account the potential of computers to contribute to mathematics learning.

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JUSTIFICATION IDENTIFIED IN MATHEMATICS CLASSROOM

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The purpose of this paper is to describe some ways of student's "pragmatic justification" identified in mathematics classroom. Previous studies, for example Harel (2007), have described various ways of justifying and proving by students. In this paper, I will emphasize that even if pragmatic justification is not necessarily mathematically valid, students consider mathematical validity as a reason of justification gradually according to the shift of activity by students from autonomous to heteronomous in mathematics classroom.

For analysis with this paper, I utilized the video data of a set of junior high school geometry lessons and some comments by students after every lesson. I extracted typical ways of justification for others, which had a necessary condition whether others were convinced to them.

In this way, some ways of pragmatic justification by students are following:

(a) Efficiency…Students tend to consider that some ideas are better if they progress problem solving efficiently.

(b) Possibility of reproduction…Students tend to consider that some ideas are valid if they could obtain same results even if they choose anything.

(c) Illustration…If students obtain an expected or coherent result by using some ideas, they tend to consider them valid. And, the way to quote counterexamples for demonstrating mistakes is included in this point.

(d) Visualization…If students could "see" and verify the result of thinking by using some ideas, they tend to consider them valid.

(e) Similarity…This point is a kind of justification based on similarity. If students recognize similarity between a concept and more familiar one, they determine that statements drawing from familiar one may be valid in similar one.

Mathematics lesson proceed with interaction among these ways. In this process, if determination whether to be efficient or not is made based on the idea by others including a teacher, for example, student will be aware of the necessity for mathematical validity. Therefore, in mathematics lesson, teacher needs to actualize the above-mentioned pragmatic justifications and needs to argue with each other.

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MENTAL IMAGES, PROTOTYPICAL FIGURES AND FLEXIBILITY

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In the present study we investigate students' mental images of geometrical figures, the extent to which mental images differentiate students' flexibility in generating a variety of different figures, and, finally, the consistency by which they are applied.

OVERVIEW

A number of studies have investigated children's mental images of geometrical figures (e.g. Clements, 1998; Tsamir, Tirosh & Levenson, 2008). As a result of geometry teaching at school, students' mental images of geometric forms often have a prototypical figure as a point of reference (Clements, 1998), which may hinder the understanding of a concept (Tsamir et al., 2008). Even though a number of studies investigated the mental images of geometrical concepts, little attention has been given to the way children's mental images may influence their flexibility.

This study investigates fifth and sixth graders' mental images of geometric forms, the extent to which they influence students' flexibility in constructing a variety of figures and the consistency by which they are applied. In terms of this study, flexibility refers to the ability to shift from the prototypical examples of geometrical figures to a variety of non-prototypical examples.

A total of 201 students in Cyprus, aged 10-11, participated in the study. A test which consisted of two parts was used. In the first part, students were asked to construct a figure of a triangle, a square and a rectangle. Based on the assumption that the figures students construct reflect their mental images of geometric shapes, we used this task to examine if the initial mental images of students were prototypical examples of the concepts. The second part aimed to investigate students' flexibility by asking them to construct multiple and different examples for the triangle, square and rectangle. The results showed that most of the students constructed prototypical figures for all geometrical shapes. Prototypes were found to hinder students' flexibility only in the case of triangles. Finally, evidence was provided for students' consistency in drawing prototypical or non prototypical examples for different geometrical shapes.

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CLEO'S EXPERIENCE OF AN EXTERNAL ASSESSMENT PROGRAMME SET WITHIN A REAL-LIFE CONTEXT

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THE STUDY

This paper reports on a qualitative study carried out with a Grade 9 mathematics learner. The purpose of the study was to explore the experiences of the learner with respect to the Common Tasks for Assessment, an external assessment programme that they completed for their summative assessment, which was grounded in 'real-life' context.

A review of recent research reveals many criticisms of national assessment tasks set within real life contexts. In her article, Boaler (2003) focuses on a low-income school in which the learners made incredible achievements but however were labelled as 'under performing'. The judgement of 'under performance' was based on the results of the standardised tests, which were based on 'real-life' contexts. Boaler argues that the standardized tests and similar tests throughout the United States "stack the decks against language learners, and students from minority ethnic and cultural groups and low-income homes" (p. 502). Cooper and Dunne (2004) have presented similar criticisms of certain national assessment tasks in the U.K

The study utilised an interpretative qualitative case study approach. In this paper, we present a case study of Cleo which was drawn up from the analysis of 5 videotaped classroom observations, Cleo's responses to 6 CTA activities, her written reflections on the CTA and one semi-structured interview.

The findings reveal that Cleo: was disadvantaged by the teachers' inconsistency in marking; missed crucial information which appeared amongst contextual information; did not understand the complexities and specialised language of the context; and consequently resorted to 'number grabbing' techniques. The findings support Murray's (2003) statement that the 'so-called' real-life contexts are not necessarily accessible to all learners.

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AN ANALYSIS OF THIRD GRADERS' FUNCTIONAL THINKING

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Algebra is a very important area in mathematic education so it is necessary for all students. Teaching early algebra for primary school students needs different approaches from that of traditional algebra. Among them, the introduction of function would offer an opportunity to learn algebra (Carraher & Schliemann, 2007). Whereas emphasizing the importance of function, there have been little studies about early graders' functional thinking process. This study is intended to suggest implications on how to approach early algebra by investigating students' functional thinking

A qualitative case study was conducted. At first, 28 third grade students were asked to solve a task which can be completed only when they focused on the relationship between two varying quantities. After analyzing their solutions, 3 students who were seemed to engage themselves in functional thinking were selected for clinical interviews. Each student was interviewed three times with different tasks. We scrutinized students' thinking process in terms of three aspects by Smith (2008): 'engaging in a problematic within a functional situation', 'creating a record', and 'seeking patterns and mathematical certainty'.

The results showed that students understood the functional situation. However, they tended to rely on their intuition when asked to guess the result. For instance, when asked to compare which is a better way between doubling something and tripling it and then taking away 7, one student responded, "The former is always better because the latter is taken away anyway." With regard to creating records, students came up with diverse representations such as descriptive writings, pictures, and expressions. The records sometimes were an obstacle to solving problems, in particular when they made inappropriate records for the situation. Students were also quite successful in operating with concrete numbers, but had difficulties finding a pattern and making a generalization. They also tended to focus on the changes in each quantity in place of the relation between the corresponding two quantities. This study reveals that third graders have possibilities to participate in functional thinking, so it is necessary to give them an opportunity to experience the functional situation.

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SECONDARY MATHEMATICS TEACHER-LEADER PROGRAMS: TRENDS & FUTURE RESEARCH

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This short oral session provides a forum for international exchange and a foundation for collaborative publications among international scholars engaged in the development of secondary mathematics teacher leaders. Two questions focus this study: (1) What specialized knowledge and skills are needed for secondary mathematics teaching and teacher leadership (KSMTL)? (2) What do we know about program design, content, and leadership-development practices for secondary mathematics teacher leaders, and how can we join to learn more?

BACKGROUND, THEORETICAL FRAMEWORK AND METHOD

Building on the 2004 PME Research Symposium on the Nature of Mathematical Knowledge for Secondary Teaching (Stacey and Doerr, 2001), this study examines program design principles, integrated mathematics content and pedagogy strands, and leadership development tasks for secondary mathematics teacher-leader programs. Modifying Perkins and Simmons' (1988) knowledge framework, this study employs five categories of mathematical understanding (content, concept, problem solving, epistemic, and inquiry) to identify mathematical KSMTL. One documentary account of program development between a university and its school-district partner provide the basis for comparative program analysis.

RESULTS, DISCUSSION AND EDUCATIONAL IMPORTANCE:

How do we translate educational research and local needs into responsive curricular practice with global impact? Using one university/school district's "9-12 Mathematics Instructional Leadership Program" as exemplar, the study describes how research on secondary mathematics student learning shaped program content. Summative analysis compares exemplar program philosophy and courses (Mathematical Reasoning, Modeling, Calculus Applications, and inquiry-oriented courses on developing algebraic, geometric and stochastic thinking) with global approaches to secondary mathematics teacher leadership (e.g., Hofstein and Even, 2001).

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ANCIENT KNOWLEDGE, UP TO DATE THEORIES AND MODERN APPLICATIONS IN THE MATHEMATICS CLASSROOM

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Mathematics educators have been emphasizing the importance of integrating the history of mathematics in the curriculum (Arcavi, 1987; Kline, 1972). "Knowing the past is the key for understanding the present" (J. Dewey in "Democracy and Education"). The integration of computers and historical aspects contribute to the interest and understanding in teaching and learning mathematics. We will describe a way of adapting ancient Greek knowledge and ideas to modern mathematics, using up to date theories of learning as well as up to date technologies.

In this presentation we will describe a way of adapting ancient Greek ideas to modern mathematics using computers in a constructivist way. Educational theories were already developed by the ancient Greeks. The fundamental issues they dealt with are still relevant in education today (Guri, 1985). Thales, Socrates, Aristoteles, and other Greek philosophers (600-400 B.C) believed that exchange of opinions is the main issue in teaching, stressing the process and not only the outcomes. Now a day, believers of constructivism see the primary role of teaching in encouraging students to construct their knowledge by active inquiries and engagement in dialogues.

Many mathematical topics which have already engaged the greatest mathematicians from the dawn of history such as Archimedes, Euclid, Pythagoras and Heron were since, and must be (in our opinion) taught in school mathematics across the curriculum. Computers enable us today to incorporate these topics in modern mathematics.

In our speech we will present a constructivist way of teaching these ideas starting at the upper grades of primary school and up to college mathematics. The learning takes place in a mathematical computer laboratory which enables active involvement of the student guided by the teacher (Breuer and Zwas, 1993). Some of the topics to be discussed are: Euclid's algorithm for primary school students, Archimedes' methods for computing the number Pi for middle school students, Heron's method for computing the square root and Archimedes' ideas for limits in modern calculus courses.

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AN ANALYSIS OF TEACHER'S QUESTIONING IN THE MATHEMATICS CLASSROOMS IN GERMANY AND JAPAN

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The previous studies of international comparison of mathematics classrooms identified characteristics of mathematics lessons in Germany and Japan. Japanese lessons have a coherence of contents, for example, and the process to reach an answer is considered as most important part of the lesson. The TIMSS 1995 Videotape Classroom Study (Stigler & Hiebert, 1999), which analyzed eight-grade mathematics classes in Germany, Japan and the United States and the TIMSS 1999 Videotape Classroom Study (Hiebert et al., 2003), which extended the earlier study and seven countries participated, among others, identified with a resemblance among participating countries while instruction in Japan seemingly unique.

The purpose of this study is to explore the characteristic of the Japanese and German mathematics lessons by reconsidering the similarities and differences identified in the previous studies from the participants' perspectives. For this aim, this study analyzes the mathematic lessons in a fine-grained level by focusing on teacher's questioning at the phase of introducing the mathematical concept from the data of Japanese and German lessons. This study intends to explore the similarity and difference between German and Japanese mathematics lessons by analyzing the videotaped sequence of ten consecutive mathematics lessons in both countries, which is a subset of the data in the Learner's Perspective Study (Clarke, Keitel & Shimizu, 2006).

The results of analysis reveal that the German teacher behaves in the classroom with a focus on student's logical thinking and constitutes lessons by aiming that students can accomplish a procedure precisely, while the Japanese teacher pays attention to ideas behind the answer and constitutes lessons by having students reflect on what they did. This study suggests that the similarity of how to introduce concepts and the difference in the elicitation-response sequence between German and Japanese classrooms identified by the earlier study need to be reconsidered by attending to the meaning constructed by the teachers and learners.

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ANALYZING STUDENT'S GESTURES THROUGH OPEN APPROACH

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This study was a part of research project and development of students' Mathematical Thinking through Lesson Study and Open Approach*. The open approach was a teaching process in urban school of Khon Kaen by posing the open-ended problem, group working, whole class discussion, and summarizing lesson respectively. It gave more chances for students to interpret the problem and represent various mathematical ideas to solve the problem using both verbal and nonverbal languages. Gesture, nonverbal language, was an important way for communicating mathematical ideas of students. In traditional classroom, mathematics teachers viewed that the gestures were not important. Gestures were completely synchronized with the description of the problem solving technique. They also showed flexible roles in solving problems (Edwards, 2005). McNeill (1992 cited in Edwards, 2005) defined gesture as "movements of the arms and hands … closely synchronized with the flow of speech" and classified gestures in different categories: *deictic*, *metaphoric*, *iconic*, and *beat* gestures.

The purpose of this study was to analyze the students' gestures through open approach according to McNeill (1992) three of 4th grade students during the 1st semester of 2007 academic year. There were collaborations in planning, observing, and reflecting among researcher and colleague. The audio and video tape recordings were used during the problem solving sessions. Then, they were transcribed into Protocol Analysis. The participants were interviewed after problem solving session and transcribed into protocol.

Data were analyzed based on McNeill's (1992) approach. The results indicated that open approach gave more chances for students to interpret the problem and represent various mathematical ideas in expressing various gestures representing their mathematical ideas.

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EXPLORING STUDENTS' STRATEGIES IN COMPARISON OF AREA TASKS

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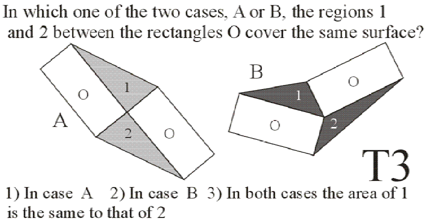
Conservation of area is a key concept for the understanding of the measurement process. The research concerning the concept of area has been mostly confined to problems involving the known formulae of the area measurement. The procedural and rule-dominated (formula application), rather than conceptual and relational, character of students' thinking appeared to be prevalent. Furthermore, visual perception seemed to be another factor determining students' responses.



An example of a task used in the study

We studied the choice and adequacy of students' strategies, as well as visualization effects, in problems concerning conservation of areas. For this purpose, some special tasks have been constructed, that could be dealt with a variety of methods. For example, in the task pictured left, one can employ visual estimation, or measurements and computations using formulas, or even euclidean deductive argumentation without direct area computation.

The data reported here, concerning four of the tasks, derived from interviews conducted with 21 twelfth graders and university students of Mathematics.

The fraction of the participants that adequately completed a geometrical deductive inference was small. The majority of the students held the view that the proper way to deal with the problems was the geometrical method, but the precariousness of their background knowledge did not support the method. We could characterize the informal methods of the students as "cut-and-paste" procedures with crude and unwarranted approximations. It could be said that the most frequent intuitive notion affecting students' attitudes has been the view that area-equivalence coincides with congruence. This notion sometimes acted on implicitly and sometimes was expressed directly. The educational experience of the subject pulled to the opposite direction but this strong inclination counteracted and gained the upper hand when the conditions permitted. The visual set of the problem seemed to exercise a powerful effect on students' strategy choice. In many cases they attempted to prove that triangles couldn't have equal areas as non-congruent, in the same time used the "cut-and-paste" method to establish area conservation.

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"DRAMA IN EDUCATION" IN TEACHING MATHEMATICS TO 10TH GRADE'S STUDENTS

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It is generally accepted that traditional methods of teaching and learning Mathematics have failed. For this reason researchers have turned towards alternative approaches amongst which is Drama based instruction in Mathematics.

'Drama in Education' is a highly structured pedagogical procedure which uses exercises and techniques of theatrical art. The ''as-if'' world that it creates gives the context for teaching a concept, an idea or an event. Within this context, the teacher creates frames containing scaffolding that allow legitimate peripheral participation of learners in roles that are developmentally appropriate (Andersen, 2004).

Existing literature in preschool children as in 1th- 8th grade students concerns the effects of Drama based instruction in students' constructing their own knowledge and in students' attitudes, beliefs and metacognition. The research results are very encouraging in students' understanding and retention of mathematical notions and in creating positive attitudes towards mathematics (Duatepe, 2004).

This research, which has been conducted in twenty six 10th grade students, concerns the effects of Drama based instruction, in students' beliefs about Mathematics. The research has been conducted during the project 'Integrating 10th grade curriculum through Mathematical Literature'. In this project, the reading of the literary work "The Sand Reckoner" by Gillian Bradshaw, which refers to the life and work of Archimedes, gave us the framework to unify most of the lessons of the curriculum. It also gave us the pretext to perform in Mathematics' lesson, among other activities and drama based activities as Dramatization, Round table discussion with role-playing and Radio-broadcasting. These activities focused on students' adopting Mathematics as a cultural and human construction and on bringing in life the central role of the historical and social context in the nature of Mathematics. From the study of their answers to our questionnaire given at the end of the school year, we confirmed that all these activities helped the majority of children in modifying their beliefs of the nature and the role of Mathematics.

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EVALUATIING TEACHER PRACTICE AND LEARNER PERFORMANCE IN SOUTH AFRICA'S FIRST NATIONAL MATHEMATICS NCS, EXAMINATION.

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South Africa saw the exit of 340000 Grade 12 mathematics learners who had just completed writing the first NCS(National Curriculum Statement) examination in 2008. It was to be the culmination of 12 years of the new mathematics curriculum. This research examines the expectations of curriculum innovators, classroom teachers and learners. Despite these expectations the results of the learners gives a cause for concern as regards the teaching and learning of mathematics in South African Classrooms.

The last 12 years in South Africa has been spent on transforming the curriculum into one that is high skills and knowledge. The mathematics curriculum has been internationally bench marked and looks at core skills and knowledge that is consistent with many countries. Other principles of problem solving, critical thinking, modelling and reasoning seemed to permeate through the curriculum. These principles were celebrated by many and they welcomed the new found "spirit" in the curriculum. Mathematics teachers as always took on the great challenge of engaging with the new mathematics content in a determined and enthusiastic demeanour.

The performance of the learners in the examination has evoked national concern and despite the overtures of the national department to provide 2 exemplar papers and examination guideline documents, the results in mathematics has still been polarised between the weak and able students.

This research attempts to examine the reasons for the poor results in the subject mainly looking at how teachers engaged with the new content and whether their pedagogic approaches enhanced successful teaching and learning. The marks obtained in the examination scripts, interviews with teachers and learners will provide the core substance of this research.

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MULTIPLICATIVE REASONING IN GRADES 5 TO 7

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During middle grades' school mathematics students are required to transfer from whole numbers to rational number system. Students must go beyond the whole number idea, in which a number expresses a concrete quantity, to understand numbers that are expressed in relationship to other numbers. Notable changes have to happen in the student's thinking process. From literature on proportional reasoning we know that students have to move from qualitative thinking and additive reasoning that characterizes the whole numbers, to quite different multiplicative reasoning which is necessary for proportional reasoning (Steinthorsdottir, 2005). Ability in proportional reasoning is needed for understanding of many topics in mathematics, such us percentages, ratio, similar triangles and linear functions, and it is a cornerstone of middle school mathematics. But such reasoning it is notoriously difficult for many learners and it develops slowly over a number of years.

The paper presents results of the study that was conducted in one of the comprehensive schools of Tallinn, Estonia. Current study is a part of longitudinal study which began in autumn 2007 and one of the aims of research is to investigate the development of proportional reasoning in students. During the school year 2007/2008 were tested 5^{th}, 6^{th} and 7^{th} grade students, in total nine classes (three from each parallel class) and according to plan the same students will be tested during the next two school years. Tests on proportional reasoning for 6^{th} and 7^{th} grade consist of 21 exercises and there are three different types of tasks: missing value, numerical comparison, and qualitative prediction and comparison. Tests for 5^{th} grade include 13 exercises from tests for 6^{th} and 7^{th} grade.

