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Research Reports

Research Reports
Eli - Lee

A STRUCTURAL MODEL OF PRIMARY SCHOOL STUDENTS' OPERATIVE APPREHENSION OF GEOMETRICAL FIGURES

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This study investigated the role of various aspects of figure modification proposed by Duval (1995), i.e., mereologic, optic and place way, on the operative apprehension of geometrical figures. Data were collected from 125 sixth graders. Structural equation modelling affirmed the existence of three first-order factors indicating the differential effects of the types of figure modification and a second-order factor representing the operative apprehension. The findings revealed interesting differences in students' performance on mereologic, optic and place modification tasks. Furthermore, students displayed greater consistency in applying the mereologic and the optic figure modification rather than the place figure modification. Teaching implications and suggestions for future research are also discussed.

INTRODUCTION AND THEORETICAL FRAMEWORK

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one because in a figure various relations of an object with other objects are depicted (Mesquita, 1996). However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998). Geometrical figures are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Therefore, the double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (e.g. Mesquita, 1998).

Duval (1995) distinguishes four apprehensions for a "geometrical figure": perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its

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specific laws of organization and processing of the visual stimulus array. Particularly, perceptual apprehension refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. Sequential apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. Discursive apprehension is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through operative apprehension that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one makes the figure larger or narrower, while the place way refers to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. In a problem of geometry, one or more of these operations can highlight a figural modification that gives an insight to the solution of a problem.

Even though previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Mesquita, 1996; Kurina, 2003), the cognitive processes underlying the four apprehensions for a "geometrical figure" proposed by Duval (1995) have not been empirically verified yet. Recently, Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2009, in press) have confirmed a three level hierarchy about the role of perceptual, operative and discursive apprehension in geometrical figure understanding. In this paper, we present a part of a study that focused on analyzing the cognitive processes underlying the various kinds of geometrical figure apprehension. In particular, this study investigates the role the mereologic, the optic and the place modifications exert on operative figure understanding of primary school students. This knowledge may be useful in understanding students' operative apprehension processes of geometrical figures and in providing teaching implications for the improvement of students' geometrical understanding.

Specifically, drawing on Duval's (1995) theoretical model, the study sought answers to the following three research questions:

(1) To what extent does students' performance on geometry tasks reflect the structure of the theoretical model delineated above, and especially the differential effects of the

various types of geometrical figure modification, that is, the mereologic, optic and place way on the operative figure understanding?

(2) Are there any differences in students' performance on using each of the three types of geometrical figure modification?

(3) How consistently do the primary school students apply each of the three types of geometrical figure modification?

METHOD

The study was conducted among 125 students, aged 11 to 12, from primary schools in Cyprus (Grade 6). The a priori analysis of the test that was constructed in order to examine the research questions of this study is the following:

1. The first group of tasks includes task 1 (M1), 2 (M2) and 3 (M3) concerning students' mereologic way of modifying a given figure.
2. The second group of tasks includes task 4 (O4), 5 (O5) and 6 (O6). These tasks examine students' optic way of modifying a given figure.
3. The third group of tasks includes task 7 (P7), 8 (P8), 9 (P9) and 10 (P10) that correspond to the place way of modifying a given figure.

Representative samples of the tasks used in the test appear in the Appendix. Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students' answers to the tasks were codified with M, O and P corresponding to mereologic, optic and place way, respectively, followed by the number indicating the exercise number.

In order to explore the structure of the various operative apprehension dimensions a second-order confirmatory factor analysis (CFA) model was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > 0.9, RMSEA < 0.06. Paired *t*-tests were used to examine whether there were significant differences in students' performance on each of the three operative apprehension dimensions. Finally, the hierarchical clustering of variables (Lerman, 1981) was conducted using the statistical software C.H.I.C. (Bodin, Coutourier, & Gras, 2000). Thus, a hierarchical similarity diagram of the sixth graders' responses to the tasks of the test was constructed. The similarity diagram allows for the arrangement of the tasks into clusters according to the homogeneity by which they were handled by the students.

RESULTS

Figure 1 presents the results of the elaborated model, which fitted the data reasonably well [x^2(32) = 35.228, CFI = 0.961, RMSEA =0.029]. The coefficients of each factor were statistically significant. The errors of variables are omitted.

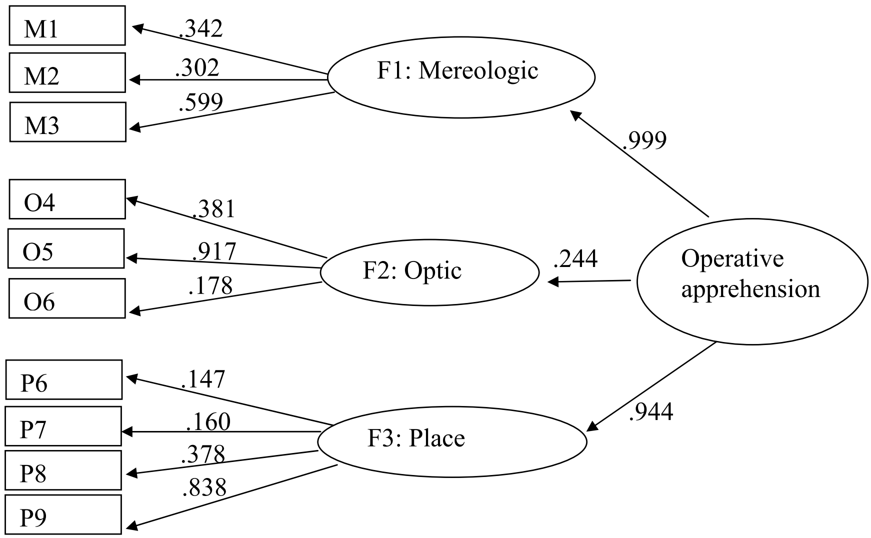


Figure 1. The CFA model of operative apprehension

The second-order model which is considered appropriate for interpreting operative apprehension, involves three first-order factors and one second-order factor. On the second-order factor that stands for operative apprehension the first-order factors F1, F2 and F3 are regressed. The first-order factor F1 refers to the tasks which correspond to the mereologic way of modifying a given figure, the first-order factor F2 refers to the optic modification tasks and the first-order factor F3 refers to the place modification tasks. The factor loadings reveal that the mereologic and place types of modification are the primary source explaining students' operative apprehension of geometrical figures. That is, they are highly related to the operative apprehension. However, as concerns the first research question, the results indicate that all three ways of modifying geometrical figures have a significant effect on operative figure understanding.

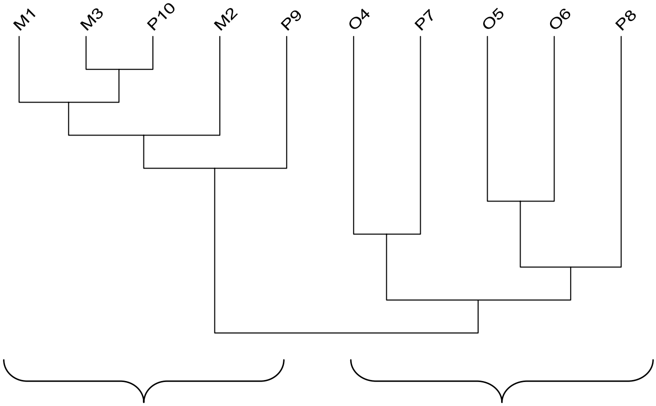
Regarding the second research question, we used the set of tasks that loaded uniquely to each of the three factors under consideration (i.e., mereologic, optic and place modification of geometrical figures) to calculate students' performance on each of the three figural modification types. Table 1 presents the means and standard deviations of students' performance on each modification type of geometrical figures. As can be seen in Table 1, students' performance on the place modification tasks ($\overline{X}$=0.66, SD=0.24) was higher than their performance on the optic modification tasks ($\overline{X}$=0.61, SD= 0.26), but the use of the t-criterion for paired samples revealed that this difference was not statistically significant ($p>0.01$). In contrast, students' performance was significantly lower on the mereologic modification tasks ($\overline{X}$=0.28, SD= 0.29) than their performance on the other two types of modification tasks ($p<0.01$).

Types of geometrical figure modification	$\overline{X}$ *	SD
Mereologic	0.28	0.29
Optic	0.61	0.26
Place	0.66	0.24

Table 1: Mean scores and standard deviations on each type of figure modification

*maximum score=1

Figure 2 presents the similarity diagram of the sixth graders' responses to the tasks of the test. Two similarity clusters are identified. Cluster 1 involves students' responses to all the mereologic modification tasks (M1, M2, M3) and two of the place modification tasks (P9, P10). Cluster 2 is comprised of students' responses to all the optic modification tasks (O4, O5, O6) and the other two place modification tasks (P7, P8).



Cluster 1 Cluster 2

Figure 2. Similarity diagram of sixth graders' responses to the test

Thus, concerning the third research question, the two clusters suggest that students displayed consistency in applying respectively the mereologic way and the optic way of modifying geometrical figures. This is not the case, though, for the place way of modifying geometrical figures, as the responses of the students to the four corresponding tasks are split into the two existing clusters. A number of the place modification tasks were approached similarly to the mereologic modification tasks, while the rest of the place modification tasks were tackled similarly to the optic modification tasks.

DISCUSSION

This study investigated the role of various aspects of modifying a given figure, i.e., mereologic, optic and place way, in the sixth graders operative apprehension. Structural equation modelling affirmed the existence of three first-order factors indicating the differential effects of the ways of figure modification and a second-order factor representing the operative apprehension. This finding lends support to Duval’s (1995) conceptualization of the cognitive processes underlying operative figure understanding and suggests that, in order to develop operative apprehension during mathematics instruction in primary school, emphasis should be given on the three types of figure modification, which provides figures with a heuristic function.

Furthermore, the results revealed that differences existed between the students’ performance in mereologic, optic and place modification tasks. Particularly, the sixth graders performance on the place modification tasks was similar to their performance on optic modification tasks. In contrast, their performance on the mereologic modification tasks was significantly lower than the other two types of modification tasks. The weak performance on these tasks may have been caused by the fact that they required more complex figural processes relative to most of the other tasks. That is, the students needed to understand the division of the given figure into parts and their combination in another figure and proceed to calculations of specific areas (e.g. M3) or estimations of the figures’ perimeter (e.g. M2) in order to provide a solution to the corresponding tasks.

The similarity diagram showed that students’ consistency varied across the three types of geometrical figure modification. Whereas students exhibited consistency in the mereologic modification tasks and the optic modification tasks respectively, they applied the place way of modifying geometrical figures in a rather fragmentary way. A number of the place modification tasks (P9, P10) were approached similarly to the mereologic modification tasks, and the rest of the place modification tasks (P7, P8), were tackled similarly to the optic modification tasks. This finding suggests that although it is the place modification that gives insight to the solution of the corresponding tasks (Duval, 1995) some additional operations need to take place so that students successfully reach the ultimate solution. These additional operations may have common characteristics with the figural processing which is required in either the mereologic modification tasks or the optic modification tasks. Specifically, in the first case, the place modification tasks P9 and P10 did not require only the understanding of the position or orientation variation of the figures, but also the combination of figures in another figure (reconfiguration), which is a characteristic of the mereologic type of geometric figure modification. Moreover, both mereologic and place modification tasks (P9, P10), involved measurement or estimation concepts (e.g. perimeter) and processes in addition to the spatial processes. In the second case, both optic and place modification tasks entailed principally spatial skills and specifically the comparison of figures of the same form which differed either in their position and orientation because of rotation (P7, P8), or in their magnitude (Fischbein

& Nachlieli, 1998), because of enlargement (O4) or variation of distance from a reference point (O5, O6).

Finally, it seems that there is need for further investigation into the subject. It would be theoretically interesting and practically useful to examine whether the validated structure of geometrical figure operative understanding and students' deficiencies revealed in this study, that is, the difficulty students met in the mereologic modification tasks and their limited consistency in applying place modification tasks, remain invariant with development and learning at school. The effects of intervention programs, which aim to develop students' abilities in modifying a figure, on the operative and other apprehensions for geometrical figures or in geometry problem solving, could also be investigated in future studies.

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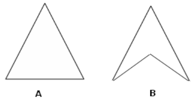
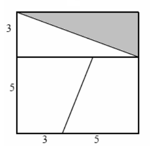
APPENDIX

1. Underline the right sentence: (M2)



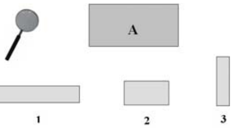
a) Fig. A has bigger perimeter than Fig. B
b) Fig. A has equal perimeter with Fig. B
c) Fig. A has smaller perimeter than Fig. B

2. This figure is a square.

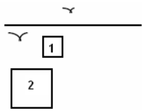


Calculate the shaded area. Explain your answer. (M3)

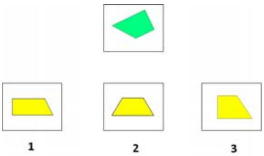
3. Vassilis constructed a rectangle in his writing book. Shape A is the rectangle as it looks through a magnifier. Circle the picture that shows the rectangle, as it is in Vassilis writing book. (O4)



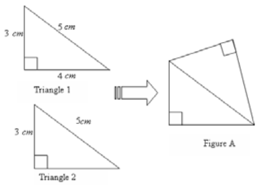
4. Paris is looking the box 1 and 2 in the horizon. He says that the box 1 has exactly the same size with box 2. Is his opinion right? Explain your answer. (O5)



5. Maria must match the cards with the same shape. Circle the yellow card that has exactly the same shape with Maria's card. (P7)



6.Theodosis combines Triangle 1 and Triangle 2 making Figure A. Calculate the perimeter of Figure A. (P9)



THEORETICAL BASES IMPLICIT IN THE ABBACI AND CIPHERING-BOOK TRADITIONS 1200-1850

Nerida Ellerton — Illinois State University
M. A. (Ken) Clements — Illinois State University

Mathematics education practices are very different today from 200 years ago, when the predominant modus operandi was the so-called "ciphering-book approach". In this paper we explore theoretical bases of a tradition that defined mathematics education practices in Europe, and subsequently also in America, for over 600 years. The ciphering tradition emerged from European Abbaci traditions, and was refined in the late 17th century by a committee which included Sir Isaac Newton, so that it applied to the teaching and learning of mathematics associated with navigation and surveying. After analysing 172 ciphering books prepared between 1695 and 1880 and originating from 8 different nations, we identified a theoretical base for the ciphering tradition. Here we show how this base can be linked to Abbaci traditions.

INTRODUCTION

This paper begins by defining a "ciphering book" and then draws attention to the influence of the European *Abbaci* tradition on mathematics education practices during the period 1200-1850. Three questions which informed the research are then presented, and the principal data set that was analysed in order to answer those questions is described. Then follows a brief analysis and summary of a theoretical model on which the ciphering book tradition appeared to be based. Finally, the three research questions are answered.

DEFINITION OF A CIPHERING BOOK

During the period 1200-1850 mathematics teaching and learning was usually consistent with what came to be known as the "ciphering approach". We begin by defining a ciphering book as a book with the following properties:

1. All entries appeared as handwriting in ink, or as illustrations (which sometimes had ink outlines, and sometimes incorporated water colours).
2. The book was dedicated to setting out mathematical definitions, rules, cases, model (or "type") problems and their solutions, and exercises with solutions prepared by the person who owned the ciphering book. The book could be concerned with problems from arithmetic, especially in relation to business, from algebra, geometry, trigonometry, or with the application of mathematics to navigation, surveying, military strategy, etc.
3. The content was sequenced so that it became progressively more difficult. It also took account of the expectation that no child less than 10 years of age was normally assigned the task of preparing a ciphering book.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 9-16. Thessaloniki, Greece: PME.

4. Headings were prepared using calligraphic printing or writing.

Most of the 172 ciphering books in the principal data set summarised later in this paper dealt with just one branch of mathematics (e.g., with arithmetic or algebra or geometry or trigonometry, etc.), but some had entries from two or more of these areas. For much of the period 1200-1850, ciphering books, or their predecessors, the *abbaci*, were usually made up of unlined rectangular folio-sized paper sheets (in the 18th century "rag" paper was often used) with dimensions about 29 cm by 24 cm.

Although the concept of "completeness" is not included in the above definition, ciphering books usually provided complete treatments of several topics. The treatment of any one topic occurred on successive pages. Ciphering books were intended to serve as reference manuals for subsequent use, in later life, by owners.

LITERATURE REVIEW: THE HISTORICAL BACKGROUND

Leonardo Pisano's *Liber Abbaci*

The ciphering approach to mathematics education probably began around 1200 with Leonardo Pisano, better known among mathematicians as "Fibonnaci" (Karpinski, 1929). Leonardo's *Liber Abbaci* was a compendium of all known mathematical practices of his day. Its contents were drawn both from personal experience and from earlier Arab texts on commercial mathematics, algebra, and geometry. It became a primary source from which Europe learned how the Hindu-Arabic numeral system was applied in day-by-day affairs (Van Egmond, 1980).

The first section of *Liber Abbaci* described the Hindu-Arabic numeral system, based on numerals 0, 1, 2, …, 9, and on algorithms for addition, subtraction, multiplication and division. The second section discussed examples from commerce, such as currency conversion, measurement, profit and loss, and interest calculations. In the third section, Pisano turned to theoretical problems such as a Chinese remainder theorem, a rabbit breeding problem, perfect numbers, Mersenne primes, arithmetic series and square pyramidal numbers. The fourth section offered methods for approximating irrational with rational numbers. *Liber Abbaci* also commented on Euclidean geometric proofs, and on simultaneous equations (Gies & Gies, 1969).

The Emergence of the Italian Abbaco Tradition

The so-called *trattati* or *libri d'abbaco* are believed to have emerged from *Liber Abbaci* (De Morgan, 1847; Van Egmond, 1980). They were Italian pedagogic manuals of commercial mathematics and geometry widely used in Italian reckoning schools during 1200-1600 (Karpinski, 1929; Smith, 1908; Swetz, 1987, 1992). From the thirteenth century, Italian city-states formed vernacular schools in which commercial mathematics, accounting and writing were taught. Most problems in *trattato dell' abbaci* manuscripts related to payment of merchandise, commercial partnerships, weights and measurements, money exchange, etc. During the 14th and 15th centuries the *abbaco* genre swept through the counting houses of Continental Europe. According to Swetz (1987), in the early 17th century there were almost 50 reckoning schools in the German city of Nuremberg alone, and in each of these students worked at creating *abbaci*. In Italy the teachers were called *maestri d'abbaci,* in France, *maistre d'algorisme*, and in German territories, *rechenmeister* (Swetz, 1987). During the 17th century the tradition made its way to Great Britain and to North America. Early *abbaci* featured a rhetorical genre, with entire problems being written out in words. Thus, a writer would state "divide 35 by 7 giving 5, and add 3 giving 8", etc. Computations were *not* displayed, but were included within a single block of text so that they appeared in sentences. Towards the end of the fifteenth century, however, *abbaci* writers began to display calculations (Van Egmond, 1980). At all times, solutions to model problems were given immediately after problems were stated. The *abbaci* offered solutions to typical problems and were something that students often used later in life when they became full-time practitioners.

Van Egmond (1980) has provided comprehensive analyses of about 300 Italian *abbaci*. He argued that without business problems a manuscript was not an *abbacus* book. Typical *abbacus* books had between 100 and 250 pages of text and presented and discussed problems that had numerical, and especially place-value, connotations. These *abbaci* are to be distinguished from *abacus* boards, which were physical devices on which reckoning was carried out. Although reckoning masters and students often used *abbacus* books and *abacus* boards in the same transactions, there was no connection between the two.

Van Egmond (1980) identified different content categories content in *abbacus* books. Abbaci typically began by summarising the Hindu-Arabic numeration system and place-value, the four operations on counting numbers and quantities (including money). This was followed by vulgar fractions and the so-called "rule of three". Then came elementary business problems – from simple calculations within one currency, to finding the price of a certain commodity using the "rule of three", to converting different currencies and weights and measures. More complex applied areas followed – like loss and gain, discount, barter, fellowship, interest, equation of payments, annuities, alligation, rule of single false position, rule of double false position, gauging, and tare and tret. Then might follow recreational and geometric problems.

According to Van Egmond (1980), the quality of penmanship, calligraphy, and illustrations in the handwritten *abbaci* that he studied varied considerably. The most elaborate were colourfully decorated with the most exquisite penmanship, calligraphy, and diagrams. These were prepared by professional writing masters and sold to wealthy merchants or to students from well-to-do families. They were usually well organized from a content point of view. Students who were serving apprenticeships as reckoners, and would be expected to refer to their manuscripts for the rest of their lives, tended to take more care, and put more effort into their books than students who somehow found themselves receiving lessons from reckoning masters but never intended to make large use of reckoning skills in the future.

By the end of the 16th century the *abbaco* curriculum, combined with an algorithm-dominated ciphering approach, was widely used within Europe (Yeldham, 1936).

Business leaders wanted arithmeticians who could apply algorithms. They wanted to quantify what had happened, what was happening, and what might happen in every facet of their businesses. The algorithm practitioners had a rule, and a case, for every situation. Reckoning masters taught students important definitions, rules and cases and exemplary solutions to “type” problems in applied contexts (Swetz, 1987).

In the 16th century, many children in European nations like Italy, Holland, France, and Prussia were taught a ciphering approach to mathematics education in their community schools (Karpinski, 1929). Most children in Calvinist or Lutheran families were expected to own a “ciphering book”. Kool (1999) analysed the content of 36 15th and 16th century Dutch-language manuscripts and found that students were well trained to use the Hindu-Arabic numeration system and significant other mathematics to solve practical and recreational problems. Mathematics education in Holland was clearly based on a ciphering approach, and such was the standard that the Englishman John Child (1693), Governor of the East-India Company, argued that the superior economic performance of the Dutch nation could be attributed to the high average level of mathematical competency in the Dutch population. Given the outstanding mathematical achievements across Europe in the 17th century by scholars like Desargues, Descartes, Fermat, Galileo, Huygens, Kepler, Leibniz, Napier, Newton, and Pascal, the systems of mathematics education that generated such scholars are worthy of study. Since, by the middle of the 17th century most of the North American settlements had come to be under the control of England, the mathematics education situation in England is of particular interest.

The Royal Mathematical School, Christ’s Hospital, London

In 1677, John Newton, a strong advocate for the introduction of more mathematics into the curricula of English grammar schools (Taylor, 1954), asserted that he did not know of any grammar school in England in which mathematics was taught (Jones, 1954). The situation improved towards the end of the 17th century, however, with the Royal Mathematics School at Christ’s Hospital in London playing a major role.

Perhaps the finest ciphering book of those that we examined was produced by Charles Page, who attended Christ’s Hospital in the mid-1760s. According to Jones (1954), Christ’s Hospital was the first English school in which mathematics was made an important part of the curriculum. In 1673, King Charles II, and his advisers, recognizing that sea-power was crucially important, not only for national status but also for colonial expansion and for mercantile development, started a mathematical branch for 40 boys aged from about 10 years, at Christ’s Hospital. These boys were supported by royal finances, and a high-level advisory committee comprising the nation’s top scientists, including Sir Isaac Newton, developed the curriculum. After much trial and error a program based on a “ciphering approach” was put in place.

At a time when sea power was vitally important to nations (Mahan, 1898), the Royal Mathematical School in Christ’s Hospital quickly became the nation’s leading institution in applied mathematics, especially in relation to navigation. The program relied on two main inputs: first, a printed text prepared by a master; and second, a ciphering-book approach, whereby every boy prepared a handwritten record of his school mathematics. The content was designed to prepare students for the Royal Navy, or to become apprentice navigators or reckoners for private merchants. All graduates were required to serve as apprentices for seven years (Trollope, 1834). Christ's Hospital became known as one of the world's best training schools for navigators. As early as 1697, Peter the Great, Czar of Russia, visited the School and appointed two graduates (one was 15 years old, the other 17) to lead a training school for navigators in Russia (Cross, 2007). The influence of Christ's Hospital would be felt in North America because, after completing their apprenticeships, some graduates became schoolmasters in the New World (Coldham, 1990).

The ciphering book created by Charles Page (1764) at Christ's Hospital had 671 handwritten pages (each 29 cm by 24 cm). It included sections on Arithmetic, Algebra, Euclid, Trigonometry, Navigation, Oblique and Globular Sailing, Spherics, Astronomy, Globes, and a "Journal of a Voyage from England towards Madeira". The last section was *not* based on an imaginary journey, for Christ's Hospital sent students on *actual* journeys. An extraordinarily high standard of penmanship and calligraphy was demanded for the ciphering books. With Christ's Hospital as its model, many other schools in England (e.g., Greenwich Hospital and the Plymouth Workhouse) taught advanced arithmetic, algebra, geometry and trigonometry and trained students for apprenticeships in surveying and navigation (Watson, 1913).

THREE RESEARCH QUESTIONS, AND THE PRINCIPAL DATA SET

Research Questions

Because of the fact that the *abbaci* and ciphering-book traditions played such an important role in mathematics education, for more than six centuries, we decided to seek answers to the following three research questions:

1. What were the main features of these traditions?
2. Why did the *abbaci* and ciphering-book traditions continue to exert such a powerful influence on mathematics education, for so long?
3. Did the *abbaci* and ciphering-book traditions have a strong theoretical base?

Principal Data Set

The ciphering-book collection that we examined, manuscript by manuscript, page by page, comprised 172 handwritten manuscripts. Most of them originated in Europe or in North America between 1695 and 1890. Of the 172 manuscripts, 120 were prepared in North America between 1702 and 1862. The two oldest ciphering books were prepared in France in 1695 and 1696. Most of the 172 ciphering books were written in English, but nine were written in Japanese, seven in German and four in French. Twenty-two of the manuscripts originated in England or Scotland. Although most of the ciphering books came from different schools or institutions, there was no expectation that they were representative of nations, or schools, etc.

A THEORETICAL MODEL FOR THE CIPHERING APPROACH

There can be little doubt that around 1700 governments, and mathematicians, in many European nations were convinced that the ciphering-book approach provided the most effective mathematics education possible. It derived from the influential, powerful, *abbaco* tradition by which reckoning masters required their students to follow a well defined sequence that depended crucially on model problems, with solutions being recorded accurately and legibly in ciphering books, for future reference. The reckoning masters thought it was important that students learned to recognize the general arithmetical structures to which particular problems belonged.

Van Egmond (1980) mentioned that when he began studying *abbaci* manuscripts he could not identify any associated rational theory of education. Later, he began to see the *abbaci* as structurally quite similar. We have had a parallel experience. After studying the contents of 172 ciphering books we have identified an educational rationale on which they were based. The fundamental aim was to assist students to become independent problem solvers, for life. Students were invited to recognize important structural similarities in carefully chosen problems and to observe how powerful rules could help solve the problems. Each major problem category had sub-categories, and for each sub-category a "type" problem was chosen and solved. Students then individually tackled problems that were structurally identical to the type problems. Before they were permitted to enter solutions in their ciphering books they had to be checked for correctness. When, finally, solutions were handwritten into the ciphering books, it was required that they displayed excellent penmanship.

After repeating this sequence for all the rules and cases associated with several main categories of problems, students were asked to solve miscellaneous problems {or "promiscuous questions") that embodied structures just studied. Each was concerned with a real-life situation that was either meaningful to the student then, or might be in the future. Each student was expected to recognize the mathematical structure, solve the problem, have the solution checked, and write the solution in his or her ciphering book. It was assumed that writing correct solutions to problems compelled students to "reflect" on problem structures, and made them likely to recognize and solve structurally similar problems in the future (Van Egmond, 1980).

This theoretical rationale was known by teachers and students, but never explicitly recorded. Our schematic summary of the rationale is shown in Figure 1. The rationale might be regarded as generating a "socially oriented, structure-based, problem-solving" theory. It was "socially oriented" because "type" problems were deliberately chosen by instructors so that they would be relevant to present or future social situations of the students. It was "structure-based" because each problem was chosen because it offered the opportunity to help the student to recognise that a particular social situation involved mathematics of a particular kind, with a special structure. It was "problem-solving" because students were expected to learn to solve the problems independently. Christ's Hospital boys, for example, knew that ultimately they might have the responsibility of navigating large merchant ships, or even warships, across the Atlantic Ocean. There was a sense of apprenticeship theory incorporated in the model, with instructors assisting students to learn to solve mathematics problems efficiently. The theory also incorporated the idea that it was not enough for students to be able to apply algorithms to solve problems – they needed to demonstrate, repeatedly, that they would be able to work out which mathematics was needed when they were confronted with unseen ("promiscuous") problems for which the structures were not immediately obvious.

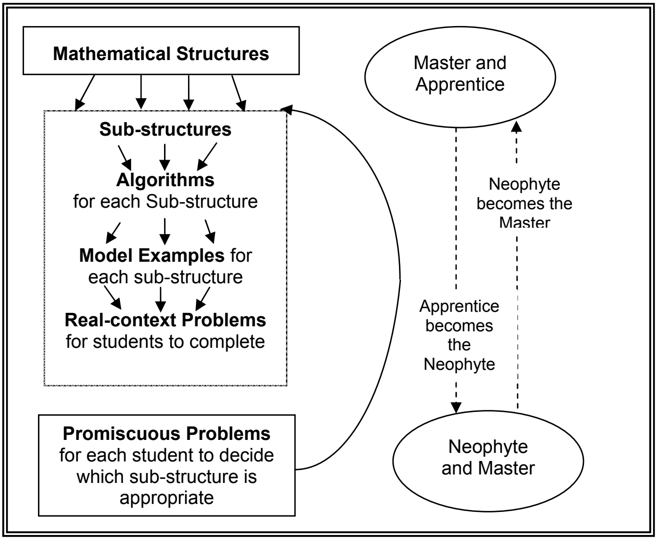


Figure 1. A socially oriented, structure-based, problem-solving model.

CONCLUSIONS

We have described the implicit theoretical bases of the *abbaco* and ciphering traditions. The traditions exerted a powerful influence on mathematics education for over 600 years because they emphasized social relevance, mathematical structure, model solutions, and careful recording of correct solutions. They made sure that an eye was kept on a student's future as well as on the present. They paid attention to individual needs. The traditions did have a strong theoretical base, one that might usefully be re-examined in the 21st century. There was an implicit danger, however, that the theory would be misused. If it were to be used inappropriately, it could generate an overemphasis on algorithms and mere copying. In fact, that is what happened … but that is another story.

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EQUATIONS IN A CONSUMER CULTURE: MATHEMATICAL IMAGES IN ADVERTISING

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Affect energises the learning and use of mathematics; a key influence comes from the images of mathematics available in society. We sought advertisements containing such 'images' (e.g. mathematical expressions, equations or graphs) in 1600 editions of UK newspapers, over two recent three-month periods. We found that 4.7% of editions included a 'mathematical' advert, compared with 1.7% found in the pilot for 1994-2003. This supports the idea that mathematical images are being used more in advertising, paralleling the increase observed anecdotally in films. The incidence varied from 8.2% among the 'quality' papers, to 2.3% in mid-market, to 0.6% among the 'populars', suggesting a correlation with the social class of the readership.

INTRODUCTION

Despite traditionally being considered to be 'invisible', mathematics has recently seemed to be 'everywhere', as a theme in Hollywood movies, as a basis for works of art, and in equations purporting to explain all aspects of human experience (e.g. Goldacre, 2008). We are interested in how the availability of different types of image of mathematics in various cultural outlets have effects on people's attitudes and emotions towards mathematics (Evans *et al.*, 2007; Mendick *et al.,* 2007). This second phase of this ongoing project built on a pilot study, which searched for adverts including a 'mathematical image' in a 'light sample' of 543 newspaper editions (sampled over the years 1994-2003, from three 'quality' titles, one mid-market and two 'populars'). We thus examined a much more representative sample of newspapers than in the pilot, to study differences among different categories of newspaper, which in the UK are related to social class profiles of the readership.

THEORETICAL BASES

Gail FitzSimons (2002) sees the *public images* of mathematics as "created and reflected both in the cognitive and affective domain and concern[ing], *inter alia*, knowledge, values, beliefs, attitudes, and emotions". She argues that

> a very strong influence on the public image of mathematics comes from the experience of formal mathematics education … [and] other influences such as stereotypes reinforced by popular media, or personal expectations conveyed explicitly and implicitly by significant others such as peers and close relatives. (2002, pp. 43-45)

For these reasons, the public images of *mathematics* and the images of *mathematics education* are exceedingly difficult to disentangle.

Our theoretical approach uses discursive perspectives (see Evans, Morgan & Tsatsaroni, 2006), drawing on work on pedagogic discourse in the sociology of

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 17-24. Thessaloniki, Greece: PME.

education (Bernstein, 2000), and on advertising (e.g. Williamson, 1978). Williamson argues that adverts refer outside of themselves and draw meanings from one or more *referent systems*, such as mathematics or other types of high-status knowledge. Bernstein's sociological theory provides the key concept of recontextualisation in understanding the construction of discourse. Pedagogic discourse is created through social processes involving selection, repositioning and refocusing of elements drawn from knowledge producing discourses. These processes entail transformations of these elements and changes in social relations. Therefore, like official pedagogic discourses, media discourses regulate the construction and reconstruction of identities and subjectivities: agencies (and agents) of symbolic control, such as advertising companies, are "concerned with the regulation of social relationships, consciousness, disposition and desire" (Bernstein, 2001, p. 21). In Bernstein's theory the pedagogic discourse assumes priority over 'unofficial' (or non-state, e.g., media) institutions and their discourses in the cultural field. However, a poststructuralist reflection on Bernstein's notion of discourse (Delamont & Atkinson, 2007) recommends greater consideration of the interconnections – and the boundaries – between the wider cultural field and the field of education.

Bernstein's brief analysis of media discourses provides insights relevant to advertising as well. First, *cultural productions*, whether oral communications in the classroom, textbooks, syllabuses, advertisements or films, are the means by which power relations translate into discourse and discourse into power – through classification and framing. Both concepts utilise the idea of boundary pointing to the importance of describing changes in its strength in the processes of recontextualisation through which (pedagogic) discourse is constructed, and enacted. Second, contrary to pedagogic discourses that form more durable pedagogical relations, media and other cultural representations contain a range of discourses that are segmentally organised, and aim to maintain, develop or change an audience niche (Bernstein, 2000, Ch. 11). Due to their segmental organisation, we cannot expect a strong, or even indirect, control over the context, social relations and motivations of the receivers/consumers of media discourses. Bernstein calls such cultural forms *quasi-pedagogic discourses*, thus indicating that they entail some form of pedagogic (i.e. social) regulation, irrespective of the ways in which messages are acquired.

Thus the problem of the kind of mathematical pedagogic identity created in formal schooling – clearly important to mathematics education researchers – would benefit from detailed empirical work on changes in the field of cultural (re)production.

RESEARCH QUESTIONS

We start from the idea that cultural representations, in particular advertisements, may play a major role in reinforcing (or challenging) long-term public images of mathematics, thereby reproducing or disturbing dominant social and educational discourses. In particular, our hypothesis is that the modes of communication created by the organization of advertising texts may work to distribute differentially forms of consciousness, identity and desire. Thus, the following set of research questions:

RQ1 To what extent – and in what ways – do advertisements use mathematics as a resource to construct their messages?

RQ2 What kind of images of mathematics, people doing mathematics, and/or teachers of mathematics can be identified in our sample of advertisements?

RQ3 How are readers / consumers affected by advertisements depicting mathematics / mathematicians? That is, how are they constructed by discourse as able to learn, or as knowledgeable or not, in mathematics? And how may this vary according to social position (e.g. class, gender, age group, general vs. specialised reader)?

METHODOLOGY

UK national newspapers are divided into 'quality', 'mid-market', and 'popular' papers, on the basis of their traditional styles of presentation and level of reporting and commentary – and their consequent attraction of readers from different social classes. We sampled four of the five 'quality' papers in the UK (*Daily Telegraph*, *Financial Times*, *The Guardian*, *The Times*), two mid-market papers (*Daily Express, Daily Mail*), and three 'populars' (*Daily Mirror, Daily Star, The Sun*). We selected two three month periods, Sept. – Nov. 2006, and Jan. – March 2008 for all papers (normally including the Sunday editions), except for *The Guardian / Observer*, where we also examined an intermediate period (July – Sept. 2007).

As indicators of a 'mathematical image', the following key words were used: mathematics, maths, mathematicians, geometry, geometrician, algebra, science/scientist, calculations, sums, equation/s, number/s, calculation/s (or related terms); and the name or picture of a prominent mathematician (such as Einstein, Stephen Hawking). Or a mathematical expression, equation, formula, or graph.

RESULTS

Here we examine each of the Research Questions in turn.

RQ1 To what extent – and in what ways – do advertisements use (images of) mathematics as a resource to construct their messages?

For each of the nine papers, we examined between 150 and 185 editions, with the exception of *The Guardian / Observer* (see above); see Table 1. The overall 'success rate' (the number of adverts found, divided by the number of editions) at 4.7%, is about three times as high as in the Phase 1 pilot (1.7%), when we examined over 500 editions from six newspapers (Evans *et al*., 2007). Part of this increase may be due to the more representative corpus we examined in Phase 2, but we consider that the increase also reflects the greater use of adverts containing mathematical images in the period 2006 to the present, compared with the earlier period, 1994-2003. We have argued, in considering media and culture more generally, that there has developed, over recent years a greater sensitivity to the meaning of mathematics as an indicator of scientificity and reliability, and a greater interest in representations of mathematics, and of people doing mathematics, also evident for instance in film production (Evans *et al*., 2007). Nevertheless, there is a fair amount of variation over the two time periods of Phase 2, with the success rate apparently decreasing from autumn 2006 to winter (Jan.-Mar.) 2008. Some newspapers (three) have seen their success rates rise between the two periods, and others (five) have fallen.

When we group the titles into the three types of newspaper, there is the expected difference in success rates. This is shown in Table 1.

Publication type	No. of editions	No. of adverts	'Success rate'	'Success rate' %
Quality	790	65	0.082	8.2%
Mid-market	344	8	0.023	2.3%
Popular	475	3	0.006	0.6%
Total	1609	76	0.047	4.7%

Table 1. 'Success rates' (incidence of 'mathematical adverts') by newspaper type.

To address the question of how advertisements use mathematics as a resource to construct their messages, we go on to RQ2.

RQ2 What kind of images of mathematics, people doing mathematics, and/or teachers of mathematics can be identified in our sample of advertisements?

We classified each of the adverts, according the main image of mathematics used in it. Since a number of the 76 adverts found in our sampling were 'repeats' (in different newspapers), now we have only 56 distinct adverts, not 76 instances; see Table 2.

Type of mathematics	No. of adverts	%
Use of key words	8	14.3
Name/picture of mathematician	0	0
Numbers, figures	4	7.1
Squares, powers etc	8	14.3
Charts, graphs	17	30.4
Simple equations	13	23.2
More complex equations	3	5.4
Other	3	5.4
Total	56	100

Table 2. Type of mathematics used in the set of distinct advertisements (n = 56).

Here we see that simple uses of keywords, such as ‘mathematical’, and slogans like ‘Do the maths’ accounted for almost 15% of the types of mathematics used. Adverts reporting simple numbers (e.g. “9 ½ / 10 for student satisfaction. No other university scored higher.”) or measurements accounted for half that number, though this category did include one advert for a bank, where a face was constructed rather appealingly from numerals (RB34, Quality, Jan.’08). Adverts using simple equations (including formulas and mathematical expressions) accounted for a second 25%. Adverts using charts and graphs accounted for 30%: only 4 of 17 of these presented real data, while the rest used fabricated data, often ‘playfully’, in their message; moreover, almost all could also be considered as simple, with the exception of one advert for an asset management firm which arguably uses a parabola in a thought-provoking way (RB49, Quality, Mar.’08). Thus, even considering the adverts using squares and powers (almost 15%, and more complex equations and other uses (5.4% each) as more challenging for the reader, around ¾ of the adverts use very simple mathematics indeed. We discuss the adverts using equations in more detail below.

Given the strong social class stratification in UK newspapers, we consider differences in the types of mathematics used for different types of newspapers; see Table 2a.

Newspaper type			Mid-mkt/Pop.	Quality	Total
Type of math used	Use of key words	Count (%)	2 (25.0%)	6 (12.5%)	8 (14.3%)
	Numbers, figures	Count (%)	1 (12.5%)	3 (6.3%)	4 (7.1%)
	Squares, powers	Count (%)	0	8 (16.7%)	8 (14.3%)
	Charts, graphs	Count (%)	1 (12.5%)	16 (33.3%)	17 (30.4%)
	Simple equations	Count (%)	3 (37.5%)	10 (20.8%)	13 (23.2%)
	Complex equation	Count (%)	1 (12.5%)	2 (4.2%)	3 (5.4%)
	Other	Count (%)	0	3 (6.3%)	3 (5.4%)
Total		Count (%)	8 (100.0%)	48 (100.0%)	56 (100.0%)

Table 2a. Crosstabulation of type of mathematics used by type of newspaper.

Because of the small numbers of ‘mathematical’ adverts found in the mid-market and popular papers, we consider the two types together. We note that 38% (3 of 8) of the adverts in mid-markets / populars use simple keywords or figures, compared with only 19% of qualities. There is a similar difference between the two groups in the use of *simple* equations. On the other hand, there is a greater use of charts in the ‘qualities’ (33% to 12.5%) and in the use of squares, powers, etc. (17% to none).

Thus adverts in the quality papers use a wider range of types of mathematics, and more demanding mathematics (charts & graphs, more complex equations).

At this point, we examine the 16 uses of ‘equations’, accounting for more than 25% of the adverts found. Most of these might more accurately be called ‘formulae’, since they involve a 'mathematical' expression (the 'left-hand side') set equal to what the reader needs / needs to do, in order to obtain what is indicated on the 'right-hand side'. Sometimes the elements that are added are icons – e.g. Christmas gifts in a Woolworths advert (RB42, Popular, Nov.'06); sometimes they are words, or a mixture of words and figures – e.g. "40min + £199 = YOUR PERFECT MORTGAGE" (RB35, Quality, Nov. '06). 13 of the adverts fall roughly into this category, including 2 of the 3 'more complex' equations. There are at most two 'equations' which are even plausibly correct in mathematical terms: one simple equation which sums the prices of pieces of furniture that you could buy with a £5000 prize draw voucher from a shop (RB40, Quality, Mar.'08), and one more complex, which contains a formula for velocity ("Give him the Lift Off Rocket and who knows what he'll grow up to be", RB2, Mid-market, Nov.06). See also our analyses of the equations used in adverts from the pilot study (Evans et al., 2007).

RQ3 How are readers / consumers affected by advertisements depicting mathematics / mathematicians? That is, how are they constructed by discourse as able to learn, or as knowledgeable or not, in mathematics? And how may this vary according to social position (e.g. class, gender, age group, general vs. specialised reader)?

Newspaper	**% Male readers**	**% ABC1 readers**	**'Success rate' %**
The Times	58.2	89.0	7.3%
The Guardian	55.7	91.7	5.8%
Financial Times	74.1	94.1	13.1%
Daily Telegraph	55.7	86.0	5.9%
Daily Express	51.3	60.8	2.3%
Daily Mail	47.8	65.4	2.3%
Daily Star	70.5	32.0	0.6%
Daily Mirror	52.8	41.9	1.3%
The Sun	56.8	36.8	0.0%
Total			

Table 3: 'Success rates' for newspapers, by gender and social class of readership.
Source: Readership data from the National Readership Survey Oct.-Nov. 2007

To address this question, various types of evidence are being produced by the present project: readings of the 'semiotics' of advertisement(s) as text, aiming to draw out the discourses that underpin them (cf. Evans *et al*., 2007); and interviews with 'typical' readers, asking for 'reactions' to particular adverts in terms of 'how it makes you feel'. Here we can begin with RQ3 by noting any relationship between the demographic characteristics of the readership of the newspapers studied and the incidence of adverts' containing 'mathematical images' during our time periods.

Table 3 shows a clear stratification of the nine newspapers into three familiar groups – 'qualities', mid-market and 'populars' – according to the percentage of ABC or 'middle-class' readers. And there is a clear correlation between category of newspaper and 'success rate' (which can be seen even more simply in Table 1 above). At first, there does not appear to be a strong relationship between gender of readership and 'success rate': the two newspapers with the highest proportion of male readers are the *FT* and the *Star*, and they have very different 'success rates', but when we look more closely (or check the scatterplot of success rate and % male readers), we see that the *Star* can be considered an 'outlier'.

Now, the correlation coefficient of success rate and % ABC1 readers is 0.862 ($p \leq .001$, highly significant) and the correlation of success rate and % male readers is 0.493, $p \leq .089$, not quite borderline significance). But the partial correlation of success rate and % male readers, controlling for % ABC1 readers, is 0.850 ($p \leq .008$, again highly significant).

Thus we can conclude that both the % of ABC1 readers and the % of male readers of a newspaper make a statistically significant contribution to 'predicting' the incidence of adverts containing mathematical images in newspapers.

CONCLUSIONS

In this phase of the research, we found a much higher incidence of mathematical images in advertisements during 2006-08 (a 4.7% 'success rate'), than in the pilot phase (1.7%, using very light sampling during 1994-2003). There was a much higher success rate in 'quality' newspapers, compared with mid-market or 'popular' ones, showing a strong correlation with the percentage of ABC1 (middle class) readers, and also with the percentage of male readers (controlling for social class of readership). Nevertheless, the images of mathematics offered by the advertisements were for the most part simplistic, and hence impoverished, for all – and the range of images offered was more restricted in the popular and mid-market papers, as compared with the quality papers. Thus, we might agree with other researchers that mathematics suffers from 'low visibility' – but to different degrees across different social groups.

Our evaluation of the level of mathematical demand made on the reader by these adverts raises questions about the value of using such means as resources in the mathematics classroom. Nonetheless, some of the adverts might be useful in two ways: first, in discussing the effectiveness of the adverts as adverts – and that of the mathematical images, including how they are or are not 'properly mathematical'; and second, in discussing how they might influence students' feelings about mathematics. More generally, however, both policy makers and researchers may have an interest in considering the extent to which cultural representations work to cancel out initiatives designed to improve the level of mathematical knowledge in the general population.

Relatedly, we note the emergence of a trend – supported by our evidence – whereby mathematics equations or formulae are recruited as global communication technologies of subjectivity, shaping desire especially for those strata of the middle classes that are the most promising clients in the global consumers' market. This emerging strategy might undercut the use of maths as a critical discourse for citizens.

Our overall approach uses mixed methods (see Evans *et al*., 2007), but the analysis here has been mostly quantitative. Nevertheless, we hope to have shown how much can be done with such an analysis, especially in the light of our concerns with social difference and social justice (Evans & Tsatsaroni, 2008). We see further research as important on several levels. First, we are extending our work (in Phase 3 of the project) by (i) 'audience research' interviews with 'typical' readers, to study reactions to selected adverts; and (ii) interviews with agency workers, on the creative process. Second, other fields of media activity can be investigated for their images of mathematics (see e.g. Mendick *et al*., 2007). Third, there is scope for cross- cultural studies, using an approach like that described here.

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EFFECT OF THE NUMBER STRUCTURE AND THE QUANTITY NATURE ON SECONDARY SCHOOL STUDENTS' PROPORTIONAL REASONING

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Research has shown that the number structure (integer or non-integer ratios between given numbers) and the nature of the quantities (discrete or continuous quantities) influence students' proportional reasoning performance. This study investigates the effect of these two variables on secondary school students' performance when solving proportional as well as non-proportional problems (with an additive structure). We found that number structure affected students' performance differently for proportional and additive problems, whereas the nature of quantities had no influence.

THEORETICAL AND EMPIRICAL BACKGROUND

The concept of proportion as 'an equivalence relationship between two ratios' is fundamental in the math curriculum. It appears in the first grades of primary school and develops further during secondary and tertiary school. According to Freudenthal (1983), the ratio is a function of an ordered pair of numbers or quantities of magnitude. There are two kinds of relationships among quantities: 'within' relationships (internal ratio), which are relationships between quantities of the same nature, and 'between' relationships (external ratio), which correspond with quantities of different nature. Take, for example, the problem '5 kilos of potatoes cost 2 euro. You want to buy 8 kilos. How much will it cost?'; if we relate kilos with euro we obtain a 'between' relationship (5 kilos / 2 euro), whereas if we relate kilos with kilos we have a 'within' relationship (5 kilos / 8 kilos). In a proportion, internal ratios are equal ($5/8 = 2/x$) and external ratios are constant ($5/2 = 8/x$).

Proportional reasoning implies not only the ability to solve different types of proportional problems, but also a good understanding of the multiplicative relationship between quantities that represent a proportional situation, and – last but not least – the ability to discriminate proportional and non-proportional situations (Christou & Philippou, 2002; Cramer, Post, & Currier, 1993; De Bock, Van Dooren, Janssens, & Verschaffel, 2007). As far as the last element is concerned, research shows that students frequently over-use linearity, i.e. erroneously apply it in non-proportional situations (De Bock, Van Dooren, Janssens & Verschaffel, 2002, 2007;

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 25-32. Thessaloniki, Greece: PME.

Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock, Evers, & Verschaffel, 2006).

The literature on proportional reasoning points at many task characteristics that affect proportional reasoning (Harel & Behr, 1989; Tourniaire & Pulos, 1985). The current study investigates the effect of two of these task characteristics on students' over-use of proportionality. The task characteristics selected for this study are the presence of integer ratios between the given numbers and the discreteness of the quantities involved in the problem.

The effect of the presence of integer ratios has been investigated by Van Dooren et al. (2006) in Flemish 4th to 6th graders. It was found that students made more errors (particularly more additive errors) in proportional problems with non-integer ratios than in proportional problems with integer ratios. For non-proportional problems with an additive structure, it was the other way around: Problems with integer ratios elicited more errors (particularly proportional errors) than problems with non-integer ratios. The differences were strongest in 4th grade and gradually diminished in 5th and 6th grade.

The discreteness of the quantities involved in the problem has also been proposed as a factor affecting students' proportional reasoning (e.g. Harel & Behr, 1989; Lo & Watanabe, 1997; Moss & Case, 1999). On the one hand, it has been argued that people can more easily visualize discrete content than continuous content (Tourniaire & Pulos, 1985) and therefore will perform better on proportional problems involving discrete quantities. On the other hand, Jeong, Levine, and Huttenlocher (2007) found that 6, 8 and 10 years old students more often used erroneous counting strategies (i.e. approaching the problem additively) in discrete situations, and more multiplicative strategies in continuous situations. Also, Boyer, Levine, and Huttenlocher (2008) found that 6-9 year old children had much more difficulty solving proportional reasoning problems when the proportions were represented with discrete units, than on problems for which the fractions were represented with continuous amounts. Using comparison problems (problems where two ratios have to be compared), they found that children's difficulties were at least partly due to their propensity to compare quantities on the basis of the number of elements rather than on the basis of proportional relations. Moreover, the effect of discrete or continuous quantities seems to vary strongly across groups (Tourniaire & Pulos, 1985). For example, large differences in performance were found in less able students and small differences were found in more able students (Horowith, in Tourniaire & Pulos, 1985). Also, Pulos et al. (in Tourniaire & Pulos, 1985) found that, in 6th grade, males performed best on discrete problems and females performed best on continuous problems, while in eighth grade the pattern was reversed. So, while this context variable seems to affect proportional reasoning, it is not well understood and deserves further research.

Our goal is to study the effect of these two variables – whether the numbers form integer ratios or not, and whether the quantities are discrete or continuous– in the success level of secondary school students. New in this study is that the effect will not only be investigated on the performance on *proportional* problems, but also on *non-proportional* problems where students are inclined to apply proportional solution methods.

Although there are numerous types of non-proportional problems, we have chosen to focus on additive non-proportional problems, because according to the literature additive reasoning is the most common error in proportional reasoning (Misailidou & Williams, 2003; Tourniaire & Pulos, 1985). So it is interesting to compare students' solution behaviour on proportional problems where proportional reasoning is correct and additive errors are expected, and additive problems where additive reasoning is correct while proportional errors are expected, and to see whether the same task characteristics affect performance on both kinds of problems.

HYPOTHESES

We expect that students will be more successful in proportional problems than in additive problems (HYP 1), due to their tendency to apply proportional methods in the latter problems too. With regard to number structure, we expect, in line with the proportional reasoning literature (Van Dooren et al., 2005, 2006), that students will be more successful in integer proportional problems than in non-integer ones (HYP 2). For additive problems, we expect (in line with Van Dooren et al., 2006) that students will be more successful in non-integer problems than in integer ones (HYP 3). With respect to the nature of quantities, although it is still controversial in the literature, according to Boyer et al., (2008) and Jeong et al., (2007), we expect that students will be more successful with continuous quantities than with discrete quantities if the problem is proportional (HYP 4) and reversely, that students will be more successful with discrete than with continuous quantities if they have to solve an additive problem (HYP 5). Finally, we expect that all above effects would disappear (or at least seriously decrease) with students' age (HYP 6).

METHOD

Participants were 558 secondary school students: 124 students in the 1st year of secondary school (12-13 year olds), 151 in the 2nd year (13-14 year olds), 154 in the 3rd year (14-15 year olds), and 129 in the 4th year (15-16 year olds).

We used a test consisting of eight experimental problems (four proportional problems (P) and four additive problems (A)) and four buffer problems. Two of the four proportional problems and two of the four additive problems referred to discrete quantities (one with integer ratios between given numbers, D-II, and one with non-integer ratios, D-NN). The other four referred to continuous quantities (again two with integer ratios, C-II, and two with non-integer ratios, C-NN). Table 1 exemplifies how the experimental variables were manipulated.

	Examples	II	NN
P-D	Peter and Tom are loading boxes in a truck. They started together but Tom loads faster. When Peter has loaded ***a*** boxes, Tom has loaded ***b*** boxes. If Peter has loaded ***c*** boxes, how many boxes has Tom loaded?	40 160* 80 **Prop**: 320 **Addit**:200	40 100 60 **Prop**: 150 **Addit**:120
A-D	Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded ***a*** boxes, Tom has loaded ***b*** boxes. If Peter has loaded ***c*** boxes, how many boxes has Tom loaded?	40 160 80 **Prop**: 320 **Addit**:200	40 100 60 **Prop**: 150 **Addit**: 120
P-C	Ann and Rachel are skating. They started together but Rachel skates faster. When Ann has skated ***a*** m, Rachel has skated ***b*** m. If Ann has skated ***c*** m, how many meters has Rachel skated?	150 300 600 **Prop**: 1200 **Addit**: 750	80 120 200 **Prop**: 300 **Addit**: 240
A-C	Ann and Rachel are skating. They skate equally fast but Rachel started earlier. When Ann has skated ***a*** m, Rachel has skated ***b*** m. If Ann has skated ***c*** m, how many meters has Rachel skated?	150 300 600 **Prop**: 1200 **Addit**: 750	80 120 200 **Prop**: 300 **Addit**: 240

Numbers are schematically represented as *a b

c x (**Prop**: Proportional answer, **Addit**: additive solution)

Table 1: Examples of problems and manipulation of quantity nature (versions D and C) and number structure (versions II and NN)

Starting from eight discrete situations (like loading boxes) and eight continuous situations (like skating certain distances), proportional or additive word problems were created by manipulating the crucial second sentence (for example, ”They started together but Rachel skates faster” in the proportional problem, and ”They skate equally fast but Rachel started earlier” in the additive one). Then numbers were inserted so that the internal and external ratios were either integer or non-integer. The problems were controlled for number size, calculation complexity (e.g., the outcome is always an integer), the context (always actions) and the position of the unknown quantity. With the obtained set of problems, 8 parallel tests were composed, with different orders.

Students had 50 minutes to complete the test and they could use calculators.

Students’ answers to each item were analyzed to identify correct solutions (scored 1) and incorrect solutions (scored 0). We also analyzed the strategies applied by the students, but due to space restrictions this analysis will not be reported here.

RESULTS

Table 2 shows the percentages of correct answers to the proportional and additive problems in the four different age groups. Contrary to our expectation (HYP 1), globally, students were more successful in additive problems (44.3%) than in proportional problems (40.3%). A repeated measures logistic regression analysis showed that this difference was significant, $\chi^2(1, N=558)=6.248$, $p=0.012$. However, there also was a significant 'model' × 'grade' interaction effect, $\chi^2(3, N=558)=99.074$, $p<0.001$, which indicated an increase in performance on the proportional problems along grades (in 1st grade 13.8% and in the 4th grade 72.3%) and a decrease on the additive problems (in 1st grade 51.3% and in the 4th grade 30.0%). Table 2 also clarifies that 1st and 2nd year students were much more successful in additive problems than in proportional problems while 3rd and 4th year students were more successful in proportional problems than in additive ones. These results suggest that, as in the study by Van Dooren et al. (2005) with Flemish students, also Spanish students with growing age increasingly tend to apply proportional solution methods to non-proportional (additive) problems.

	1st	**2nd**	**3rd**	**4th**	**Total**
P	13.8	26.5	48.8	72.3	40.3
A	51.3	54.8	41.0	30.0	44.3

Table 2: Percentage of correct answers on the proportional (P) and additive (A) problems.

Moreover, the analysis revealed a 'model' × 'number type' interaction effect, $\chi^2(1,N=558)=104.604$, $p<0.001$. As shown in Table 3, in proportional problems students were more successful in integer versions than in non-integer versions. So our hypothesis HYP 2 was confirmed. The difference between the II and NN versions was very strong and significant in 1st grade (22.0% correct answers to the II version and 5.5% on the NN version) and 2nd grade (with 33.0% and 20.0% correct answers, respectively), while it was smaller but still significant in 3rd (55.5% versus 42.0% correct answers) and 4th grades (with 75.0% versus 69.5%).

	1st	**2nd**	**3rd**	**4th**	**Total**
P-II	22.0	33.0	55.5	75.0	46.4
P-NN	5.5	20.0	42.0	69.5	34.3
A-II	46.0	48.5	37.5	27.0	39.8
A-NN	56.5	61.0	44.5	33.0	48.8

Table 3: Percentage of correct answers on the proportional (P) and additive (A) problems with integer (II) and non-integer (NN) structure.

In additive problems, students were more successful in non-integer than in integer versions. So, HYP 3 was confirmed. The difference between the II and NN versions was strong and significant in 1st grade (46.0% correct answers to the II version and 56.5% on the NN version) and 2nd grade (with 48.5% and 61.0% correct answers respectively), less strong but still significant in 3rd (with 37.5% and 44.5% correct answer, respectively) and 4th grades (with 27.0% and 33.0% correct answer, respectively).

With regard to the discrete versus continuous nature of quantities, there was no significant difference in the performance, neither as a main effect nor in interaction with grade, model or number type. Table 4 shows the percentages of correct answers on the discrete or continuous proportional and additive problems, separated out by grade. So, our hypotheses HYP 4 and HYP 5 were not confirmed.

	1st	**2nd**	**3rd**	**4th**	**Total**
P-D	15.0	26.0	49.0	71.5	40.4
P-C	12.5	27.0	48.5	73.0	40.3
A-D	50.0	54.5	41.0	30.5	44.0
A-C	52.5	55.0	41.0	29.5	44.5

Table 4: Percentage of correct answers on the proportional-discrete (P-D), proportional-continuous (P-C), additive-discrete (A-D) and additive-continuous (A-C) problems.

Finally, HYP6 was confirmed for the task variable ‘number structure’ (integer versus non-integer): The differences in the percentages of correct answers on proportional and additive problems with an integer or non-integer structure became smaller with the age. The analysis revealed a ‘grade’ × ‘model’ × ‘number type’ interaction, $\chi^2(3,N=558)=19.352$, $p<0.001$. Students gave more correct answers in the II-versions than in NN-versions of proportional problems and more correct answers in the NN-versions than in the II-versions of additive problems. And these differences decreased with the age (Table 3). However, HYP6 was not confirmed for the task variable ‘nature of quantities’ (discrete versus continuous). This could be expected because there was no effect of this variable overall, so there could also be no decrease of that difference with age.

CONCLUSIONS AND DISCUSSION

Contrary to our expectation that overall, students in our study would be more successful in proportional than in additive problems (as also observed in other studies, see for example Van Dooren et al., 2006), in the present study with Spanish secondary school students, only 3rd and 4th grade students were more successful in proportional problems, whereas students from the first and second grade were more successful in additive problems. So, the success in proportional problems increased with age whereas the success in additive problems decreased between the age of 12-14 and 14-16 years, due to the over-use of proportional methods in non-proportional situations (Van Dooren, De Bock, Janssens, & Verschaffel, 2008).

However, the main focus of this research was on the effect of two variables – the discrete or continuous nature of quantities involved in the problem and the integer or non-integer number structure – on secondary school students' (12-16 years old) solutions of proportional and additive problems in a missing-value structure. In an earlier study, Van Dooren et al. (2006) manipulated only the integer or non-integer character of the ratios in the problems; in the current study we have manipulated the two variables simultaneously.

With regard to the impact of the integer or non-integer number structure, students were, as expected, more successful in proportional problems with integer ratios than in proportional problems with non-integer ratios, whereas students were more successful in additive problems with non-integer ratios than in additive problems with integer ratios. This effect was particularly strong in first and second year students and became less influential in third and fourth year students. So, our results about the effect of number structure obtained with Spanish students replicated those reported by Van Dooren et al. (2006) with Flemish students, although in the present study the evolution of this effect with age was found in considerably older students.

With regard to the discrete or continuous nature of the quantities, we did not find a significant difference in students' performance. We remind that the literature about the effect of this variable is controversial. It may be that we did not find any effect of this task variable because we used missing-value word problems in instead of comparison problems, for which the effect has been shown in previous research (Boyer et al., 2008; Jeong et al., 2007). Another reason may be that our word problems (even the NN versions) always had integer outcomes. Maybe we would have observed a significant difference between the versions with discrete and continuous quantities if we would have included word problems with non-integer ratios *and* non-integer outcomes. In those problems, proportional solutions to discrete problems have little or no sense (e.g., one cannot load 12.5 boxes in a truck), whereas they are perfectly reasonable for the versions with continuous quantities (e.g., one can walk 12.5 meters). A final element that may explain this result is that we involved secondary school students, while in previous research participants were mostly primary school students. We are currently investigating whether the hypothesised effects manifest themselves in younger students.

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WHY COULD NOT A VERTICAL LINE APPEAR? IMAGINING TO STOP TIME

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This research study concerns 9 year-old children who experienced, through the use of a technological device, position-time graphs that represent a motion happening in a vertical plane. Attention is drawn to moments of a classroom discussion, in which children are thinking of the graphs related to vertical and horizontal trajectories. In these moments, it comes to the fore the big issue of having a vertical line as one of the graphs. We want to analyse the expectation of stopping time that arises from this issue, and grows through a recollection of the past experience with graphs given by horizontal lines. To this aim, we focus on children's words and gestures.

INTRODUCTION AND THEORETICAL FRAMEWORK

In the last decade, many perspectives on mathematical understanding have been informed by embodied cognition (Lakoff & Núñez, 2000). Different views have been developed, and new looks at the way we think and understand emerged. The bounds between perception and action on the one side, and cognition on the other side, began to be seen as porous: "we do not simply inhabit our bodies, we literally use them to think with" (Seitz, 2000; p. 24). Evidence comes from recent results in neuroscience:

> conceptual knowledge is embodied, that is, it is mapped within our sensory-motor system. […] The sensory-motor system not only provides structure to conceptual content, but also characterises the semantic content of concepts in terms of the way that we function with our bodies in the world. (Gallese & Lakoff, 2005; pp. 455–456)

The great significance progressively acknowledged to perceptuo-motor activities in thinking processes provided Mathematics Education research with insights on the nature itself of mathematics learning. Nemirovsky (2003) stresses, for example, that it is a very different experience to watch a movie, which displays a geometrical object, than it is to touch and walk around a plastic model of the same object. These experiences can be both useful, but they are not mere repetitions, even if they could reflect the same mathematical principle. In fact, one difference is that "the use of appropriate materials and devices facilitates the inclusion of touch, proprioception (perception of our own bodies), and kinesthesia (selfinitiated body motion) in mathematics learning" (*ibid.*; p. 104). In this regard, Nemirovsky guesses that:

> the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context. For instance, seeing a trigonometrical function as a component of circular motion or as an infinite sum of powers may entail distinct and separate perceptuo-motor activities. (*ibid.*; p. 108)

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 33-40. Thessaloniki, Greece: PME.

Such a conjecture is corroborated by neuroscientific results on the role of the brain's sensory-motor system in conceptual knowledge, and on its feature of multimodality:

> Multimodal integration has been found in many different locations in the brain […]. That is, sensory modalities like vision, touch, hearing, and so on are actually integrated with each other *and* with motor control and planning. (Gallese & Lakoff, 2005; p. 459)

Roughly speaking, as we encounter a new mathematical concept, we are involved in it from a perceptuo-motor stance rather than from a theoretical point of view, through formal definitions. It is not surprising if we think of the historical and epistemological roots of concepts. Most concepts passed through different steps before coming to a formal definition. Earlier steps often concern empirical issues; later steps reach a theoretical form by means of deep refinements. Take the concept of function as an example. History shows the critical role of kinematics in the birth of this concept. Famous scholars took part in it, by facing problems of kinematics. We can remember Nicole Oresme and Thomas Bradwardine and their works on qualities and intensities, Galileo Galilei and the issue of free fall, Isaac Newton and the theory of fluents and fluxions. For instance, Newton saw mathematical quantities in terms of motion:

> I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines [curves] are described, and thereby generated, not by the apposition of parts but by the continued motion of points. […] These geneses really take place in the nature of things, and are daily seen in the motion of bodies. (quoted in Struik, 1986; p. 303)

The strict relation between function and motion is apparent. Indeed, listening to history gives education researchers reasons for considering motion as a meaningful context to approach the concept of function. Modelling motion is a means to recover its roots. Nowadays, the availability of new technologies with powerful visualisation facilities affords ways of modelling motion through graphing. Such ways allow recovering the roots of concepts on the one side, and stimulating the multimodality of learning on the other side, by awakening the perceptuo-motor activities of learners.

Graphing motion: The issue of time. In interpreting a graph that represents a motion, one has to take into account various aspects. Perceivable aspects come from the experience of motion. Geometrical aspects correspond to features of motion (e.g. the constant rate of change with respect to constant speed). Definitional aspects hide the relations between variables. Intrinsic aspects are determined by convention or by technology settings (like the disposition of variables, the unity of measurement, etc.). By rule, motion graphs display functions of time (position, velocity, and acceleration vs. time). Time does not play the same role as the other variables, depending on it. The notion of time was inspected throughout history. Views can be found in Merton College's and Galilei's works. Newton, who knew these labours, first had a major new idea; he saw time quantitatively, putting it into a new place in mathematics:

> I consider time as flowing or increasing by continual flux and other quantities as increasing continually in time and from the fluxion of time I give the name fluxions to the velocity with which all other quantities increase. […] I expose time by any quantity

> flowing uniformly and represent its fluxion by an unit, and the fluxions of other quantities I represent by any other fits symbols. (quoted in Edwards, 1979; p. 210)

Newton's conception models time as flowing exactly the same for all observers, in a universal and uniform manner. Such model already considers time as a parameter; using it, we can place time on one axis in a coordinate system (by convention, the horizontal axis). Also, Newtonian time has a conventional character, in that it is not to be meant in the literal sense, but as a different quantity, through whose uniform increase or flux time is represented and measured. In this way, Newton distinguishes absolute time from relative time:

> Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year. (Cajori, 1947; p. 6)

The denominations with which we define time are seconds, minutes, hours, days, etc. Then, they are arbitrary human constructions that constitute our ordinary conception of time, different from the mathematical time. Representing time as a variable in a Cartesian plane, and then interpret it in relation to other motion variables is not simple, due to these reasons.

In this paper, we look at a segment of a classroom discussion, in which time plays a crucial role in investigating the possible shape of a motion graph. The fact that time is a variable flowing uniformly, without stopping, will be a key point of the argument.

EXPERIMENT AND METHODOLOGY

The discussion we are going to consider is part of a teaching experiment carried out between February and May 2008. The experiment took place in a primary school in the neighbours of Turin, in Northern Italy. It involved 16 children at the 4th grade (9 years) in a series of activities related to modelling motion in two dimensions. A researcher and the teacher (the authors) were present in the classroom. The children already worked the year before in activities of the same sort. They encountered motions along simple trajectories, given by horizontal and vertical segments in a vertical plane, and the absence of motion. They met both oral and written tasks, working in different manners: individually, in couples, in collective discussions. The methodology does not change from an experiment to the other one. All the written materials (texts, drawings, sketches, etc.) were collected. We also have track of all the oral productions, thanks to a moving camera and a mobile one filming all the phases of the experiment. Transcriptions of the videotapes were then produced. This set of resources has been used to analyse moments relevant in terms of sense making.

Both the experiments were aimed at introducing children to the mathematics of change and to the graph sense, by the parametric functions composing a motion in two dimensions. To this aim, a graphical approach was realised through the use of a technological device, called Motion Visualizer DV (MV DV). The MV DV works detecting colours with the support of a computer and of a webcam (or a camera). The webcam detects light coming from a coloured object moving in its field of view. The computer displays in real time: the graphs representing the position components versus time, the corresponding trajectory, and a video of motion with the trajectory superimposed (fig. 1). The coloured object moved is an orange ball on a short stick (we call it "the magic stick", as those of stories). The children never used the device before such activities, although they participated in a first teaching experiment on motion (in grade 2) using a uni-dimensional motion detector (Ferrara & Savioli, in press). At the moment of the current experiment, the classroom background includes knowledge on motion in one dimension and is informed by the previous experiment.

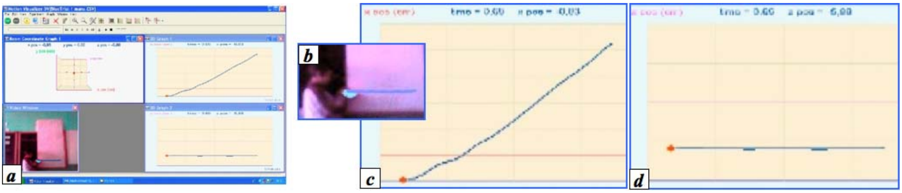

Figure 1: (*a*) The interface of the MV DV; (*b*) the video; (*c*-*d*) the graphs.

The setting of the classroom consists of: the computer, the webcam, and a projector. During the activities involving motions, children were all seated around the device. In this way, they could have a good look at the wall, where two bills where posed: one for projecting the computer screen, the other one for collecting motions. In turn, each child had the possibility to move the stick. The results shown on the computer screen (both the movement and its representations) could be replayed and watched off-line.

We made a distinction between the world of mathematics and the world of motion (metaphorically given by the computer screen and by the bill where motions were performed). We called the first world "Cartesiolandia" (in Italian), i.e. the land of Descartes; the second one "Movilandia", i.e. the land of Motion.

The dialogue we will analyse is at the very beginning of the new experiment, and aims at recovering understanding about the motions already experienced. We are interested in analysing it for different reasons. First, it represents a join with the previous activities and can allow us to see what children have really acquired in terms of graph sense. Secondly, it is a moment of social construction of knowledge, to share and compare ideas, the discussion orchestrated by the researcher (according to Bartolini Bussi, 1996). Finally, the crucial issue of time comes into the discussion: time as a flowing variable, in the Newtonian absolute sense. The thinking processes are related, on the one side, to something already observed (a recollection): in absence of motion, the graphs are horizontal lines. On the other side, they are related to something to be imagined (an expectation): the possibility of stopping time for having a vertical line for one of the graphs. The matter of time passing and never stopping is a crucial point, which has epistemological and cognitive difficulties related to the ordinary conception of time as a relative variable, and as existing inherently. The segment going to be discussed puts forward the effort of imagining a counter-intuitive situation in order to overcome conflicts with the past experience.

ANALYSIS

The children have started the experiment since a week. In the first session, they watched the video of a motion along a vertical segment and its representations on the computer screen. Also, a first written task required them to tell, represent and draw everything understood they remembered from the past activities. The second session opens with a discussion, guided by the researcher (F below), which aims at recalling competences acquired on vertical and horizontal motion paths. The analysis focuses on segments from this discussion (gestures are in italics in brackets; LF=left hand, RH=right hand, LA=left arm, RA=right arm). The researcher asked how many shapes have been seen last year (referring to motion trajectories). Three shapes are early recalled (#26-28): vertical, horizontal and oblique.

26	B	Vertical, horizontal and …
27	D, EF	oblique
28	B	Oblique, vertical and horizontal.
31	F	Oblique, how?
32	A	It is made in a diagonal [*miming with her LH in the air an oblique line*] that … when you move in Movilandia, in Cartesiolandia … you move in a certain way in Movilandia, and a line, vertical or horizontal or oblique, appears in Cartesiolandia.
33	ES	But it depends on the way you move.

Soon confusion arises, between the possible segments in Movilandia (the trajectories) and the corresponding lines that appear in Cartesiolandia (the graphs). While B is thinking of different directions of motion (#28), A is trying to recall the eventual shapes of the graphs representing a motion (#32). ES shifts attention to the fact that what one sees in Cartesiolandia is the result of what one does in Movilandia, thereby "it depends on the way you move" (#33). But the possibility of having a vertical line in Cartesiolandia, considered by A, sounds incorrect:

34	M	It cannot be vertical [speaking in a whisper]
35	F	You [referring to A] said that a line, vertical or horizontal or oblique, appears in Cartesiolandia …
36	M	No, a vertical line never appeared in Cartesiolandia.
39	F	Did a vertical line never appear?
40	M	In Movilandia we moved along vertical segments, but in Movilandia [meaning Cartesiolandia] a vertical line never, it never appeared.
41	EF	Or in a diagonal direction … or in a horizontal direction.

M remembers that he never saw a vertical line in Cartesiolandia (#34, 36). Even if he is not precise, he needs to distinguish the action of moving from the representations of motion given by the software (#40). One concerns Movilandia, the other one has to do with Cartesiolandia, so they are very different things (although they are related, one to the other). At this point, the role of the researcher is fundamental. The fact that "a vertical line never appeared" comes from a recollection of past experience with the device. M is simply remembering that the software never showed vertical lines. But F asks for an explanation of this fact, stressing the impossibility of its occurrence (#42):

42	F	Why could not a vertical line appear?
43	G	Because the glove [an orange glove used in the first experiment] … when it moves, it moves [*miming a short movement in the air with her RH*] from bottom [*indicating a position in the air with her RH closed in a fist*], since … when it appears in Cartesiolandia, the glove is always at the bottom [*indicating a position in the air again*] and then it makes the line in this way [*shifting her RH horizontally from left to right*: fig. 2*a*] as they moved, and it does not start in this way [*moving her RH twice along a vertical direction, from top to bottom*] to make … the vertical line
45	EF	To me, because in the table [*miming the axis of a Cartesian plane with her RH*], which is in Cartesiolandia, it appears … to come vertical, it does not arrive at the end of the table [*moving her open RH horizontally, from left to right*: fig. 2*b*], but it should arrive at the end [*repeating the gesture*]
47	B	There is the table [*placing his LA vertically to refer to the vertical axis*], where here there is time [*moving his RH twice along a horizontal direction, from left to right*] and here [*shifting his LA twice along a vertical direction, from top to bottom*] there is the movement you make … but you cannot, for example, in little time, say, 10 seconds, in few seconds make … be able to have such a movement on a platform, that is in a place making you understand that time passes [*placing the LA vertically, and moving the RH horizontally from left to right*: fig. 2*c*], since it would be as you stopped time [*pointing with the LH to a position in the air*] and moved [*miming a change in position with the LH jumping twice*]

Figure 2: (*a*) G's, (*b*) EF's and (*c*) B's gestures along the horizontal direction.

This is the most significant piece of the dialogue, since time comes to play a central role. The understanding process goes through three different steps. In a first step, the reason of G is still mainly phenomenological, being related to past experience (#43). Her gestures and words constantly refer to what she observed using the software. It is as if these gestures and words were pictures of the general situation of motion, which cannot be considered without looking at the act of moving the glove in a precise manner ("as they moved"). G, at least at the beginning, still refers in a confused way to the graphs in Cartesiolandia, until she mimes a horizontal line with her RH. We think that for her time is intrinsic in this gesture, and the meaning of the gesture is that time always goes on ("it makes the line in this way"). EF makes a step further (#45). She does not need to recall the motion, but she early thinks of the graph in

Cartesiolandia ("the table"). She refers to the specific constraint given by time: the fact that the lines have to "arrive at the end". This is repeated twice both in words and gestures, as to stress its importance. Time is not yet explicit in words, but its flowing is represented in the gesture. The third step, the most abstract, is made by B (#47). He suddenly speaks of time as a flowing variable and traces the corresponding axis in a gesture, also taking as a reference the vertical axis mimed through the LA in a vertical position. Then, his argument marks that he makes an effort to imagine a vertical line as a motion graph. Modelling time, by measuring it, on the horizontal axis ("a place making you understand that time passes") forces to the impossibility of such a shape. The fact is that a vertical line would represent an absurd motion where one stops time but moves, changing his/her position. The imagination process is thus completed. The sense making of the situation is shared, as witnessed by the last part of the discussion. The researcher helps the children to pass to the interpretation of the impossible line in terms of a straighforward relation between variables (#58-62):

48	F	If you stop time, it is as if time didn't change, but what does it change?
49	B	… the movement.
50	F	The movement. But in what you call table, what does it appear vertically?
51	Some	The position!
58	F	So, to have a vertical line [*miming it*] it should be … I stop time, but?
59	D	the position changes.
61	Some	Yeah [laughing]
62	EF	so, it's impossible!

FINAL REMARKS

As the analysis shows, making sense of the vertical line arises from a recollection of the past experience with various motions. It develops in an effort of imagining such a line in terms of relations between time and position. Necessarily, the effort requires to model time as a flowing variable, in the Newtonian sense. It is not a simple effort, due to the relative idea of time children have. The multimodality of their thinking processes is a picture of this attempt. Gestures and words are well coordinated with each other, and they are functional to the mathematical understanding. In particular, gestures performed horizontally serve to represent time as never stopping. Certainly, this is related to the way the data are collected. For modelling motion, time has to be measured in an interval, in which it always goes on. The possibility of stopping time is real only if the data collection is interrupted. On the other hand, there is a relevant mathematical issue in the interpretation of a vertical line as a graphical representation of motion. Stopping time entails that there is no longer motion to be modelled. Indeed a vertical line means infinitely many different positions at the same time, which is impossible in terms of actual motion. The mathematical reason of this impossibility lies in the fact that *the graph of a vertical line does no longer represent a function*: it loses the basic property of assigning to each element of a given set (i.e. the domain) *exactly one* (*one and only one*) element of another given set (i.e. the codomain). Our interpretation is that the success of the arguments by G, EF and B comes from the knowledge they acquired on the sense of horizontal lines representing motions. To understand the meaning of a horizontal line one has to know that time passes even if the object to be moved does not actually move. In mathematical terms, this means the same position at different times. And it is the analogue interpretation for a vertical line, if we think of exchanging the role of the two variables of time and position in the graphical representation. We also believe that the success of this kind of activities depends on the use of technological devices. Technology allowed us to foster the multimodality of learning and to recover the epistemological roots of the concepts, making them cognitive roots for the children.

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KINDERGARTEN CHILDREN CAPABILITIES IN COMBINATORIAL PROBLEMS USING COMPUTER MICROWORLDS AND MANIPULATIVES

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The purpose of this paper is the study of kindergarten children's capabilities while they engaged in problem solving concerning the production of combinations of one or two from a set of four objects with repetition. Two groups of children initially solved the problem using two microworlds without and with feedback and scaffolding mechanisms correspondingly. Finally, all children solved an equivalent problem using manipulatives. Video recordings of the children activity have been analysed in order to understand the effects of microworlds features in their performance.

INTRODUCTION

The interest of mathematics education research in the development of combinatorial reasoning of young children has been increasing in the last years. Basic principles and concepts of combinatorics support the introduction and development of other components of school mathematics curriculum e.g. counting, computation, probability etc (Batanero et al., 1997; English, 1991, 2005). Moreover, combinatorics constitutes a core element for other subjects of the curriculum like computer science for which discrete mathematics is considered key knowledge (Marion & Baldwin, 2007). Engaging children in combinatorial problem solving could, among others, foster independent thinking, encourage flexibility in approaches and representations, help the focus in problem structure and encourage sharing of solutions. According to NCTM standards (2000), young children should have opportunities to engage in simple combinatorial problems using tables, diagrams, list construction etc. However, combinatorics extent in the K12 mathematics curriculum is usually inharmonious with its importance while the relevant educational research is rather limited in comparison to other mathematics subjects.

The aim of this paper is to investigate the capabilities of kindergarten children while they engaged in problem solving concerning the production of combinations of one or two from a set of four objects with repetition using several learning environments, exploiting software microworlds and manipulatives.

THEORETICAL BACKGROUND

Recent research has revised the points expressed by Piaget and his colleagues (Inhelder & Piaget, 1958) about the development of combinatorial reasoning of children. According to the Piagetian view, children even in the age of 7-8 years have not developed the ability to invent a systematic strategy for the production of all the possible combinations for problems 2x2 or 3x3. Nevertheless, recent research results

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claim that through an appropriate problem framework, closer to the experiences-interests of children, and exploiting their informal knowledge, kindergarten children could perform successfully in meaningful learning activities involving combinatorial problems.

English (1991) has studied 4-9 years old children strategies in combinatorial problems (combinations of two objects with repetition) and she has described six strategies of increasing effectiveness according to children method for object selection and combination. More specifically for children 4-6 years old the research describes the following strategies: **A**. Random selection of objects without rejection of repeated combinations, **B**. Trial and error process with random selection of objects and rejection of repeated combinations. Each new candidate combination is compared with all the (already) existing ones in order to ensure that it does not constitute a repetition. **C.** Emerging pattern for the choice of objects with the rejection of repeated combinations. The children select systematically one of the two objects or backtrack in a previous strategy presenting instability in the application of the pattern. The difficulties in the production of combinations for children of preschool age include: repetition of combinations, production of permutations, and omission of some combinations, due to the lack of a systematic strategy. Furthermore, other researches have shown that the type of problem, the size of numbers, and the type of educational material influence the solutions of children in combinational problems (cf. Batanero et al, 1997).

In parallel, many researchers designing microworlds (Papert, 1980) for the improvement of mathematics education (Edwards, 1995; Hoyles et al., 2002; Stohl & Tarr, 2002) have shown that the activity of student with the microworld can be an important source of diagnostic information for the researcher and the teacher (Noss, & Hoyles, 1996).In this paper we present the main findings from a research on the reasoning and performance of kindergarten children during their engagement in a problem solving situation concerning the production of combinations of one or two elements from a set of four objects with repetition. Two groups of children initially solved the problem using two microworlds without and with feedback and scaffolding mechanisms correspondingly. A few days later all children that participated in the experiment solved an equivalent problem using manipulatives.

The research questions were as follows:

1. Are kindergarten children able to produce combinations of one or two objects from a set of four with repetition?
2. Are there qualitative and/or quantitative differences in the overall performance, as well as any errors of children that depend on the kind of microworld?
3. Does the exposure of children to the two different microworlds produce any differences during the manipulatives use?

METHOD

In order to investigate the research questions we applied a research experiment (Cohen et al., 2000) in which the kindergarten children used the microworlds designed for diagnostic purposes and then, after a while they used manipulatives to solve a similar problem in order to study any persistent differences.

The microworlds

Two similar microworlds (named μK-0 and μK-1 respectively) have been designed for the research. For the development of the microworlds we used the Scratch programming environment (MIT Media Lab, 2008), a very good rapid prototyping platform. Their user interface is depicted in Figures 1 and 2.

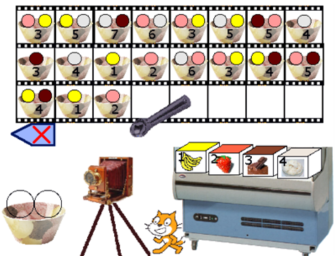

Figure 1. μK-0 (simple) interface

Figure 2. μK-1 (support) interface

Both microworlds make it possible to create of ice creams with one or two scoops choosing from four different flavors (B-Banana, S-Strawberry, C-Chocolate, V-Vanilla). User may take a photo of the ice cream and keep it along with the ice cream price in a price list - photo album, appearing on the top of the screen. The differences between the two microworlds are based on the provided feedback and scaffolding mechanisms. Microworld μK-0 does not provide any feedback or scaffolding information to the user that could help in the production of all the combinations. In contrast, the μK-1 microworld prevents the definition of duplicate combinations by informing the user about the mistake while in addition pointing out the definition of permutations. This information communicates the errors to the user and constitutes the **feedback** mechanism of μK-1. Furthermore μK-1 provides two **scaffolding** mechanisms. The first scaffolding mechanism is triggered by clicking on the cat which in response gives a sound and written hint about which flavor could contribute to the production of combinations that the user has not created yet. The second mechanism is triggered by clicking on the cloud (showing the "thought" of the cat) and thus giving away a combination that has not been defined up to that point. Both scaffolding mechanisms inform user of the end of the process in case there are no more possible ice creams. It is obvious that μK-1 provides enough cognitive tools for the user to produce all the possible combinations, while μK-0 permits the assessment of user performance under no support conditions. In the research we sought to study the effect, if any, of this supporting interactivity on the performance of the children.

Participants and process

30 kindergarten children participated in our research that took place in October and November 2008. The age of the children was from 4:09 (years:months) to 5:08 with average age 5:01. The children were organized in two random and equivalent (in terms of members, age, and sex) groups in order to work with the corresponding microworlds. The research program was organized in the following phases:

P1. Problem introduction: In the first phase the teacher-researcher introduced the problem to the children using a story, in which Nicolas, the ice cream seller, wanted to create a price list for his gelateria. The narration of the story was mediated with an electronic presentation in which the children had the opportunity to get familiarized with the symbols used for the price list in the microworlds. During the story the children were also introduced to the physical constraints of a price list (all the combinations, no permutations, no duplications). At the end, the children were asked to help Nicolas construct the price list using computer software. The mean duration of this phase was about 5'.

P2. Microworld familiarization: In this phase the children were introduced to the microworld with exhibition of their functionality and hands on practice. The mean duration of this phase was about 2' minutes for the exhibition and 1' minute for the practice (construction of the first ice cream photo). The children overcame their initial difficulties and got familiarized with the use of the software rather quickly.

P3. Problem solving: In the third phase the children tried to create the price list producing all the possible ice cream combinations. The average duration of this phase was 20' for μK-1 and 17' for μK-0. During the problem solving phase, the researcher's intervention was limited to the least possible, needed to help children manage any user interface or technical matters.

P4. Problem solving with manipulatives: Two weeks after the use of software, the same children were asked to produce all the combinations of fruit juices with one or two fruits from a set of four (A-Apple, B-Banana, O-Orange, P-Peach) with repetition. The fruits were printed paper cutouts which the children glued on a cardboard. A sample solution is depicted in figure 3.

Figure 3. Sample problem solution using manipulatives

In each phase, the children worked individually with the discrete presence of the teacher-researcher.

Data collection

The teachers' and the children's talk, their actions on the computer screen, as well as the children's faces were video recorded using screen casting software. The video has been transcribed producing an observation table for each child with all the interesting actions of the children (ice cream definition, deletion, support triggering etc) time stamped. In the case of phase 4, research data comprised of the final products of the children that have been represented in corresponding data tables showing the codes of combinations produced along with their production order. These raw data permitted the quantitative and qualitative analysis of performance and reasoning of the children.

RESULTS

The children were able to produce combinations in several levels of efficiency in all learning environments (microworlds, manipulatives) even the fact that they do not follow any systematic pattern. The following table (table 1) contains the frequency distribution for the number of combinations in the children's solutions per learning environment.

	MICROWORLDS			MANIPULATIVES		
COMBINATIONS	TOTAL	μK-1	μK 0	TOTAL	μK-1	Mκ-0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	1	0	1	0	0	0
4	1	1	0	1	0	1
5	4	1	3	0	0	0
6	4	0	4	3	1	2
7	5	5	0	1	0	1
8	1	1	0	2	1	1
9	3	0	3	8	6	2
10	1	1	0	7	3	4
11	3	1	2	1	0	1
12	1	0	1	0	0	0
13	2	1	1	1	1	0
14	4	4	0	6	3	3
CHILDREN	**30**	**15**	**15**	**30**	**15**	**15**

Table 1: Frequencies of solutions per number of combinations for the three environments

The kindergarten children using μK-1 had a slightly better performance in terms of mean produced combinations ($mean_{\mu K\text{-}1}=9.467>mean_{\mu K\text{-}0}=7.733$), but this difference was not statistically significant (Wilcoxon signed-rank test / Upper-tailed test: V=83, n.s). This seems strange because it would be reasonable to hypothesize that children using the scaffolding mechanisms of μK-1 would more easily find all the combinations. As it is going to be presented in the following paragraphs most children did not use the scaffolding mechanisms. Similarly, the slightly better performance ($mean_{\mu K\text{-}1}$=10.200> $mean_{\mu K\text{-}0}$=9.467) in the problem with the manipulatives by the children who used μK-1 was not statistically significant.

Although there was not a significant difference between the groups using the different microworlds, we found that there was a statistical significant difference in the performance of children using the manipulatives. More specifically the overall performance of children using software was significantly worse than their performance when they used manipulatives (Wilcoxon signed-rank test / Lower-tailed test: V=134, p=0.023). The difference was significant even in the level of the two different groups. It seems that independently of the microworld they used in their first activity, the children's performance is significantly better when working with the manipulatives. While a possible interpretation could be that the children had been trained by the first activity and had developed their skills an alternative explanation could be that manipulatives is a better educational environment for kindergarten children than the specific microworlds for this kind of problems.

The use of supporting mechanisms

A very small number of children exploited the supporting mechanisms of μK-1. As the following table (table 2) shows, only 5 children used and followed the supporting mechanisms.

	TOTAL COMB/NS	ASKED THE CAT	FOLLOWED THE HINT	ASKED THE CLOUD	FOLLOWED THE HINT
ch1	13	1	0	1	1
ch3	14	2	2	3	3
ch5	14	2	2	3	3
ch7	7	2	2	1	1
ch8	14	1	1	1	1

Table 2: Use of the supporting mechanisms of μK-1

It is interesting to notice that most of them had a very high performance in combinations production (ch1, ch3, ch5, and ch8). The small percentage of children that took advantage of the supporting mechanisms could explain the insignificant differences between the groups that used the two microworlds. However, this finding poses the question why the children did not use the support even though the teacher-researcher had showed it to them. A possible interpretation is that the children did not understand the information, or they did not want to get help and preferred to use their own skills.

The analysis of errors

The potential errors in these problems were the production of permutations, the repetition of combinations and the omission of combinations. The omissions are directly depended on the performance presented previously so they are not analyzed any further. Table 3 shows basic statistics for the synopsis of errors.

	PERMUTATIONS						REPETITIONS					
	SOFTWARE			MANIPULATIVES			SOFTWARE			MANIPULATIVES		
STATISTIC	TOT.	μK1	μK0	TOT.	μK1	μK0	TOT.	μK1	μK0	TOT.	μK1	μK0
Children	30	15	15	30	15	15	30	15	15	30	15	15
Median	1.5	1	2	0	0	0	4	1	8	0,5	0	1
Errors	55	26	29	9	7	2	182	58	124	54	30	24
Mean	1.83	1.73	1.93	0.3	0.47	0.13	6.07	3.87	8.27	1.8	2	1.6
St. dev. (n)	1.37	1.34	1.39	0.59	0.72	0.34	6.19	6.06	5.49	2.18	2.42	1.89

Table 3: Descriptive statistics of the permutations and repetitions

Children made 55 **permutations** using microworlds while they produced only 9 using manipulatives. This difference was statistically significant (Wilcoxon signed-rank test / Upper-tailed test: V=430, p=0.0001). There was no significant difference between the two groups of children as far as the permutations are concerned in software or the manipulatives activity. The production of permutations even with the μK-1 shows that children did not exploit the available feedback mechanism. Only three children were deleting the permutations.

Children made 182 **repetitions** using software and 54 with manipulatives. This difference was statistically significant; in addition there was a significant difference between the group used that μK-1 in comparison to the group that used μK-0 during the software activity (Wilcoxon signed-rank test / Lower-tailed test:, V=25, p=0.025). Even though such a difference seems reasonable considering the feedback mechanism of μK-1, it is worth pointing out that in our analyses we took into consideration all the repetitions produced by the children throughout the process, rather than just the ones that were present in their final solutions. So, it seems that a strong feedback mechanism is possible to have stronger results. There was not a significant difference between the two groups in repetitions during the manipulatives activity.

CONCLUSIONS

Children can engage quite successfully in combinatorial problems provided that they are presented in an authentic and developmentally appropriate manner. Kindergarten children are able to produce combinations with repetition yet without applying any systematic strategy. Despite the initial research hypotheses, the microworlds used within this research framework did not seem to influence children performance and errors except for the duplications. Furthermore there was no significant microworld influence in children performance with the manipulatives. The children participating in the experiment showed significant decrease in errors from the use of software to the use of manipulatives. This could be obtained at least because of the learning during the software use or because manipulatives are more appropriate for the children and this kind of problems. Further research could give more information on this question. In addition more questions were raised by the research about the feedback and support mechanisms and their use by the children. The existence of such mechanisms is not a capable condition for their exploitation. More research is needed about how these mechanisms could be adopted by children in order to self-regulate their learning.

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DEVELOPING A MICROWORLD TO SUPPORT MATHEMATICAL GENERALISATION

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This paper outlines the development of a microworld to address students' cognitive difficulties in expressing generalisation. We sketch five critical ideas that have informed our iterative design: (a) providing a rationale for generality, (b) supporting simultaneously model construction and analysis, (c) scaffolding the route from numbers to variables, (d) working on a specific case 'with an eye' on the general and (e) reflecting on derived expressions. The paper concludes with a summary of insights gleaned from the trials with students and plans for future development

INTRODUCTION

> I am asking whether one can identify and teach (or foster the growth of) something *other* than algebra or geometry, which, once learned, will make it easy to learn algebra and geometry. No doubt, this other thing (let's call it the MWOT) can only be taught by using particular topics as vehicles. But the "transfer" experiment is profoundly changed if the question is whether one can *use* algebra as a *vehicle for deliberately teaching transferable general concepts and skills."* (Papert, 1972, pp. 251).

The problem, as Papert eloquently states, is that the means by which generalisation is to be expressed – algebra – may not be a 'particularly good vehicle' for achieving what we are really trying to teach: an appreciation of generalisation as a crucial and powerful mathematical idea. The challenge, therefore, is to design a new vehicle with the help of digital technologies and its dynamic potential: this is the main objective of the MiGen project[i]. Its aim is to design, build and evaluate a pedagogical and technical environment for improving 11-14 year-old students' learning of mathematical generalisation. The core component of this system consists of a microworld which attempts to address the problem of generalisation by supporting students in their reasoning and problem-solving of a class of generalisation tasks.

The work presented in this paper arises from data gathered in various design experiments with 11-12 year old students from UK mathematics classrooms. Space constraints mean we are not able to present the data that provided us with the evidence for our design decisions: more detailed examples with data will be given in the presentation. Our focus in the paper is on the problem of *design*: we will outline five main principles that guided and evolved the development of the system and our research priorities. We begin by presenting students' known difficulties with mathematical generalisation. We then briefly outline our methodology, before sketching the five design principles that have emerged from the work. We conclude with some remarks regarding the effectiveness of our system and our future goals.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 49-56. Thessaloniki, Greece: PME.

STUDENTS' EXPRESSION OF GENERALISATION

Appreciating what generalisation is for or what its classical language of expression – algebra - is all about, are elusive ideas for secondary school students. Even though students are able to identify and predict patterns (Mason, 2002) they are much less able to articulate a general pattern or relationship in either natural language or – more demanding still – in algebraic symbolism (Hoyles and Küchemann, 2002). The challenge, therefore, is to design situations that are rich in affording opportunities for the construction and analysis of patterns.

Generalisation activities are often presented as numeric or pictorial sequences with tasks that encourage students to predict the number of elements in any position in the sequence and spot the rule by comparing consecutive terms in the sequence (e.g. Moss and Beatty, 2006). This "pattern-spotting" is used as a strategy to help students find the general algebraic rule for such sequences. There is a tendency to teach students abstracted techniques to find these rules or to view this method as a set of rules to follow in certain tasks (Sutherland and Mason, 2005).

There is also a tendency for students to develop rules from a few examples, which prevents them from seeing beyond the particular case or the need for their rules to be justified. This failure to see beyond particular cases, focusing instead on specific calculations, is what restricts students' thinking strategies to the specific rather than the general: and it is one instance where algebra – with its elegant and concise means to express generality based on the specific – may, once again, be an inadequate vehicle for learning to express generality.

The literature is replete with examples of this difficulty. For example, struggle to understand the idea behind using letters to represent any value (Duke & Graham, 2007) and are inexperienced with using mathematical vocabulary to express generality. It is evident from a wide range of research (e.g. Warren and Cooper, 2008) that even students who are capable of expressing a general rule through the use of words, like 'always' or 'every', but struggle to use letters and symbols.

Finally, moving flexibly between different forms of representations has been identified as a valuable skill and, moreover, one that is feasible for students (Hoyles & Healy, 1999). However, even though students are usually provided with various representations and models to assist them in their efforts to understand, justify and communicate arguments (Greeno & Hall, 1997), students tend not to make expressive links between these different forms of representations (symbolic, iconic, numeric) neither perceive their equivalence. Consequently, the challenge is to find ways through the right design of tasks and the use of digital technologies to assist students in appreciating this powerful skill. More significantly, the struggle to help students perceive algebraic (in whatever form) representation as a structural description of general relationships demands some way of creating representations which *themselves* transcend the separation of symbolic, iconic and numeric.

METHODOLOGY

The overall methodological approach is that of design experiment, as described by Cobb et al. (2003). Throughout the development of our microworld, which we have named the *eXpresser*, we have followed an iterative design process, interleaving software development phases with one-to-one and classroom sessions with year 7 students (aged 11-12 years old). In addition, we have integrated feedback from teachers and teacher educators as well as the students who participated in the studies. As part of this process we have mapped out a schedule of activities to cover a complete process of construction, collaborative exchange, expression and reflection.

We report the design implications arising from several one-to-one sessions with students and three large classroom studies in which their teacher was present. All sessions were recorded on video and later analysed and annotated with the help of the researchers' written observations during the sessions. One of our goals during these sessions was to inform our system's design and evaluate the effectiveness of the microworld in terms of students' ability to construct and analyse pictorial patterns, but also their ability to articulate the rules underpinning their generalisation of the patterns.

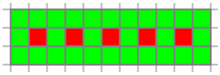


Figure 1. For any given number of red tiles (black), find a rule for the number of green tiles (grey) that are needed to construct this pattern.

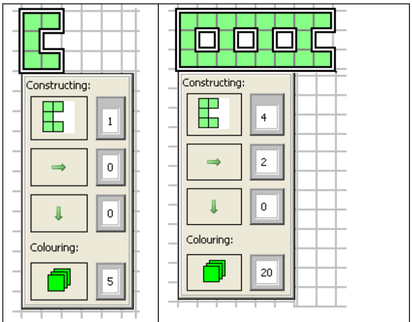
In all our studies students are first introduced to the system and its affordances through simple tasks. Subsequently, they are asked to investigate, construct and express relationships that underpin a pattern like the one in Figure 1. These types of tasks have the potential to emphasise the structural aspect of patterning rather than the purely numerical and hopefully divert students from "pattern-spotting" techniques often practiced in (UK) mathematics lessons. In addition, these tasks provide a valuable opportunity in classroom sessions where students are asked to collaborate with other students and share their constructions and expressions, thus leading to discussions about the equivalence of expressions and the generalisation process itself.

THE EXPRESSER: A MICROWORLD TO SUPPORT GENERALITY

We now outline 5 key design elements of the microworld, giving short illustrations of some.

Providing a rationale for generality

We hypothesised, that by presenting tasks that were dynamically changing, students would be less likely to focus exclusively on particular cases, and take too literally questions that ask 'how many' tiles there are in a given configuration (understandably, a perfectly reasonable strategy is simply to count!).We conducted studies (see Geraniou *et al.*, 2008) to assess the extent to which dynamically-presented tasks discourage the ‘pattern-spotting’ or ‘counting’ approaches discussed above that are associated with paper-and-pencil presentation. Accordingly, the pattern in Figure 1 is shown dynamically; it changes regularly showing a different instance of the pattern each time. This makes it impossible for students to count the number of red or green tiles while allowing them to ‘see’ the variants and invariant parts of the pattern. This presentation provided a rationale for deriving a rule that outputs the number of green tiles for any instance of the pattern, i.e. a ‘general’ rule giving concrete instantiation to the meaning of ‘any’.



(a) (b)

Figure 2a. The construction of a ‘unit of repetition’ (a C-shape) from 5 single tiles grouped together and coloured appropriately. **Figure 2b.** The construction of a pattern by specifying in the property list the number of repetitions (4) of the unit, the position of each repeated unit shape (2 across and 0 down) and the appropriate number of tiles.

Supporting simultaneous construction and analysis

In eXpresser, linear patterns are specified by indicating in their property list the number of repetitions of a ‘unit’ shape (see ‘Constructing’ part of the property list in Figure 2) and the position of each repetition. Our studies with previous versions of the system suggested that students needed a direct experience with the number of tiles needed to construct the unit of repeat of a pattern by editing properties that specify not only the construction of the pattern (i.e. ‘how’ and ‘how many times’ it is repeated) but also properties that engage them directly with the number of coloured tiles needed (see ‘Colouring’ part in Figure 2). This facilitates later the construction of a rule that suggests the number of coloured tiles required as a coefficient in an algebraic expression (later explained in Figure 4).

Scaffolding the route from numbers to variables

If students are to express generality, they need to construct dependencies within and between objects. That requires the use of an ‘algebra’ to specify relationships. For example, consider the case where a student visualises the pattern (as is usually the case in our studies) as repeated inverted-Cs wrapped around a red tile. She may first create an object of green tiles as shown in Figure 2a,b. Having also constructed another object of red tiles, she has to specify the relationship between the red and the green tiles in order to make a general pattern that can be potentially shown dynamically in the system.

From a pedagogical point of view, we had to find a rationale for students to think about the particular with an eye on the general. This is, perhaps, the crucial pedagogic challenge: the student can *only* work with a particular example, and yet the solution to a given task is supposed to be for an infinite number of invisible and unspecified cases. One solution was inspired by the "messing-up" technique borrowed from Healy et al (1994), in which failure to attend to the structural relationships of (dynamic geometry) figures, resulted in their collapse when dragged. Students were challenged to construct a solution that was impervious to "messing-up": i.e. a construction that would be valid even if the values for the unit of repeat of a pattern were changed: for constructions that are not entirely general, changing the number of (independent variable) tiles breaks the pattern. In this way, changing various parameters of the problem provides a simple yet powerful mechanism for the student to judge whether a pattern is general or not, and provides students with the opportunity to realise that there is an advantage to thinking in terms of abstract characteristics of the task rather than only specific values.

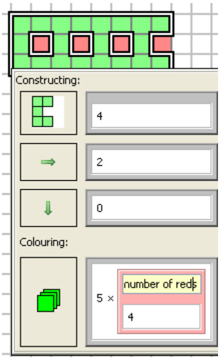


Figure 3. Placing a tied number, dragged from the property list of the "red" pattern, to the corresponding place in the property list of the "green" and after writing the correct expression for the number of greens, links the two patterns.

The eXpresser allows the creation of relationships between patterns using a feature that we refer to as 'tied numbers' (see Figure 3). This process encourages students to recognise the independent variables in the task and express relationships, reason and reach generalisations using these variables rather than using specific numbers. This stands in contrast to the standard approach in which generalisations are constructed from special cases, and the path to the variable 'n' appears as a separate (often non-negotiable) cognitive leap.

Working on a specific case 'with an eye' on the general

In our efforts to help students see the general through the particular, we designed the the system so as students see a different instance of the pattern in question *while they are constructing a specific one*. While constructing, students see how their construction would look if the values of selected parameters (i.e. the independent variables of their current construction) had different values.

This 'parallel' construction takes place in a separate window on their screens, which we name the 'parallel world' and unfolds their construction with other values (randomly chosen by the system) for the independent variables. The mechanism of parallel worlds provides another way to see the general through a particular instance the student has chosen, providing a further incentive to construct with general expressions. A general construction can of course only emerge if the student refers to multiple instances through the language that the microworld provides (tied-numbers) that are applicable across the parallel world.

Reflecting on derived expressions

The final step for the generalisation activity to be completed is the creation of an expression that gives the general rule for the activity in question, using icon-variables. As discussed in the second section, one of the main difficulties is the use of variables for expressing and operating with unknowns (e.g. *n* representing the number of red tiles in the pattern) and therefore we were conscious that using letters to denote variables was likely to prove difficult for learners. We have therefore decided to use icons to represent variables (see Figure 4).

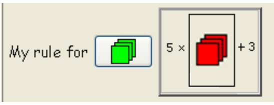


Figure 4. A final expression where students provide a rule for the number of green tiles based on the number of red ones.