The results in 7^{th} grade were especially remarkable. These students were tested two times during school year 2007/2008 (four tests in October 2007 and four identical tests at the beginning of May 2008) and the results in May showed that there was no improvement in solving tasks that require ability to see multiplicative relationships between numbers, this fact is especially worrying as meanwhile the students had studied different concepts relating to proportion, and also solved in class numerous problems on direct and inverse proportions.

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RELATIVE ITEM DIFFICULTIES OF MKT MEASURES IN THE U.S., IRELAND, GHANA, AND KOREA

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Seán Delaney (Coláiste Mhuire, Marino Institute of Education)

With cross-national studies of students' mathematics achievement, there has been increasing interest in teachers' mathematical knowledge (An et al., 2004), as well as their instructional practice across countries (Santagata and Barbieri, 2005). Notwithstanding these efforts, little is known about how teachers' mathematical knowledge relates to the work of teaching in practice. To measure mathematical knowledge used in teaching, researchers at the University of Michigan developed multiple-choice items, based on the construct of mathematical knowledge for teaching (MKT). To study the equivalence of the U.S. measures in other contexts, the authors translated and adapted measures, considering mainly the general cultural context, the school cultural context, and issues related to mathematical substance in each country. Adapted MKT measures were administered to 100 Irish teachers, 60 Ghanaian teachers, and 77 Korean teachers. Using psychometric methods, relative item difficulties from a one-parameter Item Response Theory model were compared to data from the United States. Based on the work of validating MKT measures in each country (Delaney et al., 2009), this paper addresses relative item difficulties of number and operation multiple-choice items to examine how same items were easy or difficult for teachers in the U.S., Ireland, Ghana, and Korea.

As a result, in spite of quite strong correlation of item difficulties across countries, some items were easier for teachers in one country, but more difficult for teachers in another country. For instance, asking teachers to identify the correct representation of one-fourth, the item difficulties of all four representations (area model, equal partitioning, set model, and linear model) were similar for U.S. teachers; whereas the linear representation of fraction was more difficult for Irish teachers; equal partitioning of fraction was more difficult for Ghanaian teachers; and set model was more difficult for Korean teachers. Findings from this study indicate that the mathematical knowledge used in teaching differs across countries and what factors influence such differences.

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WHY STUDENTS DO NOT PERCEIVE COGNITIVE CONFLICTS EVOKED BY COMPLEX NUMBER

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In mathematics education, cognitive conflicts can be used as a strategy to develop students' awareness to their misconceptions (Sela & Zaslavsky, 2007). The concept of complex number which was historically emerged with confusion and doubt is notorious for students' misconceptions. It is the core of conflict teaching that the students' awareness reorganizes their knowledge and leads to meaningful learning. Therefore, in complex number context, we need to examine whether students really perceive the cognitive conflict and whether the conflict can support their learning.

We focus on students' response to conflict situation. Students can not easily perceive the cognitive conflicts during the process of problem solving. When the contradiction is explicit by help of researcher, they simply want to avoid and remove cognitive conflicts. From the perspective of students, cognitive conflicts are only aliens (Duffin & Simpson, 1993). Students do not see contradictions as inconsistency but as two independent events, and simply separates conflicting elements and discard trivially one of them. In this way, they can easily avoid conflicts Cognitive conflicts evoked by complex number do not bother students, because students had rich experience of removing and avoiding conflicts. Furthermore, In order to remove and avoid cognitive conflicts, they usually use the panacea of 'because of imaginary number' which can make students stable. Because imaginary number is strange and special object, even impossible, it is needless to reflect why this calculation is not allowed and why that concept can not applied to. This study finds that students naturally try to avoid conflicts and want stability. Especially, because complex number is easy to evoke cognitive conflicts, they actively try to avoid the conflicts to the extent that the conflicts simply are ignored without further thinking. In this vein, awareness of misconception is not a sufficient condition for better conceptualization and improved performance (Flavell, 1993).

Therefore this study suggests that simply presenting conflict situation is not a sufficient condition for learning to occur. In particular, in the context of complex number which is full of mathematical contradictions students naturally try to avoid the cognitive conflicts rather than experience them.

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THE VIDEO STUDY OF MATHEMATICS TEACHING IN FOURTH GRADE IN TAIWAN

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This research was based on the video study of TIMSS 1999 to investigate the fourth-grade mathematics teaching of Taiwan, through the video studies of lessons. The methodology was video study and questionnaire survey. We refer to TIMSS 1999 video study (Jacobs, et al, 2003) and TIMSS 2003 (Martin, Mullis, & Chrostowski, 2004) in the aspect of study designing, sampling method, and data analysing. The sampling had 60 mathematics lessons from the 4th-grade in the 60 sampled schools. The lesson codes were separated into seven coding passes, included coverage codes and occurrence codes. The reliability score for each code was from 85.7% to 100%.

We report some of the main results here. At questionnaire, all of teachers in Taiwan have completed teacher training, but only 5.6% of them majored in mathematics or mathematics education. In the professional development activities, during the last two years, 63.7% of the teachers have taken the course about using new technology. 48.0% of the teachers have taken the course about classroom management and organization. 45.6% of the teachers have taken the course about mathematics materials and methods. The teachers reported spending 4.4 lessons to teach mathematics in a week. 74.4% of the teachers reported that the textbook reportedly played a major role in teachers' choices about what to teach, and 60.0% of the teachers reported that the teaching guide is important. 100% of the teachers were enthusiastic at teaching mathematics.

At the video lessons, 84.1% of lesson time was spent public presentation by the teacher or student intended for all students. 59.6% of lesson time was spent solving problems, and 40.4% was spent presenting mathematical definitions or concepts. 47.4% of lesson time was spent introducing new content, 32.7% practicing new content, and 19.9% reviewing content introduced in previous lessons.

47.7% of problems were set up using mathematical language or symbols only, and 42.7% of problems were used graphs to problem posing or solve problem.

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A COMPETENCE STRUCTURE MODEL FOR SOLVING PROBLEMS BY USING MATHEMATICAL REPRESENTATIONS

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Students use verbal, numerical, graphical and symbolic representations of functions to solve mathematical problems concerning growth and change. We state a theoretical multidimensional competence model for the cognitive processes connected with translating between representations and evaluate it empirically.

In order to assess the described competence we built a theoretical competence structure model using previous research about students solving problems and switching between mathematical representations (e.g. Kaput, 1985; Goldin, 1998; Swan 1998; Gagatsin, 2004). The model was operationalized by some 80 tasks representing all main representation and translations types (except for the algebraic one): modelling and interpreting situations numerically (SN), graphically (SG), working (predominantly) within the graphical (GG) or the numerical representation (NN).

	SN	GG	NN	GS
SN	0.64	0.25	0.24	0.32
GG		0.62	0.23	0.34
NN			0.52	0.28
GS				0.72

Latent correlations and reliabilities of the

Students' competences were measured by *item response models* which have several diagnostic benefits. We inspected the dimensionality empirically and found that the postulated 4-dimensional model (SN,SG,GG,NN) fits the data significantly better than concurring 2-dimensional (G vs. N) or unidimensional scales (ΔAIC_c= -42*).

We used digressions from model fit for further investigation of the structure of the competence: Analysis of violations of specific objectivity via *Differential Item Functioning* revealed influences of gender and spatial reasoning on the students' model fit. Exploring competence clusters via *Latent Class Analysis* yielded groups of students with specific profiles, e.g. with a strength in interpreting situations numerically

Our next goal is to develop a diagnostic instrument based on these findings that can be used by teachers for formative assessment and specific support for learners.

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COGNITIVE APPREHENSIONS IN CABRI 3D ENVIRONMENT

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It is well known that many students have difficulties in visualizing 3D figures from their 2D drawings, and in perceiving spatial relationships such as perpendicularity in the 3D space. Cabri 3D is a dynamic geometry environment (DGE) which, similar to 2D DGE, empowers students with rich opportunities to construct and to manipulate geometrical objects, to experiment with and explore mathematical ideas in a simulated 3D space. In the past two decades 2D DGE has major and continuing impact on the teaching and learning of plane geometry, and so should Cabri 3D in 3D geometry in the coming years.

In Cabri 3D, in addition to the usual *object dragging* in DGE that allows the user to drag geometric objects with the relationships defining them preserved, there is a distinctive drag mode, which we call as *perspective dragging*, that allows the user to change the viewing perspective of the 3D objects in the screen by dragging them with the right mouse button pressed and held. This important drag mode induces a dynamic appended dimension to the flat shapes on the screen, with which the users could get a sort of spatial depth of the 2D drawings of 3D figures. These two drag modes are very useful in providing engaging mathematical environments for students to strengthen their visualization of 3D figures, and to facilitate their reasoning in 3D space.

This oral presentation explores the pedagogical values of the distinctive features such as perspective dragging in Cabri 3D. We are going to present some tasks in the Cabri 3D environment the second author designed and discuss how these tasks can foster the four cognitive apprehensions in Duval's framework for analysing the visualization of geometric drawings, namely, *Perceptual Apprehension*, *Sequential Apprehension*, *Discursive Apprehension* and *Operative Apprehension* (Duval, 1995). Furthermore, a pilot study on how students visualize the concept of rotational symmetries of 3D figures in Cabri 3D will be reported. In particular, we will present how a student identified and reasoned about geometric properties through perspective dragging, which suggests some worthwhile research directions in studying the pedagogical potential of Cabri 3D.

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THE INFLUENCE OF LANGUAGE ON THE CONCEPTION OF GEOMETRIC FIGURES

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INTRODUCTION

This study deals with the possible problems which may arise when students learn the names of elementary geometric figures in the languages of English, Chinese and Korean. Saussure's idea of the signifier and the signified (1966), and Vinner's differentiation between concept image and concept definition (1991) provide the framework for data analysis in this study. A test comprising 5 items requiring grade 8 students to identify and/or define some common geometric figures was administered to 123 Chinese students from a school in Hong Kong, 104 Korean students from a school in Seoul, and 91 English speaking students from a public school in San Diego, USA. Based on the test results, 10 students from each language group were selected for in-depth interview to investigate their understanding on the geometric terms. In this paper, results of 2 out of the 5 items are reported.

RESULTS AND DISCUSSION

The Chinese term for trapezium, *ti-xing*, and the corresponding Korean term, *sadari-ggol*, mean "ladder shape", whereas the English word "trapezium" comes from the Greek word *trapeza* which means "small table". While all three terms refer to a real-life object, most English speakers nowadays who do not know the original meaning of the term trapezium. Similarly, the English word "sector" comes from a Latin root which means "cut out", while the Chinese term *shan-xing* and Korean term *buchae-ggol* mean "fan shape".

Geometric figures with names borrowed from everyday life objects are meant to help students understand the mathematics concepts through relating the figures to familiar objects, but they may also cause students to fix their attention on some attributes of the objects which may not be consistent with the definition of the figures.

The two items in the test designed for this study intend to investigate whether or not the use of word in different languages to denote the trapezium and the sector really affects students' identification of the figures and their conception of the properties of these two figures. Results of the test, together with students' responses during the interview, will be discussed in the presentation, using Vinner's theory of interaction between concept image and concept definition (Vinner, 1991; Vinner & Dreyfus, 1989) as the framework to analyse the results.

There are of course other factors which affect students' identification of geometric figures, but based on the results of this study, it is argued that the words used in naming geometric figures in different languages may have an impact on students' understanding as well. Finally, implications of the findings of the study are discussed.

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MATHEMATICALLY-BASED AND PRACTICALLY-BASED EXPLANATIONS: WHICH DO STUDENTS PREFER?

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This study investigates fifth-grade students' preferences for mathematically-based (MB) and practically-based (PB) explanations within two contexts: even numbers and equivalent fractions. MB explanations are based solely on mathematical definitions or previously learned mathematical properties, and often use mathematical reasoning. PB explanations use every-day contexts and/or manipulatives to "give meaning" to mathematical expressions. In a previous study we showed that elementary school students are capable of using MB explanations when solving multiplication tasks (Levenson, Tirosh, & Tsamir, 2004). Yet, in certain contexts, some teachers believe that the use of PB explanations is preferable in the elementary school (Levenson, Tsamir, & Tirosh, 2007). Is students' use of MB explanations limited to multiplication tasks? Our current study investigates students' preferences within additional contexts as well as the basis for these preferences.

Four fifth-grade classes, consisting of a total of 105 students, participated in this study. Questionnaires, one for each context, were handed out in order to investigate the types of explanations students generate on their own as well as the types of explanations students prefer when presented with both MB and PB explanations.

Results showed that students overwhelmingly generated more MB explanations than PB explanations for both mathematical contexts. On the even numbers questionnaire, significantly more students were most convinced by PB explanations, and preferred to use a PB explanation when explaining this concept to a friend. Regarding the types of explanations students preferred their teacher to use, there was no significant preference for either MB or PB explanations. On the fractions questionnaire, no significant preference was found for either MB or PB explanations. Educators often stress the use of PB explanations in the elementary school. This study showed that by the fifth grade there is not always a definitive preference among students for PB explanations. When reviewing the basis for students' preferences, although some students were still attracted by the "fun" quality associated with PB explanations, others showed an appreciation for the mathematics involved in the MB explanations.

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RAPID CHANGE IN PRACTICE AS A WINDOW INTO THE PROFESSIONAL GROWTH OF TEACHERS

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In my work with teachers I have observed a phenomenon of profound and rapid changes in practice. A deeper look at this phenomenon has revealed that there are nuances to it that go beyond the nature of the change itself. Consideration of the results necessitates a change in perspective when thinking about, and working in, the professional development setting.

INTRODUCTION

Using a methodology of noticing (Mason, 2006) within a diverse number of professional development settings over the last six years – from workshops to learning teams to graduate programs – I have, from time to time, noticed teachers undergoing *rapid* and *profound* changes in their beliefs and practices. This phenomenon is rare. Most teachers engaged in inservice work follow a trajectory of change that is much more pedestrian. At first such changes surprised me but as more and more of these accounts accumulated I decided to investigate further. Eventually I accumulated data on 42 instances of rapid and profound professional growth.

RESULTS & DISCUSSION

Analysis of these 42 cases showed that the professional growth of a teacher was a long term process wherein the inservice 'intervention' is but a small piece. This fits with the newly emerging paradigm where the professional growth of teachers is seen as natural (Leikin, 2006; Perrin-Glorian, DeBlois, & Robert, 2008) and teachers are seen as agents in their own professional learning (Ball, 2002).

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THE FORMATIVE ASSESSMENT DESIGN ON MATHEMATICS PROJECT-BASED COLLABORATIVE LEARNING FOR THE 5TH GRADERS

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The purpose of this study is to develop a formative assessment design for web project-based learning (WPBL) in elementary schools. Students of WPBL group were supported to monitor their own understanding and progress by the feedbacks from formative assessment system. Students' web-discussing dialogues and e-learning diaries were ratted regularly on line. The rating rubrics functioned as the supportive scaffoldings. Suggestive feedbacks were also provided.

This paper described a three-year study undertaken in two Taiwan elementary schools which supported students' learning progresses in Web-based learning environment. The formative assessment was used to determine a student's readiness for an inquiry task, to monitor performance on relevant aspects of engagement during the task (e.g., communication), and to benchmark and establish students' mathematical performance. The study incorporated clear standards to guide beginning inquiry. On-line formative assessment feedback embedded guidance for or models of how to conduct an operation within project materials. The formative assessment also provides norms for individual accountability in the group and progress for each group. There were 62 fifth graders at WPBL group (24 students of gifted class and 38 students of normal class). Another 62 students were closely matched as the control group. The students at WPBL group were learning collaboratively for 15 months. The correlation coefficients between web-discussion and reflective diary were around 0.60. Holding pre-test as covariate, students of WPBL group outperformed students of control group on criterion variables. The variances accounted by group are around 12% and 8% on spatial sense and mathematics ability respectively. These results show that collective knowledge and creativity is attainable even for the elementary school students.

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A MACRO-ANALYSIS OF STUDENTS INTUITIVE PERCEPTIONS OF THE INFINITE

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Concept of limit is the most difficult topic for students in learning calculus （Milani & Baldino, 2002; Tirosh & Tsamir, 1996）and the infinite is a significant element for understanding the limiting process and nature of calculus. However, due to its process-object duality and our finite logical scheme, relative researches have yielded inconsistent results. The present study observed Taiwanese college students' perceptions of the infinite in terms of numerical, arithmetic, algebraic, geometrical and verbal aspects.

Fifty-eight Taiwanese college engineering-major freshmen participated in this study. Six questionnaires were designed and administered to them to investigate their intuitive conceptions of the infinite. The Infinite Set questionnaire and Infinite Sequence questionnaire observed participants' conceptions regarding numerical aspects of the infinite. The Infinite Series questionnaire investigated their arithmetic understanding of the infinite. The Indefinite Limit questionnaire surveyed students' algebraic understanding of the infinite. Participants' infinite reasoning was probed by the Paradox questionnaire and the Fractal questionnaire explored how they reacted to infinite processes of self-similarity of geometrical shapes (e.g., Koch curve).

Results suggest students' use of comparison strategies were varied, subject to the representation of the task. One-to-one comparison test was used if the target is discrete infinite quantities and elements are listed sequentially by number. Part-whole relationship test was applied to continuous infinite quantities with inclusive relation. By contrasting students' arithmetic and geometrical perception of the infinite, the present study found that they were likely to do brief calculation on arithmetic infinite items whereas employed intuition to make judgment on fractals. Such a discrepancy reveals their tendency of relying on the representation of problems and ignoring potential connection between them. Students' reasoning about the infinite on Zeno's paradoxes was quite feeble. To a great extent, they kept away from key issue and resolved the contradictory argument by referring to physical laws or on the basis of realistic context.

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CONTINUOUS TRAINING IN MATHEMATICS – INFLUENCE ON THE PROFESSIONAL KNOWLEDGE OF ELEMENTARY SCHOOL TEACHERS

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This report is based on an ongoing investigation, aiming to study the contribution of participating in a Continuous Training Program in Mathematics for the professional development of elementary school teachers, specifically regarding professional knowledge. This program involves group training, supervision sessions, as well as portfolio building.

Professional knowledge covers several areas such as knowledge of the educational context, of subject-matter, of class organization and management, pedagogical knowledge and curricular knowledge (Hiebert, Gallimore, & Stigler, 2002; Ponte, 1999). Professional knowledge must be seen beyond technical knowledge. A mathematics teacher must have sound knowledge of and about mathematics, and also be able to appropriately represent mathematical ideas, so as to make them comprehensible knowledge for students (Ball, Lubienski & Menborn, 2001).

In this study we opted for a methodological approach of the interpretative kind, performing three case studies, resorting to interviews, participant observation of sessions and documental analysis for data collection.

Sara, one of the study participants, considers that the training program was mostly useful to deepen and update her mathematical knowledge. It is obvious how she associates mathematical knowledge, didactic knowledge, and curricular knowledge, particularly when she reflects upon the use of manipulative materials regarding the study of specific concepts in her teaching practice.

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EXPLORING STUDENTS' UNDERSTANDING OF ANGLE MEASURE

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Angle has been defined in many different ways and examples of them also appear to describe various contexts; (a) a corner which is sharp or, (b) two rays and a vertex as static concept of angle, and (c) a turn of one ray around a fixed point or the space between two rays as dynamic concept of angle. Previous studies revealed that students have difficulty in distinguishing between the definition of angle and its standard units of measure (i.e. degree and radian). For example, Mitchelmore (2000) reported that even 8th grade students need more help than is presently given to identify angles where one or both rays of the angle are not visible. Thompson (2008) suggested the need of developing angle measure in degrees and radians correlatively.

The purpose of this study is to explore the difficulties and its factors in the process of learning angle measure experienced by students from elementary school to secondary school with a focus on the difference between dynamic and static aspects of angle concept. For this purpose, 5th, 7th, 9th, and 11th grade students (N= 1,271) were given the set of assessment tasks which were developed under three broader categories: (a) What an angle is and what we are measuring when we measure it, (b) What standard units of angle measure actually measure when students understand an angle as a turn of a ray, and (c) How to use a protractor when we measure or draw angles.

The results showed four major difficulties in understanding dynamic and static aspects of angle concept as follows; (i) It is difficult for 5th graders to pay attention to an angle as one of the attributes of the shape with distinguishing from measuring a side of a shape, (ii) About half of the students are not able to extend the range of angle more 180 degrees or exceeding 0 to 360 degrees though they had learned angles were made by turning of a ray, (iii) Most of the 11th graders after they had learned radian measure confuse expressions of the size of angle by radian with degrees, and (iv) More than 30% of the 5th graders have difficulties in using a protractor upside down and reading scales which start from the right. Implications for teaching are discussed in terms of developing students' concept of an angle not only as static aspects in relation to fundamental geometrical figures but as dynamic aspects which is a quantity of turning of a ray from elementary school through secondary school.

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MATHEMATICS AS AN ARTEFACT FROM A CRITICAL MATHEMATICS EDUCATION PERSPECTIVE

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The notion of artefact gains special relevance when we seek to understand learning as participation in social practices as an emergent phenomenon. According to the Soviet cultural-historical school mediating artefact is a central concept. “The idea is that humans can control their own behaviour – not ‘from the inside,’ on the basis of biological urges, but ‘from the outside,’ using and creating artefacts.” (Engeström, 1999, p. 29). Artefacts reveal the reproductive as well as the productive character of human activities, contributing to the constitution of practices over time.

We see mathematics as an artefact which mediates people activities in the social world opening opportunities and putting constraints. Therefore while discussing mathematics learning we have to consider the ways the world is permeated by mathematics and how mathematical models impose on people specific forms of action and behaviour.

Considering those issues, we come into key questions raised by Critical Mathematics Education (CME). The main concern of CME is the development of *mathemacy* (Skovsmose, 1994), whose objective is not to merely develop the ability to carry out mathematical calculations, but also to promote the critical participation of students/citizens in society, analyzing and discussing political, economic, and environmental issues where mathematics plays a major role as a technological support. In this case, critique is directed at mathematics itself, as well as its use in society. Here the relationships between artefacts and power come into play as artefacts constitute means through which social power is showed and exerted.

In this presentation we offer a discussion of key issues in this topic as we travel through the following questions: i) how is mathematics understood as an artefact in the social world? ii) what are the implications of considering mathematics as an artefact from a critical mathematical education perspective? iii) how can mathematics education provide opportunities to interrogate forms of using mathematics in society?

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TEACHING ABOUT TEACHING MATHEMATICS: INCREASING PRACTICAL KNOWLEDGE

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Teaching about teaching mathematics effectively can be a challenging task. It involves assisting the pre-service mathematics teachers towards developing their pedagogical content knowledge (PCK) in mathematics. This means that the prospective teachers need to become aware on their future students' misconceptions on a variety of mathematics topics. They also need to build up a repertoire of teaching strategies that will help their students reorganise their thinking.

This study focuses on the topic of 'ratio and proportion' and examines a practical approach in eliciting and developing PCK on the topic. A diagnostic test designed with the intention to reveal the pupils' thinking in the area of 'ratio and proportion' was developed further as an instrument that investigates the teachers' knowledge about their learners (Misailidou, 2008).

The study involved two stages. During the first stage, trainees that were nearly at the end of their training were interviewed with the aim to reveal their knowledge on the teaching of 'ratio'. The resulting data demonstrate that the trainees were aware of the curriculum requirements and descriptions concerning the topic. Nevertheless, they do not demonstrate knowledge on specific misconceptions and teaching strategies. During the second stage, the same trainees were asked to complete the instrument mentioned above. The results reveal the instrument's twofold function as a tool that brought to the surface a. tacit awareness of aspects of the topic b. gaps in the trainee's PCK and a consequent fruitful reflection on these.

The PCK that is required for the successful teaching of mathematics is not only content-specific, but vary often is local to the task at hand (Misailidou & Williams, 2003). It is therefore argued here that teacher training courses would benefit by incorporating instruments which are designed with the aim to reveal and then enhance this 'local' PCK.

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GOOD MATHEMATICS INSTRUCTION: HEARING TEACHERS' VOICES

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Mathematics educators have been sought for improving the way mathematics is taught. Teachers play an important role in enabling students to learn mathematics meaningfully. There has recently been much attention to how Asian teachers teach mathematics (Leung, Graf, Lopez-Real, 2006). But many of them focus mainly on the theoretical standpoint of good mathematics teaching. There has been little research into teachers' perceptions on good mathematics teaching. Therefore, we investigated the views of effective mathematics instruction on the part of teachers. We carried out a survey with 223 elementary school teachers in Korea. The questionnaire consisted of 4 main categories with a total of 48 factors: the curriculum and teaching content, teaching and learning, classroom environment and atmosphere, and assessment.

The teachers thought that, out of the four categories, the curriculum and teaching category was the most important. In particular, they felt that restructuring the curriculum and teaching content to meet the needs of students was especially important. With regard to the 48 factors, teachers felt the followings were essential components of good teaching practice: (a) allowing students to play a leading role in their learning, (b) having students and teacher interact using mathematics within the classroom, (c) ensuring students have a good understanding of key mathematical concepts, and (d) improving students' problem-solving ability.

However, teachers thought the followings were not essential for good teaching practice: (a) the use of ICT, (b) improving students' calculation ability, (c) controlling students with special needs, (d) emphasizing human relationships, (e) managing students by classroom norms, (f) creating a good learning environment, and (g) teaching the curriculum as it were.

Some ideas teachers revealed about what would enable good mathematics teaching coincided with previous research (e.g., consideration of students' individual differences or focus on concepts). However, there were differences with regard to the use of technology and the importance of learning environment, which have been emphasized in mathematics education literature. Considering that the teacher plays a key role in implementing good instruction, this paper emphasizes us to attend to teachers' perspectives in order to initiate good teaching at the actual classroom.

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MOTIVATING TEACHERS' DEVELOPMENT THROUGH DISCUSION GROUPS

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Teachers´ training should be one of the most important concerns in educational policies. In here, we describe some of the results of a research and educational project. The purpose of the study was to observe the changes produced in their classrooms when teachers gather in weekly sessions, guided by a math educator, to discuss about content and pedagogical knowledge related to their practice. These discussion groups were implemented at different grade levels and were organized as a series of tasks, problem solving and reading materials (articles like Schwarz et al., 2006 and McDonough and Clarke, 2003). We performed initial and final classroom observations, which were analyzed according to the framework of Askew et al. (2000) on productive learning in the classroom: I. *Tasks* are challenging, meaningful and interesting; II. *Talk* facilitates learning and includes all sorts of teacher and students interactions; III. *Tools* cover a range of modes and types of models; and IV. *Relationships and norms* help towards a social construction of knowledge. For example, in one group, five teachers were followed. To condense the results of the analysis here, we assigned to each teacher, in each of these components, a level, from A ("Very good") to D ("Poor"). The table below shows the changes between the initial and final observations (if only one level is given, no change was perceived).

	T1	T2	T3	T4	T5
Tasks	D to B	A	C	D to B	C to A
Talk	D to B	B	C	D to B	B
Tools	D to B	B	C	D to C	C to B
Norms	C to A	A	C	C to B	B to A

As can be observed, three of the five teachers had substantial improvements, but the other two, not at all. Of these, the second teacher (T2) was a very capable teacher from the beginning, and the third teacher (T3) was a very traditional, rigid, directive teacher, who recommended in one questionnaire "to practice the operations daily".

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ASSESSING TEACHERS UNDERSTANDING OF THE ARITHMETIC MEAN

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This communication describes a subset of results in the field of statistics education, from a larger study that explores teacher's knowledge for teaching mathematics. It focuses specifically on the conceptual understanding of the arithmetic mean and it was developed with two clusters of teachers: 19 elementary teachers (6-11 years old students) and 18 middle school teachers (12-14 years old students) both enrolled in teacher development programmes. Results of this study show several teachers´ misconceptions and indicate that they are not prepared to teach arithmetic mean with a deep understanding for conceptual learning.

Until recently statistics education have had few importance in Portuguese curricula. They have now some improvements and statistics become more evident, what implies consequences on teacher education and in-service programmes. The statistical concept of mean is very powerful in statistics education since it summarizes information about a data set Konold (2002). Usually the emphasis is in the computation in simple situations and there is a superficial understanding of the mean by students. Students can develop the understanding of the mean through several situations for instance, fair sharing, repeated measures or finding a typical value of data set. But how well teachers understand the concept of arithmetic mean? What is the teachers´ perception of self-competence? What kind of tasks they provide to teach this topic ? Shaughnessy (1992) points out that teachers are not adequately prepared to develop concepts in students if they have their own misconceptions. So, it is important to know how teachers understand topics such as the arithmetic mean, that is vital to a thorough understanding of other statistics topics emphasized by students curricula, for instance the standard deviation. There is an agreement that how well teachers know subject matter is central to their capacity to use instructional materials wisely and to assess students´ progress. Institutions in charge of teacher education and in service teacher training should develop convenient programmes in order to develop teachers' awareness of their mathematical needs for teaching.

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ATTENTION! THERE IS A BIG GAP BETWEEN WHAT WE EXPECT AND WHAT STUDENTS ATTAIN!

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The first time that the Third International Mathematics and Science Study (TIMSS) was conducted, Iran participated in the first two populations and the result was a big shock! The first reaction to this news was "it couldn't be true!" Then, many stakeholders tried to dismiss the whole thing and cry that "there must be something wrong with these findings". However, after a while, the anger decreased and many decision makers convinced themselves that with a little more endeavour, Iranian students could perform much better in TIMSS-R. Thus, Iran continued to participate in TIMSS- Repeats up to 2007 with little or no improvement in results.

To deal with this issue in a more systematic way, we did the content analysis of the elementary mathematics textbook (at the national level). The first phase of the study showed the reasonable consistency between what was expected from elementary students in Iran and what was aimed at TIMSS. We therefore, took the matter one step further and looked at what was going on in real classrooms. Our observations shed more light to our content analysis. For instance, we realized that mathematics textbooks include "estimation"-the area that Iranian students' performances was below average and more close to the end- but just touching the concept. In fact, the textbooks have not provided appropriate opportunities for both students and teachers to tackle with the concepts and learn it in more depth. This was clearly apparent in teachers' activities in classes as well. The finding of this study make us confident to say that Iranian students did so poorly on TIMSS for two main reasons; first the inadequacy of textbooks in terms of conceptual understanding and providing real life contexts, and next, the expectations of schools form teachers to teach to the "tests". Therefore, we have to pay more attention to this fact that there is a big gap between what we expect and what students attain and this has a great effect on students' performances in various situations such as TIMSS.

Key words: TIMSS, elementary students, content analysis, mathematics textbooks, students' performances.

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A COMPONENT FOR PROMOTING THE PROGRESS OF MATHEMATICAL UNDERSTANDING

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What is power to promote or control a student's mathematical understanding? The purpose of this study is to reconsider components of a mathematical understanding model in order to indicate the power for promoting a student's mathematical understanding.

RESEARCH QUESTION OF THIS STUDY

We focus on meta-analysing the description of mathematical understanding and name that view the *value* of mathematical understanding. The aim of this study is to answer the question, "How is mathematical understanding promoted?" or "What is power to promote or control a progress of mathematical understanding? "

A FRAMEWORK OF THE SUBORDINATE COMPONENT

We argue that values are a new component of mathematical thinking needed to construct a process model of mathematical understanding. This is a 'sub-component', when we consider the levels and stages of mathematical thinking (Koyama, 1996) as the 'super-components'. The sub-component has three factors; pre-reasoning, reasoning, and mathematical proving or explaining. We show the necessity of this third component of a process model of mathematical understanding which is to represent values to promote or control a student's mathematical understanding. Then we discuss the usefulness of our framework with one of scenes from a geometry lesson.

Creating a Proposition	Justifying the Proposition	
A) Pre-reasoning	B) Reasoning	C) Mathematical Proving/ Explaining
- Having observations	- Justifying observations	- Generalizing observations mathematically
- Creating connections among some observations	- Justifying connections among observations	- Having doubts about the way of reasoning

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COGNITIVE TENDENCIES BEHIND STUDENT RESPONSES ON A CONSTRUCTED RESPONSE TIMSS ALGEBRA ITEM

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Resent research within the learning of algebra has emphasised the various sources of meaning from which students develop understanding for algebraic objects. Fundamental for algebraic learning is the source of meaning that stems from the algebraic structure itself, involving the letter-symbolic form. *"This is a structural source of meaning which also provides connections among the symbolic forms of algebra, its equivalences, and its property-based manipulation activity"* (Kieran, 2007, p. 711).

My PhD-project explores how Norwegian students in the 8^{th} and the 10^{th} grades develop understanding of algebraic objects and how different aspects of algebraic competency might be linked. Describing their equation-solving competencies is a central element. The part of the project reported here will briefly explore possible understanding and thought processes behind student responses on a TIMSS algebra item [if 4(x+5)=80, then x=]. By linking solution-strategies used and errors made, this study will illustrate one possible way of increasing the diagnostic information available in the TIMSS scoring guide, emphasising the duality of operational and structural understanding of algebraic expressions and equations (see Sfard, 1991), and the importance of seeing the abstract ideas behind the symbols in order to develop meaning for these objects.

My PhD-project applies a mixed methods design, containing analyses of written responses from 800 students, supplemented by one-to-one interviews. The test consists, among other items, of 6 linear equations with different foci.

The purpose of the presentation will be to present results from this research project focusing on the relationship between understanding and solution strategies, error types, and explanations.

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LETTER AS GENERALISED NUMBER: THE VALUE OF A HISTORICAL PERSPECTIVE

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The importance of the concept of the literal symbol, and in particular the idea of generalised number, is well documented. However, many studies have stressed student difficulties and errors in this area. In light of such difficulties, researchers have, in recent years, turned to history to inform practice. Educators have asserted (e.g., Fauvel & van Maanen, 2000) that the history of mathematics is an excellent resource for teaching. A study of historical texts (Datta & Singh, 2001) reveals that the decimal number system with place value and zero as well as many of the algebraic ideas originated in India. An interesting idea developed in ancient India and which we used in this study was that different colours were used to represent various unknowns, such as *yavat-tavat* (so much as), *kalaka* (black), *pitaka* (yellow),.... In addition we used the idea of Mason Graham and Johnston-Wilder (2005), that classification is a form of generalisation, where they suggest guiding students' attention to number and geometric patterns to detect generalities. This same method was also described by Srinivasan (1989) who proposed the use of 'pattern language' to concentrate students' thinking on variation and invariants in patterns and recommended the associated vocabulary. In this study, we addressed the question of whether a combination of historical (use of colours in place of changing numbers as in Indian history) and mathematics education research ideas (directing students' attention to classification) would help improve students' understanding of the concept of generalised number. The results show that this method of noting variation and invariants was accessible to most students and they were often able to classify them, although with varying degrees of success. It appears that historical ideas may reveal fresh approaches to student understanding of algebra and its notation.

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CONFLICTS WITH THE RADICAL SIGN. A CASE STUDY WITH PATRICIA

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This work takes its cue from the Patricia's case. Patricia, a Spanish secondary school mathematics teacher, on attempting to understand the definition of equivalent radicals expresses a conflict. She states that the equality $\sqrt[6]{3^2} = \sqrt[3]{3}$ cannot be true, since in the expression on the left the index of the root is even, so that it has two opposing roots, two solutions, whereas in the expression on the right the index is odd so it only has one root, which means that the two expressions have a different number of roots. In order to explain Patricia's conflict we took into account a framework with two fundamental references. One of these is the change in meaning of the signs when passing from arithmetic to algebra, reflected in a traditional way of teaching. Traces of this tradition are found in texts such relevant as Euler's algebra. Euler associated, for example, the expression $\sqrt{4}$ to the set of two results, $+2$ and -2, and the expression $\sqrt{a}$ to one of the two roots of a. (Euler, 1770, p. 44, p. 150). Another reference is the dual (operational/structural) nature of mathematical conceptions (Sfard, 1991). In arithmetic the expressions as $\sqrt{4}$, for example, are perceived as an indicated operation (operational conception), while in algebra the expressions as $\sqrt{a}$ are perceived as objects (structural conception). According to Sfard, the operational conception is the first to develop, whereas the structural conception needs external interventions (of a teacher or a textbook). In our study we have identified characteristic features in the $\sqrt{\ }$ sign teaching in current and representative Spanish textbooks and also, omissions that reinforce the double value (first conception) of the $\sqrt{\ }$ sign (Roach, Gibson & Weber, 2004). The data that we have obtained from the answers to a specific questionnaire and the interview with Patricia show that she has an operational knowledge of the $\sqrt{\ }$ sign and that this conception is consistent with what is shown in the studied textbooks.

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9TH GRADE MATHEMATICS TEACHERS' PERCEPTIONS ABOUT GRAPHING CALCULATOR AND GEOMETER'S SKETCHPAD USAGE IN THEIR LESSONS

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With the advent of technological developments, materials used in mathematics education are changing. New developments in technology affected both the school practices and researches in mathematics. On the other hand, the way to enhance the lessons with activities and the nature of the activities in classrooms were also altered by technology (Kaput & Thompson, 1994). Moreover, for the sake of educational goals, benefits of usage of technological tools such as graphing calculators, Geometer's Sketchpad (GSP) to students' understandings of mathematics are widely mentioned in literature (Bazzini, Bertazzoli & Morselli, 2005). In this study, the findings in an on-going study of technology use (Specifically graphing calculators and GSP) in a private high school will be discussed.

The research was carried out with mathematics teachers in the private high school named Ozel Bilkent Erzurum Laboratory High School in Erzurum Province of Turkey. The school is following an international mathematics curriculum. Since students in 9th grade are taught coordinate system, linear equations and simultaneous equations in their first semester curriculum, and since they are widely using graphing calculators and GSP in their lessons, data gathered from semi-structured interviews done with mathematics teachers of 9th grade students and students' first overall scores in mathematics lesson and then analyzed. Students' overall scores were above the average (76,6 out of 100 which is considered as "good" in our education). The findings will be presented broadly, some of which obtained from the interviews are;

- Students were aware of geometric meaning of linear or non-linear equations.
- They might explain linear and non-linear equations graphically and algebraically.
- Their problem solving and operational skills were developed when graphing calculators and GSP are complementary parts of mathematics lessons.