Through our trials, we have found these icon-variables provide a way to identify a general concept that is easier for young learners to comprehend and use in their articulation of generality. They can be operated on in exactly the same way as other expressions, i.e. copied, deleted, used in operations (such as addition) and used to define other objects, but in a dynamic way that gives immediate instantiation of the expression. Our claim, to be further validated, is that these iconic representations serve as an intermediate step that eases the articulation process and supports the transition to and from the numeric to the symbolic since the icons not only provide a representation of the concept but also the variable and its current value.

In fact, the pattern cannot be dynamically presented in the parallel world unless an expression for the number of tiles is constructed. In order to write their expression, students need to specify first what their expression is for (e.g. number of green tiles); a step designed to encourage students' reflection on their own actions. This process allows students to validate the generality of their final rule as well as a means to express their generalisations 'symbolically'. Furthermore, in classroom sessions, students are requested to share their expressions and justify them to other students. Such a collaborative activity aims at emphasising the equivalence of seemingly different expressions and discussions, which leads to reflections on the notion of generalisation.

CONCLUSIONS

The fundamental idea of the eXpresser is that patterns can be built and analysed so that the general case – "n" – is held in mind throughout the process, rather than being a final step. In traditional classrooms, it is easy for teachers to delude themselves into thinking that a sequence of carefully designed tasks will lead the student from the particular to the general, but this is not easy and the risk of students failing to reach the learning objectives is high. Our continuing attempt to build a system that simultaneously allows students to work with the particular and general is ongoing, but we believe that the five key ideas are getting us one step closer to reaching our goals.

Our alternative 'vehicle' is taking shape, though there is much still to be done. In particular, our work on the development of many iterations of the microworlds has made clear that the fidelity of a microworld to a mathematical idea is not enough: there needs to be a complementary design process of activity sequence - a sequence that exploits the potential of the microworld and incorporates opportunities for the teacher to orchestrate sharing, group tasks and reflection. Thus we are also in the process of implementing mechanisms for *supporting* both students and teachers by prompts and appropriate tools to assist before, during and after lessons.

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CAUGHT IN THE MIDDLE: TENSIONS RISE WHEN TEACHERS AND STUDENTS RELINQUISH ALGORITHMS

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Relinquishing algorithms in favour of less-conventional approaches for calculating is a challenge for both students and teachers. Analysis of a Roundtable Reflection session with Grade 3 and Grade 4 teachers reveals that tensions for teachers arise: when students automatically choose to use algorithms rather than alternative approaches; when they contemplate their class's transition to the next teacher; and when teachers think about parent/community/curriculum expectations concerning calculation processes. It is essential to address these tensions to sustain desired changes in teaching practice.

INTRODUCTION

Life in the twenty-first century requires an unprecedented level of mathematical knowledge and skill for full participation in community life, and for access to opportunities in education and employment. It is therefore of concern that recent research suggests that some Australian students under-perform in mathematics despite consistent attempts to improve mathematics learning and teaching. For example, Gervasoni, Turkenburg, & Hadden (2007) found that 30% of students beginning their final year of primary school in regional Victoria have underdeveloped arithmetic reasoning strategies, a key indicator of mathematical competence. Improving mathematics learning and teaching in the primary school is still necessary.

This paper reports on one aspect of a research program that overall seeks to discover whether a supported professional learning approach using *Self-Study* as a methodology and *empty number lines* as a catalyst for improving mental computation will challenge conventional mathematics learning and teaching and result in better learning outcomes for students in Grade 3 and Grade 4 that are sustained throughout the primary school. The aspect considered here are the tensions that arise when teachers delay teaching algorithms until later in the primary school and instead emphasise mental computation and subsequent recording of associated reasoning on empty number lines. Insight about these tensions is provided through analysis of themes emerging during a Roundtable Reflection session involving six teachers and two researchers. Revealing these tensions provides important insight for those engaged in designing professional learning programs and new curriculum that aims to improve children's arithmetic reasoning and associated teaching practice. Anticipating the tensions that teachers may experience when this is attempted, and knowing the type of support that teachers and students may require will assist new approaches to be more effective and sustainable.

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CONSIDERATIONS FOR CHANGING TEACHER PRACTICE

Teachers are often reluctant to identify and alter habitual ways of teaching. Consequently, new curriculum policy may not transfer to changes in teaching practice. This issue is important to address when introducing an innovation that involves delaying the teaching of algorithms in favour of greater emphasis on mental computation. The concepts underpinning the research reported in this paper (Self-Study, Roundtable Reflection and empty number lines) seek to address this issue.

Self-Study and Roundtable Reflection

Self-Study methodology (Berry & Loughran, 2002; Brandenburg, 2008; Bullough & Pinnegar, 2001; Hamilton & Pinnegar, 1998; Russell, 1999) and Roundtable Reflection (Brandenburg, 2008) are features of this research. Self-Study has emerged from the action research/reflective practice/ethnographic research traditions, and is an important development in teacher education research (Zeichner, 1999). An expectation of Self-Study is that teachers identify the issues and tensions they experience and unpack these using systematic reflection as a means of understanding more deeply the ways that their teaching impacts on student learning. Teachers theorise their learning through ongoing data analysis and through experiencing changes in understanding that demands enactment in practice of new learning (Loughran, 2006). Learning is named and scrutinised during Roundtable Reflections.

Roundtable Reflection (RR) involves structured pedagogical inquiry that aims to identify and challenge assumptions (Brookfield, 1995) which more often than not, are taken-for-granted. Fundamental to the success of the RR approach is the establishment of a secure, trusting environment where each participant's voice is acknowledged and respected. Each RR begins by teachers identifying and briefly describing a critical incident/interaction that has occurred in their classroom. The group then prioritises, and determines one incident to be scrutinised in detail. Focused questioning and subsequent challenging of assumptions about learning and teaching enables some confrontation and de-stabilisation of practice (Segall, 2002).

Using "empty number lines" as a catalyst for improving arithmetic reasoning

The mathematical focus for this research is mental computation and arithmetic reasoning strategies, which have been the focus of many studies (e.g., Clarke, Cheeseman, Gervasoni, et al., 2002; Fuson, 1992; Gervasoni, 2006; Steffe, Cobb, & von Glasersfeld, 1988). Not all children have these strategies available or choose wisely to fit the characteristics of a strategy to the demands of a task (Griffin, Case, & Siegler, 1994). Narode, Board, & Davenport (1993) suggest that introducing algorithms too early in schooling is detrimental to students' developing arithmetic reasoning strategies. However, it is common for Australian teachers to introduce Grade 2 and Grade 3 students to algorithms for addition and subtraction. This is contrary to curriculum guidelines that no longer deem this necessary (Victorian Curriculum and Assessment Authority, 2008). Indeed, it is now broadly accepted that the conventional focus on taught procedures for calculating can negatively impact on children's number sense (Clarke, Clarke, & Horne, 2006). It is proposed that a more effective approach is to delay the introduction of algorithms and instead focus on mental computation. An associated practice is to encourage students to record their thinking on empty number lines, so that monitoring and reflection on strategy choice may occur. This approach is widely used in the Netherlands and aims to link early mathematics activities to students' own informal counting and structuring strategies (Beishuizen & Anghileri, 1998). Monitoring strategy selection by written recording on empty number lines and through reflection on strategy choice in class discussion is essential to stimulate contraction towards higher-level strategies (Beishuizen & Anghileri, 1998). This approach seldom occurs in Australia.

TENSIONS WHEN TEACHERS LET GO OF ALGORITHMS

The research reported in this paper involved trialling a curriculum reform in which the teaching of algorithms for Grade 3 and Grade 4 students was delayed in favour of emphasising mental computation, with students' reasoning strategies recorded on empty number lines to enable monitoring and reflection of strategy choice. The research involved six teachers from two schools in a low-socio-economic area of a regional Victorian city. The curriculum reform aimed to improve students' arithmetic reasoning strategies. Assessment of students in the region at the beginning of 2008 using the Early Numeracy Interview (Clarke, Sullivan, & McDonough, 2002) showed that many of the students were using counting-based strategies rather that reasoning strategies to solve the following problems: 4+4; 2+19; 4+6; 27+10; and 10-7 (44% of Grade 3 students, n=584, N= 1314; and 22% of Grade 4 students, n=282, N =1261).

The research approach emphasised teacher professional learning using Self-study and Roundtable Reflection. Teachers met with the researchers for five professional learning sessions (June to November 2008) that considered the theoretical underpinning the approach, teaching advice and curriculum planning. An important part of each session was a Roundtable Reflection (RR) that was recorded, transcribed and analysed by the researchers. The data reported in this paper relate to the fourth Roundtable Reflection (RR4), conducted in Term 3 as part of a 2-hour professional learning session. The RR began with each teacher describing a critical incident that had occurred recently during a mathematics lesson, and then the selection of one critical incident by the group for detailed discussion and inquiry. For example, the critical incident presented by Sarah in RR4 follows.

Sarah — What I have found is that my top group automatically go to vertical addition [algorithm]….

Meg — Some of mine do too!

Sarah — They don't even attempt number lines ... they are quite happy right from the start just to quickly do vertical stuff and some of the little ones try to do it but then they are getting confused with it. So actually then we go back to the number line and they can see it so much more clearly…. Do you just let those kids go now, those top groups? Because basically….

you can't really try and teach them [to use number lines instead] can you? Because if they have been doing it [algorithms] since Grade 1 or 2….

The group chose this critical incident for detailed discussion and inquiry. It reflected a tension that most of the teachers were experiencing and were searching to resolve. Letting go of algorithms was proving difficult for both teachers and students.

Sally I would like to work on Sarah's because … I am finding that too. That there are kids that just want to do the vertical computation and…..

Meg I do too.

Sally And that's what they want to do and that's what they are good at. I think some of the kids think they are going backwards if they don't do it that way [vertical method] ….

Meg I'm finding that some of mine aren't as good at it [algorithms] as they think so they are making mistakes but I am not re-teaching it to them. I am not saying well, you know, remember this is where you trade….

Even though the tension associated with letting go of algorithms is clear in this excerpt, Meg's final statement acknowledges that some children really don't understand algorithms, a key reason for the approach being trialled in this research.

Key themes in the dialogue extract

Once RR4 was transcribed, the researchers identified the dialogue in the transcript that dealt with the critical incident chosen for discussion and inquiry. Seventy-four pieces of dialogue were extracted with each providing substantial comment about the critical incident. Each piece of dialogue was examined and any emerging themes identified. This process was repeated and the themes confirmed or refined. Table 1 presents the eight most commonly occurring themes identified in the extract.

Key Themes in the Dialogue	**Occurrences**
Children don't understand algorithms	11
Children are not letting go of algorithms	7
Using number lines for proof and for understanding	5
Timing of introducing algorithms	5
Community/parent/government expectations	5
Habitual use of algorithms or children automatically using algorithms when presented with calculations	5
Children don't understand number concepts	5
Transition to the next class/teacher	5

Table 1. Key themes in the dialogue extract

These themes highlight that some children have difficulty understanding algorithms, but also that once taught some children have difficulty relinquishing algorithms. In contrast, teachers believed that using number lines aided children's understanding and the ability to prove their calculations. Tensions for the teachers also centred on community expectations, transition to the next class and teacher, and timing for the introduction of algorithms. Some examples of the dialogue follow.

Children don't understand algorithms

Jan	When I introduced the algorithms to grade two, because they [the curriculum] said I should ... the majority of children were very mixed up and some of the answers they were getting showed they had absolutely no concept ... [that] sixty four and twenty one couldn't possibly be a thousand and something, but that is what they were writing. But since they have been using the number line, they have really increased their understanding. In fourth term, I don't know whether we should perhaps try some algorithms and see has there been a transfer of knowledge.

This extract highlights that students often do not understand algorithms or the answers produced. A further tension is revealed; even though the teacher explains that using the number line had been successful in building children's undersanding, she wonders whether algorithms should be attempted again in Term 4. This is likely due to her contemplating the expectations of the students' new teacher.

Using number line as proof and to build understanding

Sarah	One of my little ones the other day was trying to do the algorithm, which wasn't right, so then we actually went back to the number line to actually explain what she was doing.
Researcher	And what happened then?
Sarah	She could see where she had gone wrong and everything with it.

This extract highlights that the teacher chose to build the student's understanding through instruction using the number line. She also noted that students continue to use algorithms even though they may not understand them or recognise whether answers are correct. This relates also to the first and second themes in Table 1.

Timing of introducing algorithms

Leini	Sometimes you see them write answers to an algorithm that are like eight hundred above what it should even be because they don't even know [understand] the quantity.
Sally	They don't even know what this is.
Meg	Well that came out in the early numeracy testing didn't it (all agree) ... the lower numbers - they are fine, but when they had to estimate [numbers] in between [for larger numbers]….
Sally	[When] I have been teaching f{Grade] 5 and 6, I say, "now have a look at the sum, do it, have a look at your answer, is it reasonable?" You know

> and that kind of questioning as you are doing algorithms.... But it is also the timing and the scope and sequence [that] is important. But we aren't starting it off [using number lines] from the very beginning so we are caught in the middle of kids who already know how to do it [algorithms]– ... and they haven't had what we would like them to have [had] to get there [to the understanding of quantity].

This extract reveals an important tension for the teachers. Even though this research anticipated delaying the introduction of algorithms, in fact the students had already encountered algorithms in earlier classes. The teachers felt caught in the middle because they could not undo this situation, but had to deal with students who didn't have the number sense necessary to understand or successfully use algorithms.

Transition to the next class/teacher

> Jan I'm sure the Grade 3 teachers would like them to have done some vertical addition and so there is my dilemma.

Another example:

> Sally What happens if we have decided we are *out with algorithms*, and you know and it's great and there is sound philosophy behind it and we are all for it! We go to a Grade 5 teacher who goes, "This kid doesn't know how to do a division sum." You know it's that whole change, shifting.

Both examples provide insight about the tensions encountered when teachers contemplate the expectations of the next teacher. Tensions surrounding transition to the next class highlight the need for a whole school approach to curriculum change.

DISCUSSION AND CONCLUSION

Key themes that emerged from analysis of this Roundtable Reflection excerpt were: children not understanding algorithms; children not letting go of algorithms once they had encountered them; the usefulness of number lines for providing proof and for understanding computational reasoning; the timing of introducing algorithms; tensions associated with community/parent/government expectations; children's habitual use of algorithms or children automatically using algorithms when presented with calculations to perform; children not understanding number concepts involving 2-digit numbers; and tensions related to transition to the next class/teacher.

These themes provide insight about the tensions teachers face when they are implementing and sustaining curriculum change and the difficulties teachers and children have in relinquishing algorithms in favour of mental computation. We recommend that professional learning programs and school and system leaders support teachers by assisting them to identify and address the tensions that arise.

Overall, it seems that teachers in this research were *caught in the middle* between research-based innovative practice and the *tug* of more conventional practice. These Grade 3 and Grade 4 teachers were also *caught in the middle* between curriculum in the junior primary school that focused on mental computation, but also introduced vertical methods of addition and subtraction, and expectations in the senior primary school that calculation methods would be algorithm-based. One important theme that the teachers explored during the Roundtable Reflection was the optimal timing for introducing algorithms. They believed their students had encountered algorithms too early, and that this had impeded attempts to develop students' mental computation strategies. They proposed that Grade 5 was soon enough to introduce algorithms.

Meg	If [students] have got a good understanding of mental computation shouldn't it, when we go to teach them the algorithm, not be a problem?
Leini	Well that's, I suppose, the theory.

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EXAMINING THE ROLE OF LEADERSHIP IN THE SUPPORT OF MATHEMATICS – SCIENCE PARTNERSHIPS

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This case study examines the role of leadership within the school improvement process, and provides a unique view of the ways in which leadership interprets and enacts educational reform in mathematics. A three-year Mathematics Science Partnership project situated within a large urban school district provides the lens through which mathematics reform is studied. This case study consists of three parts, which (1) further explicate the nature of the gap between leadership theory and practice, (2) consider the complexities of individual and institutional learning within the school improvement process using the experience of the MSP project as the focus of analysis, and (3) suggest a process of school improvement that better facilitates essential interconnections between leadership and mathematics teachers.

INTRODUCTION

The local districts nested within the federal, state, and regional educational agencies that direct the school improvement process are challenged with monitoring and interpreting reform policies and practices in order to provide "coherent instructional guidance" (Spillane, 1998, p. 34) to school-based instructional staff. In response to this challenge, the project district encouraged the research team to create and implement school-based professional learning in mathematics to address the problem of low student achievement. The resulting Mathematics Science Partnership (MSP) project was designed to increase pedagogical content knowledge related to the mathematics that teachers must possess to teach effectively. This focused professional learning was based on the understanding that increasing teachers' understanding of content and pedagogy is considered to be a critical step in advancing student learning, particularly in mathematics (Schoenfeld, 2002, Remillard, 2005, Desimone, Smith, Baker & Ueno, 2005). Although the project work was concentrated on developing teacher content knowledge, the data collection revealed an unexpected connection between the individual and the institutional learning that occurred as a result of the district reform effort (Rorrer, Skrla, & Scheurich, 2008). The study revealed that advancing practitioners' individual learning in order to improve classroom practice is only one component of educational reform. Significant institutional learning must also occur if districts are to develop the necessary capacity to successfully enact the necessary, complex reforms needed in mathematics education.

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THEORETICAL FRAMEWORK

The guiding framework for this study is the product of a narrative synthesis generated by Rorrer et al. (2008) that analyzed the "essential roles of districts in systemic reform" (p. 313). The authors' resulting theoretical framework identifies four thematic strands that provide a basis for exploring "the complexity, interrelatedness, and nonlinearity of the district's roles and the ways that together these roles position the district as an institutional actor in reform" (Rorrer et al., 2008, p. 341). The four important roles districts assume in the work of educational reform are: (1) providing instructional leadership, (2) reorienting the organization, (3) establishing policy coherence, and (4) maintaining an equity focus. This framework served as an analytic tool that the research team used to aid our understanding of the role of leadership in the school improvement process and its ensuing affect on mathematics reform.

Research Question

The MSP project was a major professional learning component within a system-wide improvement effort that included a series of professional learning initiatives. The project district's concentration on systemic reform was in large part a response to the No Child Left Behind (NCLB) federal legislation (U.S. Dept. of Education, 2002) and analogous state requirements demanding increased district accountability for student learning. As the work of the MSP project unfolded in district high schools, it was apparent that the district served as more than the *context* in which the work would occur (Finn, 1991, cited in Rorrer et al, 2008). Rather, the project team soon realized that the district had assumed a critical role as an *institutional actor* in the learning. The district entity, which included the School Board, district-level administrators and principals, was actively engaged in the process of educational reform in a way that immediately affected the learning that occurred in individual teacher's classrooms. Thus, the research question that guides this descriptive case study is: How does the district entity and its interconnected roles, responsibilities, and relationships (Rorrer et al, 2008), facilitate or hinder school-based improvement?

MSP Project Methodology

The MSP project is invested in researching the potential for change when mathematics teachers actively engage in focused, intentional collaboration. The research model employs a learning community model and creates a unique tool for evaluating content knowledge for teaching mathematics. The research design evaluates the efficacy of the collaborative model as a means to develop the potential for reflective praxis and concomitantly, a true shift in teaching practice that results in improved student learning. The project design is exceptional in three areas: (1) the intentional development of an analytic tool to identify growth of content knowledge for teaching secondary mathematics; (2) a plan which leverages the collaborative learning community model, partner resources and generates new materials linked to State standards for broad public dissemination; and (3) a sustained focus on reflective practice situated within each teacher's building.

The project employs a quasi-experimental design with three categories of participants using the project-developed instrument. The instrument is administered to the first group during project-sponsored release time within the content collaborative community. A second group is given the instrument at the beginning and the conclusion of an outside professional development cycle. The instrument is also administered to a third group, not participating in a professional development program. Data collection is ongoing.

Data collection spans the three years of the project (We are currently in year 2.), and includes: researcher field notes, videotaped observations of individual/group work as well as study sessions and classroom lessons, teacher work and written reflections. Throughout teachers' interaction within the professional learning environment, we ask teachers their mathematical reasons for decisions (e.g. choices for tasks, questions to ask, assessment, ways of explaining, representations). Data is analyzed to reveal the degree to which participants demonstrate a deep and connected understanding of mathematics as seen in the many tasks of teaching, including: determining goals and objectives, motivating content, developing lessons (connectivity and sequencing), analyzing student methods and solutions to determine their adequacy or to compare them, unpacking mathematical ideas, procedures, and principles to make them more accessible to students, choosing representations, and designing and modifying tasks (Kahan, Cooper, & Bethea, 2003).

Our research is guided by an increasingly accepted belief that "teacher effects on student achievement are driven by teachers' ability to understand and use subject-matter knowledge to carry out the tasks of teaching" (Hill, Rowan, & Ball, 2005). However, what is not understood is how to foster teachers' ability to understand the underlying organization of the mathematics taught in schools in a manner that effectively transfers to classroom practice. Research suggests that professional development is most effective when it includes components embedded in the work of teaching and increases teachers' theoretical understandings of their work (Miller 1995). These understandings include knowledge of mathematics content and knowledge of pedagogy. Deborah Ball (2004) further identifies a third essential characteristic of effective teaching, content knowledge for teaching mathematics, and suggests that "…at least in mathematics, how teachers hold knowledge may matter more than how much knowledge they hold" (p. 332). Effective teaching requires deep understanding of procedural and conceptual knowledge, and "…the interplay between teachers' knowledge of students, their learning, and strategies for improving that learning" (Ball, 2004, p. 332). Both the project work and products focus on this third domain, researching the content knowledge needed for teaching secondary mathematics and teachers' abilities to transfer their growing understanding to practice to more effectively meet the needs of traditionally underserved students.

Professional learning communities provide the framework for collaborative professional development in a manner that expands Lave and Wenger's (1991) concept of communities of practice, to Wenger's (1998) further definition of a community, which "coheres through 'mutual engagement' on an 'indigenous' (or appropriated) enterprise, and creating a common repertoire" (Cox, 2005, p. 531). Thus, we create a context that supports the need for innovation and change, is more likely to be sustained (Cox, 2005), and result in improved teacher practice and concurrently, student learning.

FINDINGS

This descriptive case study reports on the preliminary findings of a mathematics professional development project embedded in two of the five district high schools designed around the following goals: (1) create a professional learning environment that connects and validates teacher professionalism to district/state-wide educational standards, (2) define the content knowledge for teaching (CKT) needed to successfully teach secondary mathematics, (3) create an analytic tool to measure CKT, and (4) create sustainable learning communities to increase teachers' CKT in the natural setting. Participants in the study were eighteen mathematics teachers from two district high schools, district curriculum coordinators, principals, central office administrators, and the project team including mathematics faculty representing five universities. This case study focuses on the interactions, interconnections, and roles that individuals and the district entity assumed as participants in mathematics reform.

To date, university staff has created and piloted the tool with a range of secondary mathematics teachers across four school districts state-wide. A facilitator cohort training occurred in August 2007, which provided school-based group facilitators with the necessary experience to "guide" collaborative group sessions. In year two, university faculty were embedded at the school sites in order to (1) administer the tool in schools, (2) press teacher thinking around the mathematical discussions, and (3) enact the data collection process. In particular, the research team is interested in being able to explicate claims about characteristics of and how CKT is different from other mathematics, and build an argument for the relationship to teaching practice. Of particular interest are aspects of mathematics that teachers attend to, and what norms, habits of mind, and dispositions have become apparent over time, through the work of the learning communities. This study reports on our findings through year two.

A key component to building effective learning communities is our focus on mathematical content and ability of the groups to identify a mathematically relevant and significant instructional problem that furthers their own understanding of mathematics teaching. The research team has developed and piloted an instrument to help guide teacher teams to identify a "rich" mathematical problem that lends itself to deep study of content and pedagogy and may result in a true shift in teacher thinking/practice as it "...depends not on abstraction of formulation, but on deepening the negotiation of meaning" (Wenger, 1998, p. 268). As teachers grapple with understanding a mathematical problem from within the complexity of their lived experience, they are better able to transfer newly understood mathematical knowledge to their classrooms.

Providing Instructional Leadership and Reorienting the Organization

The project team developed the MSP based on the district's interest in building school-based professional learning communities. In early conversations with the project team, the district administration clearly articulated its support for teacher-led professional communities, in which teachers would initiate the work of the communities and lead investigations, a "bottom-up" approach to professional learning. As the project work unfolded in the schools, it quickly was apparent that the district's theory of action did not correspond to its practice of enacted leadership. Rather, the district's decision to limit teachers' practice to a time-bound, district curriculum and unit assessments, as well as restrictions on professional learning time and resources represented a "top-down" effort to improve teacher understanding of content and pedagogy in mathematics. District resources, including outside and in-house professional learning sessions, coaching time, technology resources, district-created curriculum guides and assessments were aligned to this top-down emphasis and did not correspond to teachers' growing understanding of content and pedagogy. Although project teachers invested significant time and energy to their community learning, they were frustrated by the apparent gap between their individual professional learning and the district's institutional learning. The district entity's theoretical disconnect serves as a negative example of the affordances and constraints that district-level leadership provides school-based administrators, teachers and students as critical stakeholders in the school improvement process.

Establishing Policy Coherence While Maintaining an Equity Focus

The project district's policies and practice have been strongly influenced by the demands of the current standards-based reform movement. These reforms have shifted the locus of educational control "from a micro-level, top-down leadership reform managed internally, primarily by the principal, to a macro-level, top-down reform orchestrated by national, state, and district leaders" (Easley, 2005, p. 164). A fundamental problem with the current reform movement is that the large scale innovations needed to improve classroom instruction and benefit struggling students rarely penetrate state, regional and local district educational systems to the classroom level (Elmore, 2007). Critics of the reform process argue that reforms will not result in school improvement unless the change process resides in schools, in individual teacher's classrooms (Wood, 2007, Louis, Marks, & Kruse, 1996, Shen, Shen, & Poppink, 2007, Elmore, 1979-80). This is a form of deep practice that presses teachers' understanding of content knowledge and the associated pedagogical strategies that will best support student learning (Lave & Wenger, 1991, Loucks-Horsley, 1998). In contrast, failure to support teachers' understanding of the differences between the intended-enacted curriculum through a process of reflection, refinement and revision may serve to limit professional growth and ultimately, student achievement (Shen et al., 2007, Wood, 2007).

The project site, a Northwest urban school district, is thus challenged. Currently, only 64 percent of the district's students graduate on time. In fact, if the 2011 State academic performance requirements for high school graduation were applied today, well over half of the district's students would not meet State standards and would not graduate. There is a disturbing pattern of disproportionately low scores for ethnic minorities in all content areas (See Table 1). In particular, district scores on the state administered assessment show significant gaps in the area of mathematics. Realizing

Content Area	American Indian	Asian	Black	Hispanic	Pacific Islander	White
Reading	63.5%	81.0%	54.9%	82.9%	68.8%	82.2%
Math	**25.6%**	**51.9%**	**13.2%**	**33.8%**	**11.8%**	**49.1%**
Writing	67.1%	87.5%	73.0%	87.7%	66.7%	85.7%
Science	19.5%	48.1%	10.7%	33.3%	5.9%	47.7%

Table 1: 10th Grade Data by Ethnicity for 2007-08 (% met Standard)

that meeting standard is an indicator of preparation for post-secondary education and future work, more than one half of the district's students will leave school without the preparation they need to be successful. State assessment data reveals a pattern of inequity that causes central office administrators to re-evaluate the district's underlying values and educational practices. In response, the district has redefined its process of data collection in a manner that makes the data transparent and keeps the district focused on the revealed inequity and district practices that might further aid or exacerbate the problem.

The district is further challenged by the very real threat of NCLB sanctions. With five district schools already in the early stages of Adequate Yearly Progress (AYP) sanctions[1], district administrators are confronted with the need to improve schools quickly in a way that will yield measurable student achievement. The need for rapid improvement is particularly problematic since the district is facing a budgetary shortfall for the fourth year in a row due to declining enrollment and inadequate levels of State funding. The combination of NCLB sanctions and declining revenues seriously threatens the district's ability to enact the necessary deep-level institutional changes in a manner that would positively affect student learning. Added pressure has shifted the district's attention away from its own institutional learning. Increasingly, district leadership is narrowing its focus to the management of economic and policy concerns. As a result, the disparity between individual and institutional learning has intensified.

[1] Four district schools are in Stage 1 of AYP (Requiring corrective action), having not met Adequate Yearly Progress limits, and one school is in Stage 2 of AYP (The year 2 AYP designation is currently under appeal.)

DISCUSSION AND CONCLUSIONS

The current reform movement underscores the gap between leadership theory and practice. The standards-based reform movement that is in large part defined by the "No Child Left Behind Act" has heightened public awareness of problems in student achievement and challenged district and school-based administrators to transform schools so that they support the needs of practitioners and improve student learning. Unfortunately, traditional views of leadership often reduce "complex realities to simple explanations" (Rost, 1993, p 98) that are not sensitive to the particular needs and interests of the teachers in the schools. Too frequently, what many practitioners recognize as the difficulties of schooling are overlooked, ignored, or disregarded by policy makers, school administrators, educational researchers, among others in various quests for change or demonstrations of leadership.

Rorrer, Skrla, and Scheurich (2008) propose that school improvement is dependent upon an understanding that systemic change is nonlinear and complex. How well districts interpret and respond to a constantly changing set of federal and state regulations, accountability measures, and funding policies is determined in large part by the district entity's ability to successfully engage school-based practitioners in the school improvement process. Elmore (2007) further suggests that district reform is best accomplished through a backward mapping process in which district-level administrators "base their decisions on a clear understanding of the results they want to achieve in the smallest unit –the classroom, the school– and let their organizational and policy decisions vary in response to the demands of the work at that level" (Elmore, 2007, p. 5). Enacting leadership in this manner is complex, requiring that district administrators deepen their understanding and level of engagement with teachers and the daily work that occurs in classrooms. From this perspective, leadership not only provides school-based practitioners with an interpretive bridge to reform policies and practices, but also is responsible for providing the essential professional learning and resources that practitioners need to advance their individual learning. Thus, the enactment of leadership contributes both to individual and institutional learning. –And serves as a bridge between leadership theory and practice.

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DEFINING AND DEVELOPING CONTENT KNOWLEDGE FOR TEACHING

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The Mathematics Content Collaboration Communities (MC^3) has created professional development materials specifically designed to increase teachers' Content Knowledge for Teaching (CKT), the explicit knowledge used in teaching. To foster this teacher learning, the project has created a series of professional development tasks which are carefully sequenced to highlight the mathematics that teachers use in teaching. These tasks are designed to increase teachers' knowledge of mathematics in specific instances of instructional decision making and to help teachers develop more general knowledge that can be applied flexibly in new teaching situations. This paper will explore an example of secondary level, mathematical content knowledge for teaching.

THEORETICAL FRAMEWORK

To guide student thinking, teachers must understand how children's ideas about a subject develop, as well as the connections between student ideas and the important ideas in mathematics (Schifter and Fosnot, 1993). The *Mathematics Content Collaboration Communities* (MC^3) has created professional development materials designed specifically to increase teachers' *Content Knowledge for Teaching* (CKT), the explicit understanding needed to support student success in mathematics (Hill, Rowan & Ball, 2005). To foster teacher learning, the project has created a series of tasks which are carefully sequenced to highlight the mathematics that teachers use in teaching. The GAMUTs (*Guides for Accessing Mathematics Understanding for Teaching*) are designed to increase teachers' knowledge of the mathematics used in specific instances of instructional decision making and to help teachers develop more general knowledge that can be applied flexibly in new teaching situations. These materials are inherently situated in practice and can serve as important cognitive tools for teachers. GAMUTs can help teachers connect students' prior knowledge and also provide a framework for teachers to scaffold their own understanding of curriculum. Materials such as these can ". . . serve as cognitive tools to help teachers make connections between general principles and specific instructional moves—to integrate their knowledge base and begin to use their knowledge flexibly in the classroom" (Davis & Krajcik, 2005, p 7).

The research team is focused on understanding the nature of mathematical knowledge for teaching, and how teachers develop and use this knowledge. The project is striving to engage the following research questions: 1) What is the nature of secondary level content knowledge for teaching mathematics? 2) What is the nature

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of the mathematics in the teachers' participation around the GAMUT? and 3) What factors are associated with the enactment of a mathematical discussion? This paper concentrates on the first question.

Our initial mode of inquiry is to connect the features of the GAMUT to CKT. The GAMUTs provide teachers with insights about the ideas underlying the tasks and choices made for student activities (Remillard, 2000), rather than merely providing "guidelines" for teacher actions. Improving student achievement requires significant learning on the part of teachers. And yet these types of deep level changes are difficult to make without support and guidance (Ball and Cohen, 1996; Borko, 2004; Putnam and Borko, 1997; Wilson and Berne, 1999). The Teaching Commission (2004) suggests that "ongoing and targeted professional development" is needed to help teachers teach to the new standards. Unfortunately, much of "the professional development currently available to teachers is woefully inadequate" (Borko, 2004). Our project fills this gap, linking curriculum models to ongoing, intensive teacher support. More importantly, within the professional learning communities established by the project, the GAMUTs advance teachers' mathematical knowledge while promoting their autonomy (Shkedi, 1998).

The importance of developing Content Knowledge for Teaching (CKT)

A major factor in increased student achievement is a knowledgeable, skillful teacher (NCTA, 1996). In fact, Darling-Hammond and Ball (1998) conclude that teacher quality accounts for 40% of the variation in student achievement. Teacher knowledge is critical to student success. However, we have found that, "although teachers are told *not* to tell, they are given no explicit direction regarding what they could or should do instead" (Stein, Smith & Remillard, 2005, p. 352). Teacher learning includes *knowledge of mathematics content* and *knowledge of pedagogy* (Miller, 1995), with the implication that a sound understanding of mathematics paired with coursework in methods of teaching can provide the necessary knowledge base for teaching. Over the last two decades, research studies suggest that while individuals with bachelor's degrees in mathematics may have a specific kind of knowledge, they often lack what Liping Ma (1999) described as a p*rofound understanding of fundamental mathematics*, a deep understanding of basic mathematical ideas. Deepening that analysis, Hill & Ball (2004) have identified a third characteristic of effective teaching, *content knowledge for teaching mathematics*, and suggest that "…at least in mathematics, *how* teachers hold knowledge may matter more than *how much* knowledge they hold" (p. 332).

It is widely established that teachers need to possess an understanding of mathematics that goes beyond the math they teach. But, for secondary and post-secondary teachers, the nature of that knowledge remains unclear. All teachers of mathematics activate this additional mathematical knowledge when they differentiate problems to challenge students, listen to students' explanations of unconventional solution strategies to determine whether or not they are mathematically productive, and select assessment problems that are mathematically similar to the work done in class.

Mathematics teachers' effectiveness is influenced by the mathematical knowledge they possess. In the past, this knowledge was measured by the number of college mathematics courses taken (Hill, Sleep, Lewis, and Ball, 2007). While teachers' undergraduate mathematics courses support their learning up to a point, Monk (1994) found that beyond five mathematics courses, students' learning was *less* significantly affected by the number of mathematics courses teachers had taken. In fact, Adler and Davis (2006) suggest that advanced courses may encourage teachers' compression and abbreviation of mathematical knowledge. This is problematic, since *unpacking* mathematical knowledge can provide entry points for students to understand, and therefore is *necessary* for teaching. This process of unpacking serves to focus our research into teachers' mathematical knowledge as it concerns the depth, connectedness, and explicit articulation of the mathematics of teaching (Ball, 2003; Ma, 1999).

Knowing how to respond appropriately to students' questions as well as developing the ability to identify or create questions and problems targeting specific mathematical concepts actively engages the content knowledge needed for teaching (Ball, 2003). In fact, studies of elementary students' achievement have found that improving teachers' mathematical knowledge for teaching significantly affects students' learning of mathematics (e.g. Hill, Rowan, & Ball, 2005). Project materials focus on what is important for students (and teachers) to know and be able to do. This project emphasizes not just the connection to CKT, but also facilitates and supports the growth of CKT through the *application* of the GAMUTs.

However, teacher education activities presented on a single release day or even as a week-long summer institute rarely result in significant shifts in teaching practice. Stein, Smith and Silver (1999) found that they "generally result in a disconnected and decontextualized set of experiences from which teachers may derive additive benefits that is the addition of new skills to their existing repertoires" (p. 239). Adding to a teacher's collection of activities does not result in a significant change to the teacher's underlying beliefs about content nor does it result in a shift in practice. Professional learning is only effective when it deepens Shulman's (1986) *pedagogical content knowledge*, teachers' understanding of the relationship between content, and specific strategies for how to teach it.

Professional Learning Communities as the Locus of Professional Development

After reviewing several long-term professional development projects, Loucks-Horsley, et al (1998) noted that if optimal teaching improvement is to occur, work with teachers must be an adaptive process. The teacher-learner needs multiple opportunities to reflect and adjust instruction when necessary. This model is designed to be experiential, breaking professional development out of the type of isolated classroom experience that has been the historical instructional model for many teachers (Richardson, 1994). Rather, the project's design sets up a more effective professional development framework. Cochran-Smith and Lytle (1998) affirm that when teachers engage in systematic, intentional analysis of their own practice a significant shift in thinking is more likely to occur. In this way, teachers collaborate,

> …reconsidering what is taken for granted, challenging school and classroom structures, deliberating about what it means to know and what is regarded as expert knowledge, rethinking educational categories, constructing and reconstructing interpretive frameworks, and attempting to uncover the values and interests served and not served by the arrangements of schooling. (p. 249)

As teachers work together, they begin to form "professional communities …located within the workplace, offering the possibility of individual transformation as well as the transformation of the social settings in which individuals work" (Grossman, Wineburg, & Woolworth; 2000). Familiar with the contextual constraints and affordances of the local environment, teacher communities are much better able to determine which activities fit the teacher's teaching style and underlying beliefs, and are appropriate for the students and classroom setting.

Only teacher communities allow the type of peer discussion and ongoing reflection that will help support and sustain long-term shifts in practice. There is clear evidence that the most effective way to change teacher thinking is to encourage a focus on specific problems and areas of practice (Wellman & Lipton, 2004). As teachers consider problem areas in their own school setting, within their own classrooms, they are able to begin to deconstruct and reconstruct not only the curriculum, but also their thinking about practice. In this manner, the teacher community has the potential to become a powerful force for change as individual teachers become researchers of what constitutes best practices, and then enact this understanding with principals, students and the community at large.

METHODOLOGY

The research team is designing professional development materials to enhance CKT for high school teachers. At this time, methods for identifying CKT at the high school level are not clearly defined. However, drawing on Ball et al's work, we have developed a list of teaching practices that engage teachers' CKT. These include: examining the connections between representations in mathematics problems (tables, graphs, algebraic, diagrams); sequencing and connecting mathematical ideas; selecting appropriate formative assessment tasks; listening to student explanations for mathematically productive understandings and also for mathematical misconceptions; knowing mathematically related problems in different contexts, i.e. discrete vs. continuous cases and the opportunity to bridge between the two.

The GAMUTs engage practitioners around mathematical problems that we identify as "rich tasks." We define rich tasks as tasks that are generative. They are not simple dead-end exercises, but problems that raise many interesting mathematical issues, have multiple entry points, and surface ideas that are related to the fundamental nature of mathematics. The GAMUTs are purposely *layered*, where each part deepens and extends the mathematics of the prior task.

FINDINGS

For an example of CKT, consider the opening task from the *Study Buddies* GAMUT shown below (adapted from Lamon, 2006). You might want to spend a few minutes thinking about the task before reading on.

> *Part 1: Lincoln Elementary School pairs 2nd and 6th grade students as "Study Buddies" If $\frac{2}{3}$ of the 2nd graders are paired with $\frac{3}{4}$ of the 6th graders, are there more 2nd grade or 6th grade students? How do you know?*

The task requires students to compare two ratios. Most adults reasonably could be expected to complete this task, even though the reasoning necessary to complete the problem requires the solver to apply non-algorithmic thinking. This is a non-standard *proportional reasoning* task. The task qualifies as *doing mathematics* (Stein, Smith, Henningson, & Silver, 2000). The salient feature of this task is that, since the 2nd and 6th grade students are paired, two-thirds of the total number of second grade students must equal three-quarters of the total number of sixth grade students.

We can examine the progression of the *Study Buddies* GAMUT using work from one of our participant teachers, Dean. In this second year of the project, two groups of five teachers in two high schools were released one period for an entire semester to work in professional learning groups. Participants in the professional learning communities worked on GAMUT tasks for one to two days each week. The GAMUT, is comprised of ten different tasks. For the purposes of this paper, we will discuss the first and last tasks from *Study Buddies*.

For the opening task, Dean was able to support his contention that there were more second graders by using a numeric approach, choosing an arbitrary number of 2nd graders. He chose 21, 2nd graders, and calculated that $\frac{2}{3}(21)$ was 14. He then set up an equation so that three-quarters of the number of 6th graders was 14, came to an answer of $\frac{56}{3}$, and ended with 18.67. He noted that, "In this case there will be 21, 2nd graders and about 18, 6th graders. In his second case, Dean chose 300 as the number of 2nd graders, calculated that $\frac{2}{3}(300)$ was 200, and decided that there was $\frac{800}{3} \approx 267$. This led to the conclusion that there were "more 2nd graders."

We categorize Dean's approaches as numeric – the solution strategy was to select specific numbers (the choice of convenient numbers seems to be purposeful), and then compare the ratios of those numbers. Also, the calculations are independent of the context of the task. Both cases assume an approximation, since Dean rounded to the nearest integer.

Part 2 of this GAMUT asks for the *ratio* of the total number of second grade students to the total number of sixth grade students.

Part 2: If $\frac{2}{3}$ of the 2^{nd} graders are paired with $\frac{3}{4}$ of the 6^{th} graders, what is the ratio of 2^{nd} to 6^{th} grade students? How do you know?

Dean's group was given this task one week after Part 1. A feature of his solution that was a typical response for this part of the task was that, after working through essentially the same calculations as in part 1, the ratio is given. In Dean's case, he crossed out the expression $x = 1.25y$, and then wrote "Ratio of 2^{nd} graders to 6^{th} graders $\frac{9}{8}$ *or* $9:8$."

While all of the answers Dean recorded are accurate, (most teachers seem to get the "right" answer) the purpose of the GAMUTs is to attend to the differences in the mathematical representations. There are salient differences between the two tasks. (E.g. Part 2 asks for a numeric ratio and Part 1 does not.) As stated earlier, an important feature of CKT is the ability of a teacher to understand these mathematical differences.

In Part 3 of the *Study Buddies* GAMUT, a different context is used to forefront another aspect of the mathematics.

Bart and Anita Question 1: Bart and Anita work in the same bulding and meet each day at a park-and-ride lot to take a bus to work. When they meet, Bart has traveled from his house one-third of his distance to work and Anita has traveled from her house one-third of her distance to work. Who travels further to work? How do you know?

In this case, the mathematical context shifts from a discrete model (students) to a continuous representation (distance). The same ratios are in play. We have given this GAMUT to over 75 teachers during the project, and an interesting finding is that the solution methods often switch context as well. Dean's work, shown in Figure 1, uses a diagram to show each person's trip. If you can see the commute as travelling from right to left, it is clear that two-thirds of Bart's trip is the same length as three-quarters of Anita's. The project staff is continuing to research the representations favoured by both teachers and students when the context of a task is manipulated in this way.

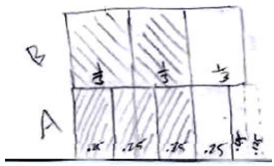

Figure 1

One of the last tasks of the *Study Buddies* GAMUT asks the teachers to quantify the difference in the two commutes.

Bart and Anita Question 2: If Anita moves into the same apartment building at Bart, by what fraction will she change her commute? Explain how you solved the problem.

This question also has generated mostly visual solutions. Dean's diagram is shown in Figure 2. Although each of these tasks contains the same

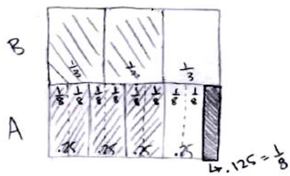

Figure 2

ratios, each provides different supports and different representations, which then elicit different ways of finding solutions. As we continue to explore and learn more about CKT and how dispositions toward unpacking mathematics develop among high school teachers, we can infer that changes may need to occur in teacher preparation and teacher inservice to appropriately prepare teachers with the kinds of mathematical knowledge necessary for teaching. Thus, it is necessary to characterize and articulate the nature of CKT at the secondary level and continue research into how teachers develop this knowledge.

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AN ANALYSIS OF THE INTRODUCTION OF THE NOTION OF CONTINUITY IN UNDERGRADUATE TEXTBOOKS IN BRAZIL

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In this paper we analyse the approach used to introduce the concept of continuity in undergraduate textbooks utilised in pre-service mathematics teacher courses in Brazil. Our analyses reveal that the privileged approach is to introduce continuity based on the definition of limit. We can also see that the preferred register is the algebraic one. Based on this analysis, we discuss some consequences for the teachers' initial education.

INTRODUCTION – TEACHER KNOWLEDGE VS TEACHER EDUCATION

Since the mid 1980's, research on mathematics teachers education has pointed out the role of content knowledge on teachers initial education. More specifically, literature has addressed both which aspects of the mathematical content are supposed to be necessary for teachers, and how such aspects are treated on the structure of pre-service courses.

Shulman (1986) distinguishes three different categories of teacher knowledge:

1. *subject matter content knowledge* (SMCK): refers to knowledge about the subject *per se*. This includes not only understanding the content of the discipline, but also the ability to establish intra and extra disciplinary connections, to explain why a question makes sense within the context of the discipline, etc.
2. *curricular knowledge* (CK): refers to knowledge about the teaching material and techniques related to a given curricular topic, their adequacy and contraindications to each specific instructional context.
3. *pedagogical content knowledge* (PCK): differs from CK, as it is not knowledge of teaching techniques but about the discipline content itself. On the other hand, it differs from SMCK because it goes beyond the understanding of subject matter *per se* and orientates SMCK *for teaching*. That is, it comprises an awareness of a meaningful order and organisation of the topics, the most useful forms of representation, analogies, examples, non-examples… "The ways of representing the subject that make it comprehensible to others" (Shulman, 1986, p.9).

Literature has given examples that show that teachers' mastery of SMCK (expressed, for example, by high grades in university mathematics courses) does not ensure sufficient PCK. Ball (1988) identifies three tacit assumptions pervading teachers initial education in the United States: (1) topics that are learnt at school are simple and commonly understood, therefore (2) prospective teachers do not need to relearn

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 81-88. Thessaloniki, Greece: PME.

them at university, and (3) the training entailed by taking college-level mathematics will prepare the prospective teacher with a deep and broad understanding of the subject matter. These assumptions implicitly imply that the ability to correctly get answers to advanced mathematical questions prepares one for teaching.

The three assumptions pointed out by Ball (more than 20 years ago) are still present in many programmes of undergraduate mathematics teachers training, at least in Brazil. As we explain in the next section, the syllabuses of these courses are usually constituted by a group of disciplines of advanced mathematics content juxtaposed to a group of disciplines of general pedagogical content (such as psychology, general didactics, sociology of education, etc.). In this model, little relation between subject matter and teaching is conveyed. As it could be predicted, little PCK is developed by prospective teachers, even the most academically successful ones.

THE TEACHING AND LEARNING OF CONTINUITY

In this paper, we address the approach to the concept of continuity of single valued real functions of one real variable in undergraduate textbooks used in pre-service mathematics teachers' calculus courses. In undergraduate courses, continuity has traditionally deserved less attention than other topics, such as derivatives and integrals (this is also the case of other concepts, as real numbers, limits and convergence of sequences and series). Contradictorily, it is assumed that students have sufficiently deep understanding of these concepts when higher mathematics topics (which depend theoretically on them) are presented. The notion of continuity is usually treated in introductory calculus courses with the support of a number of analogies that, if carelessly presented, may trigger misleading images (typically: sketching the graph without raising the pencil from the paper). Later on, it is revisited, but then within a far more formal approach, based on ε-δ definitions. Therefore, traditional teaching of the idea of continuity leaves a conceptual gap between the initial intuitive approach, which plays little role on supporting further theoretical developments, and the formal ε-δ constructions.

In the particular case of teacher education courses, this perspective has at least two potentially harmful effects – one more related to PCK and the other to SMCK. Firstly, as many authors have underlined (e.g. Cornu, 1991) some common misconceptions in advanced mathematics are attributed to lack of familiarity with limits and continuity. Secondly, since elementary real functions is one of the central topics of secondary school syllabus, solid understanding of the theoretical foundations of real functions is highly desirable for teachers. This justifies our choice of the concept of continuity to be the focus of this paper. We aim at analysing how the concept of continuity is introduced in undergraduate textbooks and whether this introduction fosters the coordination of different representations.

The concept of continuity has proven to be hard to grasp (Tall, 1992), even for teachers. The research conducted by Hitt (1994), Hitt & Lara-Chavez (1999), and

Mastorides & Zachariades (2004), to cite but a few examples, shows that a primitive idea of limit induces an obstacle in the construction of the concept of continuity. It is also shown that teachers exhibit disturbing gaps in their conceptualisation of continuity and that even their teaching might influence students' construction of the concepts of limit and continuity, producing cognitive obstacles. Thompson & Wiggins (1988) have also shown that even textbooks often introduce the concept of continuity inconsistently. To arrive to this conclusion, the authors selected a sample of more than 200 textbooks edited before 1988 and created an analysis grid to analyse the different approaches. The Methodology section shows how we adapted their research to our purposes, which also take into account the role of graphic representations.

Continuity and teachers education in Brazil

In Brazil, the certification to teach each school discipline is awarded by specific undergraduate courses, called *licenciaturas*. The *licenciaturas* are usually run by the same institutes responsible for the respective specific university courses (called *bacharelados*) but they are separated courses, with particular programmes and disciplines. In most of the cases, the syllabi of the *licenciaturas* are structured basically in two groups of disciplines, one concerning specific content (in our case, mathematics) and the other regarding pedagogy. The first calculus disciplines, in which the notion of continuity is first presented, are usually offered during the first years of the *licenciaturas*. It is important to notice that calculus does not appear in the secondary syllabi in Brazil (except for a few outstanding schools). Thus, the first contact with calculus concepts (including continuity) occurs during the undergraduate studies for the great majority of the students. This means that the introduction of continuity in the undergraduate textbooks we have selected is generally the first contact with this concept for our future teachers.

THEORETICAL FRAMEWORK

In the study of the understanding of mathematical concepts, the distinction of three dimensions has proven to be useful: epistemological, cognitive, and didactic (Brousseau, 1983). Artigue (1992, p. 47) defines them as:

- The *epistemological* dimension associated with the characteristics of the knowledge at stake.
- The *cognitive* dimension associated with the cognitive characteristics of those who are to be taught.
- The *didactic* dimension associated with the characteristics of the workings of the educational system.

At present, we are studying the didactic dimension of the concept of continuity and our first step is to analyse the ways in which textbooks introduce continuity to see whether these ways foster a correct understanding of this concept.

Regarding understanding, we follow Duval's theory of the registers of semiotic representation (Duval, 1995). For Duval, to achieve mathematical understanding, the distinction between an object and its representation is fundamental, and this makes absolutely necessary the use of different semiotic representations of a mathematical object. Not only is necessary the use of different representations, the learner must also be able to make connections between them. Duval states that to understand a mathematical concept the coordination of at least two different representations is necessary. This is due to the fact that each representation is only partial with regard to what it represents, and shows different aspects of the mathematical object.

Hence, in our analysis of textbooks, we will pay special attention to the use of the algebraic and the graphic registers to introduce the concept of continuity.

METHODOLOGY

Taking into account our theoretical framework, we wanted to analyse the way in which the concept of continuity is introduced to future teachers and whether this introduction calls for the use of different representation registers.

We chose five textbooks to develop our analyses. To select these five textbooks, we studied the official bibliography of the first calculus course of the mathematics *licenciaturas* in the four major public universities in the state of Rio de Janeiro. We selected for our study the references that appeared in more than one university, so only five textbooks filled this condition. We also verified that this set of books does not vary much across the undergraduate courses in the country (including engineering, nature and exact sciences). The references are:

A	*B*	*C*	*D*	*E*
H. Guidorizzi	L. Leithold	J. Stewart	E. Swokowski	G. Thomas
2002	2002	2002	1995	2002
Um Curso de Cálculo	*Cálculo com Geometria Analitica*	*Cálculo*	*Cálculo com Geometria Analitica*	*Cálculo*

Table 1: Selection of textbooks.

Data analysis

We constructed an analysis grid based on the one used by Thompson & Wiggins (1988), but we included items to measure the number of graphic representations used in the theoretical explanations of the textbooks. Our grid distinguishes the different definitions given for the notion of continuity (e.g. formal definitions with ε-δ, informal and visual definitions, use of neighbourhoods, use of limits). We also studied the types of examples given (algebraic, graphical, or mixed), the types of exercises (purely algebraic, without explicitly demanding graphical skills; explicitly demanding graphical interpretations and/or visual skills; demanding complementary graphic and algebraic skills). The analysis of examples and exercises is particularly important to reveal which kinds of representation registers and conversion of representation registers are more prominent in each book. Thurs, we analysed the definitions, the examples, and the exercises using the following categories of analysis.

Definitions	Examples	Exercises
(F): Uses formal definitions (symbolic or partially symbolic)	(A): Uses algebraic examples	(A): Proposes algebraic exercises
(I): Uses informal or intuitive explanations	(G): Uses graphic examples	(G): Proposes exercises requiring graphic or visual interpretations
(L): Uses the notion of limit to define continuity	(AG): Uses examples connecting algebraic and graphic representations	(AG): Proposes exercises requiring the connection of graphic and algebraic interpretations
(N): Does not present any definitions for continuity	(N): Does not present examples	(N): Does not propose exercises

Table 2: Categories for the textbooks analysis.

DATA DISCUSSION

Book *A* (Guidorizzi)

- Definition: The author presents the definition of continuity before the definition of limit. So, he does not use the idea of limit to explain what continuity is. Rather, he states the ε-δ definition and uses an intuitive explanation to sustain it: "Intuitively, a continuous function at a point *p* of its domain is a function whose graph does not present a 'jump' at *p*" (p.54). This explanation is accompanied by a picture, comparing the graphs of continuous and discontinuous functions.
- Examples: The examples presented are restricted to symbolic manipulations and applications of the ε-δ definition to verify that constant and 1^{st} degree polynomial functions are continuous.
- Exercises: Similarly to the examples, only algebraic exercises are proposed.

Besides that, three theorems (Bolzano, Intermediate Value, Weierstrass) are stated and demonstrated with the support of graphic images. However, little (or none) graphical insight is required either in the examples or the exercises.

Book *B* (Leithold)

- Definition: The definition of continuity is grounded on the notion of limit. To introduce the definition, the author takes up again one example that had been presented previously, in which the lateral limits of a function differ. Then, a function is defined to be continuous at a if it fills three conditions: (i) $f(a)$ exists; (ii) $\lim_{x \to a} f(x)$ exists; (iii) $\lim_{x \to a} f(x) = f(a)$.

- Examples: The definition is followed by a set of examples strongly based on the three conditions above. All the examples consist of discontinuous functions, for which at least one of the conditions does not hold. The examples are analysed graphically and algebraically.
- Exercises: The exercises have essentially the same structure of the examples.

Basic properties of continuous functions (such as algebraic properties) are also established through the three conditions of the given definition. The formal ε-δ definition is stated at the end of the chapter. However, no graphic interpretation or further examples to support it are given.

Book *C* (Stewart)

- Definition: The definition of continuity is also based on the notion of limit. The author puts high stress on "intuitive" analogies, such as: "The mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language"; "A continuous process is one that takes place gradually, without interruption of abrupt change"; "The graph can be drawn without removing your pen from the paper" (p. 122).
- Examples: Few examples are given. All of them consist of continuous or discontinuous functions analysed algebraically.
- Exercises: Students are asked to decide whether a given function is continuous or not, on the basis of algebraic or graphic representations.

Book *D* (Swokowski)

- Definition: Similarly to book *B*, a function is defined as continuous if three conditions are fulfilled. Condition (i) is f defined in an open interval containing a. Conditions (ii) and (iii) are the same as in book *B*.
- Examples: A set of algebraic and graphic examples is given. In some graphic examples, algebraic formulae for the functions are not given. The author uses the examples to classify discontinuities of three kinds: (a) lateral limits do not exist at a, and a does not belong to the function domain; (b) lateral limits exist but are different; (c) lateral limits exist and are coincident, but the function is not defined at a or $f(a)$ is different from the limit at a.
- Exercises: Exercises are purely algebraic and no visual or graphic interpretation is required.

Besides, some theorems are stated (such as intermediate value), but no one is formally demonstrated. Rather, they are algebraically discussed and graphically illustrated.

Book *E* (Thomas)

- Definition: The definition of continuity is also based on the notion of limit. Intuitive analogies are strongly used, for instance: "a curve that is not interrupted" (p.121). The definition is divided in two cases: points in the interior and point on the extremities of an interval.

- Examples: Most of the examples presented are used to sustain the idea that the graph of a continuous function is not interrupted.
- Exercises: Exercises are similar to the examples.

Some theorems (algebraic properties, intermediate value theorem) are stated, but without a formal definition. The three conditions set out as the definition of continuity in the books *B* and *D* are here stated as a "continuity test". No formal definition is given. Applications of continuous functions are also mentioned.

Summary

Our main research objective was to analyse the way in which the concept of continuity is introduced to future teachers and whether this introduction calls for the use of different representation registers. Our analyses (see table 3) show that the concept of continuity is mostly introduced using the notion of limit, and all the textbooks analysed use intuitive images that call for the image of "a curve drawn without removing the pencil from the paper". However, this image might produce many obstacles, especially when the function domain is not an interval.

Moreover, our analyses also allow us to see that, even if many textbooks use graphical representations, rarely are students asked to perform any activity of coordination or to produce a graph by themselves. Therefore, conversion between algebraic and graphic representation registers does seem to be satisfactorily elicited by the textbooks.

Book	Definitions	Examples	Exercises
A	F / I	A	A
B	F / I / L	A / G / AG	A / G / AG
C	I / L	A	A / G / AG
D	I / L	A / G / AG	A
E	I / L	A / G / AG	A / G / AG

Table 3: Textbooks analysis.

CONCLUDING REMARKS

This situation seems to refer to the 3rd tacit assumption mentioned by Ball (1988): training future teachers with college-level mathematics will prepare them and will give them a deep understanding of the subject matter. However, as remarked by many researchers (Hitt, 1994; Hitt & Lara-Chavez, 1999; Mastorides & Zachariades, 2004), teachers show disturbing gaps in their own conceptions about continuity, and the approaches used by textbooks do not seem to help to create adequate conceptions (in many cases, they promote the creation of erroneous images). It seems that both

SMCK and PCK of future teachers will be affected by the choices made by the editors of textbooks. In particular, we can wonder how the images developed for the concept of continuity act as obstacles for the learning of other concepts in calculus (particularly, derivatives and integrals).

We are aware that we have chosen a very small sample of textbooks, even if they are the most used in the public universities of Rio de Janeiro. Further research will use a greater sample and we will produce a deeper analysis of each category.

We have also seen that in most of the cases, the definition of continuity is built upon a previous understanding of limits, that students are expected to have. Some research about students' understanding of limits when the concept of continuity is introduced seems to be another step in our research agenda.

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ONTOLOGICAL BELIEFS AND THEIR IMPACT ON TEACHING ELEMENTARY GEOMETRY

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This paper proposes a conceptual framework to classify ontological beliefs on elementary geometry. As a first application, this framework is used to interpret nine interviews taken from secondary school teachers. The interpretation leads to the following results: The ontological beliefs vary in a broad range, denying the assumption that a similar education provokes analogue opinions. Ontological beliefs have a remarkable influence on the standards of proofs and on the epistemological status of theorems, and also on the role of drawing, constructions, construction descriptions, media, and model building processes.

THE IMPACT OF TEACHERS' BELIEFS: A FOCUS ON ONTOLOGY

In recent years, teachers' beliefs have become a vivid exploratory focus of mathematics education (Calderhead 1996). The main reason for this interest is the assumption that "what teachers believe is a significant determiner of what gets thought, how it gets thought, and what gets learned in the classroom" (Wilson und Cooney 2002). Following this idea, this article concerns the impact of ontological beliefs on teaching elementary geometry at secondary school. Especially, we consider their subtle influence on the modalities *how* geometrical issues are thought, presented, and managed.

A CLASSIFICATION OF ONTOLOGICAL BELIEFS ON GEOMETRY

The ontological background of a theory can be described as the answer of the following questions: To what *kind of objects* does the theory refer and what are the *basic assumptions* the theory claims upon this objects? Insofar, ontology is split into a *referential* and a *theoretical* aspect. This idea can be specified on the base of a particular kind of philosophy of science which is called the "structuralist theory of science", primarily established by Sneed and elaborated by Stegmüller (Sneed 1979 and Stegmüller 1985). To establish our classification of ontological beliefs, we will combine this approach with an investigation of Struve, who adopted this theory to mathematics education to analyse the influence of textbooks (Struve 1990). As a further source, the concept of geometrical working spaces is used, which was developed to classify students' handling of geometrical problems (Houdement & Kuzniak 2001).

Following the structuralist theory of science, we assume that a non-trivial (more or less scientific) theory can be described by two components, namely by its *system of axioms* and by a set of *intended applications* (Stegmüller 1985, pp. 27–42). By the set

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 89-96. Thessaloniki, Greece: PME.

of axioms, the *conceptual* and *propositional* content of a theory is given; and by the set of intended applications, the *referential* aspect of the theory is determined. In case of elementary geometry, the set of axioms normally corresponds to an axiomatization of classical Euclidean geometry. Concerning the set of intended applications, already in history of mathematics, its content was controversial. We will distinguish between three influential opinions, which seem to cover the whole range of geometrical ontology (Kline 1983): In a *formalistic* view, geometry is seen as an uninterpreted calculus without any reference, i. e. the set of intended application is regarded as empty. In an *idealistic* view, geometry refers to a world of ideal objects which fulfils the Euclidean axioms without any approximation and which do not belong to the physical world; and finally, an *applied geometry* refers to physical objects, typically with some approximation. At school, the paradigmatic real-world objects elementary geometry is applied to are drawing figures, figures produced by IGS (iteractive geometry software), and physical objects of middle dimension like balls, dice, chambers, ladders, bridges, and churches (especially the ornaments of their windows).

By this threefold distinction, the first step of our classification is given. It is only defined by a difference in the set of intended applications, taken a complete Euclidean geometry as a theoretical background for granted. To analyse teachers' or students' beliefs, this assumption is inappropriate, since their geometrical propositions may differ from the standards of an axiomatic Euclidean geometry. For this reason, we introduce a second distinction on the *theoretical* level, insofar as we discriminate between an *axiomatic Euclidean* theory and an *empirical* one. In the first case, the individual theory follows the mathematical standards of an axiomatic elementary geometry (possibly except some minor mistakes due to human fallibility); in the latter case, the individual theory lacks these standards significantly and consists of geometrical assumptions which substantially differ from a scientific view and which may be at most locally ordered, fulfilling the inferential standards of everyday discussions.

For our investigation, it is not necessary to describe the differences in the *content* of an individual empirical theory of geometry and a Euclidean one. We are rather interested in the question how the ontological difference influences the way of treating geometry on a *meta-level*, which we have initially circumscribed by keywords like standards of proving, presenting objects, or applying geometry. We claim that the differences on this meta-level are independent from the specific content of an empirical theory and only determined by its status *as* an empirical one. The main influence on these issues is already indicated by choosing the expression "empirical theory" for theories which do not fulfil axiomatic standards. Due to the lack of an elaborated axiomatic background, these theories cannot be treated in a formalistic or idealistic manner, since they afford neither a coherent calculus nor the conceptual strength to describe a world of idealistic objects sufficiently. Therefore, theories like these have to be regarded as empirical ones, which can only be denoted as geometrical, since their share the same set of intended applications with an applied Euclidean geometry and since they are used for similar purposes – for instance for measurement, for calculating lengths, angles, and areas or for formulating general theorems containing common geometrical concepts. To distinguish between these two types, we will call an applied geometry which is intended to have a complete axiomatic Euclidean background a *rationalistic geometry* and an empirical geometry without such a background an *empiristic geometry*. This is the second distinction of our classification. Figure 1 gives a complete overview.

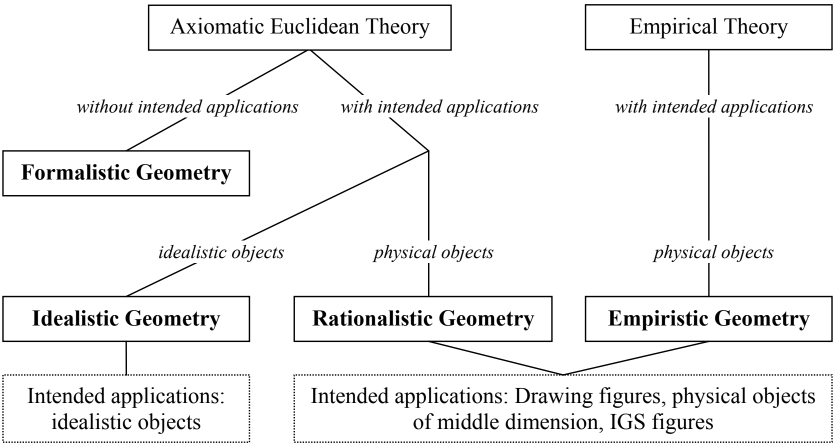

Figure 1: An ontological classification of elementary geometry

The expressions used in our classification obviously allude to 17th/18th century philosophy, not claiming to match the corresponding ideas of philosophers like Descartes, Locke, Hume, Leibniz, and Kant precisely, but claiming to follow their main intention that, from a rationalistic point of view, the basic assumptions of every empirical theory are *independent from experience* and, therefore, that they cannot be verified or falsified by experience, whereas an empiristic approach claims just the opposite: basic assumptions are *derived from experience* and have to be verified or falsified by experience. This idea is elaborated within the theory of geometrical working spaces as a tool to analyse students' behaviour in the field of proving assumptions and solving problems, since it delimits the *range of methods* which is taken as allowed. We expand the ideas of Houdement and Kuzniak (2003) to our classification (they only discriminate between three types of geometry, having no counterpart to our rationalistic one). The following table summarizes what roles typical aspects and methods of geometry play in respect of the ontological background.

	Formalistic G.	Idealistic G.	Rationalistic G.	Empiristic G.
methods of proof	purely	purely	purely	inferential

and sources of knowledge	deductive, linked to axioms	deductive, linked to axioms	deductive, linked to axioms	arguments, experiments, intuitions
role of experience, experiments, and measurement	heuristic	heuristic	heuristic and to indentify geometrical objects	basis of knowledge
status of drawing	heuristic	heuristic	an application of geometry	objects of study and validation
access to objects	by relational or constructive descriptions	by relational or constructive descriptions	by experience and measurement	by experience and measurement
experience	linked to a formal concept of space	linked to an abstract Euclidean space	linked to physical space, interpreted in Euclidean concepts	linked to the measurable physical space without a predefined geometrical interpretation
objects of intuition	internal to mathematics	linked to idealistic figures	linked to idealized real figures	linked to perceptions

Table 1: Ontological influences on geometry

It is claimed that the content of table 1 is a logical consequence of the different types of geometrical ontology; i. e. the table is guided by the assumption that if someone possesses the ontological background mentioned in the header of the table, it will be rational for him (and from an empirical point of view expectable) also to hold the statements in the related row. If this assumption is empirically traceable is one of our further tasks.

TEACHERS' ONTOLOGICAL BELIEFS ON GEOMETRY

Students' ontological beliefs on geometry are extensively investigated by two studies (Andelfinger 1988 and Struve 1990). In our terms, they both end up in the result that students gain an empiristic view, assuming that the ontological background of teachers is a formalistic or idealistic one. In this article, the presupposition that teachers form a "unified community of formalists and idealists" is taken into question.

The empirical base of our investigation consists of semi-structured interviews taken from nine teachers of mathematics who are employed at German higher level secondary schools (so-called "Gymnasien") and who teach mathematics from grade 7 to 13, that means that the age of their students ranges from 12 to 19 years. We refer to the teachers by the letters A to I. The aim of our whole investigation consists in the task to reconstruct the teachers' individual curricula of teaching geometry as *subjective theories* (Eichler 2006). For this article, the results are restricted to ontological aspects. Subjective theories are defined as systems of cognitions containing a *rationale* which is, at least, implicit (Groeben et al. 1988). For this reason, the construct of subjective theories is a tool to reveal logical dependencies within the belief system of an individual. In our case, we are focussed on the dependencies between general ontological assumptions and the specific handling of geometry, guided by the following questions: 1) What types of ontological backgrounds occur according to our classification? 2) Do they lead to the consequences which are to be expected (table 1)? 3) Are there unexpected influences which do not seem to be accidental, but also implications of the ontological background?

Following our first question, we can conclude that every type of ontological background occur in our sampling. Our interpretation leads to the following classification:

	Formalist	Idealist	Rationalist	Empirist
Teacher	A, I	B, D, F	C, G	E, H

Table 2: Ontological classification of teacher A to I

Due to the limited space, it is only possible to present a single case study per category, and it is necessary to restrict the empirical base to one significant phrase (all transcripts are translated by the author). Mr. A's ontological background is a clearly formalistic one. He regrets that time limits him to implement it extensively:

Interviewer: What do you think of formalism in mathematics?

Mr. A: I loved it at university. It is pure reasoning. […] I would like to do such a thing [at school], but that is difficult, since I only teach four lessons a week. […] Five years back, when I had five lessons to teach, I did it and I did it gladly.

By Mrs. D, an example of an idealistic view is given:

Mrs. D: The beauty of mathematics is the fact that everything there is logical and dignified. […] Everywhere else, there are mistakes and approximations, but not in mathematics. There is everything in a status in which it ideally has to be. [It is important] to recognize that there are ideal things and objects in mathematics and that, in reality, they are similar, but not equal.

It is interesting to note the subtle difference between Mrs. D and Mr. C below: Whereas Mrs. D stresses that mathematical objects are ideal and do not occur in reality, Mr. C refers to physical objects by geometrical terms without doubts, but emphasizes that some kind of abstraction is necessary, which indicates that he holds a rationalistic view of geometry, and not a empiristic one:

Mr. C: I make them [the students] search for shapes in reality and to prescind from them. Then this cone is a steeple or an ice-cream cornet. […] There are some basic shapes which are consistently occurring in life.

Since the difference between a rationalist and an empirist does not arise from a referential disagreement – they both refer to physical objects –, we omit a quotation concerning this issue and present two key phrases which show that this difference depends on the status of the geometrical theory.

Interviewer: What do you say if a student claims that he can see that something is as it is? Do you insist on a proof?

Mr. C: As far as classical proofs are concerned, I think: Yes, I do. If someone asserted in case of the Pythagorean Theorem "By measuring, the theorem holds", then something of value would disappear, […] something which is genuinely mathematical. […] If geometry just consisted of measurement, calculation, drawing, and constructing, then I would regard it as meagre.

In the context of IGS and congruence, Mr. H refers to proofs. It is obvious that he allows experience and measurement to be bearers of knowledge. Insofar, he holds an empiristic view of geometry.

Interviewer: What is your experience with interactive geometry software?

Mr. H: It is possible to demonstrate and to prove many things by such software, for example Thales' theorem. We move the third point of the triangle on the arc of the circle and observe that it [the angle] always equals 90°, and we take this as a proof. […]

All triangles are cut out and laid on top of another, and we observe that they are all equal […] and we achieve the insight that three attributes are sufficient to construct the same triangle. […] Thereby, the concept of congruence is given. What does congruence mean? That means that something can be laid on something different without overlaps. […]

We introduce pi by measuring the circumferences of circles […]. That is more exact and more concrete for the students as if we went from a quadrangle to a pentagon, to a hexagon […], and sometime, we get an infinitygon, which we call a circle. [Using the latter method,] the aberrations are significant at the beginning, and it is difficult to draw a triacontagon […]. So, it is worth to ask if this method makes sense, since for students, it will be important to solve specific things. That won't have to be exact.

At a first result, we can conclude that the ontological beliefs of teachers are more divers than assumed by Struve and Andelfinger. Especially, even the empiristic type which is supposed to be limited to students occurs twice in a sampling of nine individuals. It would be interesting, if a quantitative investigation could confirm this remarkable percentage. Due to the limits of space, we could not expose in detail that the claims of table 1 are empirically detectable. But we tried to choose quotations which make our assumption plausible and which should have shown that the ontological background is the crucial influence on the epistemological status of geometric theorems and, therefore, on the role of experience and measurement.

FURTHER INFLUENCES

The first part of our investigation was guided by a pre-defined hypothesis. Already in the quotations above, it is noticeable that ontological beliefs have an unexpected impact on further aspects of teaching geometry – for instance, Mr. H's students would presumably gain a physically defined notion of congruence and approximation and no elaborated concept of limits and irrational numbers. Unexpected impacts leads to theory construction. We will present our results in table 3, not being able to establish our claims in detail. Instead, we will quote some unconnected episodes taken from different positions of our interviews to make our deliberations plausible and to consider the differences between a formalistic and idealistic view, which was of minor interest until now – arguing for the assumption that a "community of idealists and formalist" is a fiction.

We observe remarkable differences between a typical formalist and a typical idealist in matters of content, axiomatization, constructions, model building, and IGS.

Mr. A: It doesn't matter what content we teach. The most important thing is that it is mathematics. The essence of mathematics can be found in every part of it: this consistency. […] The necessity of proof is reduced by IGS, since there are always 90° [in case of Thales' theorem]. […] I want that students solve complex problems in larger contexts […] and that they justify algorithms. […] Concerning analysis and probability theory, there are many things which cannot be proved [at school], and in geometry, I don't see this at all. […] Arguing, thinking in conceptual hierarchies, problem solving, and model building – these are the higher goals in my view.

Mr. B: On the way from a real situation to a mathematical model, […] an argumentation arises which was untypical for teaching mathematics until now. […] I regard problem solving as a very important part of geometry, […] whereas describing the real world is not in the first place. […] There are some very challenging constructions, but with IGS, there is no problem. […] In an optimistic view, I expect that, after school, a student copes with the complete mathematical contents and methods of secondary school.

This summarization of short episodes may illustrate why we have chosen the topics and assumptions mentioned in table 3. From a meta-level, the differences between formalists and idealists seem to arise from the ontological attitude that an idealist is more interested in (idealistic) objects and their properties and constructions, whereas a formalist stresses theories, conception, and deductions, which opens an access to general abilities in the field of argumentations, problem solving, and model building.

	Formalistic G.	Idealistic G.	Rationalistic G.	Empiristic G.
purpose of proofs	verify the truth, reveal logical dependencies	verify the truth, tools to remember content	verify the truth	make the truth of a sentence plausible
objects to prove	general theorems	general theorems, attributes of	general theorems	unclear

		objects		
didactic aims of proving	argumentative abilities, insights in the nature of mathematics	argumentative abilities, insights in the nature of mathematics	argumentative abilities, insights in the nature of mathematics	of minor interest
content of school mathematics	of minor importance, exchangeable in principle	important entity to learn, large amount desirable	important entity to learn, medium amount desirable	is to restrict to practically useful topics
object studies	of minor, only didactic interest	important task, no physical objects allowed	to learn the approximative use of geometrical concepts in real world situations	to achieve knowledge by experience
type of definitions	according logical standards	according logical standards	according logical standards	derived from experience
purpose of theories and axiomatization	objects of study and objects to achieve deductive abilities	tools to describe mathematical objects, different approaches desirable	tools to describe mathematical objects, of medium interest	of minor interest, possibly as a tool to solve practical problems
influence of IGS	decreases the insight in the necessity of proving	allows complex constructions, identifies (in)adequate constructions	identifies (in)adequate constructions	additional source of mathematical knowledge, introduces motional aspects
role of construction descriptions	of minor interest	most important way to access objects	of minor interest	obsolete
model building processes	motivation, occasions to learn further argumentative abilities	contains an "unmathematical" way of thinking, didactical tool	contains an "unmathematical" way of thinking,	important justification of teaching mathematics
problem solving	train argumentative abilities	to train argumentative abilities	to train argumentative abilities	to link to real-world problems
role of algorithms	tools and objects to justify	tools and objects to justify	tools and objects to justify	tools

Table 3: Assumptions on the ontological impact on aspects of teaching geometry

References

Andelfinger, B. (1988): *Geometrie: Didaktischer Informationsdienst Mathematik.* Soest: Soester Verlagskontor.

Calderhead, J. (1996): Beliefs and Knowledge. In D. C. Berliner & R. Calfee (Eds.): *Handbook of Educational Psychology* (pp. 709–725). New York: Macmillan.

OPENING THE BLACK BOX OF TEACHER LEARNING: SHIFTS IN ATTENTION

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How do practicing mathematics teachers continue to improve their teaching over time? This question was central to our review of over 100 studies on teachers' learning. This paper identifies characteristics of current knowledge about teachers' on-the-job learning based on our review, and proposes refinements to the model of teacher learning outlined by Clarke and Hollingsworth (2002). We report on data from three studies of our own, using them to illuminate the "black box" of teacher learning by exploring teachers' changing attention to, and use of, student thinking.

Like many careers, teaching requires lifelong learning, especially at a time in history when so many teachers are being called upon to re-examine their ideas about learning, deepen their mathematical understanding, and adopt new instructional approaches. Studies of teacher learning have focused on an array of areas, including changes in beliefs, knowledge, decision-making, pedagogical approaches and sense of self-efficacy and identity (c.f., Leikin & Zazlavsky, 2007; Haanula & Sullivan, 2007). We have reviewed more than 100 studies of practicing K-12 mathematics teachers in order to take stock of the field's current state of knowledge about teachers' learning with respect to their mathematics practice, and to contribute to developing a model that captures the complexities of the process.

We identified papers for review by searching three major databases using keywords including combinations of "mathematics" with "teacher learning," "teacher knowledge," and "teacher change," as well as by additional search procedures such as following up on commonly cited references. We have organized the resulting studies into three predominant categories: (1) studies that describe ideal (or intermediate) states of knowledge associated with "reform-minded" instruction but that do not investigate changes that teachers undergo in reaching these states (e.g., Wood, 1999); (2) large-scale survey studies that link measures of professional development (PD) experience or of teacher knowledge to practice (e.g., Garet et al., 2001); and (3)

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studies of the teacher learning/change process, many of which are "small-*n*" studies (e.g., Ainley & Luntley, 2007). This research report focuses on the third category, which, if synthesized across studies, might elaborate models of teacher learning by unpacking the "black box" of processes leading to teachers' growth.

We use Clarke and Hollingsworth's (2002) model of teacher learning as the starting point for articulating teachers' professional growth (Figure 1). Their model lays out four domains involved in teachers' growth. The *External* domain, represented in Figure 1 by the square, includes factors that may be shared across many educational settings, such as the mathematics curriculum, district testing policies, or PD opportunities. The domains represented by the three ovals represent areas of professional growth that are more specific to individuals. The *Personal* domain refers to individually-held knowledge, beliefs, and dispositions, such as the tendency to take an inquiry-driven, rather than evaluative, stance toward students' work; the domain of *Practice* includes elements such as engagement in practice-based discussions with colleagues and the use of instructional strategies and classroom-based assessment; and the domain of *Consequence* includes salient outcomes of instruction, most centrally, students' learning.

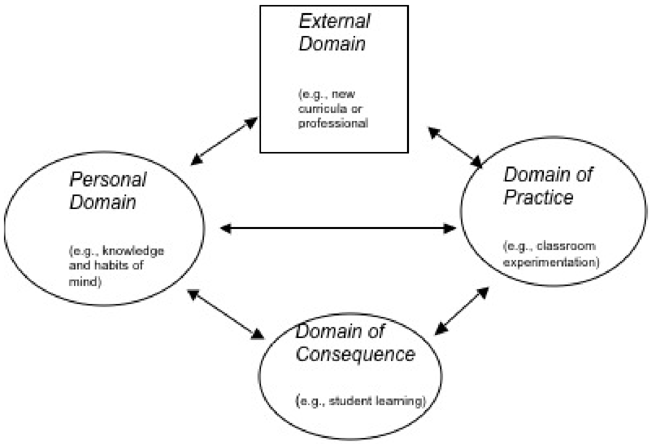

Figure 1: A model for professional growth (adapted from Clarke & Hollingsworth, 2002)

A linear view of learning posits that the External domain affects the Personal domain, which in turn results in changes to the domain of Practice. Clarke and Hollingsworth's model acknowledges a more complex interplay of influences on teachers' professional growth. It is the attention to multiple dynamic relationships among the domains that particularly captured our interest, and that we believe should be explored further in studies of teachers' learning. In this paper, we explore and extend some of the interactions identified in Clarke and Hollingsworth's model. In particular, Clarke and Hollingsworth draw on cognitive/developmental theory to posit *enaction* and *reflection* as processes for growth. We suggest *noticing* as an additional process (Ainley & Luntley, 2007; Jacobs et al., 2007; Mason, 2002; van Es & Sherin, 2008). Shifts in noticing (or attention to) elements of practice may help drive teachers' enactions and reflections and, in turn, be further sharpened by them.

In the remainder of this report, we present results from three projects, conducted in the U.S., that highlight different ways teachers developed new attentional foci. These new foci helped teachers to be more analytic about elements of their practice—in particular, students' writing, students' mathematical ideas, the mathematics underlying students' thinking, and the cognitive demands of tasks enacted during lessons.

THREE EXPLORATIONS OF TEACHERS' LEARNING

The three projects differed in a number of ways, including the participants' motivation to undertake PD, the nature of the intervention, and the dynamics among the domains of teachers' learning. Common to all three is that teachers learned to attend more carefully to aspects of students' thinking, and that their new attentional foci positioned them to further explore their practice in new ways.

The impact of new curriculum on teachers' learning: The Math and Literacy Project (MLP)

The Math and Literacy Project (MLP) was four-year collaboration between a team of university-based researchers in mathematics education and literacy education and about 25 mathematics teachers in a mid-sized urban school district. A major impetus for the project was the district's recent adoption of *Standards*-based mathematics curriculum, *Connected Mathematics Project* (CMP), (Lappan et al., 1998). This curriculum presented a number of new challenges for teachers, including the need to support the development of students' abilities to communicate mathematically.

The data reported here come from one of the schools in the study, where 5 middle-grades teachers taught 11–13 year old students. The teachers worked together for three weeks during the summer to develop instructional strategies to support students' mathematical communication. During the school year, they met together monthly to explore mathematical communication and participated in individual "lesson cycles" every 3–4 weeks. During the lesson cycles teachers worked with a project staff member to plan, implement, and debrief math lessons that supported literacy opportunities for students (Doerr & Chandler-Olcott, in press).

As this group of teachers considered the literacy demands of their new curriculum, they became more sophisticated observers of both their students' mathematical thinking and students' ability to communicate that thinking. Over the course of MLP, the teachers identified and sought to address several challenges related to literacy and mathematics.

At the beginning of the project, the teachers reported that their students struggled to express their ideas when answering CMP's extended-response questions. Teachers found that, while CMP often offered examples of the range of students' *mathematical* solutions to problems, there were no such illustrations of the range of *written* responses students might make to the extended response questions, no indications of the kinds of difficulties that students might encounter in explaining their thinking, nor suggestions about strategies that teachers might use to help students improve their writing. The teachers, in collaboration with the research team, explored ways to improve students' written communication. The group began to work with writing samples to analyze students' strengths and weaknesses in communicating their ideas and to develop new instructional strategies focused on literacy. While teachers' work was stimulated by the new curriculum (an element from the External domain), we see teachers' resulting growth in attention to, and analysis of, students' written work as an interaction between the Practice, Consequence, and Personal domains.

This work led to the first of two shifts in teachers' practice. Teachers came to see CMP's emphasis on writing as an opportunity, rather than a barrier, to students' success. Teachers began to look for writing opportunities in their mathematics lessons, often using "quick writes" at the end of class—short, informal writing guided by an open-ended prompt—both to provide students with additional writing opportunities and to gauge students' understanding. For example, one teacher used the following prompt: "When you add a positive and negative integer, you sometimes get a positive result; you sometimes get a negative result. Show that this is true." From students' responses, she realized that their understanding was not yet fully developed, and that she also needed to focus more on the needs of her second language learners. Teachers' classroom instruction shifted from virtually never having students write to making writing an increasingly regular part of students' work. In this case, changes in the domain of Practice came about through the teachers' noticing, and then analyzing, students' communication of their ideas through writing (the domain of Consequence).

We observed a similar dynamic among domains in a second shift in teachers' practice: teachers saw a need to systematically address opportunities for writing in order to support students' development as mathematical writers. Teachers found that, despite their increased emphasis on writing, the overall quality of students' written communication was not improving. To address this problem, the group developed a planning framework that could be used across the grade levels. The framework included: (1) a specific writing task, usually modified from CMP's mathematical reflections, (2) a rationale for the writing task, and (3) a description of the instructional approach to the task. (An example of a complete unit level plan will be given in the presentation.).

These writing plans helped the teachers focus on specific strategies to promote improvements in students' mathematical writing. Teachers began units by articulating explicit expectations for written work and modelling examples. Writing elements later in the plan called for work with partners and peer editing; towards the end of units, the plan called for independent student work. The focus on the development of students as writers became central to teachers' classroom practice.

The impact of artifact-based professional development on teachers' learning: The Turning to the Evidence project (TTE)

The TTE project studied middle and high school teachers as they participated in 36 hours of PD focused on algebraic thinking. Individual courses were offered in each of 4 school districts, with a total of 51 PD participants. In addition, 25 teachers served as a comparison group; these teachers completed pre/post research measures but did not participate in any PD. All teachers were volunteers. Unlike the MLP teachers, who were motivated to join the MLP because of specific challenges related to curriculum implementation, few TTE participants indicated a specific issue that brought them to the PD; they expressed a more generalized interest in improving their practice. (For a more detailed description of the study, see Nikula et al., 2006.)

Each PD course consisted of 12 sessions that followed the same format: (1) teachers explored a mathematics problem themselves, discussing their solutions and the mathematical ideas underlying the problem; (2) teachers analyzed classroom artifacts (e.g., video segments of classroom lessons, written student work, transcripts of students' small group problem solving). A major goal of the PD was for teachers to learn to attend to the mathematical thinking captured in artifacts—to engage in deeper mathematical analysis; to see strengths, as well as weaknesses, in students' thinking; and to see artifacts as tools to use to inquire into students' thinking.

Analyses of teachers' learning relied on two main data sources: (1) teachers' written responses to a pre/post-PD "artifact analysis" task and (2) teachers' discussions in two seminar sessions, one session from early in the PD and one from the end. The written artifact analysis task had three parts: (1) viewing a video segment of students explaining solutions to an algebra problem, (2) answering a series of increasingly specific questions about the video, and (3) commenting on three students' written solutions to the problem. The task was scored using a 3-point rubric for each of 5 dimensions: (1) attending to the potential in students' understanding (vs. only to deficits), (2) making comments focused on the mathematics (vs. non-mathematical content), (3) providing evidence for claims, (4) making claims of an inquiring nature (vs. prescriptive), and (5) attending to specific students (vs. "students" in general).

Results from the written task indicated that PD participants' post-program responses for the first three categories differed significantly from those of the comparison group: the PD group was more focused on the potential of students' thinking, on the mathematics, and on supporting claims with evidence from the artifacts than the comparison group. Analyses of seminar transcripts yielded similar results. In the early session, teachers' conversations focused on students' misunderstandings and also moved quickly from a focus on the thinking captured in the artifact to discussion of students in general. In the later session, participants' discussions were more detailed, more focused on specific mathematical ideas, and more mathematically sophisticated (Goldsmith & Seago, 2008). Teachers learned to notice (and discuss) more of the mathematical thinking captured in the artifacts—learning that seems to reside in the interaction between the External and Personal domains.

The impact of a professional learning community on teachers' learning: Lesson Study (LS)

The LS project includes an ongoing study of "Foothills Elementary," a K–5 school where lesson study has been practiced since 2001. Lesson study is a form of professional learning that centers on collaborative planning, observation, and analysis of a live classroom lesson, using protocols that help teachers focus on student thinking (Fernandez & Yoshida, 2004; Lewis, 2002 a,b; Stigler & Hiebert, 1999). The impetus for forming the school-wide group at Foothills came from a small group of teachers who had read about and then initiated lesson study at the school. They subsequently shared their experiences conducting a lesson study cycle at a faculty meeting, and their colleagues later voted to practice lesson study school-wide.

LS groups from Year 1 and Year 3 were studied. Two meetings of each group were coded: a planning meeting that took place before the group had finalized the lesson, and the post-lesson discussion. Written transcripts were coded sentence-by-sentence for evaluative stance (e.g., global evaluation of the lesson), knowledge-building stance (e.g., questioning, making a proposal), and sources of knowledge cited (e.g., established sources, student thinking/work, teachers' own professional experience). Inter-rater reliability exceeded 80% for all codes.

Coding revealed that teachers' *references to established sources* (including curricula, standards, research literature, named experts/programs) and *solicitation of knowledge from colleagues* increased between Years 1 and 3 (from 2% of all statements to 9%, and from 2% to 4%, respectively). From Year 1 to Year 3, we also found increased *attention to student thinking* in both the lesson plans and in teachers' discussions. The Year 3 discussions showed more than twice the proportion of references to student thinking/student work than those in Year 1 (18% in Year 1; 43% in Year 3). During the same period, global and fixed-ability evaluations (e.g., "that was a good lesson," "he's a low student") declined from 8% to less than 1% of statements during the post-lesson discussion. These results suggest that teachers shifted away from an evaluative mindset and toward a knowledge-building mindset, focusing more heavily on both knowledge sources within the school (e.g., student thinking) and external knowledge sources (e.g., research, outside expertise and programs). Analysis of individual lesson study cycles suggests that research knowledge (External domain affecting the Personal), observational practice (domains of Practice and Consequence) and discussion protocols (domain of Practice) all helped shift attention toward analysis of student thinking.

CONCLUSIONS

We have attempted to unpack the Clarke and Hollingsworth model to illuminate pathways of teacher change by asking the question, "what are the specific changes that occur as teachers improve their practices, and what experiences can promote these changes?” One way the field of teacher learning might advance is to look across case studies for shared theoretical mechanisms. We have illustrated this idea by drawing on one type of change that emerges from our collective work and our review of the literature: attention to and use of student thinking. We examine data from three empirical studies that approach this issue from different perspectives; these data suggest that attention to and analysis of student work is an important process within the “black box” of teacher improvement that deserves principled attention in future research. The data further suggest that intervention power might be gained by joining together the “change agents” found in the three separate studies: demands of the classroom curriculum, selected mathematics tasks, students’ work on those tasks, and school protocols for shared planning, observation and analysis of lessons.

In all three studies, teachers learned to attend to and better analyze important elements of their practice. Teachers in MLP learned to attend to issues of written communication in their mathematics classes. Their increased use of writing activities in class served both to strengthen students’ communication skills and to gather information about students’ mathematical understanding. Teachers in both the TTE and LS projects learned to analyze students’ thinking in more mathematically detailed ways and to take a stance of inquiry, rather than evaluation, toward both students’ work and teachers’ instructional decisions. We believe that these processes of noticing and analyzing elements of practice are central to teachers’ on-the-job learning, for they link the domain of Consequence (student understanding) with development of the Personal domain (deeper and better integrated knowledge regarding students’ mathematical thinking, a greater sense of self-efficacy, and new dispositions toward teaching) and the domain of Practice (new instructional strategies and decision-making).

The three studies also illustrate that the patterns of interaction among the domains of teachers’ professional growth are complex, and are likely to differ in different kinds of professional learning settings. MLP illustrates a strong interaction between the External and Practice domains: teachers began exploring their practice because the new curricular materials made demands on students that the teachers did not know how to support. The university partners helped to shape and guide teachers’ work, tailoring the PD experiences to the group’s needs and interests as they identified new practices and resources to help stimulate teachers’ learning. In TTE, it was the pathway between the Personal and External domains that was most strongly at play as teachers studied the mathematical thinking captured in the artifacts they analyzed. Teachers participated in a formal, carefully crafted PD curriculum in which the facilitators’ skillful enactment of the PD helped the teachers learn to attend to classroom data in new ways. Finally, the dynamic in the LS project included interactions between the Personal and Practice domains, as careful anticipation and observation of student thinking and study of relevant research built a culture of looking at student thinking and drawing in outside knowledge, as well as Practice and

Consequence domains, as analysis of the public lesson encouraged further reflection on practice.

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THE CONCEPT OF SERIES IN THE TEXTBOOKS: A MEANINGFUL INTRODUCTION?

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The concept of infinite sum (or series) is complex and contradicts our intuition, but has many applications. Despite its epistemological complexity, there are not many research results about the teaching and learning of this concept. This paper presents the results of our analyses of one of the aspects of its teaching, having a look at the way in which textbooks introduce this concept. Our results show that even if there is an evolution in the edition of textbooks, their approach to introduce series still remains quite "traditional" and restricted to the algebraic register, showing very few applications of the concept.

INTRODUCTION

The concept of infinite sum (or series) may seem quite "artificial", but has many applications. One of the easiest ones to see is the approximation of areas by means of rectangles. By using multiple and thinner rectangles, we can better approximate the area under a curve in a given interval, which gives origin to the concept of Riemann sum and of definite integral, essential to calculate the area under a curve. Other applications of series are the writing of numbers with infinite decimals (0.333… = 0.3 + 0.03 + 0.003 + …), or modelling situations such as the distribution of medicines or pollutants. Therefore, the concept of infinite sum has many applications in different areas (Physics, Economics, Biology, etc.), and accordingly, a better understanding of it would improve the training of future professionals in multiple fields.

However, due to its "mysterious halo", series are usually reduced to their algorithmic aspects, which later produce many misconceptions in understanding the concept of integral (Bezuidenhout & Olivier, 2000; González-Martín, 2006). In fact, we hypothesise that traditional teaching concentrates on studying the different techniques to conclude convergence or divergence and also on the formulae to calculate the sum of a convergent series, but that little emphasis is placed on the applications of the concept or on the construction of meaning.

To verify this hypothesis, two teams (one in Canada, one in the UK) have started the analysis of textbooks used in postsecondary education to study the ways in which the concept of series is introduced. This paper focuses on the results obtained by the Canadian team, making references to the results obtained by the UK team.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 105-112. Thessaloniki, Greece: PME.

In spite of the applications of series and their key role to understand the concept of definite integral, there are not many research results about their understanding. There are many works that study the concepts of convergence, limit, and numerical sequence. Although the concept of series can also be seen from these perspectives, not many works have focused on the concept of series itself. One of the pioneer works on the concept of convergence (Robert, 1982), already remarked that the acquisition of the concept of convergence is not made without problems and that it remains incomplete to the students while in their postsecondary studies. She also pointed out that the exercises did not let the students construct a correct notion for the convergence of numerical series. Boschet (1983) also pointed out that traditional teaching shows very few examples of graphic representation of convergence (with the existing ones fostering static representations). This agrees with our results (González-Martín, 2006), that showed that students do not have visual images associated with the concept of series.

Bagni (2005) has suggested the use of historical examples to improve the teaching of infinite series and to overcome the misconception "infinitely many addends, infinitely great sum". He also encourages the use of visual representations to give a meaning to the concept of series. In this sense, Codes & Sierra (2007) designed an activity to introduce the basic notions of numerical sequences before introducing infinite sums as the limit of a type of sequence. They based their activity on Oresme's work and used graphical representations with the help of a computer. They stated that the use of the computer and graphical representations helped some students. Subsequently they used the computer representations to reason through some of the paper-and-pencil questions. Therefore, it seems that the computer representations improved or added to the students' own representations of an infinite sum.

Fay & Webster (1985) also stated that in most calculus textbooks, little or no relation between improper integrals and infinite series is given other than the integral test for the convergence of series. The results of our previous research (González-Martín, 2006) support this and we showed that our students usually learn the concept of improper integral without making any link with the concept of series.

THEORETICAL FRAMEWORK

In our study we start from the assumption that to have a good grasp of the learning and understanding of a mathematical concept, the study of three dimensions is necessary (Brousseau, 1983). These three dimensions are: epistemological, cognitive, and didactic, and they interact.

At present, we are studying the didactic dimension of the concept of series, associated with the ways in which the concept is presented to the students in pedagogical and curricular terms (Artigue, 1992). Our first step is to analyse the ways in which textbooks introduce series to see whether these ways foster a sufficient understanding of this concept.

According to Duval's theory of the registers of semiotic representation (Duval, 1995), the distinction between an object and its representation is fundamental in mathematical understanding. To achieve this understanding, the use of different semiotic representations of a mathematical object is absolutely necessary. But not only the use of different representations is necessary; the learner must be able to make connections between them. Duval states that to understand a mathematical concept the coordination of at least two different representations is necessary. This is due to the fact that each representation is only partial with regard to what it represents, and shows different aspects of the object (for example, the information given by the expression $y = x$ and its graph is different; the graph allows to directly see that the function is increasing). It can be said, so, that a mathematical concept is constructed by tasks which involve the use of different representation registers and promote the coherent articulation between representations.

Hence, in our analysis of textbooks, and to verify our hypothesis, we paid special attention to the use of the algebraic and the graphic registers to introduce the concept of series. The choice of these two registers is motivated by the results found in our literature review, showing the impact of visualisation to better understand the concept of series.

METHODOLOGY

At present, we are working on the didactic dimension of our study. This dimension will analyse three major sources: Official programs, Textbooks, and Teachers' practices. Thereupon, we are developing our analysis of textbooks and the questions we would like to answer in this first stage are:

- Is there an evolution in the way in which textbooks introduce series?
- What representations are privileged?
- Do textbooks foster coordination between the algebraic and the graphic register?
- Is the concept introduced in a meaningful way, showing its applications?

We aim at producing an exhaustive revision of textbooks for the last fifteen years in Québec. To do so, we are choosing textbooks that appear in the official programs of the last years of postsecondary establishments of Montréal. We have only chosen establishments of Montréal, considering the biggest city of Québec as reference of what happens in the rest of the province. At present, we have analysed eight texts[1] which have been present in the programs of many postsecondary establishments and that cover a wide period of years:

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
1993	1993	1996	2000	2002	2002	2004	2008

For the analysis, trying to adhere to our questions, we constructed an analysis grid. We took into account the following elements: the space given by each textbook to series; the number, type and role of the figural representations (e.g. graphs, drawings, etc.) and the ratio of representation per page; the number and type of applications of the concept of series (e.g. real life applications, applications in other disciplines, etc.); the number and type of historical references (e.g. a simple reference to events, integration of history in the exposition of contents, etc.). Hence, our analysis grid included both qualitative and quantitative data.

To analyse the role of visual representations in the textbooks, we adapted the classification by Elia & Phillipou (2004), who studied the role of images in problems:

- *Decorative* (D): does not provide any information to solve the problem.
- *Representational* (R): represents a part of the content of the problem, but it is not essential to solve the problem.
- *Organisational* (O): helps to organise the steps for the resolution, but it is not essential to solve the problem.
- *Informational* (I): the solution of the problem is based on this visual representation. Some of the necessary data are in this image, so it is essential to solve the problem.

Drawing on this classification and to analyse the role of images in exposition of the theoretical contents, we defined the following three categories:

- *Non-conceptual* (NC): does not relate to a mathematical concept (e.g. the portrait of a mathematician).
- *Conceptualised* (C): does relate to a mathematical concept and it is used to explain a theoretical notion (e.g. a graph in the proof of a given theorem).
- *Bland-Conceptualised* (BC): does relate to a mathematical concept, but it is not essential to understand the theoretical explanations (e.g. a sketch in a margin showing the graph of the sinus function, to remind the student its shape).

The following section shows the results of our analyses, regarding our research questions.

DATA DISCUSSION

The analysis of our eight textbooks shows that series is not a content neglected by the editors. It seems that, in general, textbooks of postsecondary level allow for about 10% of their subject matter to be used for the concept of series. The minimum ratio we found is 4.27% in *B*, and the maximum of 18.14% in *E*, with an average of 11.70% of space.

Regarding the use of visual images, in the textbooks produced from 2002 there is an evolution in the number, but the ratio stays quite irregular in the whole set. We can also see that, in general, the number of graphical representations for the concept of series is quite low:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Number of images	4	3	7	3	17	29	11	7
(ratio per page)	0.104	0.146	0.109	0.081	0.318	0.598	0.202	0.143
Number of graphs	2	1	2	1	7	5	2	2
(ratio per page)	0.052	0.049	0.031	0.027	0.131	0.103	0.037	0.041
Other images	2	2	5	2	10	24	9	5
(ratio per page)	0.052	0.098	0.078	0.054	0.187	0.495	0.165	0.102

Through the eight textbooks, all the graphs appear in the theoretical part, none in the exercises (so we did not apply Elia & Philippou's categories to graphs in the problems). The most common apparition of a graph is to support the understanding of the Integral Test (stating under which conditions the behaviour of a series is the same as the behaviour of an integral), like in Figure 1. So, the role of these representations, according to our categories, is mainly *conceptualised* (only the textbook *E* uses also *bland-conceptualised* graphs).

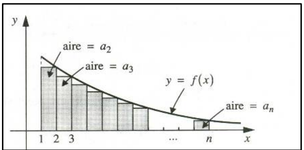

Figure 1

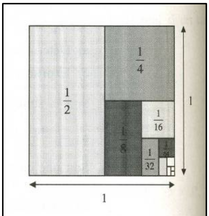

Figure 2

However, the student is never asked, in any of the eight textbooks, to produce a graph to represent a series. Inversely, the student is never explicitly asked to interpret a given graph. We think that the coordination of the graphic and the algebraic registers is absent in all the textbooks we have analysed. Moreover, a graphical representation of a series never appears before the graph shown in Figure 1 for the Integral Test. So, even if the graphs are *conceptualised*, it is possible that they are meaningless for the students, who have not seen the graph of a series before. We think that this situation does not help the student to construct a visual image of what a series may represent (indeed, the results of González-Martín, 2006, suggest that undergraduate students have no visual images associated to the concept of series).

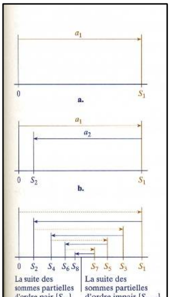

Figure 3

Another useful representation for a better grasp of the notion of an infinite sum having a finite result (see Figure 2) only appears in three of the textbooks (*A, F*, and *G*). The fact that the addition of the positive powers of ½ gives 1 can be visualised using a configuration that reminds of Oresme's configurations (see Codes & Sierra, 2007). And the representation of the behaviour of an alternate series filling the hypothesis of the Leibniz Theorem and so approaching to a given value (Figure 3), only appears in four textbooks (*C, E, F* and *G*). Moreover, in none of the eight textbooks the student is asked to produce any sketch or visual representation to interpret any other result. Again, the creation of a meaning is neglected in the textbooks, and visual representations seem to be an exception. The great majority of the theoretical contents are developed exclusively in the algebraic register, and the images are restricted to the first half of the chapter for series.

Even if the examples given in the textbooks are numerous, most of them are given in the algebraic register:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Examples	51	3	39	31	35	29	55	23
(ratio per page)	1.32	0.15	0.61	0.84	0.65	0.592	1.009	0.47
Visual	0	0	0	0	0	0	0	0
Algebraic	50	3	39	31	34	29	54	23
Mixed	1	0	0	0	1	0	1	0

Regarding applications, three of the textbooks (*B, D,* and *H*) do not show any practical application of series (besides its use to calculate integrals, in *H*). In addition to this, textbook *C* only gives as applications the writing of numbers with infinite decimals, and the calculation of the total distance covered by a ball that bounces infinitely. Amongst the applications shown in the other four textbooks, they are: calculation of areas, medicine, economics, social sciences, movement of a pendulum, and fractals. All the applications appear only in the first half of the episode for series; when the convergence criteria and alternate series are shown, no application appears. The exercises are restricted then to calculations and applications of criteria.

If we compare the number of exercises with a context, trying to give a certain utility or meaning to the concept of series, to the total amount of exercises given in each textbook, we can see that the ratio is really small:

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Exercises	56	83	285	84	302	200	40	14
With a context	6	0	2	0	13	6	3	0
(context/exercises)	0.107	0	0.007	0	0.043	0.030	0.075	0

Regarding the use of history, it is restricted to some anecdotes of mathematicians or to portraits. Besides that, Zeno's paradox appears usually as an anecdote. There is no integration of history to give a meaning or to produce a questioning in the student.

The preliminary results of the UK team (see Nardi, Biza & González-Martín, 2008), after the analysis of seven texts, go in the same direction. Texts allow little space for visual representations (eleven found in three of the seven texts, mostly of the BC type) and hardly require of the students a coordination of the algebraic and graphic registers. Moreover, none of the UK texts makes any historical reference. In addition to this, the applications shown are rare (three in two of the seven texts) and usually the concept is introduced without explaining what it is useful for (with the exception of one text that introduces the concept in terms of its relevance to other mathematical topics). These results seem to suggest that the elements we have detected in the Québec textbooks are characteristics that transcend the boundaries of this province.

FINAL REMARKS

The fact that there is little research developed around the concept of series may not encourage editors to use other than the algorithmic and algebraic approaches so far privileged by textbooks. We have seen that even if textbooks give a relatively significant space to explain content about series (more than 10% generally), the approach that they use seems to be “traditional” and the register used is almost exclusively the algebraic one, with very few graphical or visual representations. In addition to that, the series are usually introduced just as a mathematical object that meets technical mathematical needs, so students do not necessarily develop a vision of series out of mathematics or of its applications.