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GRAPHICACY AS A DIDAKTIK DESIGN TOOL

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This paper explores the concept graphicacy and puts a case for 'extended graphicacy'. Graphical representations are robust since they aid to conjure up reality and thus promote knowing relative to the different school subjects. Based on the response rate in OECD PISA graphicacy test items different graphicacy levels and students performance are identified. It is then claimed that part of pupils' performance can be explained by their experiencing difficulties orientating within different semiotic registers[1].

GRAPHICACY

The concept graphicacy can be said to have developed from the idea of graphical representation where virtual perception is the focus and the purpose is considered as the storage, understanding and communicating fundamental information (Bertin, 1967/1983). The concept graphicacy was coined as an indicator of ones ability to understand and communicate visual information. From a research design research perspective the understanding of fundamental information (or the occurrence of fundamental situation) as outlined in Bertin's definition is clearly the subject of interest.

Pisa test items

PISA uses graphicacy items as stimuli since they enrich the depth of the questions allowing for intensity (OECD 2006). The perspective of PISA on graphicacy is that it can be used to assess understanding. This implies that harnessing aspects of graphicacy can be of interest in teaching mathematics. Since context is embedded in graphicacy, it is claimed this contributes to an increase in the interaction with mathematical representations.

Conclusion

Due to limitations inherent to PISA test items caution is taken in explicitly claiming the results as a clear indication of students understanding of the concepts being tested. Preliminary results show that there is some variation in response rate depending on the mathematical tools required in use with the graphs.

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SUPPORTING MATHEMATICS PROFESSIONAL DEVELOPMENT IN A COMPLEX SCHOOL SETTING

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Professional development (PD) designed to support teachers' learning sometimes results in practices that are temporarily adopted and then set aside. Thus, involving teachers in the planning process is conceptualized as a way to personalize the professional development while promoting change. When teachers assume an active role, groups of professionals interact in new and complex ways. This paper uses complexity science to deconstruct the emergence and development of learning communities (LC) within a PD/research project that investigated English language learners' acquisition of mathematical ideas and language. Davis and Simmt (2003) identified five conditions that support the emergence of a viable learning community. These conditions are: (a) internal diversity, (b) shared goal, (c) decentralized control, (d) organized randomness, and (e) neighbour interactions.

The elementary school operated as a system with a principal who assumed responsibility for the organization. The PD/research project emerged from collaboration between the authors with Principal 1. During the conceptualization process, random ideas were valued and promoted autonomy. Internal diversity surfaced as these diverse ideas blended and merged to create the project. Then, Principal 1 accepted a new position within the school district and Principal 2 was hired.

Principal 2 formed a PD planning team which included the authors and four support staff with expertise in mathematics and English language learning. Principal 2 centralized control and limited the team's autonomy by adhering to the school district's mandates. The PD Planning team had a shared goal, but it was superficial. Members of the PD planning team had personal beliefs and agendas for the PD which sometimes conflicted with each other which led to the expression of differential power. Unlike the other school district members, Braun and Colansti had four roles in the project. They were teachers, researchers, liaisons between the school and Olson/Salsbury, and PDs who led the PD with Olson/Salsbury. They felt the tensions within the PD Planning team and their time commitment increased. The increased time impacted their work as a teacher, drew them away from their classroom, and led to frustration. We suggest that time is an essential condition for the emergence of a viable learning community that is sometime assumed or overlooked.

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MATHEMATICS: I'M OVER IT.

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A significant body of research suggests that negative attitudes are a major factor limiting the development of mathematical skills, knowledge and confidence: many students and adults do not like mathematics, they find it irrelevant and boring, and are often anxious about working with it. This paper reports on a small-scale qualitative study that investigated the perceptions of primary-aged children about their experiences of mathematics education. It describes children's changing attitudes to mathematics, and provides some insights into the connections between classroom experiences, emotions and attitudes. In particular, it seeks to:

1) Identify when negative attitudes begin to develop;
2) Describe the relationship, if any, between teachers' approaches to mathematics and their use of strategies to improve student attitudes.

The results provide exemplars of change from positive to negative perceptions across the primary years, and indicate connections between teaching approaches and attitude. Abstract, procedural teaching, with little or no variation in style overtime, was seen by the children as a main reason for their dislike of the subject.

While this study does not prove a cause and effect relationship between the teachers' approaches and students' attitudes, the results strongly support the likelihood that teachers' mathematics teaching approach influenced students' attitudes. The findings are closely aligned with Ellerton and Clements' (1994) research-based model, 'Development of Beliefs and Attitudes', that argued when students experience positive emotions and feelings to new mathematics stimuli, they go on to develop a positive attitude towards the subject. The implications for practice are potentially significant: Teachers, school leaders, and teacher educators need to be aware of the significance of attitudes and educated about effective strategies, and supported in learning how to better plan, implement and assess for the development of positive attitudes, throughout the primary years.

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THE CONFLICT EMERGE BEFORE THE GRADUATION

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The teaching profession requires professional competence which includes *knowledge of teaching* and *beliefs* (Bromme & Tillema, 1995; Llinares, 2002). The focus of this presentation is a conflict between intentions and possibilities expressed by prospective elementary teachers. To become and to be a mathematics teacher is a part of the identification-process of the individual. This process starts in teacher education and continues after graduation when students start working as mathematics teachers. The results in the presentation derive from a research project that aims to investigate the reflective relationships between *beliefs* and *knowledge of teaching* and the process of identification.

Ten prospective teachers were interviewed just before their graduation. The focus in the interviews was on *beliefs* and *knowledge of teaching.* In the interviews the prospective teachers expressed a range of contextual factors for mathematics teaching, for example tradition, class-style, other teachers and teaching material. They talked about these contextual factors as limits and named them frames for their coming teaching. They discussed how they will be able to, or not will be able to, go beyond these frames. Already before graduation they experience a conflict between their intentions as teachers and their prospective possibilities as teachers. They expressed a concern that the frames will prevent and control their desired teaching because this desired teaching is teaching that, according to the students, in different ways goes beyond these contextual factors. Going beyond these contextual factors is not something they imagine as self-evident or easy to do. They also express that they do not have any mathematics teachers as role models and they talk about teachers as “they”, consequently a group that they do not belong to themselves. The continued study will investigate how this perceived conflict and alienation are expressed when the students start working as teachers. The research will also investigate how the perceived conflict influences their identification with the profession, that is the narratives they tell about themselves as professional teachers (Sfard & Prusak, 2005).

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DEVELOPMENT AND APPLICATION OF CASE-BASED PEDAGOGY FOR PROSPECTIVE TEACHERS TO LEARN HOW TO TEACH MATHEMATICS

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Cases have been extensively used in professional fields such as law and medicine. However, it is recent to employ cases as tools in mathematics teacher education. Many issues have been raised with regard to "what is a case?", "how can we facilitate cases?", "what do teachers learn through cases?" and so on (Markovits & Smith, 2008). Against this background, the purpose of this study was to develop 'case-based pedagogy' in mathematics for elementary school teachers and to investigate how they participate in the course employing case-based pedagogy. The term case-based pedagogy is used to underline a series of pedagogical flow by which teachers analyze cases, create and implement their own cases, and reflect on multiple cases.

A total of 30 cases of elementary mathematics instruction were developed. The cases varied in terms of content area, grade level, teacher, and authorship. Each case included not only detailed description of 40-minute instruction but its related curricular materials, questions for discussion, theoretical review, focus analysis, reference list, etc. The 11 among 30 cases were used in pre-service teacher education for one semester. Sixteen teachers participated and discussed the cases extensively.

This paper focuses on two cases with different purposes. The first case was about teaching 1km with the emphasis on students' length sense. The case led hot debate among the teachers on whether or not such emphasis was meaningfully implemented. They also discussed how the teacher of the case re-constructed curricular materials and how such re-construction might influence students' learning. This case had great impact on the teachers' own instruction during the practicum period. In fact, this case was evaluated as the most impressive case. This suggests that the analyses of cases be tightly linked to a teacher's instructional decisions in the actual classroom. The second case was about teaching line symmetrical figures. This case was also influential in that it seemed to implement many desirable student-centered approaches but it failed to teach line symmetry effectively. This helps us extend the types of cases beyond exemplars reflecting good practice. This paper provides issues and suggestions for the professional development of mathematics teachers with case-based pedagogy.

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INVESTIGATING YOUNG CHILDREN'S STRUCTURAL KNOWLEDGE ABOUT SQUARES THROUGH DESCRIPTION, CONSTRUCTION AND REFLECTION

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The consensus in existing literature is that children's limited, and often appearance-based descriptions of shapes, indicate that children view shapes as a whole and lack structural understanding. This study approaches children's understandings of shapes from a different perspective, based on an alternative and more dynamic interpretation of the van Hiele model (van Hiele, 1986) and with the acknowledgement that there might be multiple ways of knowing and expressing mathematical knowledge.

Within a constructionist framework (Noss & Hoyles, 2006) the study examines the understandings young children have about the structure of shapes, and how this knowledge is expressed and used in the process of constructing squares. Fifty-two, five to six year olds, were engaged in three phase naturalistic task-based interviews. In Phase A (Description) the children were involved in classification and shape recognition activities. In Phase B (Construction) the children were asked to construct squares with the use of sticks and, in Phase C (Reflection) the children were asked to reflect on the construction process of Phase B. Even though during Phase A, the children, as supported by existing research, exhibited limited structural understanding about squares, through their involvement in Phase B, they exhibited much richer intuitive structural understandings. In Phase C, children tended to express structural understandings about squares in diverse and inventive ways.

The findings challenge the view that children's limited verbal descriptions of shapes indicate lack of structural understanding. In the process of the interviews the children articulated, through the 'language' provided, structural knowledge about squares that may be characterized as intuitive -if we share DiSessa's, definition of intuition (DiSessa 2000)- and at the same time they were able to situate their abstractions in the context of construction. Overall the findings indicate that, provided sufficiently sensitive techniques are employed, it is possible for children to express structural knowledge in diverse and often unconventional ways.

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THE EFFECT OF MATHEMATICAL MODELLING ON STUDENTS' BELIEFS

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Students' beliefs and attitudes toward mathematics affect their cognitive involvement in the learning process determining to a great extent the amount and quality of acquired knowledge (McLeod, 1992). The formulation of these beliefs and attitudes is mainly attributed to the students' experiences in the mathematics class where the way of instruction functions as a mean to communicate beliefs among the teacher and students. Mathematical modelling is believed to help students build deep and flexible mathematical knowledge by moving the focus from simple calculations and solving of equations to mathematization (Mousoulides *et al.*, 2007). The main purpose of this research was to explore the effect of a mathematical modelling experience on students' beliefs and attitudes toward mathematics. In our study, statements considered by the individual as known and stable are classified under beliefs, such as *"in mathematics statements are always right or wrong"*, *"I need mathematics for my profession later"*. Attitudes refer to positive or negative feelings toward mathematics, such as *"mathematics is boring"*, *"I like mathematics"* (McLeod, 1992).

The case study examined 53 Greek students in grade 9 (14-15 years old) with no previous experience in mathematical modelling. Half of the group was exposed to a 3x45 min. intervention based on a modelling activity from Mousoulides *et al.* (2007). A constructivist teaching approach was adapted during the lessons, in which the teacher remained in the background while students worked in groups. Data collection involved class observations, student interviews, and a pre- and a post-test on beliefs and attitudes through an instrument derived from the TIMSS-2007 Background Questionnaire. The control group was involved in the pre- and post-test only.

Results show that students in the experimental group related the modelling activity to everyday life situations, but almost all of them said that mathematics is only useful if it is necessary for their later profession. The modelling experience amended only a limited spectrum of students' affective domain.

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THE PROBLEM SOLVING PROCESS BETWEEN STATIC AND DYNAMIC

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Problem solving is a target-oriented activity. While reading a problem, one's activity is driven by the purpose of solving it, which orients all the subsequent steps. Thus, the solver tries first to make explicit his/her understanding by representing the problem through a mental model. Some problems require dynamic representations, in which the configuration changes along the understanding-solving process, while some others are static. The question we address is double folded. First, as the mental models generated by a problem in the solving process seem to be static or dynamic, is it relevant to classify problems using a kinematical criterion? Second, is such difference in problems relevant from a cognitive perspective? In other words, are students behaving differently when they face different types of problems?

To answer the first part, we examined various problems and extrapolated definitions. A *dynamic* problem supposes motion of a configuration, or transformation of the context, or some degrees of freedom which lead to more possible results. Conversely, a *static* problem does not suppose motion or variability of data. We have further analyzed several hundreds of problems and we found that this criterion is consistent: problems do classify in these two categories (with very few of them disputable).

To test our second question, we looked at the results of a multiple choice contest, with approximately 250,000 participants in grades 3 to 12. We examined the following statistics: variation of the percentage of good solutions; variation in percentage distribution for distracters; significance of the distracter with highest percentage.

We found that the difference in problem classification manifests in the evolution of the correct solution rates along students' ages: the rates for dynamic problems show a slowly increasing tendency with an upper limit or even a small decrease in the last grades of high school; meanwhile static problems do not show this type of limitation. However, it seems that school mathematics goes in favor of enhancing static thinking, a fact that can be observed in the analysis of the highest percentage distracters.

These preliminary results raise new and challenging questions, like the connection between curriculum, evaluation practices and the decay of dynamic thinking in time or as the strategies that static and dynamic problems allow to be used in problem solving.

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A FRAMEWORK FOR EXAMINING MATHEMATICS TEACHER KNOWLEDGE AS USED IN ERROR ANALYSIS

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The importance of understanding the knowledge needed for teaching has become an important topic for the teaching and leaning of mathematics (Adler & Davis, 2006). Error analysis is a basic and important task for mathematics teachers (Luo, 2004), unfortunately, in the present literature there is a lack of detailed understanding about teacher knowledge as used in it. In the present study, based on a synthesis of the literature in error analysis and teacher knowledge, a framework for prescribing and assessing mathematics teacher knowledge in error analysis is proposed, which is shown in the following table.

Dimension	Analytical categorization	Description
Nature	Mathematical	Confusion of concept, etc
of	Logical	False argument, argue in a circle, etc
mathematical	Strategical	Couldn't distinct from pattern, etc
errors	Psychological	Mentality deficiency, improper mental state
Phrases	Identify	Knowing the existence of mathematical error
of	Interpret	Interpretting the underlying rationality
error	Evaluate	Evaluating students' levels of performance
analysis	Remediate	Presenting teaching strategy

In the framework, two dimensions of the nature of mathematical error and the phrases of error analysis are concentrated on, and it will be validated by two empirical examples when it will be presented in the conference. The framework makes the preparatory step to investigate teacher knowledge as used in error analysis and will hopefully stimulate new ideas and developments on it.

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KNOWLEDGE IN/FOR TEACHING: MATHEMATICS TEACHER 'LISTENING' IN ENGLISH, FRENCH AND GERMAN CLASSROOMS

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Based on an ethnographic study of twelve teacher, this study explores teacher 'listening' in English, French and German secondary classrooms, in order to develop a deeper understanding of teacher knowledge in and for teaching. It is argued that teacher 'listening' was different in the English, French and German classrooms studied, in the sense that teachers needed different kinds of knowledge to listen 'appropriately' within their respective environments. Even similar kinds of knowledge (for example, subject knowledge) appeared to be differently situated in the different culturally figured environments.

Teaching is often perceived as 'telling' - instead, I put listening at the centre of teaching - to listen to teach mathematics, listening to get to know. The act of listening is based on student-teacher interaction, it focuses on meaning making in the classroom, and includes listening to what students say, what they write and do, perhaps their gestures; and moreover the immediate classroom environment and wider context in which the lesson takes place. Listening to teach mathematics involves knowledge of the learner/s (and their development); knowledge of the immediate classroom situation; the wider socio-cultural context; and of course knowledge of the subject matter. This provides the framework for my analysis.

For this presentation I have re-analysed, on the basis of my understandings of 'listening', selected data which consisted of extended lesson observations and interviews with twelve teachers, four in each country, plus shorter observations and interviews with an additional ten teachers in each country.

To 'listen carefully' means different things for different teachers, and working in different environments. Whereas listening for English mathematics teachers meant helping their pupils working on tasks, on their own, and attending to their individual problems, for a French teacher it meant bringing 'to the front' all different ways of solving a mathematical problem, and discussing and searching for the most suitable methods. The German *Gymnasium* teacher listened to pupils by attending to their misconceptions in a whole class discussion, and the *Hauptschul* teacher spent a large amount of time listening to pupils' personal problems.

In summary, the crucial point for listening, I argue, is the 'quality of listening', and this was differently perceived in different educational environments. Furthermore, it is argued that knowledge in and for teaching can both be applied, as well as developed through 'listening'. These will be further explored in the oral presentation.

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WHY DON'T THEY DO WHAT THEY SAY THEY SHOULD? STUDENTS' PERCEPTIONS ON LEARNING MATHEMATICS

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A number of research studies have used interviews to explore students' perceptions of what practices lead to success in learning mathematics (Kershner & Pointon, 2000; Freeman, McPhail & Berndt, 2002). Much of this research has focused on either what children perceive to be best practice (espoused theory) or what they actually do (theory-in-use). However these two theories do not always align (Perger 2007; Robinson and Lai, 2006).

There are very few studies that compare what students say with what they do. In one study that looked at both students' espoused theories and their theories-in-use higher and lower achieving students held identical espoused theories (Perger, 2007). Differences were only found when their theories-in-use were compared with their espoused theories. Higher achieving students' practice aligned more closely with their theories-in-use than the lower achieving students. When lower achieving students' were asked to justify the differences between their espoused theory and their theory-in-use a greater insight into the importance they placed on these practices as well as their perceptions of what these practices entailed was gained.

This presentation presents three case studies of lower achieving students' espoused theories and theories-in-use regarding practices that lead to success in learning mathematics. It then discusses the students' reported reasons for the differences between these two theories, and argues that giving students an opportunity to voice their own justifications leads to greater research insight into the students' perceptions of best practices in learning mathematics.

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ALGORITHMIC AND FORMAL CONTEXTUALISATIONS: EXPLORING STUDENTS' UNDERSTANDING OF THRESHOLD CONCEPTS IN CALCULUS

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In students' efforts to understand the subject some concepts may be more crucial. A *threshold concept* (Meyer & Land, 2003) can be considered as a portal opening up a new and previously inaccessible way of thinking about something. As part of a thesis two empirical studies were carried out to explore students' understanding of threshold concepts in calculus. The aim was to explore how university students use their conceptions when working with mathematical tasks. How do students contextualise concepts in calculus? What interplay among different contextualisations occurs? The first study investigated the nature of students' understandings of the concepts of limit and integral. Engineering students taking a course in calculus were asked to reflect in writing on the meaning of the concepts limit and integral. Four students also take part in a subsequent interview involving questions probing the students' understanding of the two concepts (Pettersson & Scheja, 2008). The results reveal that the students expressed their understanding in an *algorithmic context*, in which procedural knowledge was predominant. However, faced with probing questions the students appear to shift to a contextualisation foregrounding conceptual knowledge. In the second study a group of students in their first year of a mathematics programme was working with a challenging task including the concepts of function and derivative. The results reveal that these students expressed their understanding in a *formal context* where also intuitive ideas played an important role, and that probing discussions initiated shifts between the formal and the intuitive.

By using a theory of contextualisation to model the students' conceptions of threshold concepts it is made clear that the students' interpretations of a given task are not due to cognitive shortcomings, but rather ways of dealing with a learning situation at hand. The results display the contextual shifts as part of the conceptual development and that students may have a potential for developing a formal understanding of a mathematical concept previously viewed within an algorithmic context.

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GRADE TWELVE STUDENTS' UNDERSTANDING OF THE CONCEPT OF THE DERIVATIVE AND UNDERLYING REASONS FOR STUDENTS' DIFFICULTIES IN THAT CONCEPT

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This paper reports on a qualitative study carried out with students from a grade 12 mathematics class from a South African school. The main purpose of this study was to explore student's' understanding of the concept of the derivative and underlying reasons for student's difficulties in that concept. The participants comprised one class of 29 learners and their responses to May and August examination questions based on the concept of the derivative were examined. Semi-structured interviews were conducted with 4 of 29 students. Vinner's (1997) framework for conceptual and pseudo-conceptual behaviour was used.

A review of recent research reports reveals that the students' understanding of calculus concepts is very limited. Ferrini-Mundy and Graham (1991) argue that students' understanding of central calculus concepts is very primitive as they "demonstrate virtually no intuition about the concepts and processes of calculus … they diligently mimic examples and their attempts to adapt prior knowledge to a new situation" (p, 631) frequently results in very persistent and often inadequate conceptions. Tall (1992), in his discussion on students' difficulties in calculus, points out that reflecting "on the difficulties encountered by students' of differing abilities and experience, to obtain empirical evidence to build and test theories of learning to enable more fruitful learning experiences for students in calculus" (p, 12) might be useful.

The study utilised an interpretive qualitative case study approach. In this paper we present a case study of 4 students which was drawn up from the analysis of their responses to the May and August examination and one semi-structured interview. It was found that the students applied inappropriate techniques drawn from their grade 11 work on quadratic theory. Furthermore, many learners' difficulties with the test items were grounded in their difficulties with algebraic manipulation skills. One of the most interesting findings revealed from the analysis of the learners' responses was the extensive use of quadratic theory to provide answers to questions based on calculus.

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SUPPORTING BASIC MATHEMATICAL SKILLS WHILE TEACHING WITH CAS HANDHELDS

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The loss of basic mathematical skills is often feared and, allegedly, many times observed in classes where CAS handhelds or other technology are used extensively. We will show that doing "mental mathematics" regularily, with exercises from arithmetics, algebra, functions etc., can support basic skills and competencies in a CAS oriented learning environment.

Where CAS or other handhelds are being used extensively pupils prefer to use their calculators even for simple operations instead of solving them mentally. While they may still be able to solve them by hand or mentally, teachers often see this as evidence that handhelds and basic mathematical skills do not go together.

Banning handhelds from the classroom is, in the view of its many advantages and some curricular necessities, not an option. But these concerns cause many teachers to view this technology with suspicion. Moreover, the ability to master basic mathematical skills without the help of calculators is an unquestioned objective in all national curriculums, of course also where handhelds are compulsory, as it is the case in the federal state of Lower-Saxony in Germany.

Located here, CAliMERO is a five year project evaluating the use of CAS from class 7 combined with a teaching concept concentrating on sustainable learning and the development of mathematical competencies (Ingelmann 2003). One of its objectives is meeting the concerns about the alleged loss of basic mathematical skills. For this, regular exercises in "mental mathematics" were introduced in grade 8 which cover arithmetic, algebra, functions, geometry and stochastic. In grade 9, it now shows that these measures have a positive effect on basic skills and competencies. Pupils of the CAliMERO project classes, which have regularily been doing mental mathematics, show a larger increase from pre- to posttest (47,5→54,6 percentage of right answers, N=628) than pupils of the control groups (49,9→51,4, N=113). Moreover, the increase in the project classes is larger than the expected one third of the standard deviation of 17,2. This shows that basic skills and competencies can be supported with the right measures, even in a CAS oriented teaching environment.

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LONGITUDINAL NARRATIVE RESEARCH INTO BUDDING TEACHERS' PROFESSIONAL GROWTH: A CASE OF AADA

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In my presentation I report of a part of a longitudinal study, 2005 - 2008, on mathematics preservice teachers' professional growth. The target group of the study comprised six students who began their studies on a master's programme at the University of Tampere in the autumn of 2005. They were students majoring in education with mathematics as a compulsory minor subject. The data corpus consisted mainly of reflective narratives by the students and transcribed conversational interviews. The analysis methods are those commonly used in narrative research. In my presentation I report a case of Aada.

In this paper I define novice teachers' professional growth after Kagan (1992) as changes over time in the behavior, knowledge, images, beliefs or perceptions. In my presentation I will focus on a case of Aada, a student teacher, and her views of mathematics, and of learning and teaching mathematics. When I refer to views my theoretical framework comes from Wedege & Skott (2007). They distinguish beliefs, attitudes and emotions. Beliefs include self-perception, aspects of identity and confidence, and attitudes are considered more stable than emotions. I selected the categories after Beswick (2007). When Aada started her studies her views of mathematics were positive although mathematics was first and foremost related to rules and mechanical calculations. She commented that learning mathematics was lonesome and hard work. The teacher's role was to explain thoroughly and the future work was merely perceived as teacher's actions, not as classroom work. After the second semester the differences between university mathematics and school mathematics are becoming more marked. In learning mathematics she stresses understanding, but relies on teacher-led methods. At the end of the third semester she sees mathematics as a logical system where problem solving is a central part of learning, and uses more investigational teaching methods.

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SPREADSHEETS AS A TRANSPARENT RESOURCE IN LEARNING MATHEMATICS FOR TEACHING

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The international mathematics education community is giving increasing attention to teachers' knowledge of mathematics. Ball, Bass and Hill (2004) argue that a key component of teachers' mathematical work is that of unpacking. Davis and Simmt (2006) argue that a key competence for teachers is an ability to work between various representations, images, metaphors and analogies.

This study focuses on the learning of 40 pre-service secondary mathematics teachers participating in a course on introductory financial mathematics. Students use two kinds of spreadsheets in exploring the mathematics of annuity-based investments: one used an *account balance approach* and the other an *individual payment approach* which tracks the behaviour of each payment over time. The latter approach is far more powerful because it reveals the mathematical structure, it was not intuitive for students and many struggled initially to accept that the account balance can be tracked by disaggregating individual payments.

The students' use of the spreadsheets was analysed through the notion of transparency of resources (Lave and Wenger, 1991). I argue that students' ability to see and see through the spreadsheets with the individual payment structure enabled them to access and model the mathematical structure of the scenario, and thus to unpack for themselves the processes that are compressed in the annuities formulae. This enabled them to deal easily with scenarios that diverted from a perfect payment plan, such as missed payments. Simultaneously the students imposed a geometric structure onto the patterns in the spreadsheet which enabled them to compress the processes in a way that differed from the annuities formulae. In doing so, students displayed an ability to shift easily between representations. This ability was displayed during the course and six months after the course ended, which suggests that the transparency of the representation that was achieved for them through their activity, may develop mathematical knowledge that is usable for teaching.

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MENTAL CALCULUS: PERFORMANCE AND ATTITUDES IN 2ND, 4TH AND 6TH GRADE PORTUGUESE STUDENTS

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The importance of mental computation in the development of number sense is widely documented to date (e.g. Heirdsfield, 2000; Kamii, & Dominick, 1997; Varol & Farran, 2007). The last authors argue that mental computation for addition and subtraction of multidigit numbers plays an important role on teaching children how numbers work. On the other side, students' attitudes are known to have impact on their performance. In this research we meant to identify Portuguese students' performance and attitudes with respect to mental calculus through the elementary education (2^{nd}, 4^{th} and 6^{th} grades) and we followed the procedures indicated in Reys, Reys, Nohda and Emori (1995)'s work.

This research was conducted in four schools in Lisbon, and involved 166 students. Fifty tasks were presented orally and visually to children, together with a 28 items questionnaire asking about their attitudes towards mental calculus. Ten interviews were then conducted with good and average performing 4^{th} grade students, to inquire about the strategies they used.

Results show that mental computation interests the majority of students on all grades more often than written computation. Orally presented items appear to have better results than visually presented ones when 2^{nd} and 4^{th} graders are concerned. With 6^{th} graders the opposite is true. None of the children indicated an alternative way to solve the problem. The good performers used a wider variety of strategies and showed to be more at ease with mental strategies, besides being more successful, while doing so, than the average students.

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EVERYDAY LIFE EXPERIENCE IN DIALOGUE

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This presentation reports from a pilot study connected to my PhD project focusing on dialogue in mathematics learning. The schools from which the data is collected are cooperating with local companies or enterprises. It is their aim to strengthen the children's mathematical competency by locating learning in applied out-of-school contexts. My focus is on pupils' dialogue with each other, with their teacher or with other adults in the school and the workplace. I am studying how the conversations establish conditions for learning.

The pupils' (11 years old) learning is connected to collaboration with a store in their neighbourhood. The data consists of videotaped dialogues from the classroom and from the workplace. The research strategy is in the area of ethno-methodology and I use conversation analysis as a tool. Through these analyses I indentify the pupils' use of everyday life experience as references in their dialogue with their teacher. I also identify their attempt to grasp the teacher's mathematical and educational intentions. Further, I can identify how they are navigating between their everyday experiences and language and the experiences and language they regard relevant from the teacher's point of view. On the basis of these preliminary results, I question the impact of the teacher's insight into this navigation. Analysis is to be illuminated by the use of social practice theory (Lave, 1996) and dialogical approach, using Alrø & Skovsmose's (2002) inquiry-cooperation model. Lave (1999) describes the learners' movement between different social practices. This makes it relevant to investigate the differences between school mathematics and mathematics used in practices outside school. The reported research will be extended to a further study of dialogues in mathematics learning, including how the dialogues may encourage pupils' ability to see connections between learning gained in different practices.

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BROKERING AS A MECHANISM FOR THE SOCIAL PRODUCTION OF MEANING

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In this report we analyze how the brokering moves of the teacher and some students in an undergraduate differential equations class functioned as a mechanism for the social production of meaning. From an individual cognitive point of view, there are well-established mechanisms that describe how individuals build ideas. From a social point of view, however, mechanisms that describe how ideas are interactively constituted are less developed. Such mechanisms are significant because they address the complex job of teaching and teacher moves that promote the collective construction of meaning.

In his seminal work on communities of practice, Wenger (1998) highlights how brokering requires the ability to “cause learning” by introducing into one community elements of practice from a different community (Wenger, 1998, p. 109). In our analysis, we consider three different communities: the broader mathematical community, the local classroom community, and the various small groups that make up the local classroom community. The brokers in these communities are the teacher and specific students in the class. A broker, by definition, is someone who has membership status in more than one community. For example, in our case the teacher is a member of the broader mathematics community, the classroom community, and a peripheral member of each of the small groups that make up the classroom community.

We identified the following three broad categories of broker moves that contributed to the social production of meaning: creating a boundary encounter, bringing participants to the periphery, and interpreting between communities. Each of these brokering categories highlight the view that teaching and learning mathematics is a cultural practice, one that is mediated by and coordinated with the broader mathematics community, the local classroom community, and the small groups that comprise the classroom community. Our presentation will tender examples of each of these brokering moves. Data for the analysis is drawn from classroom videorecordings collected during a 15-week classroom teaching experiment (Cobb, 2000) conducted in an undergraduate differential equations course.

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ANALYZING A TEACHER'S PRACTICE FROM THE RELATIONS BETWEEN HER COGNITIONS. THE CONTRIBUTION OF A COGNITIVE MODEL TO THE UNDERSTANDING OF WHAT SEEMS TO BE HAPPENING

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We intend to share the process we are using to study the professional development of two Portuguese primary teachers, some findings from that on-going study and the potentialities of the use of the model (based on Schoenfeld, (1998)) we are using, and improving (Ribeiro, Monteiro & Carrillo, 2009), in the process of teachers' professional development (both in- and pre-service). Specifically, we present and discuss the modelling process of a classroom episode in which the teacher uses the drawing on the board (a square) for revising the different ways of writing a tenth. In our model we focus on the teacher's beliefs, goals and knowledge, in action, which are made accessible trough the type of communication (s)he employs, resources and the pupils' way of working.

The study involves two primary school teachers, one of whom is the focus of this paper, and in particular one episode in which the teacher reviews content using a drawing on the board. It combines a case study with an interpretative approach. The modelling derives from transcriptions of lesson recordings, centred on the teacher, principally audio, but supplemented with video.

We believe that this process of modelling the classes (if done with the teachers themselves) could play a critical role in the process of professional development, representing a starting point for discussions and reflections arising in the group where they take place (Sherin, 2004). It could lead also to an awareness of their own cognitions (even reflecting on others' practice) and how these influence their teaching. This model could be also a useful instrument for initiating discussions and for helping teachers identify their own mathematical critical moments and the way they deal with them.

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ANALYSIS OF THE INTERRELATIONSHIP OF TECHNOLOGICAL, PEDAGOGICAL, AND CONTENT KNOWLEDGE IN ALGEBRA TEACHER EDUCATION

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This study explores the development, advancement, and interrelations of Algebra teachers' technological, pedagogical, and content knowledge (TPCK) in the teaching and learning of Algebra I. The research describes the participating teachers' mathematical thought processes during teaching and learning tasks, each of which required a TPCK framework, throughout a 120 hour professional development program for in-service 8th grade mathematics teachers.

INTRODUCTION

Technological, pedagogical, and content knowledge (TPCK) framework lies at the core of bringing together knowledge about technology, pedagogy, and content as interconnecting factors affecting the development of effective teaching (Mishra & Koehler, 2006). Mathematical TPCK, dissimilar from knowledge of all three concepts individually, refers to the intersection of knowledge of mathematics with knowledge of technology and with knowledge of teaching and learning. The sum and intersection of technological knowledge, pedagogical knowledge, and content knowledge serve as a framework for effectual mathematics teaching and learning.

METHODOLOGY AND RESULTS

Twenty in-service 8th grade mathematics teachers from six different schools participated in a professional development project designed to advance teachers' TPCK in the teaching and learning of Algebra I. Participant journal entries and observations of participant interactions were categorized and analyzed using a TPCK content analysis framework. As a result of the project, teachers increased their level of content knowledge, identified specific activities that guided their planning and teaching algebra, and increased their knowledge and appropriate use of technology in the teaching and learning of Algebra I.

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EPISTEMIC AND COGNITIVE CONFIGURATIONS OF PRE-SERVICE TEACHERS WHEN SOLVING A MISSING VALUE PROBLEM

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Proportionality is an important integrative thread that connects many of the mathematics topics studied in grade 6-8 (NCTM, 2000). A great deal of research literature on proportional reasoning is located on strategies, errors and difficulties expressed by pupils. Nonetheless, many pre-service teachers show shortcomings in their pedagogical content knowledge to teach it (Ben-Chaim, Keret, & Ilany, 2007). In this short oral we will report the first results of a descriptive study about ratio and proportion knowledge of 60 pre-service teachers, as it is put into action while solving a missing value problem.

The identification of objects and meanings put into effect when educator and students solve a missing value problem was done using the epistemic and cognitive analysis. These analyses have allowed identifying epistemic and cognitive configurations. Using the former, the trainer was able to get an idea of the complexity of mathematical meanings underlying the solutions. Thus the trainer was able to foresee the possible strategies, errors and difficulties that could be manifested by students. This identification, according to Hill, Ball and Schilling (2008), encourage the development of trainer mathematical knowledge for teaching. The teaching model implemented made use of epistemic and cognitive configurations, that in turn, have allowed identifying some relevant issues that could be useful for the educator in his work with pre-service teachers; just to mention a few: (a) the numeric and the algebraic: manifested in the resolution of the problem by using the rule of three, (b) the verbal-symbolic modelling: that transforms the problem statement into a symbolic representation of the rule of three, (c) the symbolic-symbolic modelling: that transforms a symbolic expression (rule of three).

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TEACHERS' QUESTIONS IN MATHEMATICS TEACHING: QUANTITATIVE-QUALITATIVE SHIFT

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The contribution presents some results of research focused on creating plausible criteria for characterising mathematics teachers' questions. Transcripts of mathematical classroom talk gained through audio recordings and field notes were examined through qualitative analysis in terms of the form and function of particular utterances. Types of individual mathematical tasks presented in lessons are suggested as one of the important features of communicational situations. We can distinguish quantitative and qualitative characteristics of mathematical tasks, both defined according to the terminology used in TIMSS 1999. Quantitative characteristics are determined by the procedural complexity of the task or "the number of learner's decisions required to solve the problem using conventional procedures" (Hiebert et al., 2003). Qualitative characteristics are determined by "the kind of mathematical processes implied by the statement of the mathematical problem: using procedures, stating concepts, and making connections" (Hiebert et al., 2003).

However, one of the results of the research presented here suggests that it is not the characteristics of a particular task as such, but rather the way it is connected to other tasks used in the lesson, which plays an important role with respect to the development of learners' cognitive processes. For example, a teacher may pose several questions about the original task which differ in their quantitative or qualitative characteristics. Thus, the quantitative and qualitative characteristics of the given task are shifted. Let us call this phenomenon a *quantitative-qualitative shift*, as both types of characteristics are mutually interwoven to such an extent that it seems extremely unlikely for a shift in one of them to occur separately. In the research, two types of quantitative-qualitative shift have been identified according to its orientation: a *downward-oriented shift* which roughly corresponds to funnelling effect (Wood, 1998); and an *upward-oriented shift* where the learner's correct answer to the original question is followed by the teacher's new questions which gradually increase the procedural complexity of the task. This way the teacher proceeds from simple procedural tasks to conceptual tasks of higher diagnostic significance.

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THE NATURE OF FIRST GRADE STUDENTS' SEMIOTIC ACTIVITY ON NUMBER OPERATION IN OPEN APPROACH

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This study is a part of a three-year long professional development project implementing Lesson Study and Open Approach conducted by the Center for Research in Mathematics Education during 2006-2008. The purpose of this study was to analyse the first grade students' semiotic activity on number operation for a year-long teaching experiment. In this study, Open Approach incorporated in Lesson Study is treated as a teaching experiment and the focus of this teaching experiment was to investigate kinds of students' activity in making signs and their meanings which can be called semiotic activity (van Oers, 2000). This occurred when the students encountered with the open-ended problem situations on adding and subtracting numbers used in Open-Approach Classroom (Nohda, 2000). Ethnographic study was employed for conducting the research through working with a classroom teacher, some of the school teachers and a research team of Khon Kaen University in planning, observing and reflecting the lessons in lesson study cycle. Major data for analysis were 6 protocols of classroom teaching experiment on number addition and subtraction problems, students' written works and the researcher's field notes. In the teaching experiment, teacher provides cube blocks and ten frames of egg trays as a semiotic tool for several approaches to the open-ended problem situations. The research results revealed that the nature of first grade students' semiotic activity in Open-Approach classroom in this project occurs during students attempted to engage in open-ended problem situations by apparently constructing, using and refining schematic diagrams by themselves in order to represent the provided materials designed for fostering them to think symbolically and communicate meaningfully their own mathematical ideas on adding and subtracting quantities.