According to our theoretical framework, the coordination of at least two registers is necessary to understand a mathematical concept. However, our analyses both in Canada and in UK show that textbooks do not favour this coordination of registers for the concept of series. This might explain why some university students were unable to interpret a graph of a series (González-Martín, 2006). Unfortunately, it seems that Boschet’s (1983) remarks about the absence of graphic representations for convergence still hold for concepts like series twenty-five years later.

We are aware that the set of textbooks we have chosen is still very small to draw general conclusions. However, this set has allowed us to see some tendencies which will better guide our further analyses. But the fact that the same tendencies appear in the UK set tends to make us think that the tendencies we have observed are general. Once we have completed the analysis of a significant set of textbooks over the last 15 years, we aim at analysing how teacher practises are developed. Our working hypothesis will be that teachers tend to follow the approaches of the textbooks.

Our research is just starting, but our first results are encouraging. To complete the analysis of the didactic dimension, we want to complete the analysis of textbooks, to analyse the official syllabi of the last 15 years and their recommendations and, finally, teacher practices. We aim that these results will inform the epistemological and the cognitive dimensions of our analysis that will follow soon.

NOTES

1 They are : *A*: Charron & Parent (1993). *Calcul différentiel et intégral II*, Québec : Études Vivantes; *B*: Ayres & Mendelson (1993). *Calcul différentiel et intégral (2^{e} édition)*, Québec : Chenelière/ McGraw-Hill; *C*: Anton (1996). *Calcul intégral*, Québec : Reynald Goulet inc.; *D*: Ouellet (2000). *Calcul 2 : introduction au calcul intégral*, Québec : Le Griffon d'argile; *E*: Bradley *et al* (2002). *Calcul intégral*, Québec : ERPI; *F*: Thomas *et al* (2002). *Calcul intégral (10^{e} édition)*, Québec : Beauchemin; *G*: Charron & Parent (2004). *Calcul intégral (3^{e} édition)*, Québec : Beauchemin; *H*: Amyotte (2008). *Calcul intégral*, Québec : ERPI.

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A SOCIOCULTURAL FRAMEWORK FOR UNDERSTANDING TECHNOLOGY INTEGRATION IN SECONDARY SCHOOL MATHEMATICS

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This paper proposes a theoretical framework for analysing relationships between factors influencing teachers' use of digital technologies in secondary mathematics classrooms. The framework adapts Valsiner's zone theory of child development to study teacher learning in terms of the interaction between teacher knowledge and beliefs, professional contexts and professional learning experiences. Use of the framework is illustrated by case studies of an early career teacher and an experienced teacher.

The potential for digital technologies to transform mathematics learning and teaching has been widely recognised for some time. Research has demonstrated that effective use of mathematical software, spreadsheets, graphics and CAS calculators and data logging equipment enables fast, accurate computation, collection and analysis of real or simulated data, and investigation of links between numerical, symbolic, and graphical representations of mathematical concepts (see Hoyles, Lagrange, Son, & Sinclair, 2006, for a recent review of the field). However, integration of digital technologies into mathematics teaching and learning has proceeded more slowly than initially predicted (Cuban, Kirkpatrick, & Peck, 2001; Ruthven & Hennessy, 2002). Many studies have shown that access to technology resources, institutional support, and educational policies are insufficient conditions for ensuring effective integration of technology into teachers' everyday practice (Burrill, Allison, Breaux, Kastberg, Leathem, & Sanchez, 2003; Wallace, 2004; Windschitl & Sahl, 2002). These findings suggest that more sophisticated theoretical frameworks are needed to understand the teacher's role in technology-integrated learning environments and relationships between factors influencing teachers' use of digital technologies. The purpose of this paper is to propose such a framework and illustrate its use via analysis of sample data from secondary school mathematics classrooms. The data were collected in a three year study that aimed to understand how and why technology-related innovation works (or not) within different educational settings.

THEORETICAL FRAMEWORK

The theoretical framework for the study is the product of an extended research program informed by sociocultural theories of learning involving teachers and students in secondary school mathematics classrooms (summarised in Goos, 2008). Sociocultural theories view learning as the product of interactions between people and with material and representational tools offered by the learning environment.

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Because it acknowledges the complex, dynamic and contextualised nature of learning in social situations, this perspective can offer rich insights into conditions affecting innovative use of technology in school mathematics.

The framework used in the present study adapts Valsiner's (1997) zone theory of child development in order to theorise teachers' learning (Goos, 2005a, 2005b). Valsiner extended Vygotsky's (1978) concept of the Zone of Proximal Development to incorporate the social setting and the goals and actions of participants. He described two additional zones: the Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA). The ZFM represents constraints that structure the ways in which an individual accesses and interacts with elements of the environment. The ZPA comprises activities, objects, or areas in the environment in respect of which the individual's actions are promoted. For learning to be possible, the ZPA must engage with the individual's possibilities for development (ZPD) and promote actions that are believed to be feasible within a given ZFM. When we define these zones from the perspective of the *teacher as learner*, the ZPD represents a set of possibilities for *teacher development* influenced by their knowledge and beliefs about mathematics and mathematics teaching and learning. The ZFM suggests which teaching actions are *allowed* by constraints within the school environment, such as teachers' perceptions of students (abilities, motivation, behaviour), access to resources and teaching materials, curriculum and assessment requirements, and organisational structures and cultures. The ZPA represents teaching approaches that might be *promoted* by pre-service teacher education programs, professional development activities and informal interaction with colleagues at school.

Valsiner's Zones	Elements of the Zones
Zone of Proximal Development	Mathematical knowledge Pedagogical content knowledge Skill/experience in working with technology General pedagogical beliefs
Zone of Free Movement	Students (perceived abilities, motivation, behaviour) Access to hardware, software, teaching materials Technical support Curriculum & assessment requirements Organisational structures & cultures
Zone of Promoted Action	Pre-service teacher education Professional development Informal interaction with teaching colleagues

Table 1: Factors affecting teachers' use of technology

Previous research on technology use by mathematics teachers has identified a range of factors influencing uptake and implementation. These include: skill and previous experience in using technology; time and opportunities to learn; access to hardware and software; availability of appropriate teaching materials; technical support; organisational culture; knowledge of how to integrate technology into mathematics teaching; and beliefs about mathematics and how it is learned (Fine & Fleener, 1994; Manoucherhri, 1999; Simonsen & Dick, 1997). In terms of the theoretical framework outlined above, these different types of knowledge and experience represent elements of a teacher's ZPD, ZFM and ZPA, as shown in Table 1. However, in simply listing these factors, previous research has not necessarily considered possible relationships between the teacher's setting, actions, and beliefs, and how these might influence the extent to which teachers adopt innovative practices involving technology. In the present study, zone theory provides a framework for analysing these dynamic relationships.

RESEARCH DESIGN AND METHODS

Four secondary mathematics teachers participated in the study. They were selected to represent contrasting combinations of the factors known to influence technology integration summarised in Table 1. They included two early career teachers who experienced a technology-rich pre-service program and two experienced teachers who developed their technology-related expertise solely through professional development experiences or self-directed learning. The early career teacher participants were recruited from a pool of recent teacher education graduates from The University of Queensland, while the experienced teacher participants were identified via professional networks, including mathematics teacher associations and contacts with schools participating in other university-based research projects.

There were three main sources of data. First, a semi-structured scoping interview invited the teachers to talk about their knowledge and beliefs, professional contexts and professional learning experiences in relation to technology. Additional information about the teachers' general pedagogical beliefs was obtained via a Mathematical Beliefs Questionnaire (Goos & Bennison, 2002) consisting of 40 statements to which teachers responded using a Likert-type scale based on scores from 1 (Strongly Disagree) to 5 (Strongly Agree). The third source of data was a series of lesson cycles (typically 4 cycles per year) comprising observation and video recording of at least three consecutive lessons in which technology was used to teach specific subject matter together with teacher interviews at the beginning, middle, and end of each cycle. These interviews sought information about teachers' plans and rationales for the lessons and their reflections on the factors that influenced their teaching goals and methods. Data from these sources were categorised as representing elements of participants' ZPDs, ZFMs, and ZPAs, an analytical process that enabled exploration of how personal, contextual and instructional factors came together to shape the teachers' pedagogical practice in relation to use of technology.

The next section draws on the sources of data outlined above to illustrate use of the zone framework in comparing the cases of two teachers, Susie (early career teacher) and Brian (experienced teacher).

SUSIE: AN EARLY CAREER TEACHER

Susie graduated from the university pre-service program at the end of 2003 and found a position teaching in an independent secondary school located in a large city. Most students in this school come from white, Anglo-Australian middle class families.

Susie's responses to the Mathematical Beliefs Questionnaire suggested that her beliefs were non-rule-based and student-centred (Tharp, Fitzsimons, & Ayers, 1997). For example, she expressed strong agreement with statements such as "In mathematics there are often several different ways to interpret something", and she disagreed that "Solving a mathematics problem usually involves finding a rule or formula that applies". The beliefs about mathematics teaching and learning revealed through questionnaire responses were supportive of cooperative group work, class discussions, and use of calculators, manipulatives and real life examples.

Susie's own experience of learning mathematics at school was structured and content-based, but this was different from the approaches she tried to implement as a mathematics teacher. When interviewed she explained that in her classroom "we spend more time on discussing things as opposed to just teaching and practising it", and that for students "experiencing it is a whole lot more effective than being told it is so". Aged in her mid-20s, Susie felt she was born into the computer age and this contributed to her comfort with using technology in her teaching. Although her first real experience with graphics calculators was in her university pre-service course, she indicated that "the amount I learned about it [graphics calculators] during that year would be about 2% of what I know now". She spoke enthusiastically of the support she had received from the school's administration and her colleagues since joining the staff: "Anything I think of that I would really like to do [in using technology] is really strongly supported".

Observations of Susie's Grade 10 mathematics class provided evidence of how she enacted her pedagogical beliefs. For example, in one lesson cycle Susie introduced quadratic functions via a graphical approach involving real life situations and followed this with algebraic methods to assist in developing students' understanding. Lessons typically engaged students in one or two extended problems rather than a large number of practice exercises.

The questionnaire, interview and observation data "fill in" Susie's Zone of Proximal Development with knowledge and beliefs about using technology to help students develop mathematical understanding by investigating real life situations and linking different representations of concepts. Likewise, the Zone of Promoted Action within the school explicitly promoted technology-enriched teaching and learning. Elements of her Zone of Free Movement were also supportive of technology integration. The school's mathematics department had for many years cultivated a culture of technology innovation backed up by substantial resources. Students in Grade 9-12 had their own graphics calculators, there were additional class sets of CAS calculators for senior classes, and data logging equipment was freely available. Computer software was also used for mathematics teaching; however, computer laboratories had to be booked well in advance.

The evidence outlined above suggests that there was a good fit between Susie's Zone of Proximal Development and her Zone of Free Movement, in that her professional environment afforded teaching actions consistent with her pedagogical knowledge and beliefs about technology. Susie used this ZPD/ZFM relationship as a filter for evaluating formal professional development experiences and deciding what to take from these experiences and use in her classroom. She had attended many conferences and workshops since beginning her teaching career, but found that most of them were not helpful "for where I am". She explained: "because we use it [technology] so much already, to introduce something else we'd have to have a really strong basis for changing what's already here". Although Susie's exposure to technology in her mathematics pre-service course may have oriented her towards using technology in her teaching, the most useful professional learning experiences had involved working collaboratively with her mathematics teaching colleagues at school. The only real obstacle she faced was lack of time to develop more teaching resources and to become familiar with all of the technologies available to her. For Susie, the most helpful Zone of Promoted Action lay largely within her own school, and was thus almost indistinguishable from her Zone of Free Movement.

BRIAN: AN EXPERIENCED TEACHER

Brian had been teaching mathematics in government high schools for more than twenty years. For much of this time he was Head of the Mathematics Department in an outer suburban school serving a socio-economically disadvantaged community. In the late 1990s he recognised that the traditional classroom settings and teaching approaches the students were experiencing did not help them learn mathematics. He pioneered a change in philosophy that led to the adoption of a social constructivist pedagogy in all mathematics classes at the school. This new philosophy, expressed through problem solving situations and the use of technology, concrete materials and real life contexts, produced significant improvement in mathematics learning outcomes across all grade levels. At the start of 2006 Brian moved to a new position as Head of Department in a different school, also situated in a low socio-economic area. Here he faced many challenges in introducing the mathematics staff to his teaching philosophy and obtaining sufficient technology resources to put his philosophy into practice.

Brian's espoused beliefs, as indicated in his responses to the Mathematics Beliefs Questionnaire, were consistent with the constructivist principles that guided his practice. For example, he expressed disagreement with statements such as "Doing lots of problems is the best way for students to learn mathematics", and he strongly agreed that "The role of the mathematics teacher is to provide students with activities that encourage them to wonder about and explore mathematics". When interviewed, he often emphasised that his reason for learning to use technology stemmed from his changed beliefs about how students learn mathematics. For him, technology was a vehicle that allowed students to engage with concepts that they would not otherwise be able to access.

Observations and interviews from several lesson cycles revealed that Brian's preferred teaching approach exemplified his general philosophy in that he initially used graphical representations to help students develop understanding of concepts so they might then see the need for analytical methods involving algebra. He justified this by saying that developing an understanding of the concepts gives meaning to the algebra and students would then became more likely to persevere with algebraic methods.

Brian's knowledge and beliefs (Zone of Proximal Development) were the driving force that led him to integrate technology into his inquiry-based approach to teaching mathematics. When graphics calculators became available in the mid-1990s he attended professional development workshops presented by teachers who had already developed some expertise in this area. He later won a government scholarship to travel overseas and participate in conferences that introduced him to other types of technology resources. Apart from these instances Brian had rarely sought out formal professional development, preferring instead to "sit down and just work through it myself". His Zone of Promoted Action was thus highly selective and focused on finding coherence with his personal knowledge and beliefs.

In the seventeen years that Brian spent at his previous school he was able to fashion a Zone of Free Movement that gave him the human and physical resources he needed to teach innovatively with technology. However, when he arrived at his current school at the start of 2006 he found little in the way of mathematics teaching resources – "there was a lot of stuff here but it was just in cupboards and broken and not used, and not coherent, not in some coherent program". There were no class sets of graphics calculators and it was difficult for mathematics classes to gain access to the school's computer laboratories. Exacerbating this situation was an organisational culture that Brian diplomatically described as "old fashioned". Almost none of the mathematics teachers were interested in learning to use technology, and it appeared that an atmosphere of lethargy had pervaded the mathematics department for many years. Students demonstrated a similarly passive approach to learning mathematics, expecting that the teacher would "put the rule up and example up and set them up and away they go". Brian responded to these challenges in several ways. First, he lobbied the newly appointed Principal, who was strongly supportive of his teaching philosophy and plans for expanding the range of technology resources in the school, for funds to buy software for the computer laboratories and a data projector for installation in his mathematics classroom. Secondly, he took advantage of loan schemes operated by graphics calculator companies to borrow class sets of calculators. He also used his influence as Head of Department to secure timetable slots for senior mathematics classes to use the computer laboratories.

Brian evaluated the adequacy of his present Zone of Free Movement, or professional context, by looking through the inquiry-based, technology-rich lens created by the relationship between his ZPD (knowledge and beliefs) and ZPA (previous professional learning). He identified his priorities for re-shaping the ZFM in his new school as continuing to advocate for the purchase of more technology resources and helping his staff become comfortable and confident in using these resources. His main obstacles were lack of funds and a teaching culture that resisted change.

DISCUSSION

The research reported in this paper examined relationships between factors that influence ways in which teachers use digital technologies to enrich secondary school mathematics learning. While the findings are consistent with results of other studies of educational uses of technology in highlighting the significance of teachers' beliefs, their institutional cultures, and the organisation of time and resources in their schools, the socioculturally oriented zone theory framework offers new insights into technology-related innovation. For example, although access to technology is an important enabling factor, the cases of Susie and Brian show that teachers in well resourced schools do not necessarily embrace technology while teachers in poorly resourced schools can be very inventive in exploiting available resources to improve students' understanding of mathematical concepts.

The knowledge and beliefs that Susie and Brian hold about the role of technology in mathematics learning are central in shaping their pedagogical practice, but more important are the relationships between their knowledge and beliefs (ZPDs), professional contexts (ZFMs) and professional learning experiences (ZPAs). It was significant that Susie and Brian differed in the degree of alignment between their respective ZPDs and ZFMs. For Susie, the Zone of Free Movement offered by her school was important in allowing her to explore technology-enriched teaching approaches consistent with her knowledge and beliefs. It may be that this kind of alignment is critical in helping beginning teachers seek out professional learning opportunities consistent with the innovative practices they may have encountered in pre-service programs. On the other hand, Brian, as an experienced teacher and Head of Department, relied on his knowledge and beliefs about learning to envision the kind of professional environment he wanted to create in his school. For him, the ZPD/ZFM misalignment was a powerful incentive to pursue his goal of technology-enriched teaching and learning. These initial findings need to be tested with different teachers in a wider range of settings in order to further explicate the application of zone theory to teachers' technology related professional learning.

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VALIDATING STUDENTS' RESPONSES WITH TEACHER'S RESPONSES IN TWO-TIER TESTING

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This study aims to investigate assessment information generated by research-based two-tier tasks in 'proportion' and to document teachers' reactions to the assessment information. As part of the doctoral study, this paper discusses one Y7 teacher's reactions to her students' responses by validating their responses. 12 Year 7 girls took four similar tests each having 8 two-tier tasks in 2008. 8 of the Y7 girls and their teacher are interviewed. Analysis of students' responses reveals that: i) high math ability students are more successful in solving the two-tier tasks, and ii) students' justifications give some information about their correct and incorrect proportional reasoning. The teacher finds the two-tier tasks make the students think about why the answer is correct or wrong and make her think about why the students are thinking.

PURPOSE OF THE STUDY

In Australia, there has been a shift towards a more inclusive view of assessment (Panizzon & Pegg, 2007) over the past few years. Aligned with recent changes to syllabuses in Australia is an assessment regime requiring teachers to identify what their students 'know' and 'can do' in terms of the quality of understanding demonstrated. Teachers need an appropriate support system to use the assessment information to diagnose strengths and weaknesses of students that can help them to improve student learning by adjusting their instruction. Two-tier tasks are known for its diagnostic function providing information to teachers about students' misconceptions in science education (see Treagust, 2006). But, application of two-tier testing is not reported much in mathematics education. The structure of two tier testing is such that the first tier consists of a multiple-choice question and the second tier contains possible justifications for choices of answers.

Research-based tasks can function as instructional activities as well as assessment tools. Cramer and Post (1993) suggested that research tasks pertaining to the learning and teaching of proportional reasoning can and should be used in classrooms and argue that research can have a positive effect on classroom practice. The types of problems generated for research purposes can inform teachers of the different ways in which student proportional reasoning can be assessed. This study involves Y7 and Y8 teachers in classroom assessment using research-based two-tier tasks to investigate what they infer from their students' responses and how they use or intend to use the assessment information to organize their instruction. As part of the doctoral study, this paper discusses one Y7 teacher's (Samsu) reactions to her students' responses to the two-tier tasks. Specifically, this report attempts to answer the following two questions.

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1. What information is generated by two-tier tasks about Y7 students?
2. How do Y7 teachers react to the assessment information?

THEORETICAL FRAMEWORK

Post, Behr, and Lesh (1988) describe proportion as a watershed concept as proportional reasoning is the capstone of the elementary school curriculum and the cornerstone of high school mathematics and science. A *proportion* is the statement that two ratios are equal in the sense that both convey the same relationship. The essential characteristic of *proportional reasoning* is that it must involve a relationship between two relationships rather than simply a relationship between two concrete objects or two directly perceivable quantities (Piaget & Inhelder, 1975). This study makes use of Carpenter and his colleague's (1999) study to analyse students' written responses to two-tier tasks. Carpenter et al. (ibid.) identified four levels to describe the progression of 'ratio and proportion' understanding by investigating 21 students in a combination of fourth and fifth grade class. Students at level 1 show limited ratio knowledge. Students at level 2 are able to combine the ratio units together by repeated addition of the same ratio to itself or by multiplying that ratio by a whole number. Students at Level 3 can scale the ratio by non-integers and this allows diversity in students' strategies. Students with strong multiplication skill are able to perform direct scalar multiplication on the given ratio. Students at Level 4 recognize the relation within the terms of each ratio and between the corresponding terms of the ratios in the proportion. They are able to focus on the numbers in the problem rather than the problem context to determine an appropriate and efficient strategy. It would be interesting to find how the Y7's fit into these levels and whether they maintain the same level for all the 8 tasks given in a test.

METHODOLOGY

Qualitative and quantitative methods of inquiry are used in this study. Tools used are: Four versions of proportional reasoning paper-pencil tests, teacher questionnaire, student interview schedules, and teacher interview schedules. A range of research on students' proportional reasoning and errors informed the writing of the two-tier tasks. 16 items have been adopted with modifications from previous research studies (e.g., Karplus et al., 1983; Hart et al., 1984; Tourniaire & Pulos, 1985; van den Heuvel-Panhuizen, 1996). The two-tier tasks include four representational formats of proportional problems (See Appendix):

Numerical (numbers only) - Missing Value #1, Missing Value #2, Comparing two equivalent ratios, Arranging four fractions in ascending order,

Pictorial (context + picture) - Eel, Matchstick, Paint, Shoe

Symbolic (context + diagram) - Rectangle, Number line, Candies table, Bus circle,

Contextual - Recipe, Juice, Running Laps, Chin-ups

Pictorial and symbolic tasks do have context and thus may be considered as subsections of contextual. In this study, two types of task structures are introduced (See Appendix). They are:

Task structure 1: Select answer + write justification + select confidence level (ranging from 'not at all confident' to 'extremely confident')

Task structure 2: Select answer + select justification + write justification to safety-net question (an extension of the main question)

Four versions of tests are constructed each having 8 tasks in it. Task types and task structures are evenly distributed in tests. Samsu administered the two-tier tests to 12 of her students (G1 to G12) in the first week of September 2008 in a Melbourne independent school. Students have experience in solving number line problems prior to the testing. Samsu didn't teach fraction prior to the testing. The principal researcher interviewed 8 girls (G1 to G8) after the two-tier testing. Interview with Samsu took place after Samsu had marked the students' responses. The interviews are audio-taped and transcribed for analysis. Students' written responses are also analyzed.

Students' proportional reasoning

G5 performs better in two-tier tasks as compared to her peers. Samsu identifies G5 as a 'bright' kid. Samsu also comments on G5's justification to the bus question as "good". G5's justification to the Bus task is: '3 out of 5 students ride to school, if there were 25 students that is 5×5, five groups of students which each have 3 which ride the bus to school. So, you multiply 3×5 to get 15'. G5 appears to be at stage 3 level for this task. In her attempt to justify her choice of answer, G5:

- comprehends that there are five groups in 25 students,
- identifies that each group has 3 bus riders, and
- multiplies 3 by 5 to get the total number of bus riders.

G6's justification to the Rectangle Safety-net task was: 'yes because all that is changed is the angle we are looking at it from'. G6 didn't attempt to think the problem in proportional context and found to be in stage 1 for that task. Samsu identifies G6 as a 'less bright' kid. G2 appears to be in stage 1 and 2 and has difficulty in solving two-tier tasks. G2's 'constant sum' justification to the Rectangle task was: '12- 9 is 3 so if I add 3 to 6 it should give me the answer of the height because they are the same shape'. Samsu identifies G2 as a 'less bright' kid. G2 too reflects on her performance in two-tier tasks as, "not very well" because she couldn't remember anything during the test. G7's justification to the Shoe task was: '40% is higher so you take more off the original price to get a cheaper price'. G7 apparently couldn't interpret the shoe picture correctly and her answer to the Bus task is also diagram dependent. G7 used her general knowledge about fractions to arrange the four fractions in ascending order: '1/8 is the smallest, 2/5 is bigger but not bigger than ½, and ½ is not bigger than ¾ and ¾ is the biggest'. G6 found to be in stages 1 and 2. Table 1 summarizes twelve Y7 girls' success rates in two-tier tasks.

Students	Success rate in select answer	Success rate in write justification	Success rate in select justification	Success rate in Safety-net Question	Frequency of confidence-level in selecting answer				Frequency of confidence-level in selecting justification			
					NC	NVC	FC	EC	NC	NVC	FC	EC
G1	75	75	50	50			1	3			2	2
G2	25	25	50	25	1	1	1	1	1	2	1	
G3	50	50	75	25	1		3		1		3	
G4	75	75	100	75			3	1			3	1
G5	100	100	100	75			4			1	3	
G6	75	75	75	25	1			3	1	1	1	1
G7	75	75	100	75				4				4
G8	50	0	50	25	2		2			2	1	1
G9	0	0	0	0		1		2		1		2
G10	50	50	50	50			1	3			3	1
G11	50	50	50	25	1		1	2	1		2	1
G12	75	75	100	75				4				4

NC – Not confident at all; NVC – Not very confident; FC – Fairly confident; EC – Extremely confident. Test 1 Test 2 Test 3 Test 4

Table 1: Year 7 Girls' Success Rates in Two-tier Tasks

G1 did fairly well in the test and Samsu identifies her as 'bright'. G1's 'fairly confident' justification to MVn2 was: '28-8 =20 therefore 12+20=32'. She used a 'constant sum' approach for this missing value problem but she was successful in getting another missing value task correct. It may be due to the fact that 28 can't be expressed as a direct product of 8 where as in the first missing value task, 500 can be expressed as a direct product of 25 (25×20 = 500). When students can't see an obvious multiplicative relationship between two ratios then they find another means (such as constant sum and building up) to establish relationship between the given two ratios. G1's 'extremely confident' justification to Matchstick was: '1 match=to 1.5 paper clips and if there is only 2 more matchs then 2×1.5=3 and 6+3=9'. In her attempt to explain how she chose the answer 9, G1 uses 'unit strategy' to find the proportional relationship between match stick and paper clips. G1 found to operate in stages 3 and 4.

G4's 'constant sum' justification to the Paint task was: '3+7 is 10 paint cans so 6+4 is 10 paint cans'. For the Candies task G4's working out was: '7-2, 7-2, 7-2, 7-2, 7-2, 7-2, 7-2, 7-2, 7-2, 7-2'. Apparently G4 added all the 7's to find the answer where she was using additive reasoning instead of multiplicative. G4's justification to the Chin-up's Safety-net was: 'No, she [Kate] is the same because she would only be able to do 3 in 5 seconds the same as David because 3×9=27 and 3×15=45 and 3×3=9 and 3×5=15 so she would only be able to do 3 in 5 secs'. G4 could see the similar multiplicative relationships in both cases and thereby operated in stage 4. G8 guessed most of her answers (Candies, Chin-ups, Bus, Comparing two ratios) because she 'didn't really know it' and she admits that 'it was not a full answer' and she just wrote what she knew. G8's 'extremely confident' 'building up' justification to the Laps task: 'I compared 9 to 3 and it was 6 so I added 6 to 15 and got 21'. Samsu identifies G8 as a 'low achiever' who had levels at stage 1 for most of the tasks.

VALIDATING TEACHER AND STUDENT RESPONSES

Two-tier testing: Two-tier task according to Samsu is a different way of testing the students. Samsu says, "normal multiple choice doesn't tell whether students have a fairly good understanding and does not give the full range of thinking but looking at students' responses to two-tier tasks tells you that they have either no idea of what the concept is". Samsu says, "[two-tier task] was really good for them to actually think about the way they were solving the question". Samsu thinks that two-tier tasks actually making them think about why the answer is correct or wrong. Samsu says, "we [teachers] have actually been working a bit of time on how students handle multiple choice but they never had written down 'why' or 'this is because'". Samsu likes to go with the two-tier tasks more often because girls don't handle multiple choice questions very well and the teachers work to get the girls to discuss why did they pick an answer. Samsu observes that two-tier task doesn't help for those who can't reason out and multiple-choice is not helping them in that respect either. But, she thinks, in general, two-tier task is good for the bright and middle road of the students. Two-tier tasks give all possible methods and it tests students' thinking and they have to write a bit whereas multiple-choice just give right or wrong kind of things (G4). G5 also thinks that two-tier task has more in it such as 'I don't know' and a survey that asks students 'how confident are you with your answer'. G7 notes that she has to say 'why I chose the answer' and usually with other tests she just needs to write the answer. G7 finds a bit harder to choose 'why she did' and G8 too finds two-tier tasks are difficult items because of its complex wordings.

'Select justification' and 'write justification': Some students (eg., G2, G8) prefer 'select justification' while some students (G3, G5) prefer 'write justification'. G2 favors 'selecting the justification' because, "you don't know how to write it in words….so....if you just got the choice it is easy". G8 finds it easier to select because, "when you write you have to think more…like you have to put detail in". But, G3 says, "it is good that we get to write our own" and G5 says it is easier to write than to select "someone else's words". The 'write justification' part tells Samsu about the students' thinking processes. Samsu notes that the 'select justification' made the students to go back and change their answers. Samsu observes that "the bright ones are able to justify their answers quite well" and "the middle road students had no idea even though their comments were interesting". Samsu also observes that "the justifications given in the tasks didn't help the low achievers to choose the correct answer". For example, G8 just had written 'I don't know'.

Safety-net Questions: Samsu finds that some of her students had really no idea of the safety-net questions because the students hadn't done them before. Samsu further added that the capable students had an idea because "they were very logical and used the common sense" to write justification from what they knew and what the question was asking and "their answers were interesting".

Different representations: The multiple opportunities provided for the Y7s to justify their reasoning using numerical, pictorial, symbolic and contextual strategies was significant. Girls justify through triangulation of numerical, verbal, and pictorial strategies by making connections across different representations. In combination with effective pedagogical support, opportunities to engage with proportional problems may provide students to deepen their use of justification to validate their proportional reasoning. Samsu finds that the Matchstick task is effective in making her students think because the students are actually able to see the relationship between the paper clips and the matchsticks. Samsu is very impressed with her students' thinking because the students can actually visually see the matchsticks in relation to the paperclips and Samsu further notes that the pictorial representation helps the students. Samsu speaks from her experience, "if I put something out with pictures and visual things and they actually get it lot quicker…specially year 7 because they do a lot of that in primary…everything is done by pictures". G1 finds the numerical ones effective because, "you are using numbers and you don't have to use words like and you can use signs". G1 observes that the diagram didn't really help her as much as she thought it would be. G7 too finds the numerical ones are effective in making her to think because, "you don't have any picture to help you and stuff and you have to imagine the picture in your head". But, on contrast, G6 thinks that the numerical ones are harder because it is harder to explain using words. G6 finds the diagram/picture ones are effective as it is easier to explain with matches and "it makes sense with picture".

Results and conclusion

Analysis of students' responses to two-tier testing on 'proportional reasoning' shows that generally high mathematical ability students (G1, G3, G4, G5, G10, G12; 70.83%) are more successful in writing correct justifications as compared to the less brighter (G2, G6, G7, G11; 56.25%) and low achievers (G8, G9; 0%). Girls appeared to be at ease in selecting justifications (bright–79.16%; less bright–68.75%; low achievers–25%). Success rate of safety-net questions is found to be low (58.33%)

even among bright kids and less brighter managed to get 37.5%. Most of the students would not mind using two-tier tasks in future. G5 feels that two-tier tasks require quite a bit of time and G7 feels that two-tier tasks help her to say "how she got the answer".

Two-tier tasks throw some light on students' existing misconceptions in 'proportion'. While marking the two-tier tasks, Samsu did notice students' misconceptions and from there she would then make sure that those misconceptions are dealt with as she moves on to cover the course. Samsu notes "the two-tier tests actually nuts out and makes you think about why they are thinking". Samsu does use two- tier tasks quite often to actually reflect on their learning but doing it for question by question it becomes quite cumbersome for her. Even when information is gathered in the form of student responses to questions, it appears that teachers have great difficulty adjusting their instruction "on-the-fly" (Black & Wiliam, 1998; Kennedy, 1999). Many school staffs lack the expertise to learn from assessment results (Murnane et al., 2005). The report suggests the following:

1. Teachers may try to look at students' justifications as a way to provide students with feedback on how to improve their proportional reasoning.
2. Teachers may plan, review, and discuss with other colleagues about what they are learning from assessment information and how they are going to make use of the information in organizing their instruction.

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Appendix

Numerical

If $\frac{2}{25} = \frac{n}{500}$, then n = ?

a. 20
b. 40
c. 50
d. 477
e. I do not know
f. Other (*please write your own answer)* --------------

(*Circle the answer of your choice)*

I chose my answer because:

1. 20 times 25 is 500. Therefore n is 20.
2. 25 goes into 500, 20 times. Therefore n is 2 times 20.
3. n = 2 times 25.
4. Multiplying $\frac{2}{25}$ by 2 gives $\frac{4}{50}$. Multiplying $\frac{4}{50}$ again by 10 gives n.
5. 2 is 25–23. Therefore n is 500–23.
6. n = $\frac{2 \times 500}{25}$
7. Other (*please write your own justification)*

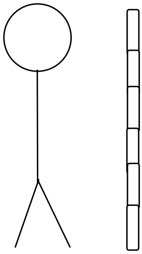
<u>Extended Safety-net Question</u>

If $\frac{4}{50} = \frac{m}{500}$, then m = ? <u>Justify your answer</u>. (*Please write your justification below)*

Pictorial

You can see the height of Mr. Short measured with 6 paper clips.

Mr Short



Mr. Short has a friend Mr. Tall. When we measure their heights with matchsticks: Mr. Short's height is four matchsticks. Mr. Tall's height is six matchsticks. How many paper clips are needed for Mr. Tall's height? *(You may use the picture above to help you find the answer)*

a. 8
b. 9
c. 12
d. I do not know
e. Other (*please write your own answer)* ------------

(*Circle the answer of your choice)*

I chose my answer because: *(please write your justification below)*--------------------
----

Symbolic

A bag contains 2 blue candies for every 7 candies that are not blue. If another bag contains 20 blue candies, how many candies would you expect to have that are not blue?

Blue	2	2	2	2	2	2	2	2	2	2
Non-blue	7	7	7	7	7	7	7	7	7	7

(*You may use the table above to help you find the answer*)

a. 15
b. 25
c. 27
d. 70
e. I do not know
f. Other (*please write your own answer)*---------------

(*Circle the answer of your choice)*

I chose my answer because: *(please write your justification below)*------------------
---

Contextual

An onion soup recipe for 8 persons is as follows:

8 onions

2 litres water

4 chicken soup cubes

12 tablespoons butter

½ litre cream

I am cooking onion soup for 2 people. How many tablespoons of butter do I need?

a. 3
b. 6
c. 24
d. I do not know
e. Other (*please write your own answer)* --------------

(*Circle the answer of your choice)*

I chose my answer because *(please write your justification below)*-------------------

---

How confident are you of your <u>answer</u>? (*Please shade your choice of confidence level*)

Not confident at all — Not very confident — Fairly confident — Extremely confident

How confident are you of your <u>justification</u>? (*Please shade your choice of confidence level*)

Not confident at all — Not very confident — Fairly confident — Extremely confident

UPPER SECONDARY STUDENTS' MATHEMATICAL BELIEFS AND THEIR TEACHERS' TEACHING BELIEFS

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The influence of classroom context on students' mathematical beliefs was analysed from survey data from 894 Finnish upper secondary school students and their teachers from 39 advanced mathematics syllabus classes. The results indicate that students of the same class tend to have similar enjoyment of mathematics and evaluation of teacher. An analysis of their teachers beliefs revealed that enjoyment of mathematics and perceived teacher quality were higher in those classes whose teacher held more progressive beliefs regarding teaching. These progressive beliefs included favouring problem solving, trying new methods and believing in every student's capasity to learn.

INTRODUCTION

Mathematical beliefs are on the one hand considered as individual constructs that are generated by individual experiences. On the other hand, beliefs are considered to be constructed socially, in a shared social context of a classroom. Which is more important? Are all beliefs constructed in the same way or are some beliefs socially constructed while some others are purely individual? What is the role of teacher's beliefs in the development of students' beliefs?

It is frequently assumed that there is a link between teachers' and their students' affect towards mathematics (e.g. Cockroft, 1982). Curious enogh, there seems to be few studies on the relationships between student and teacher affective variable. For example the review of PME research on affect (Leder & Forgasz, 2006) does not mention any such study. As an example of research relating teacher and student beliefs we can take Crater and Norwood's (1997) study of seven teachers and their 138 students, where they found out that this group of teachers practiced what they believed and that these practices affected what their students believed about mathematics.

Although Finland scored to the top in PISA achievement scores, Finland was also characterised by less favourable results on the affective measures. Finnish students' lack interest and enjoyment in mathematics, they have below average self-efficacy, and low level of control strategies. As a more positive result, levels of anxiety were also low. In Finland, affect was an important predictor of achievement. The study also revealed that gender differences favouring males in affect were larger in Finland than in OECD on average. Differences between schools in mathematics achievement are lower in Finland than in most other countries. (OECD-PISA, 2004)

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 129-136. Thessaloniki, Greece: PME.

In a study of Finnish elementary and secondary teachers' beliefs Pekka Kupari identified two types of mathematics teachers, traditional and innovative teachers. The traditional teacher emphasises basic teaching techniques and extensive drill, while the innovative teacher emphasises student thinking and deeper learning. (Kupari, 1996) Moreover, Riitta Soro (2002) found out in her study that most Finnish mathematics teachers held different beliefs about male and female students.

In this report we shall explore more deeply which aspects of mathematical beliefs are most affected by shared classroom context. We shall control the effect of individual achievement and gender. Moreover, we will explore how the group level differences can be attributed to differences in teacher beliefs. Due to space limitation we focus on the advanced syllabus students only.

THEORETICAL FRAMEWORK

In this article we consider mathematical beliefs as "an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" (Schoenfeld 1992, 358).

Op 't Eynde, De Corte and Verschaffel (2002) provide a framework of students' mathematics-related beliefs: 1) Mathematics education (mathematics as subject, mathematical learning and problem solving, mathematics teaching in general); 2) Self (self-efficacy, control, task-value, goal-orientation); and 3) The social context (social and socio-mathematical norms in the class). With regard to the social context, Op 't Eynde & DeCorte (2004) found out later that the role and functioning of one's teacher are an important subcategory of it. Fhis framework has been partially confirmed also among Finnish upper secondary school students (Rösken, Hannula, Pehkonen, Kaasila and Laine, 2007).

The origin for students' mathematical beliefs can be attributed to their individual life histories, to their shared experiences it the context of the classroom and to their experiences as a member of a social group (Figure 1).

To begin from the most general level, there are experiences that people of the same social group tend to share. These are mediated through cultural conventions and media. Such experiences would lead to formation of socially shared gendered beliefs.

Moving into more speficic, the context of the classroom provides shared exeriential basis, which influences all students in a class. This basis is the origin of shared experiences. One specific aspect of the classroom context are practices that mediate cultural conventions regarding gender and other social variables. For example, most teachers have different beliefs about boys and girls as mathematics learners (Soro, 2002). Hence, for each mathematics classroom, the shared experiences may be different for male and female students. This is represented in the model with an arrow from gender to contextually shared experiences.

Individual life histories influence how students position themselves in the classroom, the way they engage with mathematics, teacher and peers and the way they interpret their experiences in the classroom. Yet, these individual experiences are partly shaped by the shared events in the classroom and by student's gender (and other social variables). These influences are illustrated with arrows from social level and classroom context to individual experiences.

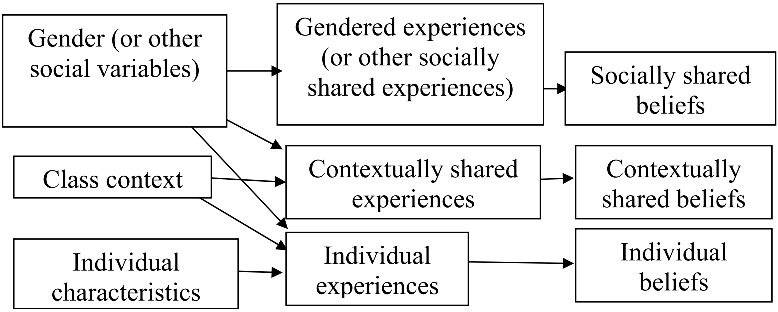

Figure 1. A model for generation of mathematical beliefs.

In this report, the focus is to look at those student beliefs where significant amount of belief variation can be attributed to classroom level, and to explore if the teacher beliefs as characteristics of a classroom context have an effect on those beliefs.

METHODS

The view of mathematics indicator has been developed in 2003 as part of the research project "Elementary teachers' mathematics" financed by the Academy of Finland (project #8201695).More information about the development of the instrument can be found e.g. in (Hannula Kaasila, Laine & Pehkonen, 2006).

The participants in our study came from fifty randomly chosen Finnish-speaking upper secondary schools from overall Finland. The respondents were in their second year course for mathematics in grade 11. Altogether 894 students and their teachers from 39 advanced syllabus classes filled in the questionnaire and gave it back.

Through an exploratory factor analysis we obtained a seven-factor solution that counts for 59 % of variance and provides factors with excellent internal consistency reliability (Table 1). A description of factor analysis as well as all components and their loadings can be found in another report. (Rösken et. al, 2007)

Name of the component	Sample item	Number of items	Cronbach's alpha
Competence	Math is hard for me	5	0.91
Effort	I am hard-working by nature	6	0.83
Teacher Quality	I would have needed a better teacher	8	0.81
Family	My family has encouraged me to	3	0.80

Encouragement	study mathematics		
Enjoyment of Mathematics	Doing exercises has been pleasant	7	0.91
Difficulty of Mathematics	Mathematics is difficult	3	0.82
Confidence	I can get good grades in math	5	0.87

Table 1. The 7 principal components of students' view of mathematics.

A GLM univariate analysis was performed on SPSS. The seven belief factors were the dependent variables, gender was a fixed factor, and class a random factor. Mathematics grade was a covariant. Because all variables did not confirm with the assumptions of normality, we made also a nonparametric Kruskal Wallis test to test the statistical significance of the grouping effect.

In order to measure the possible sources of the group effect, the mean value for each advanced syllabus group was calculated for each of the belief factors. In the next stage of the analysis, the effect of teacher beliefs to this average belief of the group was measured.

The teacher beliefs were measured using 10 likert items (Table 3). Teacher's were also asked to indicate their goals for mathematics teaching between three alternatives (rules and routines, concepts and their relations and problem solving and applications). However, this latter measure did not explain any of the observed group effects in advanced syllabus students' mathematical beliefs and, consequently, it was not used in further analysis.

While a simple correlational analysis between teacher and student beliefs did not indicate any clear explanations for group differences, the teachers were grouped into clusters. The teacher belief responses for a clustering of teachers (K-means cluster analysis). The received clusters were then subjected to an ANOVA analysis and an independent samples t-test to indicate differences in the students' beliefs.

RESULTS

The GLM univariate analysis indicated several statistically significant effects (Table 2). However, because the assumption of equal variance did not hold true in all cases we made separate nonparametric (Kruskall-Wallis) analysis that confirmed the group effects presented (see Hannula, forthcoming, for details).

The analysis indicated a strong group effect for teacher quality and enjoyment. Moreover, there was a gender and group interaction effect for enjoyment, indicating stronger group effect for female students. These results confirmed the model presented in Figure 1.

	Advanced mathematics

	Grade			Gender			Group			Gender x Group		
Student belief	F	Sig.	η^2	F	Sig.	η^2	F	Sig.	η^2	F	Sig.	η^2
Competence*	332,61	,000	,30	1,09	,301	,02	1,63	,077	,63	1,08	,355	,05
Effort*	254,72	,000	,25	,13	,717	,00	1,02	,479	,51	1,13	,278	,05
Teacher Quality*	53,34	,000	,07	5,83	,019	,10	7,26	,000	,88	1,14	,274	,05
Family Encouragement	1,20	,274	,00	,34	,561	,01	1,50	,116	,61	,73	,877	,03
Enjoyment of Mathematics	175,78	,000	,18	,30	,591	,01	2,41	,005	,71	1,49	,036	,06
Difficulty of Mathematics	254,08	,000	,24	34,27	,000	,40	1,67	,066	,63	1,24	,160	,05
Confidence	115,86	,000	,13	75,07	,000	,60	1,29	,228	,57	1,28	,132	,05

Table 2. GLM univariate analysis for advanced mathematics students (gender*group, grade as covariate). η^2 is partial η^2. *) variance in groups was not equal (Levene's Test of Equality of Error Variance)

In the analysis of teachers, the two-cluster solution provided poor explanatory power, and the number of cases in the four-cluster solution became too low for succesful analysis. The three-cluster analysis (Table 3) provideed an interesting solution, where the cluster one (12 teachers) represents the most progressive teaching approach and cluster two (6 teachers) the most old-fashioned teaching approach. The cluster three (17 teachers) teachers are in the middle in their teaching approach, and their belief in the innate talent ("some have head for mathematics ..") is the strongest.

	Cluster		
Teacher belief item	1	2	3
It is worthwhile to begin mathematical problem solving only after students master the necessary basic skills.	2	4	3
Students learn new mathematical content best through solving a related problem.	4	3	4
Boys figure out novel solutions to tasks more often than girls.	2	2	2
Girls are usually more concentious than boys in their mathematics studying.	3	3	3
All students can understand even the most difficult topics taught.	3	1	2
I oten try out new methods for mathematics teaching.	3	2	3
Some have head for mathematics while some have head for languages.	1	3	4

I teach the way I have been taught myself.	2	2	2
I often use games and play in my lessions.	2	1	2
I often teach according to teacher manual's advice.	1	1	2

Table 3. The three clusters' responses in the teacher belief questionnaire (Likert scale from 1 = disagree to 5 = agree)

Statistically significant differences between the clusters were observed in competence (sig. = .040), enjoyment (sig. = .027), and confidence (sig. = .019). In all cases the beliefs in cluster one were statistically significantly more positive than beliefs in the cluster three. Also the statistically non-significant results in other variables followed the similar trend. Morover, beliefs in cluster one were throughout more positive than beliefs in cluster two, even if the diference did not reach a statistical significance. This consistant trend suggested combining clusters two and three together. An independent samples t-test between cluster one (progressive teachers) and combined clusters (traditional teachers), produced even more clear results (Table 4), that confirmed more positive beliefs in classes taught by more progressive teachers.

CONCLUSIONS

The results of the multilevel analysis provide supporting evidence for multilevel origin of student beliefs suggested in the model (Figure 1). Most clearly the classroom context had an efect on perceived teacher quality and enjoyment.

	Mean of student beliefs		t-test for Equality of Means
Student belief factor	Progressive teachers (N = 12)	Traditional teachers (N = 21)	Sig. (2-tailed; Equal variances)
Competence	3,35	3,25	,025
Effort	3,40	3,35	,309
Teacher Quality	3,51	3,21	,023
Family Encouragement	2,97	2,85	,263
Enjoyment of Mathematics	3,32	3,01	,012
Difficulty of Mathematics	3,43	3,58	,074
Confidence	3,58	3,42	,005

Table 4. The belief averages for progressive and traditional teachers' students.

Among teachers it was possible to find different degrees of agreement with progressive ideas regarding problem solving, teaching methods and the learning potential of every student. The student belief averages in the more progressive teachers classes were more positive, especially regarding enjoyment and perceived teacher quality. Although the difference may seem small, these are results in class averages and results are very consistent. This is indicating that through choices in instruction, it is possible to create a 'culture' in the classroom that is more enjoyable.

Outside the research results presented here, it is noteworthy that although support for the multilevel model (Figure 1) and a relationship between teacher and student beliefs was found also students in the general syllabus courses, the relating variables were different there.

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RELATIONSHIPS BETWEEN SENSORY ACTIVITY, CULTURAL ARTEFACTS AND MATHEMATICAL COGNITION

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Solange Hassan Ahmad Ali Fernandes — Colégio Nossa Senhora do Rosário

To explore the construction of mathematical meanings in learning settings, we focus on the co-ordinations of speech, gestures, material objects and sensory activities in a dialogue between a mathematics teacher (researcher) and a blind student. We argue that as the student came to know the mathematical object in question (a pyramid), the teacher too engaged in a process of re-conceiving the object through the hands (eyes) of the student. We argue that in the search for a more inclusive mathematics education, we need to pay more attention to ways that students who lack access to one or other sensory field, both to create more accessible learning situations and to extend our understanding of the interplays between perception and conception.

INTRODUCTION AND BACKGROUND

It can hardly be controversial to claim that we develop and that we learn by interacting within the various biological, social and cultural systems that make up the world as we experience it. Individuals construct their own meanings of the mathematics they encounter which depend upon the ways and means through which they come in to contact with the knowledge culturally labelled as mathematics, as well as upon their individual resources – physical, visual, auditory and cognitive. And yet the precise nature of the relations between perception, cognition and culture has long been an academic battleground. We might go back to Plato who distinguished between that perceived through the senses and that perceived by the soul, judging the first untrustworthy and deceptive and the second the intellectual route to the universal forms representing the true essence of reality. A perspective countered by Aristotle, who argued that our knowledge initiates from that which comes to us first in the form of sensations from our sense organs. As we fast forward to the Enlightenment, the debate continues, with the Empiricists, defending the idea that all knowledge is a consequence of experience and the Rationalists insisting that the rules of reason are the means through which knowledge comes to the mind and soul. Forward again to the 1960s and 70s and the emergence of Mathematics Education as an academic discipline, it was Piaget's constructivist perspective, which emphasises the learner as a rational being whose activities are guided by her (mental) logical structures, which dominated research efforts. Finally, as we move into current times, the focus has moved from the individual learner to learners interacting in cultural settings. While in the past, the controversy concentrated on the relationships between experience and intellect of the individual, today cultural and semiotic considerations vie for centre stage, and yet as embodied cognition too enters onto the

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 137-144. Thessaloniki, Greece: PME.

scene, the relationship between sensory activity and knowledge remains a relatively open problem.

It is perhaps surprising given this background that so little attention within the mathematics education literature has focussed on the learning of those who lack access to one or other sensory field. In part, our work with blind mathematics learners is a response to this absence and motivated by our belief that if we are able to identify the differences and similarities in the mathematical practices of those whose knowledge of the world is mediated through different channels, then we may build a more robust understanding of the relationships between experience and cognition more generally. But this is not the only motivation. Like many other countries across the world, current Brazilian education policies legislate for "inclusive schools", aiming towards a system in which learners with particular educational needs are no longer educated in special schools, but attend regular classes. In Brazil, what this means in practice is that mathematics teachers face the challenge of teaching in classes in which, amongst the around 40 students, there might be a small number of students who are blind or deaf or have some other physical or cognitive impairment. Unlike some other countries, in Brazil, most blind students attending the regular public school system will not be accompanied in their mathematics classes by individual support workers nor will the mathematics teacher have extra classroom assistance (although they should have access to a special needs teacher responsible for translating material to and from Braille). In most cases, the mathematics teacher will also have little or no access to materials adapted for blind learners and, almost certainly, have received no training for working with students with special needs (for more details, see, Fernandes and Healy, 2007). Given this scenario, there is also a pressing pragmatic motivation for our research: if we are to create truly inclusive mathematics classrooms, it is critical that we seek to better understand the particularities of learning mathematics of all our students.

MEDIATION AND MULTIMODALITY IN MATHEMATICS LEARNING

In seeking theoretical grounding for our work, we have been influenced by the debate on the mediating role of material and semiotic tools and by approaches which recognise the situated and embodied nature of cognition. In terms of the first, which posits a reciprocal relationship between tools and thinking, we note that the Vygotskian construct of mediation (Vygotsky, 1997) has its roots in his work with differently-abled individuals. He argued that "the eye, like the ear, is an instrument that can be substituted by another" (p.83), and that, just as the inclusion of any other tool in the process of activity alters its entire structure and flow, so too the substitution of the eye by another instrument will cause a profound restructuration of the intellect. Indeed, we already know that there are some differences in the ways that blind learners process data when compared with sighted students (Ochaita & Rosa, 1995). When exploring a physical object or a raised representation on paper, it is the hands of the blind learner which act as substitutes for the eyes. Like the eyes of the sighted, the hands are moved in an intentional manner, catching particularities of the form in order to perceive – and at the same time conceive – the object, although in a slower and successive form. Whereas vision is synthetic and global, touch permits a gradual analysis, from parts to the whole. One question, then, to be addressed in this paper is how this form of exploration mediates mathematics learning, highlighting (or not) particular mathematical relationships and properties or particular ways of thinking about mathematical objects.

To some extent, the second area of influence emerges as a consequence of the first: if we accept that tools are part and parcel of cognitive activities and that the organs of the body can be thought of as mediating tools, then it follows that thinking cannot be separated from the activity in which it occurs nor can it be considered an entirely cerebral activity. In this sense, our approach concurs with that of Radford (in press), who argues that thinking "cannot be reduced to that of impalpable ideas; it is instead made up of speech, gestures, and our actual actions with cultural artifacts (signs, objects, etc.)… thinking does not occur solely *in* the head but also *in* and *through* a sophisticated semiotic coordination of speech, body, gestures, symbols and tools". Indeed, Healy and Fernandes (2008) reported how a blind student coordinated his interactions with his teacher, other blind students, unit cubes and physical representations of geometrical figures to create a sign (in the form of a repeatedly used gesture) to represent his thinking about area and volume, in a multimodal form which linked his perceptual activities with the cultural conceptions of these mathematical objects. Radford (2002) terms this process of coordination, which involves becoming increasingly aware of a cultural object, *objectification*, a construct somewhat reminiscent of Mason's (1989) view of abstraction as a "delicate shift of attention" (p.2) in which learners move between seeing and using expressions as processes or objects.

The aim of instructional situations, within this perspective, becomes the objectification of cultural knowledge, with the means being the signs and materials already impregnated as it were (at least for the teacher) with a cultural history of the objects in question. For Radford, the role of the teacher becomes that of mediating a kind of alignment between the subjective meanings attributed to the objects by the learners and their cultural meanings, by involving their students in an active re-interpretation of the signs in play. From our point of view, however, the role of the (sighted) teacher of blind students is a little more complex: they must also be open themselves to re-align their own perspectives on the mathematical objects according to the alternative perceptions of learners who process data in a manner different to that they are accustomed to. The second question we examine in this paper, then, is how the process of coordinating the physical and semiotic resources within such an instructional setting contributed to the construction of meaning for a three dimensional geometrical object, the pyramid. We concentrate particularly on the central role of one blind student's gestures in this process.

THE RESEARCH CONTEXT

This research was carried out as part of a research project investigating the processes by which blind learners appropriate mathematical knowledge[1]. The project took place in a school from the public school system of the state of São Paulo in Brazil, with a long history of including learners with visual impairments, and counted on the collaboration of group of six teachers from the school, along with a total of twelve blind or partially sighted students who participated in the various empirical activities. Here, we report on an activity in which a student we shall call André (an 18 year-old, second year high school student) attempted to construct his own representation of a pyramid with a square base. It is important to stress that before participating in the project, the blind students had had very little contact with geometrical content in general and André indicated that he had had no prior experience in constructing his own representations of geometrical figures. Before working individually on the pyramid task, he had, however, participated in four other research sessions in which he worked with 5 other blind students to construct representations of plane figures (each of 50 minutes) and a session during which he constructed individually a three dimensional model of a cube (50 minutes). The pyramid task lasted for 40 minutes. All the sessions were video recorded. The way in which the tasks were proposed and the organisation of the students aimed to stimulate dialogues between the participants, including researcher(s) who assumed the role of teacher.

For this article, we have selected, transcribed and coded episodes from a research session in which André, working with one of the researchers, made use of the various resources of the learning scenario.To illuminate the role of gestures in the coordination of resources and examine how they became tools for creating and communicating mathematical meanings, we applied the classification of the four gesture types proposed by McNeill (1992). In this classification, *Iconic gestures* (♦) have a direct relation with the semantic discourse, that is, there exists an isomorphism between the gesture and the entity it expresses, although comprehension of the iconic gesture is subordinated to the discourse which accompanies it. *Metaphoric gestures* (●) indicate a pictorial representation of an abstract idea which cannot be represented physically, for example, when we illustrate with hand movements the limit of a function f(x) when x tends to zero. *Deitic gestures* (☚) have the function of indicating objects (real or virtual) people and positions in space. *Beat Gestures* (♪) are short and rapid and accompany the rhythm of the discourse giving special meaning to a word, not due to the object that it represents but to its role in the discourse.

Representing a pyramid in three dimensional space

To construct his pyramid, André had access to a set of wooden matches intended as possible representations of the edges of the figure and plasticine (modelling clay) for

[1] We are grateful for the research grant from FAPESP (No. 2004/15109-9) which supported this project.

its vertices. Also available were a number of wooden geometrical figures, including the solid square-based pyramid. In his first attempt, André selected just three matches of the same size. These he joined using the plasticine to construct the triangular form presented in Figure 1. Possibly, he had misinterpreted the task as that of producing a two dimensional representation of the three dimensional figure – we had previously noted a tendency amongst most of the visually impaired students to offer representations of just one face of three dimensional figures when asked to produce planar representation (one of the reasons we had resolved to create this new task). To help André perceive the difference between the shape he had constructed and the solid pyramid he had been exploring, the researcher gave him a wooden representation of an equilateral triangle, asking whether a third person would be able to tell which of the two figures his model represented.

Figure 1 – André's first model

He realised that his representation was closer to the triangle than the pyramid and that he needed to attend to other elements of the original solid:

André: *I thought, like, only of this part here* (☛) (he indicates one of the triangle faces)*, its base is missing, this part underneath* (☛) (moves his hand over the pyramid's square base).

He went on to add a square to one side of the triangle, then moved his hands between the solid pyramid and the new form (Figure 2) for some time. Although not satisfied with his results, he didn't know how to proceed and the researcher intervened.

Figure 2 – Adding the base

André: *I still don't think they are equal*

Researcher: *So what can you do to make it better?*

André: *...I have no idea at all.*

Researcher: *Look, you made this base* (☛) (she puts the square base on to André's right hand), *but the other part* (☛) (places the forefinger of André's left hand on the vertice at the point of the pyramid) *is not in the same plane.*

It seemed that the notion of plane was unfamiliar to André and he continued unsure as to how next to proceed. The researcher attempted to draw his attention once more to the relative positions of the apex of the two forms. Placing André's hand on the plasticine representing the vertex on the triangle not fixed to the square base, she asked him to indicate the equivalent vertex of the wooden solid.

Figure 3 – André's iconic gesture

André did not respond with a deitic gesture, but an iconic one, running two fingers simultaneously along the two edges of one of the triangular faces from the base to the apex, as he affirmed "it's here" (Figure 3).

His gesture indicated that he was seeing the pyramid in a rather different way to that of the researcher. His view was dynamic: the apex was at the end of a process in which André's fingers became successively closer until they met.

More precisely, we now know that what André was in fact feeling was a series of rapidly decreasing squares – for us, a clear indication of how action and thought cannot be separated. Since André was conceiving the pyramid as a collection of layers of decreasing squares, his task was to some how position the top layer (the vertex) without having the layers in between. He made two attempts, first rotating the triangle until it was perpendicular to the pyramid base (Figure 4) in an attempt to construct the pyramid's height, and then, realising the apex was not over the base and unconnected to the other vertices of the base, he pushed the triangle so that it was on top of the square, with both faces once again in the same plane (Figure 5).

Figure 4 – Giving the pyramid height

Figure 5 – Placing the vertex over the base

Still unsatisfied with the result, he looked to the researcher for help. In the resulting intervention, the researcher supplied a potted history of his activities until that point, still stressing her point of view centred on the faces and edges rather than André's. Yet as she emphasised specifically the elements he had yet to represent, her intervention enabled André to complete the pyramid to his satisfaction (Figure 6).

> **Researcher:** *You found the base and you saw that we have to leave the same plane to make the point* (apex). *We moved out of the plane. Now you have put it in the same plane again. And thinking again, you have made this side* (☛) (moves André's forefinger over a triangle face). *You made this and this* (☛) (drags his finger over the edges of the solid that have corresponding matches in his model). *But there is another here... another here* (☛) (passes André's finger over the not yet represented edges).
>
> **André:** *So I have to put this back up to here* (raises the triangle face once more), *and I have to put a match here and another here* (♦) (he draws in the edge the missing edges, before adding the appropriate matches to complete the figure shown in Figure 6).

With the completed figure in front of him, the researcher asked a last question, attempting to elicit from André a move from action to articulation:

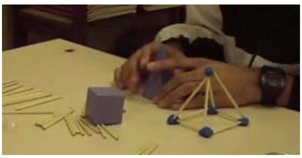

Figure 6 – The final representation

Researcher: *So how would you describe this to colleague, in a letter, who was also working with figures of this form*?

André: *I would tell them that the base is square* (☛) *and that as they move up the sides they become smaller until forming a point on top* (♦) (moves his finger simultaneously up the edges of a triangle face).

The researcher was surprised with André's description, in which he finally makes explicit his conception of the pyramid as a pile of decreasing squares. While her previous intervention had successfully drawn André's attention to her way of decomposing the pyramid into faces and edges, it is his previous conception, so difficult for him to model given the available tools, which still characterises his objectification of the geometrical solid. We could say that as the dialogue was played out there was a kind of dual process of objectification – with each of the participants emphasising different aspects of the cultural history of the pyramid, both of which are mathematically valid. The researcher, as teacher, was primarily thinking in terms of the pyramid as it appears in didactical texts for Brazilian school students and where it is important that students come to know facts such as the number of faces and vertices of different geometrical solids. André's conceptions connect to another aspect of the object's history, one that we might trace back to the Ancient Greek mathematician, Eudoxus of Cnidus, who made use of the same idea in order to prove the formula for calculating the volume of a pyramid (Huxley, 1980).

SOME CONCLUDING REMARKS

These two awarenesses of the pyramid emerged, we argue, as the two participants, attempted to coordinate the multimodal forms that emerged during the dialogue. André's gestures hold the key to understanding the particular mathematical properties that became salient to him as his hands were employed as tools for seeing, and his gradual and dynamic exploration of the solid form led him to notice rather different aspects of the pyramid than the synthetic visual feedback received when the eyes are the seeing tools in question. Since it was difficult for him articulate his view with the available modelling tools, it was necessary for him to try to make sense of the researcher's point of view and once again gestures were critical for this activity, since it was the gestures acted out by her that drew his attention to the edges that were missing from his model. It is interesting to note that these edges would in fact have been constructed as a consequence of the decreasing squares' sides had it been possible for André to model more directly how he was seeing the solid.

But it was not only André who had to make changes in his thinking about the pyramid. The researcher too was confronted by a reading of the object that she had not expected as André's description drew her attention to a property of the pyramid that she did not have in mind. Like André, she too became involved in active re-interpretations of the different semiotic signs expressed in a variety of modes, as a kind of reciprocal process of re-aligning personal, other and cultural meanings was played out.

Before we end, it is important to stress how very challenging the task of building a representation of a geometrical solid was, not only for André, but for the other blind students with whom we worked. And although our work is still at an exploratory stage – we have worked with only a small number of students and there are currently very few other studies of blind mathematics from which we can draw – the evidence that we have collected does suggest that André's decomposition of a three dimensional geometrical figures is a common strategy. From the point of view of our pragmatic considerations, we can begin to draw some lessons from our analyses. First, it is clear that as teachers (and researchers) our thinking about the tools we should provide is strongly mediated by our experiences as and with sighted learners. We did not think, for example, that the task might have been more accessible to the blind if it had been suggested that they use modelling clay alone in a first attempt to build their own model of the solid. Or indeed had we considered confection of a multitude of different-sized geometrical figures which might be used as layers. And here we identify a real difficulty in terms of any realisation of a truly inclusive mathematics classroom: unless we develop more robust understandings of alternative (though valid) ways of expressing mathematics, then blind learners, or any other learners who do conform to the supposed educational norm, will be expected to assimilate to existing practices and we will lose the opportunity to exploit the new ways of doing and representing mathematics that they can teach us about.

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THE DEVELOPMENT OF MATHEMATICAL COMPETENCE OF MIGRANT CHILDREN IN GERMAN PRIMARY SCHOOLS

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After 30 years of work related migration in Germany, migrant children are still underrepresented in schools of higher education. Differences between migrant and non-migrant children's achievement already exist at the end of primary school. In this contribution we present results of the first three years of the longitudinal study SOKKE with N = 292 students which focuses on the development of migrant and non-migrant students' competences during primary school. The findings of our quantitative study show that already at the end of grade 1 mathematical competence of students with a migrant background differ significantly from mathematical achievement of their non-migrant classmates. These disparities are preserved in grade 2 and 3 and can be explained mainly by proficiency in the language of instruction.

INTRODUCTION

About 19% of the inhabitants in Germany (15.6 million) and about 31% of primary school children have a migrant background (micro census 2005). Here, the term "migrant" or "migrant background" is not restricted to persons with a foreign citizenship but encompasses also Germans whose father or mother immigrated to Germany or persons of the so called "third generation" which mainly speak a foreign language at home.

National reports on the situation of the German educational system have repeatedly shown great disparities between migrant and non-migrant students in Germany. Students from migrant families typically attend secondary school tracks that prepare for vocational training, and they are underrepresented at schools aiming for an academic career (e.g., Herwartz-Emden, 2003). Furthermore, empirical educational research has detected disparities to the disadvantage of migrant children already at primary school level for the subjects reading, writing and mathematics (e.g. Bos et al. 2007, Tiedemann & Billmann-Mahecha 2004). Hence, the transition from primary to secondary school constitutes a central barrier for students from migrant families to be educated at more demanding secondary school tracks.

In the last decades the mathematical competence of primary school children with a migrant background in comparison to their classmates with a non-migrant background was investigated in many empirical studies. Summarizing the findings, it seems that children with a migrant background achieve a significant lower

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mathematical performance than students without a migrant background. The reasons for these findings are diverse, starting from individual language problems (see below), the role of the parents in supporting educational progress of their children (e.g. Civil, 2008) up to institutional reasons. The fact that the educational system is one determinant of the mathematics achievement of migrant children is supported by international comparisons. For example, TIMSS 2007 revealed that there is a huge variety between different countries when analyzing the success of migrant children in mathematics (Mullis et al., 2008).

In Germany several studies with primary school children indicate that already after three or four years of mathematics instruction students with a non-migrant background are one school year in advance in their mathematics performance in comparison to students with a migrant background (effect sizes up to d = 0.76, Bos et al. 2007, Tiedemann & Billmann-Mahecha 2004).

LANGUAGE PROFICIENCY AND MATHEMATICS COMPETENCE DEVELOPMENT

Although in Germany roughly 75% of the children with a migrant background were born in Germany their linguistic competence in the language used in school is noticeably lower than their classmates' without a migrant background (Bos et al., 2007). There are several reasons for this situation: For one thing, many migrant families mainly speak the language of their country of origin at home. Furthermore, the help offered for migrant children during German preschool is rarely sufficient.

Language is seen as the foundation of academic learning which may explain partly the migrant children's lower level of attainment. Obviously linguistic competencies for everyday life situations are necessary for the participation in learning processes but they are not sufficient. Already Cummins (1979) distinguished between the linguistic competence levels BICS (basic interpersonal communication skills) and CALP (cognitive academic language proficiency). Although BICS allows students to participate in lessons, CALP is necessary for the context reduced communication of abstract concepts and cognitive operations. It is also used in textbooks and by teachers, and it poses a significant challenge for migrant students.

These theoretical observations are supported by empirical results. For example, data of the German extension of PISA 2000 have shown the important influence of linguistic competence for academic attainment (Baumert & Schómer, 2001). Although studies repeatedly show that there is a significant correlation between linguistic competence and mathematical competence, we do not have much information how restricted competence in the language of instruction influences the development of mathematical competence – in particular in primary school.

The development of mathematical competence depends highly on academic learning processes. The longitudinal study SCHOLASTIK with N = 1050 children showed that deficits in the learning processes in grade 1 and 2 can be partly compensated by general cognitive abilities and preschool knowledge but their influence decreases in the second half of primary school (Stern, 1997). At the same time the role of comprehension and usage of mathematical semantics becomes increasingly important for the solution of typical mathematical problems. From this stage, one can expect that academic learning processes in the subject of mathematics are significantly affected by linguistic competence. Moreover, in comparison with children without migration background migrant children have to cope with two demands at the same time: On the one hand they have to acquire the second language up to a level that allows the participation in maths lessons. On the other hand they have to learn the specific meanings of mathematical terms, which might as well appear in a different meaning in nonmathematical contexts (Gorgorio & Planas, 2001).

Although mathematical learning processes in primary school are highly based on material and enactive and iconic representations (cf. Bruner, 1966) the acquisition of mathematical concepts and operations are mainly mediated by language based explanations. Interacting with other people through language seems to be vital when internalising operations and developing mental representations. If children are not able to use the same language as for informal learning processes out of school, they experience substantial disadvantages. For example, a study in the UK by Phillips and Birrell (1994) indicates that Asian children from families which do not speak English at home show a significant lower competence development in mathematics in the first grade than non-migrant children even though the competence development of both groups in the English language did not differ significantly.

Moreover, also a study of Abedi, Lord and Hofstetter (2001) with Hispanic grade 8 students with limited English proficiency showed that the level of understanding of mathematical concepts and processes is substantially mediated by the linguistic competence. The performance of one group of students in a mathematics test posed in their first language was lower than the performance of a comparison group of Hispanic students to whom the same test were posed in English. Since the language of instruction was English for both groups, the students learned mathematical concepts and operations in English.

Accordingly, linguistic competence in the language of instruction is not, as might be presumed, only relevant for reading and writing but also for mathematics. “Language plays a key role in the learning and teaching of mathematics, particularly in reform-based classrooms.” (Civil, 2008).

RESEARCH QUESTIONS

Although there is substantial empirical evidence for disparities between migrant and non-migrant students when learning mathematics in German (primary) schools, the development of these disparities throughout the primary school years has been barely researched so far. Based on the theoretical context described before, the study in this contribution addresses the following research questions:

- Are there differences in the development of mathematical competence between migrant students and non-migrant students during primary school years?
- What are the effects of linguistic competence on the development of mathematics achievement during primary school years?
- Is it possible to identify aspects of primary school mathematics which are highly influenced by linguistic competences of migrant students?

SAMPLE AND METHOD

Our study is part of the longitudinal study "Socialization and acculturation in areas of life of migrant children – school and family" (German acronym: SOKKE) which focuses on the development of migrant and non-migrant students' competences during primary school (1st – 4th school year). The sample of the part of the study reported here consists of 292 students from 22 classes in a large German city. About 55 % of the sample has a migrant background; the ratio of the numbers of boys and girls is nearly the same for the migrant children and the non-migrant children. The results presented here are based on the first three measurements (grade 1 to grade 3).

To assess students' mathematical competence, German mathematics tests (DEMAT 1-3) were administered at the end of each school year. The German DEMAT-tests are curriculum valid standardized tests (e.g. Krajewski, Kóspert & Schneider, 2002). For each grade there exist statistical parameters based on data of a norm sample (mean values, percentile rank etc.). Unfortunately, the tests are not linked to each other by anchor items and the norm parameters for each grade are based on different samples, so that a direct comparison of the test scores is not possible. Data for linguistic competence was collected at the end of the first school year by a standardized specific language test (SFD). We used parts of this test to assess listening comprehension[1], lexis[2] and the use of prepositions and articles. As an indicator for general cognitive abilities we used the German version of Culture Fair Intelligence Test 1 (CFT 1). The tests were administrated and evaluated by trained assistants according to the prescribed procedure for each test.

RESULTS

The comparison of the scores of the mathematics tests shows that already at the end of grade 1 mathematical competence of students with a migrant background differs significantly from mathematical achievement of their non-migrant classmates. These disparities are preserved in grade 2 and 3 (see Table 1). However, the differences disappear for all three measurements when linguistic competence in school language is controlled as a covariate in an ANCOVA. Accordingly, as expected, linguistic

[1] First, the students listen to a story (read out loud or played from media) and afterwards they have to answer several questions by multiple choice.

[2] For a given expression students have to choose a similar or synonymous expression from four alternatives.

competence in the language of instruction has a great influence on success in mathematics learning.

Score (%) (SD)	Non-migrants	Migrants	*t*-test	ANCOVA (controlling linguistic competence)
Grade 1	71.65* (19.31)	65.71* (18.86)	$t(290) = 2.56$ $p < 0.05$ $d = 0.30$	$F(1) = 0.14$ $p = 0.71$
Grade 2	57.2* (25.53)	50.12* (24.82)	$t(290) = 2.42$ $p < 0.05$ $d = 0.28$	$F(1) = 0.93$ $p = 0.34$
Grade 3	66.58** (17.39)	60.9** (15.7)	$t(290) = 2.93$ $p < 0.001$ $d = 0.34$	$F(1) = 0.26$ $p = 0.61$

Table 1: Mathematical competence of migrant and non-migrant students.

A detailed analysis of the subscales of the mathematics test revealed that controlling the general cognitive abilities (CFT 1) migrant and non-migrant students do not differ in their performance for symbolic represented items demanding the basic arithmetic operations and procedures (e.g. items like 17 - 3 - 4 - 6 = in grade 1, 24 : 8 = in grade 2 and 763 – 356 in grade 3 by the standard algorithm).

Systematic significant differences in the performance of migrant and non-migrant children were mainly found for specific mathematics subscales asking for the understanding of mental representations of mathematical concepts (e.g., assigning numbers to the number line, comparing lengths given in different units, using representations for arithmetic operations in simple modeling items).

Interestingly, these significant differences disappeared when controlling linguistic competences as a covariate. We assume that especially the generation of individual mental representations of mathematical concepts depends on language based interaction processes in the classroom, so that students with lower linguistic competence experience disadvantages in this field.

To investigate the development of mathematical competence from grade 1 to 3 we conducted a *repeated measures ANOVA* based on percentile ranks assessed within the norm sample of the DEMAT tests. We found a significant main effect of measuring time ($F(2) = 32.44$, $p < 0.001$) and a significant between-subject effect of family background ($F(1) = 8.9$, $p < 0.01$). Interestingly, there is no interaction effect between development in mathematics achievement and migration background ($F(2) = 0.13$, $p = 0.88$). This means that although there are great differences between children with and without migration background, mathematics competence of both groups develop parallel over the three-year period (cf. Figure 1).

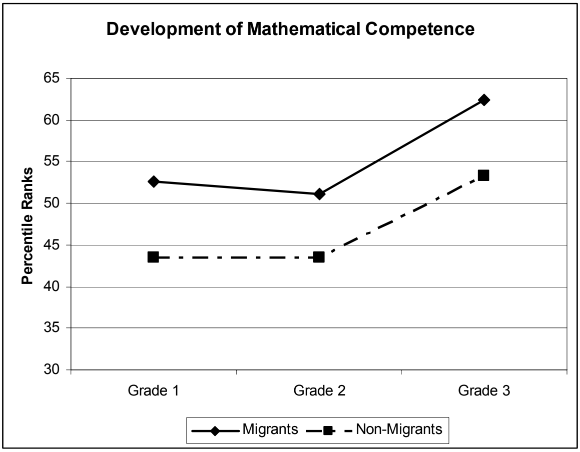

Figure 1: Development of mathematical competence from grade 1 to 3.

Based on the language skills, we chose four percentile groups for each subsample (migrant and non-migrant students) to analyze the development of mathematical competence depending on students' skills in the language of instruction. The result shows that there is no interaction effect between percentile group and development in mathematics achievement neither within the non-migrant nor within the migrant group (non-migrant students: $F(3) = 1.83$, $p = 0.15$; migrant students: $F(3) = 0.05$, $p = 0.97$). This indicates that the groups do not differ in their mathematical competence development over the three-year period although there are great discrepancies in linguistic competence.

To analyze if characteristics of the classrooms - in particular the proportion of migrant children in the class - influence the mathematics achievement of the children, we conducted multilevel analyses. The results show that in grade 1 and 2 only 6-8% of the variance of the individual mathematics achievement is explained by the class level[3]. Although the proportion of migrant children in the classes varies between 5% and 95% this class characteristic does not significantly influence the individual mathematics achievement. Moreover, the influence of the class level is in general very low.

DISCUSSION

The study presented here is part of the larger project SOKKE, investigating the competence development of migrant and non-migrant students in Germany in the

[3] Here we use the extended samples of grade 1 and grade 2. These samples encompass children which did not participate in all measurements in the study. The nested structure of the longitudinal sample does not satisfy the conditions for a multilevel analysis.

four years of primary school. In the last decade, several studies revealed that migrant students in Germany experience large disadvantages during their school career. In this contribution, we focus on the case of mathematical competence development, particularly in relation to the linguistic competence of the investigated sample.

Our study shows that children with and without migration background differ significantly with respect to their mathematical competence at the end of grades 1, 2 and 3. However, these differences disappear when their linguistic competence for the language of instruction is controlled statistically. This indicates that migrant students will achieve the same test performance if they have the same linguistic competence as non-migrant children.

As presented in Figure 1 the mathematical competence develops parallel in both subsamples despite of great discrepancies in linguistic competence in the language of instruction. Hence, the performance gap between both groups becomes neither larger nor smaller. However, this result has to be carefully interpreted since the analysis may be affected by the problem of sample mortality. As in each longitudinal study we lost a part of the sample from the first to the last measurement. This means that the result in Figure 1 only considers the students who showed a sufficient performance in school to reach grade 3 (in Germany students should repeat a grade if their performance is insufficient). We cannot rule out that only low achieving students dropped out of our sample and we cannot prove the opposite. (All other results in this contribution were confirmed with the larger grade specific sample size.)

A careful consideration of the results of the subscales of the mathematics tests revealed that migrant and non-migrant students do not differ in their performance for several types of symbolic presented arithmetic tasks (if the general cognitive abilities are controlled statistically). Especially, for the items concerning the important basic arithmetic operations and procedures, no significant differences were observed despite the differences in language performance.

Based on a detailed analysis of the items which are responsible for the language-dependent differences in the mathematics competence, we found that migrant students' difficulties are mainly caused by specific items which ask for mental representations of mathematical concepts and operations. Since in mathematics classroom mental representations are mainly mediated by language-based interaction, these findings can be interpreted as additional evidence for the essential influence of linguistic competence for the learning of mathematics in school. We consider this result as a starting point for further research and as a basis for intervention studies to foster migrant students' acquisition of conceptual knowledge.

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(RE)CONCEPTUALIZING AND SHARING AUTHORITY

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In this paper, we argue the necessity of working with mathematics teachers in re-conceptualizing teacher- and textbook- authority in mathematics classrooms. We rely on mathematics education literature and on work with a group of secondary mathematics teachers both to contend that focus on authority is needed and to illustrate the interesting contributions that secondary teachers make that would be absent without the close relationship to classroom practices that they bring.

INTRODUCTION

Questions about authority are central in mathematics and mathematics education because of the discipline's characteristic interest in truth and proof. How are truth and value established in mathematics? Who should decide what mathematical questions are worth pursuing? On what basis should these decisions be made? Though mathematics is a powerful discipline with strong traditions and expectations, including those that relate to authority, students in mathematics classrooms only experience the discipline through their teachers and other mostly textual media.

Most scholarship that investigates such issues of authority in mathematics classrooms draws on qualitative research methodologies. In a recent computer-aided *quantitative* investigation of a large body of transcripts from secondary mathematics classes, however, Herbel-Eisenmann, Wagner and Cortes (2008) also corroborated the prominence of authority in the discourse. The pervasiveness of authority issues in the discourse may seem to suggest that classrooms focus on the kinds of questions listed above, but this study of the discourse showed that authority structures were commonly contingent on social positioning which was encoded in mundane phrases in the classroom discourse.

We contend here that further research on authority in mathematics classrooms needs to be done in conversation with teachers to consider ways of developing teachers' repertoires for handling authority issues at play in their mathematics classrooms. It is time to move beyond description of socio-cultural factors related to authority (though we do not mean to minimise the valuable insights of this work), and to get past simplistic rhetoric that suggests teachers either eschew or establish authority as much as possible. Teachers are well positioned to collaborate in this kind of research because they are situated in diverse contexts, each with its own complexities, and because they alone have the authority to change positioning practices in classrooms.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 153-160. Thessaloniki, Greece: PME.

After outlining the relevant literature, we will introduce some artefacts from conversations with teachers to show how they relate to this literature and how the literature has room to develop to address mathematics teachers' needs.

LITERATURE ON AUTHORITY IN MATHEMATICS CLASSROOMS

Authority is one of many resources teachers employ for control and has been defined in an educational context as "a social relationship in which some people are granted the legitimacy to lead and others agree to follow" (Pace & Hemmings, 2007, p. 6). This relationship is highly negotiable. Students rely on a web of authority relations with friends and family members as well as with the teacher (Amit & Fried, 2005).

Teacher Authority

Educational research related to teacher authority often makes distinctions between different types of authority (e.g. Amit and Fried, 2005; Pace and Hemmings, 2007). Most relevant considering the authority questions raised above, are the distinctions made between being an authority because of one's content knowledge and being an authority because of one's position (e.g., Skemp, 1979) – teachers are "an authority [of content] in authority [by virtue of position]" (Russell, 1983, p.30). Many scholars argue that the former is more relevant to teachers because it emphasizes their ability to reach their educational goals. Although these distinctions are made for analytic purposes, Pace (2003) has shown that the types of authority become blended as participants interact in classrooms. This blending is also demonstrated in Herbel-Eisenmann, Wagner and Cortes's (2008) corpus analysis.

Skemp (1979) noted that when authority is gained by position, authority is imposed: the teacher commands, students obey, and instructions are perceived as orders. In contrast, authority by knowledge involves being more like a "mentor." The authority is vested by virtue of the person's own knowledge; instruction is sought and is perceived as advice. Rival and conflicting values complicate authority relations because they are socially constructed in the service of a moral order (Pace & Hemmings, 2006). Moral order, in this case, was defined as "shared norms, values, and purposes" (p. 21).

Regardless of what kind of authority seems to be at play, Wilson and Lloyd (2000) contend that teachers need to develop an internal sense of authority, or a sense of agency, rather than rely on external forces in order to develop their own "pedagogical authority." Wilson and Lloyd make a parallel argument for how teachers help students develop their own sense of mathematical authority. That is, the same kind of reliance on internal authority can help students learn mathematics with meaning. As Schoenfeld (1992) pointed out, however, the development of internal authority is rare in students, who have "little idea, much less confidence, that they can serve as arbiters of mathematical correctness, either individually or collectively" (p. 62).

Teachers can unknowingly undermine their intentions to develop this kind of mathematical authority in their students. For example, Forman, McCormick, and

Donato (1998) examined authority patterns in a classroom in which the teacher was working toward the vision described in the National Council of Teachers of Mathematics (NCTM) standards documents in the United States. The authors found evidence that, although the teacher wanted to solicit, explore, and value multiple solution strategies, some of her discourse practices undermined this goal. They argued that the teacher asserted her authority through the use of tacit language patterns like overlapping speech, vocal stress, repetition, and expansion. Despite the fact that three students in her class presented mathematically correct and different solution strategies, the teacher overlapped a student's explanations only when the student was not using the procedure that the teacher recently taught.

Textbook Authority

Up to this point, we have briefly considered authority relationships between teachers and students. In mathematics classrooms, however, another pervasive presence that influences what and how mathematics is taught is the textbook. Most research on authority in classrooms focused on teacher authority and briefly mentioned that the textbook may have played a role in authority relationships in classrooms (Amit & Fried, 2005). None of those authors, however, seriously considered the interactions among the teacher, textbook, and students in their inquiries, perhaps because, as Olson (1989) argued, textbooks "are taken as the authorized version of a society's valued knowledge" (p. 192).

We draw on two related perspectives about the positioning of the textbook as an authority. First, Olson (1989) argued that the separation of the author from the text as well as the particular linguistic characteristics of a textbook helped to instantiate the textbook as an authority. Textbooks, thus, constitute a distinctive linguistic register involving a particular form of language (archival written prose), a particular social situation (schools) and social relations (author-reader) and a particular form of linguistic interaction (p. 241). Second, Baker and Freebody (1989) contend that the authority of the text is the result of pedagogy. Their perspective takes as central actual classroom interactions and the authors empirically investigate how "text-authorizing practices…may be observed in the course of classroom instruction" (p. 264), as well as how these practices evolve in relation to the authority of the teachers. To illustrate these practices, they examined the kinds of questions teachers ask and the ways teachers respond to students' answers to their questions. They sought to "describe the intimate connections between talk around text and the social organization of authority relations between teachers and students. Teachers may be shown to use various practices to assign authority to the text and simultaneously to themselves" (p. 266).

The Role of the Mathematics Teacher

The NCTM carries significant authority in setting the agenda for teacher development, among other things (e.g., many textbooks for prospective teachers advertise with the claim that the text is in line with NCTM standards). The NCTM

(2000) standards documents address authority, but are underdetermined. Like the literature described above, the standards promote the development of students' authority, but they are not explicit about how this is to be done (though one could argue that many of the standards' advice for teachers would be positive supports for the development of student authority): "Most important, teachers need to foster ways of justifying that are within the reach of students, that do not rely on authority, and that gradually incorporate mathematical properties and relationships as the basis for the argument" (NCTM, 2000, p. 126).

Even if we agree that students should develop their own sense of mathematical authority, it is problematic to say that teachers need to cede their authority. From our conversations with teachers, we know that they are reluctant to entertain the idea of giving up authority, partly because of the imagined (or experienced) implications on the teacher's necessary social authority, but also because they know that their mathematical authority is necessary for teaching. Chazan and Ball (1999) confront this tension in their in depth descriptions of two situations in which they, as teachers (and expert mathematics educators), were reluctant to express their authority but realized the necessity of it. Although they suggested that the mathematics education community needs to better understand the complexities associated with the decisions they make as teachers, little has been done to date to try to better understand authority and positioning issues in a "reform" mathematics classroom. As Chazan and Ball pointed out, being told "not to tell" is not enough. Our realization of the need for mathematics educators to better understand how to share and use authority in ways most productive for student learning led us into the research that we draw upon here.

TEACHERS' VIEWS ON AUTHORITY

At the outset of our study engaging middle school and high school mathematics teachers in conversation about authority structures in their classrooms, we interviewed each teacher and asked him/her to describe his/her view of authority in his/her classroom. After asking some questions related to authority (e.g., what or whom their students see as authorities in their classrooms, how their students know something is right in mathematics, how their students know what to do in mathematics, how they as teachers know what is right and what to do in mathematics), we drew for each teacher a thick dot on a blank paper or blackboard and asked the teacher to complete the drawing to show how authority works in their classrooms. We learned that teachers have very different ways of thinking about authority, though their different ways of seeing are interrelated.

In the first of the three diagrams below, Dawn completed by drawing icons and other symbols representing the different sources of authority in her classroom (Figure 1) around the black dot representing her. An *x* represents a student, another black dot represents other mathematics teachers, an open dot a tutor, and other symbols represent textbooks, rulers and calculators. As she introduced each source of authority, she drew arrows to show where one looks for authority. For example, the arrow from a student to Dawn indicates that the student looks to her as an authority. When showing her diagram to other teachers later, Dawn noted other sources of authority as well. Her diagram represents some of the relationships, demonstrating that there are many authorities at play.

Dawn's conception of authority in her typical classroom is reminiscent of Amit and Fried's (2005) web of authority relations as she notices a variety of sources of authority. Dawn, however, draws more attention to inanimate objects as authorities – calculators and textbooks, for example. We note that even inanimate objects, such as textbooks, can be considered within human relationship by drawing attention to author choices. Author-ship is an important part of authority structures. Dawn also draws more attention than Amit and Fried to people related to the academic institutions, namely other teachers and tutors, but left family members out.

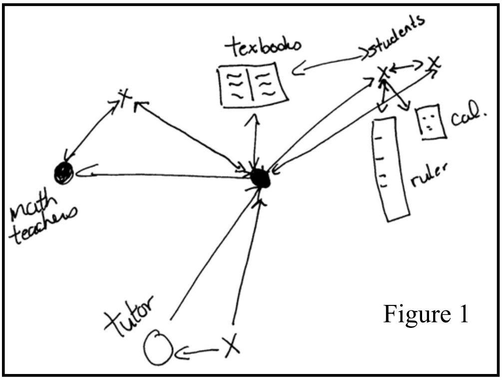

Figure 1

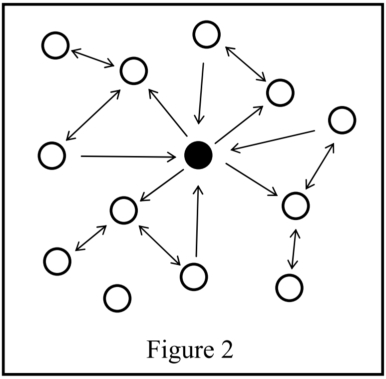
Figure 2

Jill completed the diagram (Figure 2) by drawing empty circles for students around the black dot representing her, and then arrows to show the direction of authority. (Hers was done on a blackboard, so it could not be scanned.) Her arrows are different than Dawn's (in a way they are opposites in terms of direction). Sandi talked about the arrows as showing the direction of the communication of understanding. Her descriptions accompanying the drawing of arrows show them to be more complex than representations of just any communication. For example, an arrow from her black dot to a student's open dot, indicates her showing the student something that she understands or knows, and the student understanding and accepting her knowledge. Not all students understand or accept all they hear, thus only some students receive arrows. Similarly, some students who do not understand or accept her mathematics manage to find understanding in conversation with other students, and are able to show Jill their knowledge in a way that Jill accepts.

Jill's diagram is reminiscent of diagrams in education literature showing paths of communication, though her conceptualization of the arrows is more sophisticated. She shows no external influences. She said in the interview that she tries to focus her attention on the students themselves. She listens to them and interacts with them as an individual herself, not as a representative of something beyond the reach of the students. For example, she models what Schoenfeld (1992) refers to as internal authority as she justifies the ideas she wants to communicate in terms of the experiences and prior knowledge of her students, not by appealing to a book for authorization. This is like Harré and van Langenhove's (1999) positioning theory, which focuses analysis only on immanent presences, nothing external.

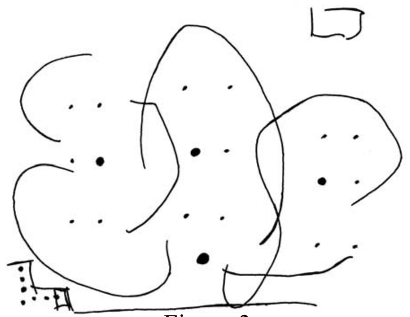

Figure 3

Mark completed his diagram (Figure 3) with a physical representation of the classroom, showing the arrangements of students, who are smaller dots, the blackboard (the straight line), a bookshelf with texts (also authoritative dots) that students can refer to, and his desk at the back of the room. The curvy lines indicate his movement throughout the room. Some students have larger dots because they, like him, are recognized as having more mathematical authority than the others. When drawing, he talked about balance. Authority should be spread throughout the classroom, he said. Thus he arranges seating plans to spread the students regarded as authorities around the room, and he himself moves around to avoid fixing authority in one place. His conceptualization is quite different from anything we have seen in the literature on authority in classrooms, yet we find his elaboration interesting and compelling. It relates to positioning, but unlike most scholarship on positioning that use physical relationships as metaphors for interpersonal relationships, his conceptualization recognizes the effect of physical positioning. We think that physical arrangements are significantly related to human interpretations of relationships in any given situation.

When each of the three teachers described their diagrams to one another, they all found each other's diagrams and explanations informative and true representations of some of their own views on authority. They attributed some of the differences to their different personal experiences and teaching situations. Because Dawn teaches mathematics in a French Immersion setting, there are two disciplines often seen in competition for priority – mathematics learning and language learning. Thus it does not surprise us that her conceptualization of authority shows awareness of multiple sources of authority. Jill has many Aboriginal students, with a culture that is very sensitive to human relations and that has a long history of tension with external colonial powers. Thus we are not surprised that her conceptualization of authority focuses on the human relationships immanent in the classroom. In addition to being a teacher, Mark is a coach who runs sports camps. His conceptualization reminds us of play sheets, and he talked about the need for every student (like every player) to follow the directions of the "coach" at the same time as they make decisions for themselves within the coach's system.

DISCUSSION

These three conceptualizations of authority represented by the mathematics teachers in our study raise a number of important issues, all of which relate to the diversity. First, analyses of authority structures tend to give only partial pictures of a mathematics classroom as each analysis takes a theoretical perspective that illuminates particular things. Thus no such analysis could address every teacher's particular concerns. Second, the scholarship has not yet exhausted the useful ways of conceptualizing authority. Mark's focus on physical positioning makes this clear to us. Third, and most important to us, the work of mathematics teachers differs significantly with their contexts. Thus, it is inappropriate to generalize about what features of authority are the most important to consider in a mathematics classroom. With these complexities, an important question remains: What can one say to mathematics teachers to help them understand better their authority relationships and to equip them to develop their practice to improve these relationships?

We repeat that further work on authority should be done with teachers and not on them because teachers can offer interpretations and identify complexities that we, as researchers and teacher educators (who no longer teach in public schools), may not see. We note that some of the most compelling examples of changing classroom discourse that resulted in empowering students can be found in literature on teachers' action research (e.g., Graves & Zack, 1997; O'Connor, Godfrey, & Moses, 1998). As mathematics educators recognize how they encode the authority structures that are implicit in their classroom practice, it becomes possible to envision alternative authority structures and to consciously choose what values we want to communicate.

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ARE SECOND GRADERS ABLE TO EXPLAIN THEIR MATHEMATICAL IDEAS?

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This study reports on how a large number of second graders performed in two explanation tasks of different kinds. We found that: a) a considerable number of them were able to produce (full or partial) explanations, b) the number of "explainers" cluster in certain classes (suggesting that the practice of explaining maybe related to classroom norms) and c) the way explanations are envisioned and understood played an important role in children responses. Suggestions for educational implications and further research are proposed.

BACKGROUND

Explanations have been studied as both instructional tools (e.g. Leinhardt, 1987; Leinhardt & Schwarz, 1997) and as key components of learning (e.g. Chi & Basok, 1989). This article focuses on the latter, more specifically on very young students' capabilities to produce explanations in mathematics, and on their nature.

What are explanations? In our view, a mathematical explanation is a description that addresses the origins/sources, entailments and connections of a mathematical idea and it is often an answer to a 'why' question, or a justification within an argument to support a central claim (Means & Voss, 1996). The ability to provide a coherent and convincing (to oneself or to others) explanation of an idea is often taken as an operative expression of the 'understanding' of that idea, and as a sign for taking up individual responsibility for one's own claims or ideas. The latter is one of the key evidences for autonomic learning.

Why explanations? Children's production of explanations has been considered as central to learning, mainly to the construction of knowledge processes, by both the cognitive and the socio-cultural perspectives. While children engage in producing explanations or in listening to them, a cognitive approach may, for example, examine how explaining "de-centers" egocentric thinking, reflects a schema or a "mental image". A socio-cultural approach may study, for example, the role explanations play in dialogues, in the enhancement of the zone of proximal development, in promoting internalization of more advanced ideas, and in the development of communication (Brown & Palincsar, 1989).

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Explaining may also enhance the awareness of the nuances of the ideas raised and discussed, by highlighting first and foremost to the “explainer” (but also to her audience), both the weak and strong points of her own reasoning with respect to a mathematical topic.

It may create opportunities for listening, discussing, imitating, and legitimizing different approaches to the underlying concepts of a mathematical topic.

It gives a prominent role to the “practice” of explaining providing spaces for exercising it in its many forms, e.g. explaining by verbalizing, by illustrating, by exemplifying, by making analogies.

Students' explaining processes provide (to teachers) a window into the mathematical knowledge and needs of the student, and into the development and expression of students' autonomic thinking, and thus it can serve as a point of departure upon which to build instruction.

When to engage in explaining? Given the important roles attributed to explaining, a simple and general reply seems to be: anytime! However, one may argue that, without a certain knowledge base, young children can rarely produce meaningful explanations, with the intended effects as the ones described above. On the other hand, one can view the production of explanations as an integral part of building such a knowledge base. This may lead us to circularity and to a subsequent impasse: how can one build a knowledge base needed to explain if one needs explaining to build such knowledge? Paraphrasing Sfard (2001), we would argue that explanations are one of the building blocks which at times scaffolds knowledge building and at times is it scaffolded by knowledge, making the task of "constructing" possible. Moreover, children also need to experiment with and learn about the “meta-level”, namely what an explanation is all about, what are its desirable components, why is it needed, when an explanation or an argument indeed explains and convinces and why. If this is so, it would seem that the sooner children are exposed to and requested to engage in the practice of explaining, the richer and more effective their learning may be. However, the production, exposition and discussion of mathematical explanations is a sophisticated activity, especially for very young children, and thus not surprisingly, except for some notable exceptions (e.g. Levenson et al. 2007), has not been very widely studied.

Thus regardless of the primary theoretical lens one may adopt, and certainly if one advocates a more eclectic approach, the process of producing and communicating explanations is important, and deserve our awareness and research efforts.

THE STUDY

We administered an achievement test to 17 second grade classes (470 students) spread over Israel. The test included items in which students were asked to explain/justify to their solutions/claims. We note that the explaining responses we analyze in this report did not emerge as a spontaneous need of the students to explain, rather explanations were explicitly requested. In the following, we analyze student responses to our requests for explanations in two items, which are quite different from each other. In our analysis, we address the following issues:

- To what extent can very young students engage in the practice of providing mathematical explanations?
- What are the characteristics of the explanations that they are able to produce?
- To what extent can the production of explanations (regardless of their nature) become a mathematical classroom norm for young students?

The tasks

We designed the following two tasks which require explanations.

<u>First task</u>

a. Insert in the empty space the right symbol: >, < or =

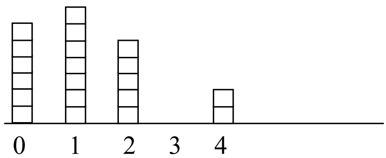
$$750 - 13 \;\boxed{\quad}\; 750 - 25$$

b. Can you answer this question without calculating the results first? Explain!

In order to make the comparison of these two subtractions without calculating, one must realize that whereas the *minuends are the same* (element 1) the *subtrahends are not* (element 2), and then to *conclude and explain* (element 3) that the more you subtract from a number the less you are left with. Thus 750-13 must be larger than 750-25, regardless of the exact result. This task requires children to move away from calculating and to focus on the production of an explanation, which is based on mathematical reasoning related to the very nature of the subtraction operation. In terms of the Realistic Mathematics Education (RME) approach, the children have to engage in *vertical mathematization*, which is the process of the reorganization of knowledge within mathematics itself, finding connections between concepts and strategies and applying them (Treffers, 1987).

<u>Second task</u>

Second grade children built a diagram to describe how many books they have read during the last month. Each kid pasted a small square in the appropriate place in the following diagram.



a. How many students in the class read one book? ________________

b. How many students are there in this class? ___________________

c. Why, do you think there are no squares above the number 3 in the diagram? Explain___________

We were specially interested in students' responses to item c, namely how students explain the meaning of a feature of a graphical representation in terms of the information it represents. In order to provide an explanation for the absence of squares above the number 3, children have to refer back to the story that is represented, and infer that since nobody placed a square there, nobody has read three books during the last month. It should be noted that, whereas in the first task students were familiar with the mechanics of subtraction, they had had little experience reading graphs of this kind. In terms of the Realistic Mathematics Education approach (RME), here the *mathematization* is characterized as *horizontal*, because it refers to moving back and forth from the world of real life to the world of mathematical symbols and representations, without referring to new mathematical ideas or connections thereof.

RESULTS

First Task. The following is the distribution (n=470) of the types of explanations provided by students in the first task:

Full	Partial	Non-mathematical	Wrong	No explanation
17%	**15%**	**13%**	**13%**	**42%**

Firstly, we note that, in spite of their young age, at least **58%** of the students attempted to engage in the production of an explanation to this item, and more than half of them gave an either partial or correct explanation. A closer inspection of the data by classroom shows that in some of the classrooms almost **80%** of the students did not provide explanations at all, with only up to **4%** correct explanations, whereas in other classrooms almost **50%** of the children provided correct explanations. In the following, we bring examples from some of the categories of responses to part b.

Full explanations:

Yoram: "*Because we start in both exercises (sides) with the same number, but if you subtract a bigger number, you get a smaller result*" (All 3 elements)

Dana: "*25 is a bigger number, therefore the result is smaller*" (2^{nd} and 3^{rd} elements)

David: "*If I do 750-13 and I do 750-25, it means that if I subtract less, the result will be bigger*" (2^{nd} and 3^{rd} elements)

Partial explanations:

Gal: "*Because there are 750 in both expressions and we subtract different numbers*" (1st and 2nd elements)

Ruti: "*Because the number 13 is smaller than 25*" (2nd element)

Non-mathematical explanations:

In this category, we included several kinds, for example those related to:

- Children's beliefs about the role of calculations, for example, Moshe replied correctly to item a, and in item b he wrote: "*No! Because one can't do an exercise without computing it"*
- tautological explanations, for example, Miri said: "*you should do it without calculations*"
- Explanations which are not connected to mathematical issues, for example, Eli said: "*Yes! I trust myself and therefore I do not think*"

Wrong explanations:

Neta: "*When you subtract more, you get more*".

Second task. The following is the distribution (n=470) of the types of explanations provided by students in the second task:

Full	Partial	Non-mathematical	Wrong	No explanation
24%	**8%**	**8%**	**16%**	**43%**

This distribution looks quite similar to that of the first task, with a slight increase of correct answers for this task – probably due to the fact that, in spite of the novelty of the topic, the request of *horizontal mathematization* may be less demanding than that of the *vertical mathematization*. As well as with the first task, there was a lot of variation among classes, some had up to **70%** of "full explainers" whereas others had less than **20%**. Here also those students who attempted to and/or were able to provide correct interpretations/explanations cluster in certain classrooms. Moreover, in some cases, there are striking similarities between the two items. For example: in class #17, **64%** of the students did not provide any explanation for item 1 and **72%** did not provide it for item 2. Only **12%** produced a full explanation for item 1 and **20%** produced it for item 2. For the students in this classroom, there was almost no difference between the vertical mathematization required in the first item and the horizontal mathematization required in task 2. In contrast, in class #8, **69%** of the students provided explanations (full-**26%**, partial-**43%**) in Item 1, and **78%** were able to interpret correctly the graphic information in item 2 (full-**61%** and partial-**17%**). This finding suggests that student production of explanations may be related to classroom practices and norms.

The following are some examples of students' responses to task 2.

Full explanation:

David: “*no one read 3 books*”

Orna: “*I think that there are no squares above the number 3 because there are no students who read three books*”

Partial explanation:

Mazal: “*No one wanted 3 books*”

Wrong explanations:

Benny: “*three students did not read books*”

Eyal: “*the third student did not read books*”

Non-mathematical explanations:

Amos: “*Because the children did not put any squares*”

Dina: “*There are no squares above the number 3*”

Naomi: “*Because the boy, or the girl, was ill*”

DISCUSSION

The data show that, in spite of their young age, many children are either fully able to explain or at least attempt an explanation involving either *horizontal* and/or *vertical mathematization*. Regarding task 1, this finding was also of interest as it contrasts with expected responses on the basis of the 'intuitive rule: 'more A $\Rightarrow$ more b' (Stavy & Tirosh, 2000), which has been found to be widespread among young children.

We also found that the number of “explainers” in some classes is considerably more widespread than in others. Although it is not fully clear from the data available to us whether we can attribute these differences to the existence and enactment of certain norms in the class, we suggest that teaching practices can certainly have had a role in supporting children to engage in the “culture of explaining”. This can be done through several activities: role modelling, argumentative dialogues in whole class and/or in small groups. A large study on how these norms emerge, develop and are supported in class is in progress. In any case, on the basis of our data we assume the existence and the sustainability of such cultures.

Our data also suggest that explanations can be integrated in many different activities or problems in which the children engage. As already mentioned, the first task requested an explanation on a topic which is central to the curriculum: subtraction. However, it requires the children to move away from the mechanics of the calculation to the reflection upon the operation and its nature, to comparison between results without performing the operation and even to some kind of hypothetical thinking. Interestingly, the context in which these explanations were required to be produced conflicted with some students' views of the mathematical culture: you are not supposed to guess, you cannot provide an answer without calculating.

The second task, although it was more novel to the children, is based on a real story upon which one can rely in order to make sense of a graphical feature of a data representation as the source for an explanation. Also in this case, for some students there were different interpretations of what an explanation is supposed to do. The question intended to probe into students' ability to interpret and verbalize a feature of the graph, which is somehow different from the others: the absence of a column in a histogram. Although we classified Naomi's question (see above) as non-mathematical, it may well be that for her it was clear that the absence of squares indicated that nobody read three books, and for her an explanation may have been interpreted as *why* nobody read three books – because they were ill. In spite of the highly speculative nature of this interpretation, when we suggest to incorporate the practice of explaining in class, whenever it is possible, it should be clear that the nature of the explanation may be: a) linked to the nature of the task (and thus it can have a different character according to the context in which it is requested), b) practicing explaining, as mentioned above, should also include a gentle and indirect introduction into what an explanation may be, should look like and why it is important to engage in it. Thus, a task analysis of the activity in which explanations are requested should be a pre-requisite for teachers and curriculum designers.