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MATHEMATICS LEARNING TRANSFORMATIONS IN MULTICULTURAL CONTEXTS

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The recognition of the influence of the pupils' socio-cultural experiences on the shaping of the mathematical meaning imposed a new framework of thinking about teaching mathematics, which draws on the relevant practices in which pupils are engaged outside as well as inside school. This perspective addresses the needs of and the demands made on all pupils in the context of school mathematics but especially of those from minority groups, whose culture differs from that represented in and through school mathematics practices and discourses (e.g., Zevenbergen, 2007). Thus, it becomes apparent that, in order to increase all children's opportunities to succeed in mathematics, social and cultural diversity needs to be acknowledged and exploited as a learning resource. To this direction, the role of classroom interaction and communication as well as pupil's collaborative activities appear to be of crucial importance (e.g., Gorgorio & Planas, 2002).

The research reported here is part of a more than ten years' project aiming at examining the transformations that mathematics learning and teaching practices go through, following the introduction of an instructional approach based on an innovative package of educational material in multicultural upper primary and secondary school contexts. The approach encourages collaboration but also independence in learning, focuses on conceptual and structural rather than computational aspects of the mathematical knowledge and respects social and cultural diversity. Here, the focus is narrowed down to the examination of the learning transformations. In particular, the presentation provides an account of the rationale, the methodologies, the data and their analyses of a series of small scale quantitative as well qualitative studies (e.g., surveys, case studies) undertaken over the years. The preliminary results reveal a number of gains made by pupils at the cognitive and social plane, but also confusion and drawbacks in their learning practices due mainly to contextual factors.

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DEFINING IN MATHEMATICS SCHOOL TEXTS[1]

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The aim of this work is to approach to the form in which defining appears in the mathematics school texts related with 16-18-year-old students level. I consider that, before thinking what the pupils learn, it should be considered what it is wanted that they learn. In this study, I will use the texts as the context that allows "to see" the messages that the students receive across them. From a research point of view, for reaching my aim I need a conceptual frame, and a methodological design that makes possible to approach the process of data collection, analysis and interpretation of the results.

In relation with the conceptual frame, studies of authors as Borasi, (1991), Mariotti & Fischbein (1997), Zaslavsky & Shir (2005) and other developed in our group (Sánchez et al (2008) have been a starting point in this part of the whole study. In particular, I take into consideration as variables: *role,* understood as representing different facets of defining, and *type*, which establishes differences inside of defining, including different systems of representation.

Respect to the design of the study, here I focus on two of four set of textbooks considered, trying to look for the similarities and differences that allow characterizing what aspects are related to the school culture or are closer the considered texts. The obtained results can be summarized in:

a) In all the courses considered defining is related to a search of common properties that characterize to the element that is defined.

b) The textual mode of representation is considered to be a vehicle with enough entity for the access to the idea of defining.

c) The use of the modes of representation is different in both publishers.

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THE USE OF MANIPULATIVES IN TEACHING MATHEMATICS TO GREEK ROMA CHILDREN

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The education of Greek Roma children is characterized by non-systematic school attendance and low participation level in all school subjects. The limited school enrollment and low achievement of Greek Roma children end up looking like a cultural given, as do the low expectations and support of Roma parents for their children's performance in school. Besides having to overcome social and cultural barriers in order to enroll in and attend school, the common low competence of Greek Roma children in the Greek language places them in a double bind: of learning the official language of instruction and coping with the curriculum.

In our presentation we will sketch the rationale behind the development of educational material that served the needs of a large-scale program conducted in 39 of Greece's 52 prefectures. The main aim of the program was the development of knowledge and improved skills in language and mathematics of Greek Roma children. The corpus of tasks in mathematics was developed around the use of manipulatives and covered areas including arithmetic, geometry and problem solving. The educational potential of instructional devices like manipulatives is to be found in carefully selected tasks that require the use of such devices. As the tasks are 'acted out' by the participants in the mathematics classroom, these instructional devices can become resources for exchanging arguments and conceptualizations (Meira 1998). Physical devices like manipulatives, then, may play an important role in building an effective learning environment for students marginalized by an educational system uncritically centered on an anachronistic monocultural and monolingual norm.

From the data that we collected, in cases where children could not bridge the gap between everyday life language (to a large extent Romani) and formal, school terminology (language of mathematics, official language of instruction), they relied on the manipulatives to communicate their conceptualizations to the researcher and the teacher. We may suggest then, that children used the manipulatives as resources to construct and communicate mathematical meaning (Moschkovich 2002) and claim a fully participatory role in the activities.

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DESIGNING PROFESSIONAL DEVELOPMENT MATERIALS BASED ON THEORIES OF TEACHING AND LEARNING MATHEMATICS

Nanette Seago

Artifacts of practice, such as classroom video, can be effective tools for bringing the work of teaching into the professional development setting. They offer the opportunity to analyze and unpack the mathematics in classroom activities, examine instructional strategies and student learning, and discuss ideas for improvement (Ball & Cohen, 1999; Borko et al., 2008). The Learning and Teaching Geometry (LTG) project is currently developing professional development materials that are designed to engage teachers in learning about mathematical similarity through the use of video cases, in which specific and increasingly complex mathematical ideas are presented within the dynamics of classroom practice. The LTG materials follow a carefully developed mathematical storyline, which has been vetted by a group of mathematicians. A core component of the materials is a set of short video clips showing teachers' and students' experiences working with similarity problems in their classrooms. The clips expose teachers to a dynamic, transformational conception of similarity, and provide opportunities for them to consider how to present such a conception to their own students.

To create an initial draft of the materials, the LTG project engaged in a six-phase design process: (1) conjecturing a sequence of activities based on a specified learning trajectory that would provide a mathematically robust experience for middle school teachers around learning and teaching similarity, (2) using a strategic videotaping process to film a number of classroom lessons in which teachers used these activities with their students, (3) selecting promising video clips from these lessons to map on to the learning trajectory, (4) designing PD modules by creating a framework for the materials that incorporate these video clips, (5) developing video case resources to promote teachers' mathematical knowledge for teaching, and (6) revising materials based upon formative evaluation data.

The focus of the presentation is to discuss the theoretical background and design of the LTG materials, the proposed learning trajectory, and the current conceptualization of the initial draft of the materials.

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FAGNANO'S PROBLEM AND THE USE OF DYNAMIC SOFTWARE AS A LEARNING TOOL

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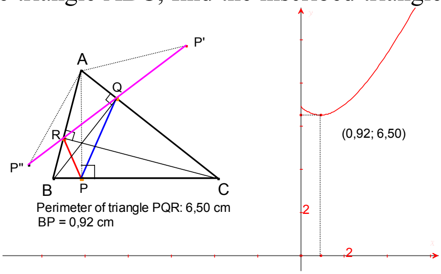
The Fagnano's Problem, is a rich problem of geometric variation that relates to the perimeter of a triangle inscribed in a given acute triangle. By means of a ruler or of dynamic software, it is possible to observe that the perimeter of an inscribed triangle changes when one moves its vertices; the problem consists in finding when the perimeter has a minimum value.

FAGNANO'S PROBLEM. Given an acute triangle ABC, find the inscribed triangle PQR with the smallest perimeter.



The students discovered that by moving the vertices P, Q and R the perimeter also varied, noticing they were dealing with a problem of variation. We apply to vertex P the heuristic strategy of symmetry (Polya, 1945): let P' and P'' be the symmetric points of P with respect to AC and AB, respectively. We draw lines from P' to Q and from P'' to R. Since P is symmetric to P' with respect to AC: PQ = P'Q, and AP = AP'; and since P is symmetric to P'' with respect to AB: RP = RP'', and AP = AP''. Therefore: PQ + QR + RP = P'Q + QR + RP'', and AP' = AP''. So, the broken line P'Q + QR + RP'' is the perimeter of ΔPQR; and is minimum when the broken line is a segment.

With help of the *Cabri géomètre* software it was possible to draw the required construction for different locations of P, Q and R and thus find the perimeter of ΔPQR, and additionally it was possible to draw the graph of the relationship between distance BP (as the independent variable) and perimeter of ΔPQR (as the dependent variable). Without doubt the use of software can contribute to the students' understanding of the problem, to developing their visualization and to confirming their ideas regarding the solution (Presmeg, 2006). The graph in Figure shows the variation of the perimeter of ΔPQR by moving the vertices P, Q and R. The minimum point on the curve corresponds to the minimum value for the perimeter of ΔPQR inscribed in the acute ΔABC.

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HIDING SHAPES

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The National Council of Teachers of Mathematics (NCTM) emphasizes developing spatial sense because "Spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world. Insights and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense." (NCTM, 1989 pp.48) Xistouri & Pitta-pantazi (2006|) found a significant positive relationship between both spatial abilities and performance in symmetry, as well with mathematical achievement.

The new Israeli curriculum for mathematics at primary school also focuses on spatial abilities and the need to develop them.

To develop spatial sense it is recommended that "children must have many experiences that focus on geometric relationships; the direction, orientation, and perspectives of objects in space; the relative shapes and sizes of figures objects; and how a change in shape relates to a change in size." (NCTM, 1989 pp.49)

We studied how Israeli students from primary school deal with spatial tasks. We adapted two of Del Grande (1990) suggestions for spatial sense activities, focusing mainly on the following aspects: visual coping, left-right coordination, and figure ground relationship. We traced and interviewed 36 students from grades one through six. According to Piaget's theory (Piaget & Inhelder, 1954) we expected that younger children will show some difficulties, while older students would succeed, but our findings showed that, as always, it is not that simple.

In our short oral presentation we will describe the activities. We will show the students' results (both correct and incorrect answers), describe various strategies they applied, and present the main false intuitions we found. We will also show the development of spatial sense that we traced, based on students' results.

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'CHANCE' - WHAT DOES IT MEAN

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Piaget and Inhelder (Piaget & Inhelder , 1954) defined “understanding probability” as the ability to produce some quantification assessment for the probability that a certain event will occur. In order to give such a quantification assessment, one should understand both 'chance' and the usage of the combinatory operations. Piaget and Inhelder also deduced that understanding "chance" is to understand total possibilities and randomness. Total possibilities events are events that either never occur (like throwing two dices and getting 14) or always occur (like throwing two dices and getting a result between 2-12). They defined 3 stages for developing the understanding of chance and probability. Only at the third stage (from the age of 11-12) one understands 'chance' and is able to use the combinatory operations. Many researchers examined 'chance', and wondered whether the subjects can identify total possibilities and random events in some statistical distributions. E.g.: Sanches & Martinez (2006) and Watson & Moritz (2003). In this research we wanted to find out what meaning do teenagers give to the word 'chance', and to what degree they connect 'chance' to some quantification assessments.

56 students (19 from 6th grade, 18 from 8th grade and 19 from 11th grade) were interviewed. Each interview included, in addition to free conversation, the following assignments: Explain what 'chance' is and to compose a sentence with the word 'chance' ; Explain what 'almost zero chance' is; What does 'almost zero chance' mean in the story of the weather man? ; Explain a newspaper quotation with 'almost zero chance'. The phrase 'Almost zero chance' was chosen because it discriminates between ‘total possibility’ event and ‘random’ event. 'Almost zero chance', when relating to events, means a random event (a very poor chance to occur, but still it may happened). However, one may interpret it as zero chance, a total possibility event (an event that will never occur).

At the short oral presentation the assignments, the subjects' reactions and answers, will be describe. Parts from the conversations will also be presented, mainly those that show interesting ideas about what 'chance' is. We will also present the findings and the conclusions.

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ANALYSIS OF A G9 STUDENT'S ACTIVITIES BASED ON SUBSTANTIAL LEARNING ENVIRONMENT (SLE) IN ZAMBIA

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OBJECTIVE OF RESEARCH

Zambian Students' low learning achievement needs to be considered. Therefore, this research aims at examining if SLE can enhance both basic skills and higher order thinking. The research question is to identify how students can grow 'mathematising', 'exploring', 'reasoning' and 'communicating' (Wittmann, 2005).

FRAMEWORK OF RESEARCH AND METHODOLOGY

Design experiment is applied for the framework. It has micro and macro cycles for development and examination through the long-term perspective. It also has three stages; (a)Design and plan for teaching and learning settings; (b)Experiment in class and ongoing analysis; and (c)Retrospective analysis(Cobb, 2002&2003, referred from Okazaki, 2007). The paper-based activity based on SLE (e.g. Number Brick) was set up for 23 times in the second cycle and students' work is quantitatively analysed.

DISCUSSION

A typical G9 fast learner's worksheets on number patterns were selected. Consequently, three characteristics are identified: *slow improvement*, *the holdback to the previous stage of understanding*, and *the instability of understanding*.

RESULT

A learner's change is both subtle and not linear in the sense that he showed improvement and regression alternately. It is concluded that four objectives mentioned above were not fully achieved; however 'communicating' and 'exploring' were identified in his activities, which gives us a positive impact on quality improvement.

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THE ROLE OF THE TEACHER IN ONLINE MATHEMATICS COURSES FOR TEACHERS

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We will discuss our recent work in *Proportional and Algebraic Reasoning*, an online 10-week, content-based courses in the master's degree program in mathematics education. In this course, we have two explicit goals: (1) supporting teachers' mathematical development and (2) using the residue from their mathematical collaboration as seeds for second-order conversations that support teachers in moving from their current understanding towards deeper mathematics, mathematical practices and pedagogical practices. To support mathematical development, we use Online Asynchronous Collaboration (OAC) (Clay & Silverman, 2009), a model for online mathematical interaction that was developed primarily to scaffold participants' engagement in legitimate mathematical practices online. OAC involves cycles of individual private mathematical work, small group discussions, and instructor led synthetic discussions of emergent issues. We will present examples of a brief segment of mathematical interaction and use this segment to demonstrate the mathematical development of individual participants and also to highlight potential learning opportunities, that although may be lost in a face-to-face class, can be capitalized on in OAC. We will conclude by presenting three "emergent prompts" that are grounded in analysis of the online interactions and our conjectures about supporting teacher development at the intersection of their existing understandings and our instructional objectives.

Our work highlights the ways in which OAC amplifies the potential impact of the mathematics teacher educators by providing them time to devise effective instructional interventions rather than the split-second decision-making required in face-to-face courses. In this way we are better able to scaffold students from their current knowledge base towards our goals of deeper, more connected mathematics.

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AN INVESTIGATION OF THE RELATIONSHIP BETWEEN WRITTEN CURRICULA AND ENACTED CURRICULA IN TERMS OF COGNITIVE DEMANDS ON STUDENT THINKING

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How teachers use problems and questions presented in curriculum materials decide what students learn mathematics. This study explored, through a survey, teachers' use of problems and questions in mathematics textbooks in terms of cognitive demands. A total of 169 elementary teachers from first to sixth grade participated in the study. In the survey, teachers were asked to rate the frequency of high-level problems (e.g., problems requiring students to represent mathematical ideas in three different ways) and teacher questions in their textbooks and then the presence of high-level problems and teacher questions in teaching on a five-point Likert scale. Based on teachers' average frequency rating, textbook use patterns from textbook to teaching were analyzed in three phases: (1) problems to problems, (2) questions to questions, and (3) problems in teaching to teacher questions in teaching.
Two noticeable findings emerged. First, there exists relatively a simple relationship between textbook problems and problems used by teachers in teaching. When teachers use problems, they at least maintain the cognitive demand of textbook problems in teaching. In other words, when teachers reported having textbook problems that require higher cognitive level, they also reported that they used problems that require the higher level. Similarly, when teachers reported having textbook problems that require lower cognitive level, some teachers reported using the lower level problems in their teaching whereas other teachers reported that they increased its cognitive level in their teaching. This suggests that the cognitive demand of textbooks play an important role in deciding the cognitive demand of problems used by teachers in teaching.

However, this pattern changed when teachers used questions. Although teachers have textbooks that provide higher cognitive demand of teacher questions, some teachers decrease its cognitive level in teaching. This pattern urged the researcher to examine influential factors on these teachers' decision-making. This study provides implications for policy makers, curriculum developers, professional developers, and teacher educators.

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THE POETICS OF ARGUMENTATION

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In linguistic anthropology, discourse forms that involve repetition of words, sounds, or grammatical units are known as poetic structures. Mathematics students use poetic structures to express deductive and inductive reasoning, and to collaboratively refine mathematical conjectures in classroom discussions (Staats, 2008). This presentation will compare the analysis of students' mathematical statements using poetic structures to methods based on Krummheuer's (1995) and Toulmin's (1958) work on argumentation. These methods share a concern with identifying knowledge construction that may not be expressed in a fully mathematical manner.

Poetic discourse can be identified in the mathematical speech of children, young adults in a university algebra class, and post-graduate mathematics students (Inglis, Mejia-Ramos, & Simpson, 2007; Staats, 2008). Inglis et al report that post-graduate mathematics students used three warrant types as they considered number theory propositions—inductive, deductive, and structural-intuitive warrants. Each of these warrant types were at times expressed using poetic structures. Poetic structures often appeared near the beginning of post-graduate students' verbal reasoning. Modal qualifiers and other logical modifiers that students used as they made their arguments more complex appeared to reduce the level of repetition. These observations suggest that poetic discourse structure is evidence of mathematical argumentation, and that it can occur at many levels of mathematical thought. It appears to be used as a sorting process through which speakers establish comparisons and contrasts among mathematical objects, and it helps speakers articulate the intuitive structures that they develop in the course of their argument. Like gesture, poetic structures may be an informal mode of reasoning that students use before they develop more formal discourses.

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A GROUNDED THEORY APPROACH ON DIFFERENT THEORETICAL PERSPECTIVES

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The discussion about different theoretical concepts within research of mathematics education is still ongoing, but has changed over time. In the beginning, the main question was whether an individual and social perspective on learning could be unified or if their ontological and epistemological foundations were incommensurable. Today, the debate has turned to a more open discussion of combining and coordinating different theoretical perspectives (Cobb, 2007). Different theories are adopted, based on a priori analysis of the suitableness and usefulness of different theoretical perspectives according to a specific research question. In this short oral, another approach is presented. Starting with a real word situation, empirical data is gathered and analysed with methods inspired by grounded theory. The results show how different theoretical perspectives come into play, based on the character of empirical data.

This approach emerged in a research project about the transitions between mathematics studies at secondary and tertiary level from a students' perspective. Based on a definition of the transition as students learning mathematics in a new setting in light of their previous experiences, empirical data from students were collected. Data were analysed with methods inspired by grounded theory (Charmaz, 2006). Conceptual categories and relations between them have been found and how different theoretical perspectives come into play have been examined. Results show that whether the transitions should be characterised as mainly an individual or social phenomenon varies during time, but also between different students. At secondary level, an individual and social perspective on learning seems more converged. At university, a challenge for the students is to cope with the more diverged and various approach on learning as an individual and social phenomenon. How different students manage this seems to be crucial for how successful students manage the transition between secondary and tertiary level. This kind of results would have been difficult to achieve if the theoretical perspectives on the transitions would have been settled from the beginning. Thus, instead of an a priori analyse of which theoretical perspectives should be used, an important part of the results is how the transition should be considered from a theoretical perspective, based on empirical data.

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STUDENTS' PHILOSOPHICAL QUESTIONS ABOUT MATHEMATICS

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This report concerns an inquiry into the nature and character of the questions that US seventh graders have been observed to pose in philosophico-mathematical discussions, and into the patterns of conceptual development associated with those discussions. Drawing from the Philosophy for Children program (Lipman, Sharp, & Oscanyan, 1978)—a distinctive curricular and pedagogical approach designed to lead students toward collaborative inquiry into philosophical questions through narrative texts and communal dialogue—the study analyzes a series of critical discussions focused on certain philosophic-mathematical questions, prompted by a text in which fictional characters explore themes related to philosophy and history of mathematics. All discussions were initiated by questions posed by the students.

METHOD

Data was collected from transcripts of video-sessions of discussions with a group of 20 seventh graders in a small private school in northern New Jersey. The discussions were conducted by a team of two facilitators on a weekly basis over the course of one school year. Each session started with reading aloud from the text in round robin fashion, followed by collecting student questions. The discussions that ensued took up one or more of the questions selected by the group for collaborative inquiry. The questions generated in the first five discussions were compared to the questions generated in the last five. The data was analyzed using a grounded theory approach—that is, categories and patterns emerged during analysis rather than as a result of the application of predetermined categories.

FINDINGS

The findings support previous results of research conducted on Philosophy for Children methodology—i.e., that careful modeling, coaching and facilitation of deliberate communal inquiry into philosophical-mathematical questions and concepts tends to help students generate questions that are philosophical rather than empirical, to target areas concerned with the structure, logic, and limitations of mathematics as a system, and to explore the relation between mathematics and human experience. Thus, we suggest that the introduction of a philosophical dimension into classroom mathematics offers the possibility of a deeper and broader epistemological approach to teaching and learning in the field.

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CONTEXT ALBANIAN STUDENTS PREFER TO USE IN MATHEMATICS (PILOT STUDY)

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In this short oral I introduce a pilot study conducted in Albania. My study is about Albanian students' preferred contexts for learning mathematics. It includes students in grades 8-10. The methods used in this pilot study consist of surveys and interviews. First, 211 students from 6 different classes (68 of them being in grade 8, 70 in grade 9 and 73 in grade 10) filled in a questionnaire containing 23 items. The items represent different contexts that could be used for learning mathematics at school such as for example: mathematics involved in lotteries and gambling, mathematics involved in making computer games, mathematics involved in managing personal and business financial affairs etc. They answered using a 4 point Likert scale, expressing their interest in those items. Then I interviewed 24 students (4 from each classroom) with the aim of exposing the reasons that lie behind their preferences. In the interviews I presented 7 cards with images representing different contexts, and I asked them to order the cards from the most preferred to the less preferred.

The meaning of context used here is what Wedege (1999) refers to as task context. The word 'context' in the dictionary has two meanings, and one of those is the linguistic one: "Those parts of a text preceding and following any particular passage, giving it a meaning fuller or more identifiable than if it were read in isolation" . Close to this meaning, it is the concept of task context, or the context connected to reality in tasks, textbooks, examples, exercises etc. Freudenthal (1991) has two distinct definitions about text and context; both have to do with context in mathematics or task context as Wedege refers to it. Hence, "text means a linguistic vehicle, in particular of word problems or text problems" (p. 69). And "context means that domain of reality, which in some particular learning process is disclosed to the learner in order to be mathematised" (p. 73).

In this presentation I inform the outcomes of my pilot study: the ranking of different contexts according to students' preferences, and the results I obtained during the interviews with some students about their reasons for having certain preferences. It seems that they connect these reasons with the actual conditions and trends of society in general and in particular with their generation's actualities and their hobbies as well.

Interviewed pupils observed this was the first time that someone asks them about what they would like to learn in mathematics, and for some of the items they had never thought before that they have to do with mathematics.

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FIVE YEAR-OLD CHILDREN CONSTRUCT, DECONSTRUCT AND TALK ABOUT PATTERNS – DESIGN AND IMPLEMENTATION OF AN ANALYTICAL TOOL

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Many educators believe that mathematical skills like generalization, abstraction, perceiving relations could be practised in a "pattern environment". Therefore, pattern exploration has been identified as being central in mathematical inquiry and fundamental in children's mathematical growth.

We were interested in studying how patterning can stimulate and shape young children's intuitions of geometrical transformations. The understanding of geometrical concepts has been the focus of a number of theories. But the relation between visual recognition of geometrical objects and actions which can lead to creation of dynamic images of those objects, need further investigation. For this reason we undertook an experiment to observe the role of manipulation in early geometry. The experiment took place in April 2008 in a kindergarten school in Rzeszow, Poland. Our study group consisted of 19 five year-old children. As a research tool we used two types of tiles. Our basic aims were to:

a) examine the extent of children's acquisition of patterns as expressed by their verbal and non-verbal communication, and

b) identify the mathematical concepts that emerged during the sessions.

In order to do so, we needed an analytical tool that would be relevant for our purposes, would account for the whole range of the phenomena observed and would fulfil a certain degree of credibility. The process of identifying, characterizing and classifying the dominant phenomena led us to three sets of categories, which are related to three different phases of the sessions, namely constructing the patterns, describing their construction to another child and deconstructing the patterns.

The results of our study show the significance of patterns as a way to engage children in a communicative activity that can be rich in mathematical concepts. They also show that children at the age of five experience difficulties in expressing verbally the mathematical features of patterns.

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PCK: A CASE STUDY OF ONE UNIVERSITY LECTURER'S TEACHING OF LINEAR ALGEBRA

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Based on the studies by Shulman (1986) I am exploring the *pedagogical content knowledge* (PCK) of a University lecturer in his teaching of linear algebra (LA) to first year mathematics undergraduates at a UK university.

Two researchers (with specialism in mathematics education) attended all lectures and tutorials for the LA module over one semester, twelve weeks. Thirty-three (fifty-minute) lectures and tutorials were audio-recorded, as well as 12 hours of interviews/ discussions with the lecturer. This data collection supported a developmental inquiry approach (Jaworski, 2006) to the study of teaching mathematics at university where the researchers and the practitioner formed a collaborative partnership. Development was not an explicit goal but all were aware of the theoretical framework of the study.

Research questions to date include "Where do teacher explanations come from?" and "How do teachers decide what to teach and how to represent it?" These kinds of question led Shulman (1986) to develop the concept of PCK. Other researchers have elaborated and broadened Shulman's original definition (e.g., Even & Tirosh, 1995). I am using the concept of PCK in the context of university mathematics teaching. For example, I apply Even & Tirosh's 1995 definition of subject matter knowledge (SMK) and knowledge about students as sources for PCK to my study.

Analysis of data has barely begun but suggests that the university mathematics lecturer has excellent SMK, of 'knowing that' and 'knowing why'. The 2nd source for PCK, knowledge about students, which includes common student conceptions and students' way of thinking, appears to be less accessible to the lecturer, mainly due to the large class size, 200+ students. In the interviews the lecturer gives details of how he is trying to access students' mathematical thinking, and how this has informed his teaching of LA. We relate his reflections to what we observe in lectures and tutorials, and seek a characterisation of teaching related to the topic of linear algebra.

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TRANSFER OF LEARNING: A COMPONENT OF THE LEARNING ACTIVITY

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This research suggests a theoretical approach for the transfer of learning within interdisciplinary school activities through the analysis of the subjects' action (13-14 years old students of the Brazilian public basic school) and of the environment where this action takes place. To characterize transfer and its relation with the learning activity in a situated perspective of learning, I take a dialectic viewpoint of the social-educational phenomenon articulating social-historical (Leont'ev, 1978; Lave, 1988) and ecological perspectives (Greeno, 1993). The methodological approach is grounded in the ethnography in education as the logic of investigation for the classroom situations (Green, 2001). In conformity with such theoretical and methodological perspectives, the empirical study helped me to describe transfer of learning as a social and historical practice that composes the *learning activity* (Engeström, 1987) in the mathematics classroom. I exemplify this notion of transfer with classroom situations on resolution of problems involving notions of rational numbers and rules on proportions, as part of an interdisciplinary activity related with the theme Water. I bring in the idea of the expansive cycles to describe the resulting notion of *learning by expanding* (Engeström,1987) originated from the subjects' practice of transfer. The notion of transfer developed in this research is not only a cognitive ability; it is, above all, a practice structured within a social-historical and cultural activity.

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FACTORS LIMITING CONFIGURAL REASONING IN GEOMETRICAL PROOF

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The aim of this research is to improve our understanding and description of pupils' behaviour when they attempt to solve geometry problems requiring a deductive proof, in a pencil-and-paper environment. We have taken as our starting-point the Theoretical Framework proposed by Duval (1998) regarding Visualisation and Reasoning processes. The data for this research consist of the answers given by 55 Primary trainee teachers to a set of 8 geometry problems in a pencil-and-paper environment. The analysis was carried out in two stages. In the first stage we tried to discover what actions of the Configural Reasoning model (Torregrosa & Quesada, 2008) had not been carried out correctly. In the second stage of the analysis we tried to identify the causes of deficiencies in the actions the student was expected to carry out. The analyses obtained enabled us to identify certain regularities which describe Configural Reasoning as a kind of reasoning which does not appear in Duval's work. Configural Reasoning is the process which the pupil uses when coordinating discursive and operative apprehensions in order to solve a problem or generate a mathematical proof. From this viewpoint, we can distinguish three possible outcomes when a geometry problem is approached: truncation, unproven conjecture and closed loop. The existence of these three outcomes in problem-solving raises questions regarding the causes (pupils' acts or omissions) which can provoke the mental-block situations which some pupil experience. More specifically, the question lies in determining what factors enhance or reduce the actions involved in Configural Reasoning: Operative and Discursive Apprehension and the coordination between them.

We have provided examples of a patent lack of the skills required both in using the two kinds of Apprehension and in coordinating them. The analyses reveal how visual predominance tends to inhibit the visualisation of the configuration which is relevant in arriving at the required proof or solution. This factor is also responsible, sometimes to a high degree, for the inoperation of the Operative Apprehensions.

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STUDENTS' INTERPRETATIONS OF PLACE VALUE REPRESENTATIONS IN THE WORKPLACE CONTEXT

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This paper discusses the complex role that workplace representations play on the transfer of learning from school to workplace context. The data came from a study that focuses on students' interpretations of authentic representations. These representations were embedded in everyday work activities of a group of technicians in a telecommunication Organization. Our main goals were to analyse students' mathematical conceptualizations of authentic cultural inscriptions and identify possible links that students' make across academic and workplace representations. Our theoretical position is that mathematics in workplace is embedded in the work context and is mediated through the tools and inscriptions (Noss, Bakker, Hoyles and Kent, 2007). Five students participated in the study for a period of eight months. The students were in the final year of their studies in a Technical Institute and they were doing their practicum in the above Organization. Our main methodological tools were semi-structured interviews and ethnographic observation. A number of these interviews were based on four different authentic situations coming from the particular workplace. One of these situations was related to place value. The base ten-numeration system was used as the structure for classifying wire pairs in a telecommunication closet. The students in their attempt to identify the position of a specific wire pair were challenged to make links among different representational systems, the concrete representation on the closet and the arithmetical one they had learned in the school context. We use the semiotic framework of Davis and McGowen (2001) to study students' interpretations based on Pierce's hierarchical trichotomy: iconic, indexical and symbolic. The results indicate different ways that the students attributed meaning to the concrete representation of the place value. Students' interpretations were mostly indexical as they associated the concrete representation to the arithmetic one they had met in school but without considering relations in the structure of these two systems. Their understanding of the place value from school was at an operational level so they could not transfer the place value structure to the new context. However, there were students who developed the "intermediate" steps required to make links between the two representational systems and provide rich symbolic associations of meanings.

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THE CONTRIBUTION OF "ALGORITHMIC-THINKING" IN DEVELOPING MATHEMATICAL THINKING PROCESSES

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"Algorithmic-thinking" is a powerful mode of problem solving, which is aimed at designing, using and analysing algorithms that solve problems, and not only carrying them out (Armoni & Gal-Ezer, 2006). This research helped to define five mathematical ***thinking processes*** *that were developed by learning"* ***Algorithmic-thinking****" and to specify the interaction between them and students'* ***achievements.***

THEORETICAL BACKGROUND

"Algorithmic-thinking" is a critical factor in executing and directing the problem solving process. Students, who use this way of thinking, obtain mind tools, gain a deeper understanding of the role of mathematics (Brunner, Coskey and Sheehan, 1998), are prepared for higher learning in mathematics', and develop their reasoning and communication skills (Davis & Barnard, 2000).

METHODOLOGY AND RESEARCH FINDINGS

Sixty-seven students in secondary school participated in a quantitative study. The research investigated whether "Algorithmic Thinking" helped learners to develop the requisite thinking processes for solving algorithmic problems, and whether it increased their expertise in the thinking processes that were activated intuitively. These students took mastery tests before and after an intervention program, and their way of thinking and achievements were assessed. The research findings revealed five mathematical thinking processes that were developed by the learning and their branching out to sub-skills. High consistency reliability was found between these thinking processes and skills. In addition, the effect of these thinking processes on the improvement of students' achievements was assessed, and a significant improvement was revealed: $F(5, 62) = 393.39$; $p <0.001$. In the presentation, I shall show these thinking processes and their sub-skills, and I shall discuss some examples from students' work in the research.

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PROOF IN ELEMENTARY MATHEMATICS: A CASE OF ALGEBRA

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Making logical inductive and deductive inferences are keynoted in different reports and research studies as the general goals of the mathematics education. That means that school students need to be exposed to proof not just as the formal process of constructing logically consistent arguments based on axioms, definitions, and theorems but also in a broader set of processes that include argumentation, justification and validation. Extensive works conducted on proving focused on classifying students' approaches along various dimensions (e.g. Bell, 1976; Harel & Sowder, 1998; Marrades & Gutiérrez, 2000). In this corpus of research, little attention has been paid to documenting curriculum approaches to proving. So, this paper concerns the approaches to proving in the elementary mathematics curriculum guidebook. Specifically, I try to respond to the questions like: What does proof mean in elementary mathematics?

All of the activity examples provided in the elementary mathematics curriculum guidebook (grades 6-8) (Talim ve Terbiye Kurulu [Board of Education], January 2008) under algebra content was the data source for this study. A total of 41 activity examples requiring active students' involvement was analysed in order to describe the practices associated with the notion of proof. The activities were categorized by two researchers (each holding doctoral degree in mathematics education) using the proof scheme framework (Marrades & Gutiérrez, 2000) with an agreement of 0.98. The different opinions were further discussed between them and categorized on agreement. Based on the analysis, three categories of *crucial experiment and generic example*: *constructive*, *analytic*, and *intellectual* are present in generating of a conjecture and devising a justification.

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MOTIVATE ME IN MATHEMATICS AND SCIENCE

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To tackle the lack of motivation of pupils in mathematics and science, the authors started a European cooperation[1] *to exchange data and views on pedagogical methods in teaching of maths and physics in European context, bring together and produce a glossary of methods to use with trainee teachers and their tutors, establish a framework by which trainee teachers and their tutors can develop methods, evaluate effectiveness of methods using action research by students and tutors, and disseminate successful outcomes by means of a collection of case studies.*

BACKGROUND

In many countries one can observe a lack of mathematics and physics teachers, as well as a lack of students in mathematics and physics teacher training. School children can hardly be motivated to be interested in maths and physics, partly because of lack of motivating material and because of lack of appropriate pedagogical methods to present such material. Partner universities from five European countries (Austria, UK, Czech Republic, Slovakia, Italy) initiated a cooperation to make trainee teachers and their tutors aware of new pedagogical methods for the learning of maths and physics, and specifically to develop appropriate methods to given materials (developed in an earlier cooperation, available at http://www.PromoteMSc.org), aimed at increasing pupils' motivation.

PROJECT DESCRIPTION

The authors collected methodology that is used in teaching in the partner countries and developed concise definitions of the different methods, taking into consideration already existing definitions in the literature. This glossary of methods has already been published and will be available at the presentation. Partners are currently conducting case studies with teacher trainees and teachers in practice to evaluate how certain methods do fit with or can be made appropriate to the pre-developed materials. Partners also established frameworks in their home institutions allowing teachers and teacher trainees to further develop methodology to this or other pre-determined materials (e.g. textbooks). The results of the case studies, as well as some materials, will also be available at the presentation.

[1] The cooperation was supported by the European Commission under its SOCRATES Comenius programme (ref. No 129572-CP-1-2006-1-AT-COMENIUS-C21). The content of this project does not necessarily reflect the position of the European Community, nor does it involve any responsibility on the part of the European Community.

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DEVELOPING THE ABILITY TO OBSERVE MATHEMATICS TEACHING: RELATIONSHIPS BETWEEN FOCALISATION OF DISCOURSE AND MODES OF INTERACTION

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The goal of the research is to characterise the role of the focalisation of the discourse and the modes of interaction in online debates in relation to the development of the ability in observing mathematics teaching. 23 pre-service secondary school mathematics teachers participated in a teaching method course focused on developing observation skills regarding mathematics learning and teaching. The course followed a b-learning methodology (Callejo, Llinares and Valls, 2008). The web-based multimedia learning environments included analysing segments of mathematics lessons (*video-clips)* using the theoretical reports provided, participating in asynchronous discussions and writing a synthesis report. We adapted Byman, Järvela and Häkkinen's (2005) interaction models in order to analyse participants' contributions within the context of the conversational chains into which they were inserted. This analysis enabled us to identify different learner-profiles in relation to the focalisation of discourse and modes of participation in the online discussion. We describe these profiles using two case studies. The two profiles in the participation of students in an on-line debate show that "focalisation of discourse" and modes of interaction between prospective teachers may be considered two complementary variables in understanding how ability evolves in observing mathematics teaching. In this sense, our results reveal that it is essential to delve deeper into the dialectic between personal and social dimensions of learning.

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ALGEBRA AT RISK IN DUTCH EDUCATION: A PILOT STUDY

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Jacob Perrenet — Eindhoven School of Education

Koeno Gravemeijer — Eindhoven School of Education

In the Netherlands, students entering higher education face problems with algebraic skills. We investigate the development of algebraic skills in pre-university education using two perspectives. From a cognitive psychological perspective, we use an instrument of the cognitive load theory to measure mental effort. From mathematics education perspective, we use distinctions of mathematics in procedural fluency and conceptual understanding. Students are assessed in these aspects by measuring performance and mental effort.

We follow individual students during a year. We have constructed paper and pencil tests for students in grade 8 till grade 11. The tests consist of 12 up to 16 mathematics items. Four schools participate. Half the items aim at measuring procedural fluency (F-items), the other half aims to measure conceptual understanding (U-items). We use scores 0 and 1 to measure performance on the items. After each item, students were asked to report their mental effort invested on the 9-points scale developed by Paas (1992). We use Paas' and Van Merriënboers (1993) efficiency to combine those measures.

First results show that students perform better and experience less mental effort on the F-items than on the U-items. Further, on nearly half of the items, performance does not increase and mental effort does not decrease with the grades. Efficiency scores are higher on the F-items than on the U-items. Also, efficiency does not increase with the grades. We expect the analysis of the whole series of assessments to provide information about where in the curriculum problems arise in the mastering of algebraic skills.

Unexpectedly, algebraic skills do not increase through the grades. A fine-grained analysis of the development of algebraic skills is needed.

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SUPPORTING YOUNG CHILDREN'S SPATIAL STRUCTURING ABILITY

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An instruction experiment was conducted to gain a better understanding of how to support children's (aged 4-6 years) spatial structuring ability for stimulating insight into numerical relations. The process of designing a hypothetical learning trajectory and conducting the instruction experiment highlighted several influences of the instructional setting on children's learning. We specifically discuss the role of the overarching context in connecting the activities' conceptual knowledge.