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YOUNG CHILDREN'S EMBODIED ACTION IN PROBLEM-SOLVING TASKS USING ROBOTIC TOYS

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19 Grade One children were observed while programming a simple robotic toy (Bee-bot or Pro-bot) to solve a spatial mapping task. Video-taped interview data was analysed using Studiocode for children's level of engagement their gesture, action, dialogue and representations of their solution process. 153 examples of gesture were classified and matched with children's drawn mathematical representations. Children's representations highlighted advanced use of iconic and symbolic notations indicating iteration of measurement, rotation and dynamic movement. Micro-analysis of the data suggests that integrated analysis of speech, gesture, action and representations is advantageous in technology rich dynamic contexts as an effective method for gaining insight into young children's mathematical thinking.

INTRODUCTION

At PME 2008 we presented case studies of how young children's exploration of a simple programmable toy demonstrated the dynamic capabilities of robotics in early mathematical development (Highfield, Mulligan & Hedberg, 2008). This report highlighted children's engagement in transformational geometry, iteration of the toy as a unit of measure and semiotic processing. The children's use of kinaesthetic motion mimicked their mathematical thinking. It was this dynamic process of embodied actions that led to further analysis of the role of gesture in a larger study investigating mathematical and meta-cognitive processes of 31 young children (aged three to seven years) while using simple robots. The present paper reports one component of this project; children's use of gesture, dialogue and representation while engaged in a mapping task requiring programming of robotic toys.

BACKGROUND

Over recent years there has been increased interest in the links between gesture and mathematical learning and thinking (Arzarello & Edwards, 2005; Lakoff & Nunez, 2000). The embodied cognition approach, as proposed by Lakoff and Nunez (2000) highlights the role of perceptuo-motor functioning in the learning of mathematics. Radford, Bardini, Sabena, Diallo, and Simbagoye assert that this role impacts beyond initial mathematics learning, "sensorimotor activity is not merely a stage of development that fades away in more advanced stages, but rather is thoroughly present in thinking and conceptualising" (2005, p.114).

Gesture, defined by McNeill as movements of the arms and hands synchronized with speech (2005), is a specific form of percepuo-motor activity that is relevant in

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mathematics. These actions have the potential to "encode meaning differently from speech" and to allow the speaker to harness "visual and mimetic imagery" to convey meaning (Goldin-Meadow & Wagner, 2005, p.234). Gesture can also be considered as a way to alleviate cognitive load, the learner "externalizing" thoughts so they can save "cognitive effort" (Goldin-Meadow & Wagner, 2005, p. 234).

Theoretical Approach

Radford et al. suggest that thinking is both mediated by and located in "the body, artifacts, and signs" (2005, p. 118). This highlights the body's role in thinking, but also places gesture in a broader context. Radford goes on to suggest that gesture must not be looked at in isolation, and that "knowledge objectification is a multi-semiotic mediated activity... a dialectical interplay of diverse semiotic systems" (2005, p. 144). In this light we must consider the interrelationships between children's dialogue, their gestures, the artifacts they use and the representations they create.

The use of technology in the development of early mathematical concepts impacts on this "dialectical interplay" by incorporating advanced and dynamic artifacts. Healy (2008) relates Papert's 1980s concept of "body-syntonicity" to the notions of embodiment and argues that the dynamic representations generated with technology provide a conduit for learners to relate mathematical thinking to their own bodies.

Given the affordances of Logo (Clements & Sarama, 1997), it follows that the dynamic movement enabled by robotic toys facilitates 'syntonic' learning where children use their body in modelling, solving and representing mathematical problems. The toy's movement within a three-dimensional plane promotes the child's use of embodied action and gesture as mechanisms for problem-solving and reflection (in such a way that may not be possible using static tools).

Thomas, Mulligan and Goldin (2002); Saundry and Nicol (2006) and Smith (2003) have each examined, described and analysed children's drawings. This work promotes children's representations as windows into their thinking, or in the case of Saundry and Nicol (2006) and Smith (2003) as insight into their problem-solving. Examination of children's representations within their situated context (Radford, 2005) allows a closer analysis the interrelationships between these representations and the children's dialogue, gestures and problem-solving.

This raises three questions: i) What forms of embodiment are evident as children engage in mathematical problem-solving tasks using dynamic robotic toys? ii) What is the relationship between embodiments and the problem-solving process, and iii) How does gesture impact on children's representations of the problem?

METHOD

Twenty-one children, from one mixed ability Grade One class (aged six to seven years) engaged in a *classroom-based robotics program* for approximately 2 hours a week over a twelve-week period. The children were pre-assessed using a semi-structured interview, ascertaining prior experience with robotic toys and programming competence, and completing a standardised test of general mathematical ability (*I Can do Maths*, Doig & de Lemos, 2000). Throughout the program four children, representing a range of abilities, were subject to in-depth study. These children were interviewed each week by the researcher, using video stimulated recall interview techniques gauging the children's reflective and metacognitive recall. After the twelve-week period a post assessment interview was conducted, where children solved five problem-solving tasks.

Robotic Toys

Bee-bots (Figure 1) and Pro-bots (Figure 2) were used predominantly in this project. Programming functions for the Bee-bot and Pro-bot and a description of children's problem-solving strategies have been described previously (Highfield, Mulligan & Hedberg 2008; Highfield, in press).

Figure 1: Bee-bot programmable toy

Figure 2: Pro-bot programmable toy

Classroom-based robotics program

During the twelve-week period the children were introduced to the robotic toys through free play and tasks designed by the teacher to elicit mathematical thinking. The children also completed a problem-solving project designed by the teacher in collaboration with the researcher, which integrated programming, measurement and mapping (spatial and measurement) skills similar to previous work in this area (Davis & Hyun 2005). This project included small group construction of a large scale, 'island' map designed in three dimensions to accommodate the toys. The children posed, modelled and solved problems using these maps.

The assessment interview

The post-assessment interview was administered to 19 of the 21 children, by the researcher, in a room adjacent to the classroom, with a duration ranging from 20 to 35 minutes. All interviews were video recorded and copies of inscriptions and drawings collated. Children used either the Bee-bot or Pro-bot toy, a large map of the task on plain or grid paper and pre-cut arrows (indicating step length of robot). The post-assessment comprised of five problem-solving tasks of increasing difficulty. Tasks ranged from simple ("Program the toy to move forward three steps") to intermediate ("Program the toy to make a square shape") and then to advanced ("Program the toy to move from 'my house' to 'your house' without popping the balloons", using a large scale map). Children recorded their solution process ("Draw what you made the robot do so that a friend would know how to solve that task") and explained their thinking ("Tell me what you thinking about?"). In this paper children's solutions to the 'House' task, are described and analysed. Children were required to program the toy to move across a map, from one house to another, while avoiding two objects (the lake and balloons) (see Figure 5).

Analysis

Interview data was collated and analysed for each of the five tasks. Children's problem-solving strategies for the 'House' task were first categorised for accuracy and the number of solution attempts. The data was then micro-analysed for types of gesture, action and dialogue (Edwards, 2003), and categorised as 'iconic', 'metaphoric', 'deictic' and 'beat' (McNeill, 2005). As most gestures were classified as 'deictic' they were re-coded into four subcategories (discrete pointing, hand sliding, hand stepping and pointing with eyes or head). Two broad categories of gesture were present; i) gesture synchronised with speech or sub-vocalisations, and ii) action with speech preceding and following the movement. Video analysis was conducted using digital editing software (*imovie*) and digital coding software (*Studiocode*), which allowed specific gestures to be identified and quantified. Further micro-analysis of children's drawn representations examined the type and complexity of symbols used to imitate dynamic movement. Children's programming of the toy was matched with their drawn representations, bodily action, gesture and dialogue allowing an examination of the situated context (Radford, 2005).

DISCUSSION OF FINDINGS

Of the 19 children interviewed, 12 chose to use the simplest robotic toy, the Bee-bot and seven chose the more advanced toy, the Pro-bot. Six children used the map with a superimposed grid, and 13 others the plain map. Only three children used the pre-cut arrows (as informal units of the toy's step length). All children were able to successfully solve the task but differed in their number of attempts to program the toy to move from one house to the other, avoiding the lake and balloons. Three children solved the task on the first attempt, six on the second, five on the third, three on the fourth and two on the fifth attempt.

Gesture and action in the problem solving process

All children used gesture in the problem-solving process, in describing their representation and reflecting on their thinking. While problem-solving the gestures used by ten children were synchronised with their speech or sub-vocalisations while eight others used hand or bodily actions that were not synchronous with speech. These non-synchronous actions were used by the children to directly assist in solving the problem, rather than as tools to convey meaning such as iterating the robot's steps. One child initially used silent action and later combined gesture with speech.

While each of McNeill's (2005) gesture dimensions (*iconic*, *metaphoric, deictic* and *beat)* were evidenced in this data, *deictic* gesture, were prominent. *Deictic* gestures, ("pointing ... locating entities and actions in space" p.39-40) dominated, with children using pointing with their hands, eyes and head. Four specific types of deictic gestures were evidenced in data: Discrete pointing - isolated pointing, E.G., "over there" (*pointing to an area)*; Hand sliding - moving the hand or head in an arc or sliding motion indicating general movement or rotation, E.G., "it went like this" (*with the hand sliding across the map*); Eye pointing - pointing with the head or eyes to indicate steps of movement, e.g., "I need one, two, three steps" (*head pointing to indicate position of steps*), and Hand stepping - using the hand to measure steps, indicating the position of movement, e.g., "Forward, forward, turn" (*hand indicating measured steps and directionality*). Another action was also evident, Acting out - moving the toy to indicate steps of motion.

Studiocode analysis revealed 153 examples of gesture or action during the problem-solving phase comprising 56 instances of eye pointing, 40 of hand sliding, 29 of hand stepping, 20 discrete points and eight examples of acting out with the toy. Eight children used one dominant type of gesture (seven children used the technique of "eye pointing" and one child used the strategy of "acting out") to solve the problem. These children persisted with their strategy even if successive attempts were needed to solve the problem. Eleven children used multiple strategies: nine began with "eye pointing" and two others by 'hand sliding', and then moved to other strategies for subsequent attempts. There was no direct correlation observed between particular actions or gestures and the efficiency and sophistication problem-solving. Further, analysis suggested that the children used gesture as a tool to manage 'cognitive load', assisting them to predict and plan the robot's path and motion.

Representations

Children's drawn representations were analysed for accuracy and classified as pictorial, iconic or notational (based on the work of Thomas, Mulligan and Goldin, 2002), or as combinations of these; iconic-pictorial (iconic with pictorial elements) and pictorial-iconic (pictorial with iconic elements). There were five 'iconic' drawings, which used lines or arrows alone to symbolise the robot's movement (Figure 3). Five were 'iconic-pictorial' and primarily depicted arrows indicating the robot's movement (Figure 4). The remaining nine drawings were 'pictorial-iconic', dominated by pictorial elements (Figure 5).

Relating children's action and gesture to their drawn representation yielded some important connections, with children incorporating different representations of dynamic movement in their drawings. Figures 3 to 5 provide evidence of advanced symbol use. These are representative of the data set for 18 of 19 children, indicating representations of movement, unit of length, directionality and rotation. An analysis of these representations is both richer and more explicit when contextualised with speech, gesture and action.

Cathryn (Figure 3) made three attempts using the Bee-bot. She initally used a plain map and then used a superimposed grid. Cathryn used "eye pointing" (without synchronous speech) and later explained that she was "looking to see how many steps to take". Cathryn's representation, classified as iconic, focussed on the robot's movement rather than irrelevant pictorial details. Her use of gesture, combined with her use of the grid indicated that she was able to accurately iterate the robot's step length. It seems that her action enabled her to accurately program the movement; her representation of equal sized steps and rotation indicates that she also has an understanding of a 90º rotation.

Teo (Figure 4) also used the Bee-bot and made three attempts to solve this task. He used three different strategies, "eye pointing", "hand stepping" followed by "acting out". Each of these strategies were gestural, combining action with speech, E.G.,"forward, forward, forward, turn..." (*moving the car on in steps; forward, forward, forward and then a rotation*). Teo's explicit gestures, using the toy to act out his intended plan, provides insight into his thinking, understanding of iteration of step length and 90° rotation. Teo's representation is iconic-pictorial emphasising start and end points, detailing the motion between these points.

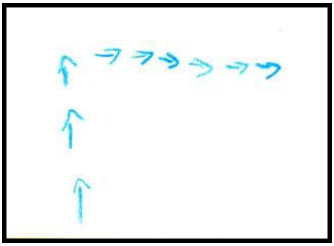

Figure 3: Cathryn's iconic representation

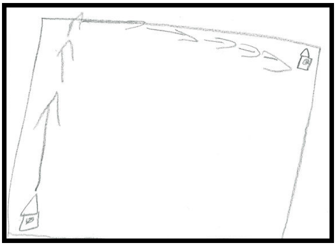

Figure 4: Teo's iconic-pictorial representation

David (Figure 5) made three attempts, before changing the Pro-bot's direction, completing the task on his fifth attempt. In each attempt he used the strategy of "eye pointing". He articulated detailed steps for the toy to take "forward, forward, forward, turn, forward, forward, forward go (*eyes pointing to indicate step length)"*. His pictorial-iconic representation is also detailed, one of five drawings to include arrows indicating rotation. He also includes notation of "bip" (sic) indicating the beeping sound made by the robot on completion.

Iman (Figure 6) used a Beebot with the grid, solving this task on his second attempt, but his representation is much less sophisicated. Iman used two strategies, "eye pointing" on his first attempt and then "hand stepping". When hand stepping Iman subvocalised "there, there, there" (*placing his hand on the grid, indicating a step and then inputting the program, one step at a time).* Iman's representation of the robot's movement is iconic as it only uses lines. It is inaccurate, but when taken in context with the accompanying speech the image reveals accurate first steps and pencil motion used to indicate direction. The image starts in the top left corner: Iman spoke as he drew: "forward, (*winding the pencil)* forward (*drawing the first line)* and forward (*drawing the second line)* and forward (*eyes looking at the grid checking where the toy moved and then drawing the third line)* and turn (*drawing the fourth line down the page)* and turn (*drawing the fifth line, right to left along the bottom of the image)* there!". The combined use of gesture, representation and speech provides greater insight into Iman's emergent mathematical language and directionality than would have been possible if we analysed any of these aspects in isolation.

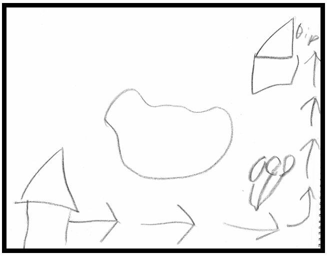

Figure 5: David's pictorial-iconic representation.

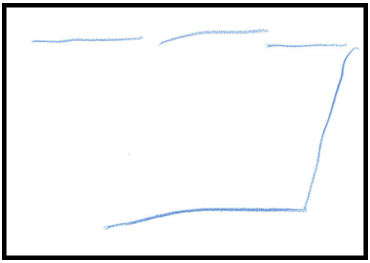

Figure 6: Iman's innacurate iconic representation.

CONCLUSIONS AND LIMITATIONS

This is a small-scale descriptive study with limited duration of the classroom program. However, the micro-analysis of the problem-solving process is valuable in itself, in determining the design and focus of further work and in developing early mathematics curriculum.

From this analysis it is clear that these children used varied forms of embodiment and gesture that impacted on the dimensions of their representations and mathematical concepts. These actions were seen in the children's planning, problem-solving, reflection and the description of their experiences. While there is some natural connection between embodiment, gesture and problem-solving using robotic tools, this inter-relationship is complex and further analysis across a range of contexts would be beneficial.

We question to what extent the robot's movement may promote the increased use of gesture and thus advantage the problem-solving and representational processes. The design of a manipulable robotic tool as a unit of measure in itself, with the capacity to rotate, affords children's syntonic processes and we suggest that the use of emergent symbols is unlikely to have been afforded by a static toy in the same problem.

This research highlights the importance of a multifaceted analysis and theoretical approach; with children's representations, speech, gesture and action providing greater insight than would be possible if these facets were examined in isolation. The project also affirms the potential of robotic toys in syntonic learning and provides possible direction for reconceptualizing early mathematics curriculum.

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SCHOOL STUDENTS' UNDERSTANDINGS OF ALGEBRA 30 YEARS ON

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In this paper, we discuss early results of a large-scale survey of attainment in algebra of 11-14 year olds in England. As part of Phase 1 of Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS), tests were administered to a sample of 3000 children in June and July 2008. The tests used items developed during the 1970s for the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981). This enables a comparison of students' current attainment with that of 30 years ago. Results suggest that the practice in England of teaching algebra earlier confers an initial advantage to students, but this increased attainment is not sustained and, by age 14, current performance in algebra is broadly similar to that of students in 1976.

BACKGROUND

Over the past 30 years, there has been a great deal of work directed at, first, understanding children's difficulties in mathematics and, second, examining ways of tackling these difficulties. Yet, there is no clear evidence that that this work has had a significant effect in terms of improving either attainment or engagement in mathematics. Indeed, children continue to have considerable difficulties with algebra and multiplicative reasoning in particular (Brown, Brown, & Bibby, 2008).

The original Concepts in Secondary Mathematics and Science (CSMS) study was conducted 30 years ago. The study made a very significant empirical and theoretical contribution to the documentation of children's understandings and misconceptions in school mathematics (Hart, 1981). In the intervening period, there have been various large-scale national initiatives directed at improving mathematics teaching and raising attainment: e.g., the National Curriculum, National Testing at age 7, 11 and 14, the National Numeracy Strategy and the Secondary Strategy. Many of these initiatives have drawn directly on the CSMS study. During this period examination results in England have shown steady and substantial rises in attainment: e.g., the proportion of students achieving level 5 or above in national tests at age 14 has risen from 56% in 1996 to 76% in 2006 and the proportion of students achieving grade C or above at GCSE[1], the examination taken at 16 at the end of compulsory schooling, has risen from 45% in 1992 to 54% in 2006. However, independent measures of attainment suggest that that these rises may be due more to "teaching to the test" rather than to increases in genuine mathematical understanding. Replication results from the science strand of the CSMS study (using a test on volume and density) suggests that, students' understanding of some mathematical ideas as well as the

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 177-184. Thessaloniki, Greece: PME.

related science concepts has declined (Shayer, Ginsberg, & Coe, 2007). Studies at the primary level indicate that any increases in attainment due to the introduction of the National Numeracy Strategy have been at best modest (Brown, Askew, Hodgen, Rhodes, & Wiliam, 2003; Tymms, 2004). Results from the Leverhulme Numeracy Research Programme suggest that any increase in attainment at Year 6 is followed by a reduction in attainment at Year 7 (Hodgen & Brown, 2007). Further, Williams et al. (2007) found that there is a plateau in attainment between the ages of 11 and 14.

METHODS AND THEORETICAL FRAMEWORK

Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) is a 4-year research project funded by the Economic and Social Research Council in the UK as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics disciplines. Phase 1 of the project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England. This is followed in Phase 2 by a collaborative research study with the teacher-researchers extending the investigation to classroom / group settings and examining how assessment can be used to improve attainment and attitudes. Comparison with the CSMS study will enable us to examine how students' understandings have changed since 1976 when the main algebra study was carried out and to update this work in the light of the considerable work in algebra over the intervening 30 years (Kieran, 2006). In addition to the algebra test, the survey consists of two further CSMS tests, Ratio and Decimals, and an attitudes questionnaire. A sub-sample of students will be followed longitudinally and tested in Summer 2009 and 2010.

Participants

In June and July 2008, tests were administered to a sample of approximately 3000 students, of whom 2044 took the algebra test: 679 Year 7 (aged 11-12), 756 Year 8 students (aged 12 -13) and 608 Year 9 students (aged 13-14), from 11 schools and approximately 90 classes. The sample was randomised and drawn from MidYIS, the Middle Years Information System. MidYIS is a value added reporting system provided by Durham University, which is widely used across England (Tymms & Coe, 2003). When the cross-sectional survey is completed in 2009 with a further group of 3000 students, the sample will be representative of schools and students in England. The original CSMS algebra test was administered to a sample of 12-15 year olds. In the current study, the tests were administered to younger students in part because formal algebra is taught earlier than it was generally in the 1970s. We note that the current 2008 sample is skewed towards higher attaining students.

The design and theoretical underpinning of the algebra test

The CSMS algebra test was carefully designed over the 5-year project starting with diagnostic interviews. The original test consisted of 51 items. Of these 51 items, 30 were found to perform consistently across the sample and were reported in the form of a hierarchy (Booth, 1981; Küchemann, 1981). For the current study, a few additional items were added. These relate to fractions (drawn from the CSMS Fractions test) and spreadsheet algebra, but are not reported in this paper. Piloting indicated that only minor updating of language and contexts was required.

The test items range from the basic to the sophisticated allowing broad stages of attainment in each topic to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding. The focus of the test was on generalised arithmetic, and in particular it looked at different ways in which pronumerals can be interpreted (Collis, 1975). Items were devised to bring out these six categories (Küchemann, 1981):

> Letter evaluated, Letter not used, Letter as object, Letter as specific unknown, Letter as generalised number, and Letter as variable.

In this paper, we focus on just ten items due to space constraints: items 3, 5c, 9b, 9c, 9d, 13a, 13b, 13d, 13 e and 13h, chosen because they typify the performance of the sample of students across the attainment range. All these items were amongst the consistently performing items that formed part of the original hierarchy.

Item 5c presented the following problem to students: "If $e + f = 8$, $e + f + g = \dots$." Here the letters e and f could be given a value or could even be ignored; however the letter g has to be treated as at least a specific unknown which is operated upon: the item was designed to test whether students would readily 'accept the lack of closure' (Collis, 1972) of the expression $8 + g$. Students tend to see the expression as an instruction to do something and many are reluctant to accept that it can also be seen as an entity (in this case, a number) in its own right. Students tend to see the expression as an instruction to do something and many are reluctant to accept that it can also be seen as an entity (in this case, a number) in its own right (Sfard, 1989). Thus, of the Year 9 (aged 13-14) students tested in 1976, only 41% gave the response $8 + g$ (another 34% gave the values 12, 9 or 15 for $e + f + g$, and 3% wrote $8g$).

A similar proportion (38%) of the Year 9 students answered item 9d correctly in 1976 (see Figure 1, below). Here the given letter, n, also has to be treated as at least a specific unknown. The corresponding facilities for the preceding two items (9b and 9c) were substantially higher (68% for each item); here the letters can be treated as objects, or at least as the names of object (i.e., the names of various sides of the given figures).

In question 13, students were asked to simplify various expressions in a and b. Some of the items could also readily be solved by interpreting the letters as objects, be it as as and bs in their own right, or as a shorthand for apples and bananas, say (eg 13a: simplify $2a + 5a$; 13d: simplify $2a + 5b + a$); however, such interpretations becomes strained for an item like 13h (simplify $3a - b + a$), where it is difficult to make sense of subtracting a b (or a banana).

In item 3, students are asked, "Which is larger, $2n$ or $n + 2$?" The item looks deceptively simple and students tend to go for $2n$, giving reasons like 'Because it's multiply'. Here the difference between $2n$ and $n + 2$ varies with n, and $2n$ is only greater than $n + 2$ when n is greater than 2. Students are more likely to see this if they appreciate that n can *vary* and if they can also *coordinate* the values of n with the values of the given expressions.

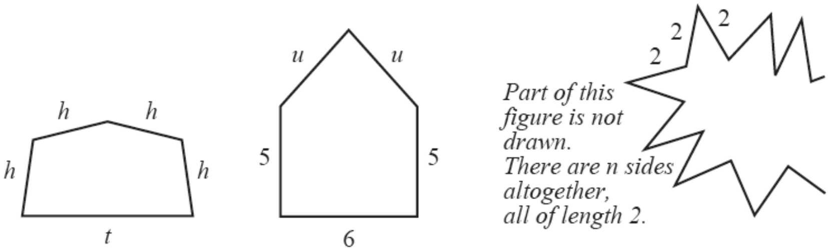

Figure 1: Illustrations for items 9b, 9c and 9d. Students are told that the perimeter of a square side g can be written $p = 4g$, then asked, "What can we write for the perimeter of each of these shapes?"

Methods of analysis

In this paper, we focus simply on facilities: the percentage of correct responses. Future analysis will be conducted using a variety of techniques, extending those used in the original CSMS study with more modern approaches, such as Rasch modelling.

RESULTS

The item facilities for 1976 and 2008 are presented numerically in Table 1 and graphically in Figure 2.

The current students tend to perform better than the 1976 cohort at age 12-13. So, on the selected item facilities presented here, at Year 8, the 2008 performance was as good or better than the 1976 performance. Of all the 30 hierarchy items, the 2008 attainment was worse on 9 items and better on 18 items with changes in facility ranging from -11% to +23%. However, by age 13-14, the attainment of the two cohorts is quite similar. So, on the selected item facilities presented here, at Year 9, the 2008 performance was as good or better than the 1976 performance on only 5 items, but worse on the 5 others. Of all the 30 hierarchy items, the 2008 attainment was worse on 17 items and better on 10 items with changes in facility ranging from: -20% to +19%.

Item	1976		2008			Change	
	Y8	Y9	Y7	Y8	Y9	Y8	Y9
13a: $2a+5a$	**77%**	**86%**	69%	**79%**	**82%**	**+2%**	**-4%**
9b: $p = 4h+t$	**58%**	**68%**	50%	**69%**	**74%**	**+11%**	**+6%**
9c: $p = 2u+16$	**54%**	**64%**	43%	**64%**	**67%**	**+10%**	**+3%**
13d: $2a+5b+a$	**40%**	**60%**	42%	**61%**	**65%**	**+21%**	**+5%**
13h: $3a-b-a$	**27%**	**47%**	25%	**40%**	**46%**	**+13%**	**-1%**
13b: $2a+5b$	**29%**	**45%**	29%	**40%**	**40%**	**+11%**	**-5%**
5c: $e+f=8$, $e+f+g=$	**25%**	**41%**	18%	**38%**	**42%**	**+13%**	**+1%**
9d: $p = 2n$	**24%**	**38%**	14%	**29%**	**36%**	**+5%**	**-2%**
13e: $(a-b)+b$	**15%**	**23%**	10%	**15%**	**20%**	**0%**	**-3%**
3: $2n$ or $n+2$	**4%**	**6%**	3%	**4%**	**6%**	**0%**	**0%**

Table 1: Facilities (items 3, 5c, 9b, 9c, 9d, 13a, 13b, 13d, 13 e and 13h), showing results for both 1976 and 2008 samples at Years 7, 8 and 9.

In general, the pattern of progression in 2008 consists of a relatively steep rise between Years 7 and 8 that is then followed by a much shallower rise between Years 8 and 9. This pattern of progression appears to be mirrored in 1976, although the rise appears to take place a year later. Data were not collected in 1976 for the younger students. However, the algebra test was administered to Year 10 students (aged 14-15). A comparison of progression between 2008 (Years 7-9) and 1976 (Years 8-10) across items 9 (Find the perimeter) and 13 (Simplify expressions) is presented in Figure 2. This suggests that the pattern of progression is similar over time: an initial relatively steep rise is followed by a much smaller rise subsequently. However, although this initial steep rise now takes place a year earlier, students' attainment does not appear to increase in the long term.

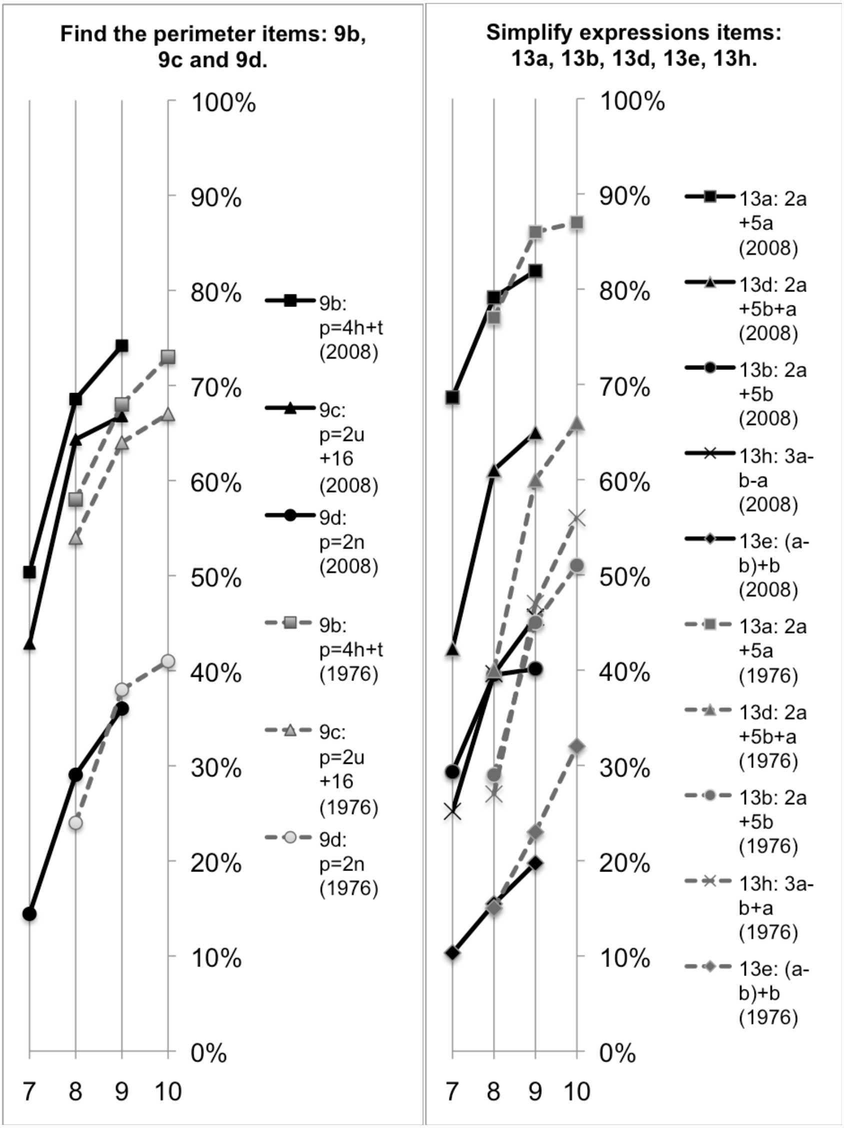

Figure 2: Comparison of facilities for items 9b, 9c, 9d in 1976 (Years 8-10) and 2008 (Years 7-9). [2008: solid black; 1976: dashed grey.]

The pattern across items is less straightforward to interpret. For example, on the simplify expressions items, current students are only slightly better than in 1976 on the straightforward simplify task, item 13d (60%: 1976 v 65%: 2008), and actually slightly worse on the less straightforward items: 13e, 13b and 13h.

DISCUSSION

In comparison to 30 years ago, in England, formal algebra is taught to all students earlier. This is partly as a consequence of the introduction of a National Curriculum (Brown, 1996). The initial results of the study reported here suggest that, whilst this practice confers an initial advantage to students, this increased attainment is not sustained. By age 14, current performance in algebra is broadly similar to that of students in 1976. Moreover, it is worth noting that sample of students tested in 2008 is in general a relatively high attaining group. Hence, the data presented here provide further evidence that increases in examination performance are not matched by increased conceptual understanding.

We emphasise, however, that we are not arguing against the teaching of algebra to younger children. Rather we suggest that earlier teaching *could* give students much needed time and space to grapple with the key ideas of algebra. We are currently in the process of conducting an analysis of current textbooks and comparing these to those of 30 years ago. This suggests a strong emphasis in current textbooks on procedural teaching. In comparison, although cursory at times and not always very purposeful, the textbooks of 30 years ago do tend to emphasise the conceptual rather than procedural.

NOTES

1. The GCSE examination is taken by virtually all 16 year olds in England.

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STUDENTS PROVIDE INFORMATION ON AFFECTIVE DOMAIN AND LEARNING ENVIRONMENT THROUGH INTERVIEW

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Students from years 7-10 were assessed through interview on a number of mathematical domains including a section on affect. They identified their liking for mathematics, aspects of the learning environment that helped them learn and gave advice for improving mathematics learning. There were differences between the year7 and 8 students and the year 9 and 10 students. The data can inform teachers about their practice, the practices within the school and the students' perceptions of classroom environments that will assist them learn mathematics.

BACKGROUND

> Affective issues play a central role in mathematics learning and instruction (McLeod, 1992, p575).

McLeod (1992) continues "If research on learning and instruction is to maximize its impact on students and teachers, affective issues need to occupy a more central position in the minds of researchers" (p575). These affective issues include values, emotions, beliefs, attitudes and self conceptions as they relate to mathematics and mathematical learning. Affect involves behaviours such as engagement, motivation, interest, persistence, and self-efficacy and contributes to mathematical success.

Interestingly the Second Handbook of Research on Mathematics Teaching and Learning (Lester, 2007) does not have a corresponding chapter on affective issues. There is still a chapter on mathematics teachers' beliefs and attitudes but no corresponding focus on students. In the section on the influences on student outcomes, Stein, Remillard and Smith (2007) note that the learning environment is an important determinant for student achievement. Student perceptions of the classroom environment can enhance our understanding of it (Athanasiou & Phillipou, 2008). Tarr et al (2008) use a standards based learning environment which acknowledges affect. Indeed standard 5 of the Professional Standards (NCTM, 1991) concerns the learning environment and the Principles and Standards (NCTM, 2000) use confidence, engagement and enjoyment in relation to students.

The change to a focus on learning environments reflects a move in research away from the more traditional quantitative studies using Likert scales, for example, to access attitudes. Recent studies have included qualitative aspects from classroom studies which consider learning more holistically, observing learning behaviours and interactions in the classroom (Alston, Brett, Goldin, Jones, Pedrick, & Seeve, 2008; Ball & Bass, 2003; Cobb & Yackel, 1998). Way, Bobis, Anderson and Martin (2008) have argued that constructs such as motivation need to be studied taking into

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 185-192. Thessaloniki, Greece: PME.

account a number of other factors from the affective domain such as self-efficacy, persistence and anxiety, thus utilising a range of approaches and taking both the learning environment and the characteristics of the student into account. Beliefs and attitudes are now accessed in a greater variety of approaches (eg drawing - Halverscheid & Rolka, 2007).

While there is recognition of the importance of affect in documents such as the NCTM standards, and while teachers collect much information on students' achievements in the cognitive domain and utilise it to inform their teaching they rarely systematically collect and use data from the affective domain. By listening to their children teachers can learn a lot about both the students' understandings and about their beliefs and motivations (Zambo, 2008). This in turn can influence the teaching approaches as well as the content needed for the individual students.

THE RESEARCH

As part of a project to improve mathematics teaching and learning in a school, interviews were conducted with a sample of students across grades 4 – 10. The interviews focussed on the mathematical domains of whole number and operations, including place value understandings; fractions; decimals; and algebra, including relational thinking, equations and equivalence. The interviews were not taped but rather the interviewer took notes following a recording sheet. Aspects of the affective domain were also included in the interview. This paper focuses specifically on the affective domain responses at years 7-10 and the implications for teaching. The teachers were particularly interested in the learning environment and what the students may say about it. A stratified random sample was chosen for interview leading to interviews of 103 students across years 7-10.

In the school the year 7 and 8 students are in a separate sub-school to the year 9 and 10 students. In the past the school has practised setting (tracking) for mathematics but, since the project began, in the past two years this has changed at years 7 and 8 so that the year 7 and 8 classes are now not set/tracked.

THE INTERVIEW

Each interview was conducted one-on-one over a 30-40 minute period. The text for the affective domain questions is shown in Figure 1. The instructions for the interviewer are in italics and the specific questions in ordinary text. The questions were open ended but within strict time constraints. The interviewer took notes recording as accurately as possible the students answers. The purpose of the questions was not only to collect data on the students' general attitude towards learning mathematics but also to elicit responses about their approaches to learning and their reflections on what would improve mathematics at school. The questions were designed through discussion with the classroom teachers and aspects they perceived that could inform their teaching. These affective domain questions were the first questions in the interview and also served to establish some rapport between the interviewer and interviewee before moving to the mathematical questions.

The interviewers were both teachers at the school and university students. All had been given training in interview techniques and in the use of the interview. Each teacher interviewed three students who were not actually in their class. This was seen as an important part of the teachers listening to the student voice. The remaining interviews were conducted by four students.

A. How do you feel about maths?
If they seem unsure of the question ask Do you like maths?

Negative answer	*Positive answer*
Have you ever liked it?	What do you like most about it?

If yes When?
What did you like (or enjoy)?
If no
Are there any things you like about it?
B. What things help you most when you are learning maths?
C. Are there any suggestions you would like to make about how we could make learning maths better?

Figure 1. Questions in affective domain section of the interview.

RESULTS AND DISCUSSION

Data from the record sheets was transcribed to a spreadsheet. The qualitative responses to the questions in figure 1 were grouped to similar types of responses. Table 1 shows the sample size and responses to the question of whether the students feel positive or negative towards mathematics. The results have been split by year level. The percentages are the percentage of that particular year rather than of the whole sample to enable comparisons.

Table 1. Feelings about mathematics split by year level.

Year level	N	Positive	Negative
7	17	12 (71%)	5 (29%)
8	23	20 (87%)	3 (13%)
9	31	19 (61%)	12 (39%)
10	32	22 (69%)	10 (31%)
Total	103	73 (71%)	30 (29%)

Overall 10 of the 73 students when expressing their liking for the subject voluntarily related it to their perceived ability commenting that it was one of their strengths or that it was easy. Two others who expressed liking did so with a comment such as “enjoy it despite struggling with it”.

Similarly 10 of the 30 who expressed dislike related it to their own lack of success and that they found it hard: "don't like 'cause it's never really clicked" "don't like – it's hard, difficult, stressful".

The initial answers tended to be very brief. Some of the students were more expansive when asked about things they liked about mathematics. The responses varied. A small number were again brief saying "nothing" or "it's all okay" while most were specific. The students were not prompted about the nature of their responses so that their answers reflect aspects that were salient to them at the time. Considering that the responses were usually only one or two ideas from each student and were not prompted, they are indicative of a much higher response on a questionnaire soliciting specific responses. Table 2 shows some of the responses with years 7 and 8 separated from years 9 and 10. Differences in curriculum emphases show in the topics mentioned with Pythagoras and trigonometry being year 9 and 10 topics and arithmetic being more a focus as years 7 and 8.

Table 2. Things year 7, 8 and year 9, 10 students liked about mathematics

What is liked about mathematics	% year 7-8	% year 9-10
Mathematical topics	85	30
Arithmetic including responses such as "number" and "multiplication"	27.5	
Fractions	12.5	5
Decimals including two mentions of percentage	10	5
Algebra and in year 10 "formula" and "equations"	30	20.5
Measurement which at years 9 and 10 was exclusively trigonometry and Pythagoras	22.5	8
Geometry divided between shapes and maps/diagrams	10	2
Chance and Data (Probability and Statistics)	5	2
Computers		2
Aspects of teaching approach	35	14
Problem solving	30	9.5
Games, activities	12.5	3
Traditional method of teaching with explanation followed by practice		2
Writing things down and worksheets (split)		5
Characteristics of mathematics and liking challenge	10	35
Challenge, logical and really having to think ourselves (sometimes implying that they do not have to think in other subjects)	5	12.5
Learning new things, things that are needed/useful, interesting	2.5	9.5
Mathematics is straightforward and there is always an answer	2.5	14.5
Context of the class including teacher	10	19
Success and easier class (streamed in years 9 and 10)	5	9.5
Teacher	5	6.5
Friends in class		5

From the responses it is interesting that for the year 7 and 8 students all topics in the curriculum were mentioned with 85% of the students mentioning liking for at least one topic as contrasting with only 30% of year 9 and 10. Problem solving was well liked by many of the year 7 and 8 students and, by their responses, is well established in the curriculum. This was one aspect of the results that drew teachers' attention, raising the question of why the levels are different.

One unexpected response at year 9 and 10 was the clearly held belief that mathematics was straightforward and that there is always an answer: "You don't really have to explain your answer. There is a definite answer (unlike English)." "There are always set answers" "I like that I like the fact that there is always one definite answer" "There's a set answer, work through steps to get one straight answer". Although some of the teachers may agree with these sentiments they did not expect this response so hearing the students has caused them to reflect on their teaching approaches.

The second question focussed on what the students perceived as assisting them most in their learning of mathematics. Table 3 shows the general student responses.

Table 3. Aspects that assist students in their learning of mathematics

What helps most in the learning of mathematics	**% year 7-8**	**% year 9-10**
Teacher giving clear explanations and one-on-one help	22.5	38
Pen& paper and writing including recording every step	22.5	14.5
Lots of examples explained		16
Friends, group work and talking about it	17.5	19
Hands on learning (including diagrams and making things)	17.5	3
Practice, exercises and homework particularly before tests	15	16
Remembering facts and formulas	10	6.5
Visual things	7.5	6.5
Reading over notes and cheat sheets and revising	5	8
Calculator/computers	5	8
Quiet so can concentrate	5	6.5
Games	5	
Working it out yourself asking questions where necessary		8
Summary books and good notes		6.5
Simple language		3
Music		1.5

Not surprisingly there was a strong focus on the teacher. This included the teacher giving clear explanations, writing clearly on the board, explaining with lots of different examples, correcting work and being available to answer questions one-on-one. The students valued one-on-one discussion with the teacher or with a tutor and valued the teacher giving them clear explanations.

Recording the steps and writing things down assisted a number of students. Part of this is also about the students taking some control of their own learning and recording as they need to but it also signals that they are aware of this as being valuable in their learning. Group work and discussing the work with others was seen as valuable in assisting learning. Five students made strong references to needing quiet so that they could concentrate as their only salient point suggesting that they perceived the normal class as noisy and in the words of one “disruptive”. In contrast one wanted music.

The final question on this section of the interview asked the students to give advice on how mathematics learning and hence teaching could be improved in the school. Table 4 shows the students responses.

Table 4. Suggestions for improving mathematics learning in the school

Suggestions for improvement	% year 7-8	% year 9-10
no change required	30	11.1
games and activities	27.5	9.5
teachers including clarity and variety of explanations at year 9 and 10 this included listening to the students	12.5	25.5
relevance and interest	7.5	14.5
more hands on	10	
smaller classes/groups	5	1.5
better notes	5	
board explanations	2.5	3
more challenge	2.5	1.5
giving students more independence	2.5	1.5
less worksheets/more interactive, group work, allow time for discussion		12.5
spend longer on each topic and allow time		8
Set/track		5
not set/track		1.5
No response	5	19

While the most common response at year 7 and 8 was games and activities at years 9 and 10 over a quarter of the students indicated a change in the approach of the teachers. The following quotes illustrate the types of responses and show the seriousness with which the students undertook the task and their comfort in saying what was salient for them at the time.

> Teacher should explain clearly, good communication skills. Interesting work in years 7 and 8. They wanted you to do your best. I felt not forgotten. Need teacher interest, relevance to real day life (yr 10).
>
> Explain it in more than one way for students who think differently. Teacher frustration - if students don't understand it will lead to student frustration. Teacher needs to find time to catch up with them and explain. Students get distracted and distract others, leading to a non-learning environment (Yr 8).
>
> Make it more interesting, applied in real life situations. Need a teacher passionate about maths who enjoys teaching it (Yr 9).
>
> Feel less negative about your teacher. Stay on topics longer and don’t move on too quick (Yr 10).
>
> Teachers should listen to student needs (Yr 9).

CONCLUDING COMMENTS

This last comment raises a very important aspect of the study. The teachers were surprised by the information they gained from interviewing the students. They listened to them in a different way to normal. They were instructed specifically not to "teach" but rather to use this opportunity to listen to students and gain as much information as possible in the short time that they had. The experience the teachers had of interviewing the students themselves made the results of the study more interesting to them. It also meant that they were positive about responding to the differences they saw between the year levels and more willing to change. The opportunity for teachers to sit one-on-one with students and listen to them is a valuable one for the teachers and the whole school.

Following the interviews the data analysis was discussed with the teachers. Particular aspects to which they wished to attend included the fact that at year 7 and 8 so many of the students mentioned mathematics itself in the form of topics and strategies while at year 9 and 10 overall there was more attention on the teacher. This reliance on the teacher was of concern. It was felt that the students needed to be given more independence, perhaps with a change in culture providing more opportunities for students to participate in decision making (Athanasiou & Phillippou, 2006).

Another aspect highlighted was the focus on problem solving at years 7 and 8 while at year 9 and 10 the students seemed to see mathematics more as set procedures with a set single answer. Also the advice from the year 9 and 10 students was clearly for more reality in the tasks studied. A team of the teachers have put strategies in place to trial as they use the information from students to inform their practice.

Having completed some interviews themselves also drew attention to misunderstandings in mathematics that led to some changes in approaches to specific topics but that is another section of the study.

Listening to students talk about their learning in mathematics and the learning environment that they see as assisting them learn can inform teaching and make a difference.

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INTENTIONALITY AND WORD PROBLEMS IN SCHOOL DIALOGUE

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Abstract. The aim of integration of word problems into mathematics education is teaching the pupils how to mathematize real situations. When solving a word problem, the pupil should go through: encoding, transformation, calculation stage and storage including verification. This requires employment of relatively demanding cognitive processes. However, what happens when solving the problem within a mathematics lesson led by a teacher is lowering of cognitive demands in consequence of the teacher's stimuli and interference. In the contribution we will demonstrate which cognitive processes should be in play and how they can be distorted in school dialogue.

INTRODUCTION

In the conditions of institutionally constituted school education, it is the curricular materials that define the set of goals on various levels of explicitness, obligation, generality and comprehensibility. Their projection in the teaching and learning processes is hindered by a number of factors (the distractive influences in teaching, low pupils' motivation, teacher's competence etc.) It is often extremely difficult to uncover why the projected aims are not achieved in education. One of the possibilities is that the original aim disappears in the process of teaching and is replaced by other aims (e.g. the teacher must finish the lesson in time, he/she wants to involve more pupils in the dialogue, he/she wants to explain the solution of the problem in such a way that it is understood by more pupils etc.) This can happen even in the case that the teacher identifies with the curricular aims and is experienced enough for their realization.

Comparison of the didactic analysis of the curricular topic "solving word problems" and realization of a selected example in teaching serve as demonstration of the fact that the teacher unconsciously lowers the intellectual demandingness and thus he/she deviates from the original aim of inclusion of the topic in the curricula. The solving process of the word problem (planned and realized) is broken down into single steps which are assessed from the point of view of cognitive processes necessary for their solution. The comparison is based on the classification according to the revised Bloom's taxonomy of cognitive processes (Anderson, Krathwohl, 2001, overview in Annex).

In the contribution we are going to discuss the following questions and statements: (a) Generally, when including a word problem in the teaching process we expect the pupil to apply the learned solving algorithms in a new situation (category 3 in the revised Bloom's taxonomy – see below). However, the solving process presupposes

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the execution of certain substeps, some of which may belong to different categories. In the contribution we ask which categories of cognitive processes the pupil should go through to succeed in completion of the individual phases of the solving process. (b) If the word problem is solved at school under the supervision of a teacher, the solving process is broken down by the teacher into smaller segments (tasks). The solution of these tasks may need only lower rated cognitive process than presupposed. It is a question what the consequences of lowering the category of cognitive processes are with respect to understanding mathematics and to pupils' behaviour when solving a word problem.

DATA COLLECTION

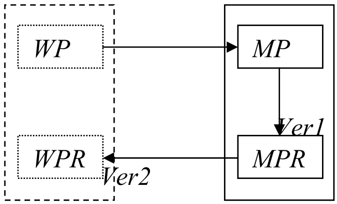
In the research reported we analyse solutions of two word problems from a two videotaped lessons taught in different schools (CZ1, CZ2). The lessons were a part of video recordings of ten consecutive lessons on the solution of linear equations and their systems in the 8th grade (pupils aged 14-15) of two lower secondary schools in the Czech Republic. The method of data collection is based on the Learner's Perspective Study (LPS) framework (Clarke, Keitel, Shimizu, 2006) and consists of the video recordings of a lesson and postlesson interviews with the teachers and a separately monitored pair of pupils. Both teachers are experienced and respected by parents, colleagues and educators, although their teaching strategies differ. In our previous analysis (Novotná, Hošpesová, 2008) we discovered that CZ2 teacher mostly focuses on the question "How?"; she approaches the solution of every problem as a new one and does not deal with it in the context of what the pupils may already know from solving other word problems. But she seems to trust her pupils' independent discovery. CZ1 teacher pays more attention to the question "Why?" and tries to plant the new knowledge on her pupils' previous knowledge.

WORD PROBLEMS, THE AIM OF THEIR INTEGRATION INTO EDUCATION IN THE CR

In mathematics education literature, the concept of a word problem is not formalized in a unified way. For the purposes of this research, we understand as word problem (in agreement with Verschaffel, Greer & De Corte, 2000) "verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement". The solving process of a problem consists of a sequence of correctly ordered operations, decisive steps etc. starting from the data and relations among them and finishing by finding the unknown(s).

Use of word problems in mathematical education

Solution of word problems belong to one of the few domains of school mathematics necessitating mathematization of situations from the assignment and the return to the semantic context after solving the mathematical model (see Fig. 1 from Odvárko, 1990):



P ... word problem
MP … mathematical problem
MPR … result of MP
WPR … result of WP
Ver …verification

Fig. 1 Schema of the word problem solution

The tasks for the pupil is to discover or construct the mathematical model. The pupil's available algorithms are of no use at this stage. The themes of the problems can but do not have to favour the solving process (see e.g. Sarrazy, 2002). Therefore, the didactical use of word problems appears to be a suitable means, often used by teachers, for developing their pupils' mathematical competences. They allow them to develop their capacity to use their mathematical knowledge outside of the domain of mathematics and hence help to recognize, understand and memorize mathematical concepts, methods and results (Blum & Niss, 1991). Moreover, they contribute to their ability to select necessary information, work in a creative way and to develop their heuristic procedures (Verschaffel, Greer & De Corte, 2000).

Word problem solving procedure – theoretical analysis

In the paper the stages of word problem solving process (Novotná, 1997) are used: (a) encoding stage (grasping the assignment), (b) transformation stage (transfer to the language of mathematics and creation of a mathematical model), (c) calculation stage (mathematical solution of the problem including mathematical verification of the obtained results), (d) storage stage (transfer of mathematical results back into the context including contextual verification of the obtained results).

For the analysis of the cognitive processes necessary for overcoming the stages of word problem solution we used the categories (in the following text in italics) of the revised Bloom's taxonomy (Anderson & Krathwohl, 2001): *1 Remember*, *2 Understand*, *3 Apply*, *4 Analyze*, *5 Evaluate*, *6 Create*.

Encoding stage: Coding of the word problem assignment is the transformation of the word problem text into a suitable system in which data, conditions and unknowns can be recorded in a more clearly organized and/or more economical form. Cognitive processes needed for this stage belong to the category *4 Analyze*. It includes *differentiating, organizing*, and *attributing*.

Transformation stage: This stage covers mathematization of the assigned problem structure. It includes two categories: *4 Analyze*, namely *organizing* and *3 Apply*, namely *implementing*.

Calculation stage: This stage covers mathematical task solution and the (explicit or implicit) execution of verification of the obtained mathematical results. It is covered by the category *3 Apply*, namely *executing*.

Storage stage: Mathematical results are transformed back to the semantic context of the assignment. They are confronted and verified in the context environment. This stage corresponds to the category *5 Evaluate* including *checking* and *critiquing*.

The process of problem solving in school teaching

In the previous text, there is an analysis of an ideal progress of the word problem solving. However, the results of our experiments carried out within the frame of LPS and other long-term observations signal that in school teaching it may happen and often happens (be it consciously or unconsciously) that the process diverts to activities demanding other categories of cognitive processes, often lower than the anticipated ones.

Our discussion is based on examples of solutions of two word problems. The first sample – a more comprehensive one – is from the school CZ1 (L3 18:53 – 22.42). The following transcript of the dialogue proves that the teacher entered the solving process (T1- T17) with the intention of sustaining the understanding of the solution of a difficult problem. The assignment of the problem was: *A 95 m long train is crossing a bridge at the speed 45 km/h. It takes 12 seconds from the moment when the locomotive drives onto the bridge until the moment when the last carriage leaves the bridge. How long is the bridge?* The transcript includes all stages of the solving process: encoding (1 – 24), transformation (25 – 32), calculation (33 – 34), storage (36 – 39).

1	T1:	We want to calculate how long the bridge is. What is the situation? Describe it, Lucka. A train drives onto the bridge and we know how ...
2	Lucka1:	... long it is.
3	T2:	And we lose interest as soon as the end of the train leaves…
4	Lucka2:	...that bridge.
5	T3:	Let’s make a drawing. The bridge (the teacher draws a line segment on the board). The train (the teacher sketches a locomotive) is driving onto it. Now we are becoming interested. And we lose our interest at this moment (the teacher is sketching a locomotive with carriages). Why is this drawing so important? Lenka.
6	Lenka1:	Because we have to calculate the trajectory.
7	T4:	Yes, the length of the distance covered. And what do we know about it, Michal?
8	Michal1:	Only the length of the train. 95 meters.
9	T5:	And there were some other data in the text.
10	M2:	There was also the speed of the train 45 km/h.
11	T6:	45 km/h (she writes it down into the drawing on the board). … This is also interesting because the units don’t correspond here. Well, let’s decide which we’ll change, Denisa.
12	D1:	The speed.
13	T7:	Yes, it’ll probably be most handy to change the speed. So how do we convert kilometers per hour to meters per second?
14	Pupil:	Divided by 3.6.

Steps 15 – 20 were devoted to clarification of conversion of speed from km/h to m/s.

21 T8: What's the result?

22 P: 12.5.

23 T9: Yes, 12.5. Let's write it down here. (She writes it into the drawing on the board.) The unit? Hanka.

24 H: Meters per second.

25 T10: Great. So the units are OK now. Now it shouldn't be a problem. How would you continue solving this problem?

26 P: The length of the train is x + 95.

27 T11: Not of the train, of the trajectory.

28 P: We know the speed, we know the time 12 seconds.

29 T12 How would you calculate it? The length of the bridge. The trajectory is …

30 Vítek3: Velocity times time

31 T13: And that's it. Dictate it to me, David.

32 David1: *X* plus 95 equals 12.5 times 12.

33 T14: Calculate it, you have the calculator. ....Peter, what's the result?

34 David2: 55.

35 T15: Look at the question. What was the question? Peter.

36 P: How long is the bridge?

37 T16: We know how. But we haven't verified the result. The result is neat, a whole number, but how shall we verify it? Adam.

38 P: I would calculate the time.

39 T17: Why not, yes. We won't do it now. We were checking each other while solving.

The teacher paid most attention to encoding. That is understandable because this word problem is complicated by the conversion of units and by the necessity to realize what represents the "trajectory" in this problem. As stated above, this stage should focus on an analysis and evaluation of the situation described in the word problem (categories *4 Analyze* and *5 Evaluate*). In this phase the teacher intervenes in the solving process ten times. In some cases she uses intonation to signal that she expects the pupils to complete her statement (T1, T2, T4, T5). These signals belong to category *2 Understand*, namely *interpreting*. Steps 15 – 20 that we have left out of the transcript and T8 - T10 are of similar nature. We regard the tasks in T3 and T7 as the most difficult moments of this stage. In step T3 the teacher herself sketches the drawing of the problem and asks the pupils to explain why the picture is important – category *2 Understand* (*explaining, constructing models*). In T6 she asks the pupils to decide which units to change – category *4 Analyze* (*differentiating*).

The most important stage of problem solving is transformation. It is necessary to state that the teacher initiated transformation already in the stage of encoding. Steps T10 and T12 are most demanding here (category *3 Apply – implementing*). Pupils react fast although the wording of their answer needs to be refined (in T11 the teacher asks for *reformulation* – category *2 Understand, interpreting*).

The calculation itself was carried out very fast in just two steps that can be labeled as application of a known algorithm, a calculation in this case *3 Apply* (*executing*).

The stage storage was again relatively brief. In T15 the teacher recalls the question from the word problem assignment (category *1. Remember* - *recalling*). What could have been most difficult was the contextual verification of the obtained result, which belongs to category *5 Evaluate*. However, in this case the pupil only traces new contexts (*4 Analyze – differentiating*) and proposes the procedure of the verification by verification by calculation, which was, however, not carried out.

We found an example of solving of a word problem dealing with movement also in the school CZ2. In this case it is a very straightforward problem from the introductory lesson to the topic "word problems dealing with movement": *The road from Adam to Eve is 5 km long. Adam has a date with Eve and sets off to meet her at the speed 6 km/h. At the same time Eve sets off to meet Adam at the speed 4 km/h. How long will it take before they meet and how many km will Adam have covered?* Due to the straightforwardness of the problem its solution was very fast, which can be clearly seen in the following transcript (CZ2 L8 22:20-24:31).

1	T1:	Jirka, what do you remember of what I've just read?
2	Jirka1:	There are Adam and Eve walking against each other.
3	T2:	That this time they are walking against each other from different places. So let's sketch it: here is Adam, here is Eve (the teacher draws a line segment with arrows in its end points on the board). I'll read the following passage from the assignment: the distance between them is 5 km. There's no reason why not to write at the top that the distance between them is 5 km. (She writes it on the board.) Let's continue: Adam is walking at the speed 6 km/h, Eva 4 km/h. Try to record it into the drawing somehow. (The pupils make arrows into the drawings in their exercise books.) Radka, do you think that Adam will have covered shorter or longer distance?
4	Radka:	Longer, because he is walking faster.
5	T3:	Could anyone say at this point how long it'll take before they meet?
6	Pupils:	Half an hour.
7	T4:	Some of you may have solved it using relative speed. What is the relative speed they are moving against each other?
8	Pupil:	10.
9	T5:	Every hour 10 km. I add 6 and 4, ten. And because there are only 5 km between them, they will meet in half an hour.

In this case the teacher intervened into the pupils' solving process only five times. Her every intervention can be perceived as the stimulus for transfer to the following stage of the solving process. In T1 she initiates encoding and she only asks for recollection of the problem situation (category *1 Remember – recalling*). In T2 she finishes the encoding stage on her own and by the question in the end of T2 she transcends to the transformation stage. The question may be regarded as an appeal for a comparison of the two speeds (*2 Understand – exemplifying*) and to expression of a simple judgment (may be a hint of 5 *Evaluate – critiquing*). T3 at the same time invites transformation and calculation stage (*3 Apply – executing*). As far as the cognitive processes needed for answering question T4 are concerned we can assume that they are an appeal to the use of algorithms that the pupils are already familiar with (*3 Apply – executing*). The storage stage is carried out in T5 by the teacher herself.

Discussion and conclusions

In the contribution we showed the difference between the suppositional level of the cognitive processes necessary for the completion of the individual stages of the solving process of a word and the reality in school teaching. We are fully aware of the fact that lowering the categories of cognitive processes happens whenever pupils do not solve the problem on their own but communicate with the teacher and/or their classmates during the solving process. The impact of such lowering may not necessarily be negative. In our point of view, it may serve as a solid base of understanding in early stages of presentation of the problem. It is however a question whether the teachers are always aware of this lowering and whether they do not stay in the position unnecessarily too long. When analyzing other solutions of word problems in our data, it became evident that especially the teacher CZ1 uses such approach whenever she communicates with her pupils about the solution of the problem. In the postlesson interview we were told that her goal is the state when all her pupils understand the solution. It turned out that the pupils expect her "help" because one of the pupils from a separately monitored pair said while analyzing the word problem: "I'm really glad that we are not alone in it (in the solving process)."

The consequence of lowering the cognitive demandingness may be that the pupils rely on the teacher's "help". It may happen that they routinely repeat the learned process, often without deeper understanding. They do not attempt to find their own suitable solving strategies. The learning process fails to work with one of the key elements – mistake, its recognition and elimination.

When analyzing the dialogues we realized the connection with frequent inclusion of the Topaze effect (Novotná, Hošpesová, 2007), which is usually motivated by the teacher's effort to have the pupils take active part in the dialogue with him/her.

Several questions arise in the context of the use of the Bloom's taxonomy for evaluation of the cognitive level of the assigned problems in mathematics education. In our analyses we lacked a category which would demand the execution of a routine calculation – its placing into category *3 Apply* (*executing*) does not correspond with its cognitive demandingness. We propose a new subategory *executing calculations* in *2 Understanding*. However, we believe that the use of the revised Bloom's taxonomy is a suitable method, especially in qualitative research.

We find it useful to employ the taxonomy in pre-service and in-service training, e.g. a pre-service teacher first carries out the "theoretical analysis" of the solving process and then compares it with the reality. A group of in-service teachers uses the analysis as the base for reflection on a taught lesson. In both groups we can anticipate a much higher degree of sensitivity in the subsequent self-reflection on one's own teaching.

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HOW EARLY CHILDHOOD PRACTITIONERS VIEW YOUNG CHILDREN'S MATHEMATICAL THINKING

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Interviews were held with 64 professionals in childcare centres. This paper reports on their responses to 3 questions seeking their perceptions of mathematical thinking in very young children. Generally, the interviewees were found to have a good sense of mathematical concepts relevant to babies and toddlers, and they cited evidence of young children's mathematical development. It is concluded that this practical knowledge would provide a strong foundation for further professional development.

INTRODUCTION

Mathematics is a core component of cognition, and "robust mathematical learning by all young children is a necessary base for later learning" (Clements, Sarama, & DiBiase, 2003, p.105). Many researchers have recognised that children bring to school powerful mathematical knowledge, skills and dispositions (e.g., Baroody, 2000; Clarke, Clarke, & Cheeseman, 2006; Ginsburg, Balfanz, & Greenes, 2000), but it is important that people who care for young children and design their activities are also aware of this. It is vital that they can identify moments with potential to lift the level of children's thinking through scaffolding (Siraj-Blatchford & Sylva, 2004).

The project *Mathematical Thinking of Preschool Children in Rural and Regional Australia: Research and Practice* (Hunting, Bobis, Doig, English, Mousley, Mulligan, Papic, Pearn, Perry, Robbins, Wright, & Young-Loveridge, 2008) comprised a team of researchers representing 10 universities from Australia and New Zealand. The project aimed: (a) to investigate views of childcare and preschool practitioners with respect to young children's mathematical development, and (b) to review recent research literature dealing with the mathematical learning and thinking of young children in order to make this information accessible to practitioners.

This report presents results from Questions 3-5 of an interview designed for the project. These questions focused on mathematical thinking in the earliest years:

> Q.3: At what age do you start to see children thinking mathematically?
>
> Q.4: Can you give an example of mathematics learning that you have observed in your Centre recently? Was this planned or incidental?
>
> Q.5: Think of a child who has a good grasp of mathematical knowledge. Describe some of the mathematical things that this child does.

Such questions are important because awareness of mathematical thinking is likely to lead to the provision of activities and opportunities and interactions that encourage development of relevant mathematical processes, concepts, and language. It is also vital to know where to start with planning professional development.

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METHOD

Preschool case study visits conducted by *Project Good Start* (Thomson, Rowe, Underwood, & Peck, 2005), survey instruments used for assessing the beliefs and attitudes of primary and/or preschool teachers (e.g., Fennema, 1989) and other instruments used to collect views on early childhood learning tools were reviewed and adapted into an interview instrument consisting of 18 questions. The questions represented 5 broad themes related to developing mathematical concepts: children's mathematics learning, support for mathematics teaching, technology and computers, attitudes and feelings, and assessment and record keeping (Hunting et al., 2008).

Using a database of Day Care and Preschool Centres provided by the Australian National Childcare Accreditation Council, a sample of 60 facilities was chosen to include those catering for children from remote, rural, low socio-economic backgrounds, and Indigenous communities. The centres represented those of Australian states of Victoria, New South Wales, and Queensland, proportionally according to their populations, and 64 interviews with their staff were recorded. NVivo™ was used to handle the transcribed interview data. Analysis sought frequencies of types of responses and representative examples. Most of the interviewees had a 2-year child care diploma. All planned their "room's" or the centre's programs of activity, which involves writing formal activity programs.

RESULTS AND DISCUSSION

When children begin to think mathematically

Responses were grouped into three broad categories: those who thought children start to think mathematically at less than 12 months, between 12 and 24 months, and older than 2 years. In the <12 month category (n=30), 6 practitioners believed mathematical thinking began from birth, and 13 believed preschool children's mathematical thinking is evident in babies. A practitioner said:

> I am in the nursery so I see them problem solving, as in if something falls over, or how do I pick it up, and how will that sort of work. Yeah, I think from when they are born, they are solving problems and that sort of thing. (Peninsula-Gippsland VIC)

In the 12-24 month category (n=18), 10 practitioners believed mathematical thinking was present in children aged 12-18 months. Reasons offered included:

> Children like to start counting when you put their socks on and you say one and two and they try and copy that, or going down a couple of steps and counting each one as you go. One of our mums was just telling me about her one year old the other day that the little girl has learnt to go "one more, one more", and she goes like this so you know that is something that she has picked up on her own—and at one year old (Tamworth NSW).

In the >24 months category (n=18), 7 said mathematical thinking begins by age 3, 5 thought it to be about aged 4, and the others at school age.

> We don't have mathematics as such. They learn to recognise numbers but counting and sums are for the school. (Toowoomba, QLD)

Overall, a total of the 58 practitioners (88%) thought that mathematical thinking starts before the age of 3, and many identified mathematical activity in babies and toddlers. Such a finding is surprising, considering that evidence of infant and toddler mathematical thinking is not incorporated in the preparation of certified practitioners or the meagre in-service opportunities available in Australia. Hopefully the task of providing a theoretical rationale for the mathematical development of preschool children would be made easier due to the fund of personal experience and observations early childhood practitioners can draw on.

Examples of recently observed mathematics learning

Responses were classified into 5 broad categories, based on the *Principles and Standards for School Mathematics*' (National Council of Teachers of Mathematics, 2000) content strands (see Table 1).

Category	Response	Frequency	Example
Number		**40 (32%)**	
	Counting	23	… he will count how many steps. He can count by twos, too.
	Basic operations	11	… bouncing on the beam outside singing *Five Little Monkeys* … one was falling off so they were subtracting
	Fractions	2	When cooking I will say whether it is half, whether it is full, quarter
	Naming numerals	2	… their numbers, their number recognition, counting and sharing.
	Ratio & proportion	1	I bought these shapes (for the) projector … little colored see-through shapes
	Sharing	1	… sharing out the drinks, the playdoh
Algebra		**25 (20%)**	
	Classifying	9	… a sorting box with two different sorts, … round counters with 5, 6 colors and (others) were square things.
	Ordering	7	… he knows to collect certain things and put them all in order.
	Matching	4	… you match the shape with the other shape on the board.
	Patterns	4	I said, "You have blue, blue, red, red, blue, blue. What's next?" She said "Red, red".

	Grouping	1	Color grouping or similar symbols or (other) things—grouping them together.
Geometry		**23 (18%)**	
	Shape recognition	9	… a couple of my little 2 year olds … picked out the circles from the collage.
	Block building	8	… we were measuring how many of the smaller blocks made the larger blocks.
	2-D shapes	4	… I will put out a square or a round container, so it is working out what is going to fit.
	Making 3-D shapes	1	They … built in three dimensions, forming cubes and pyramids.
	Position & orientation	1	… we have been asking … where they want to put their hand-printed leaf—whether they want to put it up high, down low, or in the middle.
Measurement		**33 (26%)**	
	Volume	14	We say "Put these things into a box. How can you make them fit?"
	Length	9	I was … saying "You have bigger feet than me" (joking), and they have said, "No, mine are smaller than yours".
	Weight	5	… (using) scales and weights, how we measure things, and how we make estimations, and whether things are going to be heavier or lighter.
	Comparing and ordering	3	Children will often measure themselves and each other and within a little group they'll sort of work out who is the smallest and who is the tallest.
	Estimation	2	We have great discussions on whether this object will sink or not, so there is that estimation there.
Data analysis		**1 (1%)**	
	Graphing	1	We have been talking about what pets we have … children to stand up if they had a certain pet … and then they're counted and then put that up.

Other		**4 (3%)**	
	Maths activities on the computer	2	Objects and numbers—I have seen them do it with blocks and cars—put them in sizes.
	Problem solving	2	We have an apple tree board and the number at the bottom of the trunk can change. If I put a number 5 in and then I change it to a 7, a lot of the children are picking up that they don't have to remove all of the pegs—they can just add 2 more and so they are counting on from 5, and counting "6 and 7".

Table 1: Observations related to 5 content strands in mathematical learning

Practitioners interviewed were able to provide examples of both incidental and planned mathematical activities across the breadth of the 5 major content strands that form the school mathematics curriculum, even though they were talking about children aged well under school age. Their practical experience fits well with the findings of researchers who have examined young children's mathematical thinking and learning (e.g., Ginsburg, Cannon, Eisenband, & Pappas, 2006). This knowledge would provide a strong foundation for development of more formal understandings of early child development in terms of mathematical knowledge, skills and language—through in-service education, further study, or professional reading distributed by relevant bodies.

Activities of children demonstrating "a good grasp of mathematical knowledge"

Responses here were categorized as content oriented or process oriented. Table 2 shows the variety of content related categories, and Table 3 shows the types of process related categories mentioned.

Category	Response	Frequency	Category	Response	Frequency
Number		**47 (48%)**	**Geometry**		**19 (20%)**
	Counting	20		Block building	6
	Addition	6		Completing puzzles	3
	Identifies numbers	7		Recognizing shapes	3
	Subtraction	3		Representing shapes	3
	Sharing equally	2		Arranging	2

				shapes	
	Writing numerals	2		Drawing in proportion	1
	Fractions	2		Translating from 2D to 3D	1
	1:1correspondence	2	**Measurement**		**7 (7%)**
	Ordering numerals	1		Measuring	4
	Rote counting	1		Reading digital time	1
	Mental arithmetic	1		Volume	1
Algebra		**23 (24%)**		Weighing	1
	Making patterns	9	**Other**		**1 (1%)**
	Sorting and classifying	7		Engineering	1
	Ordering	4			
	Matching	3			

Table 2: Content-related activities of children with “a good grasp of mathematics”

Category	Frequency	Example
Problem solving skills	5	She is exceptional at doing puzzles … figuring out where things go and trial and error and all that sort of thing and she has been doing that since the toddler room.
Persistence	2	…. and they keep going and stay on the task … instead of losing interest they will actually keep going.
Explaining	1	…. with Sam doing the numbers he was showing the kids the counting, explaining to the others how to count.
Noticing	1	… would notice the difference and say “That is bigger than there” or “That won’t fit in there”, so they are thinking about it before it is happening, during and after.
Interpreting	1	Talking about the graph (explaining) circles representing family members, lines connecting members who love each other.
Trial and error	1	A lot of is that—basically, they do a lot of trial and error.

Well-developed vocabulary	1	A lot of language is involved with those children. They can explain things, describe things to you. They have the language, the mathematical language, when talking.

Table 3: Process-related activities of children with "a good grasp of mathematics"

The interviewees generally demonstrated a creditable awareness of children who seemed to have a good grasp of mathematics from a content perspective. They seemed less aware that mathematical proclivity could be demonstrated by means of processes children use as they engage in mathematical activity, so this is an area to focus on for professional development. While these processes will improve with opportunities in kindergarten and school and as their language develops, they also learn at a young age from carer's expectations, modelling and feedback when they show signs of these characteristics. Such characteristics can be developed in the right environment (Perry, Dockett, & Harley, 2007; Frakes & Kline, 2000; VanDerHeyden, Broussard, & Cooley, 2006). It is important to support this development in young children because approaches to learning demonstrate positive relationships with later growth in mathematical skills (DiPerna, Lei, & Reid, 2007).

CONCLUSION

In relation to early childhood practitioners' views of young children's mathematical thinking, the project *Mathematical Thinking of Preschool Children in Rural and Regional Australia: Research and Practice* has provided evidence that there is a general awareness that such thinking starts at a very early age. Many practitioners responsible for childcare were able to identify a wide range of mathematical concepts and skills, and were able to give examples of occasions when these had been observed. Some were able to give evidence of individual children who seemed to be developing strengths in this area, although such evidence was more content-related than process-related. It is clear that there is a good foundation for further professional development in the field, and such activity is important because good mathematics curricula and teaching in the earliest years can close equity gaps (Clements & Sarama, 2008).

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COUNTING VS. MEASURING: REFLECTIONS ON NUMBER ROOTS BETWEEN EPISTEMOLOGY AND NEUROSCIENCE

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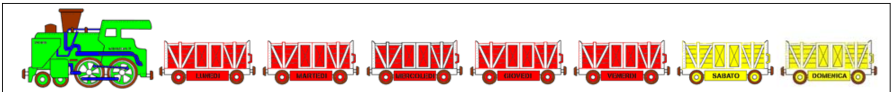
Abstract*. Pointing on some neuroscience results, concerning the presence in our brain of two distinct prelinguistic systems for processing numbers, we compare different views on the cognitive roots of numbers, in epistemology, psychology and math education. The conclusion is that the two distinct processes of counting and measuring should be better integrated in early math education. To this purpose, an example is reported and commented, of a revealing cognitive behaviour of very young children.*

PROLOGUE

A K-degree class episode. The 4 year-old children are engaged in an activity to discover how many days in a week are school-days. Of course, at the beginning their counting competence is very poor. So, how to count the passing days? The idea is to put "something" into a box every day, and then to count these things instead than days. Their first attempt is to put a handful of construction blocks every day, but on Friday they realize that this way is quite inadeguate. After a long trouble, they decide to put each day the same "quantity" into the box, namely two construction blocks. Then, on Friday, they turn the box over and count the blocks "two by two".

During the activity, Camilla says: "*Let's take all the week days and put two of them away*". Notice that a "week train", like in Figure, with five red wagons (school days) and two yellow wagons (holidays) is standing on the class wall.



INTRODUCTION

Recently, the research domain of neurosciences is expanding at extraordinary speed, and nowadays the need for taking into account its results is widely recognized by educational researchers. For example, in (Tall, 2004), the author identifies these researches, from those concerning innate numerical competencies to those on subjects engaged in complex cognitive tasks, as an emerging strand in cognitive theory. Only two years later, Campbell comes to imagine the birth of a new area of educational research, the *educational neuroscience*, "*that is both informed by the results of cognitive neuroscience, and has access to the methods of cognitive neuroscience*" (Campbell, 2006, p. 260).

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Our aim is less ambitious, as we only try to utilize some neuroscience results in order to interpret learners' cognitive behaviours and to design didactic mediations that really rely on learners' cognitive resources, and, more generally, to partially re-read or revise classical cognitive models. As a first step in this direction, in a previous paper (Iannece *et al.*, 2006) we have shown how some Houdé's results based on brain imaging research suggest a critical reflection on some assumptions of the classical cognitive psichology concerning logical thinking. Here, we face one of the more debated question: the birth and the development of the concept of number at school. For the number concept we refer to a structural view of numbers, in the sense of (Sfard, 1991), and to the "number sense" as described in (Arcavi, 1994).

The existence of two "primitive" systems for processing quantities, revealed by neurophysiology studies, suggests to re-think how the number concept develops in children. After a short description of the two systems, we propose a comparison of some of the principal cognitive and epistemological theories about numbers, in order to show how their views correspond to one or another of the two core systems. Then, we argue about the possibility/necessity that the two natural ways of processing numbers, rooted in our brain systems, develop together, thanks to a careful didactic mediation, as two sides of the same coin. To this purpose, in the last section we present a gestalt schema, where the two aspects of numbers and some of their fundamental properties are captured in a unified way; then we illustrate how such a schema can work as a semiotic mediator for a structural grasping of numbers, and, moreover, how the awareness of the brain way of working can be a powerful tool in the hands of teachers to understand and guide their pupils.

THE DIALECTICS BETWEEN THE DISCRETE AND CONTINUOUS ASPECTS OF NUMBERS

The discussion about the cognitive roots of numbers develops along our whole cultural history, involving several different research domains. In the sequel, starting from some neurophysiology results, through a short analysis of some epistemological and psychological theories, we argue that consolidated views about the process of acquisition of the number concept can and should deserve further reflections.

Neurophysiology

Nowadays, it is widely acknowledged that in the human brain, but also in the brain of many superior animals, there are two distinct systems for processing numbers (see, e. g., Feigenson *et al*., 2004). Within an evolutionary model of the brain (Changeux, 2002), both these core systems are pre-linguistic resources, developed along mankind history, via an "epigenetic" process, as effective tools for interpreting and acting on the external world, in order to guarantee the survival of the human species.

The first system is specialized in recognizing the numerosity of small groups of objects (say, up to four), by the so-called "subitizing", while the second one provides "*an analogical representation of quantities, in which numbers are represented as distributions of activation on the mental number line*" (Dehaene, 2001, pp. 10-11[1]). What is specially interesting is that the second system is activated not only for comparing and manipulating continuous quantities, but also for perceiving and processing discrete quantities in an approximate way[2].

Indirect evidence of the existence of this sort of "mental line" rests on a big amount of observations (ranging from animals, to babies without linguistic competencies, to adults normally able in symbolic performances, to pathological cases) of two special phenomena: the distance effect and the size effect. In the words of Dehaene, "*the distance effect is a systematic, monotonous decrease in numerosity discrimination performance as the numerical distance between the numbers decreases. The size effect indicates that for equal numerical distance, performance also decreases with increasing number size. Both effects indicate that the discrimination of numerosity, like that of many other physical parameters, obeys Fechner's Law.*[3]" (*ibid.*, pp. 6-7). According to a Dehaene's metaphor, our "mental number line" works like an *accumulator*: that is, a schema that both allows a static comparison of quantities and identifies numbers as dynamic results either of storing separate arbitrary units or of adding/subtracting two approximate quantities. A picture like the following

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well captures, in our opinion, the fundamental properties of the "accumulator". In fact, it shows that:

a) the additive structure is embodied in number sense;

b) counting and measuring are two strongly intertwined processes;

c) the cardinality of any state and the effects on it of additive transformations are simultaneously recognized;

d) the mental number line is nonlinearly compressive, with pairs of numbers lying closer together as their magnitude increases.

To sum up, we are provided with two different sources of number sense, two irreducible perceptive "moves", that can be contrasted with two complementary aspects of the reality, the discreteness of objects and the extension of magnitudes[4]. To

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[1] The pages refer to the PDF version downloaded on January, 10, 2008, from the website http://www.unicog.org/biblio/Author/DEHAENE-S.html.

[2] Recently, a suggestive hypothesis has been formulated (Walsh, 2003), namely that the second brain system is based on the same mental circuitry that has been elaborated by the human species to perceive and conceptualize time and space (the "locus of a common magnitude system").

[3] Also known as Weber-Fechner's Law, states that the magnitude of a sensation is proportional to the logarithm of the intensity of the stimulus causing it. It has been recognized also in the discrimination of numerosity since many years (see, e. g. (Changeux & Dehaene, 1993)).

[4] According to an evolutionary model of the brain, that is our philosophical option, it is almost obvious that our mental structures appear adequate, not to say "isomorphic", to reality structures.

trust these results means to recognize that at the origin of the number concept there are two distinct but correlated counting processes: counting discrete objects as a linguistic evolution of subitizing and counting by measuring. Moreover, the latter would be predominantly activated, because it is language-independent and is spontaneously used in everyday experience to evaluate distance, time and so on.

Psychology and Math Education

As said before, the cognitive roots of numbers have been investigated from several points of view. Psychological studies, form Piaget onward, as well as math education studies, prevalently assign, with minor differences, a primitive and priority role to natural numbers, due to the original action of counting. But it is also possible to recognize a different approach, in such a way that we could speak of a duality which resembles that observed at a neuronal level.

Here we refer, for the first approach, among others, to (Sfard, 1991) and (Lakoff & Núñez, 2000). In her cognitive reconstruction of the number concept within a dialectic process/object, Sfard proposes a schema where the counting process constitutes the starting point, whereas the measuring process appears only in a later step, when rational numbers are generated. On their part, also Lakoff & Núñez focus on natural numbers, since they assume the subitizing as the root for number concept. Then they utilize four *grounding metaphors* to build the whole arithmetic. One of these metaphors (the *measuring stick metaphor*) relies on spontaneous activities of measuring that could allow to introduce a wider range of numbers.

A different approach is followed by V. V. Davydov. In (Davydov, 1982) the genesis of the number concept is rooted in the experience of measuring continuous quantities. The notion of quantity comes from comparison of elements of a given class (e. g., lenghts of segments, amounts of water, weights, etc.), while measuring means to relate a given quantity with a part of it, assumed as a unit. Counting itself may be conceived as the particular measuring process of discrete objects, whence the sequence of natural numbers appears as just an example of quantity. But a deeper acquaintance with quantities allows children to enlarge their knowledge of numbers to include integers, rationals and reals.

Therefore, Davydov suggests that the practice with the properties of the quantities should precede in the early education the practice with natural numbers. He proposes activities where, once recognized and expressed an order relationship between two

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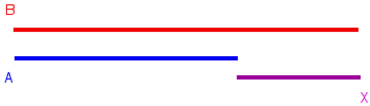
On the other hand, if a pure rationalistic point of view is accepted, where the existence of an external world is simply denied or declared inaccessible to us, the problem doesn't even occur; whereas, for an empiricist the problem is crucial. However, though being aware of the delicacy of the term *reality*, and of the fact that to speak of a resonance between individual cognitive resources and the real world (and the mathematics codified structures, too, as we did within our model of cognitive dynamics (Guidoni *et al*., 2005), and as we do here), would require an analysis of that term, we cannot go here into philosophical details (but see, e. g., (Eco, 1997) for a very subtle and detailed analysis of the argument).



quantities, the attention is focused on the quantity that has to be added to the smaller to obtain the larger one. He suggests also to transform a generic quantity into a segment, as an effective "intermediate strategy of graphic representation" (a semiotic mediator, in Vygotskijan words), something like the picture aside, where the crucial point is to recognize that X = B – A. The space metaphor allows to visualize the relationship between the two quantities A and B and to interpret subtraction as the formal description of the process of comparing A and B, rather than as a decrease: in a sense, thanks to the representation adopted, the object itself and its symbolic expression "*relate directly to the properties of the object. In a school subject, intermediate means of description have crucial significance because they mediate between a property of an object and a concept*." (Davydov, 1982, p. 237).

A similar view is proposed by Gelman & Gallistel, who point on neuroscience results. The authors identify the real numbers as closest to the originary perception of space: "*The evidence from experiments that probe the properties of numerical representations in non-verbal animals and humans suggest that there exists a common system for representing both countable and uncountable quantity by means of mental magnitudes formally equivalent to real numbers. These mental magnitudes are arithmetically processed without regard to whether they represent countable or uncountable quantity.* […] *Then the real numbers are the psychologically primitive system, not the natural numbers. The special role of the natural numbers in the cultural history of arithmetic is a consequence of the discrete character of human language, which picks out of the system of real numbers in the brain the discretely ordered subset generated by the nonverbal counting process, and makes these the foundation of the linguistically mediated conception of number*" (Gallistel *et al.*, 2006, p. 270).

History and Epistemology

Depicting an outline of the history of numbers, Sfard says "*much time elapsed before mathematicians were able to separate numbers from measuring process and to acknowledge that the length of any segment represent a number even if it cannot be found in the 'usual' way*." (Sfard, 1991, p. 12). In the sequel, we will try to read in a slightly different way the history of numbers, recognizing in it the effort, rather than to separate numbers from measures, to integrate two complementary aspects of numbers. In our opinion, this is a warning that a similar effort has to be made in the learning process in order again, not to separate, but to knowingly integrate them.

The dichotomy discrete/continuous has been pervasive in the whole western cultural history. Here we claim that in the history of mathematics numbers are hardly separable from geometry foundations. We can recognize at the origin of civilization two independent mathematical needs: measuring space and counting discrete objects. The Pythagorean school thought that natural numbers (and their ratios) could satisfy not only the latter but also the former need, until the discovery of incommensurable segments destroyed this claim. As a consequence, numbers and geometry divorced, and Euclid could build his monumental "geometry without numbers". For more than two thousand years arithmetic and geometry were treated as separate domains; but the need of conceiving numbers as measuring, not only as counting tools, slowly re-emerged. It is interesting to report a passage from Newton's *Arithmetica Universalis*: "*Per* numerum *non tam multitudinem unitatum, quam abstractam quantitatis cuiusvis ad aliam eiusdem generis quantitatem, quae pro unitate habetur, rationem intelligimus.*"[5] (Newton, 1761, p. 2). The XIX century ultimate stage of this process, the so called Arithmetization of Calculus, culminated with the construction of real (and complex) numbers[6] in terms of natural numbers, by means of successive steps: the hierarchy today predominant in the formal presentation of the various kinds of numbers; also referred to by Sfard in her schema. But in our opinion, this formal reconstruction, if succeeds in reducing several notions to one, doesn't exactly correspond to the cognitive roots of the various kinds of numbers. The priority given to natural numbers, either conceived as members of a special sequence (Peano) or as cardinals of finite sets (Cantor), emphasizes the discrete side of numbers, putting in shadow the continuous one. In other words, the reconciliation between arithmetic and geometry is successfully accomplished from a formal point of view, but is incomplete from a cognitive one.

It is perhaps noteworthy to notice that there is also a different formal approach to numbers. In fact, A. Kolmogorov, and other mathematicians, among which H. Lebesgue, following Newton's precursory idea[7], reversed the traditional path, giving priority to the domain of real numbers, wherein the rationals, the integers and the natural numbers are viewed as particular subsets.

So, also at an epistemological level, one could consider the two above approaches as the formal counterparts of the two primitive cognitive attitudes of our brain.

DISCUSSION AND CONCLUSIONS

We agree with Sfard in recognizing a repeated dialectics between processes and objects in the development of numbers. But we think that the relationships between the various steps are less linear than in her schema: namely, we recognize at the very beginning not only the process of counting, but also that of measuring: so to say, two legs for number sense-making.

The short analysis presented in the previous Section, about psychological, educational and epistemological views, shows that, in all cases, there is a choice of one of the two horns, discrete and continuous, with perhaps a predominance of the former, as primitive root of the number concept. In the wider frame provided by

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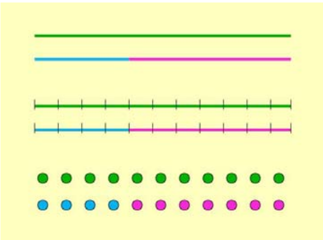
[5] "By a *number*, rather than a multitude of units, we intend the abstract ratio of any quantity to another one of the same kind, assumed as unit". (Our translation)

[6] It is not a case that the crucial idea in this process is Dedekind's translation of a geometrical property, the continuity of the line, into a numerical one, for rational cuts.

[7] The idea of real number as a ratio of two magnitudes has been also contended by G. Frege in his *Die Grundlagen der Arithmetik*, as a criticism to Dedekind and Cantor's theories.

neurophysiology studies, the two alternatives, corresponding to the two basic neural systems for perceiving and elaborating quantities, appear as two sides of the same coin. Just as the two brain systems cooperate to support our understanding and our action in the world, we believe that the two corresponding aspects of numbers should be developed together. According to Davydov's perspective, "*the ultimate aim of instruction in mathematics should be clear from the very beginning*" (Davydov, 1982, p. 230), the integration should be pursued in math education when firstly dealing with natural numbers. In fact, the counting process, starting from subitizing, rapidly develops thanks to the discrete feature of the language, allowing, after successive steps, the exact manipulation of rational and real numbers. At the same time, the measuring process helps to immediately perceive natural numbers as particular members of a wider numerical domain, favouring a structural view of numbers and, moreover, induces an early development of abilities recognized in (Arcavi, 1994) as markers of "number sense", like the ability to approximately estimate a given quantity, etc..



For these reasons, a crucial educational problem is how to design and realize an early and effective strategy of cultural mediation, able to create solid links between the discrete and continuous aspects of numbers. The schema reported in the figure aside, that generalizes and synthesizes all the previous schemata, seems to be a useful tool in this direction. Here, the discrete structure (the bullets in figure) induces the choice of a unit of measure, that uniformly subdivides the line, rectifying the nonlinear compression of the accumulator. On the other hand, the correspondence between the discrete structure and the discretized one projects onto the former the structural properties, pointed out by Davydov for continuous quantities. Therefore, the schema represents not only a semiotic mediator of the cognitive interference of the discrete and continuous aspects of numbers, but also a resonance mediator, in the sense of (Guidoni *et al.*, 2005) between natural cognitive resources and disciplinary contents, in particular the additive structure. It is a useful tool in the hand of teachers both to interpret pupils' behaviours and to plan class activities.

Let us now go back to our Prologue. We focus our attention on two points:

a) the children spontaneously choose two blocks as a unit of measure;

b) Camilla's reasoning follows the path: "consider all, and take two away".

Well, in both behaviours we can see that what is working is the "mental number line" system, since a global perception of quantity seems to prevail over the use of one-one correspondences, typical of the counting process. In particular, at the basis of Camilla's reasoning, it is recognizable the above schema, expressed in her experience by the week train. The fact that she autonomously utilizes the schema while engaged in a problem solving situation, seems to us a strong marker of its validity.

A final remark concerns the teacher's reaction. In her previous similar experiences she had forced children toward the correspondence one day-one block. This time, she was firstly astonished, but then, adopting the point of view suggested by neurophysiology studies, thanks to a training path within a research team, she realized that her pupils were anyhow building their number sense, even if not in the "usual" way.

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HOW DO UNDERGRADUATE STUDENTS GENERATE EXAMPLES OF MATHEMATICAL CONCEPTS?

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In this paper we discuss the strategies adopted by successful undergraduate students when asked to generate examples of mathematical concepts. In earlier work Antonini (2006) found that expert mathematicians used three different strategies and that they were able to switch between them when required. In our sample we also found the three strategies identified by Antonini, however the trial and error strategy – where examples recalled from memory are checked against the required properties – was by far the most commonly used approach. The most notable difference between the behaviour of the experts in Antonini's study and that of the undergraduates described here was the students' apparent inability or unwillingness to switch strategies when their initial approach proved ineffective.

EXAMPLE GENERATION STRATEGIES

It is widely recognised that understanding and using examples is an important component of learning mathematics, and consequently the role of examples in mathematics teaching has come under intense scrutiny in recent years (e.g., Bills et al., 2006). Following Zaslavsky's (1995) suggestion that asking learners to generate their own examples of mathematical concepts may be an effective teaching strategy, Dahlberg & Housman (1997) gave mathematics undergraduates a definition of a novel mathematical concept and asked them to study it for a few minutes. They found that students who spontaneously generated their own examples of the concept during this period "learned a significant amount" compared to those who did not spontaneously generate examples. This finding has been widely cited in the mathematics education literature and has been used as the basis of pedagogical advice to practitioners in higher education (Mason, 2002; Meehan, 2007; Watson & Mason, 2001, 2005).

Given this body of literature that advocates that learners generate examples as a teaching strategy, it is not surprising that researchers have become interested in the methods by which examples of mathematical concepts can be generated, and of the factors which influence whether such generation attempts are successful or not. One particularly important development in this regard was Antonini's (2006) classification of the strategies used by mathematics researchers when generating examples. Our goal in this paper is to investigate whether the example generation strategies used by the (relative) experts in Antonini's study are also used by undergraduate students (relative novices), or whether there are significant differences

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to the strategies used by undergraduates. We begin by briefly reviewing the classification of strategies reported by Antonini.

Antonini (2006) asked expert mathematicians – meaning postgraduate research students – to generate examples of four different mathematical concepts (one question, for example, concerned giving an example of a commutative but non-associative binary operation). He found three different strategies in use by his participants, which he labeled '*trial and error'*, '*transformation*' and '*analysis*'. We now discuss concisely each of these strategies in turn.

An individual adopting the *trial and error* strategy searches for an example which satisfies the given criteria from amongst a collection of recalled examples of a broader category, testing each in turn to see if they meet the more stringent criteria of the question. For example, a trial and error strategy on the "commutative but non-associative binary operation" question might consist of testing several binary operations to see if each is commutative and non-associative. Watson & Mason (2005) defined an individual's example space as the collection of examples of a particular concept or object that they have access to at a particular time. Clearly, an individual's success in the trial and error strategy will, to a large extent, depend upon the size of their example spaces for the broader category (i.e., in the above example, how many binary operations they can recall).

The *transformation* strategy identified by Antonini (2006) consists of a participant modifying an example which satisfies some of the requested properties by performing one or more transformations on it until it satisfies all the requested properties. If one attempted the above question using this strategy, one might find a non-associative binary operation (for example subtraction) and continually redefine it until it was commutative.

The final strategy observed by Antonini (2006) was denoted *analysis.* An individual following this strategy assumes that the required object exists, and deduces what other properties it must necessarily have until they are able to recall an object with the required properties, or have a procedure for constructing such an object. Antonini suggested that the analysis strategy was only enacted by the experts in his study when the other two strategies had failed, that is to say that his participants seemed to use it is used as a last-resort strategy.

Successfully implementing the strategies discussed by Antonini (2006), especially the transformation and analysis strategies, may be quite demanding. Given the recent recommendations about using learner generated examples as a teaching strategy in higher education (e.g. Mason, 2002; Meehan, 2007), an important question raised by Antonini's work is: do undergraduate students use the three expert strategies for generating examples? Do they deploy alternative strategies? Answering these questions was the purpose of the analysis reported in the remainder of this paper.

METHOD

The data reported in this paper come from the pilot of a larger study concerned with the role that learner generated examples can or should play in higher education pedagogy. The pilot study consisted of task-based interviews with nine undergraduate students (second and third years) studying at a highly ranked UK university. Each participant had been successful in their school-level studies, achieving an A grade (the highest possible) in their A Level Mathematics examinations (the qualification awarded to 18 year-old school leavers in England and Wales).

In view of the non-expert status of our participants, we did not use the same instrument as Antonini (2006), but instead developed a novel approach. Each interview began with the student being presented with a definition of a (to them) novel mathematical concept (adapted from Weber, Brophy & Lin, 2008):

> Let $f:\mathbf{R}\to\mathbf{R}$ be a real-valued function. Let $A\subseteq\mathbf{R}$. Then f is *preserved on* A if and only if $f(A)\subseteq A$. In other words, f is preserved on A if and only if $a\in A\Rightarrow f(a)\in A$.

They were then given a series of eleven example-generation tasks, designed following the strategies advised by Watson & Mason (2005). For example:

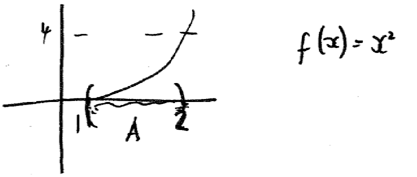
> Let A be the open interval (1,2) and B be the open interval (2,3). Find an $f:\mathbf{R}\to\mathbf{R}$ such that f is preserved on A but not on B.
>
> Let $f(x)=x^{13}$. (i) Find a set A such that A has two members and f is preserved on A; (ii) Find the largest $a\in\mathbf{R}$ such that f is preserved on $\left[-\frac{1}{2},a\right]$.

The order in which the example-generation tasks were presented was randomised for each participant. The interviews followed the clinical interview procedure of Ginsburg (1981), that is to say that the interviewer intervened where necessary to prompt the participant to verbalise their thinking, or to ask for clarification if required. The example-generation section of the interview lasted for around 25 minutes, and it is these data that we draw upon in this paper.

The interviews were video-recorded, and analysed following the quasi-judicial method developed by Bromley (1986). Interview episodes were coded according to Antonini's (2006) categorisation of strategies, providing patterns of behaviour that were then compared to the theoretical propositions in the literature. In the next three sections we discuss how each of the strategies observed by Antonini appeared in our study, with particular reference to the differences between the behaviour of the undergraduate students in our study and the behaviour of the experts in Antonini's study.

THE TRIAL AND ERROR STRATEGY

Trial and error was the strategy used most often by the students in our sample. Indeed, in nearly all instances it was the first (and only) strategy employed. When students enacted the strategy they used their own example space to find mathematical



objects which could fit the properties of the requested example. For instance, if the object to be generated was a function, the students tended to consider straight lines, parabolas and sine function – all very familiar to them from their previous studies. When the trial and error strategy was implemented students started either from the graphic or the algebraic register and worked on this, sometimes moving from the graphic to the algebraic register, to verify that what they saw in the graph was correct. In the example here Abbie first drew a familiar graph (a parabola) and then checked whether the standard parabola ($y=x^2$) fitted the requirements of the exercise. At the end of the inspection she was satisfied that her example was correct. From the analysis of the implementation of the trial and error strategy we distinguished two different methods: some students checked that the object they found was correct and satisfied all the requested proprieties, and some did not. For example Abbie, when required to produce a function $f:\mathbf{R}\to\mathbf{R}$ that is preserved on $\mathbf{R}$-$\{0\}$ said:

> So $f(0)$ cannot equal 0… Maybe…So… what could that be… when is $f(0)$ not equal zero…Maybe when it is 1 over something…[*she writes f(x)=1/x*] sounds about right. I think… well… maybe not but…it feels all right to me because if you put 0 and then it doesn't exist … that is anything that I can think of.

Other students did check that the example fulfilled the required properties. John, when asked to produce an example of a function not preserved in A=(1,2), said:

> So just $f(x)=x+10$ … in the interval (1,2) 1 is 11 and 2 is 12 so that is not preserved on A … yes… 11 and 12 are not in the interval A… it is not preserved.

It is interesting to note that when the trial and error strategy did not lead to a mathematical object that satisfied the required properties the students seemed unable to continue and hence abandoned the exercise. In the following example Abbie had been asked to find $f:\mathbf{R}\to\mathbf{R}$ such that f is preserved on **N** but not on **R**-**N**:

> Ok, preserved on the naturals so…[*marks the natural numbers on a graph*] ok, f is preserved on the… that is weird…I really don't understand the definition…[*goes back to read the definition*] … so $f(a)$ is contained in A… maybe I am thinking $f(a)$ needs to be within … the same outcomes than this rather than just matching….but I am not sure…$f(a)$ say like…. This would work but all of these as well… if you had $y=x$ then they would be all within here and then it would work but on the reals as well. I have no idea, can I come back to this?

By using the trial and error strategy the students often did arrive at correct conclusions by checking their initial response. We can see this in the following extract. Josh was given the function $f(x)=0$ if x=0 and $f(x)=\frac{1}{x}$ if $x\neq 0$ and asked to find a set A of 5 elements with f preserved on A:

> So anything greater than 1 is going to give an answer less than 1because it grows bigger… something like $f(2)=½$ so if you have any kind of set from 1 upwards the answers are going to get smaller and they will not be in the set… they won't be preserved I don't think. So… I need to try something which is a bit … more inclusive…If f… $f(½)$ will go the other way… ½ is going to be 2 …so those two. I draw the graph bigger to get this bigger in my head… just on the positive bit if I have $f(½)$ gives 2, and $f(2)$ gives ½ in between here are the answers you want to range from […] so if I pick five members again I need to get the ones that correspond.. if I have 2 in the set I also have to have ½ in the set, they preserve each other [*Josh proceeds to successfully complete the task*].

Overall, out of 29 instances of example generation observed in the study the students used the trial and error strategy 23 times.

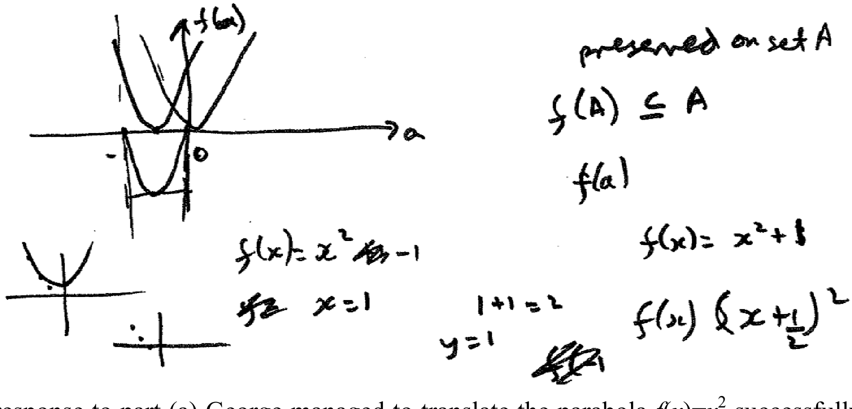
THE TRANSFORMATION STRATEGY

The transformation strategy was used in our sample 5 out of 29 times, always on well-known mathematical objects. In all cases, when this strategy was used, it was used successfully. Here is George's reasoning:

1. (a) Let A be the closed interval $[-1, 0]$. Find an $f : \mathbb{R} \to \mathbb{R}$ such that f has a local minimum in A and is preserved on A.

   (b) Let B be the open interval $(1, 2)$. Find $f : \mathbb{R} \to \mathbb{R}$ such that f has a local minimum in B and is preserved on B.

D



In response to part (a) George managed to translate the parabola $f(x)=x^2$ successfully to $f(x)=\left(x-\frac{3}{2}\right)^2$ so that it had the required properties. George selected this strategy first, without using the trial and error strategy before. This tendency of the students to stay with the strategy chosen, appears to form the most important difference to the expert behavior as reported in Antonini (2006), we will discuss this further later.

Another example of the transformation strategy is occurred when John was asked to find $f:\mathbf{R}\to\mathbf{R}$ preserved on A=(1,2) but not on B=(2,3):

> Taking the sine curve again the intervals have length only 1 so you would need half of the sine curve and it is shifted up by 2… shifted by 2 in… yes… shifted by 2 and halved. The same sine going from -1 to 1 you want it going from… 1 to 2 so it needs shrinking by a factor of 2… that is $\left(\frac{1}{2}\sin x\right)+2\ldots$ so it is between 1 and 2 and it is not preserved on B because B is here between 2 and 3… I think that works.

THE ANALYSIS STRATEGY

The analysis strategy was observed only once in our sample, during the interview with George. He was asked to find a function $f:\mathbf{R}\rightarrow\mathbf{R}$ strictly decreasing and preserved in A=(1,2).

> Preserved on A means it has got to be in that box [*pointing to his graph*], strictly decreasing means that it has to go something like that [*draws on the graph*] … so it goes down… so… to cross the point between it is going to be … *y*=*mx*+*c*… *c* is between 1 and 2, so make it 1.9. The gradient has got to be negative … so it has got to be… if that is 1.1 and that is 1.9 so…that point there is one point along … 0, 1.9 and m is above that, one line so that [*carries on with the calculations of the gradient*] so $y=-\frac{2}{5}x+1.9$ … and just check that…[*checks by substituting numerical values in the equation of the line*] that's right.

Here George deduced what other properties will necessarily be satisfied by an object of the type required, and eventually constructed a solution from these enlarged specifications.

DISCUSSION

From the data presented above we can see that the strategies implemented by the experts in Antonini's (2006) study were also implemented by the novices in our own study. Indeed we could not find any substantially different strategy implemented by our students. However, there were important differences in how strategies were used, and how examples were constructed. When using the trial and error strategy the undergraduates often did not explicitly check that the remembered object fitted the properties required. When they did explicitly check, they tended to do this by reasoning pictorially or algebraically. Most often however they were, in some sense, already implicitly convinced by their choice. This is in contrast with the expert behaviour reported in Antonini (2006) where the experts checked their choices until either they were convinced that they were correct or otherwise decided to switch strategy.

When using the transformation strategy our students resorted to examples in their example space (Watson & Mason, 2005) and used standard objects they were very familiar with (we have seen above the use of parabolas and trigonometric functions). Not surprisingly, the analysis strategy was used very infrequently – only once – in our study. It seems reasonable to suggest that this strategy required students to think abstractly about the object to be generated, and about the properties that such an object needs to satisfy. This is something students seem to find very difficult.

The most striking difference that we found between the experts and undergraduates, however, was the apparent inability of the undergraduates to switch strategies. Antonini (2006) reported that the experts in his study mostly started from the trial and error strategy but quickly determined whether or not this strategy would result in the required object. After a very short inspection of the problem the experts would decide whether they need to change strategy, and switched to the transformation or analysis strategies. In contrast the students in our sample never switched strategies. When they found that a strategy was ineffective they tended to simply abandon the exercise. These considerations point to a lack of flexibility in the use of strategies, or in other words, the lack of a meta-mathematical skill that experts seem to have developed: namely to recognise the effectiveness of their chosen course of action and switch to a different course of action if necessary. These observations resonate with Schoenfeld's (1985) finding that, while tackling open-ended problems, undergraduate students tend to pick a single problem-solving path which they never deviate from. Mathematicians too, have noted the importance of the ability to switch between different proof production strategies unable to proceed with the initial chosen strategy (Iannone, in press; Weber & Alcock, 2004).

CONCLUSION

Here we extended the work of Antonini (2006) by studying the strategies that undergraduate students adopt when generating examples of mathematical concepts. Referring to the classification of strategies used by experts in Antonini's study, we found that students essentially implement the same strategies as the experts, but in a somewhat different manner. When using the overwhelmingly most common strategy – trial and error – the undergraduates often did not check that the object they found fitted the required properties. When using the transformation strategy students resorted to well known examples and transformed them to fit the required properties. The analysis strategy was used only once in our sample, suggesting that this strategy is the most demanding to implement. However, the most important difference between the behaviour of the undergraduates in our sample and the behaviour of the experts in Antonini's study was the students' apparent inability to change strategy, even when they had convinced themselves that their chosen strategy is ineffective.