THE INSTRUCTION EXPERIMENT

Children's ability to grasp spatial structure appears essential for developing numerical insight which underlies (higher-order) mathematical abilities such as ordering, comparing, generalizing and classifying. Examples of spatial structures are finger patterns and domino-dot configurations. The benefit of learning to apply spatial structure to mathematical problems is evident, for instance, when reading off a quantity (i.e., seeing six as three and three), when comparing a number of objects (i.e., one dot in every one of four corners is less than the same configuration with a dot in the centre), when continuing a pattern (i.e., repeating the structure), and when building a construction of blocks (i.e., relating the characteristics and orientation of the constituent shapes). Following the principles of design research, a hypothetical learning trajectory was designed and an instruction experiment was conducted with a kindergarten class to study how young children's spatial structuring ability may be stimulated to support the development of insight into numerical relations.

AN OVERARCHING CONTEXT: ANT AND ITS TOOL BOX

From the observations of the first trial of the instructional sequence, it appeared that an overarching context could help the children make practical and theoretical connections between the activities themselves and between the insights that they gained from participating in the previous activities. An appealing context motivated the children and contributed to creating a shared vocabulary about spatial structuring. As such, we introduced the figure Ant and its Tool Box. Ant appealed to the children, and its Tool Box played an important role in bridging each of the five activities. The Tool Box contained objects that represent basic spatial structures such as finger patterns, large dice (i.e. dot configurations) and egg cartons (i.e. double-structures). The teacher used these objects to guide the children in exploring and comparing spatial structures to make spatial structuring and insight into numerical relations the topic of discussion throughout the instruction experiment.

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USING THE BAR AS A MODEL FOR SOLVING PROBLEMS WITH DIFFERENT REPRESENTATIONS OF RATIONAL NUMBERS

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This communication draws upon the results of a teaching experiment held in one 5th grade class, during two months. The proposal consisted in a sequence of tasks, with several contextualized problems, which gradually covered the five meanings of rational numbers (quotient, part-whole, measure, operator, and ratio), and promoted the use of its different representations. This study aims to show how students: i) use the informal knowledge they possess about rational numbers, as they are solving problems and using the bar as a model, and ii) are starting to use the double number line in the course of this experience, articulating the different representations of rational numbers.

The number bar and the percentage bar are among the models that have been described as enhancers of the development of rational number's concept, in studies under Realistic Mathematics Education theory (Van den Heuvel-Panhuizen, 2003). The work with this model allows students to explore connections between numbers and reinforces the understanding of the relationships between fractions, percentages and decimal representations (Van Galen et al., 2008). In order to provide models that aid students to represent concrete situations, the bar was one of the main models introduced by the tasks proposed, which gradually became transformed in a double number line. The initial results of this study allow to understand how the students, interacting in small groups, begin to use the bar as a model of a situation and how it becomes a model for thinking and solving problems (Van den Heuvel-Panhuizen, 2003). Among the aspects taken into account in the analysis of the results of this study are issues as the sequence of the tasks, the interaction between students and the role of the teacher in institutionalization moments. This research follows the interpretative paradigm, and draws upon the observation of lessons (with video recording) and analysis of written records of students as strategies for data collection.

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BASIC IDEAS AND THE ROLE OF COHERENCE AND DIFFERENCE FOR SECONDARY MATH CURRICULA

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This presentation will advocate a more balanced perspective on the global role of difference and coherence for secondary mathematics curricula from a philosophy of education point of view.

When discussing (students pre-)conceptions and local ideas, elaborating on differences, discontinuities and 'conceptual change' has become common sense in research on science and mathematics education. On a global perspective – when talking 'basic', 'overarching' or 'fundamental' ideas – most research in mathematics education still emphasizes the coherence of mathematical knowledge and stresses the continuity between mathematical thinking and thinking in everyday life. This presentation will address four hypotheses, based on a thorough analysis of the history of "basic ideas" as a research category (Vohns 2007):

- Usual definitions of a 'basic idea' have driven quite far away from everyday use of the term 'idea' and sometimes seem hardly understandable outside the community which uses this concept as a research category.
- Furthermore, mistaking basic ideas behind structures, mathematical concepts and techniques for structures, concepts or techniques themselves is a common issue in research on basic ideas.
- The role of coherence and continuity on a global perspective has been stressed a lot since Bruner (1960). Since then "coherence" and "continuity" have undergone quite some change interpretation wise, but not much has been done to overcome the hypothetical nature of Bruner's claims on the importance of coherence and continuity as an educational goal in itself.
- Therefore, a switch from elaborating on coherence and continuity to reflecting on coherence and differences is suggested and corroborated by taking into account recent works on philosophy of mathematics education from Germany and Austria.

The presentation will conclude presenting "quantity" and "form" as starting points for reflecting on basic ideas for the last four years of secondary education.

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YOUNG CHILDREN’S APPROACHES TO SOLVING CONCEPTUALLY LINKED ADDITION PROBLEMS

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The research of inter- and intra-individual differences in arithmetical development has revealed notable discrepancies and dissociations between procedural, factual and conceptual knowledge, and has supported the view that arithmetical ability involves the variable development of different components (e.g. Dowker, 1998; Cowan, 2003; Canobi, 2005).

The present pilot study explored the conceptual, procedural and factual knowledge that 6-7 year old children of different abilities in arithmetic employ when solving conceptually linked numerical addition problems. The focus of the study was particularly on studying and comparing different children’s ability to use and combine all three aforementioned types of knowledge in their problem solving approaches either spontaneously or after explicit prompt.

Thirty three children of different abilities in arithmetic, as assessed by their teachers, took part in individual problem solving sessions, each lasting 30 minutes. The children were presented with pairs of addition problems which were related by the principles of commutativity, associativity and additive composition.

Children’s approaches to the first of each pair of problems were indicators of their ability to use calculation procedures or factual knowledge to work out the answers. Children’s approaches to the second problem were indicators of their attempt and/or ability to use their conceptual knowledge in order to derive the answer from the first problem solution (erroneous or not) either spontaneously or after an explicit prompt. The paper discusses the similarities and differences observed in the problem solving approaches of an able child and a child who is underachieving in arithmetic. The analysis of these two cases shows notable differences in the mastery and use of procedural knowledge but similar levels of factual and conceptual knowledge accompanied by noteworthy differences in the spontaneous use of these latter two types of knowledge in problem solving.

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WHAT SECONDARY MATHEMATICS MENTOR TEACHERS THINK AND ENACT FOR MENTORING MENTEE TEACHERS

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This paper describes the forms, principles and strategies mathematics mentor teachers instruct mentee teachers' teaching. The case study method, including interviews, observations, and mentor-tutor conferences, was used as the major approach to investigate the mentoring principles and strategies of mathematics mentors. Mentors may intervene in mentees' teaching when practice teaching incidents occur and mentors encounter the conflicts and challenges of their beliefs and values. We are interested in whether mentors actually intervene in teaching of mentees when teaching incidents occurring, and what the underlying principles to enact teaching interventions and mentoring decision-making are.

We found that almost all case mentor teachers adopted the one-to-one mentoring form different from the *co-mentoring* form of mentoring addressed by Jaworski & Watson (1994). However, in our cases, there appeared to be many-to-one mentoring form like co-mentoring. But the depths and extents of their mentoring strategies and behaviours don't match with real co-mentoring, so we call it as *group mentoring*. We classified the forms of group mentoring to 3 categories: *collaboration*, *cooperation* and *separation*. We distinguished the manners of mentors' teaching interventions into three major categories: *active*, *passive*, and *no intervention*. Two subcategories *direct* and *indirect intervention* are also salient within *active intervention* category. And we could find that the main principles of mentors' interventions are *concerning teacher self-esteem*, *caring about students' learning* and *learning to solve problems* and so on. We preliminarily concluded the explicit principles of mentoring to six forms including *doing then gaining*, *active leaning*, *providing resources*, *students' learning*, *accumulating experiences* and *respecting mutually*.

Mentors should be able to clarify the misconception about teaching and mentoring competencies, at the same time, university tutors could appropriately assist mentors to understand and cultivate relevant mentoring competencies. We may view incidents of teaching interventions as the catalysts to advance mentors' mentoring competencies. We expect that both mentors and mentees can learn-to-see in mentoring (Furlong & Maynard, 1995) and empower their own professional growth through the co-learning cycle of teaching and mentoring.

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MODEL BASED CONSTRUCTION OF FRACTION COMPARISON

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Many students have difficulties developing a quantitative notion of fraction in general, and to compare fractions in particular. Research shows that fraction comparison plays a significant role in creating a "fraction sense" (Cramer et al., 1997). We investigate how students use models while constructing knowledge about fraction comparison. Specifically, our students use a computerized model of fraction bars while constructing the “complete-to-whole rule” (CTW): "the smaller the complete-to-whole of a fraction, the bigger the fraction".

Twelve pairs of 4th grade students carried out a sequence of fraction comparison tasks with the purpose of constructing and consolidating the CTW. We used two theoretical frameworks to analyse the process of knowledge construction: Realistic Mathematics Education (RME) and Abstraction in Context (AiC).

RME originated with the Freudenthal group. In RME, the term "model" refers to situation models and mathematical models developed by the students themselves. At first, the model is used as Model-Of a situation (MO) that is familiar to the students. By a process of generalizing and formalizing, the model may become an entity on its own and be used as Model-For mathematical reasoning (MF) (Gravemeijer, 1999). We analyse the role models play in students’ mathematical reasoning about the CTW.

AiC allows the researcher to trace processes of knowledge construction throughout a succession of activities by identifying the epistemic actions of Recognizing, Building-with and Constructing (Schwarz, Dreyfus, & Hershkowitz, 2009). Constructing actions are central during the stage of emergence of new knowledge. We analyse students’ epistemic actions when constructing the CTW.

Our dual analysis of the transcripts revealed a close connection between the MO to MF transition (RME) and Constructing Actions (AiC), such as combining model-based visual reasoning with numerical reasoning while constructing the CTW.

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FIRST-GRADE TEACHERS' AWARENESS ON DISADVANTAGED STUDENTS' MATHEMATICAL LEARNING DIFFICULTIES

Wu, Su-Chiao (Angel) Chang, Yu-Liang (Aldy)

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The integrity and fulfillment of in-service teachers' professional development have a great influence on how they re-obtain and re-advance their own content and pedagogical content knowledge, as well as their awareness on students' mathematical learning difficulties. Especially the aspect of teacher's awareness, it is an essential element of how a teacher recognizes students' learning problems instantly and correctly and how to solve them effectively in the whole teaching process. However, few studies have looked at teachers' awareness on disadvantaged students' mathematical learning difficulties. Besides, the beginning period (i.e. first grade) of students' mathematical learning is so important that will influence their future learning attitude and achievement in mathematics. Thus, the purposes of this study are to explore: 1. the circumstance of how first-grade teachers are aware of disadvantaged students' mathematical learning difficulties; 2. possible elements that influence teachers' awareness. A theoretical framework extracted from the Gestalt psychology and its definition of "awareness" is used in this one-year case study. A single-case holistic design is employed. Participants were four first-grade mathematics teachers (and their twenty disadvantaged students—students with low socio-economic status and of new immigrant and low mathematical achievers) purposefully selected from an elementary school in Chiayi, Taiwan. Data were collected through intensive interviews, classroom observations, teachers' instructional portfolios (e.g. homework and test), and students' learning portfolios (all documents graded by teachers), and then analyzed by the editing analytic technique. Based on the analysis of the findings, a continuous process of how these teachers are aware of disadvantaged students' mathematical learning difficulties is composed of three elements: analysis, diagnosis, and action. "Analysis" is to analyze students' development and learning capabilities and discover their mathematical learning difficulties (cognitively or psychologically), which is based on interactions within and outside of classrooms and the process of grading students' learning portfolios. Diagnosis" is to diagnose possible reasons why these difficulties are formed, which is mainly from individual interviews with students and parents. "Action" is to select and execute corresponding instructional strategies and/or counselling plans to assist their mathematical learning and/or advance their psychological state. In addition, possible factors that influence these first-grade teachers awareness on disadvantaged students' mathematical learning difficulties are as followings: teaching methods, questioning skills, and classroom management strategies used, knowledge of learners and of strategies for psychological counselling, mathematical content and pedagogical content knowledge, and capability of self-reflection. Affected by these factors, multiple paths exist in this aware process for these teachers.

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THE INTERACTION AMONG DRAGGING ACTIONS, DYNAMIC REPRESENTATIONS AND THOUGHT EXPERIMENTS IN DYNAMIC GEOMETRY ENVIRONMENT

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In recent years, many studies begin to focus on how actions of dragging affect students' reasoning or explorative processes. Arzarello et al. (2002) argued that the action of dragging supports the production of conjectures: users can explore and observe the change when dragging the figures, which in turn allows them to discover the invariant properties of objects. The main actions in DGE are produced by dragging geometric objects with mouse and forming the dynamic change of geometric objects. Dynamic representation is produced by dragging and the dynamic behavior (DB) of objects. DB is not ignored when we explore the influence of dragging on learning.

However, individual mathematical thinking has an effect on dragging because dragging is produced by individual. Tall(1996) argued that although modern computers provide an enactive human interface with manipulable visual display and symbolic facilities, it still needs the mind of a mathematician to perform thought experiments to decide what is important and what needs to be proved. Therefore, individual internal thought experiments also need to be considered.

The aim of this study is to analyze and explore the interaction among dragging actions, dynamic representations and internal thought experiments when college students explore geometry with dynamic geometry environment (DGE). A semi-structured interview was the main methodology of the research, and the subjects were two college students majoring in mathematics. Sound and video recordings were both performed to collect the original data, and a qualitative analysis was conducted to interpret the interaction between these three factors.

From the result, we know that the three aspects need to be considered when analyzing the explorative process in dynamic geometric environment. One important thing is the influence of dynamic representation on students' internal thought experiments. Because different dynamic representations are produced by different drag modalities and every student has different interpretations for dynamic representations. Whether or not it encourages students to do thought experiments, find geometric properties, and even make proof, are related to their mathematical knowledge. We suggest that We should do more case studies case to clarify what drag modalities are produced by thought experiments and what dynamic representations affect internal thought experiments.

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VALIDATING A CONDITIONAL STATEMENT: ROLE OF EMPIRICAL EXAMPLES

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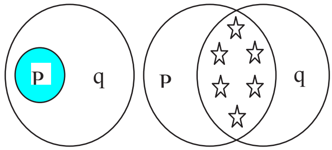
Research has reported students' different misconceptions of deduction. They are incorrect use of empirical examples in making mathematics proofs (Chazan, 1993), the misunderstanding of p➔q and q➔p as the same ones, and the confusion of the independent relations between validity of conditional statements and truth of premises and conclusions (Yu, Chin, & Lin, 2004).



This study proposes a model (see diagram) for interpreting students' behaviours with respect to deduction. We hypothesize that students use two types of rules to infer conditional statements p➔q. We also imply that students' misconceptions described above may also originate from the perception of empirical examples. As the above diagram shows, the left vine diagram refers to the situation that students use logic rules to verify conditional statements. The vine diagram demonstrates students' inferences of p➔q by examining the relations between the premise p and conclusion q. Students may also recognize that the deduction means to check whether or not the premise p is a subset of q. On the other hand, the vine diagram at right side illustrates students' behaviours in judging p➔q by checking the empirical examples in the intersection of sets p and q. If students verify p➔q by checking the empirical examples of the intersection, they will view p➔q and q➔p as the same statements because the intersection of sets p and q are the same. These students may also conclude a conditional statement as "uncertain" (sometimes correct and sometimes incorrect) if they find both supportive and counter examples.

We also designed three incorrect conditional statement items to examine the proposed model. These items are constructed with the same mathematics content but the empirical examples one can find from these items are different. Our survey results showed that students' performance was significantly related to the empirical examples existing in the conditional statements. Based on the survey results, we further suggest the proposed model should be used as guideline to design instructional intervention for improving students' understanding of deduction.

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AN INVESTIGATION OF STUDENTS' CREATIVITY AND MATHEMATICAL PROBLEM POSING IN CHINA

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Although it is widely accepted that high problem posing abilities are highly related to creativity and problem posing activities will help to foster creativity, the correlation has been found in only a few fields, such as art, but not investigated in mathematics. This study examined the relationship between creativity and mathematical problem posing abilities in the context of Chinese culture, using the Torrance Tests of Creative Thinking (TTCT) and a mathematical problem posing test. 135 students were selected from two locations, in a large city and a smaller northern city.

Despite the mathematical achievements of Chinese students in international comparisons, educators in China have maintained that Chinese students lack creativity in mathematics, although there is a prevalent philosophy of "teaching with variation." According to Silver (1997), inquiry-oriented mathematics instruction that includes problem solving and problem posing tasks and activities can assist students in developing more creative approaches to mathematics, including core dimensions of creativity, namely, fluency, flexibility, and originality (Torrance, 1966). Therefore, the purpose of this study is to investigate the relationships between creativity and mathematical problem posing abilities of mathematically strong high school students in the context of two high school Chinese cultures:

1. Is there a significant relationship between creativity and mathematical problem-posing abilities of grade 12 students in two Chinese locations?
2. Are there differences in the creativity and mathematical problem-posing abilities and thinking of high school students in the two locations?

Data generated using the Verbal and Figural parts of the TTCT (Torrance, 1966) and a mathematical problem posing test created for this research, were compared using descriptive statistical methods, and the results were augmented by data from interviews with selected students. The preliminary results suggest that there are differences in the two samples and that the research design has the potential to provide significant answers to the two research questions. It is anticipated that full results will be available for the presentation.

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REVISITING CANTOR'S DIAGONAL METHOD: A CASE OF MATHEMATICAL KNOWLEDGE IN USE

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Mathematical knowledge used in teaching has attracted interest of many researchers, but was mainly explored considering teaching at the elementary school level. This report attends to mathematical knowledge used in teaching at the University level.

We present a story about a student suggesting reconsideration of Cantor's diagonal method and the relationship between real and natural numbers. Our story is situated in the course 'Foundations of Mathematics' offered in a Master's program in Secondary Mathematics Education that is designed for practicing secondary mathematics teachers. The goal of the course was to introduce students to several fundamental ideas and 'big theorems' in mathematics, which either were long forgotten or were not encountered in students' undergraduate studies. The idea of infinity and Cantor's method of corresponding infinite sets (e.g. Dunham, 1990) were among the fundamental ideas and big theorems explored in this course.

Following a discussion of different infinities and the theorem that real numbers and natural numbers have different cardinalities, one student suggested a correspondence between sets of real and natural numbers that he argued was one-to-one, as follows:

> Start with numbers that have only one (non-zero) digit after the decimal point, and correspond them to the first nine natural numbers, that is: 0.1→1, 0.2 →2, … 0.9 → 9 . Then look at numbers with 2 digits after the decimal point, avoiding those with 0 at the end, and correspond them to the natural numbers from 10 to 99:
>
> 0.01→10, 0.02 →11, … , 0.09 →19, 0.11 →20, … , 0.99 →99
>
> Then take all the numbers with 3 digits after the decimal point, avoiding those with 0 or 00 at the end, and correspond them to the next 'bunch' of natural numbers, and so on. This method presents an 'ordering' of real numbers, and so their cardinality is $\aleph_0$.

This suggestion created an initial experience of 'disturbance' (Mason, 2002) for the instructor. We shall present his teacher's response to this unexpected classroom situation and analyse it through the lens of 'knowledge for teaching' (Ball & Bass, 2000). Our contribution is in extending research on teachers' knowledge and on students' struggle with ideas related to infinity.

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