If example generation is to become a substantial part of higher education pedagogy (as recommended by some), it seems that lecturers will need to become aware of, and develop methods of addressing, the limitations of students' abilities to complete such tasks.

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THE DEFECTIVE AND MATERIAL CONDITIONALS IN MATHEMATICS: DOES IT MATTER?

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In this paper we discuss the relative merits of two different ways of understanding conditional statements of the form 'if p *then* q*'. We demonstrate that there is no relationship between an ability to draw modus tollens deductions and having a material conditional understanding, as proposed by Durand-Guerrier (2003). Instead, we suggest that the so-called defective conditional understanding is widespread among high achieving mathematics students. We argue that, despite its name, adopting this understanding does not prevent students from drawing valid logical deductions, or from being successful in university-level mathematics examinations.*

DIFFERENT CONCEPTIONS OF "IF *P* THEN *Q*"

Logical implication is fundamental to mathematical proof, and thus of major concern to mathematics educators at all levels. However, it is well known that students find dealing with conditional statements – statements of the form 'if p then q' – to be counterintuitive and difficult (e.g. Hoyles & Küchemann, 2002). One possible reason for these difficulties arises from the different meanings that can be given to such statements. In this paper we focus on two such meanings, the so-called material conditional and defective conditional conceptions.

In formal logic courses university students are taught the formal concept definition of implication which is captured by the truth table shown in Figure 1(a): 'if p then q' is true in all cases except where the antecedent (p) is true and the consequent (q) false.

p	q	if p then q
T	T	T
T	F	F
F	T	T
F	F	T

(a)

p	q	if p then q
T	T	T
T	F	F
F	T	I
F	F	I

(b)

Figure 1: The material (a) and defective (b) conditional truth tables (T denotes 'true', F denotes 'false' and 'I' denotes 'irrelevant').

This understanding – known by logicians as the *material* conditional – leads to some oddities such as statements like "if 3 is even, then π is irrational" being logically true.

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Quine (1966) noted that this understanding of the conditional is not generally used in day-to-day life. He wrote that "'if p then q' is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent" (p.12). Wason (1966) described this alternative understanding of the conditional – one focussed on its meaning as a conditional affirmation of the consequent – as the *defective* conditional. Here the conditional is deemed to be irrelevant in the cases where p is false; the corresponding truth table is shown in Figure 1(b). Other researchers have used various other terms for this understanding, such as the 'hypothetical conditional' (Mitchell, 1962) or the 'common understanding' (Durand-Guerrier, 2003).

Given these two different understandings of a conditional statement, an obvious and important question arises: which is most appropriate for the mathematics classroom? Typically one would expect that the most appropriate concept image for a student to have is one which matches the relevant concept definition (Tall & Vinner, 1981). But Hoyles & Küchemann (2002) disagreed, and directly answered the question in favour of the defective conditional:

> "We claim that when studying reasoning in school mathematics, the [defective conditional] is a more appropriate interpretation of logical implication than the [material conditional], since in school mathematics, students have to appreciate the consequence of an implication when the antecedent is taken to be true" (p. 196).

Durand-Guerrier (2003) disagreed with Hoyles & Küchemann, and gave two arguments in support of her position. First, that a material conditional understanding would be required to understand definitions such as that of a diagonal matrix:

> An $n\times n$ matrix $[a_{ij}]$ is diagonal if and only if for every i from 1 to n, and every j from 1 to n, if $i\neq j$ then $a_{ij}=0$.

The crucial case here is that of $i=j$; the material conditional "if $i\neq j$ then $a_{ij}=0$" is true in this case regardless of whether $a_{ij}=0$ or $a_{ij}\neq 0$, and so, as Durand-Guerrier pointed out, the definition accurately describes our concept image of a diagonal matrix (i.e. where entries are zero everywhere except the diagonal, where they may be either non-zero or zero). However, the same is true of the defective conditional: it is not false in the case $i=j$, but rather irrelevant. That is to say that when $i=j$, nothing can be concluded about a_{ij} from the defective conditional. Thus, in our view at least, a defective conditional interpretation of "if $i\neq j$ then $a_{ij}=0$" does seem to give a concept definition which matches the appropriate concept image of diagonal matrices: when $i\neq j$ we know that $a_{ij}=0$, but when $i=j$ we don't know anything as the conditional is irrelevant.

Durand-Guerrier's (2003) second argument against Hoyles & Küchemann's (2002) position was more complex, and related to the kinds of deductions that can be made using each type of conditional. From a material conditional two valid deductions are possible: modus ponens (deducing q from "if p then q" and p) and modus tollens (deducing $\neg p$ from "if p then q" and $\neg q$). However, Durand-Guerrier suggested that the second of these can not be made from a defective conditional:

> With [the defective conditional] understanding of implication, it is no more possible to interpret the Modus Tollens without using the contrapositive. Indeed, the Modus Tollens conclusion is that antecedent is false; if one accepts only implications with true antecedent, one must use the contrapositive and apply Modus Ponens to it. However, the equivalence between a conditional statement and the corresponding contrapositive requires material implication (p. 29).

Essentially Durand-Guerrier suggested that the modus tollens deduction cannot be made using a defective conditional understanding. With the material conditional understanding two routes are open to reasoners: they may either simply know the modus tollens deduction and apply it directly, or they may convert the conditional into its contrapositive (i.e. convert 'if p then q' into 'if $\neg q$ then $\neg p$') and then apply modus ponens. Neither of these routes appear to be viable if you have a defective conditional understanding.

However, it could be possible to make a modus tollens deduction via a third route, by using modus ponens and an informal contradiction argument. Suppose a reasoner is given the defective conditional "if p then q" and the statement $\neg q$. They might suppose p, conclude q by modus ponens, notice that q contradicts the given statement $\neg q$, and so conclude that their supposition p was incorrect, concluding $\neg p$. This admittedly rather long chain of reasoning seems to be entirely accessible to someone who has a defective understanding of the original conditional. Nevertheless, it might well be the case that the length of this chain of deductions hinders students from accurately making the modus tollens deduction, in which case we might agree with Durand-Guerrier's (2003) arguments against Hoyles & Küchemann (2002).

In this paper we report data from an experiment which directly investigated of whether having a defective conditional conception hinders making logical deductions and, in particular, making modus tollens deductions.

METHOD

The data reported in this paper come from a wider study which investigated the development of logical reasoning skills across the first year of undergraduate mathematics study. Participants were 33 first-year undergraduate students studying mathematics (either mathematics, or a joint degree with a significant mathematics component) at a highly-ranked UK university. All the students had been highly successful during their school mathematics studies: other than the two overseas students in the sample, all participants had been awarded A grades in both A Level Mathematics and A Level Further Mathematics (this represents the highest possible achievement in mathematics for 18 year-old school leavers in England and Wales).

Students participated in two sessions of data collection during the course of their first year studies, once at the very beginning and once at the end. In both sessions participants worked individually through a booklet of tasks, all designed to interrogate logical reasoning behaviour. In this paper we report responses to the two tasks relevant to the debate between Hoyles & Küchemann (2002) and Durand-

Guerrier (2003): the conditional inference task, and the truth table task. In each case we used abstract versions of the tasks to avoid the well-documented confounding effects of realistic/mathematical content (e.g. Stylianides, Stylianides, & Philippou, 2004).

Conditional Inference Task

The conditional inference task used was identical to that used by Inglis & Simpson (2008). In both sessions participants were given 32 problems of the form:

> This problem concerns an imaginary letter-number pair. Your task is to decide whether or not the conclusion *necessarily* follows from the rule and the premise.
>
> *Rule:* If the letter is not G then the number is 6.
> *Premise:* The number is not 6.
> *Conclusion:* The letter is G.
>
> ☐ YES (it follows) ☐ NO (it does not follow)

The inferences tested were balanced: half were valid – modus ponens and modus tollens – and half were invalid – denial of the antecedent (concluding $\neg q$ from $\neg p$ and 'if p then q') and affirmation of the consequent (concluding p from q and 'if p then q'). Following Inglis & Simpson (2008), the presence of negated statements in the rules were rotated (e.g. the rules of the form 'if p then $\neg q$', 'if $\neg p$ then q' and 'if $\neg p$ then $\neg q$' were used in addition to 'if p then q'), and half of negated statements were represented explicitly (e.g. "not 3") and half implicitly (e.g. "8" rather than "not 3"). Participants took the task twice, once at the beginning of the year and once at the end, thus ended up with a score out of 64 together with subscores for each of the four tested inferences. These profiles gave an identification of how fluent participants were at drawing each of the four inferences tested, and of overall inferential fluency.

Truth Table Task

In the second session of data collection we also collected participants' responses to 32 tasks of the following form (adapted from earlier studies, Evans & Over, 2004):

> This problem relates to a card which has a capital letter on the left and a single-digit number on the right. You will be given a rule together with a picture of a card to which the rule applies. Your task is to determine whether the card conforms to the rule, contradicts the rule, or is irrelevant to the rule.
>
> *Rule:* If the letter is not E then the number is not 1.
> *Card:* D1
>
> ☐ card conforms to the rule ☐ card contradicts the rule ☐ card is irrelevant to the rule

Two cards representing each of the four lines of a truth table (Figure 1) were included, and the positions of negated statements in the conditionals were rotated. Each participant's responses were scored twice: first, with respect to how consistent they were with the material conditional, and second, with respect to how consistent they were with the defective conditional. Thus each individual ended up with a material conditional profile score (out of 32) and a defective conditional profile score (out of 32). The higher the profile score the more consistently the participant was using that understanding of the conditional.

To provide a means of controlling for general cognitive skills, during the first session of data collection participants took Part 1 of the AH5 intelligence test (Heim, 1968). This test, designed for high achieving adults, contained 36 items in the categories ‘directions’, ‘verbal analogies’, ‘numerical series’, and ‘similar relationships’. It has been widely used by other researchers interested in individual differences in logical reasoning abilities (e.g. Newstead et al., 2004).

RESULTS

Our primary goal was to determine whether having a defective conditional understanding hinders inference-making, and in particular, making the modus tollens inference. To address this issue we calculated participants’ defective and material conditional profiles on the truth table task (i.e. the number of cards classified according to each conception). Twelve participants consistently adopted the defective conditional understanding (i.e. all their responses were in line with Figure 1(b)). In contrast no participant consistently adopted the material conditional understanding.

The two difference conditional profile scores were correlated with the number of each inference correctly classified on the conditional inference task. If Durand-Guerrier’s (2003) suggestion that a material conditional is required to make the modus tollens deduction we would expect a negative correlation between the defective conditional profiles and modus tollens scores. If, on the other hand, Hoyles & Küchemann’s (2002) suggestion that the defective conditional is “more appropriate” for the mathematics classroom, we might expect positive correlations between defective conditional profiles and each of the inference scores. The various correlations between each of the key indicators are shown in Table 1.

	Def	Mat	MP	DA	AC	MT	All
Def	1	-.54**	.48**	.68**	.60**	.03	.63**
Mat	-.54**	1	-.22	-.57**	-.34	.12	-.39*

Table 1: Pearson correlations between the Defective (Def) and Material (Mat) profile scores, and the number of each inference correctly categorised (respectively modus ponens, denial of the antecedent, affirmation of the consequent and modus tollens). Significant correlations are denoted by *p<.05 and **p<.01.

Participants’ defective conditional profiles were strongly positively correlated with performance on the modus ponens, denial of the antecedent and affirmation of the consequent inferences (all at $p < .01$), i.e. those with a higher defective conditional profile score tended to classify MP, DA and AC inferences more accurately. However, there was no significant relationship with the modus tollens deduction. The material conditional profiles were negatively correlated with performance on the denial of the antecedent inference, and overall inferential performance. That is, participants with a higher material conditional profile tended, overall, to be more inaccurate on the conditional inference task. There was, however, no relationship between material conditional profiles and accuracy at classifying the modus tollens deduction.

Further analyses revealed a significant positive correlation between defective conditional profiles and AH5 intelligence scores (r = .40, p = .021), but no relationship between material conditional profiles and AH5 scores (r = -.03, p = .86). As well as being related to defective conditional profile scores, AH5 scores were found to be borderline significantly related to overall deductive competence on the conditional inference task (r = .33, p = .063). Consequently, it may have been the case that participants' AH5 scores represented a potential confound. That is to say that any relationship between the different interpretations of "if p then q" and deductive competence merely reflected mutual relationships with intelligence. To test for this possibility we repeated the correlation analyses, this time controlling for AH5 scores. The resulting partial correlations are shown in Table 2.

	Def	Mat	MP	DA	AC	MT	All
Def	1	-.57**	.42*	.62***	.53**	.10	.58**
Mat	-.57**	1	-.21	-.60***	-.35*	.12	-.40*

Table 2: Partial correlations between Defective (Def) and Material (Mat) profile scores and the number of each inference correctly categorised, controlling for AH5 scores. Significant correlations are denoted by *p<.05, **p<.01, ***p<.001.

After controlling for AH5 scores, essentially the same pattern of correlations emerged. Participants with high defective conditional profile scores tended to be more proficient at accurately categorising (as valid or invalid) modus ponens, denial of the antecedent and affirmation of the consequent deductions. Those with high material conditional profile scores tended to categorise fewer deductions correctly. There was, however, no relationship between either of the two conditional profile scores and the number of MT deductions correctly assigned. This was not because of a floor effect; participants were able to tackle the MT component of the task. Performance on this section was respectable, with a mean of 10.3 inferences correctly categorised (out of 16; SD = 3.0).

DISCUSSION

Recall that Durand-Guerrier (2003) argued that the material conditional was necessary to make the modus tollens deduction. Our results call this assertion into question. We found no relationship between accurately categorising modus tollens deductions and the interpretation of "if p then q" participants adopted. However, having a defective interpretation – where "if p then q" is deemed irrelevant when p is false – was associated with higher scores on every other measure of deductive competence we took (the valid/invalid categorisation of modus ponens, denial of the antecedent and affirmation of the consequent deductions), and with higher AH5 scores. Given this, if our data supports one interpretation of "if *p* then *q*" over the other, it would seem to be the defective conditional ahead of the material conditional.

It is somewhat surprising that defective conditional profile scores were correlated with every deductive competence measure apart from that relating to modus tollens. One possible explanation for this relates to the "informal contradiction argument" route to making modus tollens deductions discussed earlier. This involves rather a long chain of reasoning, so it seems reasonable to propose that the success or failure of such a chain might be more related to factors such as the concentration or motivation levels of the participant rather than would be the case for the modus ponens, denial of the antecedent or affirmation of the consequent deductions. Further investigations would be required to determine exactly which factors influence success or failure at making modus tollens deductions.

Some caution is needed in the interpretation of these results, due to the relatively small number of participants who adopted the material conditional. As noted above, no participant did so consistently, and indeed all but two of the participants had higher defective conditional profiles than they did material conditional profiles. This observation suggests that the defective conditional understanding of the conditional is adopted by the majority of high-attaining mathematics undergraduates, and that it does not prevent them from being relatively successful at drawing logical deductions.

This consideration also suggests a reason for why we found positive correlations between having a defective conditional understanding and performance on the modus ponens, denial of the antecedent and affirmation of the consequent inferences. If the large majority of participants adopted a defective understanding, then those with lower defective conditional profile scores (and hence probably higher material conditional profile scores) would be those participants who found it difficult to apply their understanding consistently across the task. If this interpretation were correct we might expect a correlation between defective profile and AH5 scores (which, indeed, was the case, $r = .40$, $p = .021$). Under these assumptions, the defective conditional profile score could be interpreted as a measure of consistency throughout the task.

It is possible that the participants in this study were atypical of university level mathematics students, and that their near uniform adoption of the defective conditional understanding of the conditional did harm their mathematics achievement. To test for this possibility we obtained each participant's first year examination marks for the four core mathematics modules taken by each student. Although there was no correlation between students' average marks and their material or defective conditional profile scores (rs = 0.04, 0.09 respectively), the group as a whole had very high levels of achievement. The mean first-year examination mark obtained by the sample was 76% (SD=12%), well above the 70% threshold for the top grade.

CONCLUSION

Hoyles & Küchemann (2002) suggested that the defective conditional was the most appropriate to develop in the mathematics classroom, as it allows students to appreciate "the consequence of an implication when the antecedent is taken to be true". Durand-Guerrier (2003) disagreed with Hoyles & Küchemann, by suggesting that the material conditional was necessary in order to be able to make modus tollens deductions. In this paper we have empirically demonstrated that there is no connection between having a material conditional understanding and making modus tollens deductions. Furthermore, we have shown that high achieving mathematics undergraduates almost uniformly adopt the defective conditional understanding, and that this does not seem to adversely affect either their ability to draw inferences from abstract conditional statements, or their performance on university mathematics examinations. In sum, mathematically, there appears to be no disadvantage to holding a defective understanding of the conditional.

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ENGAGEMENT IN THE MATHEMATICS CLASSROOM

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Engagement and how it relates to identity is an area within mathematics education that has recently received attention. In this paper, it is proposed that students engage in mathematics when their perceptions of what they want to achieve has not yet been realised. These perceptions are affected by students' views about mathematics, the context of the moment and the students' feelings about being able to do mathematics.

INTRODUCTION

Mathematics, as a critical filter to a variety of education and career opportunities, is deemed to be an important subject of study by the school community and society (Leder, Pehkonen, & Törner, 2002). Consequently how mathematics is learnt is of interest to many levels of society. Over two years, I studied the mathematical journeys of high school students generating many themes relating to the students' affective responses to mathematics and their identities. Emerging strongly was the importance of students' engagement in mathematics. In this paper, I explore student engagement in terms of its theoretical relationship with affect and identity and the engagement-related data.

AFFECT, IDENTITY AND ENGAGEMENT

The study of affect in mathematics education can be loosely genealogically mapped in three phases. Prior to the mid 1980's, research in this area was dominated by the psychology-based study of mathematics anxiety and attitude, and the quantitative measurement of these constructs. Pivotal to the next phase was McLeod's (1992) definition of the affective domain as a "wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition" (p. 576). He included as the main components in the domain: beliefs, attitudes and emotions; and these, often with values (Goldin, 2002). In the most recent phase, developments in educational, psychological and social psychological research have encouraged a wide variety of multi-disciplinary theoretical perspectives into the study of affect (Hannula, Evans, Philippou, & Zan, 2004). Qualitative methods are now deemed important because of the need to explain the behaviour of the participants in their natural setting and to understand the meanings they have constructed (Creswell, 2003). There has been an push to understand the influence of both the individual and the collective in learning and differences in students' socio-cultural backgrounds, and consequently a renewed interest in the notion of identity (Sfard & Prusak, 2005). Sfard and Prusak (2005) dispute any process of objectifying the construct of identity, or defining it as *who one is,* instead viewing identity as dynamic "…constantly created and recreated in interactions between people" (p.15). Importantly, they equate

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 233-240. Thessaloniki, Greece: PME.

identities with reifying, endorsable and significant stories about a person told by others. A person therefore has multiple identities that surround them. Sfard and Prusak (2005) split a person's multiple identities into two sets of *actual identity* (*I am, he is* - stories about the actual state of affairs) and *designated identity* (*I should be* - a state of affairs expected to be the case now or in the future). When there is a perceived and persistent gap between a student's actual and designated identities, there is likely to be a sense of unhappiness in that person and Sfard and Prusak describe learning as providing opportunities for closing this gap. Op 'T Eynde, De Corte and Verschaffel (2006) importantly link learning and engagement, viewing students' learning to be a "form of engagement that enables them to actualise their identity through participation in activities situated in a specific context" (p. 194).

The construct of engagement has been researched increasingly in mathematics education, often in association with constructs such as motivation, confidence, and self-efficacy. Williams and Ivey (2001) use the term 'motivation for engagement' to describe the degree to which students choose to actively participate in the classroom activities available to them. Yair (2000) similarly describes a student's disengagement as alienation from instruction. Boredem is often associated with disengagement but is under-researched in mathematics education. Models for boredom such the under-stimulation, forced effort or resistance models described by Larson and Richards (1991) are interesting. For example, in the 'forced-effort' model students have high levels of boredom in situations that require the most mental effort, deal with abstract material, and low control, similar to the research classrooms here.

THE RESEARCH

In order to understand high school students' affective responses to the learning of mathematics, qualitative data was collected on thirty-one students aged 13-15 years using observation, student and teacher interviews, parent questionnaires, assessments, reports, and written responses to questionnaires and metaphor exercises. The analysis of the data included the development of themes using a grounded theory approach, the recognition of identifying stories using Sfard and Prusak's framework and associated indicators of affective responses guided by the work of Evans, Morgan and Tsatasroni (2006). Indicators included expressed feelings, the use of metaphors, and endorsable behaviours.

STUDENTS ENGAGEMENT IN MATHEMATICS

As the data began to be analysed, elements of the students' journeys emerged. The *students' views* about mathematics encompassed both mathematics and mathematics classrooms. The students' *designated identities* were the expectations that they had of themselves learning mathematics, and their *actual identities* were evidence to the students of their present position and that they were fulfilling their designated identities. The significant narrators of these identities were the student themselves, their parents and siblings, their mathematics teachers, and their peers. Many of these identities were related to being able to do the mathematics both to the level expected of their class and in relation to their peers. Surrounded by mathematical identities and signalling gaps between students' designated and actual identities is both their positive and negative *affect*. In this paper, a student's *engagement* in the mathematics has been focussed on and how, filtered through the context of the moment, it interrelates with a student's views, their designated and actual identities and feelings.

In this research a student was deemed to be *engaged* when he or she was committed to doing the mathematics. Their engagement took many different forms and was individual to and dynamic within each student and situated *in the context of the moment.* In this research, it was too simplistic to describe a student as either disengaged or engaged. If one student was observed to have their head down and be working steadily, and another to be chatting socially while they completed their work, it was possible that both of the students were engaged. Rather than engagement being viewed as a dichotomy, this research viewed engagement to fluctuate along continua between fully engaged to not engaged with different textures and layers. The indicators of student engagement displayed in this research through their involvement in mathematical discussion was the amount of mathematics they did and the depth to which they did it, perseverance, the asking and answering of questions in whole-class activities, and asking for help (with qualifications). Students disengaged from the mathematics through behaviours such as writing slowly, ruling up a page, going to the toilet, sorting equipment, talking socially, or not committing to the mathematics except at a very superficial level.

In this research, the students engaged in the mathematics in order to fulfil the expectations created by designated mathematical identities and to close the gap between these identities and their actual identities. However, the complexities of the mathematics classroom through a myriad of factors relating to the students' views, and the context of the moment meant that it was not always simple to see a clear connection between actual and designated identities. Each student had a unique and dynamic set of identities, both social and mathematical in nature, and individual patterns of engagement. In this short paper, I have generalised these individual patterns to give a class level description. However, any data given is triangulated, endorsed by the student, and representative of the class.

Students' views on school mathematics

The students viewed school mathematics as unique from other subjects in terms of its nature; the importance placed on the subject; and its classroom routines. In general, they believed mathematics to be a subject viewed with importance by their teachers and parents, with a strong logical structure, strict rules, and only one answer. Compared to other subjects, students felt they were expected to have more cumulative content knowledge in mathematics and needed to *think* for a larger proportion of the time. This resulted in many of them finding mathematics more difficult than other subjects and this often affected their engagement.

> In maths it's harder and you have to think more than in other subjects (Joanna$_{\text{EndofYear2006}}$).

> [I] talk more in maths [than in other subjects] … I just can't be bothered doing it. [It's easier] to talk than making my brain hurt (Bridget$_{\text{Interview2006}}$).

The majority (twenty seven) of students viewed mathematics neutrally or negatively. Negative feelings in mathematics varied in frequency and intensity than other subjects. All students directly associated their feelings with their engagement.

> [Negative feelings] happen more in maths so it affects us more (Katrina$_{\text{GroupInterview2007}}$).

> I can't be stuffed. I feel like just sitting down and talking [long pause] it's like, it's maths and you just GRRRRR! I just don't want to be there. I'm just sitting there mucking around (Ruth$_{\text{GroupInterview2006}}$).

The classroom routines in mathematics were different to other subjects, which had more group work and class discussions. There was generally a starter activity, the students were introduced to new content knowledge or reviewed content from the previous year through teacher-talk with a small number of questions directed at the class, the students were shown an example, they were given notes and then exercises from the textbook. Doing textbook work usually took up the main proportion of time.

Boredom was discussed in relation to the routines of the classroom and the repetition of the content.

> Notes, textbook blah blah blah (Debbie$_{\text{GroupInterview2007}}$).

> Maths is like a boring, tedious person that is somehow necessary in our lives (Lola$_{\text{EndofYear2006}}$).

The students described the feelings that resulted as tiredness, lethargy and sometimes anger and frustration. This had the consequences of reducing the engagement in terms of the amount of work they did, the depth to which they did the work, how much they tried, and the time they spent socialising.

> [If it is boring] I don't attempt some of the work (Nicola$_{\text{Interview2006}}$).

> I kind of talk a little bit more [in maths] because I kind of get bored. I do work pretty well in most classes. I still do work in maths, but just not as well (Paul$_{\text{Interview2006}}$).

The students' views of mathematics often resulted in negative feelings and a consequent lack of engagement. This loss in learning opportunities may have impacted the students' ability to fulfil their designated identities and possibly increased the gap between these and their actual identities.

The context of the moment

There were a number of other factors that influenced students at the time of engagement, which is described here as the *context of the moment*. These contextual factors affected students' ability to meet the expectations associated with their designated identities. Some of these were external to the classroom and others related to the school mathematics classroom.

The research found that students' lives external to the mathematics classroom continued within the classroom, directly affecting the students' engagement. At 13-15 years, the students had entered adolescence with mercurial hormones and strong social identities, and during the research period, they moved from Year 10 into Year 11, the first year of national assessments, reinforcing achievement-related designated identities and causing tension with social expectations. The research showed that students' engagement generally increased for a time at the beginning of the year because they were aware of designated identities associated with their new class placement. Physical features of the individual classroom such as space, desk design and layout, lighting, and heating were also factors affecting students' engagement, which emerged in the research (Ingram, 2008).

Within the context of the mathematics classroom, not only did the mathematics teachers set the classroom routines, they were significant narrators of students' identities through the dialogue of the classroom. The teacher set expectations for both the class and the individual students, and attempted to motivate students to fulfil these designated identities through engaging in the tasks. By recognising and reporting on students' progress, they also narrated students' actual identities and class positioning. The students viewed their teacher to affect their engagement through their level of control and monitoring, their pedagogical style, their manner when asked questions, and how well they explained concepts.

> "Mr Murray … could not control our class and … I did one page of work [for the term] because I was just talking all the time (Tracey$_{\text{Interview2006}}$).

Within the classroom, the adolescent social dance continued, and the students were influenced by their social as well as mathematical identities. It seems that that who a student sat *near* was related to how they felt about the mathematics they were doing, the amount of mathematics they did, and the quality of their participation in mathematical discussions. Seating arrangements resulted in increased engagement when two conditions operated; other students' behaviour did not negatively affect the student and the student liked and felt comfortable with the others they were sitting with (see Ingram, 2008).

Being able to do the mathematics

Being able to do the mathematics was an important phrase used by the students. For the students, success in mathematics meant being able to do the mathematics problems to the level expected of their class placement and in comparison to their peers. Doing mathematics is visual because of its nature. Being able, and being *seen* to be able to do a specific problem was important, because it was evidence to the students, their teachers and their peers (an actual identity) of the student's ability to meet their designated identities. Many of the students' feelings related to being able to do the mathematics and these feelings were associated with their engagement.

> [If you're feeling good] you just like tackle problems like it helps you like go for it really (Corrina$_{\text{GroupInterview2007}}$).

When doing a specific task a student's engagement was often indicated by their perseverance. Some students who had negative feelings and/or felt they were not able to do the mathematics did not attempt the problem or only superficially engaged in it. Not attempting a problem or not persevering is an actual identity; evidence to the student that they are not meeting their designated identities, or for Ruth and Moira below, it might be their actual identities already meet their designated identities associated with their class positioning and being able to do the mathematics.

> I just don't feel like trying because I know I'm going to get it wrong ($\text{Ruth}_{\text{Interview2006}}$).

> If it's really hard I'll try and think about it and if I don't get it like I'll go just straight to the answer yeah…I feel kind of stupid cause everyone else in our class is like real smart and does everything and gets everything done but I just sit there and I'm the only one that needs to work it out ($\text{Moira}_{\text{Groupinterview2006}}$).

Getting help from peers or the teacher is a common strategy when a student becomes confused with a problem and could be viewed as an indicator of engagement. Ironically, those who felt better able to do the mathematics seemed to ask for help more and often referred to it as *discussing* the mathematics with other people. Asking for help was an indicator of student engagement when the student invested energy into clarifying a problem and was committed to understanding rather than simply getting the correct answer.

> Well, I'll try and do [a problem] first, then I'll probably ask if I don't understand it …and then I just ask and ask and ask until someone finally gives me an answer that I understand ($\text{Amanda}_{\text{Interview2006}}$).

> When I come across a hard problem I'll talk to some other people about it … see what their methods are and stuff ($\text{Katrina}_{\text{Interview2006}}$).

Students who felt better about being able to do the mathematics also asked and answered questions more during whole class activities; actual identities visual to the teacher and peers.

Asking for help is complex as it can be a form of disengagement, particularly if the student is unprepared to engage or is dependent on help. Some students were comfortable asking for help and were prepared to alert others to their class position, possibly because they simply did not feel able to do the mathematics and their designated identities reflected this.

> I get frustrated with it…can't you just tell me what it is? Get someone else to do it for me… once it gets a bit more complicated I get the teacher to help ($\text{Robyn}_{\text{Interview2006}}$).

> [If the problem is hard] I just don't bother doing it because I'm too lazy. I'll ask Katrina, or if she doesn't know I ask the teacher sometimes… if I really need to know, then I'll ask the teacher. If it's not like completely necessary than I don't bother ($\text{Susan}_{\text{Interview2006}}$).

To add to the complexity, asking for help was a visual clue that students were not able to do the mathematics. The seating arrangement, their teacher's manner and quality of explanation also influenced whether or not they asked for help.

> I don't like always asking for help cause Ms Brown tries to explain it and I don't get it. If I keep asking when I don't understand I feel really stupid ($\text{Moira}_{\text{GroupInterview2006}}$).

DISCUSSION AND CONCLUSIONS

Sfard and Prusak (2005) write that *learning* takes place to close the gap between a student's actual and designated identities. Because of the nature of the research, engagement, rather than learning, emerged. For the students in this research engagement could be seen as a mediating factor between the interaction of identities and affect, and learning. If the instructional material is appropriate, the classroom environment is conducive, and the teacher is proficient, engagement with the activities of the classroom may be an indicator of learning. When a student engages they are gaining opportunities to fulfil their designated mathematical identities and to be better able to do and understand the mathematics. This improves feelings surrounding mathematics and can perpetuate future engagement.

While students saw mathematics as an important subject, they generally regarded mathematics neutrally or negatively compared to other subjects because of the difference in its nature and routines, comparisons that impacted on their engagement. Boredom strongly impacted students' engagement in the mathematics as it appeared that they found mathematics to be mentally very demanding with a large amount of abstract content to understand but with little ability to make decisions about their own learning. Students in this research found that small variations in classroom routines improved the success of the mathematics lesson in terms of their level of boredom or interest, and resulting engagement. Many of the students' feelings related to being able to do the mathematics and their feelings had a direct impact on their engagement. This idea of being able to do the mathematics has links to the concepts of confidence (Burton, 2004) and self-efficacy (Williams & Ivey, 2001) and incorporates the importance of a student being able to do the mathematics visually. Peers have an impact on a students' level of engagement, which is consistent with the research of Ryan and Patrick (2001). Seating plans may address tension student experience when attempting to balance mathematical and social needs and minimise student disruption and distraction and maximise comfort (Ingram, 2008).

A student's engagement in mathematics is affected therefore by their views of mathematics, factors relating to the context of the moment, and their feelings surrounding being able to do the mathematics. This affects a student's ability to fulfil the expectations associated with their designated identities through engagement. A student's affect, engagement, and identities, in this research, were interdependent.

A mathematics teacher perhaps could get to know the students in their class and learn about the constantly changing mathematical identities that surround each student and how these identities are balanced with the student's social identities. Using this knowledge of a student's dynamic identities as a lens, a teacher might then understand and monitor the student's affective responses and their engagement in the mathematics, seeking indicators of engagement relating to their output, their help seeking, their behaviour during whole-class activities, and interactions with their peers.

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META-REPRESENTATION IN AN ALGEBRA CLASSROOM

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We report knowledge that one U.S. Algebra I teacher and her 8th-grade students used when generating and evaluating equations to model word problems. In addition to knowledge that has been reported in previous research on the teaching and learning of algebra—for instance, different interpretations of the equal sign—the teacher and her students used knowledge associated with meta-representational competence. Past results on meta-representational competence have come from studies based on interviews or experimental instruction. The present study extends these results to more conventional classrooms, adding to an accumulating body of evidence that knowledge associated with meta-representation plays an important role when people use inscriptions to solve problems across a variety of contexts.

INTRODUCTION

The present study focuses on one purpose for which algebra is used, representing and solving problems about the surrounding world. Such activity has always been central to mathematical thinking and continues to be an important focus of algebra instruction. We examined instruction in one Algebra I classroom and asked the following question: What knowledge did the teacher and students use when generating and evaluating equations to solve word problems? We found that the teacher and students made different judgments about how to harness algebraic notation for solving problems. What made sense to the teacher made less sense to the students, and vice versa. We were able to explain this phenomenon, in part, by identifying different criteria that the teacher and the students used for evaluating algebraic expressions and equations. Our result extends research on meta-representational competence (see below for an explanation).

BACKGROUND

Much of the algebra research conducted during the 70s and 80s focused on difficulties that students encounter when using equations and graphs to accomplish tasks (see Kieran, 1992, for a detailed review). For instance, Küchemann (1981) reported that students can avoid working with letters altogether, can interpret letters as short hand references for objects, or can interpret letters as standing for specific unknown values. Kieran (1981) reported that elementary and middle school students tend to interpret the equal sign not as a relation between two expressions but rather as a signal to compute an answer. Clement (1982) and Booth (1981) documented difficulties that students have generating algebraic representations of problem situations.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 241-248. Thessaloniki, Greece: PME.

Further studies have examined students' capacities for building their own graphs, equations, or other inscriptions to accomplish tasks (e.g., diSessa, 2002; diSessa, Hammer, Sherin, & Kolpakowski, 1991; Izsák, 2003, 2004; Meira, 1995). For the present study, the most important results are those that have to do with meta-representational competence, first reported by diSessa et al. (1991). These researchers described a sequence of lessons during which sixth-grade students "invented" graphing. The students, who were participating in an elective after-school class, were asked to consider a situation in which a motorist sped through the desert, stopped for a drink of water, and then drove away slowly. They created approximately 10 different "motion pictures" that communicated key aspects of the motion such as going fast, going slow, and stopping. As part of the activity, students evaluated each other's approaches and refined several.

DiSessa et al. (1991) coined the term *meta-representational competence* to refer to students' capacities to invent and critique various graphical representations of motions. Most relevant to the present article was the result that students marshalled approximately a dozen criteria when evaluating each other's work. Examples included *transparency* (representations should need little or no explanation), *appropriate abstractness* (representations can omit nonessential aspects of problem situations), and *consistency* (conventions should not be adjusted to accommodate features unique to a particular problem situation). DiSessa (2002) reported further results about students' criteria for graphical representations.

Results about meta-representational competence raised questions for further research. First, do knowledge resources associated with meta-representation occur only in conjunction with graphical representations? Izsák has demonstrated that the answer to this question is "No" by extending results on students' criteria to algebraic representations of motion (2003, 2004) and to area models for whole-number multiplication (2005). Second, do knowledge resources associated with meta-representation play important roles only in experimental classrooms and interview-based studies? The present study provides initial evidence that the answer to this question is also "No" by examining the role that criteria played in the reasoning of a teacher and her students around more conventional word problems and in a more conventional classroom. Thus, the present study contributes to an accumulating body of evidence that knowledge resources associated with meta-representation play an important role when people use inscriptions as tools for solving problems across a variety of contexts.

THEORETICAL FRAMEWORK

Figure 1 summarizes the theoretical frame that we used. Although Izsák (2003, 2004) has used a similar frame, we did not force it onto our data. Rather, as we analysed our data, we began to see how the frame helped us understand differences between the teacher and her students. The frame highlights coordination of fine-grained understandings that include criteria. It does not describe a unidirectional sequence of problem-solving steps.

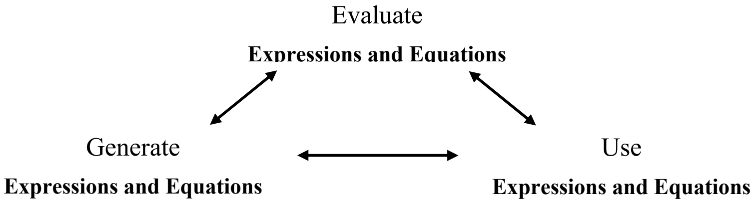

Figure 1: Theoretical Frame.

To *generate* algebraic representations of problem situations, one must select and relate quantities from the problem situation. Various understandings can come into play. These might include understanding that just a subset of salient quantities are sufficient to build an equation, conceptualizing a rate appropriately, or combining intensive and extensive quantities (Schwartz, 1988) appropriately. How one generates algebraic representations of problem situations interacts with ways one anticipates using those representations to solve a problem. Different ways to *use* an equation include guessing and checking, manipulating symbols to isolate a variable, or estimating. When solving familiar types of problems, one night generate and use equations smoothly. In novel situations (e.g., when first learning to generate equations that can be used to solve problems), one needs to monitor, or evaluate, how the solution is going. To *evaluate* an equation, one uses criteria. We present examples in the analysis that follows.

DATA AND METHODS

The present study focuses on a sequence of four lessons from a larger study entitled *Coordinating Students' and Teachers' Algebraic Reasoning* (CoSTAR). The project was set in one public middle school in the Southeastern United States. The goal was to study knowledge used by teachers and students during lessons in which they participated together.

The lessons at the center of this report were part of a unit on writing and solving linear equations from *College Preparatory Mathematics* (CPM) (Sallee, Kysh, Kasimatis, & Hoey, 2002). The goals stated in the materials include helping students "represent word problems with algebraic equations" and helping students "learn to solve linear equations with manipulatives and the fundamental laws of algebra." The present analysis is restricted to the first half of the unit, which concentrates on the first goal. (The second half of the unit concentrates on the second goal.) The teacher, Ms. Jennings (all names are pseudonyms), had been teaching mathematics for 10 years and had taught eighth grade for all but 2 of those years. The class consisted of 26 students between 13 and 14 years of age (13 boys, 13 girls, 17 white, 9 non-white) who had been placed in algebra based on their success in seventh-grade mathematics.

We videotaped lessons during the same class period every day to record continuous instruction in natural settings. We recorded all eight lessons from the enacted unit, which spanned a 3-week period. When recording lessons we did not interact with Ms. Jennings or the students about the content. At the same time that we recorded classroom instruction, the third author conducted weekly interviews with Ms. Jennings and with two pairs of students from the same class period. The interviews began during the second week of lesson taping and lasted approximately 50 minutes each. One pair of students consisted of two girls, the other of two boys. The interviewer used a combination of selected lesson excerpts and tasks. The tasks were either the original tasks in the lesson excerpts or extensions of those tasks. Students worked the tasks using pencil and paper and watched the lesson excerpts on a laptop computer. The overarching goal was to gather data on the students' understanding of the content and how those understandings shaped their interpretations of the lesson excerpts. After interviewing the two pairs of students, the third author interviewed Ms. Jennings. He showed the same lesson video excerpts presented to students. He also showed related video excerpts from the student interviews. The overarching goal was to gain access to Ms. Jennings' interpretations of the lessons, her understandings of her students, and the pedagogical decisions she made. We transcribed all of the interviews and relevant portions of the lessons. We then analyzed both the lesson and the interview data to infer knowledge that the teacher and students used to make sense of the word problems and of each other during the lessons.

RESULTS

The CPM unit instructed students to develop guess-and-check tables for each problem and write equations that expressed resulting patterns. Students had past experience generating guess-and-check tables, but they did not have past experience with manipulating algebraic symbols to solve equations. We summarise the first three lessons briefly and present more details from the fourth lesson. During these lessons Ms. Jennings made statements that provided direct evidence for two criteria that she used. Both were normative criteria that one would expect a person with experience in the domain to have. One was *single variable* (algebraic equations should be expressed in one variable), and the other was *balance* (two sides of an equation should refer to the same aspect of the problem situation). The class discussion of the first problem in the unit illustrates her use of the first criterion. The problem stated:

> The length of a rectangle is 3 centimeters more than twice the width. The perimeter is 60 centimeters. Use a guess-and-check table to find how long and how wide the rectangle is, and write an equation from the pattern developed in the table

Using w for width and l for length, students were able to generate three correct expressions for the perimeter, $(l + w)(2)$, $2l + 2w$, and $l + l + w + w$. Ms. Jennings acknowledged that these were correct, but suggested to the students, "Let's make life a little bit simpler for ourselves. Just use one. Can we name the length in terms of the width?" The students struggled to respond to Ms. Jennings request. Data from the first lesson and the interviews made clear that students had a hard time understanding the idea of substituting an expression for a variable.

During the first three lessons, we observed several more examples in which students generated equations that expressed correct relationships among quantities, but used more than one variable to do so. In each case, Ms. Jennings evaluated the students' algebraic representations based on their use of more than one variable. Furthermore, to justify this evaluation, she made frequent comments that equations in one variable would be easier to solve.

In terms of Figure 1, Ms. Jennings used the single variable criterion in conjunction with her understandings for generating and using equations. She sought to generate equations by expressing all quantities in terms of one variable, because she anticipated the second half of the unit during which students would learn to isolate variables by manipulating symbols. The arrows emphasize the coordination of the these understandings. During these same lessons, students understood that Ms. Jennings was critiquing the number of variables in their expressions but did not see advantages to equations in just one variable. They struggled with the technique of substitution necessary for generating such equations and were not skilled at manipulating symbols to isolate variables.

During the fourth lesson we observed clear conflicts between criteria that Ms. Jennings and her students applied to algebraic representations of problem situations. The class worked on the following problem:

> Antony joined a book club in which he received five books for a penny. After that, he received two books per month, for which he had to pay $8.95 each. So far, he has paid the book club $196.91. How many books has he received?

As in previous lessons, students followed the instructions to develop a guess-and-check table and then generate an equation. Students first worked with a partner and then participated in a whole-class discussion. During that discussion, students proposed five equations to represent the situation. This was a much greater variety of algebraic representations for one problem than in previous lessons. All of the proposed equations contained just one letter, so no direct references were made to the single variable criterion. Furthermore, all of the equations were at least partially correct in that students consistently multiplied intensive and extensive quantities appropriately. For example, they multiplied the cost per book times the number of books, and they multiplied the number of books per month times the number of months.

Amy explained her approach first. She had solved the problem using a guess-and-check table with columns for number of months, number of books, and a check for $196.91. She guessed 11 months and calculated $196.90 for 22 books. She then added one penny and five books to get $196.91 for 27 books, correctly. A few exchanges later, she proposed the following equation and explained that x stood for the number of months:

$$x \cdot 2 + 1 = 196.91. \qquad (1)$$

Ms. Jennings had the class calculate the left-hand side to demonstrate that the result was not 196.91 and therefore, by implication, the equation was not balanced.

Carl commented that the cost of the books should be included and proposed:

$$(x \cdot 8.95) + .01 + (x + 5) = 196.91. \qquad (2)$$

Carl's partner, Tim, had generated a similar equation that included a term for 27 books on the right-hand side. Tim had apparently balanced both the amount of money and the number of books. That sense of balance was now lost, and Carl struggled to explain Equation 2.

Nest, Greg challenged both Equation 1 and Equation 2. The following exchange provided evidence that Ms. Jennings and Greg held conflicting perspectives and that those perspectives were shaped, at least in part, by different criteria. Greg stated:

Greg: I don't think none of that's right because,

Jennings: None of it.

Greg: Because you are equaling it up to a 196.91.

Jennings: Okay.

Greg: But see, and what you're trying to do is find, figure out how many books you got, not the amount of money that you have. So I said that it was 196.91 divided by x plus 5 equals 27.

Ms. Jennings wrote Equation 3. Greg's explanation of equation was unclear because his assignment for x was not stable. At one point, he said it stood for the 22 books. Ms. Jennings responded by saying, "If I knew the whole amount of money, and I was dividing by a number of books that I don't know, what would I be finding?" Greg answered, 8.95. Ms. Jennings pointed out that the problem already stated the price per book, implying that the result of the division would recreate given information.

$$(196.91 \div x) + 5 = 27. \qquad (3)$$

Greg rejected Equations 1 and 2 due to the units of \$196.91. He used what we call the *final units* criterion (each term in an equation should be expressed in the same units as those of the final requested quantity). In contrast, Ms. Jennings used the criterion that equations should be constructed with *given information* in the problem.

Maria offered the first correct equation. She let x stand for the total number of books that Antony received, not just those that he received for \$8.95 each:

$$(x - 5) \cdot 8.95 + .01 = 196.91. \qquad (4)$$

Like Amy and Carl (Equations 1 and 2), Maria multiplied intensive and extensive quantities appropriately. Students called Equation 4 into question, however, objecting that it was inconsistent with their guess-and-check tables. Ms. Jennings also called the equation into question. She apparently did not see quickly that, when adding the penny, Maria restored the initial five books using a different quantitative unit.

After further discussion during which Cathy made reference to the final units criterion to explain why she preferred Greg's Equation 3, Maria proposed:

$$196.90 \div 8.95 + 5 = x. \qquad (5)$$

This equation expressed a correct calculation for the total number of books that Antony received. One student questioned where the one penny was, but then several students, including Cathy, agreed with audible excitement that Maria's new equation did work. Ms. Jennings also agreed that it was correct. In a subsequent interview, Maria used the final units criterion to explain why she preferred Equation 5 over Equation 4.

This was the first time over the course of the four lessons that Ms. Jennings and her students agreed that a particular equation was a good algebraic representation of a problem situation. They reached consensus around Equation 5 not so much because they shared a common perspective but rather because they could each bring to bear a coordinated set of understandings for generating, using, and evaluating equations to model the book club problem. From Ms. Jennings point of view, Equation 5 satisfied all the criteria she had used over the course of the four lessons. It contained only one variable (*single variable*), was balanced (*balanced*), and used only numbers contained in the original problem statement (*given information*). From the students point of view, Equation 5 satisfied the final units criterion.

CONCLUSION

The present study contributes to accumulating evidence that knowledge associated with meta-representation plays an important role when people use inscriptions as tools for solving problems. In particular, the present study demonstrates that criteria can play an important role in more conventional instructional settings. Key evidence came when Ms. Jennings and her students considered equations that related quantities correctly but then evaluated those equations in different ways. These differences were consequential because Ms. Jennings and her students only reached consensus at the end of the fourth lesson when considering an equation for the book club problem that satisfied simultaneously the most visible criteria that we observed.

The present study also contributes insight into knowledge of algebra that teachers need to be effective. Just a small portion of the algebra has examined teachers, but Kieran (2007) noted that research on algebra teachers is beginning to increase. In particular, students are more likely to perceive mathematics as a sense-making activity if notions they find salient, like the final units criterion, are engaged rather than ignored. This suggests, in turn, that it is important for teachers to developed the capacity to recognize and discuss students' criteria for representations. So doing should help students learn to harness inscriptions as tools for representing and solving problems about situations.

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CHARACTERSING THE TEACHING OF UNIVERSITY MATHEMATICS: A CASE OF LINEAR ALGEBRA

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This paper focuses on university mathematics teaching where the topic is linear algebra. The research team includes two mathematics educators and a mathematician who collaborate to study the teaching approach and the issues it raises for teaching-learning at university level. We see university mathematics to constitute a community of practice in which the practitioners are those who do mathematics. Such a perspective draws sociohistorically on established practices in doing, learning and teaching mathematics within a university. The paper offers an interpretation of these theoretical perspectives in relation to a first year course on Linear Algebra. We look at how teaching is constructed within the particular setting, with a critical eye on the learners, on learning outcomes and on the tensions experienced by the lecturer in satisfying student needs and mathematical values.

INTRODUCTION

The research reported in this paper is a case study within a broader project. The project focuses on mathematics teaching at university level and seeks to explore and characterise such teaching. The topic in focus is *Linear Algebra* which is taught as a year-long module in the first year of a three- or four-year undergraduate mathematics programme. In this research we seek to characterise the teaching of linear algebra particularly, and to use this topic to gain insight into university teaching of mathematics more broadly. The case study is collaborative between three researchers: a mathematician who teaches linear algebra and two mathematics educators who observe and analyse. It takes place within a *School of Mathematics* (SoM) which includes a *Mathematics Education Centre* (MEC). Mathematicians and educators (with considerable overlap) teach mathematics and undertake research into mathematics and into mathematics learning and teaching. Research questions include:

1) What is the nature of linear algebra teaching in this module?

2) What issues are raised when *linear algebra teaching* becomes a developmental focus?

THEORETICAL PERSPECTIVES

We draw on social practice theory (Lave & Wenger, 1991) to perceive university mathematics as a community of practice in which the practitioners are those who do mathematics at all levels including students and mathematicians (as in Hemmi, 2006). Within the community, participation varies according to the particular role of the participant. Thus an undergraduate student has a different role from a graduate student which differs from the role of a research mathematician. We draw

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 249-256. Thessaloniki, Greece: PME.

particularly on the perspective offered by Wenger (1998) to characterise *identity* within a community of practice and the nature of *belonging* to such a community. The differing identities of mathematicians and students are especially relevant to seeing mathematics learning as a process of enculturation into established practices whose socio-historical dimension extends beyond the particular institution as well as being rooted in local traditions and cultural perspectives (Cole, 1996; Wertsch, 1991). In our case, the goals of a *Mathematics Education Centre* within a *School of Mathematics,* to engage in research into mathematics education at university level, are especially significant for the research we undertake. We seek to know more about the practices in which we and our colleagues engage in order to have the power to develop them in informed ways.

Surveying research into the teaching and learning of linear algebra, Dorier and Sierpinska (2001) write “It is commonly claimed … that linear algebra courses are badly designed and badly taught, and that no matter how it is taught, linear algebra remains a cognitively and conceptually difficult subject” (p. 255). Various authors offer suggestions for teaching in ways that encourage conceptual understanding (e.g. Berry, et al., 2008). We do not have space here, however, to review a considerable literature that recognises issues in linear algebra and ways of addressing the difficulties students face. Here our focus is on how teaching is constructed and how it recognises and takes into account the experiences of students. Unsurprisingly, we come up against student difficulties. Our aim is to characterise teaching and look towards approaches to understanding teaching better in order to address student difficulties and promote teaching development.

METHODOLOGY

Our methodology is *developmental* (Gravemeier, 1994). It includes traditional elements of case study and ethnography but goes further to establish a collaborative process in which insider and outsider researchers both study the practices, processes and issues in mathematics teaching-learning and use the research as a basis for understanding and reconsidering the practices involved (Jaworski, 2003). In this case relationships between insiders and outsiders may be seen to constitute a “clinical partnership” (Wagner, 1997, p. 15) involving inquiry in “jointly defined work” in which activity is open to scrutiny by all. In practice, research team members take differing roles such as the one who (mainly) designs and performs the teaching (the lecturer) and the ones who (mainly) observe and work with the data (the observers).

Data collection includes observation of lectures and tutorials, (audio-recorded), and subsidiary observation of small group tutorials (SGT - involving 6-9 students and tutored by other members of the SoM). A key feature of data collection, fitting with the collaborative spirit of the project, involves recorded team meetings between the three of us immediately after a lecture (standing outside the lecture theatre, walking through the campus, or over lunch), as well as more formal meetings where we sit in an office. These provide insight into established modes of planning and also allow us to capture an immediacy of perception and to gain insight into issues as they arise. Analysis involves a data reduction process in which qualitative data are summarised, categorised and coded and transcriptions are made of data relating to key elements in categories on which research focuses. Detailed analysis of extracted key elements follows to address particular research questions which evolve alongside analysis. Student perceptions have been sought through two surveys and focus group meetings are planned. The research is ongoing and we report from preliminary analyses.

ELEMENTS OF A MACRO PERSPECTIVE ON TEACHING

We draw here on our data broadly, as well as on administrative information concerning organisation within the SoM and university regulations on teaching and examining. The linear algebra module is one of two year-long modules taken by all first year students in mathematics programmes (the other is calculus). The module is taught over 24 weeks (2 semesters) with two lectures and one tutorial each week and a cohort of 240 students, of which up to 180 attend lectures regularly. One of our team is the lecturer for the first semester; there is a different lecturer in the second semester. The two lecturers collaborate on the year-long design of the module and prepare a joint examination at the end of the year. The first semester offers an introduction to linear algebra and the second semester a more abstract treatment.

In the first semester, the lecturer prepares notes-with-gaps which are placed on LEARN (a virtual learning environment) for students to access in advance of a lecture. A purpose of the ‘gaps’ is to encourage students to attend lectures and complete the notes with solutions of key examples presented in a lecture. The lecturer’s design of the module includes choice, sequencing and written presentation of content, choice of examples, a weekly problem sheet, and preparation of assessment tasks which include on-line tests and tutor-marked coursework. Tutors of the SGT are sent problem sheets and are required to mark coursework. They are also personal tutors for students in their group, so they have access to student progress and student experiences of learning and teaching.

Issues emerging to date involve sequencing of content, choice of examples, emphasis on mathematical language and insight into student understanding. We focus on key elements of lectures and tutorials, including use of examples and metacommenting, and on perceptions of thinking with respect to the teaching of university mathematics and linear algebra in particular. Initial analysis suggests two significant features of this thinking which we refer to as *didactical challenge* and *didactic tension*.

ELEMENTS OF A MICRO PERSPECTIVE ON TEACHING

Analysis here is principally of data from project meetings, backed up by data from lectures and tutorials and minimally from SGTs and two student surveys. The meetings provide an opportunity for the lecturer to talk about his design of the module, his current teaching and perceptions of students’ learning and issues arising thereof. The two observers ask questions and offer observations or perceptions.

The lecturer talking in expository and didactic mode

Typically, in a lecture, the lecturer introduces material and works through examples, following the notes which students are asked to print and bring to a lecture. He *presents* some examples with *mathematical comments* and *metacomments* (see below). With other examples he invites students to tackle the example while he circulates and interacts with some students on the periphery of the lecture theatre. Our observations show that there is a buzz of talk while this happens: some students do not work on the example, rather seeming to wait for the lecturer to resume his exposition; others get involved with the example individually or in small groups.

Discussion in meetings has focused on responses of students to the examples and the lecturer's perception of students' understanding related to the material of the lecture. Often the nature of this discussion includes the lecturer talking about his own conceptions of the material of the lecture, of his didactical thinking with regard to this material, of his perceptions of students' activity and of his decision-making in constructing notes, examples and assessment tasks. The example below, of the lecturer's talk, shows *expository mode* (talking about his own conceptions of the material) in normal text and *didactic mode* (talking about his construction of the teaching of the material) in italic text.

> *Thursday is about defining the characteristic polynomial, understanding that its zeroes are the eigenvalues, and I'll show an example of an eigenvalue that has algebraic and geometric multiplicity 2*. Algebraic multiplicity, meaning this is the power with which the factor lamda minus eigenvalue appears in the characteristic polynomial, and geometric multiplicity is the number of linearly independent eigenvectors. *And these are the important concepts for determining if a matrix is diagonalisable because, for that, we need sufficiently many linearly independent eigenvectors.* Now if an eigenvalue has algebraic multiplicity larger than 1, that means there are correspondingly fewer eigenvalues. So, in principle, we can fail to find as many eigenvectors as we need in that case. On the other hand, if an eigenvector has algebraic multiplicity 3, the geometric multiplicity can be anywhere between 1 and 3. If it's 3, we are fine, if it's less than 3, we're missing out at least one linearly independent eigenvector. And in such a case the matrix would not be diagonalisable. *And that's the big observation that we need to get at next week, that a matrix is diagonalisable if and only if all the geometric multiplicities are equal to the algebraic multiplicities.*

The distinction between expository mode and didactic mode is not clear cut. The sentence in italics in the middle of the quotation, might also be characterised as expository mode. However, it seems here that the lecturer is *meta*-commenting on the material: i.e. expressing his value judgment regarding important concepts that need to be appreciated, rather than just articulating mathematical relationships. This seems to relate to didactic judgments in terms of what needs to be emphasised for students. We observe that such statements in meetings correspond to what we have called meta-comments, or meta-mathematical comments in lectures. Such comments address what students need to attend to, either in terms of their work on the mathematical content (meta-comments -- A) or of their understanding of the mathematical content (meta-mathematical-comments -- B). Examples A and B follow.

> A: First of all, … if I give you an equation system, this gives you a recipe to decide if that equation system is consistent or inconsistent. You transform it to echelon form and you check if there is such a special row that makes the system inconsistent.

> B: But it's important that you be able to understand the language that we're using and to use it properly. So please, pay attention to the new terms and the new ideas that we're going to introduce over this chapter.

We are emphasising this difference in modes of talk about the material of the module to contrast thinking about teaching (the didactic mode) with thinking about mathematics (expository mode). In meta-comment A, the lecturer draws students' attention to the nature of the mathematics and how they work with it. In met-mathematical comment B, he draws their attention to the processes of working with the mathematics and strategies that can lead to understanding. Both of these are "didactical" approaches on the part of the lecturer.

The lecturer reflecting and raising issues

In our meetings, we discussed frequently the kind of feedback that the lecturer received from students as to their understanding of the module material. In a lecture theatre with 180 students, feedback is not easy to recognise or interpret. For example,

> I do think, however, that didn't go very well because for many students, many students aren't sufficiently familiar with the idea of a linear transformation. We have discussed that many times, that a linear transformation is a function that is defined by a matrix. But my impression is that very many students haven't absorbed that idea of reading a matrix as a function. And whenever I talk about the transformation that is defined by a matrix in a small group tutorial, or when going around in class, quite often I get a blank stare. Now that being as it is there seems not much point in trying to express that function in a different basis, so that is … probably most students haven't really absorbed that section.

What is going on behind the "blank stare" is of course hard to interpret. The lecturer talks of the students having not "absorbed that section", referring to a section of the notes. This raises questions as to what it means for students to "absorb" material, how such absorption is thought to occur, and how the didactical process of module design relates to what students make of what they experience. However, the lecturer has to use whatever clues he can pick up from students. His impression is that students struggle with more conceptual material. Thus, his design of teaching has to take account of such difficulties and what is possible in the time allocation. For example,

> Usually people [lecturers at this level] do diagonalisation of matrices on the level of conjugation with an invertible matrix … and then we try to see if there is such a matrix. The disadvantage for that, I think, is that it's difficult to motivate. On the other hand, if the way to motivate it requires these abstract concepts that are so difficult to get across at this level, then it might well be that's the way to go. I don't know yet what I'm going to do about it next year. But the two alternatives I can see, either I can go back to leaving out the basis expansion stuff and just do conjugation with matrices, without providing

> much motivation at that point, and relying on [the other lecturer] to do more about basis expansions in his part of the course, which he will. The alternative is, I can do a lot more on linear transformations and try to get that concept across, which will require that, somehow, I find quite a bit of space in my module which will be very difficult to do.

Thus there are issues in what to offer and how to offer it that relate to the mathematics, to what the students need and what is possible in the available time. These are familiar issues in linear algebra (see Dorier & Sierpinska, 2001).

Issues in the lecturer's didactical decision-making

Discussion in the team has made clear that the lecturer tries out approaches to his teaching that he has described as "experimenting". An example of an experiment has been to give students some exploratory work to do in a lecture in order to get students to try their own approach before the lecturer offers a more formal explanation. Asked by one of the observers about this experiment, the lecturer replied:

> That's one of my experiments and I think largely it has gone well. At some points I realised I need to find different ways of phrasing the questions in order to make them more accessible. One example of that was the introductory example of, on subspaces, where I had asked students to find solutions to a homogeneous equation system with unknown coefficient matrix, given that they know a couple of solutions that I've given them. That was one question where I saw quite clearly that some of the students found it very easy, and some of the students didn't have the slightest idea even if they tried. And so at that point, because the concepts that come out of this example are so important for everything in Linear Algebra, because they lead to the ideas of linear combinations and linear independence, all that, because that is so important I think it would be good if I could come up with a way to make this example more accessible to students. As it were, to put in a couple of stepping stones for students who can't take it in the way in which I presented that. … And I am not sure how to do that.

This statement indicates that the lecturer is experimenting and (sometimes) being satisfied with the outcomes of his experiments. Experimenting has the additional effect of drawing the lecturer's attention to other issues with which it is not always clear how to deal. He asked the observers on one occasion "have you got any ideas?" thus opening opportunity for didactic discussion and wider consideration of issues.

One area of issues has emerged in recognition of students who can engage with *abstraction* and appreciate *concepts* in abstract relationships and students who remain at a more *computational* level.

> In effect, I'm saying they have mastered the material on that computational level. I am very happy that they have. But of course I would want them to be able to go further than that and put things into context a little bit more. … I had asked students to check if a given number is an eigenvalue of a matrix and if so, find the eigenvectors. And they look up how I did that in the examples and then they know they have to write down 'A minus lamda-I' and put the zero next to it. On the computational level most students can do that but then, of course, the way I would like them to think about it is, I do this calculation because I've got the eigenvalue equation and this is what it means for a number to be an

> eigenvalue, that I check if there is a non-zero vector that satisfies that. My suspicion is that for most students it's 'I do this calculation because the lecturer did that calculation in a similar example', which is again, on the computational level they can do it but most are probably not thinking about that material the way I would like them to think about it.

"The way I would like them to think about it" is on a conceptual or abstract level rather than at the more instrumental or "computational" level. So a didactic challenge for the lecturer is what approach might result in students thinking conceptually. He expressed a tension in thinking about his approach:

> I wouldn't want to go to a system that means we are in effect only teaching the top half of the class that's coming in. On the other hand, I do think we should challenge our students to adopt, well, more abstract, certainly more conceptual, views of things because that's where the power of mathematics comes from, … if you solve the coursework problems and exam problems on the level of 'if the question says this, that's the calculation you do', you have no way of adapting that to even slightly different circumstances than what the standard set of circumstances is, to which the typical exam problem is geared.

In the first sentence above, he made reference to a system in which a course presented at a highly abstract level overall resulted in the drop-out of half the students. Nevertheless, while it seems inappropriate to introduce linear algebra through a largely abstract/conceptual approach, it also seems unsatisfactory if students achieve only computational facility without appreciating conceptual relationships. It may be that the conceptual understanding comes in the second semester, but nevertheless there are issues here for the two lecturers to consider further in their overall design.

The tension highlighted here might be called a *didactic tension* (Mason, 2002), emphasising outcomes in which teaching approaches lead to form rather than substance. Students learn to walk the talk rather than talking the walk. They achieve a form of instrumental facility rather than a relational understanding (Skemp, 1976. These are concerns recognised widely in mathematics education. How they are dealt with at this level however is not well understood.

Concluding remarks

Linear algebra is generally a basic topic in an undergraduate degree programme and there are expectations in university mathematical culture as to what a linear algebra course will achieve in terms of students' knowledge and facility. Conversations within the SoM reveal such expectations and teachers designing a course do so as part of such a cultural position. Teachers and students form identities in relation to established cultures and mathematics learning for students can be seen as enculturation into established practices and ways of being mathematical.

We are trying to make sense of what it means to design and teach at this level, and how design of teaching relates to outcomes for students. Revealing issues as in our examples above enables us to address how such issues are tackled and to open up dialogue – a teaching discourse – within the mathematical culture. Within our particular environment, a School of Mathematics with a Mathematics Education Centre, we see an essential part of our community of practice to be to encourage such a discourse to enable us to address collectively how we work with students and to develop practice in mathematics teaching. Awareness of didactical challenge and a didactic tension can illuminate practice more broadly. Research such as we describe here starts to open up an inquiry process into the teaching of university mathematics. It brings practitioners together to inquire into issues of mutual concern. Its outcomes both inform and contribute to the development of an inquiry community where research into teaching becomes a regular part of teaching practice and the community becomes more knowledgeable about its teaching (Jaworski, 2008).

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ARITHMETIC EQUALITY STATEMENTS: NUMERICAL BALANCE AND NOTATIONAL SUBSTITUTION

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Numerous studies have investigated the benefits of teaching young children that the equals sign means "is the same as" and presenting a variety of statement forms such as a+b=b+a and c=a+b. However, an important and overlooked aspect of equivalence relations is that of replacing one term with another, which implies a "can be substituted for" meaning of the equals sign. I report a trial with a pair of primary pupils working on a computer-based task that requires viewing equality statements in terms of both numerical balance and notational substitution. I present screenshots and transcript excerpts to illustrate how they articulated and coordinated balance and substitution in order to achieve the task goals.

INTRODUCTION

Young children tend to view the equals sign as a place-indicator for an arithmetic result, as in 2+3=5 (Behr, Erlwanger, & Nichols, 1976). This view can prove exclusive and stubborn (McNeil & Alibali, 2005), and can lead to later difficulties with equation solving (Knuth, Stephens, McNeil, & Alibali, 2006). Numerous studies have demonstrated that such difficulties can be reduced if young children are taught the equals sign means "is the same as" and are exposed to a variety of statement forms, such as 5=2+3, 2+3=3+2 and so on (e.g. Baroody & Ginsburg, 1983; Li, Ding, Capraro, & Capraro, 2008; Molina, Castro, & Castro, 2008). However, a full conception of equivalence relations involves not just numerical sameness but also the substitution of equivalent terms (Collis, 1975). In fact the notion of substitution underpins the transitivity, reflexivity and symmetry that define mathematical equivalence (Skemp, 1986). This paper reports a trial with a pair of primary children as part of a wider study into the pedagogic affordances of a computer-based task in which the equals sign means both "is the same as" and "can be substituted for".

STRUCTURAL APPROACHES TO NOTATING TASK DESIGN

Approaches to notating-task design can be referential or structural (Kirshner, 2001). Referential approaches, such as modelling, import symbol meaning from external objects. In structural approaches meaning instead arises from the relationships between symbols. In practice, however, symbols are often implicit referents to abstractions such as numbers and arithmetic principles (Dörfler, 2006). For example, to present 2+3=3+2 as a question of numerical balance is to ask for a comparison of the number referenced by each side. Pedagogically the statement can be considered a referent to the (abstract) principle of commutativity, by which a learner might satisfy herself of numerical balance without knowing the actual number on either side.

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Dörfler has argued this implicit role of symbols as referents to otherwise inaccessible abstractions is widespread but unacknowledged in mathematics education, and is a factor in many learners' sense of alienation.

> [they] often fail to get close to those genuine objects which mathematics purportedly is all about, they believe they lack the necessary abilities to think 'abstractly', and they are convinced that they do not understand what they are expected to understand. They want to reach through the representations to the abstract objects but without success. (p.100)

Dörfler suggests learners should be presented with notating tasks in which symbols and their transformations are themselves the "very objects" of mathematical study. Learning symbolic mathematics can then be seen as an exploratory, empirical activity in which learners make discoveries and test hypotheses. For example, if 2+3=3+2 is presented as a rule for making the substitutions 2+3 → 3+2 and 3+2 → 2+3 it becomes a reusable tool for transforming notation, rather than a closed question of balance that is discarded once assessed. The commutative property of 2+3=3+2 can be observed as an exchange of numerals when the statement is used to transform arithmetic notation.

TASK DESIGN

In this section I describe a computer-based task in which learners are presented with sets of inter-related arithmetic statements for making substitutions, akin to simultaneous equations in algebra. The software presents a sequence of "puzzles" comprising equality statements and a boxed term at the top of the screen (Figure 1). The goal is to use the statements to make substitutions in the boxed term (45+16) in order to transform it into the equivalent numeral (61). We might begin by selecting the statement 45=32+13 and use it to transform the boxed term 45+16 → 32+13+16; we might then use the statement 32+13=13+32 to transform 32+13+16 → 13+32+16; and so on until 61 appears in the box. (Note that substitutions are reversible due to the symmetry of equivalence relations).

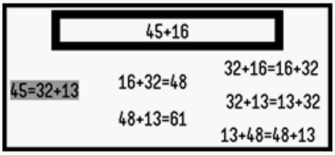

Figure 1: Screenshot of the computer-based task.

The task supports several novel pedagogic affordances which have been investigated in previous trials with pairs of 9 and 10 year old children (Jones, in press). For example, the task requires searching for multiple occurrences of numerals and terms in order to determine where substitutions can be made, and this is quite distinct from viewing statements as isolated questions of balance.

The affordance of interest here is that arithmetic principles can be observed as transformations of symbols rather than as relationships between the abstract numbers they reference. The substitutive effect of 32+13=13+32 can be observed as the commutation of numerals played out on the computer screen. Likewise the substitutive effect of 45=32+13 can be observed as partitioning the numeral 45. (Note that the statement is partitional only if viewed in terms of replacing the numeral 45 and might as readily be written 32+13=45).

In previous trials all the children readily articulated and exploited commutative transformations (“swap”, “switch round”) when solving puzzles, and about half articulated and exploited partitional transformations (“split”, “separate”). This is not to claim that a child who observes the transformation 32+13 → 13+32 on a computer screen and describes it as “swapping the numbers round” is necessarily drawing on the principle of commutativity; she may simply be indifferent to conservation of quantity (Baroody & Gannon, 1984). This was explicitly tested in one set of trials in which some puzzles contained false equalities, as in 77=11+33. It was found the children did not comment on the imbalance of such statements and their presence had no effect on how they worked with and talked about the puzzles (Jones, 2008). This is because the task promotes a “can be substituted for” meaning for the equals sign *instead of* an “is the same as” meaning. Analogous to algebraic symbol manipulation, there is simply no advantage to considering numerical balance or conservation of quantity when working towards the task goal of transforming the boxed term into a single numeral.

In the remainder of the paper I report a pilot trial of a variation on the task that seeks to overcome this disconnect between numerical balance and notational substitution. Instead of working through presented puzzles the children were challenged to make their own using provided keypad tools. This necessitates coordinating sameness and substitutive views of the equals sign because when inputting a statement learners must ensure it is both numerically balanced and capable of making substitutions. (The software allowed false equalities to be inputted but displayed them using the ≠ symbol, and they could not be used to make substitutions). Moreover, the children were given access to +, × and − operators (in previous trials the presented puzzles contained only + operators) and so needed to consider the conservation of quantity across transformations. For example, 2+3=3+2 could be used to make a substitution in 2+3+10 but not in 2+3×10.

FOCUS

The purpose of the pilot trial was to gain initial insights into how children might coordinate balance and substitutive views of equality statements when making puzzles. Any conclusions drawn from a single trial are necessarily limited but can provide novel insights in light of the predictions inferred from previous trials. Two forms of evidence for children coordinating numerical balance and notational substitution were sought: (i) the complexity and functionality of their puzzle designs; (ii) their discussion when working together on puzzle making. In previous trials substitution was articulated by comments such as “we can use that [a+b=b+a] to swap them [a and b] round”, and so on. In this trial, assuming the children succeed at making coherent puzzles, we should expect such comments to be constructively combined with discussion about the numerical balance of statements and, when the ordering of operations is a factor, discussion about the consistency of the boxed term's value across transformations.

METHOD

The method used was paired trialling and qualitative analysis for evidence of talking about mathematics in novel ways (Noss and Hoyles 1996). The participants were two talkative primary school children from the same class (Yusuf, male, 9 years; Sasha, female, 10 years). Both had been involved in a previous trial some one month prior and were familiar with the puzzle solving task but not the keypad gadgets or puzzle making task.

I introduced the keypad gadgets, challenged the children to make their own puzzles, and remained present throughout the trial to ask for verbal elaborations (e.g. "Why do you think that didn't work?") and offer encouragement. Initially I provided keypads with only + operators and later introduced × and then − operators. The total trial lasted 43 minutes. Data were captured as audiovisual movies of the children's discussion and on-screen interactions, and transcribed and analysed for articulations of the numerical balance and substitutive effects of equality statements.

FINDINGS

After being shown how to use the keypad gadgets, Yusuf and Sasha made eight solvable puzzles during the trial (Figure 2). The first puzzle (a) contains two statements both of which are substitutive with regard to the boxed term, but only one of which is required to solve it (i.e. produce a single numeral in the box). The second puzzle (b) contains four statements, one of which is redundant (30+2=2+30). The next three puzzles (c-e) contain between two and five statements, all of which are required to solve the puzzles. The final three puzzles (f-h) all have more than one type of operator in the boxed term and all contain one or three redundant statements that play no part in solving them. (The redundant statements are 5×5=25, 10×5=50 and 50=10×5 in puzzle f; 50+8=58 in puzzle g; and 8×6=48 in puzzle h).

The children therefore successfully made complex puzzles comprising multiple statements that were both balanced and substitutive with regard to the boxed term. The presence of redundant statements in all the puzzles where the boxed term includes more than one type of operator (f-h) also suggests the children experienced some challenges ensuring the conservation of quantity across transformations where the ordering of operations was a factor.

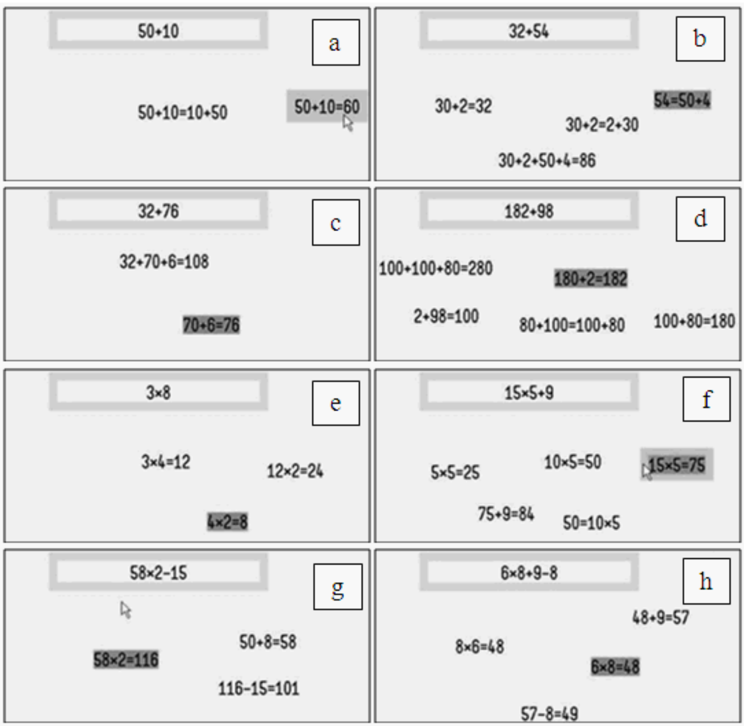

Figure 2: Puzzles made by the children.

The remainder of this section presents three transcript excerpts to illustrate how the children's discussion provides evidence that they considered and coordinated numerical balance and substitutive meanings for equality statements.

The first excerpt is from the start of the second puzzle (Figure 2b) and illustrates the children attempting to make a statement that is both balanced and substitutive. Yusuf had put 32+54 in the box and wanted to partition the numerals (turn 1). Sasha suggested 30+2 for one side of the statement (turn 2) and Yusuf inputted 30+2=2+30 without giving a clear reason (turn 5). He had entered a numerically balanced statement but, as Sasha pointed out (turns 6-8), it was not substitutive with regard to the boxed term. Following this Yusuf attempted to make the statement 32=23 (turn 10), which is substitutive but imbalanced. Sasha then suggested a statement that is both balanced and substitutive (i.e. 30+2=32; note that Sasha intended this statement to partition the numeral 32 suggesting a right to left substitutive reading). ("R" is the researcher).

1 Yusuf: I want to split them two *[the numerals in the boxed term]* apart.

2 Sasha: Let's do 30 add 2.

3 Yusuf: You can't do that bit because that's 2, that's 30 plus 2.

4 Sasha: Oh yeah.

5 Yusuf: *[unclear]* We can do two more sums, innit? *[pause]* 2 plus 30, so we want, and then we just place that there. *[enters $30+2=2+30$]* Then...

6 Sasha: I don't think that one works.

7 R: Why not?

8 Sasha: Because it doesn't give 30 add 2 in the box.

9 R: Do you see what she means, Yusuf?

10 Yusuf: Mm-hm. Wait, 32 then put *[unclear]* over here, what's it called? 23, and then... *[inputs 32 and 23 at the keypad, giving $32\neq23$ on screen]*. I don't care if the equals sign's not there but I know how to do it. Does the equals sign have to be there? *[attempts to select $32\neq23$ to make a substitution but finds he can't]*

11 R: Sasha, have you any ideas how to help?

12 Sasha: Um, maybe you can make that sum and then put the answer on the other side, and then, make something that equals 32 and make the other one.

13 R: Okay. Yeah, I think I understand but show us what you mean.

14 Yusuf: 30 plus 2 you're saying, and what else? Okay, you get that but... *[inputs $30+2=32$ and transforms the boxed term $32+54 \rightarrow 30+2+54$]* Yeah because we can split that *[32]* into that *[30+2]*.

The second excerpt is from the start of the sixth puzzle (Figure 2f) and illustrates the children's awareness of the need for conservation of quantity across transformations. They had entered $15\times5+9$ in the box and Yusuf suggested entering $5+9=14$ to make a substitution but then decided against it (turns 15-17). When prompted both children indicated that $15\times5+9$ and 15×14 would give different results (turns 18-20).

15 Yusuf: 5 plus 9, no you can't do that. 5 plus 9, because that would make 14. 15 times 14. No. *[laughs]*

16 R: Sorry, what do you mean "no"?

17 Yusuf: Swap it round *[i.e. replace 5+9 with 14 in the boxed term]* makes 15 plus 14. *[he presumably meant 15* times *14]* So that won't work.

18 R: Why not?

19 Yusuf: It's going to be different.

20 Sasha: *[talking at same time as Yusuf...]* Because then you get another answer.

The third excerpt is from the start of the seventh puzzle (Figure 2g) and illustrates the children's attempts to substitute the term $2-15$. Yusuf had entered $58\times2-15$ in the box and Sasha suggested the statement $15-2=13$ (turns 21-25). Yusuf pointed out that it could not make a substitution in the boxed term because the 15 and 2 are commuted and attempted to input $2-15=15-2$. He seemed doubtful of its validity and was unsurprised when it appeared on screen as $2-15\neq15-2$ (turn 26). He then suggested the $\times$ operator in the boxed term was the problem (probably alluding to conservation of quantity across transformations – turn 29) but Sasha pointed out that the statement was imbalanced (turns 27-28 and 30-32).

21 Sasha: I think 15 takeaway 2.

22 Yusuf: 15 takeaway 2?

23 Sasha: Yeah, and then ...

24 Yusuf: Are you sure?

25 Sasha: ...do 13

26 Yusuf: No, first we got to switch that *[i.e. 15−2 in Sasha's suggested statement]* around. *[pause]* Let's do 2 takeaway 15. *[inputs 2−15=15−2]* Let's see if that equals, er, recognises it, first. *[the statement appears on screen as 2−15≠15−2]* No, the equals sign isn't there so we can't do it.

27 R: Why isn't the equals sign there?

28 Sasha: Because 2 takeaway 15 is minus 13.

29 Yusuf: *[talking over Sasha...]* Because the top *[boxed term]* has a times. So you can't do, because that one is...

30 R: Sasha, say that again I don't think Yusuf heard it.

31 Sasha: Um, because 2 takeaway 15 equals minus 13.

32 Yusuf: Yeah, that's true.

CONCLUSION AND FURTHER WORK

The task offered an exploratory structural approach to notating in which the statements made by the children were rules for further mathematical activity, rather than static expressions of arithmetic principles. Consistent with Dörfler's (2006) vision of symbolic mathematics learning, the children's focus was on patterns of actions within the puzzles, notably commuting and partitioning numerals. They made predictions about the effect of potential substitutions in an exploratory and reflective manner that took account of the *substitutivity* of statements with regard to the boxed term (turns 6-8 and 26), the *numerical balance* of statements (turns 27-32), and the *conservation of quantity* across transformations (turns 15-20 and 29). This enabled the children to confront and discuss issues such as the ordering of operations in 15×5+9 (turns 15-20) and the non-commutativity of 2−15 (turns 21-32) in a way that was purposeful and meaningful to them.

In this manner the children engaged with a duality of "is the same as" and "can be substituted for" meanings of the equals sign. This stands in contrast to the wider literature that reports young children engaging exclusively with the sameness meaning, which supports discerning statements by truthfulness, and previous trials from this study (Jones, 2008) in which they engaged exclusively with the substitutive meaning, which supports viewing statements in terms of potential transformations of notation, akin to algebraic equations. It is tentatively speculated that such a duality of meanings for the equals sign might help with the transition from arithmetic and algebra, although this conjecture requires further investigation.

The outcomes of this pilot trial have been used to design and conduct further trials of children making their own puzzles. Analysis so far supports and builds on the findings reported here.

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ON FUNCTIONS: REPRESENTATIONS AND STUDENTS' CONCEPTIONS

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The main research questions of the present study are: a) does the way students conceive a function depend on its representation and b) are the procedures used by the students related to their conceptions about the notion of function? To this purpose a test was administrated to 190 students (17-years old). Functions were represented numerically, algebraically and graphically. Students' answers were analysed according to the procedure used and the way the function involved were conceived. The main results were as follows: a) students' conceptions depend on the function's mode of representation, b) in the case of the graphical representation of a function, conceptions and procedures used are related and c) when a function is represented numerically or algebraically, conceptions and procedures are not related.

INTRODUCTION

Research on conceptions about the notion of function is an important issue for mathematics education. Pupils and students have difficulties in conceptualizing the notion of function. The epistemological complexity of the concept (Sierpinska, 1992) and the diversity of the representations used (Hitt, 1998) are the two main factors that influence the understanding and learning of functions. Previous research on students' conceptions in the case of graphical representation of functions reveals three different approaches to conceive a function: the geometrical, the algebraic and the functional (Kaldrimidou & Ikonomou, 1998). In the same context, students used three different procedures to draw a graph: the point-by-point, the step-by-step and the holistic procedure.

In the present study we try to extend this previous research when functions are represented algebraically and numerically. The main research questions were as follows: a) does the way students conceive a function depend on its representation and b) are the procedures used by the students related to their conceptions about the notion of function?

THEORETICAL FRAMEWORK (THE CONCEPT OF FUNCTION AND ITS REPRESENTATIONS)

As we have already mentioned, one of main factors that influence the understanding and learning of the concept of function is related to the diversity of the ways of representing a function (algebraic or symbolic, numerical or tabular, graphical) and the difficulties that students have in establishing connections among them (Sfard, 1992, Hitt, 1998, Gagatsis & Shiakalli, 2004, Elia et al., 2007).

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 265-272. Thessaloniki, Greece: PME.

Concerning the ways of representing a function we can argue that each way of expression highlights some aspect of a function and provides information regarding other aspects of this function that can be thus determined (via transformations or the use of appropriated procedures). But, each of the aforementioned expressions of a function cannot explicitly describe it in its entirety. Instead, they complement one another, and each one is a way of representing one or another aspect of the function. This becomes evident if we consider the terminology regarding functions. We talk about the formula of a function, its graphic representation. Consequently the means of representing the function cannot be deemed to be a symbol of the function and it does not stand for the function.

Many researchers emphasize the role that the flexibility in establishing relations among different ways of representing a function could play in the conceptualization of the notion of function (Artigue, 1992, Even 1998).

Examining, handling and interpreting a function in the context of the different modes of its representations require different cognitive frames of treatment of information. For instance, the algebraic expression of a function is propositional, in the sense that it carries information linearly by way of sentences that can be read one after the other. While in the graphic representation of a function data are given in a holistic way. Thus, the transition from one mode of representation to the other is not a simple translation, as Janvier (1987) points out; but it is a conversion, a transformation of semiotic representations (Duval, 2002). Students' difficulties either to establish similarities of data given in different modes of representation (Hitt, 1998), or to converse a function from one mode of representation to another (Elia et al. 2007) constitute indices for the phenomenon of compartmentalization. According to Duval (2002) compartmentalization characterizes the students' confusion of an object and its representation, which leads to consider two representations of the same object as two different mathematical objects.

However, students' conceptions about the notion of function and the procedures they used are neither homogeneous neither unique. Previous research (Kaldrimidou & Ikonomou, 1998) on students' conceptions in the case of graphical representation of functions reveals three different approaches to conceive a function: a) the geometrical conception (the curve is considered independently of the system of axes; the characteristics points are marked on the curve with a single letter), b) the algebraic conception (the curve is considered only in relation to the X-axis; the curve is separated in parts, which are defined by the characteristics points set by the value of the independent variable in the X-axis) and c) the functional conception (the curve is considered in relation to the two axes; the points have two coordinates). In the same context, students used three different procedures to draw a graph: a) the point-by-point (the graph focuses on intersection points, minima and maxima); b) the step-by-step (monotone property, concavity, table of variation) and c) the holistic procedure (use of global characteristics, the category of the function, i.e. parabola, and change of system of coordinates).

In the present study we try to extend this investigation when functions are represented also algebraically and numerically. The main research questions were as follows: a) does the way students conceive a function depends on its representation and b) are the procedures used by the students related to their conceptions about the notion of function?

METHODOLOGY

To this purpose a test was administrated to 190 students of grade 12 (17-years old). The test consisted of seven tasks. The present study focuses on the analysis of the group of tasks in which students were asked to give as much information as possible about the involved function. Functions were represented numerically (task #4), algebraically (tasks #3 & #6) and graphically (tasks #1 & #5). The functions involved in the tasks #3 & #5 (represented by the formula and the graph of a polynomial of 3rd degree respectively) were more familiar to the students than those involved in the tasks #1 (unusual graph) & #6 (defined by a formula with two branches).

Students' answers were analysed according to two variables:

- The first variable, named "conceptions", concerns the way students conceived the function involved. The geometrical conception (GC) was assigned a score of 1, the algebraic conception (AC) was assigned a score of 2 and the functional conception (FC) was assigned a score of 3.
- The second variable, named "procedures", concerns the procedure used by the students. The point-by-point procedure (PPP) was assigned a score of 1, the step-by-step (SSP) was assigned a score of 2 and the holistic procedure (HP) was assigned a score of 3.

Below, examples of students' responses are provided.

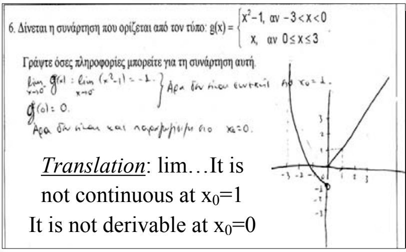


Fig 1: (FC, HP)

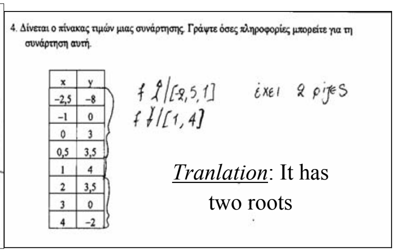
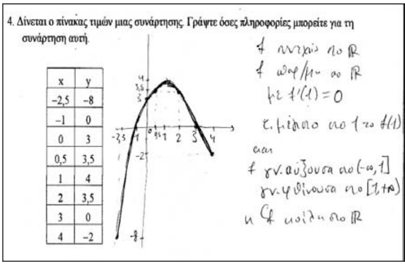
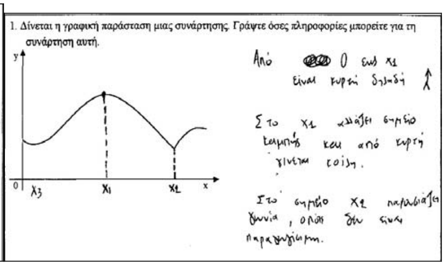


Fig 2: (AC, PPP)



Translation: f is continuous in R,
f is derivable in R and f'(1)=0.
It has a max at 1, the f(1).
f is increasing in (-∞, 1], decreasing in [1, ∞)

Fig 3: (FC, PPP)



Translation: From 0 to x_1 it is convex
At x_1 it changes and it becomes concave.
At x_2 it is not derivable

Fig 4: (AC, PPP)

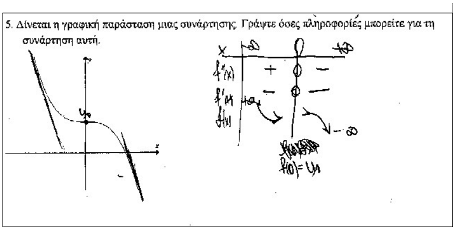
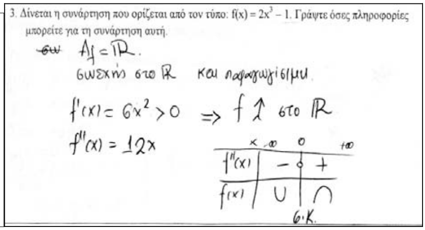
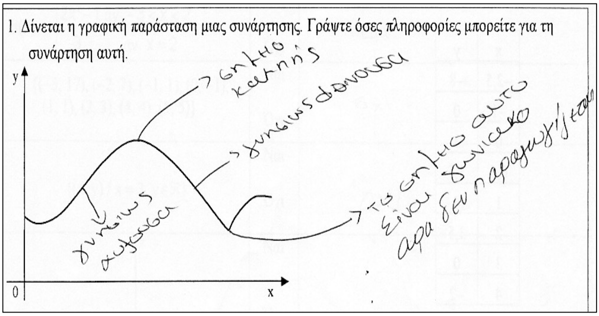


Fig 5: (GC, SSP)



Translation: f continuous and derivable

Fig 6: (GC, SSP)



Translation:
Raising, point of inflexion.
Increasing, it is not derivable at this point

Fig 7: (GC, PPP)

RESULTS

a. Conceptions

Concerning the way students conceive a function, we notice that in the algebraic and the numerical contexts the majority of them project an algebraic conception, focusing their description on the values of the independent variable. However, when the function is represented graphically, the majority of the students express a geometrical conception especially in the case of the more familiar function (see table 1).

	T. #1	T. #5	T. #3	T. #6	T. #4
GC	52.1%	77.1%	20.8%	19.8%	16.7%
AC	38.5%	19.8%	64.6%	53.1%	55.2%
FC	9.4%	3.1%	14.6 %	27.1%	28.1%

Table 1: Distribution of students' answers according to the variable "conceptions"

Moreover, factor analysis on the variable "conceptions" reveals three main factors, which explain 26.1%, 23.3% and 21.9% of the variance respectively. The first factor includes the tasks in which functions were represented algebraically, the second one the task in which functions were represented numerically and the third factors the tasks in which functions were represented graphically.

Conceptions	F1	F2	F3
#6	.879		
#3	.852		
#4		.781	
#1			.836
#5			.767

Table 2: Factor analysis of the variable "conceptions"

Based on the above results, we can argue that the mode of representation of a function plays an important role in the way that students conceive it and describe it. Moreover, the majority of the students don’t change category of conceptions in the context of the same mode of representation of a function (55% in the case of graphic representations and 60% of them in the case of a function represented by its formula).

b. Procedures

Concerning the procedures used by the students, the majority of them prefer the point-by-point manipulation in all tasks except the task #3 of a familiar function represented by its formula. It is also noticeable the very low percentage of the use of an holistic procedure in the case of a function represented with an unusual graph.

Procedures	T. #1	T. #5	T. #3	T. #6	T. #4
PPP	61.5%	49.0%	26.0%	57.3%	42.7%
SSP	37.5%	18.8%	52.1%	12.5%	20.8%
HP	1.0%	32.3%	21.9 %	30.2%	36.5%

Table 3: Distribution of students' answers according to the variable "procedures"

Factor analysis on the variable "procedures" reveals three main factors, which explain 24.4%, 24.2 and 20.3% of the variance respectively. The first factor includes the tasks in which functions were represented graphically (familiar) and algebraically (non familiar), the second one the tasks in which functions were represented numerically and algebraically (familiar) and the third the task in which the function involved had an unusual graph.

Procedures	F1	F2	F3
#5	.858		
#6	.692		
#4		-.736	
#3		.732	
#1			.965

Table 4: Factor analysis of the variable "procedures"

Based on the above results, we can argue that the procedures used by the students are not connected to the mode of representation of the function. Only 40% of the students used the same type of procedures in the tasks in which functions were represented graphically. In the case of a function represented by its formula this percentage becomes 35%.

The very unusual form of the graph of the function involved in the first task, the lack of experience to treat a function by its table of values (task #4) and the very familiar formula of the function involved in the third task could explain students' responses.

c. Conceptions and procedures

However, it seems that there is a relation between students' expressed conceptions and the procedures they used.

	T. #1	T. #5	T. #3	T. #6	T. #4
χ^2	56.672	3.204	59.720	60.243	19.974
df	4	4	4	4	4
As. Sig. (2-sided)	.000	.524	.000	.000	.001

Table 4: χ^2-test between the two variables "conceptions" and "procedures"

In all tasks, except task #5 (graphical representation of a familiar function), the two variables "conceptions" and "procedures" are related. In task #5 the familiarity with the involved function (polynomial of 3^{rd} degree) helped the students to recognise it and to report about it in a general and consistent in the same time way. In fact, 35% of the students recognise the type of the function represented by its graph and they use this data in order to give their response without reference either to specific points or to calculations.

CONCLUDING REMARKS

Summarizing, we can argue that findings revealed the complexity of the phenomena regarding the concept of function.

Concerning the main questions addressed in this study, the following points should be noticed:

1. Conceptions and procedures observed in our previous research in the context of graphic representation of function were also identified in the case of the two other representational contexts, the algebraic and the numerical.

2. Expressed conceptions in tasks focusing on function's description seem to be influenced by the representational context. A geometrical conception is more often expressed in the case of a function represented by its graph, while an algebraic conception is associated with functions represented by their formula or by a numerical representation. While the majority of the students don't change conception in the context of the same mode of representation, an important number of them expressed a different conception in the same representational context.

3. The degree of familiarity with the type of function involved seems to be a factor that plays an important role in the choice of procedure by the students.

4. Conceptions and procedures seem to be interrelated. However, the familiarity with the involved function contradicts this finding.

Moreover, the interrelation between conceptions and procedures seem to be in contradiction with the fact that procedures used by the students are not related to the mode of representation; while conceptions appear to be influenced by the representational context.

This contradiction could be seen as index of inconsistency in students' responses and, therefore, it could be interpreted as an indication of students' "incompetency in flexibility handling different modes of representation, which is a main feature of the compartmentalization phenomenon" (Elia et al., 2007. p.549). However, at the same time, it could be seen as evidence of the students' flexibility in using different procedures according to the difficulty and the unfamiliarity level of certain tasks.

The points raised above provide evidence on the complexity of the issues related to the study of teaching and learning the concept of function. Hence, they suggest that a deeper investigation is required, focusing on a variety of types of function, in order to carefully analyze the role of the presentational context in the construction of the relevant notion.

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CULTIVATING METACOGNITIVE AWARENESS IN A COMMUNITY OF MATHEMATICS TEACHER EDUCATORS –IN-TRAINING WITH THE USE OF ASYNCHRONOUS COMMUNICATION

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This paper reports a study of an online discussion forum among educators-in-training (e-i-t) for the teaching of mathematics with the use of digital media. The discussion is about the design, implementation and evaluation of two lessons in a school unit, aiming at familiarizing students with the use of exploratory software. The transcripts' analysis for the cognitive and metacognitive presence showed that the first lesson's implementation established socially shared metacognition among the paricipants.

THEORETICAL FRAMEWORK

The discussion analysed in the present study was developed on an asynchronous communication platform (claroline), in the context of a broader nation-wide teacher educator training program in Greece, for the use of digital technology in the teaching and learning of their respective subject. Perceiving teacher education as "a systemic, life-long professional development activity addressing teachers' epistemologies, practices, pedagogies and subject-related knowledge" (Kynigos, 2007), the course not only aimed at educating the selected experienced teachers in the pedagogical use of exploratory software and communication technologies, but to motivate them on issues related to mathematics pedagogy as well. In this context two of the main objectives of the program were the development of reflective stances towards mathematics teaching and, generally, towards the teaching profession, as well as the collaboration among them.

Supporting teachers on the new roles created due to the integration of technology into the teaching of mathematics presupposes the understanding of the knowledge and the practices which compose the ongoing practice. Thompson (1992) suggests that it is not useful for researchers to distinguish between teachers' knowledge and teachers' beliefs. Instead they should examine how these relate to their experience, as the relationship between beliefs and practice is dialectic, not a simple cause–and-effect relationship. Subsequently, the focus of the present study is on the teachers' knowledge and epistemology, as well as on teachers' interaction throughout the use of computer collaborative environment. According to Ernest (1989) ***teachers' mathematical knowledge*** includes knowledge *of mathematics*, *of other subject matter*, *of teaching mathematics* (*pedagogical knowledge* and *curriculum*

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 273-280. Thessaloniki, Greece: PME.

knowledge), *of organization for teaching mathematics, of the context of teaching and of education*. Issues regarding ***teachers' epistemology*** are: 1) *Conceptions about the nature of mathematics* (problem solving view, platonist view, and instrumentalist view) 2) *Models of teaching mathematics* and *models of learning mathematics* 3) *Principles of Education*. In our forum the participants' conceptions regarding the role of technology in the teaching of mathematics contributed to the approach of their knowledge and became a "window" on teachers' epistemology (Hoyles, 1992).

One of the matters in question was whether and in what extend their epistemology and their knowledge have changed or been developed throughout the teacher education program's culture, the interaction among them and the implementation of their ideas in practice. According to Jaworski (2006) discussion, negotiation and argumentation throughout inquiry and investigation practices are three fundamental elements that support the development of mathematics teachers. Consequently, the inclusion of community in the picture of learning is considered to be necessary. Some of the e-i-t created the community we refer to. It was of great importance to determine the elements of their communication, the means and the content, in order to find out if there was anything useful for mathematics teacher education. Jaworski (2006) defines a ***community of inquiry*** (CoI) as the space in which the participants grow into and contribute to the continual reconstruction of the community through ***critical reflection***. This is one of the elements that differentiate a CoI from a ***community of practice*** (CoP), as CoI involves a reflexive relationship between a CoP and its activities: CoP's activities are subjects of inquiry and reflection. Asynchronous communication platforms constitute a reflective space which favours the development of critical discourse and the manifestation and provocation of teachers' epistemologies (Makri & Kynigos, 2007). They can provide collaborative learning experiences through interaction and independence as well. Thus, they are connected with the possibility of establishing CoI through them.

From the beginning of the online discussion we examine, the participants set as a goal the design of lessons with the use of exploratory software and its implementation in the classroom. By definition, this goal includes the three teaching stages that Artzt (1998) describes: preactive, interactive and postactive. We were interested in the kind of discussion in each one of them and mainly in the specific cases, in which teachers step back, start to consider and look critically at events and processes after they have occurred. According to Jaworski (1998), this stage involves a ***metacognitive awareness*** in which knowledge and action are linked and the teacher moves to a position of reflecting-on-action. By the term ***metacognition*** Kuhn (2000) describes the cognition that reflects on, monitors or regulates first-order cognition. It encompasses knowledge of cognition (declarative, procedural, conditional) and regulation of cognition (which includes a number of regulatory skills, such as planning, monitoring and evaluating) (Schraw, 1998). Knowing about declarative knowledge (knowledge as product), procedural knowledge (knowledge as process) and personal (or other persons') epistemology, is broader than metacognition: it's what Kuhn (2000) defines as '***metaknowing'***.

Specifically, as far as the teaching is concerned, we consider as ***metacognitive components*** the teachers' commentaries about processes associated with the teaching of the lesson, regarding their *goals, beliefs, knowledge, planning, monitoring, regulating, assessing and revising (*Artzt et al, 1998). In our study we investigated 'planning' during the preactive, 'monitoring' during the interactive and 'evaluating' during the postactive stage of each lesson.

THE RESEARCH

The context

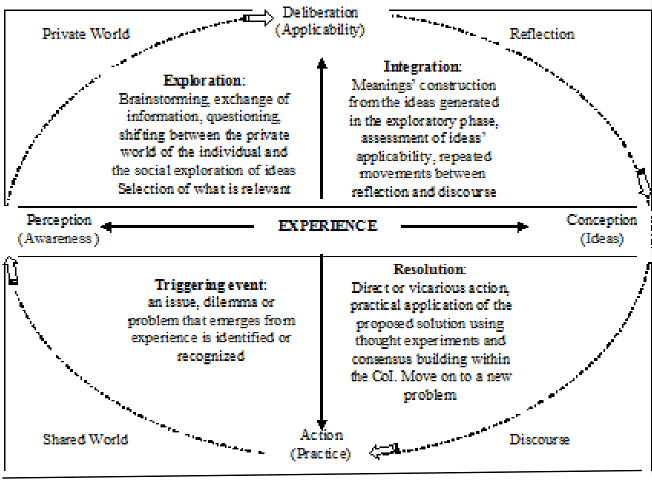
During the teacher educator training program a combination of approaches was used: 1) Authentic discussion sessions 2) Small group discussions on the preparation of teaching materials and scenarios 3) Techniques of increasing reflection: ***scenarios*** were considered as the central objects for reflection, the priming for teachers' engagement with teaching design away from curriculum limitations and tools of teacher's role empowerment 4) Software engineering and construction of other curriculum materials for the students 5) Participation into communities, using ICT as a means of expressing knowledge, conceptions, reflection and professional identity. In this context the program designers created 4 main online areas, where e-i-t were invited to post not only their epistemologies, pedagogical conceptions and practices, but also to express their views on mathematical topics. 20 discussions were created, that were seen 654 times in total, lasted 25 days and the maximum number of messages in each of them was 17. The discussion, on which we focus here, was introduced by one of the e-i-t. Its theme was initially related to secondary school textbooks, lasted for 4 months, included 289 messages and was seen 3487 times. The participants were four (out of seven) e-i-t, two e-i-t of other disciplines, a university professor and a Ph.D.-student-researcher. The discussion content comprised 4 themes: 1) Systemic 2) Preparation and implementation of lessons with the use of exploratory software (Geogebra) in mathematics, at a school unit 3) Preparation of a multi-disciplinary lesson including mathematics and Greek language-History, production of material for scenario's creation 4) Preparation-implementation of two lessons aiming at the familiarization of the students with Geogebra, worksheets production. Specifically, we focus on the 4th thematic section: four e-i-t and the researcher participated and 74 messages were exchanged, that were separated from the whole discussion body for analysis.

Research questions (R.Q.) - Method

We investigate the nature, the possibilities and the extent the communication tool can be exploited in mathematics teacher education, as a means of empowering teachers' pedagogical content knowledge, communities' creation and development of metacognition (1st R.Q.). The forum discussion shaped the basis of what is studied, this was not predetermined and known in advanced. Thus, at a second level and throughout the discussions of the participants, the design of two lessons which aims at familiarizing the students with the use of Geogebra emerged. The process of designing is tasted, reconsidered, improved and revised by the participants in real classroom conditions; We study how the teachers' knowledge and epistemologies are reflected throughout the iterations of teaching *design experiment* (Cobb et al, 2003) as well as how some of the metacognitive components (planning, monitoring, evaluating) impact the design and its implementation in the classroom during the 3 teaching stages (preactive, interactive and postactive) (2^{nd} R.Q.).

Data analysis tool - Coding

a) For the content analysis we used the practical inquiry model of Garrison et al (2001), because its four phases are analytically describing the cognitive presence within a CoI (figure bellow).



The two dimensions are:

1. Deliberation – action: This axis defines the reflection and collaboration properties of asynchronous learning. This process iterates between thought and action and unifies the private and public worlds of inquiry.

2. Perception – conception: This axis represents the transition between the concrete and abstract worlds.

19 out of the 74 messages were extracted (category "else") and the rest 55 messages were classified into one of the four phases of the model. The classification of the messages was based on indicators for cognitive presence connected with socio-cognitive processes, corresponding to each one of the phases.

b) In order to investigate the metacognitive presence, we created our own model: the 55 messages classified as metacognitive or not metacognitive. Each of the 32 metacognitive messages fulfilled the following 3 criteria (Hurme et al, 2008): 1. It was related to the ongoing discussion, 2. It had an intention to interrupt, change or promote the progression of the joint problem solving process, 3. It offered an explicit explanation as to why the group should take another feature of the problem into account. The metacognitive messages were further classified as "P: planning", "M: monitoring" and "E: evaluating", by creating descriptors for each one of the above skills, based on definitions of Artzt et al (1998) and Schraw (1998).

Results – Discussion

From the categorization of the 55 messages per Garisson's model phase we have the following findings: Phase I - Triggering event: 11 messages (20%), Phase II - Exploration: 12 messages (22%), Phase III - Integration: 7 messages (13%), Phase IV - Resolution: 25 messages (45%)

We observe that the lowest message percentage is found in the 3rd phase and the highest in the 4th (which is not in accordance with other researches' findings that used the same tool analysis). The participants exchanged a number of messages in the 2nd phase, based on their personal conceptions and experiences. Through these we distinguish evidence of their epistemology as well as elements of their pedagogical content knowledge. Below are cited extracts relating to the 1st lesson's preactive stage (R= Researcher):

John: … We'll present each tool (of Geogebra) twice and ask them to repeat the process twice….. how many and which tools we will show them will depend on which tools we might need for the two scenarios "derivative" and "integral". For this reason we must have previously prepared the scenarios on the computer….

R: I suggest that during the two-hour lesson of familiarization we'll use worksheet (WS), with the basic functionalities of Geogebra recorded…. As the students experiment and work on these functionalities they record their understandings on the use of the tools….

John: ……. if we had time we could prepare a WS, with detailed instructions for a brief practice on the tools (but we have not)… The WS's orientation should be clear on the specific tools which will be used in the two scenarios and not on the discovery of the software functions ……And why should students record their understandings? ...

Maria: …. I agree with R's idea. Having a WS is like having a guide ….

Aris: …. it's better to distribute the manual of Geogebra to the students before the lesson, so we'll have more time for practice... Then we can answer the students' questions...

Regarding their conceptions for the role of technology in the teaching of mathematics: John considers that he has "to show" to the students only the tools he believes that they'll need for the main lesson. He deprives them of the possibility of using the tools as part of their mathematical thought and following their personal paths towards the problem solving. Moreover, the teacher himself is deprived of the possibility to receive unexpected –surprising solutions on behalf of his students and makes teaching absolutely teacher-driven, thus predictable. Aris believes that students can be familiarized with Geogebra, by having studied the manual on their own, and discuss their questions - doubts with the teacher later. He seems to consider that it is enough for the students to know the whole range of the available tools, without going deeper in the use of one of them. Regarding their pedagogical knowledge: John seems to consider as necessary the initial performance by the teacher, involving explicit descriptions, followed up by repeated practice from the students. This might have its origins in the instrumentalist view. This origin explains his conceptions about the role of technology in the teaching of mathematics, as he insists on the existence of a single 'correct' method for solving a problem, the one he has on his mind. Regarding to the use of WS, he doesn't consider it as necessary, because the lesson is teacher-centered. Aris seems to consider that the WS can be replaced by the manual. His conceptions might have their origins to the Platonist view, as he talks about explanations. After long discussions, an agreement was reached and R finally designed something between a brief manual and a WS. Its parts are either initiated by the teacher (the menus) or constituted the object of students' investigations through activities (the toolbox), since it is there where the dynamic character of the software lies. Due to that fact in the sections of Toolbox and Insertion students are asked to register their findings. The extracts presented below, took place after the implementation of the 1st lesson (1st lesson's postactive stage – 2nd lesson's preactive stage):

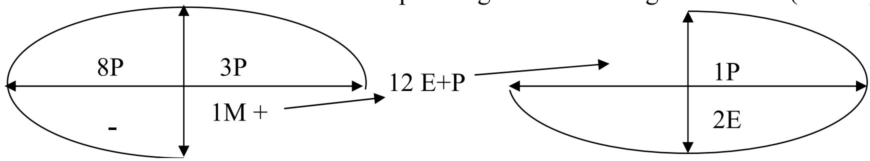
Maria: …As our aim was to introduce the software to the students it was useful for each student to use a separate computer. But I would like them next time to work in pairs of two in order to have the opportunity to cooperate and communicate… Finally, a WS is useful and necessary for the next lesson …..

John: …… if the three teachers were more coordinated during the lesson, the students would gain more benefits ….. Maria it is obvious that there will be an extensible WS … that we will have discussed it all together… after the end of the second lesson we will gather the students' opinions about the lesson and try to interpret their comments and reach general conclusions and teaching proposals.….

Aris: ……indeed if they worked in groups of two we could notice if the cooperation between them was effective……

By the implementation of the teachers' ideas new issues emerged, such as the importance of collaboration among students in small groups, the consideration of students' conceptions into the next lesson's design the co-ordination between teachers and the inclusion of extensible worksheets. Some issues were neither discussed, nor agreed upon, in the sense of reaching a synthesis of ideas, before the implementation: a) the use of the WS, b) the management of teaching (students' ways of working, co-ordination between teachers). A common terminology was not established as well. All the above considerations explain the low percentage of messages exchanged in the 3rd phase proportionately to the ones of phase 4.

As far as the metagognitive components is concerned we notice that the 1st lesson planning was mainly based on the ***individual participants' metacognitive experiences***. Consequently, the biggest part of these metacognitive messages (8P) is found in the 2nd phase of Garisson model and only 3 messages in the 3rd phase of integration (3P). This fact indicates that the metacognition which emerged during the 1st lesson's preactive stage to ensure the cognitive goals' achievement was not the result of discussion, negotiation and argumentation, thus, it was independent of the community. Both analysis models being used are in accordance as far as the *lack of* *opinion integration* is concerned before the implementation of the 1st lesson (preactive). The highest percentage of metacognitive messages (13) appears in the 4th phase of the model (interactive, postactive). It needs to be mentioned that only one of these refers to the lesson's monitoring (1M). The rest 12 (46% of the metacognitive messages) are exchanged in this phase. In these messages we can see (figure bellow) the evaluation of the 1st lesson and the planning of the 2nd being interwoven (12E+P).



The 1st lesson's implementation seems to have established a common terminology and a desire for a common evaluation. The metacognitive intervention of a community's member is acknowledged by the other members, it is developed and it is used meaningfully regarding the evaluation (Hurme et al, 2008). Thus, we can consider that a ***socially shared metacognition*** is being established within the community during the 1st lesson's postactive and the 2nd lesson's preactive stage.

Conclusions

Taking the above into account we conclude that the asynchronous communication platform constituted a belief-clarification-space and it contributed to the development of metacognitive discussions. The written publication of the participants' ideas and conceptions so as to be comprehensible by the rest of the team during all the conversation's phases encompasses metacognition. Similarly the written publication of a teacher's conception regulating how the rest of the team can be included is also a metacognitive process. For this specific community it constituted a space of the teachers' practices written publication as well. This is of great importance due to the fact that the publication of the practices' elements can contribute to revealing the beliefs that are dialectically connected with those practices.

The element which differentiated the participants' conceptions was the practical implementation of their ideas. Many of them were revised and completed for the second lesson's planning. The shared process 'planning-monitoring-evaluating' ***in real classroom conditions*** contributed to the transition from individual metacognition to socially shared metacognition. Simultaneously, the teachers' spontaneous and enthusiastic participation in the community seems to be connected with: 1) ***the program's culture influence on them***, which exploited asynchronous platforms not only as a means of communicating from distance, but as a means of expressing ideas and provoking reflection 2) the fact that from the very beginning the community was ***practice oriented***. This element is of great importance for mathematics teacher education, since it highlights the need of teachers for the immediate theory's connection with practice. It also emphasises on the possibility affecting their conceptions throughout practice, as well as the opportunity of acquiring collaborative cognitive experiences and socially shared metacognition. Our findings show that it can be cultivated within the contemporary professional development programs throughout: a. the teachers' initiation in computer collaborative environments and development of an inquiry culture of their ideas throughout discussions and b. the possibility of ***immediate innovations' implementation*** in the teaching practice giving teachers a more active role and developing an inquiry culture of their practices. The focus of our study was on three of the teaching's metacognitive components: planning, monitoring and evaluating. The possibility of teachers developing 'metaknowing' including knowledge about their own (or other teachers') epistemology is quite significant as well and has to be further investigated.

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WHEN AT-RISK STUDENTS MEET NATIONAL EXAMS IN SECONDARY MATHEMATICS

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This article reports on a project attempting to raise the number of at-risk students in secondary schools who pass the Israeli Matriculation Exam in mathematics. The project seeks to strengthen the links between theory, curriculum design and practice in regard to mathematics teaching and learning in low-track settings. A professional role of a counsellor, specializing in low achievement in mathematics, is introduced, and the work model of such counsellors within schools is described. Preliminary results from nine schools are presented and discussed.

INTRODUCTION

Failure in national compulsory exams at the end of high school is, in most countries, a serious impediment to enrolment in higher education and lucrative employment. In Israel, a prerequisite to entering universities, colleges, and many jobs is the eligibility to a Matriculation Certificate (MC). High school graduates receive the MC if they pass a series of final exams in several obligatory subjects, among which is mathematics. Recent data show that although 83% of the graduate cohort (age 17-18) take these exams, only 55% of the cohort are entitled to the MC, and that failing to pass the Matriculation Exam in mathematics is the single most common barrier that prevents students from acquiring this desired certificate (Israeli Central Bureau of Statistics, 2006; Shye et al., 2005). The mathematics final test may be taken in one of three levels: high, intermediate or low, and high school students are streamed to three tracks accordingly. Approximately 50% of the students study in the low-level mathematics track, however among those entitled to the MC, we find only about 20% who graduated this track. These data indicate that many Israeli students can be defined as "at-risk mathematics students", i.e., students' whose likelihood to fail the low-level mathematics final exam, and thus become disqualified for the MC, is alarmingly high. Moreover, in vocational schools supervised by the Ministry of Industry, Trade and Labor (rather than the Ministry of Education), the current policy allows schools to stream students into non-matriculation tracks, thus many students' learning trajectories do not include the opportunity to study towards the MC finals.

This state of affairs brings forward considerations of equity. Recent research on low achievement and failure in mathematics is linked to social issues such as disadvantaged communities, civil rights and inequalities associated with ethnicity and social class (e.g., Martin, 2000; Moses & Cobb, 2001; Secada, 1992; Swanson, 2006). Thus, we believe that any attempt to advance at-risk mathematics students should bear in mind the wider context of this phenomenon.

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The project described in this report is aimed at raising the number of at-risk students who pass the MC final exam in mathematics, while taking into account a range of cognitive, social, and behavioural factors which may characterise these students. The report will introduce the rationale of the project, its roots in previous studies on low achievement in secondary school mathematics, and the model of work developed. Components of the project which may be replicated in contexts other than the Israeli one will be highlighted. Finally, preliminary results from an evaluative study on the effectiveness of the project will be presented and discussed.

THE SHLAV PROJECT

The name SHLAV is an acronym of the Hebrew words for "Improving Mathematics Learning". The project began in 2004, and in 2007 has shifted from its pilot phase, conducted in 2 schools, to an extended mode of implementation, operated in 9 schools. As of Sept. 2008 the project runs in 17 schools spread all over Israel, mainly in peripheral areas or low-income neighbourhoods (including 4 schools of the Arab and Druze sectors). The target population is defined as “low achievers”, "at-risk students" or "non-matriculation bound students", despite variations between terms which will not be elaborated here. However, it should be noted that at this stage the project is not intended for students studying in special education settings, i.e., we try to advance students whose thinking skills are considered normative (including those with minor, but not severe, learning disabilities). It should also be noted that in Israel, ability grouping in mathematics is the common practice in secondary schools, from eighth grade onwards. Therefore, target students would usually be grouped in lower-track mathematics classes. The project acts within this given situation, hence we will not refer here to the important issues of whether and how tracking affects students, or how different settings may or may not alter students' learning characteristics (as widely discussed in the literature, e.g., Boaler et al., 2000).

Rationale

The rationale underlying this project is embedded in the need to establish, in regard to the issue of low achievers, more solid links between theory and curriculum design on the one hand, and classroom practices on the other hand. The project's team came to realize that although there is a considerable progress in understanding learning processes of low achievers in secondary mathematics, and although suitable learning materials exist for these students, these inputs do not necessarily translate into quality teaching practices in low-track settings. The project therefore seeks to fill this gap through direct contact with students and teachers, as explained in the following.

Theoretical background

In recent years we witness a substantial increase in the literature concerning mathematics teaching and instruction for at-risk students. Most studies address this issue at the primary education level (e.g., Lumpkins et al., 1991; Baker et al., 2002). Moreover, studies that do relate to adolescent students often concentrate on basic arithmetic skills (e.g. Ben-Yehuda et al., 2005). Nevertheless, research focused on teaching advanced mathematical content (such as algebra and calculus) to low-track students, is gradually accumulating (Arcavi et al., 1994; Chazan, 2000; Karsenty et al., 2007). Looking at this literature, the emerging general picture is that researchers strongly suggest tailoring curriculum, learning materials and instructional strategies to students' observed characteristics. Among these characteristics, as pointed out by Chazan (1996, 2000) and Arcavi et al. (1994), are short lived memory for mathematical procedures, short concentrating periods, difficulties with reading and writing in a mathematical language, poor note-taking and homework habits, and low frustration threshold. At the metacognitive level, it was found that low-achieving students have difficulties in planning, self-monitoring and reflecting on their actions in the course of handling a given mathematical task (Cardelle-Elawar, 1995). Lastly, but especially important, is students' inclination to view mathematics as an esoteric subject, detached from their common sense experience (Karsenty & Arcavi, 2003).

Empirical evidence offered by the above mentioned studies support the contention that at-risk students can bring forward valuable mathematical ideas, provided that instruction matches their characteristics and legitimizes any thinking tool they have available (Chazan, 2000, Arcavi et al., 1994). Using learning materials adapted to students' points of strength (see below) may yield significant informal mathematical products on the part of students. When such products are accepted, listened to and appreciated by teachers, students are more likely to strengthen their confidence in their abilities to do meaningful mathematics, and eventually cope with tasks at the expected MC level, as shown by Karsenty et al., 2007. Evidently, the need to listen and build on students' informal ideas is recognized as a desired approach for *all* students (Arcavi & Isoda, 2007). However, for at-risk students this course of action could be crucial, to the degree of making the difference between success and failure.

The learning materials

From 1991, the Science Teaching Department of the Weizmann Institute of Science in Israel has developed new learning materials for low-track mathematics students in grades 10-12, in line with the requirements of the low-level MC mathematics exam. Endeavouring to build a sound understanding through long-term student exposure to meaningful activities, materials sought to (a) engage students' common sense and real-life experiences; (b) base learning on qualitative, visual and graphical reasoning; (c) integrate multiple representations as different ways to envision the same ideas; and (d) minimize technical manipulations and heavy notations. An important element of the design was the attempt to link students' thinking to their acting (for instance, by using various physical devices), in order to help them make sense of what they do. The tasks encouraged visual and numerical approximations, and the use of informal reasoning rather than technical and symbolic treatments (see Arcavi et al., 1994). After successfully tried out in classrooms and approved by the Israeli Ministry of Education, materials were published and introduced to teachers. The SHLAV project maintains the approach presented in these materials and uses them as resources.

The work model

As said, the SHLAV project was conceived in light of the recognition that in the interactions between research, design and practice regarding mathematically at-risk students, the weak links seem to concern communications between the practice component and the other two components; hence the project should aim to strengthen these communications (see Figure 1). Our experience showed that in the absence of a supportive framework, teachers in low-track settings often find it too difficult to implement new ideas in their classrooms, and tend to preserve a system of traditional teaching which frequently clashes with the needs of at-risk students.

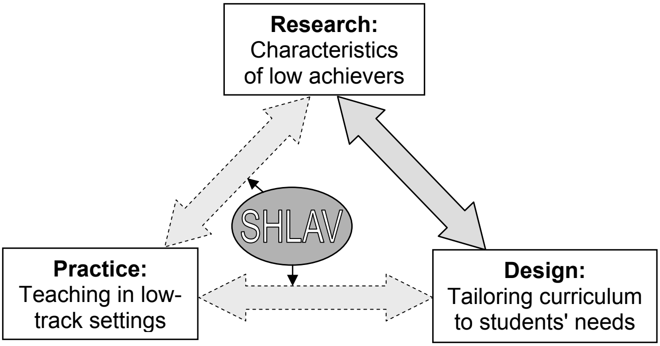


Figure 1: Placing SHLAV within the "research-design-practice" interactions

Thus, we created a model for a professional function, namely, a counsellor focusing on difficulties and low achievement in mathematics. The counsellor is an experienced teacher with a solid academic background, trained specifically for this function. Each specialized counsellor is assigned a secondary school, with a low proportion of students who pass the Matriculation Exam in mathematics, where he or she works for a full day every week. Main activities carried out as part of the counselling are:

1. Identifying at-risk students who can benefit from the project's intervention. A special test was devised for this purpose, focusing on basic number sense, visual reasoning, and the ability to make sense of information presented through several representations, rather than prior acquired knowledge. If possible, identified students are grouped by the school system to form separate mathematics classes.

2. Instructing the mathematics teachers of low-track classes. The counsellor performs the following actions as part of instruction:

a) Conducts staff meetings where difficulties of students are discussed; learning materials are presented and adapted; ideas are exchanged regarding affective and social concerns (e.g., attendance, motivation, unsupportive home environments); models assisting in understanding certain abstract mathematical ideas are suggested; teaching strategies that promote participation of low-track students are offered. These meetings form a professional forum on low-track teaching, yet they also serve as a "support group" for teachers, where doubts, frustrations, successful minutes, etc. can be shared. The counsellor, in turn, gains insights on issues which need further academic investigation or modifications of materials.

b) Observes lessons, and then meets with the teacher for a personal session, during which strengths and weaknesses of the lesson are discussed. Here too, feedback may relate to a variety of concerns, including cognitive, affective, and social issues. Decisions are made together in regard to future courses of action, whether these relate to a certain student or to the class as a whole.

c) Advices teachers on matters of assessment. The counsellor seeks to raise teachers' awareness to the idea that classroom assessment procedures, designed for low-track students, should also be adapted to their characteristics (Chazan, 2000).

3. Running teaching sessions in small groups with students requiring special support.

4. Organizing and supervising the work of volunteers who serve as mathematics tutors for students, during or after school hours.

In sum, the work model comprises a combination of activities: identification, staff instruction, personal feedback, assessment, and tutoring. By directing these activities and modifying them according to the particular needs of the school, the counsellor serves as a channel of communication between theoretical and practical insights. Although this model was developed in a specific context of the Israeli MC requirements, we believe that since the model is simple and general in nature, it may be adapted in low-track settings other than the Israeli ones.

Preliminary results

An evaluative study follows the SHLAV project, investigating both quantitative data (i.e., students' achievements) gathered by exams, and qualitative data (e.g., teachers' and students' perspectives), gathered through interviews and questionnaires. In addition, selected answers of students to mathematical exam items are analysed. Reported herein are results of Unit 1 of the mathematics Matriculation Exam, taken by the project students on July 2008, at the end of 10^{th} grade (Units 2 & 3 of the exam will be taken at the end of 11^{th} and 12^{th} grades, respectively). Table 1 presents data from nine schools. Six schools are supervised by the Ministry of Industry, Trade and Labor (ITL); the project in these schools included students who were originally non-matriculation bound. The other three schools are regular high schools, where the project was implemented in low-track classes. As can be seen, of the total number of 174 project students, 157 (90%) took the exam, and 129 passed (82% of examinees). The average grade of the students who passed was 86.6 (*SD*=13.82); the average grade of all examinees was 76.8 (*SD*=25.85). Comparing the results with the national 2007 data, which show that 75% of MC examinees took the mathematics exam, and that the average grade in Unit 1 was 70.9, these preliminary results appear to be promising. However, there are differences in results between the two types of schools: In the regular schools all the project students (n=80) took the exam, and 92%

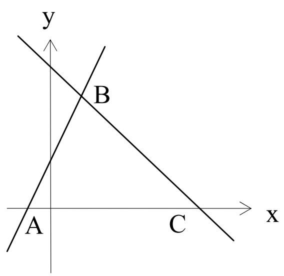
of them passed, with an average grade of 89.5 (*SD*=12.5), whereas in the ITL schools 82% of the project students (n=94) took the exam, and of them 71% passed, with an average grade of 82.6 (*SD*=14.6). There is also a considerable difference in the number of students who attained a grade of 80 or more (71% vs. 39%).

School	Students' classification	No. of students in the project	No. of students who took the MC exam (Unit 1)	No. of students who passed	Average grade of students who passed	No. of students with grade 80 or above
#1	N-M-B	27*	23	15	90	12
#2	N-M-B	17	13	8	89.1	7
#3	N-M-B	15	12	11	81.3	8
#4	N-M-B	13	13	6	69.1	1
#5	N-M-B	12	11	11	81.2	7
#6	N-M-B	10	5	4	73.8	2
#7	LT	46*	46	40	88	31
#8	LT	18	18	18	96	16
#9	LT	16	16	16	85.1	10
	Total:	174	157	129	86.6	94

Table 1: Students of SHLAV: Data on achievements in the mathematics MC Exam
N-M-B = Non-Matriculation Bound, LT = Low Track, * = 2 groups

These differences reflect a fundamental disparity between the two systems. Regular schools are assessed by the percentage of eligibility for the MC; taking Matriculation Exams is generally presented as a norm. Thus, these schools are more likely to exploit opportunities for advancement such as the one offered by SHLAV. In contrast, students in ITL schools are not always encouraged to take MC exams. Moreover, in some ITL schools the project introduces a counter-normative option. It appears that despite the prestige associated with the MC, non-matriculation bound students may find the idea of taking the mathematics MC exam overwhelming, on top of being demanding. Several students who withdrew from taking the exam had a good chance of passing it, according to their achievements during their participation period (this was especially the case in school #6). This observation strengthens the premise that SHLAV counsellors should further maintain instructional activities focusing on affective factors such as motivation and self-image of students.

Lastly, we present a student's answer to a typical item of the Unit 1 exam, which exemplifies students' preference to use graphical approaches instead of formulas.



The exam item requested students to match between two given equations, $y = -x + 5$ and $y = 2x + 2$, and the two lines shown in the blind graph. Students were then requested to find the coordinates of points A, B, C and the length of AC.

Conventional approach to this task would usually involve solving equations such as $2x + 2 = -x + 5$ and $2x + 2 = 0$. The answer shown in Figure 2 relies solely on the graphical representation. It can be seen that the student strived to locate the lines on the coordinate system as accurately as possible, using the squared paper to construct the lines through the properties of the slope as 'rise (or drop) over run', and the y-intercept. He then used the sketch to read the coordinates of points A, B, C and the length of AC (the students' final answers are omitted here).

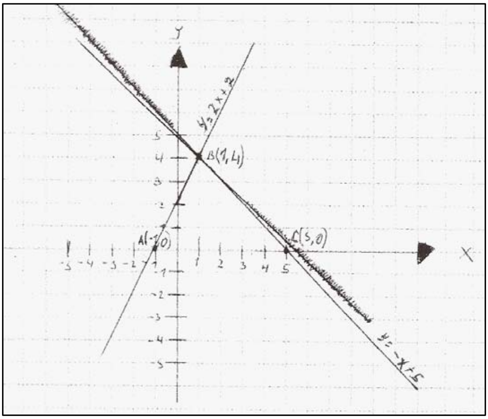


Figure 2: A students' answer to the analytic geometry item

SUMMARY

The novelty of the SHLAV project is in creating a unique framework where the inputs of previous work, including research on low achievement and curriculum design, could be exploited to address the needs of low achievers and their teachers from inside the school system. The initial results suggest that the created function of a specialized counsellor, working in low-track settings, contributes to the progress of mathematically at-risk students and improves their chances to meet the national exam. Further research is needed in order to confirm this suggestion, and to learn to what degree the model may be replicated in contexts of other national exams systems.

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EXPLORATORY TALK FOR PROBLEM SOLVING

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In this study we analyse the development and promotion of exploratory talk as a discursive tool to facilitate collective and individual reasoning in 33 grade 6 Greek children as they worked with peers in small group discussions in Mathematics. Teachers' involvement was kept minimal, without providing direct information. Interactions with peers in small group discussions were audiotaped, transcribed, and analyzed in a variety of ways. The key work in peer groups was elaborating weak or incomplete ideas until they improved. Students' discourse complexity varied within peer groups, and some groups accomplished higher level strategies than others.

INTRODUCTION

While students work in peer groups a natural discourse takes place amongst them that distributes the learning process. Different modes of talk emerge in the process that has distinct characteristics. According to Mercer (1995), these modes of talk are disputational talk, cumulative talk and exploratory talk. Disputational talk is characterised by disagreement and individualised decision-making and notably consists of short exchanges of assertions and counter-assertions. Cumulative talk occurs when speakers build positively, but uncritically, on what others say and it is characterised by repetitions, confirmations and elaborations. In exploratory talk partners engage critically but constructively with each other's ideas. Statements and suggestions are offered for joint consideration. Challenges and counter-challenges are justified and alternative hypotheses are offered. Compared to the other two discourse types, in exploratory talk knowledge is made more publicly accountable and reasoning is more visible in the talk, while progress emerges from the eventual joint agreement reached (Mercer, 1996).

The study reported here is part of a larger project designed to investigate the process of social interaction and knowledge construction of grade-6 students, while working in small groups in various assigned tasks, including mathematics (as presented here), social studies and history. In the part presented here, we examined the nature and sophistication of exploratory talk, as a discursive tool to facilitate collective and individual reasoning of students, without the guidance of a teacher. We documented specifically, the exploratory talk of sixth-grade students in two classes engaged in a mathematical task that was designed to facilitate peer talk in small groups.

THEORETICAL BACKGROUND

In our research we used a socio-cultural framework to understand learning, which emphasizes cultural tools and interpersonal relationships and resources, as well as a

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 289-296. Thessaloniki, Greece: PME.

focus on activity systems as the unit of analysis (Pressic-Kilborn et al. 2005). Researches underline that sociocultural models of learning are promoted through collaborative groups, the use of open-ended activities for learning situations and an encouragement of active participation in learning (Edwards, 2005). Among all the cultural artefacts which mediate human action in the form of instruments and signs, the sociocultural perspective emphasises the central role played by language (Rogoff, 2003; Mercer, 2000; Vygotsky, 1962, 1978). Throughout development, language serves as a key mediator for individuals to negotiate meanings and construct knowledge jointly. This process in turn is crucial for promoting progress through 'zones of proximal development' (Vygotsky, 1978), facilitating appropriation by the individual of diverse cultural artefacts and practices.

Research has shown that students working in small groups, in mathematics can spend much of their time figuring out how to complete a science task rather than gaining higher order understandings about it (Tzekaki 2007). Cognitive conflicts that emerge over mathematics problems can degenerate into social conflicts resolved through social dominance or teacher intervention. Also, not all children are equally engaged in group tasks, in part because their level of motivation, engagement and understanding is linked to the behaviours of the group leader (Vosniadou & Vamvakoussi, 2006). Lower status and minority children participate less in cognitively challenging activities such as building explanations than do majority children of higher socioeconomic and academic status (Vershaffel, & Vosniadou, 2004). Some researchers have therefore concluded that the educational and developmental potential of classroom conversation amongst pupils is being fruitless (Galton & Williamson, 1992). This appears to be particularly true in Greece as, until recently, there was little evidence of meaningful discussion in the classrooms, which were described in terms of teacher-centeredness, pupil passivity and rote learning. However, the new National Curriculum (2003) is underpinned by notions of constructivism, together with a fresh set of textbooks for all subjects call for teachers to promote environments where both learners and teachers can interact, discuss and exchange ideas.

In a study carried out in Greece by Tzekaki, Kaldrimidou and Sakonidis (2000) talk and interaction analysed in primary school children. They found that, children did not use much talk, of any kind, to solve problems jointly, but could be successfully trained to do so. We believe that at the time children lack group-working experience, suitable material to work with and training to achieve appropriation of exploratory talk. Since then the new National Curriculum in Greek education together with the new textbooks provided the necessary ground for the implementation of exploratory and constructivist teaching and learning approaches.

The present study came after a period during which children worked in peer groups and were prompt to deal with problem solving activities through group discussion without the intervention of the teacher. The research questions of this study are:

(a) What are the patterns of verbal interaction within mathematical knowledge construction in peer discussions? (b) Does exploratory talk facilitate collective and individual reasoning in mathematics?

RESEARCH METHOD: CONTEXT AND DESIGN

Participants

Thirty three children (seventeen boys, sixteen girls), aged 11-12 years (sixth grade) participated in the study. They lived in Thessaloniki Greece, and come from a variety of socioeconomic backgrounds and family structures. All students come from two urban schools, one at he east side of the city with the socioeconomical status of the kids families being middle class while the second school, at west side, in an underdeveloped area of the city where students came from poor families and were generally of a very low achievement. All students had worked with peers in small groups since the beginning of the school year.

The study took place in the regular classroom with the class teacher. The classroom was fitted with modular furniture and equipment, including computers. Both teachers had more than 10 years of teaching experience, a lot of experience with the constructivist way of working and they were regarded as master maths teachers as they were co-authors of the new “mathematics text book 6th grade”.

Material

Educational material and procedures were designed to encourage children to use exploratory talk. The material was developed by adapting culturally, linguistically and to maths context the activity “Planning a trip” designed by Kubinova and Stehlikova (2006). To deal with the task, students had to read information from two graphs showing the changes in mean day and night temperature within a month and choose a particular time period (a week) which fulfils some conditions. The material included texts, graphs and answer sheets.

Teachers supervised group work while trying to minimize their interventions by answering only clarifying questions. They responded to questions such as “Is it right?” or “Is there any other way of doing it?” giving answers such as “What do you think?”, or “Let’s discuss it with the other teams”. When the work was finished, a member of each group presented their solution to the rest of the class, while the others examined their accuracy and sometimes corrected or rejected solutions.

Data Sources

Data sources include observations of group work, audiotape recordings transcriptions and artefacts representative of each group’s writing and drawing from the task.

Data Analysis

An overview of the analyses is presented in Table 1. The sequence of analysis steps was not predetermined but rather emerged inductively through interaction with the data (Miles & Huberman, 1994). In general, coding schemes were gradually refined through interaction with several transcripts. Once the codes could describe all of the data satisfactorily, the coding schemes were established and all of the transcripts were recoded using the final schemes. Each step of analysis is described in detail within the following sections.

Analysis Procedure	Unit of Analysis	Codes
Step 1: modes of discussion	Conversational turns within episodes: Transitions between speakers within interactions	Knowledge construction, logistical (procedural), off task
Step 2: types of statements	Statement units: The smallest meaningful codable unit of speech within a turn	codes within conceptual, metacognitive, question–query categories
Step 3: Create Episode maps	Episode: One or more topic units united by a common mode or purpose	Knowledge construction
Step 4: distinguished interaction patterns	Interaction sequences: A series of turns bounded by statements that initiate a new level of focus	Consensual, responsive, elaborative
Step 5: Relate groups' interaction patterns to the solutions	All interaction patterns and solutions achieved of each group	Codes defined for statement types, interaction patterns and solution

Table 1: Overview of Analysis Steps, Units of Analysis, and Codes

Step 1: Differences in modes of discussion among groups

The first step in analysing the transcripts was to distinguish between the different modes of the groups' discussion. Three main modes were identified: *knowledge construction* (i.e., when the discussion topic concerned mathematical knowledge construction), *logistical* (i.e., when the discussion topic concerned procedural aspects of the task such as what colour markers to use), and *off task* (i.e. irrelevant).

Step 2: Types of statements

By analysing the statement units, in knowledge construction mode, three main categories of statements emerged: *conceptual statements, metacognitive statements, and questions–queries.* Conceptual statements were observations, ideas, conjectures, inferences, and assertions about the possible solutions. Metacognitive statements were of three types: regulatory statements that directed action on the task (e.g., "Now let's do the night temperature"), evaluative statements that assessed the group's degree of progress or understanding (e.g., "We don't get this one at all"), and statements that communicated the goals of the task according criteria that the solution should meet (e.g., "Every explanation you make has to relate to the graph"). Questions and queries comprise a single statement category because their function was similar (e.g., "Does someone feel comfortable at night while the temperature is 16C?").

Step 3: Episode maps

Episodes are constructed from statement units united by a common mode or purpose. Arrows lead to and from the participant columns to indicate conversational flow, or who is contributing to and building on the substance of the discussion. The maps, then, were a tool for portraying the dialogue data graphically.

Step 4: Interaction Patterns

Statements that initiate a new level of focus are considered as an *Interaction sequence*. Interaction sequences are units of dialogue that begin when a student makes a conceptual or metacognitive statement or pose a question or query. At least one statement from another student must follow the initiating statement to comprise an interaction sequence. Analyses of episode maps revealed three types of interaction sequence patterns: *consensual, responsive and elaborative*. An interaction sequence is coded as consensual when one student contributed a statement to the discussion and another student expressed agreement. Responsive interaction sequences were those that a student posed a question or query and at least another student responded. Elaborative sequences were those that all students made multiple contributions with statements that built on or clarified another student's prior statement.

Step 5: Relation of Interaction Patterns to solution

Interaction patterns maps were constructed to depict the chronological process and the content of collaborative cognition and to highlight sequencing of pattern types.

RESULTS AND DISCUSSION

To show how all groups allocated their time and attention, the total number of turns each group spent in each of the three major discussion modes were tallied.

The percentage of conversational turns each group spent in each discussion mode during the mathematical task is shown in figure 1.

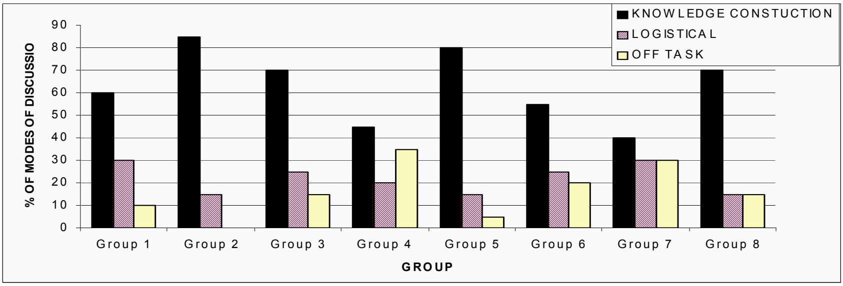

Figure 1: Percentage of modes of discussion among groups

As shown in figure 1 the amount of conversation time dedicated to mathematical knowledge construction differed between groups. As the task and the total amount of time allocated for group work were identical for all groups, this difference points to differences in the characteristics of the groups themselves (e.g., prior knowledge, conceptualization of the task, goals, sociocognitive skills, interpersonal relations, tenacity, etc.). All eight groups were considered valid for the research as all their members were engaged in knowledge construction interactions. Groups 2, 3, 5 and 8 were considered most effective groups since the were highly involved in the mathematical task (i.e. solving the problem), as can be evidenced by the amount of time spent on productive, knowledge construction interactions compared with logistical and off task interactions.

The three types of interaction patterns identified within this study were described in the Data Analysis section as *consensual, responsive, and elaborative*. How did the relative frequencies of these three types of interaction compare? Table 2 presents the total interaction sequences within episodes that were coded as each pattern type.

	Group1	Group2	Group3	Group4	Group5	Group6	Group7	Group8
consensual	7	4	10	4	6	7	3	10
elaborative	13	20	15	8	19	11	7	16
responsive	9	10	12	7	10	8	7	13

Table 2: total interaction sequences within episodes

Consensual patterns were the least frequent in episodes. This type occurred when one student took central role to think through an idea while others made passive or encouraging remarks. More consensual patterns can be found in groups where less participation recorded from all the group members. Perhaps this pattern is more characteristic of one-to-one exchanges than of interactions in groups in which more speakers participate.

Elaborative interaction sequence patterns usually appeared in groups that all members tended to be more generative. These groups did not have a leading person that dominated and continually focused the elaborations. Students were participating freely expressing their ideas, without the presence of a leader.

Responsive patterns occurred relatively often in most groups. However responsive patterns often were only a few turns in length. In six out of eight groups, only one student at a time responded to the questioner. It seems that questions or statements posed by peers that led to simple responsive patterns typically were less demanding and comprehensive than questions posed by teachers.

Considering that groups 1, 2, 3, 5 and 8 reached the ideal solution to the task problem and reasoned very well to support it, group 6 managed to get close enough and groups 4 and 7 reached to a solution quite far from the ideal and also had difficulties expressing their arguments, we can conclude, regarding to how patterns of interaction relate to reasoning levels groups achieved, that the most productive pattern of interaction was elaborative. Although multiple consensual or responsive sequences could also build towards higher forms of reasoning, these seemed to be less productive patterns. It seems that the more they talked with one another engaging in a knowledge construction discourse, the higher their reasoning complexity.

SUMMARY AND CONCLUSIONS

Despite having two schools that varied notably in students' achievement and socioeconomic levels, we did not notice differences in the interaction patterns recorded while discussing the given task. Because all groups expressed a clear desire to present a solution that would impress their peers and teacher, there were not gross differences in motivation among the groups that could account for how they allocated their time. Apparently, Greek students working in groups have the potential to engage in sustained and complex mathematical reasoning.

The dialogue of groups that focused more on conceptual than on logistical or off-task discussions had the following characteristics: presenting provocative ideas articulately, being able and willing to ask for clarifications and then interpreting and building on peers' ideas. In groups that members failed to acknowledge and engage with one another's ideas, their dialogue was disjointed and unproductive. When collaborative productivity was low, students were looking to the teacher for assistance rather than trying to create their own ideas.

Exploratory talk was used most frequently when students held a shared understanding of contextual conditions and when they placed a higher value on the collaborative cognitive process than on individual output. A significant feature of the discourse was the way students identified issues, considered and evaluated each other's contributions and adapted their own views accordingly. Moreover, students not only considered and evaluated task material, but also formulated questions themselves.

This study has implications for how groups are formed for collaborative knowledge building. The high challenge level of a task such as building a high level strategy could easily be so overwhelming that it kills all motivation to engage in it. However, a challenge shared by several people is less frightening so long as at least some of the group members have confidence, ideas, and strategies for tackling the challenge. The ability to sustain intellectual work during times when it seems like no progress is being made is crucial.

The data indicate that when students collectively recognize collaborative ground rules they exercise self-regulation, display self-determination and a desire to keep trying with a task. Group interaction involves a combination of pupils thinking aloud, being open to each other's ideas, and collaborating in the expression of shared meanings.

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MATHEMATICAL CREATIVITY THROUGH TEACHERS' PERCEPTIONS

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This study examines elementary school teachers' conceptions of creativity, focused in mathematics. The study was conducted among 47 elementary school teachers in Cyprus, using a questionnaire of four open-ended questions. The results revealed that while teachers acknowledge the importance of their role as individuals and as professionals in fostering mathematical creativity, they report several factors of the educational system that inhibit the manifestation of mathematical creativity, excluding themselves from accepting any responsibility. Implications for researchers and policy makers are outlined.

INTRODUCTION

The year 2009 was declared as the Year of Creativity and Innovation by the European Community. This fact reveals that policy makers globally have started to recognize the importance of creative thinking as an investment in their country's future (Craft, 2007). Nevertheless, any effort to foster creativity in classrooms will ultimately depend on the teacher. Thus, teachers' conceptions and knowledge of creativity are of great importance to researchers and policy makers and should be taken into account in developing students' creativity (Runco & Johnson, 2002).

THEORETICAL FRAMEWORK

Creativity as a construct

Creativity is a complex construct and as such it has been defined in several ways. Torrance (1995) defines creativity as a product of fluency, flexibility, originality and elaboration. Fluency is the ability of producing many ideas (Gil, Ben-Zvi & Apel, 2007). Flexibility refers to the number, the degree and the focus of approaches that are observed in a solution (Gil, Ben-Zvi & Apel, 2007). At the same time, the term originality refers to the possibility of holding extraordinary, new and unique ideas (Gil, Ben-Zvi & Apel, 2007), and elaboration refers to the ability of extension, improvement or format of an idea (Mann, 2006).

Teachers' perceptions on the enhancement of creativity

According to Best and Thomas (2007), one of the components to ensure creative teaching in mathematics is teachers' professional and personal domain. Thus, the theoretical framework of the present study is organized based on the abovementioned distinction. Professional domain refers to teachers' role and actions during teaching in order to enhance creativity. Horng and colleagues (2005) argued that teachers

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should serve more as facilitators, learning partners, inspirers or navigators than as lecturers.

In a study conducted by Shriki (2008), teachers believe that a creative environment should include open-ended activities and non-routine problems that give students freedom to apply imaginative ideas and find novel methods or solutions. Similarly, teachers believe that the avoidance of assignments focusing on rote thinking facilitates the development of students' creativity (Fleith, 2000). Furthermore, the use of a variety of teaching methods and aids such as technology, consist of key factors that improve students' creativity (Horng et al., 2005). Moreover, teachers believe that students' cooperation with classmates of similar interests fosters creativity (Fleith, 2000).

The personal domain includes a variety of personality traits such as self-confidence, openness to experience, fantasy orientation, imagination and flexibility of thoughts (Horng et al., 2005). Gardner's (1994) and Horng and his colleagues' (2005) research studies emphasized the communicative nature of the teacher. In particular, professional relationships and interactions with colleagues were mentioned by teachers as aspects that contribute to the development of teachers' creativity.

Perceptions on the barriers inhibiting creativity

Despite the fact that most teachers acknowledge the importance of creativity, still many of them do not include it in their teaching. Specifically, teachers identified the following factors which hinder creativity: the use of one right answer, no mistakes, ignored ideas, competition, evaluation, and insufficient knowledge (Fleith, 2000; Shriki, 2008). Other inhibiting characteristics mentioned by teachers include strict discipline, drill work, emphasis on curriculum and lack of time due to various external pressures such as covering the syllabus and helping students succeed in exams (Fleith, 2000; Shriki, 2008). Consequently, teachers tend to emphasize memorization and rote thinking in teaching rather than creativity.

PURPOSE OF THE STUDY

Despite the importance of fostering creativity in elementary school settings, the literature review indicates that little attention has been paid to teachers' conceptions regarding creativity (Diakidoy & Phtiaka, 2001). The existing body of research focused on teachers' views concerning characteristics of creative individuals, practices or environments. This study adds to the research literature some evidence about teachers' perceptions on mathematical areas which encourage the development of students' creativity by providing specific examples of tasks.

Therefore, the present study purports to: (1) reveal teachers' perceptions with respect to the characteristics and practices of a creative teacher, (2) recommend mathematical areas considered by the teachers to be more suitable for facilitating creativity, by presenting specific examples of creative mathematical tasks and (3) reveal teachers' perceptions regarding the opportunities offered by the Cypriot elementary educational system to foster mathematical creativity.

METHOD

Sample - Procedure

Data were collected through an anonymous questionnaire that was administered to 47 randomly selected teachers in elementary schools in Cyprus. These teachers varied in their teaching experience from 1 to 19 years. Most of them (55.32%) had long teaching experience (10-19 years). The questionnaire was disseminated to teachers by hand or through e-mail and it was returned in the same way.

Instrument

The research instrument consisted of four open-ended questions. Teachers were asked not only to answer the questions but to justify their responses (see Figure 1). The justifications and the examples provided by teachers were analyzed through content analysis (Weber, 1990) i.e., the collected data were broken down and coded into different concepts/labels. After clustering these labels, categories were generated, thus making it possible to emphasize general themes and draw conclusions about teachers' conceptions regarding mathematical creativity.

1. What sort of characteristics and practices would you identify in a mathematically creative teacher?
2. To what extent do you use creative activities/strategies during mathematics teaching? Are there any mathematical areas that are more suitable for fostering creativity and others that are not appropriate?
3. Please provide an example of mathematical activity that you consider to be creative.
4. What do you think about the opportunities offered by the Cypriot elementary educational system to foster mathematical creativity? Justify your answer.

Figure 1: Items of the questionnaire.

RESULTS

The results of the study are presented according to the four items of the questionnaire.

Question 1: Characteristics and practices of a mathematically creative teacher

In an attempt to describe a mathematically creative teacher, participants provided a variety of characteristics that could be categorized in two groups: personality traits and professional abilities (Table 1). These categories are in accord with Best and Thomas's (2007) notion of teachers' professional and personal domain. Regarding professional abilities, the participants focused on the variety of activities, teaching methods and manipulatives used during teaching (53.19%). Teachers believed that certain pedagogical approaches such as inquiry learning (51.07%), the use of open-ended tasks (38.30%) and differentiation of teaching according to students' needs (21.28%) are effective ways in working towards creativity. Concepts such as cooperative learning (17.02%), use of technological tools (17.02%) and realistic mathematics (12.77%) where mentioned by teachers as strategies necessary to differentiate the curriculum for creative students. Teachers put less emphasis on their mathematical knowledge (12.77%) and their role as facilitators (8.51%).

In addition, teachers considered originality (55.32%), flexibility (42.55%) and imagination (36.17%) as the most important characteristics that referred to personality traits. Furthermore, teachers perceived the openness to new ideas (28.79%), perseverance (21.28%), and divergent thinking (19.15%) as the factors that foster creativity. A percentage of 10.64% of teachers suggested communication, either with students or with other teachers, as a characteristic of a creative teacher. Finally, very few of them (8.51%) proposed critical thinking as an important factor that contributes to creativity.

Personality traits	N (%)	Professional abilities	N (%)
Originality	26 (55.32)	Use of a variety of activities, teaching methods and manipulatives	25 (53.19)
Flexibility	20 (42.55)	Investigation-discovery of knowledge	24 (51.07)
Imagination	17 (36.17)	Use of open-ended tasks	18 (38.30)
Open mindedness	14 (28.79)	Opportunities for all students-Differentiation	10 (21.28)
Perseverance	10 (21.28)	Combination of cooperative and individual work	8 (17.02)
Divergent thinking	9 (19.15)	Use of technology	8 (17.02)
Communicative nature	5 (10.64)	Use of realistic problems	6 (12.77)
Critical thinking	4 (8.51)	Mathematics knowledge-efficacy	6 (12.77)
		Teacher as facilitator- coordinator	4 (8.51)

Table 1: Characteristics and practices of a mathematically creative teacher.

Question 2: The use of creative activities/strategies during mathematics teaching

The second question requested teachers to note whether they use creative activities/strategies during mathematics teaching and to mention mathematical areas they consider to be more suitable for facilitating creativity. The majority of teachers (65.96%) stated that very often they use creative activities during their daily lessons and believed that all mathematical topics are suitable for fostering creativity (78.72%). In their own words:

"I believe that creativity can be applied in all mathematical areas, because creativity characterizes both students and tasks. Thus, all mathematical activities could be considered to be creative, if the teacher wishes to. For example, by modifying a mathematical task (think of another method, find as many solutions as you can) you could change the aim of the activity. On the other hand, students can work with an activity in their own way, indicating original solution paths."

The following extracts show the emphasis placed by teachers on their role in developing creative activities for students; specifically, they focus on the availability of suitable activities and manipulatives:

"We can use creative activities in all mathematical areas, as long as there is enough time and the appropriate manipulatives are provided."

"Since the teacher has a collection of appropriate tasks that promote originality and flexibility, all mathematical areas can foster creativity."

"If teachers know well the mathematical content, then creativity can be found in all mathematical areas. Systematic preparation is needed."

In total, teachers acknowledged problem solving (12.77%) and geometry (12.77%) as the most popular mathematical areas to apply creative tasks. A teacher referring to problem solving reported that

"Through problem solving, students' imagination and flexibility is developed, until the proper solution is found."

In contrast, 17 teachers (36.17%) thought that algorithms and routine problems are not recommended for creative tasks:

"In my opinion, during teaching the algorithms for the four operations, you can not apply creativity, since what you teach is mechanistic."

"Less or even no creativity can be applied in closed activities where the way of solution is given."

Question 3: Examples of creative mathematical activities

The mathematical examples provided by teachers were divided into two categories; creative and non-creative mathematical tasks. Torrance's features of fluency, flexibility, originality and elaboration had to characterize the tasks in order to be considered creative. For the present study, only indicative creative mathematical tasks will be presented.

A significant proportion of teachers (40.43%) provided examples of creative mathematical activities derived form "Numbers and operations" such as the following task, which allows for multiple solutions and novel responses:

"Look at the numbers: 23, 20, 15, 2. Which number does not belong here? Why?"

Participants also provided creative examples from "Geometry" (27.66%) such as covering an area in different original ways. In addition, eight teachers suggested examples related to problem solving and problem posing. Some indicative open-ended problems are the following:

"A bear weights 500Kg. How many children weigh the same as a bear?"

> "We can give students information about airport tickets, accommodation etc. Then we can ask them to decide which will be the best choice for their vacations and to justify their answers."

Question 4: Opportunities for mathematical creativity offered by the Cypriot elementary educational system

The majority of teachers (83%) consider that mathematics, as presented and taught in Cyprus, are not connected with creativity. Thus, they provided several barriers that inhibit mathematical creativity, as shown in Table 2.

Barriers	N (%)
Focus on the answer than in the process	14 (29.79)
Content to be covered	13 (27.66)
Restricted time	12 (25.53)
Textbooks and activities provided	8 (17.02)
Teachers' training	7 (14.89)
Suitable teaching material	2 (4.26)

Table 2: Barriers inhibiting mathematical creativity

As shown in Table 2, the pressure of covering content (27.66%) in limited time (25.53%) causes teachers to focus on the right answer (29.79%), ignoring the process followed and to limit in exercises provided in textbooks (17.02%). A teacher's view that summarizes the abovementioned barriers follows:

> "Teachers usually adopt the textbook activities, without differentiating them and without challenging the students. They are more interested in the presentation of typical solutions. They are not particularly concerned in helping students broaden their thinking; instead they insist on preparing them for examinations or cover the suggested content form the material."

Another barrier for not fostering creativity in mathematics is the lack of in-service training (14.89%) as well as the lack of suitable material (4.26%):

> "In my opinion, teachers have not been informed about the importance of creative activities and their use in mathematics. I believe that training courses that involve definitions of creativity, examples of mathematical creative activities, as well as evaluation methods are required."

DISCUSSION

The aim of the present study was to investigate teachers' perceptions regarding the characteristics and the practices of a mathematically creative teacher, in order to foster creativity in students. Moreover, factors of the Cypriot elementary educational system which teachers conceive as barriers toward mathematical creativity have been outlined.

With regard to the characteristics and the practices of a mathematically creative teacher, Cypriot elementary school teachers proposed both professional and personality characteristics. The proposed professional characteristics include general pedagogical approaches, such as cooperative learning, learning through inquiry and the use of technology, verifying similar findings (Fleith, 2000; Horng, et al., 2005). Moreover, teachers perceive that the use of open-ended tasks, differentiation and the variety in activities, teaching methods and manipulatives can contribute to the manifestation of creativity. It can be assumed that teachers consider that through teaching, creative potential might be maximized in all students.

With respect to personality traits, teachers mainly referred to Torrance's characteristics (1995) of creative behaviour, namely originality and flexibility. In addition, imagination and the ability to being open to new ideas were also mentioned, enhancing the findings of Horng and colleagues (1996). Nevertheless, professional relationships and interactions with colleagues were not strongly perceived to be a possible facilitator to creativity from Cypriot teachers, despite related findings (Gardner, 1994; Horng et al, 2005). It can be deduced that teachers focus only on themselves as personalities and individuals and do not extend to work relationships which include third persons.

Regarding teaching creatively, the majority of participants believed that they apply creative activities while teaching mathematics. Despite our intention to suggest mathematical areas considered to be more suitable for fostering creativity, teachers noted that all mathematical areas can serve to this end. When teachers asked to provide specific examples, problem solving was considered to be an ideal way of fostering creativity, verifying Shriki's findings (2008). At the same time, teachers perceive that the teaching of algorithms does not allow creativity. Based on the abovementioned results, it is argued that teachers believe that what really matters in relation to mathematical creativity is not the mathematical content itself, but rather the teacher and his/her practices.

Even though teachers were asked to express their opinion regarding opportunities offered by the Cypriot educational system to foster creativity, their responses focused on specific barriers inhibiting the implementation of creative mathematical activities. Mathematics curriculum in combination with textbooks represents the main obstacle, enhancing Shriki's findings (2008). In an attempt to cover mathematical content in restricted time, teachers ignore the use and influence of mathematical activities in students' understanding. Teachers' responses might be explained by the lack of pre-service and in-service training related to creativity and by the inadequacy of mathematics curriculum and textbooks to provide opportunities to foster creativity in educational settings. For this reason, new mathematical textbooks should be developed that will give the opportunity for a creative implementation of mathematical content. In addition, further research is required for the development of related teacher training.

This study offers researchers an insight into teachers' perceptions of mathematical creativity and the way it is incorporated in everyday teaching. Summing up, it can be deduced that the results of the present study give strong evidence to support that although elementary school teachers consider themselves as a key factor in developing mathematical creativity, at the same time they do not accept any responsibility as one of the reasons of hindering creativity. Instead, they focus only on aspects of the educational system.

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INVESTIGATING REPRESENTATIONAL FLUENCY IN MODELING PROCESS: THE EXPERIENCE OF PRE-SERVICE TEACHERS WITH A CASSETTE PROBLEM

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As a part of larger research on pre-service teachers' mathematical modeling abilities, this study investigates pre-service teachers' representational fluency, and its effect on modeling process in one modeling activity, that of cassette problem. The data is collected by students' individual and group written responses to the mathematical modeling activity, video-taped group discussions and classroom observation by the researcher. This study is conducted during the spring semester in 2007 and thirty three pre-service teachers were the participants of the study. The data showed that pre-service teachers have difficulty in transition between different modes of mathematical representations and this difficulty has an influence on modeling process.

INTRODUCTION

Since modeling process involves a series of develop-test-revise cycles each of which requires different ways of thinking on the nature of givens, goals and different solution steps while only the process from givens to goals is stressed in traditional problem solving approach (Lesh and Harel, 2003), mathematical modeling has been accepted as an alternative to traditional problem solving. Opportunity for applications of mathematics in meaningful real life situations has been another argument of mathematics educators for explaining the importance of modeling (Verschaffel, Greer and De Corte, 2002). There exists some different point of views on mathematical modeling in the literature (e.g.,Verschaffel et al.; Lingefjard, 2000; Lesh and Doerr, 2003). However, the studies on mathematical modeling reveal a common agreement of researchers from different perspectives on that there exists a process in modeling including multiple cycles (Zbiek and Conner, 2006). Modeling cycles with different diagrammatic representations, in general, includes sub-processes such as defining a mathematical model, solving, interpreting the solution, validating and testing the model, and using the model to explain or predict real life phenomena (Lingefjard, 2000; Lesh and Lehrer, 2003). Representational media and systems are accepted as an important component of modeling cycle in the diagram representation of Lesh and Lehrer (2003). The importance of representational fluency in mathematical modeling is also stressed in model-documentation principle which is introduced by Lesh and Doerr (2003). The usage of mathematical representations such as tables, graphs, and drawings is an important component of the documentation process.

Although, representational fluency in modeling literature is stressed as an important component of modeling cycle and as an important factor in documentation process (Lesh and Doerr, 2003), more evidences from empirical studies are needed to

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understand the importance and role of representational fluency in modeling process. Lack of transition between and within different modes of representations may be an important factor affecting the modeling process. In this study, we try to investigate, how transition between different modes of representations affect pre-service teachers' performances and so the modeling process. Findings of this study may contribute to the research area of mathematical modeling as well as the construct of representational fluency on which research studies is suggested (Zbiek, Heid, Blume and Dick, 2007). The research questions investigated in this study are;

1. What modes of representations appear in modeling process of pre-service teachers?

2. How does the transition between different modes of representations appear in and affect modeling process?

THEORITICAL FRAMEWORK

As stated before, there exist approaches to mathematical models and modeling in the literature. Models and modeling perspective on mathematics problem solving, learning and teaching proposed by Lesh and Doerr (2003) is adopted in this study. This approach comes with comprehensive and novel ideas for all aspects of mathematics education such as one essential idea appears for problem solving in mathematics. They suggest mathematical modeling activities (*model-eliciting* in their terms) instead of traditional word problems. Modeling activities involve sharable, manipulatable, modifiable, and reusable conceptual tools (e.g., models) for constructing, describing, predicting, or controlling mathematically significant systems (Lesh and Harel, 2003). According to Lesh and Harel (2003), multiple modeling cycles exist in modeling activities that require the usage of multiple representational systems. Also, sharable, modifiable and reusable properties of constructed conceptual tools (e.g., models) in modeling activities necessitate using representations in a flexible way. In this context, the role of representational fluency, the definition of representational fluency and what research studies are conducted on it are some of the questions that come into mind.

Representational Fluency

The ability to establish meaningful links between different representations and to translate from one mode of representation to another has been defined as *representational fluency* (Lesh, 1999). According to Zbiek and others (2007), the term *representational fluency* is more than the translation between different modes of representations. They defined representational fluency as follows:

> Representational fluency includes the ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations. (p.1192)

Some research studies are conducted, in technology integrated context, on representational fluency in specific content domains as algebraic equations (Suh and

Moyer, 2007) and derivative (Santos and Thomas, 2001). In the function content domain, Elia and others (2007) evidenced students' incompetence in flexibly handling different modes of representations. Also, Elia and Gagatsis (2006) showed strong effects of using different modes of representations on pupils' problem solving performances. In this study, *representational fluency* is investigated in modeling process viewing it as transition between representations by also considering mathematical meanings between and within each mode of representations which is concurrent by the definition of Zbiek and others (2007).

DESCRIPTION OF THE STUDY

This study has anti-positivist paradigm with interpretive approaches (Cohen et al., 2000, p.22). Case study is used as a research strategy to make an in-depth examination of transition between representations through modeling process in this study.

Participants and Procedures

The study is conducted with last year 33 pre-service mathematics teachers in a state university in Istanbul. 33 pre-service teachers worked on a modeling activity in a lesson period of 60 minutes. This study is conducted as a part of the "Teaching Methods in Mathematics Education" lesson in 2007 spring semester.

Pre-service teachers were asked to explain mathematically changes in the radius of both reels of a tape cassette. The time schedule is planned as follows: In the first 25 minutes, all pre-service teachers studied on the problems individually. After that, papers including individual responses are collected. During the second 25 minutes of the section, small groups consisting 3 or 4 persons are formed, and pre-service teachers studied on the same problem by a group work. One group is observed and videotaped by the researcher in the section, and important points of group discussion are noted by a group reporter in other groups. At the end of this second 25 minutes of the section, papers showing the details of group solutions and group reports are collected. This group reporting was followed by a whole class discussion (one of the groups presented their solutions in the rest of the classroom) in the last part of the lesson. The primary aim of individual working before group work, and group work before classroom discussion was increasing the quality and efficiency of the group discussion and classroom discussion. Since all participants thought about and proposed a solution for the problem before the group work, they started to group discussion with a detailed understanding of the situation given in the problem.

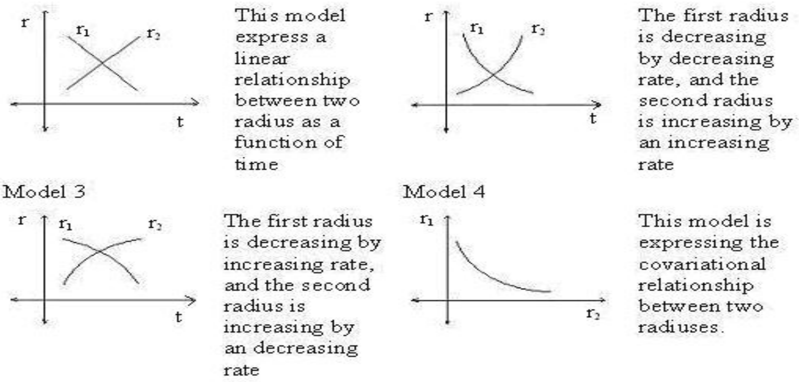
Data Sources and Analysis

Pre-service teachers individual and group written responses to the modeling activity, video-taped group discussion and field notes of the researcher are the main sources of data. Participants studied on the following modeling activity:

Activity: Consider an ordinary cassette placed in a tape recorder. When the tape is played, it is transferred from one reel to the other at a *constant* speed. Try to explain the change in the radius

of the roll of the tape on both reels mathematically. Adapted from Lingefjard (2000)

The analysis of the responses indicated the following categories for the modes of representations preferred in modeling process: Pictorial representation in which pre-service teachers drew the picture of a tape cassette, verbal expression in which pre-service teachers provided verbal explanations such as "one radius is increasing while the other one is decreasing", graphical representations as one of the types seen below, and algebraic representations such as "$r_1.r_2$=constant". These representations also categorized according to mathematical meanings they conveyed with the following 4 models.



- *M1 (PC)*: The representation fits the Model 1. Correct picture of situation (PC) is used for pictorial representation.
- *M2 (PW)*: The representation fits with Model 2. Wrong picture of the situation (PW) is used for pictorial representation.
- *M1-M2*: One part of representation fits with Model 1 while the other part fits with Model 2.
- *M3*: The representation fits with Model 3
- *M2-M3*: One part of representation fits with Model 2 while the other part fits with Model 3.
- *M4*: The representation fits with Model 4.
- *NR*: (No response) or this mode of representation does not exist in the solution.
- *NA*: The category is not applicable for that mode of representation

Different representations observed in written responses are classified with one or more of these models according to their mathematical meanings. By this way, we do not only focus on transition from one mode of representation to another, but also mathematical meanings within any mode of representation.

RESULTS

The results from individual and group studies on the problem were summarized in the Table 1, Table 2 and Table 3 in this section.

N=33	Pictorial		Verbal		Graphical		Algebraic	
	Percent	N	Percent	N	Percent	N	Percent	N
M1 (PC)	39	13	46	15	12	4	18	6
M2 (PW)	30	10	0	0	3	1	0	0
M1-M2	NA		0	0	3	1	0	0
M3	NA		33	11	3	1	3	1
M4	NA		0	0	0	0	6	2
M2-M3	NA		6	2	12	4	0	0
NR	30	10	15	5	67	22	73	24

Table 1: Frequencies of individual studies

Table 1 show that 69% of pre-service teachers provided pictorial representation where 39% and 30% of these representations are correct and wrong picture of the situation respectively. However, 85% of participants provided a verbal explanation; 46% of these expressions a linear relationship, 33% of these verbal expressions fits with M4, and 6% of participants' expressions evidence their confusion between M2 and M3. 33% of participants provided a graphical representation where 12% of them fit with M1, 12% of them have confusion between M2 and M3. 27% of participants provided an algebraic representation, but 18% of them express a linear relationship, 3% fits with M3, and 6% fits with M4. Specifically focusing on M3, we observe 33% of participants provided a verbal expression in this model, but only 3% of them could provide a graphical and algebraic representation in the same model.

N=8	Pictorial		Verbal		Graphical		Algebraic	
	Percent	N	Percent	N	Percent	N	Percent	N
M1 (PC)	50	4	37,5	3	25	2	25	2
M2 (PW)	50	4	0	0	0	0	0	0
M1-M2	NA		0	0	0	0	0	0
M3	NA		37,5	3	25	2	25	2
M4	NA		0	0	0	0	0	0
M2-M3	NA		25	2	25	2	0	0
NR	0	0	0	0	25	2	50	4

Table 2: Frequencies of group studies

As can be seen on Table 2, all groups provided a pictorial representation where half of these representations are correct and the other half is accepted as wrong. Also, all groups provided a verbal expression, but 37.5% of them are expressing a linear relationship and 62.5% of them are expressing M3 or M2-M3. 75% of groups provided a graphical representation where 25% of these graphs express M1, 25% of them express M3, and 25% of them express M2 and M3 within the same graph. 50% of groups provided an algebraic representation where 25% expressing M1 and 25% expressing M3.

	Pictorial	Verbal	Graphical	Algebraic
Individual	69	85	33	27
Group	100	100	75	50

Table 3: Comparison of representations provided by individual and group studies

On Table 3, when representations provided without considering mathematical meanings is carefully examined, a decrease in percentage of representation provided from pictorial to algebraic in both individual and group studies is observed. This is also evidence which shows difficulty of transition from one mode of representation to another.

Furthermore, an excerpt from group discussions is provided on showing how transitions between verbal explanation and graphical representation occur. The conservation below shows how pre-service teachers express their thoughts by using different modes of representations. The discussion is carried out by Group5 with 4 members.

An excerpt from the discussion of Group5:

Student 29: I think the empty-reel side of the tape will spin faster than the other reel

Student 5: The constant speed is already given in the question.

Student 29: Radius of the empty-reel increases and other one decrease by the time...

Student 11: Although *constant speed* emphasized in the question, this is constant for only one reel, the speed will change and increase eventually for other one.

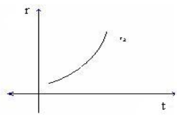
Student 29: I think we have to focus on only changes in the radius of the reels. There is no change in the speed; the radius of both sides is changing

Student 5: Radius is increasing by a decreasing rate...

Student 40: One of the radiuses is increasing by a decreasing rate while the other side is decreasing by an increasing rate.....

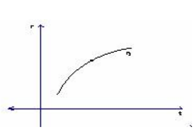
Student 11**:** Wait a minute, I am confused

Student 29**:** Oh! I see, I can explain you on graph.



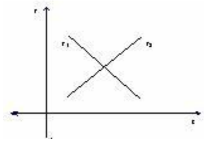
Student 5: the graph is like that for the increasing reel

Student29: No. This is not true. This graph is explaining increasing by an increasing rate of change....I think the correct graph will be like that. This graph is expressing that the radius is increasing by a decreasing rate of change.



Student 11: Let's try to visualize the conditions of both radiuses by giving numerical values at different times

full side	empty side
30	10
20	20
10	30



Student 5: If we think by this way, then the graph will be like that

In a little part of a whole discussion seen above, pre-service teachers try to interpret the situation by using verbal and graphical representations. In the discussion, pre-service teachers first realize the inverse relationship between two radiuses easily and they explain this covariation verbally. After that point, the difficulty translating the mathematical meaning of verbal expression into graphical representation is observed. Pre-service teachers' have confusion among three types of graphical representation. One more thing observed in the discussion is pre-service teachers' lack of conceptual understanding of *angular* and *linear* speed concepts.

DISCUSSION

Cassette activity revealed that pre-service teachers have difficulties in transition between different modes of representations. Inconsistencies between and within different modes of representations, with respect to mathematical meaning they conveyed, are also observed in this study. In some cases, changes in mathematical meanings of representation from one mode to another and some confusion within a representation observed. Verbal representation is used as reference representation and number of representations is considerably decreased from pictorial to algebraic continuum. This might be interpreted from social cognitive perspective. Chronological order of speaking, drawing and using symbols in a human life shows the most used order as well. As expectedly, group studies comparing to the individual ones affected students use of representations and success, particularly communicating by speaking they shared ideas, but it still did not help them so much showing representational fluency in their work.

Although pre-service teachers' performances were out of the scope of this paper, it can be argued that difficulty of transition between representations affected their performances in the activity and also the modeling process. It seemed us that representational fluency plays an important role not only during documenting the produced model, but also in all phases of modeling process. Furthermore, pre-service teachers' understandings of "rate of change" and "linear and angular speed" concepts which are diagnosed in modeling process can also be the key point in transition between different modes of representations. For in-depth understanding of the whole modeling process, representational fluency between and within different phases of modeling cycles and role of conceptual understanding in representational fluency can be investigated for further studies.

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TACIT MODELS, TREASURED INTUITIONS AND THE CONCEPT OF LIMIT

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We compare and contrast cognitive processes that accompany different manifestations of persistence of tacit models and intuitions that coexist with students' logical reasoning in topics that relate to the continuous. The students are highly trained in mathematics. We encounter expressions of the persistence and impact of intuitions and tacit pictorial models as described by Fischbein (2001) in which a situation of conflict is created. But we also observe another, very particular expression of persistence of tacit models in which the tacit pictorial model continues to interfere in the student's reasoning process but does not prevent her from reaching a feeling of logical consistency.

INTRODUCTION

The starting point of this research is the following sentence in Fischbein (2001, page 316):

> The main remark …is not the existence and influence of tacit models in our reasoning in the domain of actual infinity. The main remark is, in our opinion, the persistence and impact of such pictorial models even in individuals already highly trained in mathematics, aware of the abstract nature of mathematical objects.

In my research on students' conceptual thinking in relation to notions that relate to the continuous like the notion of limit, I noticed the persistent influence of treasured intuitions. I encountered situations described by Fischbein in which students claim: "Formally, you seem to be right but visually, intuitively, it seems to be unacceptable that…." (Fischbein, 2001 page 311). In such situations, the learner is aware of the internal tensions between his intuitive reaction and the formal knowledge. The learner is aware of a situation of conflict even if he is not aware of the exact origin of his mental models and the extent of their influence on his reasoning. This paper describes a part of a qualitative study which aims to compare and contrast different manifestations of the persistence and impact of intuitions and tacit models. I observe situations in which the learner has a feeling of contradiction, of paradox, but I also observe some other characterization of persistence of tacit models in which the tacit pictorial model continues to interfere in the student's reasoning process but does not prevent her to reach a feeling of logical consistency. In order to better appreciate the analysis of the different manifestations of persistence and impact of mental models, I first describe in the next section some facets of the dynamic interaction between formal and intuitive representations.

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THE DYNAMIC INTERACTION BETWEEN INTUITION AND FORMAL REASONING

Fischbein (1978) finds that students' intuitive conceptions of limiting processes tend to focus more on the infinity of the process than on the finite value of the limit. Tall and Tirosh (2001) point out that the never- ending struggle with the potential infinity of the process proves to offer a serious obstacle to students' understanding of the limit concept. Fischbein (2001) analyses several examples of tacit influences exerted by mental models on the interpretation of mathematical concepts in the domain of actual infinity. He describes the concept of mental models as mental representations which replace, in the reasoning process, the original entities. The models may inspire and support correct mathematical inferences with regard to some mathematical properties but they may also lead to wrong conclusions with regard to others. Fischbein (2001) deals with models about which we are not aware, which replace tacitly some of the original components of the reasoning process. The model is partially different from the original, and it brings with it also properties which are not relevant for the original. As a consequence, the influences of the respective tacit models, being generally uncontrolled consciously, may lead to contradictions. As an example, Fischbein deals with the pictorial models of statements related to the infinite sets of geometrical points on a segment "Although we know perfectly well that mathematical points have no dimensions, we continue to think tacitly, unconsciously, in terms of small spots" (Fischbein, 2001, page 315). There might be some contradiction between the abstract-formal paths of reasoning and the interference of the intuitive-figural model in the reasoning. Even so, Fischbein (1982) considers the intuitive structures as essential components of productive thinking. Moreover, he claims that the role of intuitive structures does not come to an end when analytical (formal) forms of thinking become possible. His view is that the intuitive and the analytical forms of knowledge are complementary and deeply intercorrelated like two facets of a unique, mental productive behavior. Fischbein adds that even at the level of purely formal concepts, productive thinking needs such forms of intuitive, sympathetic representations. He emphasizes the need to synthesize into one mental structure the formal understanding of a mathematical statement with a full conceptual control and the intuitive acceptance of the statement.

THE LEARNING EXPERIENCE

Helping students towards this synthesis of the formal and the intuitive into one mental structure, as described by Fischbein, was a source of motivation for my research on students' understanding of central notions in analysis like the limit concept. A description of the learning experience can be found in Kidron (2008). In the present paper the research study will be described only through the demonstration of the role and influence of tacit models and treasured intuitions. New excerpts appropriate to this perspective will be presented and analyzed. Helping students towards the synthesis of the formal and the intuitive into one mental structure, we emphasize the importance of both the formal definition and the intuitive representations of the mathematical concept. For example, we deal with the notion of limit in the concept of derivative. If our aim is to describe what is being defined in f '(x) as $\lim_{h\to 0}$ (f(x+h)-f(x)) / h, we may try to develop visual intuitions that support the formal definition. In this case we create the impression of a potential infinite process of (f(x+h)-f(x)) / h approaching f '(x) for decreasing values of the parameter h. By means of animation, the students visualize the definition of the derivative. There might be other effect as well: The dynamic picture might create a mental image of the derivative and reinforce the misconception that one can replace $\lim_{\Delta x\to 0} \Delta y / \Delta x$ by $\Delta y / \Delta x$ for Δx very small. To develop students' understanding of the concept definition of the derivative as a limit, it was crucial for my empirical study to find a counterexample that will demonstrate that one cannot replace the limit "$\lim_{\Delta x\to 0} \Delta y / \Delta x$" by $\Delta y / \Delta x$ for Δx very small and that omitting the limit, will significantly change the nature of the concept. I found the counterexample in the field of dynamical systems. The mathematical model is a differential equation dy/dt = y' = f(t,y) and we encounter again the derivative $y' = \lim_{\Delta t\to 0} \Delta y/\Delta t$. In a dynamical process that changes with time, time is a continuous variable. Using a numerical method to solve the differential equation, there is a discretization of the variable "time". In the counterexample that will be described in the following (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical solution obtained by means of discrete numerical methods. The important point that we should notice is that using the analytical solution the students use the concept definition of the derivative as a limit "$\lim_{\Delta x\to 0} \Delta y / \Delta x$". On the other hand, using discrete approximation by means of the numerical method, the students use the mental image "$\Delta y / \Delta x$ for small Δx". The two solutions, the analytical and the numerical, are totally different: I analyzed the effect of this situation of conflict between the students' mental model and the concept definition. The learning experience was carried in an innovative differential equations' course given to first year college students. The logistic equation dy/dt = r y(t) (1-y(t)), y(0) = y_0 was introduced as a model for the dynamics of the growth of a population. An analytical solution exists for all values of the parameter r. The students were introduced to the numerical solution by finding the algorithm: $y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$ for Euler's method. The numerical solution is totally different for different values of $\mathbf{\Delta t}$ as we can see in the graphical representations of the Euler's numerical solution of the logistic equation with r = 18 and y(0)=1.3. With $\mathbf{\Delta t}$ =0.1 the solution tends to 1 and looks like the analytical solution. We slightly increase $\mathbf{\Delta t}$ and the process becomes a periodic oscillation between two, four levels... As a consequence of other slight changes in $\mathbf{\Delta t}$ the logistic mapping becomes chaotic. Then, periods 3, 6, 12.. appear and the fact that a small change in a parameter causes only a small effect, does not necessarily imply that a further small change in the parameter will cause only a further small change in the effect.

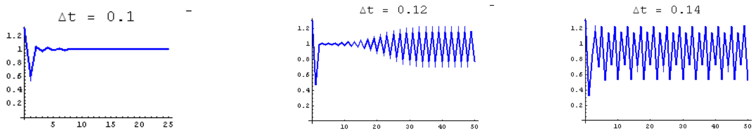

Fig. 1 Graphical representations of Euler's numerical solution of the logistic equation

The students were given a written test that was designed to elicit their thinking processes and intuitions concerning the limit in the definition of the derivative. Some of the students were also interviewed and invited to explain their answers. In the following, we present excerpts from the written test and interviews that demonstrate different aspects of the pertinence of students' intuitions and tacit pictorial models in relation to the limit concept in the definition of the derivative. In this paper, we observe the thinking processes of two students highly trained in mathematics, Miri and Cindy. We concentrate on the interaction between their intuitive representations, their tacit pictorial models and their logical reasoning. We observe essential differences in their cognitive processes.

DIFFERENCES IN COGNITIVE PROCESSES

Awareness of a conflict: Miri

In a question in the written test, the students were asked to express their opinion about the following statement "If in Euler's method, using a step size $\mathbf{\Delta t} = 0.017$ we get a solution very far from the real solution, then a step size $\mathbf{\Delta t} = 0.016$ will not produce a big improvement, maybe some digits after the decimal point and no more". The question was given to the students before being exposed to the logistic equation. We discern some uncertainty in Miri's answer:

> In my opinion, if the answer we get is very far from the analytical solution, that is, if $\Delta t = 0.017$ is not small enough, then $\Delta t = 0.016$ will not produce a big improvement. Although, each change, each reduction of Δt brings us closer to the true solution, yet due to some primary intuition (which might be wrong) it does not seem to me that it will provide a good approximation.

After being exposed to the logistic equation Miri was asked to characterize the source of error in Euler's numerical method. She observed the algorithm and reacted:

> There is some confusion: in the notion of the derivative at a point, Δt should tend to 0 and in our case Δt does not tend to 0; even so, we compare between the two cases. Nevertheless, my answer is not a definite answer. My answer is mistaken because even with smaller and smaller Δt, with Δt which tends to 0, we still observe a wild behavior of the solution. This is strange!

In the first part of her answer, Miri answered correctly that the source of the error resides in the way the derivative is defined in the numerical method. She expressed very clearly her understanding of the need for the limit in the concept definition. Then, she decided that her answer is mistaken! She expressed a kind of conflict between her cognition reached through her understanding of the formal definition of the derivative and her intuitive reaction to the plots of the numerical solution. In the interview, Miri expressed her view of the limit as a monotonic process:

> The smaller Δt the more we are expected to see the numerical solution becoming close to the analytical solution. I expect a gradual improvement with a new stage definitely better than the previous one but here I see that this is not the case! What is the meaning of the words "it tends to"? it goes and comes close to it - becoming closer. $\Delta t \to 0$ is a process and therefore $\lim_{\Delta t \to 0} \Delta y / \Delta t$ is a process.

Miri expected that with smaller and smaller values of $\mathbf{\Delta t}$, the graph of the numerical solution will steadily approach the graph of the analytical solution. In fact, the plots indicate otherwise. Then she related to the source of error in the numerical method:

> $\frac{\Delta y}{\Delta x}$ in $\lim_{\Delta x \to 0} \Delta y / \Delta x$ is an entity and you cannot multiply by Δx. I wonder why Euler wrote $f'(x)\ \Delta x$. Is it permitted? Could you please explain me in words the meaning of $\lim_{\Delta x \to 0} \Delta y / \Delta x$. The first question is what is a limit? It has to be something dynamic. You know you have this arrow. I see here a process of getting close. The smaller Δx the better $\frac{\Delta y}{\Delta x}$ reaches its real meaning in a monotonic way.

Miri considered $\frac{\Delta y}{\Delta x}$ (with $\Delta x \to 0$) as an entity in which $\frac{\Delta y}{\Delta x}$ is not seen as a quantity (Δy) divided by another quantity (Δx). She also expressed very clearly her potentially infinite process view. Then, she was asked to consider the two equalities:

0.3+0.03+0.003+….=1/3

1/3=0.3+0.03+0.003+…

Her reaction should not surprise us (see Fischbein, 2001)

> The two equalities are correct but the second one is more correct.

No situation of conflict between the cognition reached through a logical analysis and tacit models: Cindy

Unlike Miri, Cindy had no hesitation while expressing her opinion:

> A difference of 0.001 might be crucial if one crosses the equilibrium solution and passes from one slope field to a different one. Like the example we had in class…..

Then she answered correctly that the source of the error resides in the way the derivative is defined in the numerical method:

> The error is in the way the derivative is used. The concept of derivative establishes the slope at a given point (underlined twice) [t, t+dt] where dt represents an infinitely small quantity. In Euler's method, instead of dt we substitute Δt as an approximation and this is an error (underlined twice). Every slope is for the given point only and not for a given domain.

I interviewed Cindy in order to understand how she succeeded to characterize the source of error without using the word "limit". In the interview, she was self confident and drew during the entire interview. She explained the phenomenon of the two period by using the notion of equilibrium solution of a differential equation for which the right–hand side of the differential equation (dy/dt) vanishes along this line.

> Look at a given plot field, suppose you have a line which represents an equilibrium solution. The slope marks are decreasing above the line (dy/dt is negative) and they are increasing below the line (dy/dt is positive). There is a tendency from below and from above towards the line. Suppose that because a very small change, you got down from above and you cross the equilibrium line. Now you are in a zone with a positive slope and you will be thrown up and therefore you cross once more the equilibrium solution but this time in the reverse direction and so on.. that the reason you get the period 2.

Cindy mentioned a figure (Figure 2) from an example given in the lecture that influenced her. In the lecture, students were explained that the solutions cannot cross the equilibrium solutions (as a consequence of the uniqueness theorem). They used this information to check the behavior of numerical approximations of solutions. Figure 2 was present in Cindy's mind a long time after the computer was turned off.

> My explanation is only by the fact that the solution crosses an equilibrium solution. With the accumulative effect of the error I could not explain the behavior of the solutions and the period 2, 4. In the absence of an equilibrium solution, the error would have increased but there would not have been crucial changes. The error increased but we noticed it only because the equilibrium solution.

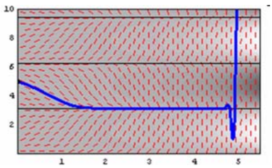

Figure 2 $\frac{dy}{dt} = e^t \sin y, \quad y(0) = 5$ (Euler's solution for $\mathbf{\Delta t = 0.1}$)

To my question if we can compare the situation with the straw that breaks the camel's back, Cindy laughed then she reacted seriously:

> The straw is the replacement of dy by $\mathbf{\Delta y}$ .But if there was no equilibrium line, the camel would not have fall down!

I asked Cindy: where else can you see a similar phenomenon as the one you describe with the crossing of an equilibrium solution? She answered:

> You have the same phenomenon with some differential equations. Sometimes, you slightly change the initial value and the behavior of the solution changes enormously.

I asked her again where else can you see a similar phenomenon. She answered: "in the derivative". She drew the secants and the tangent and added "When you look at dt as $\mathbf{\Delta t}$ you miss something! Cindy differentiated clearly between dt and a small quantity $\mathbf{\Delta t}$. However, it did not prevent her to draw dy and dt (Figure 3). She wrote dy/dt = f(t, y). Then, she multiplied the two sides of the equation by dt to obtain dy and wrote $dy = dt \;\; f(t,y)$ .Cindy was not reluctant neither to draw the infinitely small dt nor to multiply the entity dy/dt by dt in order to obtain dy!

Figure 3 drawing the infinitely small

DISCUSSION AND CONCLUDING REMARKS

My first thought was that a student who answered correctly that the source of the error resides in the way the derivative is defined in the numerical method, has a conceptual understanding of the derivative as a limit and the synthesis between the intuitive representations and the formal definition is complete. In fact, as we have seen in the previous section, the situation was different and even if the student answered correctly, there was still some persistence of tacit models. In the following, we compare and contrast Miri and Cindy cognitive processes. We have already noticed Miri's uncertainty and primary intuitions in opposition to Cindy's self confidence and refined intuitions. We also realized that for both, tacit models coexist with a remarkable logical reasoning. The essential difference resides in the fact that unlike Miri, Cindy is not aware of a situation of conflict between her formal thinking and her intuitive representations. Miri has not yet done the synthesis, into one mental structure, of the formal understanding of the derivative as a limit and her intuitive interpretations. She is fully aware of a situation of conflict and expressed her need for such a synthesis. Being aware of a situation of conflict is crucial in order to realize the influence of intuitions and tacit models on our reasoning "the student should be aware of his tacit mental conflicts in order to strengthen the control of the taught conceptual structures over the primary intuitions" (Fischbein, 1987, p.205). In the tacit spot model described by Fischbein in which we continue to think on mathematical points in terms of small spots such a situation of conflict appears: "a feeling of difficulty, of contradiction, of paradox appears of which we cannot rid ourselves" (Fischbein, 2001, p.315). Fischbein emphasizes that in spite of the awareness of the abstract nature of the geometrical objects even individuals already highly trained in mathematics continue to think in terms of the pictorial models. For Cindy, the situation is essentially different. It seems to me that when she claims that the source of error in the numerical method resides in the way the infinitely small dt is replaced by a small Δt, she does not only express her awareness of the abstract nature of the mathematical point[t, t+dt]. She does not only express her understanding that the drawn segments dt, dy are pictorial representations of abstraction non representable mentally as such like in the tacit spot model. She does much more: she articulates a beginning of a synthesis between her formal reasoning and her intuitive representation of the infinitely small. She has already done integration of different representations, in different contexts, of her refined intuition that a very small change Δt in a cause might be crucial and might produce a very large change in an effect, and therefore she declares that one cannot use the infinitely small dt as a quantity Δt. As a consequence, Cindy has built a formal view of the infinitely small as differentiated from a small Δt whatever its smallness. Even so, the synthesis between the formal and the intuitive representations is not complete. She draws the infinitely small dy and dt: the mathematical objects dy and dt exist and dy/dt is not considered as a symbol, as an entity (dy/dt) but as dy divided by dt (in contrary to Miri's reasoning; in fact, it was by contrasting their two cognitive processes that I could characterize Cindy's specific manifestation of her tacit models). When asked about the source of error in the numerical method, she claimed explicitly that the infinitely small dt cannot be used as a quantity Δt. This fact did not prevent her later, when she was not asked direct questions concerning dt and Δt, to draw dt and use it tacitly as Δt and all this with a feeling of logical consistency which can mislead even the researcher herself! She is confident in her combining of her tacit pictorial models with her understanding of the limiting process. It does not lead her to a situation of conflict maybe in the same way that in the early days of the differential calculus even Leibnitz himself was capable of combining such ideas (of the infinitely small) with a thoroughly clear understanding of the limiting process. In such cases like for Cindy, because her feeling of logical consistency, it might be difficult to find out the existence of tacit models and their influence on the reasoning process. On the other hand, the synthesis between her formal reasoning and her intuitive representation would not be complete without her awareness of her tacit models and their influence on her reasoning. As researchers and mathematics educators we should give special attention to this characterization of persistence of tacit models which do not prevent a feeling of logical consistency especially in individuals highly trained in Mathematics.

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DEVELOPING TEACHER AWARENESS OF THE ROLES OF TECHNOLOGY AND NOVEL TASKS: AN EXAMPLE INVOLVING PROOFS AND PROVING IN HIGH SCHOOL ALGEBRA

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This research report focuses on the teaching practice of a 10th grade teacher, who participated in a research project involving the use of CAS technology and algebra tasks that were novel to this teacher. The classroom lesson that is analyzed centered on a proving problem embedded within a factoring task that had been engaged in the day prior. A two-fold analysis is presented, the first one focusing on the proving activity, the second one drawing on and connecting the classroom observations with the content of a follow-up interview with the teacher. His reflections during the interview highlight both the new awarenesses that emerged for this teacher during his teaching, as well as the factors that enabled these new awarenesses.

In their review of the emerging field of research in mathematics teacher education, Adler, Ball, Krainer, Lin, and Novotna (2005) have argued that we need to better understand how teachers learn, from what opportunities, and under what conditions. The findings that we recount in this research report provide a compelling case for the particular opportunities and conditions under which the knowledge and teaching practice of one particular teacher of mathematics evolved.

THE CONTEXT OF THE PRESENT STUDY

When our research group developed the project underlying the present study, we decided that the use of new technologies (i.e., Computer Algebra Systems – CAS) for the teaching of algebra would be one of its principal components. Another was the design of novel tasks that would both take advantage of the technology to further the growth of algebraic reasoning and also focus on the interplay between algebraic theory and technique. The theoretical framework that underpinned the research project, one that we refer to as the *Task-Technique-Theory* frame (see Kieran & Drijvers, 2006, for details), draws upon Artigue's (2002) and Lagrange's (2002) adaptation of Chevallard's (1999) anthropological theory of didactics.

The project also involved collaboration with local teachers. The teachers were our practitioner-experts who, within an initial workshop setting, provided us with feedback regarding the nature of the tasks that we were conceptualizing. After modifying the tasks in the light of the teachers' feedback, we requested that, at the beginning of the following semester, they integrate the entire set of tasks into their regular mathematics teaching and that they be willing to have us act as observers in their classrooms. Throughout the course of our classroom observations, which

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occurred over a five-month period in each class, we also offered ongoing support to the participating teachers. In addition, we conducted interviews with some of them immediately after observing certain lessons that we had thought might be considered pivotal moments in their practice. The following narrative concerns one such pivotal lesson, taught by the teacher Michael.

MICHAEL'S STORY

Some Background

At the time of the present study, we had already observed 15 of Michael's classes – classes where he had integrated the CAS-supported tasks that had been created for the research project. Michael was a young teacher whose undergraduate degree and teacher training had been carried out in the U.K. He had been teaching mathematics for five years, but had not had a great deal of experience with using technology in his teaching, except for the graphing calculator. He was a teacher who, along with encouraging his pupils to talk about their mathematics in class, thought that it was important for them to struggle a little. He liked to take the time needed to elicit students' thinking, rather than quickly give them the answers.

Our observations of Michael's class had started at the very beginning of the Grade 10 school year. The students in his class had already learned the basic techniques for factoring a difference of squares and certain trinomials, and had solved linear and quadratic equations. While they had used graphing calculators on a regular basis in the past, it was only at the start of our project that they became familiar with symbol-manipulating calculators (the TI-92 Plus calculator). They had never before done any proving, either in geometry or in algebra. This report concerns the lessons that involved the 'x^n – 1 task', the last component of which was a proof problem. We observed, and videotaped, these lessons. The day after the close of the proving activity, the first author interviewed Michael.

The x^n - 1 Task

The design for this task was an elaboration of earlier work carried out by Mounier and Aldon (1996) with slightly older students. The first part of our task activity, which included CAS as well as paper and pencil, aimed at promoting an awareness of the factor $(x-1)$ in the given factored forms of the expressions x^2-1, x^3-1, and x^4-1, as well as leading to the *generalized* form $x^n-1=(x-1)(x^{n-1}+x^{n-2}+...+x+1)$. The next part involved students' *confronting* the paper-and-pencil factorizations that they had produced for x^n-1 (first with integer values of *n* from 2 to 6, and then from 7 to 13), with the completely factored forms produced by the CAS, and in *reconciling* these two factorizations (see Figure 1). An important aspect of this part of the activity involved reflecting and *forming conjectures* (see Figure 2) on the relations between particular expressions of the x^n-1 family and their completely factored forms. The final part of the activity (see Figure 3) focused on students' *proving* one of these conjectures. This proving activity is the central feature of our analysis.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Figure 1. Task in which students confront the completely factored forms produced by the CAS

Conjecture, in general, for what numbers *n* will the factorization of $x^n - 1$:
i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x+1)$ as a factor?
Please explain.

Figure 2. Task in which students examine the nature of the factors produced by the CAS

Prove that $(x+1)$ is always a factor of $x^n - 1$ for even values of *n*.

Figure 3. The proving task

Our Classroom Observations of the Proving Component of the Task

After students had completed the first two parts of the x^n-1 activity, they were faced with the proving segment of the task. They worked mostly within small groups, for about 15 minutes. Some were using their CAS calculators. Getting students into this proving task was not straightforward for the teacher, as they had never before engaged in such activity. However, with Michael's encouragement, students did make progress. When he sensed that the majority of them had arrived at some form of a proof, he initiated whole-class discussion, with various students sharing their work.

Proof 1: A general approach based on the difference of squares. Paul was the first to be invited to come to the front of the class and to present his 'proof':

Paul: Ok. So, my theory is that whenever x^n-1 has an even value for *n*, if it's greater or equal to 2, that one of the factors of that would be x^2-1, and since x^2-1 is always a factor of one of those, a factor of x^2-1 is (x+1), so then (x+1) is always a factor.

The teacher then asked: "Is everyone willing to accept his explanation?"

Dan subsequently came forward with what he considered a counterexample, $x^{12} - 1$, to Paul's proof. He proceeded by factoring $x^{12} - 1$ as $(x^6 + 1)(x^6 - 1)$, the latter of which he refactored as $(x^3+1)(x^3-1)$. He then factored (x^3+1) – a sum of cubes – which yielded the sought-for (x+1) factor. He argued that the presence of x^2-1 was not a necessary component of the proof because he (Dan) had shown that, for some even values of *n,* the factoring of x^n-1 does not have to end up with a difference of squares. A sum of cubes could result, and it too would yield (x+1). This led immediately to many students' voicing disagreement. Many of the other students, including Paul, contended that Dan's was not a counterexample, after all. They argued that $x^{12}-1$ could, in fact, produce x^2-1 if it were factored differently:

Paul: Isn't x^6+1 a sum of cubes? … So couldn't you also do the x^6-1 as the difference of cubes [one student says "yeah"] and that's x^2-1.

Commentary on Proof 1. While Paul had seen that x^6-1 could be viewed as a difference of cubes, and thus that x^2-1 was a factor, he did not seem able to link this particular example with his general affirmation that for all even *n*s in x^n-1, one would always arrive at x^2-1 as a factor. Yet, he was quite close. Could he see that x^6-1 was equivalent to $((x^2)^3 - 1)$, even if he had never expressed it in quite this way? This might then have been generalized to expressing x^n-1 for even *n*s as $((x^2)^p$-1) where $n = 2p$. And so because x^n-1 has (x-1) as its first factor, similarly $((x^2)^p$-1) has x^2-1 as its first factor, and thus (x+1) as a factor.

Proof 2: A proof involving factoring by grouping. The second approach to the proving problem was offered by Janet. Janet's proof, which she and her partner Alexandra had together generated, was based on their earlier work on reconciling CAS factors with their paper-and-pencil factoring (for the task shown in Figure 1). They had noticed that for even *n*s, the number of terms in the second factor of x^n-1 (when factored according to the general rule) was always even. Janet argued, as she presented the proof, using x^8-1 as an example, that it would work for any even *n*:

Janet: When *n* is an even number
Teacher: Write it on the board, show it on the board.
Janet: [she writes "x^8-1" and below it: $(x-1)(x^7+x^6+x^5+x^4+x^3+x^2+x+1)$]
Teacher (to the class): Ok, listen 'cause this is interesting, it's a completely different way of looking at it, to what most of you guys did. Ok, so explain it, Janet.
Janet: When *n* is an even number [she points to the 8 in the x^8-1 that she has written], the number of terms in this bracket is even, which means they can be grouped and a factor is always (x+1).
Teacher: Can you show that?
Janet: [she groups the second factor as follows, $(x^6(x+1)+x^4(x+1)+x^2(x+1)+1(x+1))$]

Commentary on Proof 2. Janet's proof, which was generic in that it embodied the structure of a more general argument and was a representative of all similar objects (Balacheff, 1988; Bergqvist, 2005), was one that seemed to be understood and appreciated by most of the students in the class (see Weber, 2008, in this regard). Janet had been able to explain how the terms of the second factor (the factor beginning with the x^7 term) could be grouped pair-wise, yielding a common factor of

$(x+1)$. Her proof appealed to her classmates' common experience in factoring by grouping and led to insights that had not occurred to them before.

Mariotti (2002, 2006) has argued that there is no proof without theory. Similarly, Mariotti and Balacheff (2008) have emphasized that the proving process necessarily starts with the production of conjectures before moving on to proof. In this regard, it is noted that the two-lesson sequence that was devoted to the x^n-1 task involved an interplay between theory and technique, with the development of student conjectures throughout. The ideas that the students generated during the proving task were those about which they had been conjecturing through the entire factoring activity.

The Subsequent Interview with the Teacher, Michael

A 35-minute interview with Michael took place at the close of the proving activity. It inquired into a range of issues, including his views on the research project, as well as his impressions of the most recent activity involving the x^n-1 task.

Low expectations at the start of the project – Extract 1.

Interviewer: Do you now see this technology as playing a different role in your class from the time before the project started?

Michael: Yes. Before it started, I hoped it would be good, but my expectations were not that high about it. I certainly have been very pleasantly surprised with what's happened.

The role played by the tasks and the technology – Extract 2.

Interviewer: How would you describe the impact on the students of this project both mathematically and technologically?

Michael: I think the biggest impact, and the thing I've been most happy with, is the way you guys have designed the activities. It's the way that we've challenged their [the students'] thinking and actually made them think about a process that maybe they knew how to do, but made them think about why they're doing it that way. And I think that's what the calculator has helped them to do and helped them to really, really look at whether they understand the material. … That's something we don't do enough of in mathematics; I think we should do and I really like to do it. … The learning through the technology was amazing. But, the technology is nothing by itself. The amount of work that you put into these activities; that's why they were so successful. … And it's been really good to see how the kids have developed these [the tasks] and worked with them.

Change in his teaching – Extract 3.

Interviewer: Has this project affected your style of teaching in any way?

Michael: I think it's made me think more, or made me realize that what I like is making them think a little bit more. And I think I did that anyway, … but it just made me, just consider a little bit more: Can I let them come through this themselves, let them try this out themselves a little bit more, which I think I always did – but just seeing these activities work, it's made me realize there's more scope to it than I have done in previous years. There is much more scope to let them really go and really know the material properly.

Pushing students to go farther mathematically – Extract 4.

Interviewer: Has the project altered your view of the nature of the mathematics content that can be taught at this level?

Michael: Yes. Because some of the things that you had in those activities I wouldn't have touched. Such as, especially the last activity [the x^n-1 task], you know there's no way I would have gone anywhere with that. It was way beyond anything that they need to know, but just doing that activity was such a fulfilling experience for, not just for me, I spoke with some of the kids afterwards, and they really enjoyed it. They really did! Just going way beyond what they needed to do [in the program] and they were all able to do it. The really nice thing about that activity is that, at the end of it, everyone had something. Even if they didn't all have as nice a little proof as Janet and Alexandra, all of them had worked someway along the lines to get to something. So, so yeah, it certainly opens up things and they couldn't have done that without the technology. So, so for sure is the answer to your question.

Increasing student involvement and promoting learning – Extract 5.

Michael: With this technology, learning goes much further, it is much more involved. … It gives them the extra level of ability, and it involves more students. It gets them into it a lot more. … they could discover things themselves. That is a valuable effect.

Michael had not had high expectations at the outset of the project. This makes the results all that much more interesting. Clearly, one of his strongest impressions of the project was the way in which the tasks and the technology pushed the students to go much farther in their mathematical thinking – so much so that he wanted to continue using the tasks and technology the following year. He also wanted to share the lesson videos of the x^n-1 task with colleagues, just so that they could see what is possible.

ANALYSIS AND DISCUSSION

As stated by Michael, it was his participation in the research project, a project involving technologies and tasks that were novel to him, that led to new awarenesses. These new awarenesses constituted change in his knowledge of mathematics and his knowledge of mathematics teaching and learning, both of which were reflected in his practice of teaching algebra. Mason (1998) has pointed out that it is one's developing awareness in actual teaching practice that constitutes change in one's 'knowledge' of mathematics teaching and learning. While Michael did participate in our professional development workshop prior to his integrating the novel tasks and technology into his teaching, it was his actual practice with these materials that had the greater impact regarding his 'developing awarenesses' regarding mathematics teaching and learning. As we gleaned from the interview, Michael developed at least three new awarenesses:

* An awareness of what students at this grade level can accomplish mathematically – given appropriate tasks (the task aspect was considered very important) – as well as the realization that they can go further mathematically than expected (Extracts 3, 4).
* An awareness of the role that technology can play in the mathematical learning of students (Extracts 1, 2, 4, 5).
* An awareness regarding the culture of the class: it changes when technology is present – students become more involved; they are more autonomous (Extract 5).

Several factors were found to enable the emergence of these awarenesses: a) access to the resources and support offered by the research group; b) use of technologies and tasks whose mathematical content differed from that usually touched upon in class; c) the quality of the reflections of his own students on these tasks; d) his disposition toward student reflection and student learning of mathematics; e) his attitude with respect to his own learning. The first two factors relate principally to the role played by resources 'from without', while the remaining three could be said to be 'from within' in that they concern the given teacher and his students. However, it was in the interaction of the two dimensions that teacher awareness and change were promoted.

Had it not been for the 'from-without' factors, that is, the access to the resources and support offered by the research group and, consequently, the use of technologies and tasks whose mathematical content differed from that usually touched upon in class, then the 'from-within' factors, such as, the quality of the reflections of his own students on these tasks, would not have been put into play. Similarly, had it not been for 'from-within' factors, such as Michael's disposition toward student reflection and student learning of mathematics, as well as his attitude with respect to his own learning, then the 'from-without' factors related to the research team's contributions would not have taken root and flowered. Both types of factors supported each other in a mutually intertwining manner.

This is of interest from a theoretical perspective. It suggests firstly that the integration of novel materials and resources that have been designed to spur mathematical learning is more likely to be successful when the teachers who are doing the integrating see clearly that these resources are having a positive effect on their students' learning. Secondly, the novel materials and resources have a greater likelihood of producing this positive effect on student learning when the teacher doing the integrating engages in teaching practices that encourage student reflection and mathematical reasoning. The synergy between the two types of factors was found to be a major force in the development of Michael's professional awareness, and one that constituted change not only in his knowledge of mathematics and mathematics teaching/learning, but also in his practice.

In conclusion, we wish to emphasize one issue. Much of the research related to teachers' learning from their own practice emphasizes teachers' planning of their interactions with students, followed by their subsequent reflective analysis of these interactions. Considerably fewer studies (exceptions include, e.g., Leikin, 2006) follow the path that we did where the majority of the planning of the instructional interaction with respect to the mathematical content and the task questions to be posed to the students is elaborated in advance by the research team in partial collaboration with the participating teachers. This, we feel, added a dimension to the study that does not often come into play in research on teaching practice. The positive nature of the reflections shared by Michael during the post-lesson interview suggests that the integration of resources coming from without can be a powerful stimulus to teachers' learning from their own practice.

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THE NOTION OF NUMBER SENSE IN RELATION TO NEGATIVE NUMBERS

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Conceptual understanding of negative numbers includes development of number sense, however, what this involves in the extended number domain needs to be clarified. Number sense components previously identified for natural numbers are here explored in relation to negative numbers. Empirical data from interviews with sixth grade students show that a highly developed number sense for natural numbers does not necessarily imply a sufficient number sense for negative numbers, indicating that number sense issues need to be an explicit part of teaching negative numbers.

INTRODUCTION

The notion of Number Sense has for the last decades been much used to describe knowledge about numbers. At the start it depicted an innate, intuitive ability to correctly judge numerical quantities. As it became a notion more widely used among researchers and educators it gradually became more and more inclusive. Today Number Sense is used in curricula and teaching materials and is included in the mathematical framework underlying the construction of the international tests in TIMSS (Mullis *et al.*, 2007), where it says about the number content domain consisting of whole numbers, fractions, decimals, and integers:

> The number content domain […] includes understanding of place value, ways of representing numbers, and the relationships between numbers. […] students should have developed number sense and computational fluency,…" (p.24)

Much has been written about number sense in relation to whole numbers and fractions, but what the notion implies concerning negative numbers needs to be clarified. The aim of this paper is to give some examples of how the notion of Number Sense could be understood in relation to negative numbers. What is a 'developed number sense' in relation to negative numbers? How does the acquired number sense for natural numbers influence the sense making of negative numbers?

THEORETICAL BACKGROUND

Number sense is today a widespread concept that, although no more than half a century old, permeates literature on mathematics education for primary school. The term is said to have come from Tobias Danzig in 1954 and was elaborately by Dehaene (1997), who describes number sense as a product of the human brain and of a slow cultural evolution. He defines it as an 'imprecise estimation mechanism'. In two of the often used handbooks of research on mathematics teaching and learning, number sense is treated in relation to or as a part of a chapter on estimation. The original, rather narrow definition of the concept has by many mathematics educators,

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curriculum writers and researchers been broadened (eg. Gersten & Chard, 1999; Kaminski, 2002; Reys, 1998). Number sense is by these depicted as "an acquired 'conceptual sense-making' of mathematics" (Berch, 2005, p.335). If we are to use the concept when speaking about numbers that are not natural numbers it is necessary to go along with the broader view, while also considering features such as intuitions about quantity and magnitude, counting and subitizing to be important ingredients in the sense-making process.

There are many different views on what composes number sense. An often referred to framework for examining basic number sense was proposed by McIntosh, Reys and Reys (1992). In their framework number sense has three interconnected components: knowledge of and facility with numbers, knowledge of and facility with operations, and the application of that knowledge in computational settings. Since then many others have come up with lists of components and descriptions of number sense. When using the notion of number sense related to percent Gay and Aichele (1997) mention four skills from the NCTM standards: 1) having *well understood number meanings*; 2) having *developed multiple relationships* among numbers; 3) recognizing *the relative magnitudes* of numbers; and 4) knowing the *relative effect of operating* on numbers, and then they also include *developing equivalent expressions*. I will for the purpose of this paper make use of the large and inclusive set of components compiled and analysed by Berch (2005) and from that set extract those components that seem especially relevant in relation to integers. These are, without any order of preference;

1. Elementary abilities or intuitions about numbers and arithmetic.
2. Ability to make numerical magnitude comparisons.
3. Ability to recognize benchmark numbers and number patterns.
4. Possessing knowledge of the effects of operations on numbers.

Having a mental number line in also often considered a component of number sense. The number line is a very powerful mathematical representation of numbers, and several studies have shown positive results of work with an empty number line when it comes to developing knowledge of and facility with natural numbers. When the negative numbers are introduced the number line is extended to the left of zero. Students who have an easily accessible mental number line to relate to should be able to correctly identify the relative size of numbers (Peled *et al.*, 1989). I consider the number line not as a separate component of number sense but rather as a tool with which to express or develop number sense.

In research literature about learning negative numbers mainly three important aspects seem to have been identified. The first aspect is an understanding of the numerical system and the relative size of the numbers, including the number zero (i.e. Ball, 1993). Ball shows that the absolute value (the magnitude) of negative numbers is very powerful, given priority over the relative value in conflict situations. This aspect relates to the first three components in the above list. The second aspect is how well students understand the arithmetic operations (Chacón, 2005; Vlassis, 2004). Sfard (1991) indicates that the interiorization of negative numbers is the stage when a person becomes skilful in performing subtractions. The third identified important aspect is the meaning of the minus sign. The same sign is used both as a sign of operation and as a sign indicating the nature of the number (Gallardo, 1995; Vlassis, 2004). The second and third aspect relate to component four since understanding representations of numbers and operations are a part of understanding their effects.

METHOD

In this paper examples of mathematical reasoning are taken to illustrate the notion of number sense when the domain of numbers is extended from natural numbers to integers. As part of an on-going longitudinal study where the same students are to be interviewed several times over a period of three years, 21 students in grade six (12 years old) were individually interviewed. Each interview lasted for about 30 minutes and was audio-taped and transcribed. The excerpts used here were translated into English. The notation (…) is used to indicate a pause and […] to indicate excluded words. In the excerpts 'Int.' means interviewer. The interviews were semi structured with a combination of open-ended questions about numbers and symbols and tasks to solve using think-aloud-protocols (Boren & Ramley, 2000).

DESCRIPTIONS OF NUMBER SENSE COMPONENTS IN RELATION TO NEGATIVE NUMBERS

The four extracted components of number sense are here illustrated by excerpts where a student either has a number sense that is insufficient (transferred from natural numbers) or is on the way to develop a number sense for negative numbers. We shall see that students often struggle when they have to leave established ideas or intuitions of numbers in order to make sense of negative numbers.

Component 1: Elementary abilities or intuitions about numbers and arithmetic.

When the students in this study were asked: "Which is the smallest number there is?" the most common answer was "*zero point zero, and a whole lot of zeros and then a one*" (0.000…1). Some of the students answer "*zero…but that is not a proper number*" or "*well, zero…but that doesn't count, it isn't worth anything*". A possible interpretation of these answers is that the students conceptualize numbers as quantities, something substantial. Numbers are first order representations of real objects. A number can be infinitely small but it has to be more than nothing. This is an intuitive conception of number in accordance with the concept of number before the 19th century. When zero first appeared it was not as a number, but as a placeholder, a symbol in a positional system indicating the absence of a particular type of number. (20 means 2 tens and *no* units). These students have an intuition about zero that has been sufficient for all situations, in reality and in mathematics, they have encountered so far. It is, however, not sufficient if they are to understand

and master algebra. According to de Cruz (2006) natural numbers fit closely with our intuitions and are therefore easy to learn and easy to transmit, whereas zero and negative numbers are counterintuitive. The intuitive understanding of number as a quantity can be seen in the interview with Anna when she is asked to explain what a number is:

Anna: it's a whole, well it's an ordinary digit*. One that doesn't have any sign, minus or plus. (…) in front if it or, something. […]

Int: If you take a number, if you say that a number is a digit without any sign in front of it, in that case is 0.5 a number?

Anna: Not really, it is more like, it is really only half a number. It is really not a whole, and a number is just a whole.

* Notice that in Sweden the word for digit (siffra) is used in everyday language to mean number, e.g. "look at these digits" meaning "look at these number"

Although the decimal number 0.5 is not unknown to Anna (she has no problem adding decimal numbers) it seems as if the meaning of numbers to her is tantamount to natural numbers.

Subtraction is the operation that is mathematically connected to negative numbers. When asked about subtractions the students all stated that the result of a subtraction can only be smaller than what you started out with. They typically explained subtraction as *"making smaller"*, *"taking away*", *"reducing"* or "*what is left over"*; also an intuition from the domain of natural numbers that need to be developed in relation to negative numbers. Malin is on her way to develop this idea. In the excerpt she is discussing the size of the result of a subtraction:

Int: [you say] it is always smaller? You can't think of any situation when you subtract and the result is <u>not</u> smaller?

Malin: well…if you only have zero from the start

Int: if you have zero from the start?

Malin: yes. It can't be smaller… (...)

Int: no? […] But can it be larger?

Malin: no. Not if you subtract. […]

Int: if you take zero, minus something?

Malin: then nothing happens.

Here Malin shows that she believes subtraction makes smaller, but if you have zero to start with it can't be smaller (since nothing can be smaller than nothing), but it can't be larger so in the end she states that nothing happens, perhaps interpreting 0–5 as 5–0 because she cannot make any sense of 0–5.

Component 2: Ability to make numerical magnitude comparisons.

For natural numbers the relative size and multitude of numbers coincide, but for negative numbers they divert. The magnitude of a negative number is the absolute value. Malin is uncertain about magnitudes. When asked which is the larger of 1 and -2 she chooses 1 and is asked why:

Malin:	because there is no minus there.
Int:	no? […] so you feel that this [-2] is smaller than that [1]?
Malin:	yes. Because there is a minus in front of the two.
Int:	okay? Kan you say a number that is smaller than this one [-4]?
Malin:	minus three.

Stating which of two numbers that is the larger when one is positive and one is negative was easy for Malin, but when both numbers were negative the magnitude aspect took over. Elke has a more developed number sense in this respect but is still struggling with these two dimensions when asked to say a number smaller than -4:

Elke:	(…) mm (…) it depends how you think. But, when I first look at it I think minus, well, five, six or seven or so.
Int:	that would be smaller than minus four?
Elke:	yes
Int:	mm? But what other way do you mean you could think?
Elke:	minus, tree or two or so. […]
Int:	yes? In what way, does one think, or why could one think that it is smaller?
Elke:	because, when you count, four is, sort of, if you count one, two, tree, four, then it is, four is the largest number.

Elke has realised that the magnitude aspect (absolute value) and the relative size (order) of the numbers are two different things, but she is uncertain which one to rely on to answer the question. A well developed sense of magnitude comparisons making use of a mental number line is expressed by Hans when he has compared the size of -16 and -5. He chooses -16 *"because it is, it is further from zero, on the minus side"* Hans does not refer to the absolute value and seems to know that the relative size of a number is determined by its place on the number line and its relation to zero, which also indicates that he sees zero as a benchmark number.

Component 3: Ability to recognize benchmark numbers and number patterns.

The most important benchmark number in relation to negative numbers is of course zero, and the pattern of symmetry on both sides of zero. In the following excerpt George is discovering the idea that if you add and subtract the same number they cancel each other. George is asked to calculate 30+12-5+5-12=__ and 'think aloud' when he does it:

George:	30 plus 12, 42. 42 minus 5, is 37. 37 plus 5, 42. 42 minus 12, 30.
Int:	mm? Does it seem okay that it makes, that the answer is 30? When you started off with 30 and then it is such a long calculation?

George: (…) yes because (…) or wait. 30, plus (…) Yes. First there is 30. Then we plus* 12. Then we minus 12. Then we plus 5, then minus 5. That neutralizes everything.

* Notice that in Swedish the nouns plus and minus are sometimes used as verbs meaning 'to add' and 'to subtract'

Becoming aware of the idea of cancellation and expressing the result as a 'neutralization' could be a starting point for seeing the symmetry of opposite numbers.

Component 4: Possessing knowledge of the effects of operations on numbers.

Knowledge of the effects of operations on natural numbers is not automatically transferred from natural numbers to negative numbers. Subtraction in particular is different since there is not closure for subtraction in the domain of natural numbers. There is no solution to the equation $4 - 5 = x$ within the domain. Just as mathematicians of the past had great difficulty accepting a negative answer, since nothing can be less than nothing, so do students of today. When the students in the interview were asked to give a quick response to the task 2 – 5 =__ a vast majority of the 21 students in the class gave "*three*" as the answer. A bit further into the interview the task 3 – 7 =__ was given and the students were asked to consider it and talk about how they would do the calculation. On this task the common answer was "*you can't do it*" or "*you can't take seven away from three*". It seems as if the intuitive idea of subtraction as the difference between two quantities is the one that first comes into mind, and then it is the difference in absolute value that makes sense. The idea that subtraction means taking something away presupposes that there is something to take it away *from*. In that case 3-7 or 0-5 will not make sense.

Tomas is reasoning about the expression -2 which he does not recognise. Although he must have encountered for example negative temperatures, in the context of mathematics he interprets the minus sign only as a sign of subtraction. He is shown -2 and asked if that is okay to write and what it could mean.

Tomas: Well then you have to have something in front […]

Tomas: for example eight.

Later, when shown the expression -6-2 he says:

Tomas: no, or (…) well it is, it ought to be, yes, it ought to be, really. But you don't have anything to subtract with. It's a bit strange.

He interprets the minus sign as an operation sign and doesn't know how to subtract if you have nothing to subtract *from*.

IMPLICATIONS

We have seen in the examples that sometimes the intuitive or acquired numbers sense from the domain of natural numbers becomes an obstacle when encountering negative numbers. A developed and sufficient number sense for natural numbers does not necessarily imply a sufficient number sense for negative numbers, which suggests that number sense issues need to be an explicit part of teaching negative numbers. When numbers or operations fail to make sense it is often helpful to start reasoning about them, making students aware of differences and similarities between different kinds of numbers. Although the components brought up in this paper do not cover the whole concept of Number Sense, the data shows us that the notion of Number Sense and the components described are applicable to negative numbers and could be useful for designing tasks and teaching strategies aimed at developing number sense as well as for assessment. Important to note is that the different components overlap and interrelate and that making them explicit is a way of fulfilling the ambition of creating a well-organized conceptual network that enables a person to relate number and operation, also when the number domain is extended from natural numbers to integers.

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ARE GIFTED STUDENTS AWARE OF UNJUSTIFIED ASSUMSIONS IN GEOMETRIC CONSTRUCTIONS?

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Studies have shown the gifted have special ability in mathematical reasoning. However, there have been relatively few studies on the diagrammatic reasoning of the gifted. On the other hand, researches on unjustified assumptions in students' geometrical reasoning and the interpretation of Pappus' idea of analysis have been reported. We may expect that gifted students are aware of the possibility of making unjustified assumptions while solving geometrical construction problem by analysis. In this study, sixteen gifted students (in the 7th grade) were asked to construct geometric figures being given conditions. Eleven of them employed analysis to make the required figure. Findings show that there are different tendencies in generating diagrammatic objects, interpretation on diagrams, and dealing with unjustified assumptions.

INTRODUCTION

Studies on the characteristics of mathematically gifted students (Krutetskii, 1976; Presmeg, 1986; Sriraman, 2004; Lee, 2005) have shown that gifted students efficiently utilize problem-solving strategies such as justification, simplification and visualization when necessity arises, and they solve problems progressively. Lee (2005) found that gifted students have the tendency to advance into higher-level reasoning through reflective thinking on their earlier line of reasoning. Sriraman (2004) reported that gifted students have thinking behaviours such as justification, generalization and formalization, corresponding to those of mathematicians. However, there have been relatively few studies on the diagrammatic reasoning of the gifted.

Dvora and Dreyfus (2004) describe that students make unjustified assumptions while solving geometrical problems because of the direction in reasoning that moves from what is required to what is given or vice versa, and the visual affirmation based on the given diagram. We may expect that gifted students are more likely to be aware of unjustified assumptions and investigate analytical construction. In this study, we want to examine how mathematically gifted students notice and overcome unjustified assumptions in their analytic investigation of the construction method. To achieve this objective, this research focused on the following three questions:

- What kinds of diagrams do mathematically gifted students draw in their construction process?

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- What kinds of interpretations on their diagrams do gifted students make in planning construction?
- What kinds of unjustified assumptions are gifted students aware of?

THEORETICAL BACKGROUND

Pappus described analysis as the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis (cited in Behboud, 1994). According to the classical interpretation of Pappus's idea of analysis, a reasoning process of analysis is a linear chain of propositions from the thing sought to the given, and a chain of propositions should be reversed from the given to the thing sought (Hinttika & Remes, 1974; Behboud 1994).

Hinttika and Remes (1974) criticized the classical interpretation of Pappus' idea of analysis. They claim that taking the given and the thing sought into consideration simultaneously is more important than concentrating only on the linear chain of propositions in reasoning. They also emphasized studying the interrelations of geometrical objects in a given configuration for an efficient analysis.

From ancient times, drawings have been used for the investigation of figures in geometry (Proclus, 1970: 45, 162). According to Fischbein (1993), figures have two aspects, visual image and concept. Drawings act as concrete material providing visual images, and because of this, the roles of diagrams in learning geometry have been studied actively (Dvora & Dreyfus, 2004; Zodik & Zaslavsky 2007; Lin & Wu, 2007). Geometrical reasoning can be regarded interaction of figural and conceptual aspects (Mariotti, 1995: 103). Therefore, diagrams in geometrical reasoning should act as concrete beings of visual images interpreted conceptually.

> A visual image is not a faithful reproduction of a diagram. Rather, it is a highly interpreted conceptualization of the diagram (Dreyfus, 1995: 12).

According to Dreyfus (1995), diagrams are given meanings by one who draws them, and he should deepen his understanding of intended meanings of the diagram by seeing logic visual structures behind the diagram in geometrical construction. Therefore, one might not take notice of unjustified assumptions if one does not have a deep understanding of intended diagram or not interpret diagrams with proper concepts.

METHOD

The participants of this research are eleven (S1-S11) among sixteen 7th graders (13 years old) who are receiving education at an attached academy for the gifted at a university. Eleven of them were focused since they employed analysis to construct geometric figures. The gifted academy provides students with education programs of about 100 hours one year. Mathematics programs that are three hours long each deal with various fields such as algebra, geometry, probability, and so on. The focus of education at the academy is improving students' abilities on problem-solving, reasoning and justification.

For this experiment, we asked the students to figure out solutions for four construction problems for three hours. Two of the problems are typical, and the other two can be regarded relatively new to gifted students, which are modified from Polya's *Mathematical Discovery 1(1962)*. Analysis of data, for this paper, focused on problem 3 on which the students worked the most actively.

[Problem 3] Construct a triangle being given the three medians. Explain your answer.

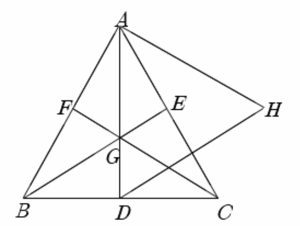
Sixteen students worked in groups of four students and were observed by four research assistants. All the responses of the students were audio- and video-taped; and their worksheets were also collected. Two researchers initially investigated how each student solved the problem by looking through students' worksheets and audio- and video data. Researchers then classified students' solutions, and three other researchers examined the classification. The results in this study were not classified and organized by a conventional framework but have been derived through an inductive method based on the responses of the students (Denzin & Lincoln, 1994; Goetz & LeCompte, 1984).

FINDINGS

We found a few features through observation of the students who worked with the construction problem. One feature is related to objects that they generated for solving the problem. The other is concerned with how they interpreted and operated them during solving the problem.

Generation of Diagrammatic Objects

The students were active in diagrammatic objects generation based on their analysis on the given and the required. Interestingly, eleven among sixteen students took the problem as if it had been solved from the very beginning. They drew the required triangle in which the three given medians are punctually assembled. Two kinds of diagrams drawn by the students are detailed below.



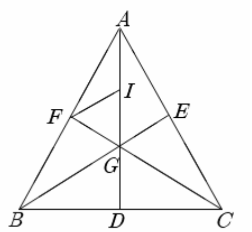


Figure 1 Figure 2

Eight students (See Figure 3) drew *Δ ADH* with the given elements, i.e., three medians of *Δ ABC* by parallel translation of these three medians as shown in Figure 1. They drew *Δ ABC*, the required, before constructed *Δ ADH*. Then, they made medians on *Δ ABC*, and moved median *FC* to *AH* and median *BE* to *DH* by parallel translation.
Three students (See Figure 3) first drew the required *Δ ABC* as well as shown in Figure 2. They added medians to it and marked point *I* trisecting *AD*. Lastly, they drew a line parallel to median *BE* starting from *F* (or *I*) to *I* (or *F*) and found *Δ FIG*.
As we can see in Figures 1 and 2, the students commonly considered the unknown elements as the known and represented the required objects and the given together, which led them to generate new objects that were constructed with the given. The following shows what kinds of objects the students made in their construction in order.

O_1 : the required object (*Δ ABC*)
$\downarrow$
O_2 : the given object (three medians *AD, BE, CF*)
$\downarrow$
O_3 : the object created with the given (*Δ ADH, Δ FIG)*

Argumentations and Unjustified Assumptions

	Parallel Translation			Similarity
Argumentation	*Arg. 1-1*	*Arg. 1-2*	*Arg. 1-3*	*Arg. 2-1*
Students	S4, S5, S6	S7, S8, S9	S1, S2	S3, S10, S11
Figure	Figure 1			Figure 2
Unjustified assumptions in generating diagram	Two segments that are moved by parallel translation meet at one point. (For example, they need to justify why the two segments *AH* and *HD* meet at the point *H*.)			One segment made by joining two points is parallel to the other. (For example, the students need to verify why the segment *FI* is parallel to the segment *BG*.)
Unjustified assumptions in argumentation	Three points that are positioned by constructions are collinear. (For example, it is needed to be examined why *A*, *F*, and *B* are collinear.)			Three points that are positioned by constructions are collinear. (For example, it is needed to be explained why *A*, *F*, and *B* are collinear.)

Figure 3. Students' approaches and Unjustified Assumptions

The students actively elaborated how to make argumentations for connecting the given and the required. One group who drew Figure 1 focused on the idea that the given three

medians can be relocated to the right position by "parallel translation". The other group more focused on a small triangle "similar to the required triangle". Some students noticed the possibility of occurring unjustified assumptions and some did not. Figure 3 shows the details of the students' approaches and expected unjustified assumptions.
Two triangles, *Δ ADH* and *Δ FIG*, are seen as stepping stones to the students (Polya, 1962). They can construct the required triangle from these new objects made of three known sides. The gifted students' argumentations are categorized into four as follows:
Arg. 1-1. In Figure 1, on the *Δ ADH* that consists of three medians, two sides *AH* and *DH* can be moved in reverse. In other words, *AH* and *DH* can be transferred to *FC* and *BE*. Finally, if one joins three points *A, B,* and *C,* then constructs the required *Δ ABC*.
Arg. 1-2. In Figure 1, one can figure out that *Δ ADH* and *Δ ABC* are similar. The ratio of similarity of the two is 3:4 using the fact that the point *E* is the centroid of *Δ ADH*. One can construct *Δ ABC* using the fact that *Δ ADH* and *Δ ABC* are similar.
Arg. 1-3. In Figure 1, on the *Δ ADH,* one can construct parallelogram *AHCF.* Then segment *FB* is parallel to *AF* and has equal length. Finally, one obtains *Δ ABC* by joining points *B* and *C*.
Arg. 2-1. In Figure 2, one extends three sides of *Δ FIG* that has one-third of the given length and constructs *AD, FC,* and *BE*. Eventually, one obtains *Δ ABC* by connecting three points *A, B,* and *C*.

Interpretation on Diagrams

We observed that there were two patterns in which the gifted students interpreted their diagrams so that they made a construction.

Interpretation based on the linearity only. As Hintikka and Remes (1974) proclaimed, being stuck to the linearity in reasoning did not lead some students to find the required analysis. Some students made the reversed reinterpretation on the order of the generated objects when made an argumentation for construction. *Arg. 1-1* and *2-1* are examples of this interpretation. The flow of interpretation based on the linearity only is drawn below.

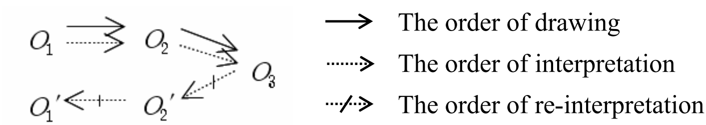
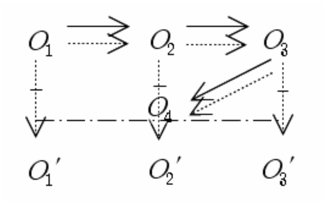


Figure 4. Flow of interpretation based on the linearity only

Interpretation based on the whole structure of the relations. Some students not only focused on the linearity, but also the whole structure of the relations between objects. The relations between objects were represented as conjectures, assumptions, and properties between the required (O_1), the given (O_2), and the new objects (O_3). Some students created new objects (O_4) and kept seeking the relations. *Arg. 1-2* and *1-3* belong to this interpretation. The flow of interpretation based on the whole structure of the relations is shown in Figure 5.



Each interpretation is related

Figure 5. Flow of interpretation based on the whole structure of the relations

Dealing with Unjustified Assumptions

The students who made an interpretation of diagrams based on the whole structure of the relations tended to avoid unjustified assumptions in their constructions and argumentations. For example, S1 and S2 made *Arg. 1-3* communicating on how to draw the required triangle (*Δ ABC* in Figure 6) from the new object made up of the given (*Δ AB'C'* in Figure 6). The following is a part of a conversation between S1 and S2.

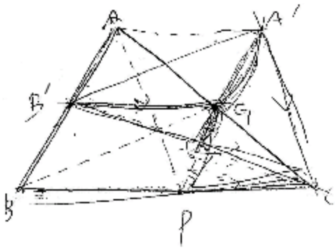


Figure 6.

1 R: Can you explain how you drew *ΔA'B'C*?

2 S2: I moved the three medians to here (pointing to the segment *B'A'*) and here (pointing to the segment *A'C*).

3 S1: Then, how to reverse it?

4 S2: Well, *G* becomes a midpoint because *AC* is the side of *ΔABC.*

5 S1: That's right. Then, let me examine the segment *AC*. We get two segments with the same length, *CG* and *AG*. Then, *G* must be the centriod of *ΔA'B'C*.

6 S2: Okay, that's it. Now, we can do.

7 S1: Then, are you going to move *C* to *P* so that we can move *A'C* to *AP*?

8 S2: Well. *G* is the centroid, so *A'G* and *GP* has the same length.

9 S1: Then, *AC is* okay. Thus, we can move segment *A'C* to segment *AP* and segment *CP* to segment *CB* using the fact that these have the same length.

S1 and S2 interpreted the midpoint *G* of *AC* is the centroid of *Δ A'B'C'* first. Then, they reinterpreted it, and could see the relationship between *Δ A'B'C* (O_3) *and* one side *AC* (an element of O_1) of *Δ ABC* (O_1) through the medium of point *G* instead of, like in *Arg. 1-1*, constructing *BG* and *CB'* of *Δ A'B'C* (O_3). Based on this reinterpretation, they jointed *C* and *G* (as the centriod of *Δ A'B'C)* and constructed *A* on the line going through *C* and *G* (as the midpoint of *AC*). As a result, they could avoid the unjustified assumption. In other words, they did not need to work more to prove that *C, G,* and *A* are collinear. S1 reinterpreted *A'C* (an element of O_3), *A'P* (an element of O_4), and *CP* (an element of O_1) in a parallelogram *AA'PC* when constructed point *P* (e.g., 7 & 9). This reinterpretation equipped him to deal with the unjustified assumption that *C*, *G*, and *A* are collinear. In other words, unjustified assumption did not happen by constructing other points on the line going through two points after constructing two points.

The students who only focused on the linearity in drawing and reasoning (*Arg. 1-1*, *Arg. 2-1*) did not notice unjustified assumptions emerged during their constructions and argumentations. They were not aware of the possibility that the moved endpoints of two medians do not meet at one point (*H* in Figure 1). Also, they missed to argue why the three points *A*, *F*, and *B* are collinear. The students ignored they might use unjustified assumptions only based on the visual affirmation.

CONCLUSION

Interpretation of diagrams based on the linearity only led the gifted students to have unjustified assumptions in their geometric constructions by analysis without awareness. They looked into figures with visual images not proper concepts, which led them to stay in the visual level in geometrical reasoning. Whereas, the students who interpreted diagrams with the whole structure of the relations either avoid occurring or reducing unjustified assumptions, which led them to make more solid argumentations than the students who stuck to the linearity of reasoning.

As Hinttica and Remes (1974) pointed out, focusing only on the linearity in analysis is not useful in geometrical reasoning. In this study, more than half of the gifted students focused on it, and reached to unjustified assumptions. These students need to learn how to interpret diagrams based on the whole structure of the relations. In other words, they have to see diagrams in two aspects: visual image and concept. Further study is needed on the causes of gifted students' tendency of interpretation focused on the linearity in reasoning not the concepts of figures.

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PERCEIVED PARENTAL INFLUENCE ON STUDENTS' IDENTITY, MATHEMATICAL DISPOSITIONS AND CHOICES IN TWO CULTURES

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The influence parents have on their children's dispositions to study further maths in higher education was investigated by comparing two medium size datasets in the UK and Cyprus. Some common patterns were identified in students' perceptions of parental influence such as parental support and pressure. However, the statistical analysis showed no significant effect of 'student perceptions of parental influence' on Cypriot students' dispositions to study further maths (after students' prior maths achievement and motivation to learn mathematics are accounted for). The qualitative data analysis demonstrates important parental influences, which vary cross-culturally. We conclude that students' perceptions are cultural constructions, and doubt they are an accurate representation of the 'reality' of their parents' influences.

BACKGROUND LITERATURE AND THEORETICAL FRAMEWORK

The problem of students dropping out of mathematics, especially advanced mathematics, has become one of the major contemporary concerns of educators, parents and politicians about mathematics education (Ma, 2001). Students' dispositions towards mathematics influence their decisions to choose advanced maths at school and to pursue further studies in mathematically demanding courses in higher education (HE). The present study aimed to investigate parental factors affecting students' dispositions to study further maths and the role of parental influence in particular. There is a large body of literature emphasizing the importance of parental influence and its impact on students' attainment and attitudes to mathematics (e.g. Fan & Chen, 2001).

Despite the vast research on parental involvement in primary school (e.g. Campell & Mandel, 1990) there is a scarcity of research on parental influence on adolescent students. We hypothesise the impact of parental influence on students' decision making to study maths in HE is crucial. Furthermore cross-cultural research on perceived parental influence has attracted the interest of many researchers (e.g. Elliot, Hufton, Illushin & Willis, 2001). Mau (1997) remarks that the degree of parental expectation which is perceived by students differs between cultural/ethnic groups and has a direct impact on children's academic performance. Specifically Cao, Forgasz and Bishop (2005) compared Chinese and Australian students' perceptions of parental influence, and concluded that Chinese students (indigenous or immigrant) have stronger perceived parental influence than Australian students.

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In a comparative study of Asian and Caucasian Americans parents, Campbell and Mandel (1990) divided parental influence into four elements: parental pressure, psychological support, parental help and parental monitoring. High levels of parental pressure, help and monitoring were found to be dysfunctional. Moreover they found that Asian Americans pressured and monitored their children more, but offered very little help. Caucasian Americans exerted low levels of pressure, help and monitoring but supplied more psychological support. Apparently different parenting practices appeal to various ethnic groups of parents. The present study aimed to investigate how students coming from two different cultural contexts, UK and Cyprus, perceive parental influence and to what extent parental influence affects their dispositions to study further mathematics in HE.

METHOD

The ESRC-TLRP project "*Keeping open the door to mathematically demanding courses in further and higher education*" investigated students' dispositions to study mathematically demanding courses in HE and the effects of different socio-cultural backgrounds (Williams et al, 2008). The study drew on a large sample questionnaire survey of up to 1700 students reporting their dispositions and performance in mathematics on three occasions. For the qualitative part of the project over 40 students were interviewed on up to four occasions, regarding their views about mathematics and their aspirations for higher education. An additional sample of 300 students from Cyprus participated in a comparable survey and interviews with 22 of the students were conducted once for a PhD (by the first author) in progress.

The ESRC-TLRP project had one item on parental influence whereas a scale for measuring parental influence (PAR) was constructed for the PhD study. Some items were adopted from Marchant et al. (2001) i.e. "*My parents encourage me to do my best at school"* and some from the qualitative data from the ESRC-TLRP project i.e. "*My parents think it is more important to be happy than to worry about grades"*. The scale had 7 items with 4 point Likert-type responses ranging from 1 = Strongly disagree to 4 = Strongly agree. The scale's reliability was investigated using the Rasch model. There were no significantly misfitting items on the scale and other explorations (e.g. DIF) revealed good measurement properties (Kleanthous, 2008).

The qualitative data of the ESRC-TLRP project and the PhD study were coded in Atlas-ti. An open coding approach was adopted in order to identify the core categories of parental influence that emerged from the interviews. Our method of analysis draws on Gee's (1999) concept of cultural models as 'unconscious explanatory theories rooted in the practices of socioculturally defined groups of people' (p.60). We draw on our analysis of cultural models to explore cultural differences in the ways students express their dispositions towards maths.

RESULTS

The quantitative data of the ESRC-TLRP project regarding parental influence were analysed using descriptive statistics (Davis & Pampaka, 2008). Cultural differences were evident between White British and ethnic minority students who indicated particularly strong parental expectations for university. White British students were the only group with significant numbers stating that their parents have no expectations of them going to university. Additionally, minority ethnic students stand apart from the norm in their tendency to say they are powerfully motivated by family, e.g. making their parents 'proud' of their achievements.

The PAR scale that was constructed for the PhD study was used for modeling/explaining Cypriot students' dispositions (DISP) to study further mathematics in HE. Various regression models were built in the statistical package R with PAR and other variables included in the questionnaire as explanatory variables. Students' prior math achievement (MACH), motivation to learn maths (MOT), maths self-efficacy (SE), maths course at school and socio-economic status (SES) were included in the final model.

DISP ~ MACH + MOT + SE + PAR + MATHS.COURSE + SES

	Estimate	Std.error	t- value	p-value
(Intercept)	-1.479072	0.674327	-2.193	0.0301 *
MACH	0.178119	0.039143	4.551	1.24e-05 ***
MOT	0.805316	0.129420	6.223	6.54e-09 ***
SE	0.114814	0.138436	0.829	0.4085
PAR	0.004725	0.154264	0.031	0.9756
MATHS.COURSE	-0.640622	0.292117	-2.193	0.0301 *
SES[T.low]	-0.458178	0.275362	-1.664	0.0986 .
SES[T.medium]	-0.443425	0.249913	-1.774	0.0784 .

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Table 1: Regression parameters for the final model built in R

Surprisingly 'students perception of parental influence' (PAR) was not statistically significant in any model for predicting students' dispositions to study further maths (DISP) whereas other variables such as students' prior maths achievement (MACH) and motivation to learn maths (MOT) were statistically significant. In order to check for multicollinearity effects between the explanatory variables of the model the GVIF statistics was calculated for each variable. No multicollinearity between the variables was noted. Thus, if there is a parental influence at work, it is not hidden in the MOT , SE (Self-efficacy) or MACH in the model of Table 1.

The qualitative data of the study provide some explanations as to why PAR was not statistically significant in any of the models. A Cypriot student, Stauroula explains why she wasn't influenced by her parents to choose Advanced maths:

> Interviewer: Did they tell you choose Advanced maths?
>
> Stauroula: No they never told me that because they knew I would never choose Advanced maths even if they beg me to. They know I don't like maths so what can they say? I wouldn't do well if I had chosen Advanced maths.

Similar patterns were noted in the UK sample by students who argued that they were not influenced by their parents as to which maths course to choose. For example Michael said: "If I was one of (those) that (say): 'Oh my parents told me to do further maths, they want me to do well'. I am not really interested in the subject."

Regarding their choices for future studies in HE, students in the UK coming from minority ethnic groups made their parents influence more evident in their interviews than White British students. Mohammed is an Asian ethnic minority UK student and when he was asked for a person or event that most influenced him he said:

> My parents….Because they were very keen for me to go to University and give me a lot of support and I basically want my parents to be proud of me, for them it would be an honour for me to go to University.

On the other hand White British students tended to argue that their parents would support their decisions and 'it is their choice' if they are going to pursue further studies or not (perhaps explaining why some of them say their paents had 'no expectations' of them going to HE). The same trend appears in the sample from Cyprus where almost all students argued that their subject choices and their choices of university course were entirely theirs, as in "I will always do what I want." While indigenous (mainly middle class) students from both samples denied their parents' influence on their choices yet it appeared that parental influence was mediated by the working experience parents provided for their children in order to get them interested in a certain area. Thus the sociocultural capital of the family resources the students' identity formation in a 'supportive' but not 'pressured' way. An UK student example:

> Interviewer: So dad encouraged you to do maths and science, was he?
>
> Lee: Yes. He just interested me, I don't know why… He works in a chemical firm so just in general conversation he just explains the chemical processes, like maths things. I don't know just interests me…

Similarly Georgia, a female student from Cyprus who has chosen to become a doctor, argues this was her own choice. Nevertheless her mum being a nurse provided her with the opportunity to visit the hospital and observe surgery:

> Georgia: I think I was a bit influenced by my mum who is a nurse because I went and saw some things that made me want to follow medicine.
>
> Interviewer: You mean you went to the hospital?

Georgia: Yeah and I observed an operation and some other things that made me choose what I want.

A common way students in the two samples perceived parental influence was in terms of 'opposing forces' of pressure or support. Amanda from the UK explains:

Amanda: My mum and dad definitely want me to go to university. I'm the first out of my family to go to university so they keep, *not pressuring me but supporting me (our emphasis)* and saying like, you know, you've got to do well and you've got to get into university so they support what I am doing.

Other forms of parental influence which were common in the two datasets were parental advice and encouragement. These forms of parental influence seem less forceful, and a reasonable explanation might be students' urge for autonomy as a form of adolescent agency. Adam (UK) and Alex (Cyprus) explain 'parental advice':

Adam: They were like, it's up to you. They'll give me advice and stuff and what they think I should do … but at the end of the day, it's up to me. They respect that it's up to me and *it's my choice* you know.

Alex: They tried to advise me but they stopped at advice. *It's my choice*.

Interviewer: What did they say?

Alex: Electrical Engineering because the money is more and now with the EU you can easily find a job. Just advice as parents. I will take it into account but I will always do what I want.

Some cultural differences were identified in the cultural models students' used to express their dispositions towards mathematics. Students from the UK, both White British and ethnic minority students, often refer to mathematics as being hard and boring. On the contrary students from Cyprus claimed that mathematics cultivates the mind and sharpens human thought. It might be argued that this cultural model is rooted in the Greek culture of Cyprus and is being used by students as a cultural tool to express their identity. Christos (Cyprus) elaborates on this cultural model:

> I think maths makes you smarter, your mind works, you are more practical. In the way someone learns how to think it is important. I think it develops human thought.

CONCLUSIONS AND DISCUSSION

A surprising finding of the present study was that perceived parental influence did not have a statistically significant effect on Cypriot students' dispositions to study further mathematics. This finding seems to contradict other researchers' claims (Aunola et al, 2003) that parental aspirations have an impact on students' attitudes towards mathematics. These contradictory results with the literature might be due to the fact that in the present study parental aspirations were reported by students. If parents had self-reported their aspirations for their children's future education, maybe parental aspirations would have been proven statistically significant as the literature suggests (Neuenschwander et al, 2007).

The present study brought to light different ways in which children perceive parental influence mostly through the qualitative data collected with semi-structured interviews. Various kinds or parental influence have been identified such as parental pressure on the one hand, and parental encouragement and support on the other hand. Other researchers have also noted different forms of parental influence such as behavioral control and pressure in contrast to psychological support (Campell & Mandel, 1990). The diverse ways students perceive parental influence might therefore explain why parental influence did not have a statistically significant impact on their dispositions as the quantitative data analysis showed.

A plausible explanation why students said they were not influenced by their parents, can be attributed to their age. At this age adolescents urge for autonomy, as some students mentioned they are now *becoming almost independent* and they are having their *own revolution*. This finding adds additional support to Neuenschwander et al's (2007) argument that "as students seek more autonomy from their parents in adolescence they begin to reflect upon and to disagree with their parents' attitudes and beliefs. We expect that parents' expectations may lose some of their predictive power across adolescence" (p.601).

More importantly than this though, students in adolescence go through an identity formation period which might explain why they will assert they are not influenced by their parents in their choices. Beyers and Goossens (2008) argue that identity formation is a dynamic process of person-context interactions, and parents are part of this context, even in late adolescence: "parenting and identity formation are dynamically interlinked, and underscore that parents keep being an important source of socialization for their developing children, even in late adolescence" (p.165).

Although these teenagers tried to preserve their identity as autonomous and independent personalities, by denying that they were influenced by their parents, this influence might well be unconscious, mediated by values as acquired at a younger age. Beyers and Goossens (2008) stress that a combination of a warm and close relationship with parents, and parental encouragement of autonomy is associated with healthy identity development. "Parents might react positively when the adolescent makes an autonomous choice of his or her study or career, rather than actively encouraging the adolescent to make such a choice" (ibid, p.169). Thus, the 'uninfluenced' student unconsciously makes the 'right' decision for their parents by autonomously making their own decision.

Cao, Forgasz and Bishop (2005) argued that there are some cultural differences in the way students perceive parental influence. In their study Chinese students had 'stronger' perceived parental influence than Australian students and they argue that "education has always been considered the most important path to success in Chinese culture, and parents pay particular attention to their children's education" (p.214).

Our findings are consistent with their findings as far as immigrant students are concerned. They argue that "among immigrant families in Australia, for whom English is not their first language, parents recognize that education is vital for success in the new society, therefore they strongly encourage their children and have high expectations of them" (ibid). The same trends were noted in the UK from immigrant or ethnic minority students who argued that their parents have high expectations of them to study in HE.

In addition to this our study used qualitative methods as well to investigate parental influence as Cao et al. (2005) suggested. Our statistical analysis showed that perceived parental influence was not statistically significant but the qualitative data provide evidence of parental influence which is subtle, and often 'denied'. Students coming from middle class families in the UK and Cyprus might deny their parents' influence on them because it is socially and culturally unacceptable to admit being subject to parental influences: "its up to me ... I can say no if I want". On the contrary ethnic minority students (and some first-generation-HE indigenous UK students) accept and use a cultural model of strong parental influence, 'making my parents proud'. Thus, the differences in culture Cao et al. (2005) identified may in part be accounted for by what is culturally acceptable for students to admit or deny, and the correspondence of these perceptions with their parents' actual aspirations or expectations – or their actual influences – may be questioned.

We found the concepts of social and cultural capital suggested by Bourdieu and Passeron (1990) useful here for investigating and understanding perceptions of parental influence. We view this as one component of the reproduction of educational inequality, the influence of family background on school experience and educational achievement, and differences in school–family relationships between social classes. In our study parental influence was mainly evident when students told of their parents provided work experience, private tutoring, and overseas study which influenced their opportunities and hence choices of studying mathematically demanding courses in HE. How parental influence is mediated by social capital and the ways parents use cultural models to communicate their aspirations to their children while promoting their autonomy still deserves to be further investigated.

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PRIMARY TEACHERS' EVALUATION DISCOURSE IN MATHEMATICS: A FOCUS ON POSITIONING

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The study reported here focuses on the positions adopted by teachers within the evaluation discourse. A four-dimensional model describing this discourse in terms of structural and interactional elements was used to read the arguments raised by eight primary teachers in the context of semi-structured interview constructed for the purposes of the study. The results of the analysis of the text provided by one of these teachers show the adoption of partly compatible positions within the respective discourse, reflecting tensions between performance and competence pedagogical models.

INTRODUCTION

Teachers' assessment of pupils' performance is a central educational process, constantly present in school life, with far reaching consequences for students' educational and professional development. It also affects teachers, whose professionalism and effectiveness are often judged on the basis of how successfully they assess their pupils' attainments. This is particularly evident in mathematics, where a positive outcome of the evaluation process is often seen as a prerequisite for academic and employment success.

The mathematics curriculum developments experienced in many countries in the last few decades have been marked by serious challenges to assessment methods. In the climate of reform occurred, particular emphasis was placed on engaging students in problem solving and on developing conceptual understanding, a noticeable departure from the more traditional focus on accuracy and procedural skills (Van de Walle, 1998). As a consequence, new assessment methods emerged and were encouraged in the effort to make the implementation of the new approaches possible. These alternative ways of assessing pupils' mathematical attainments are characterized by a tendency to shift away from conventional written or/and oral testing towards more open and allowing varied and complex responses assessment processes.

These new approaches to assessment rely to a great extend upon a view that the evaluation of pupils' mathematical productions constitutes a predominately social process and, therefore, can be only understood in relation to the power structures developed in the classroom, the school and the wider society (Morgan, 1998). Within such a perspective, the differences between teachers' assessment practices and outcomes can be attributed to the fact that there is not necessarily a one-to-one relationship between a student's text (written oral or behavioral) and the meanings

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constructed by the teacher as its reader (Kress, 1989). On the contrary, these meanings are determined by the features identified in the text by the reader. Thus, given that these features vary according to the pedagogical discourse within which the text is interpreted, teachers proceed to assessment subjectively (Morgan et al, 2002).

In the still limited research available on the highly interpretive and contextualized manner teachers read their students' texts to judge their mathematical attainments, Morgan's and her colleagues' work is a marked exception. In particular, the researchers drew on the late developments of Bernstein's theory, which offer a language for describing the pedagogic mechanism which is responsible for the reproduction of the social inequality (the assessment process being part of this mechanism), to study the way teachers read students' texts when assessing their written work. They argue that in doing so, teachers rely on resources which arise from their "personal, social and cultural history and from their current positioning within a particular discourse" (Morgan & Watson, 2002). These resources, either individual or collective, might come from different and even contradictory discourses and the way in which teachers are positioned within these discourses can lead to different assessment of a student by a particular teacher in different times and contexts (Evans et al, 2006).

Refining their framework over the years, Morgan et al (2002) came to suggest the following categories for the possible resources utilized and the positions adopted by teachers, when assessing pupils' texts respectively: (a) *Resources*: 'teachers' personal knowledge of mathematics', 'their beliefs about the nature of the subject matter', 'their expectations of how mathematical knowledge can be communicated', 'their experience and expectations of students and classrooms in general and of individual students in particular', 'their linguistic skills and cultural background'; (b) *Positions*: 'teacher-examiner, using own/professional criteria', 'teacher-examiner, using externally determined criteria', 'teacher-advocate, seeking opportunities to give credits to students' and 'teacher-adviser, suggesting ways of meeting the criteria'.

Morgan et al (2002) recognized that there is a relationship between teachers' positioning within the evaluation discourse and their assessment practices. In fact, they argued that the resources available to the positions adopted by the teachers-assessors signal "different relationships to students and to external authorities as well as different orientations towards the texts and the task of assessment". On the basis of this, Morgan and her colleagues proposed a four-dimensional model for the positions adopted by teachers as they evaluate students' work. The first dimension of this model concerns two (pre-) dominant structural positions of the evaluation discourse: teachers speaking with the voice of the official/ legitimate or of other discourses. The second is related to two oppositional forms of practice which define teachers' orientations and strategies: towards the text (performance model) and towards the student (competence model). The third and fourth dimensions represent 'strategies' in relation to the focus directing teachers' actions and the degree of generality of the judgment made respectively. In particular, with respect to the former, the focus can be either on what is absent (performance model again) or on what is present (competence model again), while the generality level of the teachers' judgment can be either localized or specialized.

The above are depicted in the form of a systemic network (Bliss, et al, 1987) in figure 1, offering a description of the discourse of evaluation in its complexity, i.e., in terms of structural (positions adopted) and interactional (discursive resources used to deploy strategies) features.

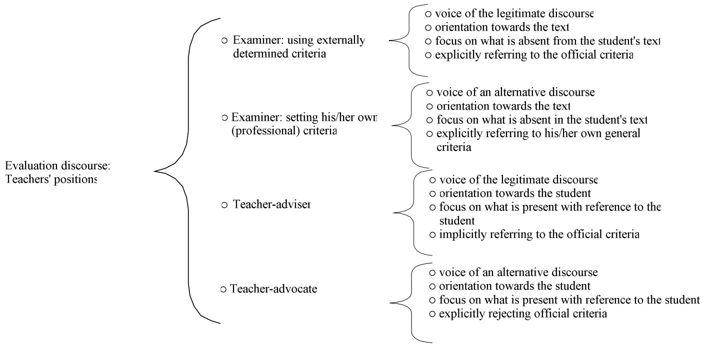

Figure 1. Teachers' positions within the mathematics evaluation discourse (Morgan et al, 2002).

Morgan et al (2002) advocate that the above model allows us to identify and explain incompatibilities in teachers' positions within the evaluation discourse, emerging in the context of an interview, on the basis of contradictory demands and tensions but also strengths created by the combination of not necessarily consistent 'values' along the four dimensions of the model. The study presented here is an attempt to exploit the model in the Greek educational context.

THE STUDY

The study constitutes part of a larger research project, which aimed at investigating the pedagogical discourse adopted by Greek primary teachers with respect to the nature of mathematics as well as the learning, the teaching and the assessment processes taking place in its context. In this report, the focus is narrowed down to a study of the discourse of assessment employed by primary teachers in the context of a relevant interview. In particular, the research question pursued concerns the study of the positions adopted by the teachers of the sample within the evaluation discourse in an interviewing situation, bearing in mind the model suggested by Morgan and her colleagues.

The data utilized are the responses of eight primary teachers (5 females and 3 males) to the questions of a semi-structured interview constructed for the purposes of the larger project, which were related to assessing pupils' mathematical attainments. The subjects were all graduates of Pedagogical Academies (two year courses), had varied teaching experience ranging from 5 to 25 years and attended a number of in-service training programs, while some had participated in innovative projects in education.

Due to the limited space, we focus below on one of the eight teachers, Nikitas, who had 16 years of service (i.e., he was almost in the middle of his professional career) and was teaching to a 4^{th} grade class at the time of the interview. Nikitas has a BSc in Mathematics and has participated in a series of intervention projects especially related to New Technologies. Thus, he can be considered as an experienced and professionally active teacher, with a strong mathematical background.

RESULTS AND DISCUSSION

For the purposes of the analysis, the transcribed text of Nikitas' interview (taking place over 4 sessions of about 2 hours each after school) was carefully read, concentrating on the parts where the issue of assessment in mathematics emerged either deliberately or accidentally. The aim of this reading, carried out independently by the two researchers for validity reasons, was to identify the positions adopted by Nikitas within the assessment discourse along the dimensions of the model suggested by Morgan and her colleagues. To this direction, an interpretive process was followed: identification and coding of the relevant content in the teacher's words; enrichment of the exemplification as more text was read; noticing compatibility and dominant orientation(s) as well as sub-categories in the data.

The above analysis revealed that Nikitas adopted, in general, two out of the four possible positions of the model: teacher-assessor using his own (professional) criteria and teacher – adviser. In the following paragraphs, these two positions are separately elaborated, using extracts from the interview to exemplify the features related to these positions.

a. Teacher – examiner using own (professional) criteria

Extract 1: I want the student to be in the position to understand many different exercises… Thus, I check whether s/he can deal with a variety of tasks through a series of standardized indexes... This is one thing. Another is (to be able to follow) a second way … They often resent this, possibly because they are not used to it.

Extract 2: I consider it very important for a child to explain, especially in mathematics…I think it is crucial to say in mathematics that 'this is so, because' and the thinking formulated to be correct, accurate. I cannot be indifferent, simply say 'bravo'! I will repeat it, I will put an emphasis on it, so that it will become an example for the rest of the class to follow.

Extract 3: Unfortunately, we have misinterpreted memorization. We see it as a very bad thing. This is wrong!...We should discuss this with the students and (ask them) to *memorize some rules, e.g. 'the formula for the area of a triangle'. It simply mustn't be done in an oppressive manner.*

Extract 4: I will certainly give problems of average and high difficulty level. I will also give a very difficult exercise… So, I can distinguish the very good from the excellent student … Mainly because I want to send a message to my students that the person, who deserves it, should occasionally stand out. They shouldn't be all flattened!

In all the above extracts, Nikitas utilizes an unofficial discourse, adopting, however, the values of the traditional pedagogic discourse (emphasis on students' performance), approving of varied solutions, rule memorization, 'correct and accurate' explanations and dealing successfully with exercises of varied degree of difficulty. Orientation is clearly towards the text, identifying the elements which are acceptable by him but absent from the text (e.g., a second solution). Nikitas endorses explicitly criteria that are compatible with his own professional values, e.g., thinking formulated accurately, varied mathematical activity, exemplary mathematical behavior.

b. Teacher - adviser

Extract 5: I consider the emotional bond important. If you convince the children for what you believe in, they will accept a grade of 8 (out of 10) or even 6, because they tell themselves: 'the teacher loves us and understands that we cannot manage the same way as the student who got 10'. It is not bad for a child to realize where s/he stands, where s/he is left behind.

Extract 6: I am always touched by hard working and stable students… not by intelligent ones, who go up and down.. I always tell my students… I am not interested in your mistakes. I will help you correct them. But I will always argue with you if you don't do your exercises. They can understand at this age that being consistent with their work is very important! If they work consistently, they will correct their mistakes.

Extract 7: I help the weak student, often as a favor, in order to improve his/her performance. I won't do this for the indifferent or the lazy student. I will simply sign his/her work, but I will explain … in private so I can be biting... But then, s/he might understand your intention, that is, you don't want to humiliate him/her, but to clearly indicate your limits, to tell him/her, 'look, you are not up to what I consider ideal for you and me and for your own good you are assessed this way'!

Extract 8: Look, I am very much influenced by students' behavior. I have caught myself, particularly this year that I teach to a 'salty' (sic) class, when I assess their written tests, to be susceptible and help them with their grades. That is, I avoid being rigid with what I see, but I like taking a longer view.

In the preceding extracts, Nikitas speaks with the voice of the official discourse, referring to identifying where a student 'stands' mathematically, consistency and stable behavior, features discussed in the official assessment documents. However, he is oriented towards the student and what is present in the text. To this direction, he invests on the emotional relationship with his students, particularly the ones having difficulties with mathematics, to appropriately exploit the effects of the grading system and encourages effort and hard work. It is apparent that for Nikitas, the task of adviser is interpreted according to the criteria offered by the official discourse, used only implicitly though (e.g., stable behavior is not an explicitly evaluative reference).

To fully appreciate the adoption of the two positions by Nikitas, we need to consider the way in which the evaluation discourse was produced for him. Morgan et al (2006) argue that an evaluation discourse consists of official as well as unofficial discourses, all of which are products of recontextulisations carried out by agents operating on various fields.

The official discourse of evaluation in Greece is shaped by the Pedagogical Institute (PI), a government agent, in fact the only one, responsible for providing assessment guidelines, procedures and criteria. These appear in few official documents and are supposedly taken into account in the textbooks used, the same all over the country, produced under the supervision of PI. The above imply an assessment discourse which encourages a position that 'speaks strongly with the voice of the official' (Morgan et al, 2002). However, the laconic and general character of the official evaluation arguments and the vagueness of some of them, accompanied by a weak control system of the assessment measures' implementation seem to grant to the teachers the power to judge as to the content and the process of assessing. This, apparently, weakens the official voice, allowing for interpretation at a local level. As a result, Nikitas sometimes speaks with the voice of an alternative discourse (extracts 1 – 4) and some other times with the voice of the official discourse (extracts 5-6).

As mentioned above, a constitutive element of the discourse of assessment is the mathematics education knowledge projected in the practice of a number of agents operating in various contexts, such as researchers contributing to a teachers' training course. This knowledge becomes resource which the official or other discourses draw on to prescribe assessment processes and pedagogical relationships. In Greece, the status of the mathematics education field is still fairly low (the state intervenes centrally on all educational matters; the relevant community is still very young, with minimal influence on educational policies and a short history in primary education departments only). In addition, there is a continuous shift in the field between a 'performance' (emphasis on knowledge, on the product/text) and a 'competence' (emphasis on students) model, because of the two traditions still competing with each other. The preceding situation makes the distinction between the official and other evaluation discourses unclear and the positioning precarious, thus providing a basis to explain Nikitas' movement between the two positions. His strong mathematical background, in particular, could be seen as contributing to predominately switching to the position of teacher-assessor setting his/her own (professional) criteria, as it seemingly directs him in placing emphasis on knowledge, that is, in relying on the product/text (performance model).

CONCLUDING REMARKS

The analysis presented in the previous section indicates that the dominant position adopted by Nikitas within the mathematics evaluation discourse is constituted on the basis of professional criteria, mostly related to elements which are absent from pupils' productions. This performance oriented position contrasts with the official purpose of assessment, i.e., to improve everybody's mathematical attainment respecting his/her own experiences and investing on them. This is because, judging pupils' productions according to the absence of certain standard, ill-described, rather unique, compatible but not necessarily tautological to the official characteristics constructed in the professional field, does not serve the interests of less privileged children. In other words, Nikitas' professional development trajectory appears to have initiated him predominately to a 'performance' pedagogical model, reinforced by his BSc studies in Mathematics.

The adoption of the teacher-adviser position too, although to a lesser degree, might be attributed to Nikitas drawing from resources originated from populist or emancipatory modes, which value students' everyday experiences and in which teacher and pupils develop a relationship of trust. These modes became prevalent in the Greek educational context in the last decade because of their extensive exploitation in the context of a number of innovative projects and in-service training courses running all over the country, the introduction of new mathematics textbooks for the state primary and high schools and the new Master's courses initiated for the first time in almost all Greek Primary Education University Departments. Nikitas' acknowledgment of these new modes appears to lead him to occasionally and hesitantly opt for a 'competence' model, which, however, does not seem to defeat his dominant orientation towards a 'performance' model. This reflects a tension possible to create incompatibilities and contradictions with respect to the official assessment criteria.

In nowadays demanding educational and social environments, teachers are continuously forced to shift their instructional and assessment practices in ways that we do not understand. The study described here, focusing on teachers' relationship to the discourses at play when assessing, is an attempt to contribute to this understanding and points to the need for further investigation of this relationship in various contexts.

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TEACHERS' MATHEMATICAL KNOWLEDGE: THE INFLUENCE OF ATTENTION

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This paper reports on some findings from the project 'Analysing the Transition from Secondary to Tertiary Education in Mathematics'. A key variable in the school to university transition is the teacher/lecturer, and here we deal with data analysing secondary teachers' responses to four mathematics questions. Elsewhere we consider a comparison of teacher and lecturer knowledge, preparedness and teaching style etc, but this paper tracks the ability to use mathematical knowledge. We hypothesise that this is a function of what we pay attention to, as described in Mason's discipline of noticing. The results reveal that many teachers fail to notice the necessary conditions for problems that imply that procedures are not always applicable. Possible reasons for this along with implications for student learning are discussed.

INTRODUCTION

The transition from school to university is coming under increased scrutiny in the light of growing concerns about decreasing numbers of students opting to study mathematics at university and beyond (e.g., the ICMI Pipeline Project), and their apparently decreasing levels of competence (Smith, 2004). While the increasing numbers and diversity of those attending higher education institutions are an issue, a lack of essential technical facility, a decline in analytical powers and a lack of appreciation of the place of proof in mathematics on the part of undergraduates haver been cited in the USA and UK (US National Commission, 2000; Smith, 2004). The importance of this transition period is also increasingly reflected in research papers and international forums (e.g. Brandell, Hemmi & Thunberg, 2008). One possible reason for such transition problems may be in the mathematical emphasis at each level. A developing theory by Tall (2008) suggests that mathematical thinking exists in three 'worlds': an embodied world where we make use of physical attributes of concepts, combined with our sensual experiences to build mental conceptions; a symbolic world where the symbolic representations of concepts are acted upon, or manipulated; and the formal world is where properties of objects are formalized as axioms, and learning comprises the building and proving of theorems by logical deduction from axioms. There is evidence that school mathematics draws heavily on the symbolic world while in university the emphasis is more on the formal world (Stewart & Thomas, 2008; Novotna & Hoch, 2008). Thus a schoolteacher may need to draw on more than one kind of thinking, and integrate them too. The scope and nature of the mathematical knowledge needed for teaching has been a focus of research by Deborah Ball and her colleagues, who have described in detail the concept of *mathematical knowledge for teaching* (Ball, Hill & Bass, 2005; Hill & Ball, 2004), which extends Shulman's pedagogical content knowledge, combining

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three kinds of subject matter knowledge with three kinds of pedagogical knowledge. One use of this knowledge for teaching is described as: "Teachers do not merely do problems while students watch…They must choose useful models or examples. Doing these things requires additional mathematical insight and understanding." (Ball, Hill & Bass, 2005, p. 17). The emphasis in this paper is the teacher's ability to choose suitable problems for Grade 13 students and the insight needed to do so.

Mason has proposed that when we look at mathematics the focus of our attention may vary depending on whether we are *looking at* the symbols or *looking through* them. His idea is that we need to structure our attention, to know what we are aware of, and he describes a number of elements that we may focus attention on, including: the whole, the details, the relationships between the parts, the properties of the whole or the parts and deductions (Mason, 2008). One way in which a teacher may focus student attention is by asking questions (Mason, 2000). The style of these questions, which will often be oral, but may include written questions, is crucial if attention is to be focussed on what is to be learned. As Mason says:

> …the style and format of the questions used by lecturers and tutors profoundly influence students' conceptions of what mathematics is about and how it is conducted. By looking at reasons for asking questions, and becoming aware of different types of questions which mathematicians typically ask themselves, we can enrich students' experience of mathematics. (Mason, 2000, p. 97)

This paper addresses some reasons why teachers decide certain questions might enrich student experience of mathematics. Such a process involves aspects of *knowing-about* that Mason and Spence (1999) describe, namely *knowing-that* (factual), *knowing-how* (techniques and skills), or *knowing-why* (ability to restructure actions). We are confident that the vast majority of teachers who teach calculus have excellent knowing-that and knowing-how mathematics knowledge. Hence the question is whether this kind of knowledge tends to structure the attention in such a way that a procedural emphasis develops, obscuring detail it may be crucial to notice.

THE STUDY

This study is part of a much larger research project entitled 'Analysing the Transition from Secondary to Tertiary Education in Mathematics', involving teachers, lecturers and students, that employs questionnaires, interviews and observations. A questionnaire was sent to all 350 secondary schools in New Zealand to be completed by all teachers who teach calculus to Grades 12 or 13 (age 17-18 years). The questionnaire was posted, complete with a stamped addressed return envelope and teachers were given three weeks to answer. After this a follow-up copy was sent by email to remind teachers to reply. Using this approach we received a total of 178 responses. There are no figures available on the total number of calculus teachers in the schools, which vary in size from fewer than 30 students (small country school) to 3000 (inner city), but we estimate the response rate at about 30% of the population. In this paper we present and analyse teachers' responses to four related questions from the questionnaire. The first two questions were considered to be fairly routine assessment questions while the others looked like routine assessment questions but require solvers to notice that certain conditions need to be paid attention to before applying a standard procedure, rule or technique. The four questions were:

> Please try the following questions and then comment on whether you think each of them is a suitable assessment item for a Year 13 [age 18 years] calculus class.
>
> 1. If $\int_1^a (2x-1)dx = 2$, find the possible values of a.
>
> 2. Sketch the graph of a function $f(x)$ such that it is continuous on $0 \le x < 3$ and $3 < x \le 5$, $f(3)=1$, and $f'(x) > 0$ for $0 \le x \le 5$, $x \ne 3$. Does $\lim_{x \to 3} f(x)$ exist for your function?
>
> 3. Find the derivative of the function $y = \ln(2\sin(3x) - 4)$.
>
> 4. Find the integral $\int_{-1}^{1} \frac{1}{x} dx$.

Of course not all teachers attempted the questions or provided comments on them. The non-responses for each question were often because the teacher had not taught Grade 13. In each case the majority of teachers decided whether the questions were suitable without providing working. It is difficult to know whether they did any, and if so whether it was correct. Based solely on the curriculum questions 1 and 2 are suitable for assessment and 3 and 4 are probably not.

Table 1. A Summary of the Question Responses

Question	S	SC	SNC	NS	NSC	NSNC	Not Sure	No response
1	73	33	12	2	2	6	0	50
2	68	31	2	15	6	2	0	54
3	71	2	36	1	12	3	1	52
4	42	3	22	23	23	9	2	54

Key: S=suitable no working; SC=suitable from correct working; SNC=suitable from incorrect working; NS=not suitable no working; NSC=not suitable from correct working; NSNC=not suitable from incorrect working.

Questions 1 and 2 results

As mentioned above Question 1 was not designed to test understanding of integration procedures. We expected that all teachers of calculus would be able to solve this correctly. In fact of the 128 who responded 92.2% thought it was a suitable assessment question, and 35 gave a correct answer and 18 an incorrect one. Of these 18 some made calculation errors and a number gave only the answer $a = 2$, obtained 'by inspection' in one case. However, interestingly, 11 paid careful attention to the limits and rejected one of the solutions $a = -1$ on the grounds that it was negative. They all had the opinion that if a definite integral is positive then the upper limit is greater than the lower limit, and so, even though some had correctly calculated a as negative, they then proceeded to reject it. It also appears that some found the actual concept of an integral $\int_p^q f(x)dx$ where $p > q$ difficult to accept. In the case of the teachers whose responses and explanation are shown in Figure 1, their mathematical belief has caused them to pay close attention to the sign of the solution for a, and hence to reject –1 as a solution. In doing so some also rejected the question as an assessment item, since it is 'too confusing', as the first teacher responds.

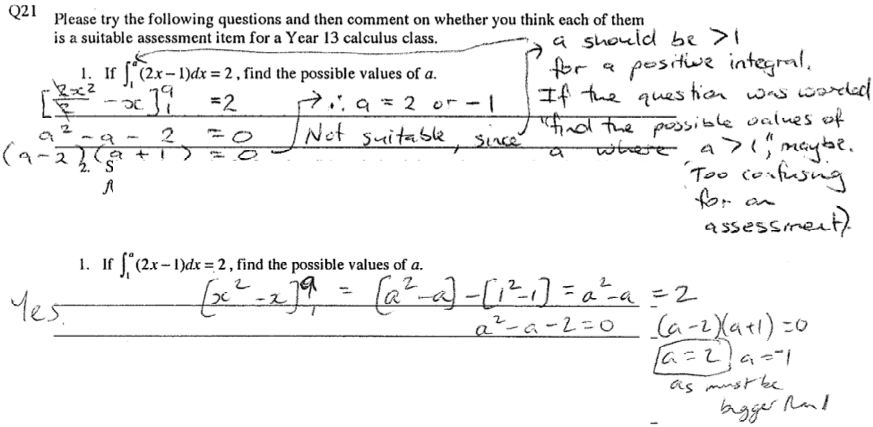

Figure 1. Responses to Question 1 rejecting one solution.

For Question 2 81.5% of the 124 who responded thought it was a suitable assessment question, based on 37 correct answers and 8 incorrect ones. The important thing in this question is to pay attention to what happens at $x = 3$. If one assumes a single standard polynomial-type function, such as a quadratic, as the exemplar then it is possible to get the answer that the limit exists even though the function is not continuous at $x = 3$, as in Figures 2a and b. However, by paying close attention to the local point $x = 3$ the teacher producing 2a has not satisfied the global condition that the gradient be positive. On the other hand it is possible to obtain graphs of functions where the limit does not exist, as seen in Figures 2c and 2d. Thus this question requires one to pay attention to both global and local properties in order to succeed. We note that, sometimes due to the open nature of the solutions, the teachers were split over whether this would be a good assessment item. Some said yes because it 'requires some thought from students', 'tests students' understanding of the concepts of limits, continuity and differentiability', and 'it requires students to understand discontinuities', while others answered no since 'no perfect answer' (see 2b), 'open-ended', 'the concept of limits need not be this complicated' and 'attempting to draw the graph could confuse students'. Overall 101 said it was a good assessment question and 23 that it wasn't, but the replies may depend on what the teachers paid attention to.

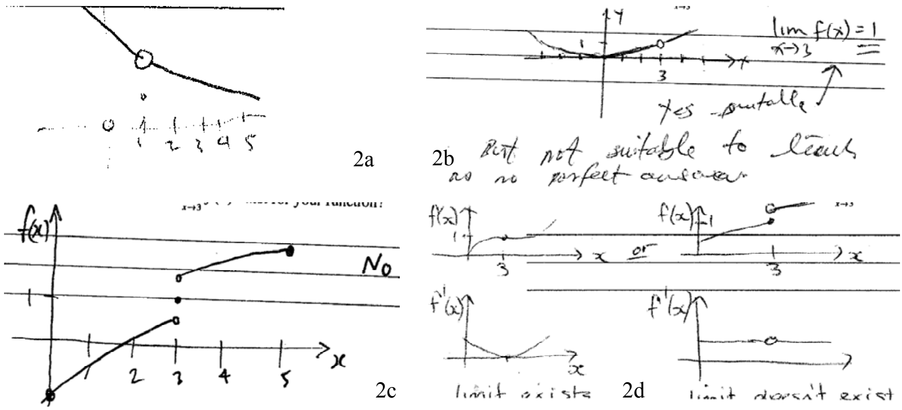

Figure 2. Responses to question 2 showing a range of solutions.

Question 3 and 4 results

101 (80.2%) teachers thought Question 3 was a suitable assessment question. Knowing the domain of the log function, and the range of the sine function should have been enough to solve it. Since calculus teachers should know the range of the sine function and the domain of the log function the question was not about their knowledge but about what we pay attention to, or notice. Yet, even though this is knowledge they have most of the teachers failed to solve it, with only 14 presenting a correct solution and 39 (31.0%) giving an incorrect one. The question is why? Question 3 had been deliberately set so that it was important to check the domain of the function and notice that the function is not defined because the argument of the log function is always negative. Hence the chain rule is not applicable in this case and the derivative does not exist. This may make it an unsuitable question in a primarily procedural assessment context. However, many teachers either “solved” it using the standard technique, or wrote that the question was a “good”, ”straightforward”, ”easy”, or ”suitable” question for assessment.

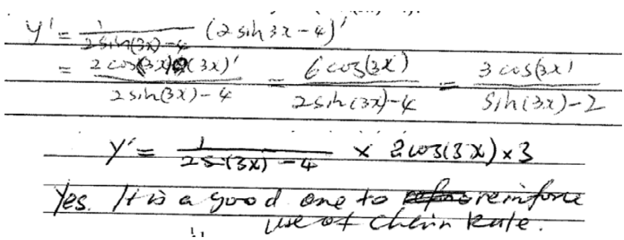

Figure 3. Two examples of incorrect responses to Question 3.

In Figure 3 we see two examples of solutions that apply the procedure without paying attention to the domain of the function, and one mentions that the question will "reinforce use of the chain rule". There is an implicit understanding here that if the procedure 'works' and produces an answer then everything is fine. In contrast we see in Figure 4 examples of two teachers who, by paying attention to the domain, have noticed that there is a problem and hence reject it as a suitable assessment item. In the first case the procedure has been applied too, but in the second consideration of domain has been the first option.

3. Find the derivative of the function $y = \ln(2\sin(3x) - 4)$.

$\frac{dy}{dx} = \frac{6\cos 3x}{2\sin 3x - 4}$ However $2\sin(3x)-4$ is always negative, and it is impossible to have ln (negative no.), so not a good assessment question.

3. Find the derivative of the function $y = \ln(2\sin(3x) - 4)$

$2\sin 3x - 4 < 0 \ \forall x \in R$; $y = \ln x$; $x > 0$

Hence not a good question

Figure 4. Responses to Question 3 showing recognition of the function domain.

For Question 4 only 67 (54.0%) teachers thought it was a suitable assessment question. The suggested integral is an improper integral and in this particular case it is undefined. What we expected from teachers was a statement that the function $y = \frac{1}{x}$ is not continuous on the interval [–1, 1] (or not defined at the point $x = 0$). Therefore the integral in not a definite integral and the Newton-Leibniz formula is not applicable, making the question lie outside the school curriculum. Again we may assume that while every teacher who teaches calculus knows this, what we pay attention to can influence actions. When there is a standard procedure that can be applied does paying attention to this cause us to fail to notice other important detail such as the fact that the function is discontinuous on the interval and we come to an incorrect conclusion? In this question 31 teachers (25%) who attempted Question 4 'solved' it with a version of $\int_{-1}^{1} \frac{1}{x} dx = \ln|x|\Big|_{-1}^{1} = \ln|1| - \ln|-1| = 0 - 0 = 0$ (see the examples in Figure 5). Some of these thought that the use of the absolute value would be a problem for students.

4. Find the integral $\int_{-1}^{1} \frac{1}{x} dx$. $= \left[\ln|x|\right]_{-1}^{1}$ $\ln 1 - \ln|-1|$ $= 0$

4. Find the integral $\int_{-1}^{1} \frac{1}{x} dx$. $= \left[\ln|x|\right]_{-1}^{1} = 0$

Figure 5. Responses to question 4 showing use of a standard procedure.

Unlike Question 3, where most of the teachers forgot to check the domain, in Question 4 quite a number of teachers noticed that the function $y = \frac{1}{x}$ is not defined at $x = 0$. They often sketched the graph, and then used this to notice that the areas on [–1, 0] and [0, 1] were the "same" because of the symmetry and concluded that the integral is zero. Some typical remarks made by those adopting this stance were: "By symmetry of areas intuitively the answer is zero"; "Zero is correct answer – use of symmetry + common sense gives answer as zero"; Basically they relied on intuition to answer. Only a few responses recognised the issue of domain or continuity (see Figure 6), and some of these made comments that showed a measure of uncertainty or ambiguity in their conclusions: "An excellent question – there are two answers – undefined or zero"; "Numerically gives 0 but undefined because of discontinuity"; "Comes to 0 this way but actually not defined since $y=\frac{1}{x}$ is discontinuous at $x=0$."

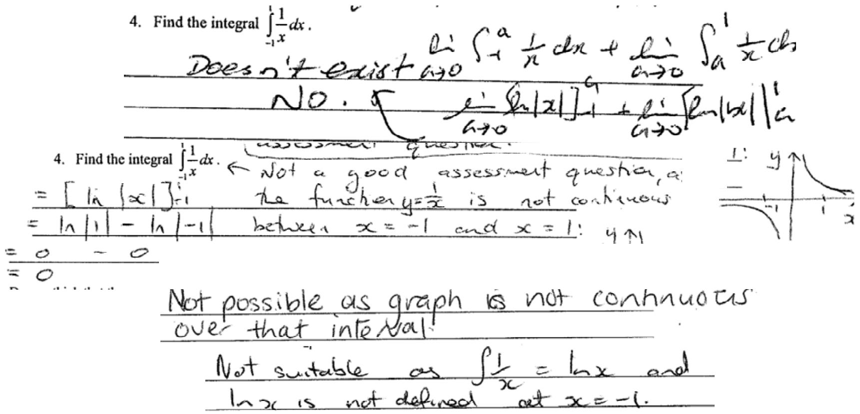

Figure 6. Responses to question 4 showing attention paid to continuity or domain.

CONCLUSION

The analysis above does raise some issues. When evaluating mathematical questions we can pay attention globally or locally, and we can consider properties and relationships, and all of these aspects are important to develop. What influences what we pay attention to? What can help us to attend to important details etc?. Mason (2008, p. 11) suggests that we need to be "developing strategies for directing or focusing attention in a way which is pertinent to the learning", especially towards recognition of properties and reasoning with them. One such strategy is the use of carefully thought-out questions. In his analysis of the teaching process Schoenfeld (1999) describes how teacher beliefs (latterly orientations, including values, etc) lead to goals and the use of appropriate knowledge to guide actions, such as questioning, in pursuit of the goals. Someone who is primarily a symbolic world thinker (Tall, 2008) with a belief that procedural skills of primary importance may be more likely to pay attention to issues that are relevant to procedures and tend to apply these without further thought, rather than considering properties. While other representations, such as graphs, may help structure attention, as they did for some teachers, it may be that emphasising procedures, intuition, 'common sense' or pictures can lead one to miss things it is important to notice, preventing one appreciating the need to justify, prove, verify a solution in mathematics. This seems to have been the case for a number of our teachers. However, in the transition to university study of mathematics students need to move toward formal thinking and this research suggests that explicit training in the discipline of noticing could be a useful addition to school mathematics teaching, and might help smooth the transition.

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MAKING A DIFFERENCE: DISCURSIVE PRACTICES OF A UNIVERSITY INSTRUCTOR

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This study is part of a larger project that investigated the impact of instructional practices on university students' learning in pre-calculus classes. The students in one section consistently outperformed the students in the other sections and led us to investigate whether discursive practices used by the instructor promoted student learning. The theoretical framework is based on Truxaw's and Sfards' ideas about the functions of discourse and the subsequent development of mathematical ideas. Qualitative analysis using constant comparative and matrix methods indicates that how an instructor invites student participation and responds to students' comments determines the function of the discourse.

INTRODUCTION

Mathematics educators throughout the world have called for change in the teaching and learning of mathematics (e.g., Franke, Kazemi, & Battey, 2007). An image of teachers and students vocalizing and exchanging ideas while constructing knowledge and solving problems materialized. Social interactions, class structure, and curricula are thought to combine to create the unique discourse of a mathematics classroom that allows students to simultaneously develop their communication skills and mathematical thinking (Sfard, 2008). Research in elementary (e.g., Hufferd-Ackles, Fuson, & Sherin, 2004) and secondary schools (e.g., Frade & Machado, 2008) describe discursive practices that support the creation of learning communities. This research body suggests that (a) asking questions that require students to reflect and generalize, (b) modelling questions and responses, (c) nurturing students' use of mathematical language and encouraging higher level cognitive skill development, and (d) communicating a belief that mathematics is accessible to all promotes a learning environment that supports students' learning.

The development of discourse that supports students' learning is complex and difficult. High school teachers struggle to develop a discourse that reflects research suggestions to improve student learning. This may be the result of having learned mathematics as an observer while the instructor explained what to do. In universities, mathematics courses tend to be lecture driven, fast paced, and content intensive. Typically, 50% of students enrolled in entry level university mathematics classes in the United States fail (Cohen, 2006). Developing successful discursive practices can be intensely challenging. University instructors are typically recognized for their advanced knowledge of mathematics, not for their understanding of the art of teaching and the science of learning.

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This report is part of a larger study in which we investigated student achievement in four sections of a pre-calculus class and four different instructors' actions to describe instructional practices that may support the learning of university students. We found that students in one section outperformed the students in the other sections (Cooper & Olson, 2008). Building from these findings, this study investigated how the instructor used discourse to support student learning.

THEORETICAL FRAMEWORK

The discursive practices employed in a mathematics classroom are unique and situated (Sfard, 2008). Social interactions, class structure, and curricula combine to create this unique discourse. Students along with the teacher shape the classroom culture. However, the discursive practices of the teacher shape this culture through implicit and explicit means (Turner et al, 2002).

Discourse is theorized to have two functions. First, univocal discourse allows an individual to accurately transmit information to another person (Lotman, 2000). Second, dialogic discourse creates new meanings as individuals exchange ideas (Wertsch, 1998).

Classroom discourse typically follows a three-part exchange beginning with a teacher initiation, followed by a student response, and then the teacher's response or follow-up (Kumpulainen & Wray, 2002). Teachers who maintain control of the classroom discourse in this way often articulate a belief that the teacher's role is to transmit knowledge to students. When K-12 teachers' beliefs about their role change to that of a facilitator in the development and exchange of ideas, Kumpulainen and Wray found that their discourse pattern also changes from triadic exchanges to one in which students interact with and question each other.. These teachers encourage collaboration during problem solving, use students' questions to further explore mathematical ideas, and model discourse and vocabulary that students can use to discuss mathematics.

However, Truxaw (to appear) found that "simply engaging students more actively in classroom discourse is not a panacea for improving mathematical achievement." (p. 18). A grade eight teacher used triadic exchanges to promote meaning-making by inviting students to hypothesize, justify, and make sense of mathematics. Truxaw theorized that triadic exchanges are not limited to conveying a teacher's knowledge; they can also help build students' understanding. We theorize that how a teacher invites student participation and acknowledges acceptable responses impacts the function of the discourse. These practices become norms and can be analysed as univocal or dialogic. From this perspective, we describe how a university instructor used triadic exchanges that resulted in the creation of dialogic discourse whose function was to promote the creation and exchange of new ideas.

METHOD

This study took place in a large university in United States serving 18,000 students. A placement examination identified entering students who needed prerequisite skills to be successful in calculus. The students were randomly assigned to a section of pre-calculus. The student demographics (scores on the placement exam, gender, career aspirations) in Pam's (pseudonym) section were similar to those of the students in the other sections. Quantitative analysis indicated that students in Pam's section outperformed the students in the other sections and this disparity increased over time (Cooper & Olson, 2008).

Qualitative methods with a single case-study (Merriam, 1998) were used to identify and describe the discursive practices that Pam used during her instruction. Data were collected from five sources during the fall semester. These sources included (a) weekly reflections, (b) monthly video recording on two consecutive days, (c) field notes made by the videographer, (d) informal interviews, and (e) semi-structured follow-up interview. Constant comparative methods (Merriam) were used to code, analyse, and collapse the data to identify emergent patterns of discourse (Truxaw, to appear). These patterns were then displayed in a conceptual matrix with illustrative examples in each cell (Miles & Huberman, 1994). Initially, our analysis focused on interpreting the discourse captured on the videotapes and the results were triangulated with analysis of Pam's reflections, videographers' field notes, and the interviews.

RESULTS

The discourse between Pam and her students was fluid, like a conversation over a cup of coffee with a friend who had multiple voices. When she asked a question, many students independently responded with individual voices that overlapped in a way that made it impossible to discern the number of students answering. Field notes indicated that approximately half of the class responded to her questions.

Pam articulated two functions of discourse while reflecting on her practice. First, she described herself as listening carefully to what students said because she wanted to know what they understood. Second, she used this understanding to guide the construction of new knowledge during the class. These goals suggested that she valued students' meaning making processes and used their responses to make instructional decisions that went beyond merely affirming a correct answer. The following excerpts from a five minute episode are typical of the triadic exchanges between Pam and her students. It illustrates how she invited student participation and guided the development of mathematical ideas.

Pam	So we had done, have we finished example 1 last time you guys?
Student	(Several students answer. One voice over-powered the rest.) No, you had done that one but we didn't

	get to the last part and that you wanted to do it.
Pam	Oh yeah, we did everything except part c. Okay, the directions were. Actually, it was a fraction, right?
Students	Yeah.
Pam	Can you tell me what it was?

Pam relied on students to remember what they accomplished in the previous class (lines 1-2). By depending on them to help her, she invited their participation. They responded (lines 3-4) and the triadic exchange ended with an affirmation (line 5). With her memory refreshed, Pam asked students if she was correct (line 6). Students responded (line 7), and she further invited their participation by asking them to dictate the problem (line 8). Her invitations to participate continued throughout her instruction and colloquial expressions like "you guys" and "oh yeah" made the discourse informal and one in which all could voice an idea. After writing the problem on the board, Pam began to guide students to a solution.

Pam	Okay, let's do this one together. Anybody have any ideas of how to get started? I know that there are lots of different ways to go with these.	Simplify the expression in terms of sine and cosine $\frac{\cot\theta}{\csc\theta-\sin\theta}$
Students	(Several students offered suggestions.)You could put it into tangent. You could write it as sine. You could square the bottom.	
Pam	The directions actually said to go ahead and put it in terms of sine and cosine first and let's see what happens. So if we do that, what do we put in for cot θ?	
Students	(Overlapping voices) Cos θ over sin θ (Bold indicated the substitution that Pam wrote.)	$\frac{\cot\theta}{\csc\theta-\sin\theta}=\frac{\frac{\cos\theta}{\sin\theta}}{\ldots}$
Pam	Good. And similarly, what would we put in for cosecant θ?	

In lines 12-13, Pam invited her students to give suggestions for how to begin solving the problem. She encouraged them to think independently by acknowledging that problems can be solved in multiple ways (line 14). Students' ideas gushed forward (lines 15-17). Some led to an efficient solution and others had detours. Rather than addressing each suggestion individually, Pam referenced the directions (lines 18-19) and modelled using directions to guide the problem solving process. Then, she prompted students' curiosity, "…let's see what happens" and implied that they would collaborate and embark on an exploration with an unknown solution (lines 19-20). In this way, the process of solving the problem was valued and students continued to be involved in the process rather than just getting the right answer. Pam used a leading question (line 20-21) to solicit students' prior knowledge. She implicitly communicated to her students they were all members of a community by using the terms "together", "let's" and "we" in which everyone would be successful because everyone had the knowledge to solve the problem. Students responded with enthusiasm (line 23). Pam reinforced their response (line 24) and then continued to guide them through the algebraic steps (lines 24-25).

After further simplifying the expression with triadic exchanges similar to lines 20 to 25, Pam enthusiastically responded, "I just love it that you said sine theta over sine theta because that is exactly one. Good." In this way, she encouraged students to use language correctly when expressing mathematical ideas. Then, she encouraged students to reflect on what they had learned in a previous lesson, "Now be opportunistic. There is something that we can simplify down in there." Pam implicitly communicated to her students that they had the prerequisite knowledge and trusted them to remember. The implicit message was clear. Students had the knowledge, they could use it, and they could participate in the final steps of the problem. After simplifying the expression, Pam said, "There were different ways to do that, I have the knack to choose the longest one possible." She implicitly indicated that she enjoyed the problem solving process and that it did not matter if they used a different strategy. Long or short, they were all valued. Pam wrote a new example on the board and asked, "How are you guys feeling about it? Do you want me to do it with you or do you want to do it alone?" In this way, she addressed their affect and let the students determine the next set of events. In a chorus they responded, "Do it alone." She gave them several minutes to work on it by themselves before collaboratively solving the problem.

DISCUSSION

Pam communicated her belief that all students could learn mathematics. She gave encouragement and affirmation while weaving in explanations of algebraic procedures. She developed collaboration by providing opportunities for students to share ideas, working with groups of students during office hours, and talking about mathematics before and after class. Students gained skills and confidence as they were expected to actively engage in doing mathematics during the class. Reflecting on her teaching, Pam said that she was attentive to students' questions and "avoided saying something that would make them sorry that they added a question." Thus, students interacted with her continuously, slowly gaining competence using mathematical language to discuss mathematical ideas. Pam illustrated a student's development of language through the following student request for clarification, "I understand how to get tangent written in terms of cosine, but I don't know why the quadrant is important." In the past, students said, "I don't get it." Now, she surmised that students could articulate what they were struggling to understand because they were more comfortable using mathematical language. Pam believed that students' learning and engagement increased as they developed their ability to talk about mathematical ideas.

Truxaw (to appear) found that triadic exchanges can be used to effectively build students' understanding. In a university entry-level mathematics course that is defined by a set of procedures and knowledge, instructors often rely on a univocal function for discourse. They resort to a lecture driven course to "cover" the material and in doing so reflect a belief that students must be shown how to do the symbolic manipulations before they can do it independently. Pam believed that students possessed many tools and could develop all the tools they needed. Thus, the purpose of her discourse was different. She was no longer the transmitter of information, she became a guide who reminded students of what they knew. In this role, she invited students' participation and encouraged them to engage in problem solving independently and with her or another student. Pam created a classroom culture with norms that reflected her beliefs, expectations, and values. Like Frade and Machado (2008), we found that classroom discourse impacts the learning community. We further suggest that an instructor's beliefs, expectations, and values are reflected in the discourse. Furthermore, how the instructor invites student participation and acknowledges acceptable responses determines the function of the discourse, either univocal or dialogic. Additional research is needed that investigates the teaching and learning of post-secondary students in entry-level mathematics courses in a way that satisfies the growing demand for individuals who are able to solve complex problems in our world.

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KOREAN 5TH GRADERS' UNDERSTANDING OF AVERAGE

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The purposes of this study are to examine Korean 5th graders' understanding of average and to add additional discussions to previous studies. For the goal of this study we made up of a serious of questions related to the concept of average by reviewing previous researches and presented 35 students in Grade 5 with them. We conducted mini-interview through the medium of the students' responses after their completing questionnaire. The results of this study show that the students have the different interpretations between of the meanings of the word 'average' itself and in the colloquial use of it; many students did not focus on that average in the context of everyday life is from a sample when interpreting the average; many students could explain average of a decimal number using algorithm but avoided a decimal number as average; a half of the students did not consider variation when finding average.

INTRODUCTION

Many researchers (Cobb & Moore, 1997; Shaughnessy, Garfield, & Greer, 1996) have argued that exploratory data analysis (EDA) should be focus of statistics instruction in lower grades because it is concerned with patterns and trends in data sets without probabilistic statements of argumentation. Along with it, suggestions about which ideas should be weighted in early instruction of data analysis have been presented. For example, Cobb (1999) and Bakker, Koeno, and Gravemeijer (2004) considered distribution as core idea of statistical reasoning. According to them, average is important concept that enables data as individual values and distribution as conceptual entity to interact. Wild and Pfannkuch (1999) and Reading and Shaughnessy (2004) viewed variation as core idea of statistical reasoning. They argued that students should learn averages as central tendencies in proper to have understanding of variation and inappropriate teaching of average makes students misunderstand variation. Konold and Pollatsek (2002) suggested that core idea taught to students should involve coming to see statistics as the investigation of noisy processes – processes that have a signal which we can detect if we look at sufficient data. According to them, a main purpose of finding average is to represent such a signal in the noise of individual values.

It might seem obvious that the idea of average is important for students to learn statistics and to understand concepts related to statistical reasoning. Ideas associated with average, in particular the arithmetic mean, have been placed in statistics in mathematics curriculum for over a century (NCTM, 1989, 2000; Watson, 2006). However, some researchers (Cai, 1995; Cruz & Garrett, 2006; Pollatsek, Lima, & Well, 1981) criticized the teaching and assessing of averages in mathematics curriculum for

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highlighting calculation rule or algorithm. In this study we examine how is Korean 5th graders' understanding of average and consider some implications for teaching and assessing of the concept of average.

BACKGROUND

Watson (1997, 2006) proposed structure of statistical literacy that involves various aspects related to statistical reasoning. According to her, the beginning of consideration for students' understanding of average is to investigate their understanding of the meaning of the word "average": students on the basic level need to understand statistical terminologies such as average, sample, and so on; students on the second level interpret and understand the meanings of statistical terminologies in the context; students on the highest level criticize statistical statements.

Mokros and Russell (1995) sorted characteristics of students' understanding of averages into five: average as mode, as algorithm, as what is reasonable, as midpoint, and as mathematical point of balance. Students who view average as mode have a characteristic that they consider average as "the most", not as representative of the data set as a whole; students who view average as algorithm think that average is to figure out arithmetic mean; students who view average as what is reasonable believe that the mean of a data set is not one precise mathematical value, but approximation that can have one of several values; students who view average as median look for a "middle" of a data set; students who view average as mathematical point of balance look for a point of balance to represent the data and take into account the values of all the data points.

The ideas of average are very powerful in statistical reasoning since averages including mean, mode, and median are used to summarize information about a data set. Almost data sets that we face in the context of everyday life are almost samples, not populations. Like it, if the data set is a sample from a population, we need to consider such a situation when interpreting statistics of sample and inferring about its' population from them since we should be provided information for its' population from statistics of sample (Shaughnessey, 2007). Therefore, it is important that students know this fact and consider it when interpreting averages from samples. This study investigates whether students notice spontaneously that averages in the reports from newspaper or TV are information on samples.

Most of researches (Cruz & Garrett, 2006; Mokros & Russell, 1995; Watson, 2006, 2007) on understanding of properties of the arithmetic mean used questions in which students interpret average of a decimal number and reported about students' understanding of arithmetic mean. In this study we tried to another method in which we investigated whether students accept average of a decimal number or not.

As the mathematics curriculum involves data handling and various aspects of statistical investigations the idea of average needs to be linked to other components of the investigation process such as sampling, variation, and so on. In particular average needs to be connected to spread and variation within the data set to reflect the central tendency (Watson, 2006).

In summary, the research questions are as follows:

- How is students' understanding of the word "average"?
- Do students consider where average is from, a sample or a population?
- How do students deal with average of a decimal number?
- How do students control variation when finding average?

METHOD

Participants

The participants of this study were 35 students in Grade 5. They have been receiving instruction for gifted students at the Institute for Science Gifted Education attached at a university and are categorized as upper 5 percent group. We can anticipate that the students have experienced average within various context of everyday life. And they have already learned arithmetic mean in mathematics curriculum.

Questionnaire

Questions for investigation of students' understanding of average are from previous researches (Konold & Pollatsek, 2002; Watson, 2006). They were translated into Korean for presenting to students. The students completed questionnaire and were organized mini-interview through the medium of their responses on the questionnaire. Questions are as follows:

1. Explain the word 'average'. What do you think the word 'average' mean?

2. A research study found that "Korean primary school students watch an average of 3 hours of TV per day."(From Watson, 2006)

(a) What does the word 'average' mean in this sentence?

(b) How do you think they got this average (3 hours of TV per day)?

3. Let's say you are watching TV, and you hear: "On average, Korean families have 1.7 children." How can the average be 1.7, and not a counting number like 1, 2, or 3? (From Watson, 2006)

4. The number of comments made by 8 students during a class period were 0, 5, 2, 22, 3, 2, 1, and 2. What was the typical number of comments made that day? (From Konold & Pollatsek, 2002)

5. A small object was weighted on the same scale separately by nine students in a science class. The weights (in grams) recorded by each student were 6.2, 6.0, 6.0. 15.3, 6.1, 6.3, 6.2, 6.15, 6.2. What would you give as the best estimate of the actual weight of this object? (From Konold & Pollatsek, 2002)

Figure1. Questions for average

RESULTS AND DISCUSSION

Understanding of the word "average".

Table 1 shows the summary of Question 1 and 2-a. We can see that many students give different meanings to average on Question 1 and 2-a. On Question 1 for examining students' understanding of the word "average" itself, students who view average as algorithm, median, mathematical point of balance, mode, and what is reasonable are 52.9%, 29.4%, 14.7%, 5.8% and 5.8%, respectively. On Question 2-a for investigating students' understanding of average in the context of everyday life, students who view average as algorithm, mode, what is reasonable, median, and mathematical point of balance are 41.1%, 41.1%, 11.7%, 8.8%, and 2.9%, respectively.

Table 1. The summary of Question 1 and 2-a

Responses	Question 1*	Question 2-a*
Average as mode	5.8%	41.1%
Average as algorithm	52.9%	41.1%
Average as reasonable	5.8%	11.7%
Average as median	29.4%	8.8%
Average as balance	14.7%	2.9%

* The sum of percents is over 100 because there were students who gave a few meanings of average.

On Question 1, the majority of students view average as algorithm or median, and the minimum number of them view average as mode or reasonable, but on Question 2-a, the number of students who view average as mode and reasonable increased and the number of students who view average as algorithm, median, and balance decreased. Students in Korea work with the limited number of data (e.g., 15~20) when learning average, which enables them to think average as algorithm, median, and even balance. But the context on Question 2-a makes students consider a large number of data, which leads them to come up to mode and reasonable.

According to previous researches (Cobb & McClain, 2004; Bakker, 2004; Bakker, et. al, 2004; Reading & Shaughnessy, 2004) on statistical ideas such as distribution and variation, reasoning of a large number of data helps stimulate students' meaningful statistical reasoning. But it seems not to have had a great effect on reasoning related to average. Many students did not view average as median and balance on Question 2-a. We propose that students should be given opportunities dealing with a great number of data in mathematics curriculum when learning average.

Considering for being average of a sample.

Table 2 shows the summary of Question 2-b. Only 25.7% of students took a notice of that average in the context of everyday life is from a sample and 28.5% of them thought that it is average of a population. 45.8% of them focused on only the method of calculation, and did not consider from where is the average.

Table 2. The summary of Question 2-b

Responses	Noticing as average of a sample	Noticing as average of a population	Focusing on only the method of calculation
Students (%)	25.7%	28.5%	45.8%

Average is used to summarize information about a data set. Average from a sample is important since it reflects its' population. If students did not notice spontaneously that averages from the context of everyday life provided by newspaper or TV are from samples, they should be taught. In other words, statistics education should connect average and sampling for students.

Dealing with average of a decimal number.

Table 3 shows the summary of Question 3. The majority of students explained about how average become to be a decimal number using the algorithm of arithmetic mean. In other words, they did not understand average of a decimal number conceptually. For example, student 1 said, "A decimal number is possible because average is what the number of children in families are divided by the number of families". 22.8% of students provided the reason why average is a decimal number with proper interpretation. For example, student 2 said, "Average of 1.7 does not mean it is really 1.7 persons, it is an approximate estimate."

Table 3. The summary of Question 3

Responses	Using algorithm	Using proper interpretation
Students (%)	77.2%	22.8%

Table 4 shows the summary of Question 4. Data on Question 4 are the numbers of comments by 8 students. We asked the students figure out average of the data, which is a decimal number. 44.1% of the students provided average of a decimal number, but the others avoided average of a decimal number. For example, student 3 said, "I got 4.625 by summing all data and dividing it by 8. But the answer is 5 because it is impossible that the number of comments is a decimal number." Five students were categorized as "Using proper interpretation" on Question 3 and at the same time as "Avoiding a decimal number as average" on Question 4. And eleven students were belonged to "Using algorithm" on Question 3 and at the same time to "Accepting a decimal number as average" on Question 4. Only three students was in "Using proper interpretation" on Question 3 and at the same time in "Accepting a decimal number as average" on Question 4.

Table 4. The summary of Question 4

Responses	Accepting a decimal number as average	Avoiding a decimal number as average
Students (%)	44.1%	55.9%

We can conclude that there is a psychological gap between understanding average of a decimal number and accepting a decimal number as average to many students. We need to study on how to help students fill up the gap.

Controling of variation.

Table 5 shows the summary of Question 5. 32.3% of the students tried to control variation within the data set. To control the variation, students excluded an extreme value, or employed mode: student 4 said, "I got average of eight data except for 15.3. I think 15.3 was estimated inproperly because it is very different from the others."; student 5 said, "I think the answer is 6.2 because 6.2 is the most frequent." A half of them had a disregard for variation. They summed nine data, divided it by 9, and found out arithmetic mean. There were students who gave up finding average. For example, student 6 said, "I can't know it. If the weight is between 6 and 7, it is possible. But one of them is more about 9 than the others."

Table 5. The summary of Question 5

Responses		Students
Controling of variation	Excepting an extreme value	23.5%
	Employing mode	8.8%
Disregarding of variation		17.6%
Giving up of finding average		50.0%

Disadvantage of arithmetic mean is that it is likely to be affected by extreme value, which is one type of variation. When learning average, students may learn to control variation within data set if extreme values are dealt with meaningfully, which enable them to deepen their understanding of average. It can help students have a deep understanding of average and variation if average is linked to controlling variation in statistics curriculum.

CONCLUDING COMMENTS

We investigated Korean 5th graders' understanding of average and discussed the results relating to other aspects of statistical ideas. Readers should remind that the participants of this study are the gifted and are categorized as upper 5 percent group. As we see in the results of Question 1 and 2-a, students mainly view average as algorithm or mode when dealing with a large number of values and do not stimulate intuitions of median and balance. Students should be provided the experience on discussing average of a great number of data in schooling. Many students did not notice spontaneously that averages from the context of everyday life provided by newspaper or TV are from samples. Students need instruction for average linked to sample and sampling. We can find there is a psychological gap to many students between understanding average of a decimal number and accepting a decimal number as average. The more detailed study on it is needed and research on how to remove the gap from students. Most of students did not control variation within data set when finding average. Students should be taught average and variation together.

In summary, average is core idea in statistics and is connected to various aspects of data handling and statistical investigation. Statistics education in schooling should consider it and reflect it to mathematics curriculum.

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AN ONLINE ICT ENVIRONMENT TO SUPPORT PRIMARY SCHOOL STUDENTS' SOLVING OF NON-ROUTINE PUZZLE-LIKE MATHEMATICAL WORD PROBLEMS

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The paper addresses a quasi-experimental study on problem solving by primary school students. In the study, problem solving is understood as solving non-routine puzzle-like word problems which require dealing with multiple, interrelated variables. It is investigated whether an online ICT environment in which the students can gain experience with covariating variables in a game context can contribute to the students' problem-solving achievement. The effectiveness of the ICT environment is examined with a pre-test-post-test control group design, which includes 195 sixth-graders from ten schools. The data analysis showed that the online learning environment had a significant positive effect on students' problem-solving performance. In addition, we found a relation between writing down the solution procedure and problem-solving success.

BACKGROUND OF THE STUDY

In 2004, the POPO project was set up with the purpose of learning more about non-routine mathematical problem solving by high-achieving Dutch primary school students. A paper-and-pencil test consisting of non-routine puzzle-like word problems revealed that, despite their high scores on a general mathematics test, these students did not demonstrate high performance in problem solving (Van den Heuvel-Panhuizen & Bodin, 2004). After realizing that even very able students encountered difficulties with non-routine problems, we wondered which learning environment could help primary-school students to improve their problem solving performance and especially their ability to solve non-routine word problems that entail dealing with multiple, interrelated variables. Moreover, even though the students' scribbles on the scrap paper gave us important information about their solution strategies, we were left with questions about their solution processes.

In the iPOPO study we attempt to answer these questions with the help of ICT: the students are offered an applet that might support their learning of problem solving, and by connecting this applet to special software that registers students' actions, we are able to monitor the students' problem-solving processes. Before the study that is reported in this paper, we carried out a pilot study with twelve high-achieving fourth-graders from two schools. The students worked in pairs with the applet for 30 minutes

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 385-392. Thessaloniki, Greece: PME.

following a particular scenario. A paper-and-pencil test on non-routine problem solving that was administered to all students before and after working with the applet showed that in one of the schools the students' performance improved in the post-test. With respect to the strategies used in the ICT environment, the collected data showed that the students used the applet as a tool to reason about the relationships between the variables and to describe these relationships in general terms.

THEORETICAL FRAMEWORK

Problem solving

Problem solving is a primary goal of mathematics teaching and learning and is considered to be the essence of mathematics (NCTM, 2000). However, due to various reported deficiencies (Verschaffel et al., 1999) students fail in solving problems typically defined as non-routine (Kantowski, 1977) and teachers encounter difficulties in supporting the development of the students' problem-solving competency. Despite the growing body of research literature in the area (Lesh & Zawojewski, 2007; Schoenfeld 1992; Schoenfeld, 1985), there is still much that we do not know about how students attempt to tackle mathematical problems and how to support students especially in solving non-routine problems. These are problems that require higher-order thinking and go beyond standard procedural skills (Kantowski, 1977). This means that non-routine problems cannot be solved by directly applying a fixed procedure or algorithm, but require both a profound analysis of the problem situation and strategic thinking. The answers to these problems cannot be found in a straightforward way, and entail dealing with multiple, interrelated variables. Therefore they may offer a worthwhile experiential base for bridging arithmetic and algebraic thinking.

In fact, students' informal strategies and notation when solving such problems can provide important entry points for learning algebra. According to Carraher and Schliemann (2007, p. 690) "problem contexts [can] constitute essential ways to situate and deepen the learning of mathematics and generalizations about quantities and numbers". Moreover, contexts pose the challenge of generating abstract knowledge about mathematical objects and structure from experience and reasoning in particular situations (ibid). However, the shift from arithmetic to algebraic thinking requires "engaging students in specially designed activities, so that they can begin to note, articulate, and represent the general patterns they see among variables" (Schliemann et al., 2003, p. 128). Word problems that involve two variables with one degree of freedom have the potential to support the transition from arithmetic to algebra (Sadovsky & Sessa, 2005).

ICT as a tool to facilitate the learning of problem solving

ICT has proved to be a very powerful tool in developing reasoning abilities applied to strategic thinking and problem solving (Bottino et al., 2006). What is more interesting about these tools though, is that they can be used not only at school, but also in extracurricular and home activities (ibid).

Furthermore, the "computer's capacity for simulation, dynamically linked notations, and interactivity" (Rochelle et al., 2000, p. 86) can provide a powerful environment for the development of sophisticated concepts. ICT can provide students with unique opportunities to experience the covariation of variables in a meaningful context and to describe and represent the relations among them. Through manipulating the variables in a dynamic computer environment, students are actively engaged in problem solving, which may increase their motivation and enhance their learning of mathematics (Brown et al., 2005). Thus the use of ICT can foster active learning and higher-order thinking (Smeets, 2004). Following the activity principle of Realistic Mathematics Education (Van den Heuvel-Panhuizen, 2001), we believe that ICT can provide students with a learning environment in which they will be able "to construct mathematical insights and tools by themselves" (ibid. p. 55).

Writing-down strategy information and success in problem solving

Writing in mathematics supports students' thinking because "it requires them to reflect on their work and clarify their thoughts" (NCTM, 2000, p. 61). According to Pugalee (2004), the awareness and self-regulation that writing promotes "appears to play an important role in students' selection of appropriate information and strategies while solving a wide range of mathematical problems" (p. 28). On the other hand, research has substantiated the tendency of Dutch primary students not to write anything down when solving non-routine word problems (Van den Heuvel-Panhuizen & Bodin, 2004). This characteristic might be rooted in teachers' and students' beliefs on what counts as a legitimate solution to a problem and what mathematics is in general (Schoenfeld, 1992). By not writing down their solution process, Dutch students might miss a fundamental opportunity to develop their problem-solving competency.

Research questions

The present study aimed to answer the following two questions:

1. Does the online learning environment have a significant positive influence on students' performance in the solution of mathematical non-routine problems?
2. Is there a relation between writing down the problem-solution procedure and problem-solving success?

METHOD

The online ICT environment

The environment that is developed to give students experience in dealing with multiple, interrelated variables consists of the dynamic Java applet called *Hit the target*. The screen displays a target, a score board, a board where the student can fill in the game rule and a board where the student determines the number of hits and missed shots. The features of the applet are dynamically linked. During the shooting the total score on the scoreboard changes according to the number of arrows shot and the game rule. This means that the applet provides instant feedback to the students.

The students involved

The study includes 195 sixth-grade students from ten primary schools in Utrecht. The students of the five experimental schools were invited to log into a website and play *Hit the Target*. The students could visit the website outside of school during a three-week period. In this period they also received three sets of instruction in school, where they were given questions they had to answer using the applet. These varied from finding the pair of hits and missed shots that produce a particular score to generating a general rule by systematizing all solution pairs. During the class sessions the students' answers were discussed and emphasis was placed on determining rules and regularities. Special software was used to keep track of the students' work in the ICT environment. In a following paper we will report on that.

Pre-test and post-test

The pre-test is a paper-and-pencil test consisting of seven non-routine puzzle-like word problems. The pre-test contains seven word problems which are also included in the post-test. In addition to these, the post-test contains three more word problems and three bare-number problems, structurally related to the pre-test problems. All problems require that the students deal with interrelated variables. The test sheets contain a work area where the students had to show how they found their answers. The students' responses were coded according to the correctness of the answer and the presence of a solution process. However, in this study we only include the students' responses to the seven items of the post-test that are identical to those in the pre-test and not the whole range of the 13 post-test items. The Cronbach's alphas for the seven items of the pre-test was 0.79 and that of the post-test was 0.86, which means that the internal homogeneity of both tests is good.

RESULTS

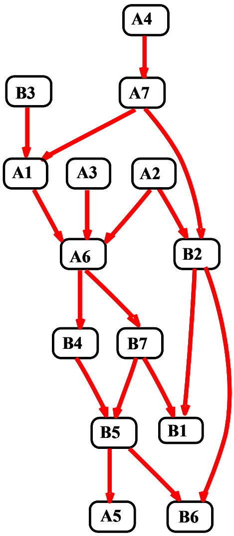
The means and standard deviations of the students from the experimental (n=104) and the control group (n=91) on the pre-test and the post-test are shown in Table 1.

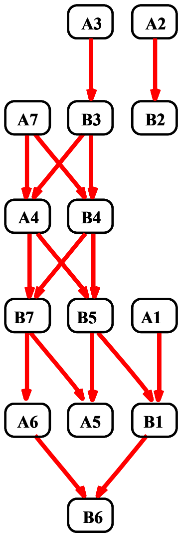
Group	Pre-test scores in % (7 items)		Post-testscores in % (7 items)	
	M	SD	M	SD
Experimental group	36	32	47	36
Control group	27	28	29	33

Table 1: Means and standard deviations of the experimental and the control group on the pre-test and the post-test

To test whether there is an effect of the treatment, the ANOVA test for repeated measures was used. This revealed a significant interaction effect between treatment (online learning environment, no online environment) and time (before the treatment, i.e. pre-test; after the treatment, i.e. post-test) ($F(1,193)=10.13$, $p<0.01$, eta square =0.05).

The implicative analysis (Gras, 1992) enabled us to move beyond the comparison of plain scores. Figures 1 and 2 present the implicative relations of the responses to the problems of the pre-test and the post-test by the students of the control and the experimental group, respectively. Figure 1 (control group) involves mainly implicative relations of the form A*i* →B*i*. That is, students who succeeded in solving a problem in the pre-test could provide a correct solution to the identical problem in the post-test. This suggests that their initial performance on the problems played a determining role in their later problem-solving performance. Figure 2 (experimental group) shows that there are more implicative relations between different tasks of the pre-test and the post-test. This reveals that the pattern of responses in the experimental group shows more coherence than in the control group. Furthermore, the responses of the students in the experimental group for most of the tasks of the pre-test are at the upper end of the implicative "spectrum", while the responses to almost all the problems of the post-test are at the lower end of the "spectrum". In other words, it can be inferred that this change in pattern may be a result of the treatment.





A1-A7= Responses to the problems of the pre-test
B1-B7= Written solution procedures to the problems of the pre-test

Figure 1: Implicative diagram of the responses to the problems of the pre-test and the post-test (control group)

Figure 2: Implicative diagram of the responses to the problems of the pre-test and the post-test (experimental group)

To explore the relationship between writing down the problem-solution procedure and problem-solving success, we carried out a correlation analysis and a hierarchical analysis. Overall, students' writing down the solution procedure was found to correlate significantly with their problem-solving performance on the pre-test (r=0.28, p<0.01). In the experimental group we found in the post-test data a correlation between writing down and success of 0.43. When we controlled for the effects of writing down and success in the pre-test, the partial correlation was 0.33. In the control group we found in the post-test data a correlation between writing down and success of only 0.15 and the partial correlation was 0.07, which was even smaller.

To explore the consistency with which students wrote down the solution procedure and provided a correct answer to each of the problems, we carried out the hierarchical clustering of the corresponding variables (Lerman, 1981). The similarity diagram of the students' responses and the written solution procedures to the problems of the pre-test by all students revealed relationships between writing down and problem-solving success in certain problems (Figure 3).

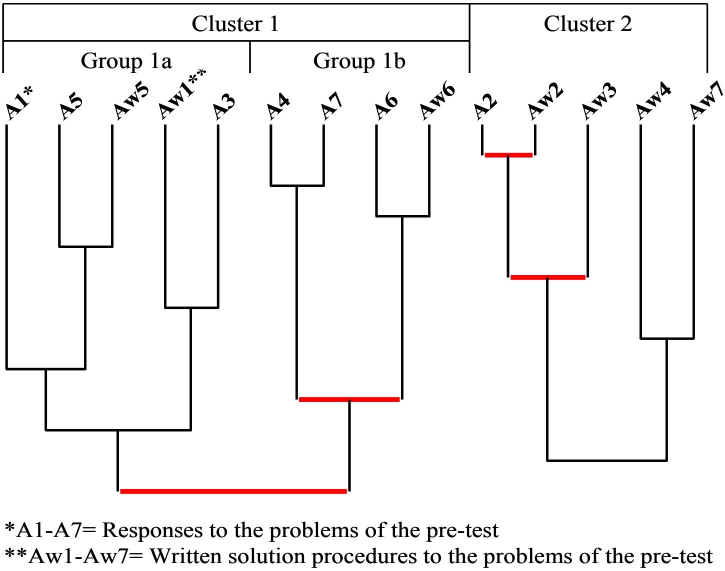


Figure 3: Similarity diagram of the responses to the problems and the written solution procedures at the pre-test by all students

As can be seen in Figure 3, we found two separate similarity clusters. Cluster 1 involves all the variables corresponding to students' success and written solution procedures to the problems A1, A5, and A6, while Cluster 2 comprises mainly of the variables standing for writing down the solution procedure for problems A2, A3, A4, and A7. Cluster 1 consists of two groups. Group 1a comprises mainly of the variables concerning problems A1 and A5. Group 1b, which is weakly linked to group 1a, includes the variables representing success for the problems A4 and A7 and the variables for problem A6. The structure of the similarity diagram suggests that, solving the problems successfully and reporting the solution procedure are strongly connected for some problems (A2, A5, and A6), but not for others. Moreover, problems A4 and A7 are connected to each other, for both success and the presence of a written solution. This means that the students treat problems which share the same context in a consistent way. It seems that problem-solving success and writing down are positively related, but that this relation is not present in all problems.

DISCUSSION

In this study, which is part of a larger project involving students from grades 4, 5 and 6, our main question concerns the effect of an online ICT environment on the students' performance in non-routine puzzle-like word problems. Additionally, we examined the relationship between writing down the solution procedure and problem-solving success.

The ICT environment had a significant positive effect on problem-solving performance, but the effect size was rather small. The implicative analysis showed that in the experimental group there are more and different implicative relations between the tasks of the pre-test and the post-test than in the control group. Furthermore, we found a significant positive correlation between writing down the solution and problem-solving success. This finding supports to a certain degree the idea that reflection as indicated by writing down the solution procedure helps children to find a correct answer. However, the relation was rather weak which might be the result of our coding. An incomplete or wrong solution procedure, even a single written trace of a reasoning process was coded as writing down a solution procedure.

Furthermore, we have to be aware that the students' tendency not to write down their solution is a deep-seated characteristic related to their beliefs and attitudes towards problem solving. Students very often think that what counts in mathematics is finding the right answer quickly and by mental calculations. Consequently, problems which require more than a straightforward solution can evoke great discomfort or even aversion in these students. We do not know to what extent this occurred in this study but it is certainly an important issue for further research.

Additionally, in this study we assume that all the students of the experimental group worked in the online environment. This may not be true, because students' participation in the study was encouraged, but remained voluntary. This means that there might be students that did not log into the online environment at all, or did not follow our instructions. Further analysis of the students' log data is needed to investigate the use of the online environment and how this specifically influenced the students' problem solving ability.

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PUPILS' EXPLAINING PROCESS WITH MANIPULATIVE OBJECTS

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The purpose of this study is to analyze how pupils can explain why a statement is true by utilizing manipulative objects. A pair of 5th graders was asked to conjecture statements on properties of some two-digit additions and explain them with coins. As a result, they were able to invent explanations of why the statement was true because they intentionally continued to coordinate their enactive and symbolic representations. Besides, the mathematical structure of coins facilitated pupils' coordination of the representations. Pupils' activity is also discussed in terms of mathematical understanding, and some important implications for teaching are mentioned.

INTRODUCTION

Proof and proving are at the heart of mathematics and should therefore be learned from elementary school mathematics (Stylianides, 2007). Though the main function of proof in mathematics education is to explain why a statement is true (Hanna, 1995), some researches show that it is difficult for pupils to explain it. According to Monoyiou, Xistouri & Philippou (2006), few 5th and 6th graders (only 3.4 %) could describe the general reason of "the sum of two odd numbers is even". Stylianides (2007) also reports a class where a 3rd grader explained why the above statement was true, but some pupils were not convinced by her explanation, and objected to her explanation.

However, it may be that pupils can express deductive reasoning better if they are allowed to utilize manipulative objects. Based on Piaget's researches (cf. Piaget, 1953), Semadeni (1984) points out that it is difficult for concrete-operational children to reason deductively using only words and symbols. And as primary-school proofs, he proposes "action proofs", which consist of performing "certain concrete, physical actions (manipulating objects, drawing pictures, moving the body etc.)" (Semadeni, 1984, p. 32). In addition, Miyazaki (1992) shows the process by which a 6th grader proves the generality of his conjecture with manipulative objects. Perhaps, pupils would be also able to explain more easily why a statement is true utilizing manipulative objects. Therefore, the focus of this study is to analyze how pupils can explain why a statement is true by utilizing manipulative objects.

THEORETICAL FRAMEWORK

Though there are many functions of proof in mathematics, their function in mathematics education is primarily to explain why a statement is true. De Villiers (1990) points out five functions of proof in mathematics: verification, explanation, systematisation, discovery, and communication. According to Hanna (1995),

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 393-400. Thessaloniki, Greece: PME.

explanation is very important function in mathematics education, because we can deepen mathematical understanding by explanation. It is therefore valuable for pupils to learn to explain why a statement is true. Moreover, as Mancosu (2001) shows, the nature of explanation continues to be a "hot topic" in the philosophy of mathematics.

Pupils will feel the need for explaining if they conjecture statements by themselves and feel some surprise or doubt from the conjecture. For example, Hadas & Hershkowitz (1998) conclude, from students' activities in which dynamic geometry software was used, that the need for explanation was heightened by "a surprise caused by the contradiction between the conjectures and what students got (or could not get) while working the dynamic tool" (p. 32). Hence, the design of tasks which cause "surprise" is important for motivating pupils' explaining.

This explanation can be represented by not only formal proofs (written in mathematical language or symbols) but also informal proofs (in which one also uses manipulative objects or diagrams). Hanna (1990) distinguishes between "proofs that explain" and "proofs that prove", and shows a geometric proof of the sum of consecutive natural numbers as an example of the former. Miyazaki (1995) also argues that it is possible to explain by proofs based on concrete actions, and discusses what students should do in order to do that. Further, according to Piaget (1953) or Semadeni (1984), concrete-operational children are able to perform deductive reasoning better with manipulative objects. Therefore, if pupils are allowed to access manipulative objects, it should become easier for them to explain why a statement is true.

When one utilizes manipulative objects, thinking explanation through relating enactive and symbolic representations becomes important in terms of deepening mathematical understanding. Hiebert & Carpenter (1992) state that a "mathematical idea, procedure or fact is understood ... if its mental representation is part of a network of representations" (p. 67). They argue that one way of constructing such a network is by making connections between different representation forms of the same mathematical idea. However, relating enactive and symbolic representations in proving has not been examined so far. For example, in Miyazaki (1992, 1995), pupils seek proofs only in enactive representation. Miyazaki (2000), who deals with transition from proofs based on concrete actions to algebraic proofs, also discusses one-way relation, that is, translation of enactive into symbolic representations. It is therefore valuable and necessary to analyze pupils' explaining process from the viewpoint of whether and how they relate enactive and symbolic representations.

METHOD

A pair of 5th graders, Hina and Yui (pseudonyms), who belong the same class in a public elementary school was interviewed. The reason why the pair was chosen is that, according to their teacher, they can communicate with each other actively. Further, in the interview they had access to coins, papers and to only one pencil, because Balacheff (1988) suggests that pupils are likely to express their thinking more naturally in such environment. Their teacher said that their grades in Mathematics are Hina (excellent) and Yui (average), and that they have little experience with explanation. He also told that they do not often utilize concrete materials in their learning of mathematics; for example, they rarely use manipulative objects in considering methods of calculations and sometime use papers in learning geometrical figures.

First, they were asked to calculate "the sum of a two-digit number and the number where the original number digits are reversed" (for example, 32+23=55) in some cases, and conjecture statements from the results. Next, they were given coins (Japanese 100-yen, 10-yen and 1-yen coins), and asked if they could explain why the statements were true by utilizing the coins. The interviewer (the author) intended to intervene only if the two pupils seemed to reach a deadlock or if their thinking was ambiguous. The interview was recorded, and the transcripts were analyzed through the video recode.

RESULTS

On the above two-digit additions, Hina and Yui conjectured two statements. The first one (statement A) was that "(when the sum remains a two-digit number) the tens digit is equal with the ones digit [1)]" (for example, 32+23=55). The second one (statement B) was that "when carrying occurs, the sum of the hundreds digit and the ones digit is equal to the tens digit" (for example, 97+79=176, 1+6=7). Because the two pupils felt more "surprise" at statement B, their enactive and symbolic explanations of why statement B is true will be the focus of what follows.

Grasping the Statement with Manipulative Objects

Firstly, Hina and Yui placed 10-yen and 1-yen coins as Procedure 1 of Figure 1 in order to represent 39+93. At that time, they placed 10-yen coins and 1-yen coins on the left row and the right row respectively with consciousness of forms of "calculations with figures". For, when they previously examined the reason why statement A was true, Hina said "we did in the case of calculations with figures, so we tried like that", and they arranged coins on 23+32 in the same form as Proc. 1 of Fig. 1.

Next, Hina said "after all, it was same as before" and confirmed that the number of 10-yen coins was equal with the number of 1-yen coins, referring to the case of statement A and the commutative law of addition. She then stated that carrying occurs when there were more than ten coins, and they rearranged coins as Proc. 2 of Fig. 1. After that, Hina said "oh, the opposite, this, this comes to hundreds" and moved a set of 10-yen coins from the center row to the left row (from Proc. 2 to Proc. 3 of Fig. 1). They would represent this set as a hundred (but actually there were only nine 10-yen coins in this set). Thus, when they moved the set at this time, they also seemed to be conscious of "positional notation".

Moreover, they exchanged ten 10-yen coins for a 100-yen coin (Proc. 4 of Fig. 1). They then regarded a set of ten 1-yen coins as equivalent to a 10-yen coin, and verified that the sum of this set and two 10-yen coins was equal with the sum of a 100-yen coin and two 1-yen coins (both were three). However, as they said "why..., um...", they felt embarrassed and could not find the reason why the two sums became equal.

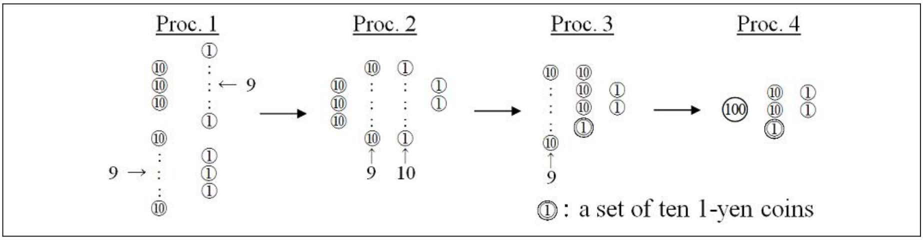

Fig. 1: The process in which the pupils arranged coins on 39+93

Explaining why the Statement is True

After Hina and Yui arrived at the arrangement shown in Proc. 4 of Fig. 1, they broke the arrangement and similarly rearranged coins on 48+84 as Fig. 2. Hina then referred to the arrangement and calculated 48+84 with figures, through explicitly separating the tens and ones places (the left-hand side in Fig. 3). According to their teacher, they do not usually write such forms in calculations with figures. This novel approach is essential to their explanation.

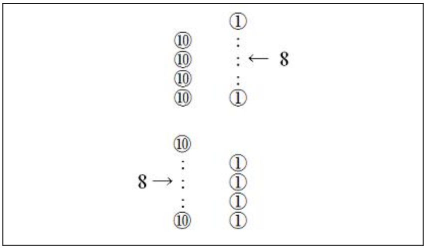

Fig. 2: Pupils' arrangement of coins

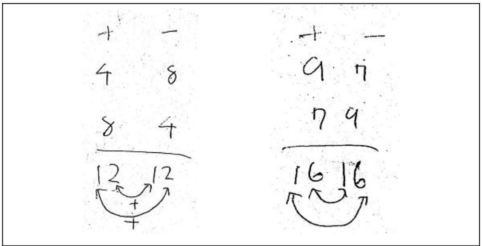

Fig. 3: Pupils' calculations with figures [2)]

Next, they added arrows to the calculation like the left-hand side in Fig. 3, and found that the sum of 2 of 12 in the tens place and 1 of 12 in the ones place was equal with the sum of 1 of 12 in the tens place and 2 of 12 in the ones place. After that, Hina proposed to consider other examples, and they verified that they could apply the same thinking to the case of 97+79 (the right-hand side in Fig. 3). Further, they regarded 97+79 as "a generic example" (Balacheff, 1988) of all two-digit additions, and described their explanation of the general reason why the statement B was true (Fig. 4).

It is clear that pupils' novel reconstruction of the form of the calculation with figures contributed to their invention of such explanation. Moreover, they could produce this novel symbolic reconstruction because they have consciously related positional notation or their calculations with figures (symbolic representation) and their arrangement of the coins (enactive representation). For example, their attention to calculations with figures led them to place 10-yen coins and 1-yen coins on the left row and the right row respectively in Fig. 2. Then, referring to the arrangement of coins, they calculated 48+84 by explicitly separating those in the tens and ones places (Fig. 3). Furthermore, their symbolic reconstruction is precisely matched by their insightful re-positioning of the coins shown in the move from Proc. 1 to Proc. 4 in Fig. 1 although they did not mention this correspondence explicitly.

くり上がった場合，十の位は，一の位のくり上がった数
と十の位のくり上がってない数をたしたもので，
百の位と一の位をたしたものと順番だけいれかえた
ものなので，答えは同じになるから。

When carrying occurs, the tens digit is the sum of one carried from the ones place and the number left in the tens place. Because the sum is reverse to the sum of the hundreds digit and the ones digit [3], the two sums become equal.

Fig. 4: Pupils' explanation why the statement B is true [4]

Explaining why the Statement is True with Manipulative Objects

After the two pupils described their explanation in Fig. 4, the interviewer asked them, "How about using coins?". They placed coins on 84+48 as Proc. 1 of Fig. 5, and told that both the numbers of 10-yen and 1-yen coins were twelve. They then transformed the arrangement from Proc. 1 to Proc. 5 of Fig. 5, and stated that the sum of 10-yen coins left (two) and a set of ten 1-yen coins represented the tens digit. Moreover, as the reason why the sum was equal with the sum of a set of ten 10-yen coins and 1-yen coins left (two), they mentioned three facts; both the numbers of 10-yen and 1-yen coins left were equal (two); both the numbers of a set of ten 10-yen coins and a set of ten 1-yen coins were equal (one); and they could therefore apply the commutative law of addition. At this point, they regarded a set of ten 10-yen coins and a set of ten 1-yen coins as a 100-yen coin and a 10-yen coin respectively, and explained why the sum of the numbers of 100-yen and 1-yen coins was equal with the number of 10-yen coins. Further, the three facts matched with the previous symbolic explanation.

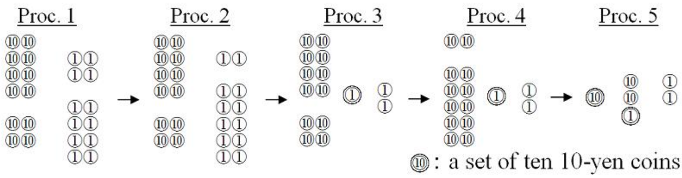

Fig. 5: The process in which the pupils arranged coins on 84+48

Then, the interviewer asked them whether they could apply the same thinking to all other cases and, if so, why they could apply. Hina answered, referring to the calculation of 97+79 in Fig. 3, "in other cases, only the order is reversed too, so I think the same thinking can be applied". Her answer seemed to mean that the above three facts hold true in all cases. Therefore, when Hina considered the arrangement of coins on 48+84 (Fig. 5), she would regard the coins as a generic example of coins which represented all two-digit additions, and show the general reason why the sum of the numbers of 100-yen and 1-yen coins became equal with the number of 10-yen coins, with awareness of the previous symbolic explanation.

DISCUSSION

Unlike other studies on pupils' explanation with manipulative objects (Miyazaki, 1992, 1995, 2000), Hina and Yui could explain why the statement B was true by shifting from an enactive representation to a symbolic representation, and vice versa, many times. For example, they grasped the statement B, which they conjectured in symbolic representation, as "the sum of the numbers of 100-yen and 1-yen coins is equal with the number of 10-yen coins" in enactive representation. At that point, they arranged coins with consciousness of calculations with figures and positional notation in symbolic representation. On the other hand, referring to the arrangement of coins, they explicitly separated the tens and ones places in the calculations with figures, and could therefore invent symbolic explanation of why the statement B was true. Moreover, when they produced enactive explanation with coins, the enactive explanation corresponded to the previous symbolic explanation (for instance, the sum of 10-yen coins left and a set of ten 1-yen coins represented tens digit.)

One of the most important reasons for the structure and fluency of their explanations was that they continued to coordinate their enactive and symbolic representations quite explicitly. As Hiebert & Carpenter (1992) discuss, relating various representations leads to deepen mathematical understanding. However, Duval (2006), who discusses mathematical comprehension in terms of registers of semiotic representations, points out that if there is no structural correspondence between two representations, translation "is for many students an impassable barrier in their mathematics comprehension and therefore for their learning" (p. 123). The two pupils in this study transformed the arrangement of coins in order to match it with positional notation, and, in turn, invented novel notations in calculations with figures to match the arrangement of coins. In this way, they intentionally continued to coordinate two representations through transforming the content of one representation in order to match it with that of the other representation. Because of their coordination, they would be able to explain why the statement B was true and deepen their understanding.

Another reason is that the mathematical structure of the coins used (the ratio of a 10-yen and a 1-yen coin is 10 : 1) matches the structure of the two-digit numbers. In this study, Hina and Yui conjectured statements on digits in the decimal notation system. Japanese 100-yen, 10-yen and 1-yen coins can be used to match the decimal notation system, *and* the numbers of coins correspond to digits in the system. It seemed that this structural correspondence had strong effect on pupils' coordination of enactive and symbolic representations. Conversely, if 50-yen and 5-yen coins had been utilized as well, or if 10-yen coins had been not available, such environments might have been obstacles for pupils' coordinating and explaining. Therefore, such promising results appear to be possible only if the structure of the manipulative objects is able to match the structure of the mathematical activities that pupils are engaged in.

From the result of this study, we can get two implications for teaching. Firstly, when pupils try to explain with manipulative objects why a statement is true, teachers should look for ways to encourage them to coordinate enactive and symbolic representations.

This coordination would promote pupils' explaining and deepen their mathematical understanding. Secondly, if possible, teachers should prepare manipulative objects whose mathematical structure matches with expected pupils activities. Such manipulative objects could encourage pupils' coordinating and explaining.

CONCLUDING REMARKS

Starting with an enactive model which they were able to reconstruct verbally and symbolically, a pair of 5th graders in this study could explain symbolically why a mathematical statement was true. Subsequently, they were able to translate their symbolic explanation by reference to variants of their enactive models. This coordination was valuable in terms of mathematical understanding, and was facilitated by the mathematical structure of manipulative objects. These findings were influenced by the particular subject matter and the choice of manipulative objects. It is therefore necessary to examine these findings more thoroughly by interviewing other pupils and/or using other subject matters and manipulative objects.

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Footnotes

1) Parenthesis by the author. Pupils' Japanese was translated to English by him.

2) "+" and "-" at the top of the calculation are Japanese characters. "+" means the tens place and "-" means the ones place, respectively.

3) If this clause is literally interpreted, the orders of the two sums are not reverse but the same. Though the two pupils intended to describe the reason why the commutative law of addition could be applied, they seemed not to care the orders in writing this clause.

4) Before writing this sentence, the two pupils stated that if the tens and ones place were calculated separately, they could get the same two two-digit numbers because of the commutative law of addition, for example 9+7=7+9=16.

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TOWARDS A COMPREHENSIVE FRAMEWORK OF MATHEMATICAL PROBLEM POSING

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This theoretical essay presents a consolidated theoretical framework for analysing mathematical problem posing. The main feature of the suggested framework is that it builds upon broadly accepted problem solving models, and, simultaneously, includes theoretical constructs that are identified as specific for problem posing. The framework consists of four facets: resources, problem posing heuristics, aptness and social context in which problem posing occurs. The framework contributes to the existing research literature by consolidating findings on particular aspects of problem posing that have been explored so far and suggests a research agenda for further advancing the field.

FOCUS OF THE PAPER

It is widely acknowledged that problem solving and problem posing are central activities in doing and learning mathematics (e.g., Schoenfeld, 1985; Silver, 1994). During several decades, these activities attract keen attention of mathematics education research community. As to problem solving, there are several comprehensive frameworks that account for the complexity of the phenomenon in different ways (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985, 1992). These models postulate existence of relatively well-defined problem solving phases and attributes and describe how they come to cohere. As to problem posing, there are many studies that explore in depth its particular aspects (cf. Christou et al., 2005). For instance, one branch of research deals with problem posing abilities and processes (e.g., Christou et al., 2005; Harel, Koichu & Manaster, 2006); another – with learning opportunities inherited in posing problems (e.g., Crespo & Sinclair, 2008; Lowrie, 2002). There are also studies utilizing problem posing as a window in students' and teachers' understanding of mathematics (e.g., English, 2003; Mestre, 2002; Silver et al. 1996) and studies exploring particular problem posing strategies (e.g., Brown & Walter, 1983).

With the growing number of empirical and theoretical studies, research on problem posing currently enters the stage in which research on problem solving and mathematical thinking was about two decades ago. In 1992, Schoenfeld characterized that stage as the one in which "[the identified categories] provide a coherent and relatively comprehensive near decomposition of mathematical thinking... But the research community understands little about the interactions among the categories, and less about how they come to cohere." (Schoenfeld, 1992, p. 361). Christou et al. (2005) made a similar comment about the state of the art in contemporary research on problem posing. In their words,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 401-408. Thessaloniki, Greece: PME.

"Notwithstanding the extent of research into students' thinking in problem posing ... [research] does not provide the kind of coherent picture of students' problem posing thinking that is desirable for current approaches to instruction." (p. 151). Thus, there is the emerging need for a framework (or frameworks) that would consolidate our knowledge on particular aspects of problem posing and suggest an adequate research agenda for further advancing the field.

The goal of this paper is to present and theoretically substantiate such a framework. The main feature of the suggested framework is that it builds upon broadly accepted problem solving models, and, simultaneously, includes theoretical constructs that are identified as specific for problem posing. We argue that the framework is applicable to a wide range of mathematical domains, different types of problem posing tasks and social settings in which problem posing occurs.

THEORETICAL BACKGROUND

Presentation of the suggested framework requires elaborating on two interrelated themes: (i) ubiquity of problem solving (PS) processes in problem posing (PP), and (ii) applicability of PS models in modelling PP.

Ubiquity of PS processes in PP

Close connections between PP and PS skills and performance are recognized by many scholars (e.g., Christou et al., 2005; Silver et al., 1996; Mestre, 2002). Despite this recognition, PP and PS are frequently considered as complementary yet different in nature activities that should be modelled within separate conceptual frameworks. Three examples below show that separating the conceptual frameworks of PS and PP is not always justifiable.

<u>First example</u>: Silver et al. (1996) pointed out that such PP processes as *reformulating*, *conjecturing* and *question generating* are ubiquitous in PS (see also Silver, 1994). To empirically study the relationships between PP and PS, Silver et al. (1996) introduced, individually or in pairs, 71 mathematics teachers to a set of conditions by which a billiard ball can be shot from one corner of a rectangular table and end at a particular pocket. They asked the subjects to write down as many questions appropriate to the task as they could. Following this, the teachers were asked to solve some of their own problems and then to generate additional ones. Silver et al. had in their possession only written responses of the subjects. They classified them as "questions," "conjectures" and "statements not appropriate for the task" (p. 298), and then excluded the latter group of responses from the further analysis. The analysis of two former groups of responses enabled the researchers to conjecture that the processes of *analogical reasoning, random goals generating, constraint manipulating, chaining* and *systematic variation* are likely to be involved in the subjects' PP in the chosen context. It is interesting to note here that at least some of these processes are not exclusive for PP as they are also involved in PS (cf. Mestre, 2002).

Second example: More recently, Christou et al. (2005) suggested an empirical taxonomy of processes involved in processing quantitative information in PP. The taxonomy consists of: *editing* and *selecting* quantitative information, *comprehending* quantitative information by giving it meaning, and *translating* quantitative information from one form to another. Christou et al. (2005) justified the existence of the four processes by factor analysis of the responses of 143 middle school students to a set of PP tasks, where every task was designed so that it would correspond to only one process.

Similarly to the previous example, one can argue that the Christou et al.'s PP taxonomy is not extrinsic to some PS taxonomies. For instance, Sternberg & Davisdon (1983) have identified three major cognitive components of insight PS. The components are: *encoding* the given information, selective *combination* of the encoded information into an integrated solution path, and concurrent *comparison* of the solution path with possible solution structures attained in past learning. Gorodetsky & Klavir (2003) extended the Sternberg & Davisdon's (1983) taxonomy by incorporating two additional components: *retrieval* and *goal directedness*. *Retrieval*, in their terms, refers to the activation of concepts and procedures that enable the interpretation of a given problem, and *goal directedness* is associated with continuous self-regulating and tuning during PS. It is reasonable to consider that some of these five components can come into play also in many PP situations.

Third example: Harel, Koichu & Manaster (2006) reported a study, in which 24 in-service mathematics teachers were asked to think aloud while making up a story problem whose solution may be found by division of two given fractions. Mixed qualitative and quantitative analysis of the interview protocols revealed two kinds of ways of thinking that seem to be responsible for the resulting problems posed by the subjects, namely, *coordinating* and *utilizing reference points* of four types. The researchers found that success in doing the chosen PP task was associated with coordinated approach and utilizing a particular reference point. Once more, we would like to make a point that *coordinating* and *utilizing reference points* are not exclusive for PP as these strategies manifest themselves also in various PS contexts.

Applicability of PS models in modelling PP

We now extend the presented arguments and argue that PP can be seen as a special case of PS, and moreover, that it is natural to look at PP through the PS lenses.

Kilpatrick (1985) defines PS as a process of moving from a given state to a goal state, when a motivated person is engaged. NCTM (2000) defines PS as "engaging in a task for which the solution method is not known in advance" (p. 52). In sum, a PS task is a task including a given state and a goal state, for which the method of achieving a goal state is not known in advance by the person(s) engaged in it. Let us also recall that PP is frequently defined as engaging a person in a task which goal is to generate a new problem from a given set of conditions (e.g., Silver, 1994).

Consistently with these definitions, PP is a special case of PS in the following meaning. A given state is a PP stimulus, i.e., some set of conditions and instructions from which PP process is to be started. A goal state is formulation of a new, at least for the posers, problem that satisfies some pre-defined specifications. For example, it should be interesting for a particular category of students (e.g., Lowrie, 2002), or its solution should involve a particular operation or technique (e.g., Harel, Koichu & Manaster, 2006). Finally, the methods of formulating new problems are, as a rule, not known in advance to problem posers. This is true, at least, in a school setting (e.g., Silver et al., 1996; Christou et al., 2005).

The above chain of arguments has an important implication. Namely, it advocates considering PS models as a natural source for modelling PP. Specifically, we chose to focus on a broadly accepted model suggested by Schoenfeld (1985; 1992) and refined by Carlson & Bloom (2005) and other scholars. Schoenfeld (1985; 1992) suggested characterizing PS in terms of five categories of attributes: *the knowledge base*, *PS strategies*, *monitoring and control*, *beliefs and affect*, and *practices*. These attributes appear in most of the recent models of PS, sometimes in a slightly modified form. For instance, Carlson & Bloom's (2005) suggested a comprehensive model of expert PS that shows how *resources*, *heuristics*, *affect* and *monitoring* come to play at different PS phases. To our knowledge, these attributes are still not adapted, at least explicitly, as parts of a comprehensive model for characterizing PP.

THE SUGGESTED FRAMEWORK

The suggested framework for characterizing PP consists of four facets: *resources*, *heuristics*, *aptness* and *social context* in which problem posing occurs. Thus, the framework concerns both processes and products of one's PP.

PP Resources

We consider *resources* of three types. First, mathematical knowledge base (in sense of Schoenfeld, 1992), including: mathematical facts, definitions, algorithmic procedures, routine procedures, and relevant competencies of mathematical discourse. Second, PS competences, including: PS schemes, PS heuristics and classes (cf. *retrieval* and *comparison* processes mentioned above). The third resource is a stimulus for PP. Following Stoyanova (1998), we distinguish three main kinds of stimulus: a request to pose new problem(s) based on a solved problem, a request to pose questions based on a given story or set of conditions, a request to pose a problem with no constraints about its content, where the posed problem would be interesting for a particular category of problem solvers.

PP Heuristics

We refer to a PP heuristics as a systematic approach to question generating based on encoding, comprehending, selective analysis or translating a PP stimulus. Some general PS heuristics, like *generalization, problem decomposition* or *creating a model* (Schoenfeld, 1985), are also PP heuristics. Heuristics that come from PP research include: *systematic variation*, (Silver et al., 1996), *what-if-notting* (Brown & Walter, 1983), *chaining* (Silver et al., 1996) and *targeting at particular solution* (cf. Koichu, 2008). Specifically, *systematic variation* is referred to creating a new problem on the basis of a given or previously posed problem when one critical aspect of the problem is held constant and other critical aspects are varied systematically. The famous *what-if-notting* can be seen as a *systematic variation* governed by a specific question. *Chaining* refers to creating a new problem based on an answer (or an element of solution) to a previously posed problem. Depending on the context, *generalizing* can be interpreted either as a particular case of chaining or of what-if-notting. *Targeting at a particular solution* refers to a PP process governed by one's decision to appropriate the problem formulation to using a particular theorem, solution or mathematical approach. Sometimes this heuristics are prescribed by a PP stimulus, and sometime come as a free choice of the problem posers.

Considerations of Aptness

This facet of PP is rooted in two categories known from the PS literature: *beliefs* and *self-regulation*. Goldin (2002) interprets *beliefs* as one's multiply-encoded cognitive/affective configurations, to which the holder attributes some kind of truth value. *Self-regulation*, in terms of Schoenfeld (1992), accounts for cognitive processes aimed at assessment of an entire solution or a particular PS steps (cf. also the aforementioned *comparison* and *goal-directedness* processes). These aspects are interrelated and, when considered together, help us in understanding a cognitive/affective mechanism that governs a problem poser's convergence to a particular problem formulation. An essential feature of this mechanism is that problem posers seem to *apt* or appropriate problems that they make to different implicit and explicit requirements, in accordance with their beliefs and considerations about the importance of these requirements.

The framework deals with four kinds of aptness considerations: aptness to a PP *stimulus*, *self-aptness*, aptness to the potential *assessors* of a problem poser on the basis of the posed problem (e.g., a teacher or a peer who, to the poser's knowledge, will formally or informally assess the problem poser's skills), and aptness to the potential *solvers* of a posed problem. Note that the first three types of aptness considerations appear, to different extents, in many PS situations, whereas the last type seems to be exclusive in PP. Note also that different types of aptness not necessarily come along.

Social context in which PP occurs

PP occurs in a mathematics classroom and in a mathematician's office, individually or in groups, in a test or in an interview setting. As a rule, PP contexts are dynamic in nature, and it is reasonable to assume that understanding the complexity of PP requires understanding the complexity of the context. For example, the dynamics of work on a challenging task in a group of four may differ from that in pairs (e.g., Sela, 2008). Another example: dynamics of work in a group of peers may develop in different ways depending on familiarity or social relationships of the members of the group. Not much is known about the effects of social contexts on PP behaviors and products. By introducing the *Social Context* facet in the suggested framework, we call to explore its role as it is done in PS (cf. the *Practice* category in the Schoenfeld's model). So far, many studies treat social context of PP as a background parameter rather than as a variable on which the scope of the results depends. An example of dealing with this issue in PP was provided by Silver et al. (1996), who, in particular, compared the quantity and quality of problems posed by the subjects working individually or in pairs.

CONTRIBUTION AND IMPLICATIONS

This essay contains description of the suggested framework and its theoretical justification. The paper, however, does not contain examples of implementation of the framework in analysing empirical data (we plan to do it elsewhere). This is because of space constraints and our wish to present all the components of the framework in some detail. Thus, we can present here our view on (potential) contribution and implications of the framework only by using general criteria developed for assessment of theories and models in mathematics education.

Dubinsky & McDonald (2001) pointed out that a model or a theory can: (1) support prediction, (2) have explanatory power, (3) be applicable to a broad range of phenomena, (4) help organize one's thinking about complex interrelated phenomena, (5) serve as a tool for analysing data, and (6) provide a language for communicating ideas that go beyond superficial descriptions (p. 275).

We believe that criteria 2, 4 and 6 are met since the framework embeds PP in the context of the related models, which have claimed to have the explanatory power, communicative value and usefulness in organizing thinking on PP and PS. The framework is constructed with the purpose to be applicable to a broad range of problem posing situation, and thus, it hopefully meet criterion 3. Criterion 5 is met since categories for analysing PP processes for better understanding problem posing products can naturally be deduced from the framework. We also feel that the framework encompasses most of previous research in the area, and, more importantly, shows directions for future research. For instance, more research is needed to understand the interplay among different types of aptness and among aptness and the rest of the PP facets. Exploration of the role of social context of PP requires conducting studies in which the performance on the same PP task is analysed, say, in small groups, when approached individually and in a mathematics classroom. Thus, the predictive power of the framework (criterion 1) can be assessed only in time and only on condition that a system of well-focused empirical studies will be designed and conducted within the suggested framework. We currently work in this direction.

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THE SOCIAL CONSTRUCTING OF MATHEMATICAL IDENTITIES

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This research examines peer collaborations in a middle-school setting. Social dynamics during peer collaborations may perpetuate the development of particular mathematical identities of students, and may lead to productive positioning or productive silencing of students. Productive positioning occurs when students are allowed to exist within the group with limited responsibilities to the group and despite the wider needs of the group. Productive silencing occurs when attempts by students to become legitimate participant in the group, are met with resistance. Socially constructed perceptions of students in mathematics were found to influence the productive positioning and productive silencing amongst students.

INTRODUCTION

Peer collaboration has been widely endorsed in both research and in policy statements pertaining as being an effective pedagogical practice for learning mathematics (Expert Panel on Student Success in Ontario, 2004; National Council of Teachers of Mathematics/NCTM, 2000; Ontario Ministry of Education, 2005). The endorsement has occurred despite concerns that research about collaborative learning has lacked rigorous analysis and despite suggestions that collaborative learning may not produce any significantly enhanced results in terms of achievement over other types of learning (i.e., individual-, teacher- centered) (Anderson, Reder, & Simon, 1996; Stacey & Gooding, 1998).

Few studies have identified, that, despite the requisite structuring or scripting by the teacher for collaborative learning, non-productive learning still occurred which resulted in incorrect information sharing, inadequate peer support, limited learning opportunities, and peer oppression (Sfard & Kieran, 2001; Sfard, Nesher, Streefland, Cobb, & Mason, 1998; Sinclair, 2005). One outcome when students engage in non-productive collaborations during mathematics is that they may be actively constructing identities for themselves based on the level and the nature of the participation they are able to achieve (Lerman & Zevenbergen, 2004). Consequently, there may be serious implications for students when collaborative learning is ineffective. This research is concerned with the way in which mathematical identities are impacted when collaborative learning that, although approached with the requisite structuring or scripting, is nevertheless ineffective for some students. Therefore, the question guiding this research is: *What are the implications, in terms of mathematical identity, for those students who have ineffective cooperative learning experiences?*

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THEORETICAL FRAMEWORK

The theoretical framework for this research draws on Lave and Wenger's (1991) "social practice theory of learning" (p. 35). Two interdependent concepts are the basis of the social practice of theory: situated learning and legitimate peripheral participation. Situated learning is described by Lave and Wenger as learning "involving the whole person rather than 'receiving' a body of factual knowledge about the world; activity in and with the world; and the view that agent, activity, and the world mutually constitute each other" (p. 33). Legitimate peripheral participation is described by Lave and Wenger as the participation of learners amongst communities of other learners in ways in which permit individuals to simultaneously participate while at the same time acquire knowledge and skills that will enable eventually full participation.

According to Lave and Wenger, learning involves "the construction of identities" (p. 53). They define identity as "long-term, lived relations between persons and their place and participation in communities of practice" (p. 53). Lave and Wenger propose that "identity, knowing, and social membership entail one another" (p. 53). The construction of identities amongst a community of learners may potentially perpetuate systematic inequalities that exist in group work and potentially limit access to learning (Cohen, 1994, p. 23). *Preconceived notions by students and by teachers of other students may contribute the construction of identities is (Berger, Cohen, & Zelditch, 1972).*

LITERATURE REVIEW

Barnes and Todd's (1978) seminal research with secondary students on peer collaborations suggests that knowledge is a negotiable commodity between students only when students are fully and willingly engaged in the learning task. Barnes and Todd say that students are more likely to achieve fruitful discourse (i.e., open, collaborative discussion and argumentation) when they are able to take independent ownership of their learning and this occurs outside of the immediate range of the teacher.

Cohen's (1994) proposes from a meta-analysis of research pertaining to collaborative learning that: Skills required for small group settings require explicit instruction, in that they are not an automatic consequence of cooperative work (p.7); the negotiation of individual roles may be more effective if originating from within the group rather than from the teacher (p. 21 and p. 22); productivity with ill-structured problems, in that they have no correct answer or problems may be routine, is dependent upon the interactions of the group (p. 8); in ill-structured problems, distinct divisions of labour among students may inhibit interactions (p. 11); low achievers benefit from being in mixed ability groups (p. 11).

Finally, Johnson and Johnson (1989, 1992, 1994) identify five basic elements of effective group work or peer collaborations: (1) individual accountability, (2) social and academic appraisal of the groups' efforts (i.e., process as defined by Johnson and Johnson), (3) collaborative skill, (4) face to face interactions, and (5) positive interdependence. Positive interdependence implies a willingness on the part of the students to accept accountability for one another's learning. As Johnson and Johnson suggest, one or more elements may be missing with the exception of positive interdependence for group work or peer collaborations to be effective. Johnson and Johnson also pointed out that these elements must be both taught and included in teachers' pedagogical choices.

METHODS

Data were from a year-long study investigating peer collaborations, in an eighth grade classroom, in a large urban setting. The teacher of the classroom had been teaching for more than 11 years. Data sources included: video data that was transcribed and subsequently coded; sociometric questionnaires completed by the teacher and all students within a group before each task asking them to identify (1) the leader, (2), the best mathematics student, (3) the weakest mathematics student, (4) the least likely to talk, and (5) the most likely to have good contributions to the group's collaborations; 19 participant interviews; peer focus groups; and whole-class discussions. During the interviews, I asked students to (a) talk about their perceptions of their involvement in the group, (b) identify challenge that they experiences, and (c) their perspectives on their own mathematical identities. I asked similar questions during the focus group sessions.

In coding of the data, I focused on students who appeared disconnect from the group. By disconnected, I was identifying episodes of video that showed students working independently and not interacting with the group, instances where students were subjected to open animosity from other peers, and instances where students were predominantly off-task. My coding was validated against students own reports about their own participation and that of other students' participation (Sleeter, 2001).

Prior to the task detailed in this paper, students discussed during a whole-class discussion lead by the teacher the ways in the groups could structure their work in order to maximize effectiveness, and how tasks could be distrusted between groups. Also discussed were the ways in which they could support each other during the investigation. The teacher engaged in some role-playing as well to model potential interactions. The task itself was related to optimization. Working in groups of five, students asked to design a box, out of a given sheet of paper, resulting in the largest volume. Students had three 70-minute mathematics periods to complete the task. During the second half of the final 70-minute period, students presented their findings to the whole class. Students were evaluated both individually, in the form of a short quiz, and also as a whole group. Additionally, each group had to present to the class their appraisal of the productivity of the peer collaborations.

The qualitative findings that follow are reported as a continuous string of events as if one episode. In actuality, data stemmed from two different videos, captured over three problem solving days, which totaled approximately 120 minutes of actual peer collaboration and video. Similar findings as those reported in this paper emerged during each peer collaboration analyzed throughout the research. The episode, *Mitchell's Cube*, was selected because it had not been detailed in other analyses emerging from this year-long research (Kotsopoulos, 2008).

RESULTS

Mitchell was a 13 year-old male. His first language was English and both of his parents were professionals and had completed post-secondary education, although his mom currently stays at home with Mitchell and his siblings. The other group members, Alice, Ella, Joanne, and Will had similar backgrounds. Mitchell was identified as the low achiever in his group in the sociometric questionnaire even though he consistently performed in the 90th percentile in his class in overall mathematics achievement.

Mitchell's group was lead by Ella. Ella, at the beginning of the task, immediately assigned various dimensions to the two other girls within the group regarding the constructions they should undertake. Will, the other male in the group, independently took up the task of cutting tape and paper for the others in the group, which he did for the duration of the task and without any suggestion that he should be participating in other ways (i.e., with a construction of a box or calculations to support another's construction). He did this job for the full 3-70 minute videos. On some occasions he was also engaged in other off-task behaviour independently and with others in the group.

With no discussion or input from his group members, Mitchell started to create a cube-shaped object. He appeared to recognize that, the higher the sides of the box (not exceeding a cube), the larger the potential volume. His attempts to construct a cube were first met with resistance from Alice, another student in the group, who observed he was constructing a model different than the rectangular shaped models that the other three girls in the group were making. Alice suggested that he could continue building his cube. However, he should also build a shape similar to the others being built by the girls in the group. Despite Alice's suggestion, Mitchell continued to work on his cube as the others in the group created rectangular shaped boxes from the pieces of paper.

As Mitchell progressed in his work building a cube, intermittently there was some discussion on the relevance of his work between the girls:

Ella: What is he doing?

Alice: He's doing a random thing, but no one actually knows…

Mitchell's contributions were dismissed by his peers. In addition, his activities were seen as counterproductive to getting the task done as can be seen in this next exchange:

Alice: Mitchell is *still* doing it.

Ella: Mitchell's not doing anything!

Mitchell: I am doing something!

Alice: You keep telling yourself that, Mitchell.

In spite of the negative discourse around Mitchell's activities, on three occasions his peers made the observation that there might have been some potential in Mitchell's cube.

Regardless of the girl's observations regarding the potential of Mitchell's cube, the general sense amongst them was that Mitchell was not actively supporting the collaborative efforts. Alice asked to this effect, "Mitchell, are you actually going to do something?" Two-thirds through the problem solving session, the following exchange occurred, now including Mitchell:

> Joanne: Mitchell's looks like it would have big volume.
>
> Alice: But Mitchell's looks weird!
>
> Mitchell: No, it's going to be a cube.
>
> Will: How come you think everything he does is weird? [*He says this to Alice*]
>
> Alice: I know, look at the way he's doing it, look!
>
> Mitchell. Look!
>
> Joanne: Different!
>
> Alice: And everything he does do is weird.
>
> Mitchell: Yea, that's what you think.

Alice then outright asked Mitchell to stop his work:

> Alice: Mitchell, you can stop making these [sic] little, whatever …

To which Mitchell responded:

> Mitchell: I guess you guys are dumb.

The negative discourse about his mathematical thinking resulted in Mitchell stopping his investigations of the cube-shaped box. Mitchell joined Will, in the cutting of tape pieces used by the others for constructing their boxes. At the conclusion of the session, Mitchell's cube was not used in the overall results of the group. Rather, Mitchell's cube was used as the group's garbage container, which he was then asked to dispose of himself.

During the focus group session, I asked the members of Mitchell's group about the cube Mitchell was creating. I asked why Mitchell's ideas were not explored or acknowledged. Mitchell said this: "It was just a cube. That was one of the ones I was working on but everyone thought it was weird." I indicated in response to his comment that I thought Mitchell was onto something that the group did not explore to which Mitchell responded, "others in the group didn't care."

DISCUSSION

Mitchell's experiences demonstrate how an individual's contributions and thinking can be dismissed based upon other students' perceptions of their mathematical abilities. There were a number of occasions where Mitchell's peers noticed the potential of his cube. However, given that the cube was coming from Mitchell, his ideas were ultimately discounted or more literally and sadly, actually trashed. Mitchell, as a consequence was subject to what I refer to as *productive silencing*.

Productive silencing

Productive silencing occurs when attempts by students to become legitimate participants (Brown, Collins, & Duguid, 1989; Lave & Wenger, 1991), in the group, are met with resistance. The silencing is productive in the sense that it limits or prohibits access to knowledge construction and thus actively keeps an individual moving within a social and academic trajectory. Therefore, the silencing is productive in that the trajectories are maintained and reinforced and can be conceived of as 're'productive in a sense (Bernstein, 2000; Bourdieu, 1991).

Students may begin to believe other's discourse about their ability to participate and learn. For productive silencing to occur there is an element of resistance on the part of the individual who is being silenced. That is, this individual does want to gain legitimate peripheral participation, at the very least, and this individual is motivated to learn contrary to what might be assumed by others.

Low achievers were oft characterized by their peers and by their teacher in the interview data as "lazy," "not doing anything to help," "doing nothing to help themselves [sic] learn," and so forth. These perceptions were not supported by the video data. In each of the videos students identified as low achievers continuously tried to work through calculations and some worked in isolation and in silence. Attempts to offer contributions by the low achievers were consistently shutdown throughout the data set. As Lerman and Zevenbergen (2004) suggest, the limited participation of students in the group process regulates their ability to view themselves as able learners of mathematics and is an outcome of productive silencing. Indeed, many of the low achievers in this research questioned their potential to learn mathematics during their interviews.

Productive positioning

Will, who cut paper and tape for the entire duration of the collaborations, remained completely disengaged from the task. There was no resistance on behalf of the members of his group with respect to his minimal contributions and he does not resist his role, largely because he has chosen it. His learning was not compromised in that he was able to achieve a 73% on the individual quiz. Will was subject to what I refer to as *productive positioning*.

Productive positioning occurs when individuals are allowed to exist within a particular social setting with limited responsibilities to the group and despite the wider needs of the group. To further explain, it occurs when a student's lack of meaningful participation is not challenged by members of the group or is overtly or covertly excused. In addition, the students themselves do not challenge their roles within the groups. Therefore, there is no form of resistance on behalf of the student or the other members of the group with respect to their participation. These students continue to be secure in their social and academic positioning within the group and thus the positioning is productive because it perpetuates particular academic and social trajectories. The sociometric questionnaires were less effective in predicting productive positioning. For example, in another episode, detailed elsewhere, a low achiever was permitted to exist unchallenged because the group implicitly understood the limitations of the student who was identified in the class as academically delayed (Kotsopoulos, 2008).

CONCLUSIONS AND EDUCATIONAL SIGNIFICANCE

The constructs of productive positioning and productive silencing are useful elaborations of Lave and Wenger's (1991) work. The vast majority of the peer collaborations fall beyond the scope of the teacher's attention. Consequently, productive position or silencing can be very potent – particularly in how students come to view themselves as mathematical doers. Thus, the importance placed upon peer collaborations amongst peers with respect to the learning of mathematics may have to be carefully rethought. More research is needed to determine (a) the factors affecting productive position and productive silencing, (b) whether these factors can be accounted for in teacher and learning, and (c) the unintended learning that occur when collaboration does not facilitate learning.

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INTEGRAL AS ACCUMULATION: A DIDACTICAL PERSPECTIVE FOR SCHOOL MATHEMATICS

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In this paper we investigate the idea of accumulation as basis for learning the notion of integral. We present a content domain analysis, the results of a questionnaire, and first impressions from piloting an instructional unit concerning the didactical perspective of accumulation. The findings show the didactical and epistemological potential of accumulation and leave us with a serious instructional challenge.

INTRODUCTION

Learning the concept of the integral is an important part of the high school mathematics curriculum in many countries, including Israel. Indeed, it isn't possible to imagine modern scientific culture without integrals. Along with its relative, the derivative, the integral forms the core of a mathematical domain that is a language, a device, and a useful tool for other fields such as physics, engineering, economy, and statistics. Moreover, the concept of the integral represents a philosophical idea for understanding the world: contemplation of the totality of the small parts of a whole enables conclusions regarding the whole in its entirety, as well as its internal structure and properties. The idea of integral emerged and grew from within physics, from the attempt to invent a mathematical tool that enables describing, analyzing and explaining physical phenomena such as motion, mass and work (Newton 1686/1989).

The concept of integral subsumes two main ideas: integral as a limit of some sum (definite); and integral as antiderivative (indefinite). Historically, the idea of definite integral shed light on many problems in mathematics, physics, and astronomy; it is an important tool in different fields of science. The idea of indefinite integral led to the development of a new field of analysis – methods of integration of functions – that is the core (or at least was the core until technology put numerical methods at the forefront of calculus) of differential equations. We consider understanding the link between these two main ideas a central aim of the instruction of integral calculus.

A common approach at the high school level presents the antiderivative (indefinite integral) as formally ’undoing‘the derivative. It then uses antiderivatives to compute areas (definite integral). And then it stops. The reason why antiderivatives can be used to compute areas, the connection between the definite and the indefinite integral is rarely considered at the high school level, at least in Israel. So the question arises whether integration can be taught in such a way that the connection *is* established.

Research shows that many high school and university students do not acquire comprehension of the concept of the integral but are satisfied, in the best case, by

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formal techniques for the solution of problems. For example, Orton (1983) observes difficulties with the integral $\int_a^b f(x)dx$ when *f(x)* is negative or *b* is less than *a*. Mundy (1984) reports problems with integrals such as $\int_{-1}^{1} |x+2|dx$. Grenier, Richard and Legrand (1990), and Bagni (1999) claim that the traditional study of calculus in high school allows only a limited conception of the integral. Thompson (1994) argues that ”students’ difficulties with the FTC stem from impoverished concepts of rate of change and from poorly developed and poorly coordinated images of functional covariation and multiplicatively-constructed quantities“ (p. 229). Thomas and Hong (1996), and Belova (2006) present evidence showing that the ways in which students are currently learning integrals often leaves them lacking conceptual understanding: Students can manage assignments requiring the calculation of simple integrals but fail in assignments which require comprehension. Students may be familiar with elementary integration skills but lack conceptual comprehension.

In this study, we report on the results of a conceptual questionnaire about integration; we then discuss an approach to teaching the integral that stresses the connection between definite and indefinite integral; and we report first results from piloting an instructional unit based on this approach.

THE QUESTIONNARIE

A questionnaire with eight conceptual questions about integration was administered to 117 advanced level grade 12 students in Israel from five different schools who had recently finished studying integrals according to the official curriculum. The students worked on the questionnaire in their classrooms under supervision and were allowed to use their textbook and a calculator. Here, we discuss three of the eight questions.

Question about area (based on a question from Rösken and Rolka (2007))

(a) Write an integral allowing the calculation of the area of the rectangle given in the figure.

(b) Calculate the area of the rectangle with the help of the integral that you wrote in (a).

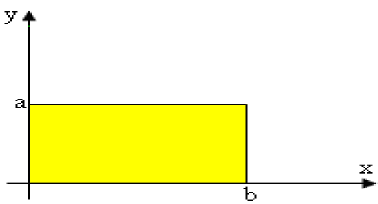

Figure 1

Our results are consistent with the results of Rösken and Rolka. Approximately 42% of the students correctly answered both, a and b, and 15% gave no answer. More interestingly, among the remaining students, a minority of 14 students (12%) did set up the correct integral $\int_0^b adx$ but made a wrong calculation such as $\int_0^b adx = \left.\frac{a^2}{2}\right|_0^b = \frac{b^2}{2} - 0 = \frac{b^2}{2}$, without apparently being bothered by the fact that the result was different form the area *ab*.

The majority (36 students, 31%), did not set up the correct integral. They produced answers such as

$$\int_a^b adx, \int_0^b axdx, \int_a^b f(x)dx, \int_a^b f(a)dx, \int_0^b (a-b)dx, \int_a^b F(a)-F(b)+c\,.$$

Question about a property

Claim: If the continuous function $f(x)$ is negative on [a,b] ($f(x) < 0$), then the definite integral of the function on this interval is also negative ($\int_a^b f(x)dx < 0$).

Is the claim correct? Explain your answer.

Only 11 students (9%) answered this question correctly, providing a variety of explanations. The vast majority (68 students, 58%) answered that the integral will be positive and consistently explained their claim by "integral is area and area is always positive" (with small variations). The remaining students did not answer the question.

Question about existence

Consider the graph of the function $y = e^{(-x^2)} - 0.1$ given in the figure. Does the integral $\int (e^{(-x^2)} - 0.1)dx$ exist? Explain your answer.

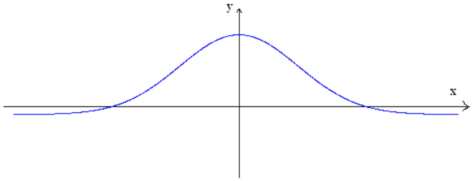

Figure 2

This question was presented only to 96 students. Only four of them (4%) provided a correct answer together with an acceptable explanation; these explanations were related to area (e.g., "There is an area between the graph and the x-axis – so the integral exists"). The remainder either did not answer at all (46 students, 48%) or provided a variety of inadequate explanations such as

- "The integral exists - Why not?"…
- "I can't calculate it, so the integral doesn't exist…"
- Pseudo-calculations

The questionnaire aimed to collect empirical evidence about written responses to questions about the integral concept by advanced level high school students who have learned the concept of the integral according to the common approach. The evidence is consistent with previous research findings and shows that students' comprehension of the integral concept is low. This raises the question whether it is possible to improve this situation and how?

PERHAPS ACCUMULATION…

Our hypothesis is that the mathematical idea of considering the integral as an accumulation function (Thompson, 1994; Thompson & Silverman, 2008) can be made the basis of a high school level unit on integration that has the potential to lead to a deep comprehension of the integral concept and at the same time allows developing computational skills. Here accumulation is understood in a plain sense: an accumulating sum that has a large number of very small terms. The basic idea is that of Riemann integration, leading in parallel to the definite and the indefinite integral, as well as their connection by the Fundamental Theorem of Calculus.

Accumulation: mathematical notion

According to Thompson & Silverman (2008), the concept of accumulation is central to the idea of integration. Integration as accumulation is at the core of understanding many ideas and applications in calculus, e.g. curve length, volumes of bodies, work, etc. There are two facets of accumulation: a) we accumulate a quantity by getting more of it; b) in case we don't have information about some whole thing, we look for and accumulate information about small parts of the whole. These two *intuitive* facets of accumulation are directly linked to the following *formal* mathematical definition of the integral as an accumulation function: Suppose a function *f(x)* is defined on an interval [*a*,*b*] and continuous on it. Then the function $F_a(x)$ is defined as follows:

$$F_{\Delta x,a}(x) = \sum_{i=0}^{\left[\frac{x-a}{\Delta x}\right]} f(a+i\Delta x)\Delta x,\ a \le x \le b \quad \text{and} \quad F_a(x) = \lim_{\Delta x \to 0} F_{\Delta x,a}(x),\ a \le x \le b.$$

$F_a(x)$ is a function of x, also defined on [*a*,*b*]. It is called the *accumulation function* or *integral* of *f(x)*, and often denoted by $F_a(x) = \int_a^x f(t)dt$.

Accumulation in school: what is it good for?

The idea of accumulation function has close connections with the main calculus ideas, function and derivative. The obvious one is that an accumulation function is simply that, a function. The idea of accumulation contributes to a coherent understanding of rate of change (Thompson 1994; Carlson, Persson & Smith 2003). When something changes, something accumulates. When something accumulates it accumulates at some rate. So, accumulation and its rate of change are two sides of the same coin. Understanding this deep mutual relationship helps to see the strong connections between the main calculus concepts of derivative and integral.

Three reasons in favour of using the accumulation approach in school curricula are:

- The idea of accumulation allows to combine the concepts of definite and indefinite integral in a natural way and to establish the connection between the (combined) concept of integral and the concept of derivative.
- The idea of accumulation is closely linked to the applications of the integral.
- The idea of accumulation allows to later introduce generalizations of the integral in a natural way, e. g. Riemann-Stieltjes integrals, Lebesgue integrals, Lebesgue-Stieltjes integrals, and Denjoy (gauge) integrals.

We believe that conceiving the integral as accumulation might help students answer conceptual questions such as those posed in the questionnaire correctly and with understanding. The area question, for example, was designed to examine whether students were able to build the integral needed for calculating a rectangular area, and use their understanding of this integral for actually calculating the area. The idea of accumulation might lead them to see the area as accumulation of rectangle areas of fixed height *a* and width Δx and thus an accumulating sum of terms $\sum a\Delta x$, and to use this form to set up the correct integral as well as calculate it correctly. Seeing the integral as accumulation might also help students seeing area as only one particular quantity that can accumulate by means of an integral and admitting negative as well as positive quantities to accumulate. The question about property then becomes almost obvious because every accumulating term is negative: a negative value of $f(x)$ multiplied by a positive value of Δx. Finally, the question about existence might be more accessible to accumulation students than to common students since for every closed interval in the function's domain, the value of the accumulating quantity is defined and may be calculated and summed up. If we vary the end of the interval we obtain a function, known as the antiderivative or indefinite integral. Accumulation students are thus given opportunities to experience that the existence of the integral as accumulation depends only on the properties of the given function (such as continuity) and not on whether or not they know an antiderivative.

INTEGRAL AS ACCUMULATION: INSTRUCTIONAL CONSIDERATIONS

One of our main goals is to develop a unit of instruction on the integral that is based on the idea of accumulation. A first version of the unit has been designed and implemented with a group of three advanced level grade 11 students who have not yet met integrals in class, in the form of a sequence of task-based teaching interviews (Goldin 2000).

Our overall conclusion is that students can learn accumulation. Within four meetings of about 60 minutes, the students successfully completed tasks such as the following:

- To approximate, by measuring and/or by using known formulas, areas as in figure 3 or volumes as in figure 4 (figures show the students' work).

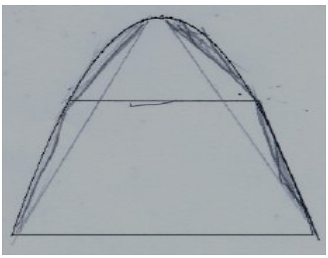

Figure 3: Area

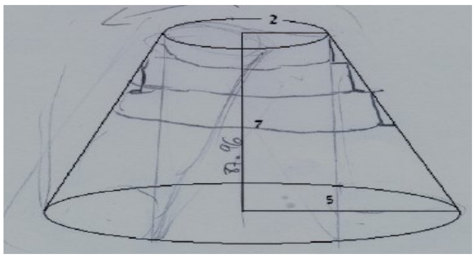

Figure 4: Volume

In both cases above students came up, after minimal hints from a researcher, with the idea to change an unknown calculation into a known one.

- To estimate the accumulation value of a positive increasing continuous function (not shown) and of a sign-changing decreasing continuous function (figure 5).

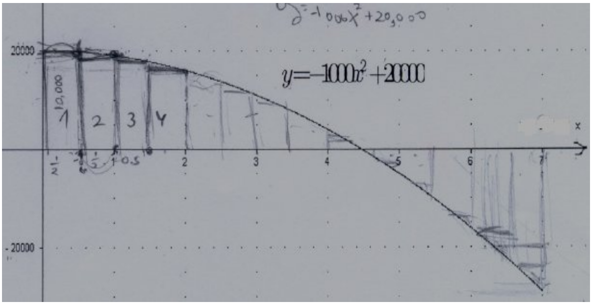

Figure 5: Estimating accumulation

- To calculate the accumulation value of a given linear function (figure 6) by using algebraic manipulations.

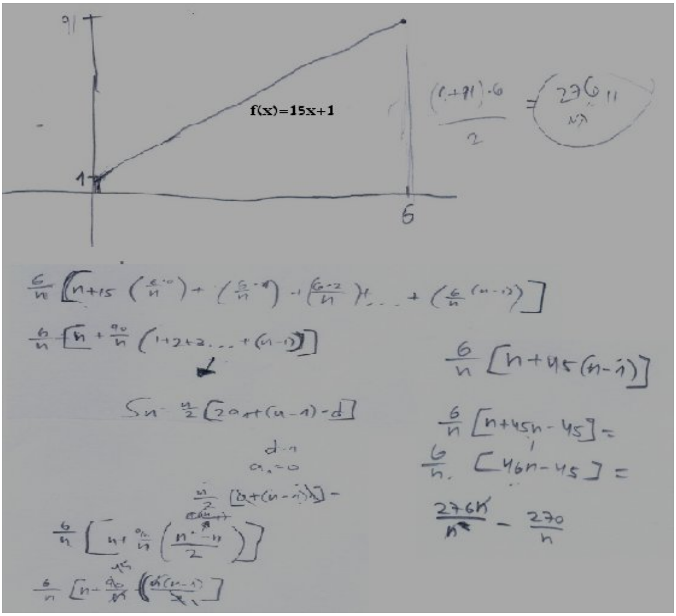

Figure 6: Calculating the value of accumulation

The students carried out this calculation by themselves after a similar guided activity with a quadratic function. We note the informal way in which the limit was carried out:

> The term $276-\frac{270}{n}$ actually equals 276 because $\frac{270}{n}$ becomes zero when n is infinitely large.

The students verified the result by geometrical considerations (area of the trapezoid).

- To calculate definite integrals of a given linear function (positive, negative or sign-changing on a closed interval) by geometrical considerations.
- To calculate the definite integral $\int_0^6 (x^2+1)dx$ as accumulation value by using algebraic manipulations and to use the result of this calculation for finding the value of the definite integral $\int_0^6 x^2 dx$.

CONCLUSION

Over the years, the mathematics education community has tried with varying measures of success to describe how students learn about the concept of integral and point to better ways of teaching it. The fact remains that many students in the upper grades of high school, including advanced level students, experience difficulties with the integral concept. The common approach to the integral at the high school level enriches students' mathematical culture by affording them opportunities to use the integral in a formal way, mainly to calculate areas. But in our view, the educational potential of the integral concept is much more significant. The realization of this potential depends on the achievement of a deep comprehension of the concept. This begs the question whether an approach to teaching the integral concept exists, which supports the development of such comprehension without neglecting technical skills. In this paper, we have suggested that such an approach exists and brought forward evidence to support this suggestion.

Significant comprehension of mathematical concepts is at the heart of mathematics learning. This proposition is particularly relevant for the integral. The idea of calculus in general and the idea of the integral in particular were born from human attempts to understand the world, from applications (Newton 1686/1989). In some way, the integral *is* the application. So, in our eyes, there is no way to understand integrals without understanding the strong connection between the mathematical concept and its applications. The heart of this connection is the idea of accumulation. Hence, the central aim of our research is to design a unit based on accumulation aimed at deep comprehension of the integral concept, and to investigate the resulting knowledge constructing processes. The first evidence we started to gather from a pilot implementation gives us reason to hope that we will be able to propose a suitable research based didactical approach to teaching and learning the integral concept at the advanced high school level.

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IMAGE OF AN IDEAL TEACHER PAVING THE WAY FOR FORMATION OF MATHEMATICS TEACHER IDENTITY

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A student has a central role in constructing personal teacher identity during academic teacher education. Here, the formation of teacher identity is approached from an individual perspective with a special focus on the image of an ideal mathematics teacher as a part of the development process. Based on experiences, teacher students construct a personal understanding of being a good teacher. This ideal image directs identity formation only if it is linked with the personal development tightly enough. Teacher identity formation is seen as interplay between the present and the ideal images of oneself as a teacher. Identification with teaching profession takes place through both images. Furthermore, profiling the process is possible by viewing separate characteristics of being a teacher. One student case is presented to illustrate how the ideal image paves the way for personal development.

INTRODUCTION

Finnish mathematics teacher education is claimed to be academic for two reasons. Firstly, the underlining idea of the study programme is that all teachers should have expertise in different knowledge domains, especially in mathematics as well as in pedagogical content knowledge regarding teaching and learning mathematics; teacher students are expected to gain their knowledge base through separate and rather theoretical university courses. Secondly, access to practical experiences in a classroom is limited. Altogether, during the pre-service teacher education, there are approximately 20 supervised teaching practice lessons in mathematics, in which a student takes a role as a teacher with full responsibility. In the context of such academic teacher education, the process of becoming a mathematics teacher is not guided by hands-on experiences situated in 'real school life' but largely by academic learning processes. Therefore, teacher students need to be able to use their imagination in order to identify themselves as teachers along academic studies.

Here, the focus is on the early steps of teacher identity formation, on 'becoming a teacher' during pre-service teacher education. In general, the formation of teacher identity is seen as an on-going process of change that involves interplay between internal processes of an individual and external processes, which take place in social interaction with others (e.g., Bohl & van Zoest, 2002; Côté & Levine, 2002). However, even if identity formation is seen as a socially embedded process, individuals have an essential role in constructing their own professional identity; they are the ones who actually transform into being a teacher. In this paper, teacher

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identity formation is approached from an individual perspective regarding internal processes of development.

THEORETICAL BACKGROUND

The theoretical framework is focused on the developmental process that persons involve themselves in when constructing their own professional identity. The starting point is an intention that the process is directed towards becoming a good mathematics teacher.

Characterisation is one way to address what it means to become and to be a professional teacher, both individually and collectively. The image of a good teacher can be described through two categories: professional competences regarding expertise in certain knowledge and skills, and affective qualities that a good teacher should possess (Fajet, Bello, Leftwich, Mesler & Shaver, 2005). On one hand, teacher knowledge and skills needed in the teaching profession have been one starting point for profiling teacher identity (e.g., Bohl and van Zoest, 2002; Smith, 2007). Here, in Finnish educational system, teacher knowledge and knowledge building is considered as a core element for being a professional teacher. A mathematics teacher should have expertise in subject knowledge, mathematics especially, in pedagogical content knowledge regarding teaching and learning mathematics as well as in general pedagogical knowledge. On the other hand, in addition to cognitive characteristics, affective aspects, like attitude towards as well as a commitment to the teaching profession, emerge (Hodgen & Askew, 2007).

Teacher students become acquainted with socially shared conceptions of a good teacher through social interaction with others, for example during their own time in school as well as in pre-service teacher education. Different experiences embedded in social interaction are seen as a core element in the process (Beijaard, Meijer & Verloop, 2004). Individuals reshape their own personal understanding of being a teacher by giving a meaning to personal experiences, and by filtering socially shared ideas (e.g., Beijaard et al., 2004; Côté & Levine, 2002). Therefore, only some socially shared ideas are internalised as a part of an individual *image of an ideal mathematics teacher*; i.e. the view of what is seen desirable and valuable to aspire as a teacher (Arnon & Reichel, 2007; cf. Sfard & Prusak, 2005). Here, the image of an ideal teacher is not seen as a comprehensive overall picture of a good teacher, but rather as a collection of separate features associated with being a good teacher.

An individual constructs an understanding of becoming and being a teacher on two levels. Teacher identity formation can be seen as interplay between the present state as a teacher and the image of an ideal teacher. On one hand, an individual reflects the present state of oneself as a teacher (’How am I at the moment?’), which is manifested in a certain time and place, for example in teaching practice. On the other hand, an individual constructs an image of an ideal teacher that manifests the aspired state as a good teacher (’How would I like to be in future?’) (Beijaard et al., 2004). Furthermore, because teacher identity formation is seen as an on-going process, both the present state as a teacher and the image of an ideal teacher are continuously progressing. Formation of teacher identity is seen as a continuos process of filling the gap between the present and the ideal images, and therefore, the ideal image is seen to pave the way for individual development (Arnon & Reichel, 2007; Sfard & Prusak, 2005).

Especially during academic teacher education, students need to be capable of and willing to imagine themselves as teachers as a part of the developmental process. Like Arnon and Reichel (2007) state, personal goals are in accordance with the image of an ideal teacher but do not necessarily cover the ideal image in all aspects. However, the ideal image directs the personal development only if it is linked with the developmental process tightly enough. Here, the focus is on the role that the image of an ideal mathematics teacher has as a part of individual development. The research question is how the image of an ideal mathematics teacher directs the formation of teacher identity.

METHODS

The research uses a qualitative approach based on interview data (Stake, 1995; Patton, 2002). Altogether 18 teacher students took part voluntarily in three semi-structured interviews during their educational study year 2005-2006 at the Department of Applied Sciences of Education in Helsinki. The following themes were discussed in all three interviews: conceptions of good mathematics teachers and teaching (the image of an ideal mathematics teacher), identification as a teacher, and expectations and aims for the studies. Furthermore, the starting point and the background of a student were also discussed in the first meeting. To improve construct validity, written work, like students' reflective portfolios as well as feedback questionnaires on educational studies, were also used as a source for augmentation. In this article, the description of the case John is presented. He was selected as a typical mathematics teacher student, who shows development during his studies and who is able to conceptualise his developmental process in the interviews.

Starting with the themes of three separate interviews described above, the interview data was analysed following the principles of the analytic induction (Patton, 2002). In the final stage of content analysis, descriptions of the cases were juxtaposed and compared. Member checking was used when research participants were asked to review the case descriptions for accuracy and palatability (Stake, 1995).

RESULTS

Here, the focus is on one student case, John. The process of development is profiled through four main categories associated with being a professional mathematics teacher: expertise in teacher knowledge domains, and the affective aspects related to being a teacher. Especially, the interplay between the image of the present state as a teacher and the image of an ideal teacher is of interest.

John seems to hesitate with future plans regarding his teaching career. However, mathematics has always inspired him greatly, and mathematics matters to him the most; a teaching profession is only one option amongst other options for an occupation in future. Despite the uncertainty, John states that he is willing and well suited to become a good mathematics teacher.

> I will act as a teacher as long as I learn what it is like to be a teacher and I familiarise with school world… but I want to act as a translator as well… it provides some kind of certainty in my life that if this choice [to be a teacher] is wrong for me as a future career, then I also have another option… (autumn)

In general, John can be portrayed as an independent and self-directed student. Already at the beginning of his pedagogical studies, he had a clear vision about being a good teacher as well as about how to become such a teacher. He emphasises that there is not only one way to act as a teacher but multiple ways depending on one's own personality.

> All teachers should teach in their own way, it is… you have to invent your own way of thinking… but like I do… I think differently about these matters than how they want us to think here [in teacher education], like they teach us here… (spring)

According to John, *mathematical content knowledge* is the basis for being a good mathematics teacher: the ideal mathematics teacher should have ability to think like a mathematician. A teacher should be enthusiastic about mathematics and should strive to improve mathematical competence when required. In general, mathematical competence is not widely reflected, as it seems to be self-evident in the core of being a good mathematics teacher.

> One of the important aims [in mathematical studies] is to learn to think in a certain way, like a mathematician, that when you have a mathematical problem you know how to attack it like you should… then there is calculus, it is good to know that as well, but the most important thing is to learn the mindset … (spring)

After experiences in taking a role as a teacher in the first teaching practice, John highlights that the solid mathematical base is a tool for planning and implementing lessons, as well as an aid to be able to handle dynamic classroom situations. Pupils have a right to rely on correctness in teaching, and therefore, a teacher's mathematical knowledge should be mathematically exact and true.

> …that you do not teach something incorrectly… I didn't understand to bring this up… of course you have to master the content in order to be able to teach it… (autumn)

In general, John does not reflect his own mathematical competence in detail. According to the interviews and portfolio assessment work, he finds his own starting point good enough in order to be confident. John states that his personal strength is his willingness to find out and to develop his own mathematical knowledge when needed. Altogether, regarding expertise in mathematical content knowledge, the present state and the ideal image seem to be parallel with no special gap between the images.

Expertise in *pedagogical content knowledge* is intertwined with the skill to provide enjoyable and suitable learning opportunities in mathematics for pupils. According to John, learning mathematics takes place through one's own active thinking. A good teacher considers the individual starting points of pupils carefully and is able to differentiate between pupils according to their abilities. It is on teacher's responsibility to support this learning process by simplifying the central topics. A teacher needs to master the field of teaching and learning mathematics on a general level and to highlight the most essential issues.

> There are pupils, like, on different levels and I think the teacher owes them, who are more skilful... they have to be given the opportunity to learn, even if there were others in the class, who would miss something in the lesson...this would kind of take into account the heterogeneity within the class... (December)

At first, John reflects issues relating to pedagogical content knowledge starting from the viewpoint of the learner and the role of a teacher in the learning process. At the end of the studies, besides a learner-centred starting point, John specifies his view of a good education with three central aims: to gain basic calculation skills, to develop mathematical thinking, and to apply mathematical knowledge in the real world.

John has doubts about his own competence regarding pedagogical content knowledge. He realises that he has no practical tools for carrying out lessons in the way he would like to, e.g., regarding a good level of abstract in instruction. It is challenging to make pupils see the beauty of mathematics or to create positive learning situations. After the first teaching practice, he is able to set more detailed personal aims. He considers creativity in planning the lessons as a special strength.

> I seem to come up with ideas very well, that I have ideas for implementation and for methods and I can implement a lesson that has a good content… and certainty for implementing the lesson, that I could carry it out in a way that I planned it… (December)

In the end, despite the need for further development, John feels that he has found his own way of teaching mathematics. Altogether, the image of an ideal teacher is reshaped gradually alongside the developmental process without any major changes. Furthermore, John reflects his present state as a teacher from the perspective of the ideal image, and finds out that further development in practical skills is needed in order to be able to implement lessons as intended.

According to John, a good teacher should master general knowledge about learners, learning and teaching. A solid base of *pedagogical knowledge and skills* is the ground for creating a positive atmosphere in the classroom as well as for differentiation. He highlights the meaning of classroom management in general for guiding pupils' learning according to the goals of learning. Besides, John talks about the importance of real interaction with pupils. General pedagogical knowledge is of help in this matter.

> One should have at least a basic understanding of human psychology, about mechanisms of learning, there is good contemporary literature also in Finnish (December)

The image of an ideal teacher regarding general pedagogical knowledge does not change much during the study year. However, the views of his own competence change slightly along practical experiences. Practical issues, like giving instructions and implementing lessons according to a planned timetable, emerge from practical experiences. The gap between the present and the ideal image is quite substantial. The themes of concern are the same, but because of a high level of abstraction, the ideal image of a teacher is far from being reached.

> … I should have guided activities in a more effective way, more directly and give more precise instructions, to have concrete models… oral instructions are not enough… I was too optimistic about oral instructions… a valuable lesson (December)

In addition to cognitive aspects, John reflected *affective aspects* associated with being a mathematics teacher. Teachers should find their own personal way to perform as a teacher, find a real interest in the teaching profession, and should engage with doing their best through a reflective and an analytical approach. Besides, John emphasises the ethical perspective associated with the teaching profession.

> …when you act as a teacher on the basis of your own starting points, you have to make decisions and think… there is no sense in making people think in a way that doesn't work for them … it is a richness to have different kinds of teachers (spring)

However, despite a relatively clear view of the ideal teacher, the affective aspect is not a part of his present image of himself as a teacher. John sees that it is in his nature to be reflective and to be willing to learn new things. Mostly, he does not identify such requirements with his own developmental process as a teacher but rather with the role as a student.

DISCUSSION

In this article, one student case, John, is presented for discussing the meaning of the image of an ideal teacher for the process of teacher identity formation. Naturally, a motivational background of a student plays an important role in the developmental process. A student needs to be willing to become and to be able to imagine oneself as a mathematics teacher. Identification with being a mathematics teacher takes place on two levels, through the image of present state as a teacher and through the image of an ideal teacher, which is considered desirable and valuable as a good teacher. At the beginning of the formation process, a student should be able to imagine oneself as a teacher in the future, like in the case with John, and to identify at least partly with the image of an ideal teacher as a personal guideline. Gradually, along with educational studies and practical experiences, students internalise their role as a teacher and begin to feel like a 'real' one. At this point, being a teacher is not only a matter of a designated state but also the present state of teacher identity (Sfard & Prusak, 2005).

Everyone has in general some kind of an image of a good teacher, most probably also in mathematics. The image of an ideal teacher paves the way for identity formation only if it is linked with an individual process (Arnon & Reichel, 2007). John had a relatively clear vision of being a mathematics teacher that was reshaped during his studies. The wideness of the gap between the present and the ideal image is consequential. On one hand, the ideal image directs the developmental process as if it is something reachable from the viewpoint of the present state as a teacher. In John's case, the vision of pedagogical knowledge and the skills of a good teacher was described on a highly abstract level. At the same time, the view of his present competence was in close connection to his practical teaching skills. The personal goals regarding this matter were distant from the ideal image. However, regarding mathematical competence, the ideal and the present image seemed to be closely connected, which consequently reflected the minimal gap between the ideal and the present state as a teacher. John did not have a real need for development in this respect or for setting any further goals.

On the other hand, through characterisation of teacher identity, it is possible to profile an individual process on both present and ideal levels. John did not link the ideal image with his personal developmental process as a whole. The ideal image comprises of several aspects of being a good teacher, and these diverse aspects were combined with personal development in different ways. In John's case, mathematical competence as well as pedagogical content knowledge had essential roles, but affective aspects, even though described on the ideal level, were not a part of the personal process. In teacher education, students should become aware of the state of their teacher identity on both the ideal and the present level. On that basis, it becomes possible to set aims for further development.

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VARIATION WITHIN, AND COVARIATION BETWEEN, REPRESENTATIONS

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This is a theoretical paper in which we conjecture that some difficulties in learning mathematics may be due to the need to coordinate covariation between different representations, not merely to 'match' individual states. We use research within variation theory on learners' understanding of the infinity of numbers in an interval to draw attention to the need to coordinate variation of decimal representation with position on a numberline. We problematise this relationship and use a small empirical study to hypothesise how learners might relate these implicitly and explicitly. This covariation lens augments the usefulness of variation theory in thinking about mathematics pedagogy and learning.

INTRODUCTION

In a series of learning studies, Runesson and Kullberg (Kullberg, 2007; Runesson & Kullberg, in press) developed ideas concerning teaching students about the density of rational and decimal numbers. Over a series of cycles of co-planning, teaching, observing and reviewing with a group of three teachers they constructed a list of core ideas which, when presented to students, appeared to bring about a change in their understanding of the infinitude of numbers in any interval. These ideas, called *critical features*, were then offered to two teachers who were not in the original study, as the basis for their lesson planning. In this paper we look not at this process, but at the critical features from the perspective of a learner who is expected to develop understanding of the mathematical idea by *discerning variation* in what is made available to experience by the teacher and other students in the *space of learning* (Marton & Booth, 1997; Marton, Runesson, & Tsui, 2004).

CRITICAL FEATURES AND VARIATION

The theoretical framework, variation theory, describes how people learn, perceive and experience the world around us. Variation theory originates from phenomenography (Marton & Booth, 1997) and is influenced by the idea of learning as differentiation (Gibson & Gibson, 1955), though its roots can be traced back to Aristotle. Gibson and Gibson argue that for a child to identify an object, he must be able to identify differences between it and other objects. From this it follows that the way we experience something is a function of features we notice or discern at the same or nearly the same time. A variation (change) in a feature makes it more noticeable than if it remains invariant, and so more open to being discerned and experienced as a feature. For instance, variation in representation of numbers between fractions and decimals makes it possible to discern a finite decimal number

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as a part/whole relationship, because the ‘part’ and ‘whole’ are shown in the decimal digits. Every concept or phenomena has particular features that are critical for learning. Marton *et al* (2004) argue that the critical features must, at least in part, be found empirically, for instance through interviews with learners and through analysis of learning the specific content in classrooms. The critical features developed in the earlier study (in grade 7) for the density and continuity of number are:

- Decimal numbers, seen as numbers (on a number line)
- Rational decimal numbers seen as a form of expressing rational numbers (where fractions and percentages are other forms)
- Decimal numbers seen as part/parts of a whole
  - e.g. 0.97 as 97 centimetres of a one metre ruler
  - the interval can be divided in smaller and smaller parts (e.g. hundredths, thousandths)

The theory suggests that for learning to take place, students need to understand and appreciate these features. In variation theory, this means that *dimensions of variation* have to be opened up in lessons so as to allow or prompt learners to see what these features mean. Watson and Mason have developed this idea further to make it more applicable in mathematics (2005, p.5). While one possible dimension of variation of number is the size of the written digit, this dimension is unconventional and it does not afford the possibility to explore continuity, so we restrict the concept to *dimensions-of-possible-variation* which do not change the focus of learning. Watson and Mason also introduced the notion of *ranges of change* to allow the dimension to be explored in different ways by learners, and *range-of-permissible-change* to show that these are constrained by the need for mathematical meaning.

HOW TEACHERS ENACTED THE OBJECT OF LEARNING

The two ‘new’ teachers in Kullberg’s study enacted the critical features in different ways despite coplanning, and this will be reported elsewhere, but both initiated activity by asking learners the key question ‘how many numbers are there between 0.17 and 0.18?’ and both offered decimal notation, fraction notation, and a numberline as tools for discussion of this question. Looking at these representations from the perspective of available dimensions of variation we realised that they are very different, and the learners’ task is to see the static representations as equivalent, that is to relate 0.175, 175/1000, and a point mid-way between points marked 0.17 and 0.18 on a line. Acceptance of three disparate signs for the same thing can either be done without meaning as a practice to be internalised, or meaningfully through dynamic relationships between the variation in each representative form. Even if one has doubts about whether variation theory tells the whole story of learning, this insight does lead to something quite useful, because the ranges of change in each of the three representations are so very different. In decimal notation, one dimension of variation is a single digit in the decimal, and the range of permissible change is the digits 0 to 9; on the line the range of permissible change is physical position; in vulgar fraction notation there are two ranges that have to be coordinated, the numbers in the numerator and in the denominator. Considering the first two of these, the learner has to understand and appreciate how variation in the digits relates to change in position on the line, how cycling through digits, and extending the string, matches the simple action of moving a point.

SEMANTIC AND SYNTACTIC ACTIONS WITH NOTATION

It is widely accepted in several traditions that it is spurious to try to separate concept from representation. However, relating representations as if they have the same meaning seems to require some kind of isomorphism that allows manipulation of one to 'match' manipulation of the other (Vergnaud, 1998). Wertheimer (1945) had a sense of this when he showed that some algebraic representations of area express spatial arrangements while others do not. Similarly, Dorfler (2005) sees the internal structures of diagrams as direct expressions of mathematical relationships. He also claims that algebraic notation is diagrammatic in the same sense.

In Confrey and Smith's (1994) development of the idea of covariation they reject the need for obvious isomorphism and instead talk about how change in one variable shows up as change in another variable. Note that the distinction between dependent and independent variable is situated when you think in terms of covariation, since you can sometimes vary several different variables and detect related variation in others. In their work, all the variables are quantities so a table of ordered data pairs enables comparison to be made on the basis of numerical change. If we regard each of two representations as providing a variable, we can see that what might be required to understand how representations are related is to look at how small changes in one representation are mirrored by small changes in the other.

But in the situation we describe a small change in position might correspond to large change in the number and nature of digits after the decimal point, or the denominator of a vulgar fraction. Thompson's view of covariation (e.g. 2002, p.205) points to the importance of coordinating actions on each variable with an 'operative image' so that one can act out how the two are related. He also shows how even the necessary idea that a point on a line represents a value is itself a potential obstacle, so that 5th and 8th grade students have to work quite hard with his image of bunny hops to make sense of the duality of point and line. This observation, combined with the difficulty Confrey and Smith's college students had in coordinating data, suggests a further difficulty – that changes in decimal and fraction representations of very close numbers can be visually dramatic, while the shift on the line is tiny. Thus number might be perceived as 'more discrete' than continuous movement on a line.

Covariation and a 'covariation approach' (Confrey & Smith, 1994) are examples of when several *dimensions of variation* are opened up at the same time for students to experience. In a 'covariation approach' to a task involving two variables, variation in two dimensions is used to produce a graph. For instance, Confrey and Smith show that in regard to functions $y=f(x)$, a covariation approach could entail 'being able to move operationally from to y_m to y_{m+1} coordinating with the movement from x_m to x_{m+1} ' (p. 137). In our example, about the density and continuity of number, it would entail coordinating different representations of a decimal number, as well as both the range of change within the same representation and between representations.

COVARIATION IN STUDENTS' REPRESENTATION OF NUMBERS

Dufour-Janvier *et al* (1987) identify difficulties with early use of representation of the numberline on students' later learning concerning the density and continuity of number. The early use of a discrete numbertrack as 'stepping stones' with gaps between counting numbers is a long way from the concept of the density of the real numbers as illustrated by a continuous numberline. 'It is hardly surprising that at the secondary school so many students say that between two whole numbers there are no numbers, or at most one. Nor should there be much surprise that they also have great difficulty placing a number if they cannot associate it with the gradation already given on the line' (Dufour-Janvier, Bednarz, & Belanger, 1987, p. 117). Furthermore, when using multiple representations of the same concept it is expected that the learner will be able to grasp the common properties and will succeed in constructing the concept. The learner should also be able to 'reinvest the knowledge acquired, in contexts embedding different aspects of the same concept' (Dufour-Janvier *et al*., 1987 p. 112). From a variation theory perspective this means that to be able to grasp common properties students must discern similarities and differences between the representations. This implies covariation.

To explore these ideas of covariation further, we provided some representational tasks for year 7 and 9 students in one school. To persuade teachers to participate the questions were such as could be offered in normal class time by the usual teachers, and the questions had also to be of use to the teachers's planning. We received completed scripts from 49 year 7 students (11/12 year-olds) and 51 year 9 students (13/14 year olds). The questions which will be reported on in this paper are those that gave us some insight into how students coordinated covariations between numberline and decimal representations of number.

Relating decimal numbers and a scaled interval of a numberline

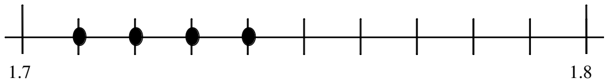
Three kinds of task each offered a scaled interval from 1.7 to 1.8, with ten subintervals between, and asked students to:

i) identify numbers represented as points on the numberline segment which coincided with scale marks

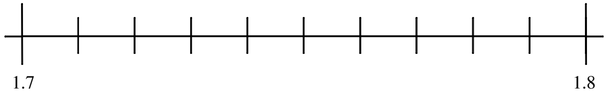
ii) represent specified numbers on the numberline segment, which would coincide with scale marks

iii) generate some numbers themselves and place them on the numberline segment.

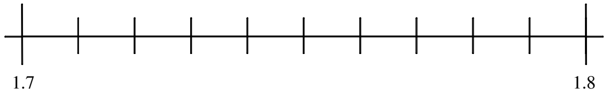
(i) Write down the numbers marked on this numberline in decimal notation



(ii) Represent these numbers on a numberline: 1.75, 1.76, 1.77, 1.78



(iii) Write down three decimal numbers between 1.7 and 1.8. Show where they would be on this numberline:



Almost all year 9 students and ¾ of the year 7 students did task (i) successfully. Almost all year 9 students did task (ii) successfully, but only half of year 7 did so. This difference in response suggests that 'reading' the scale is easier than representing given numbers on a scale, and the relationship between the two (the Cartesian duality) is not fully coordinated. For task (iii), however, when asked to write their own numbers on the same segment, almost all year 9 students were successful and 5/8 of year 7 were also successful. Since some year 7 students were better at labelling points correctly than placing particular numbers they may have used this approach for the hybrid task (iii).

A few year 9 students used three decimal places and numbers 'off' the scale marks for task (iii) to demonstrate full understanding.

Some of the failed attempts to relate decimal numbers and scaled numberline representations were illuminating. Several year 7 students labelled the line: 1.17, 1.27, 1.37 … instead of 1.71, 1.72 …. We take this to mean that they appreciate variation along the line, and knew that this related to variation in digits, but not the meaning of those digits. There were other, similar, failed attempts which suggested partial coordination of variation, and these students were all unsuccessful in task (ii).

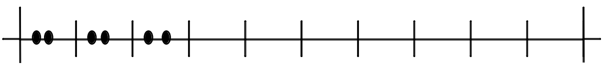
Relating decimal numbers and 'unscaled' segments of numberlines

Two tasks asked students to work with unscaled lines. We had also asked students to say which questions they found hardest, and all students who selected particular questions as hardest chose one of these, as predicted by Dufour-Janvier *et al.* Two students explained their choice as 'there were no lines to help'. It is also the case that these two tasks could not be done with only two decimal places, so they were inherently harder for that reason too. These tasks had far fewer successful responses in both years, but for our purposes we were interested in what students tried to do and the plausible reasoning behind these attempts. It is often the case that intelligent unsuccessful attempts reveal more about what is needed to understand a concept than is shown by successful attempts.

Task a: Represent these numbers on this line: 1.7, 1.71, 1.701, 1.7001

For this task, students were given numbers expressed as decimals and asked to put them on an unscaled numberline. The decimals were chosen to expose classic difficulties with place value, and only 14 students completed it correctly, but for our purposes we were interested in what students did rather than what they failed to do. 41 other students attempted this task. 20 students, half of the failed attempts, took the numbers in the order given and placed them along the line equally spaced. This was the most common approach. The next most common approach, taken by 11 students, was to order the numbers correctly, but space them equally along the numberline, showing knowledge of variation in digits and numbers of decimal places, but uncoordinated with positional variation. Of particular interest were 5 students who understood something about positional variation and how it related to the variation in digits and places, but ordered the numbers incorrectly. This attempt at coordination suggests that covariation is nearly understood, but the underlying sense of numerical value has not been brought into the task. 5 other students ordered the numbers correctly, and knew not to space them equally, but the ways they were placed showed little understanding of relative position. All students who attempted this task had been successful at the earlier task of generating their own numbers and placing them on a line, so we can assume that the coordination task is, for them, not solely syntactic.

Task b: Estimate the numbers shown on this numberline in decimal notation:



For this task, some attempts split the unscaled segments into fractions and expressed points as, for example, $1.7\frac{1}{4}$, $1.7\frac{1}{2}$, or as $1.7\frac{1}{3}$ or other simple fractions. One student combined knowledge of decimal notation with this approach and offered labels such as: 1.71½. This suggests knowledge of variation of position, expressed as vulgar fractions, not coordinated with variation of digits in decimal notation. Other students used three decimal places but only a few of these managed to produce plausible estimates. Responses like 1.701, 1.702 for the first two points were fairly common among this group. They showed knowledge of variation of digits but had not coordinated this with variation of position.

Some of the students who did not use more than one or two decimal places in this task did appear to understand further places in *Task a*, so we cannot assume that the use of further places was unavailable.

A 'COVARIATION' PERSPECTIVE ON REPRESENTATIONS

The covariation literature points us towards relating change in one variable to change in another, and in the topic we have been investigating this means relating the change in one symbolic variable to the change in another.

Two aspects of the data support the conjecture that covariation augments understanding of representation. Firstly, the difference between the responses of year 7 and 9 students shows something more has to be understood than labelling a scaled numberline segment in order to complete the relationship between decimal numbers and the numberline representation. Placing given numbers correctly on a line, i.e. representing numbers, is harder than interpolating missing values on a scale. The interpolation of missing values could even be a learnt exercise that is unrelated in the students' mind to decimal number.

Having established that the two-way relationship develops over time, we then established, by identifying the plausible thinking behind failed attempts, that students do indeed have to coordinate positional variation with digital variation and that this is a non-trivial process. From a variation theory point of view we can see that students have not yet discerned all the critical features of number. The representations offer the means to discern critical features, and covariation offers a systematic context for near-simultaneous discernment of their variation. Even when relatively successful, students ignore the underlying value of the numbers they are representing by focusing on matching individual cases rather than relating variation in different dimensions. Fractional representation relates more easily to static, isolated, special cases depicted by chopping segments of the line than to continuous number.

FURTHER RESEARCH

The research constraints prevented follow-up interviews at this stage, but our intention had only been to confirm some conjectures and set the scene for further investigation. Furthermore, Thompson's work on covariation with college students (2002, p.203) showed that they typically found it very hard to talk about relations between rates of change although they could operate successfully with them to some extent at the level of examples and algebraic representations. We conjecture that, to get young adolescents to talk about relationships between digital and positional representations, we would have to know more about their likely perceptions in order to prompt them. This research prepares the ground for such enquiries.

Our hypothesis is that learning to coordinate different representations of mathematical variation is not merely a matter of translation, but also of understanding covariation. We further suggest that dynamic approaches can play a substantive supporting role.

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LOWER SECONDARY SCHOOL STUDENTS' UNDERSTANDING OF ALGEBRAIC PROOF

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Secondary school students are known to face a range of difficulties in learning about proof and proving in mathematics. This paper reports on a study designed to address the issue of students' cognitive needs for conviction and verification in algebraic statements. Through an analysis of data from 418 students (206 from Grade 8, and 212 from Grade 9), we report on how students might be able to 'construct' a formal proof, yet they may not fully appreciate the significance of such formal proof. The students may believe that formal proof is a valid argument, while, at the same time, they also resort to experimental verification as an acceptable way of 'ensuring' universality and generality of algebraic statements.

INTRODUCTION

Evidence from a range of research studies indicates that across the world, secondary school students have difficulties in following and constructing formally presented deductive proofs, in understanding how such proofs differ from empirical evidence, and in using deductive proofs to derive further results (for recent reviews, see, for example, Mariotti, 2007). As part of a wider research initiative, we have researched such issues in the case of geometrical proofs (see, for example, Kunimune, 1987; Kunimune, Fujita and Jones, 2009). In this paper we address the issue of students' natural cognitive needs for conviction and verification in algebraic statements.

In what follows, we first provide some background from related existing research. We then outline our theoretical framework which seeks to capture secondary school students' understanding of algebraic proof. This leads to the presentation of our results in terms of how students in lower secondary schools perceive 'proof' in algebra through an analysis of data from 418 students (206 from Grade 8, and 212 from Grade 9) collected in Japan in 2005.

STUDENTS' UNDERSTANDING OF ALGEBRAIC PROOF

Of the range of research studies on students and algebraic proof, we highlight two studies that are particularly pertinent. Healy and Hoyles (2000) surveyed high-attaining 14- and 15-year-old students about proof in algebra and found that students simultaneously held two different conceptions of proof. On the one side, the students viewed algebraic arguments as those they considered would receive the best mark from their teacher. On the other side, empirical argument predominated in students' own proof constructions, although most students were aware of the limitations of such arguments.

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Similarly, Groves and Doig (2008) report that while over 35% of Year 8 students (aged 13) recognised the need for a logical mathematical explanation to prove Goldbach's conjecture, over 60% "believed that it was enough to show it true for at least 1000 randomly chosen numbers or as many as possible, or to find one number for which it was not true" (p 345).

Such studies illustrate the need to continue studying students' cognitive needs for conviction and verification in algebraic proofs.

THEORETICAL FRAMEWORK

Extensive research in algebra education (for recent reviews, see, for example, Kieran, 2006) suggests the following issues are relevant to students' understanding of algebraic proof:

- Cognitive gap between arithmetic and algebra
- Understanding of algebraic symbols

For example, consider the statement: 'Sums of three consecutive whole numbers such as 2, 3 & 4 or 7, 8&9 are always multiples of 3'. To 'prove' this statement (that is, to verifying its generality) a secondary school student might do one of the following:

- Use arithmetic examples, sometimes with large numbers, and check results. For example, the student might say 'I tried 4+5+6=15=3x5, 12+13+14=39=3x13, 23+24+25=72=3x24, and so one, and I found the answers are always multiples of 3.
- Use algebraic symbolisation to provide an argument which might say three consecutive numbers can be expressed as 'x', 'x+1' and 'x+2'; the sum is '3x+3'. Now '3x+3 = 3(x+1)'. This shows that the sum is always a multiple of three.

Given our focus on students' understanding of proof, and the knowledge that the transition from experimental/empirical verification to formal proof is not straightforward, in our research we capture students' understanding of proof in terms of the following two components: 'Generality of proof' and 'Construction of proof' (see, Kunimune, 1987; Kunimune, Fujita and Jones, 2009). In our work on students' understanding of algebraic proof, we refer to these two aspects of proof and proving as: 'Construction of algebraic proof' and 'Generality of algebraic proof'.

The first one of these, 'Generality of proof' recognizes that, on the one hand, students have to understand the generality of proof (including the generality of algebraic symbols, with, for example, 'x' as generalised number), the universality and generality of proved algebraic statements, the difference between formal proof and verification by examples, and so on. The second of these two components, 'Construction of proof', recognises that, on the other hand, students also have to learn how to 'construct' deductive arguments in algebra by knowing sufficient about definitions, assumptions, proofs, theorems, logical circularity, and so on.

In Table 1 we characterise the nature of the two aspects of student proof and proving in algebra: 'Construction of algebraic proof' and 'Generality of algebraic proof' using ideas related to the cognitive gap between arithmetic and algebra, and student understanding of algebraic symbols.

Construction of algebraic proof	**Generality of algebraic proof**
To follow, or construct, algebraic proof, students might have to: • Understand what is required to show/explain in given problems • Understand assumptions and conclusions in statements • Represent given word problems by using algebraic symbols, interpret algebraic results etc. • Undertake fundamental algebraic manipulations; for example: 3x+3 = 3(x+1), 2x+y+3x-6y = 5x-5y and so on	To appreciate or understand why formal proof is necessary, students might have to: • Understand the universality and generality of statements which are represented by algebraic symbols • Understand the universality and generality of algebraic symbols • Understand the universality and generality of proof • Understand difference between formal proof and experimental verification (inductive approach)

Table 1: The two aspects of students' understanding of algebraic proof

Given these two aspects of student proof, our theoretical approach, informed by our work on proof in geometry (see, for example, Kunimune, 1987; Kunimune, Fujita and Jones, 2009) is to characterise four levels of students' understanding of algebraic proof. This characterisation is presented in Table 2. We argue that this framework captures the increasing complexity in students' attempts at construction of algebraic proof and generality of algebraic proof.

METHODOLOGY

The research design involved a survey of secondary school students' understanding of algebraic proof. A sample of relevant questions, and the corresponding marking scheme, is provided in Appendix A.

	Construction of algebraic proof	Generality of algebraic proof
Level 0	At this level, students do not understand what they have to explain.	At this level, students do not understand what they have to explain.
Level I	At this level, students explain their argument without using any	At this level, students do not understand neither why algebraic

	algebraic symbols	proof is necessary nor empirical verification is not enough to verify the universality and generality of algebraic statements
Level II	At this level, students start using algebraic symbols in their argument, but their use is incorrect	At this level, two things occur: a) students start recognising that empirical verification is not enough, but do not understand why they have to use algebraic symbols b) students start understanding why algebraic proof is necessary, but do not recognise that empirical verification is not enough
Level III	At this level, students use algebraic symbols properly to prove statements	At this level, students can understand why algebraic proof is necessary

Table 3: levels of students' understanding of algebraic proof

FINDINGS AND DISCUSSION

Our data is from 418 students (206 from Grade 8, and 212 from Grade 9) surveyed in Japan in 2005. The results for 'Construction of proof' are given in Table 4.

	Level 0	Level I	Level II	Level III	N
G8	64%	6%	10%	20%	G8=206
G9	29%	4%	14%	53%	G9 = 212

Table 4: results for 'Construction of proof'

As can be seen from Table 4, some 70% of Grade 8 students are at the Level I or below; that is, these students use empirical examples to verify the statements (Level I) or do not know what to do (Level 0). The results of Grade 9 students are superior. This is likely to be because students study more algebraic manipulation and proof in Grade 9. Nevertheless, 33% of students remain at either Level 0 or I, which implies that the teaching of algebraic proof could be improved in Grades 8 and 9.

The results for 'Generality of proof' are given in Table 5.

	Level 0	Level I	Level II(a) and (b)	Level III	N
G8	15%	36%	4%&26%	19%	G8=206

G9	11%	23%	3%&24%	39%	G9 = 212

Table 5: results for ‘Generality of proof

These results suggest that at Grades 8 and 9, students begin pondering the difference between empirical verifications and proof. This is, as indicated above, because students study more algebraic manipulations and proof in Grades 8 and 9. In general, more students are at Level II-b) than Level II-a). This implies that the students start understanding why algebraic proof is necessary in Grade 8, yet, at the same time, they do not recognise that empirical verification is insufficient for mathematical proof. Furthermore, 58% (11+23+24) of Grade 9 students remain at Level II-b), Level I or Level 0. These findings are very similar to the findings for a parallel study on geometrical proof (Kunimune, 1987, 2000; Kunimune, Fujita and Jones, 2009).

Thus, Grade 8 and 9 students are achieving in terms of ‘Construction of proof’, but not necessarily in terms of ‘Generality of proof’. There is a gap between the two aspects. This means that students might be able to ‘construct’ a formal proof, yet they may not appreciate the significance of such a formal proof. They may believe that formal proof is a valid argument, while, at the same time, they also believe experimental verification is equally acceptable to ‘ensure’ universality and generality of algebraic statements.

We now compare students’ Construction of proof (CoP) and their Generality of proof (GoP) at Grade 8 and Grade 9, see Table 6.

Grade 8 totals	**15%**	**36%**	**30%**	**19%**	**100%**	**N=206**
CoP III	0%	1%	6%	12%	19%	
CoP II	0%	3%	4%	3%	10%	
CoP I	0%	4%	2%	1%	7%	
CoP 0	15%	28%	18%	3%	64%	
Levels	GoP 0	GoP I	GoP II	GoP III	Total	

Grade 9 totals	**11%**	**23%**	**27%**	**39%**	**100%**	**N=212**
CoP III	0%	4%	14%	35%	53%	
CoP II	1%	4%	6%	3%	14%	
CoP I	1%	2%	1%	0%	4%	
CoP 0	9%	13%	5%	1%	29%	
Levels	GoP 0	GoP I	GoP II	GoP III	Total	

Table 6: compare students' Construction of proof (CoP) and their Generality of proof

The results in Table 6 show that, on the one hand, progressions from CoP I and CoG I to CoPII and CoG II are observed in Grade 9, when students study more algebra than in Grade 8. In addition, students are introduced to ideas of 'proof' in geometry, and this is likely to contribute to students' awareness of formal proof. On the other hand, in Grade 9 some 18% (=14+4) of students at CoP Level III are, at the same time, at GoP Level II or I (indicated in gray). This suggests that the teaching of algebra that the students might have experienced might have particularly emphasised the 'Construction of proof' aspects of algebra.

In general, more students are at Level II-b) than Level II-a). This implies that students start understanding why algebraic proof is necessary in Grade 8, but do not recognise that empirical verification is not enough. Furthermore, half of the Grade 9 students remain at Level II-b) or below. These findings are very similar to those that we have found with geometrical proof (Kunimune, 1987, 2000; Kunimune, Fujita and Jones, 2009).

CONCLUDING COMMENT

The Grade 8 and 9 students that we studied are achieving in terms of 'Construction of proof', but not necessarily so in terms of 'Generality of proof'. There is a gap between the two aspects. This means that students might be able to 'construct' a formal proof, yet they may not appreciate the significance of such formal proof. They may believe that formal proof is a valid argument, while, at the same time, they also resort to experimental verification as an acceptable way of 'ensuring' universality and generality of algebraic statements.

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Appendix A

Survey questions

Q8 Consider three consecutive whole numbers and their sums, e.g. 2+3+4=9=3x3, 7+8+9=24=3x8, and they are always the multiples of 3. In fact, if you consider any three consecutive numbers, then their sums are always the multiples of 3. Explain this.

Q9. See the calendar below carefully. You might notice that the sums of the three numbers in the boxes are as three times as the middle numbers (e.g. 2+9+16=27 = 3x9). Explain this is always true.

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Q10. Read carefully the following three statements which explain a statement 'A sum of two odd numbers is an even number'.

Student A: 1+1=2, 3+3=6, 1+3=4, 3+7=10. So, I think a sum of two odd numbers is an even number.	Accept/Not accept
Student B: Let one odd number be 'm', and the other 'n'. The sum is 'm+n', and 'm+n' is an even number. I think a sum of two odd numbers is an even number.	Accept/Not accept
Student C: Let 'm' and 'n' be whole numbers. Two odd numbers are '2m+1' and '2n+1' The sum is (2m+1)+(2n+1) = 2m+2n+2=2(m+n+1). As 'm+n+1 is a whole number, therefore 2(m+n+1) is an even number.	Accept/Not accept

Table 4: survey questions of students' understanding of algebraic proof

	Construction of proof	**Generality of proof**
Level 0	Q8 & 9 • No answer • Does not make sense	Q10 • No answer

	• Copy questions • Wrong explanation	
Level I	Q8 & 9 • Explanations with concrete examples, figures or words • Incomplete explanations with words • Explanations with concrete examples and arithmetic calculations	Q10 Answers such as: A: Accept; B: Accept; C: Not accept or A: Accept; B: Accept; C: Accept
Level II	Q8 & 9 • Incomplete or incorrect explanations with algebraic symbols	Q10 Level II(a) A: Not accept; B: Accept; C: Accept Level II(b) A: Accept; B: Not accept; C: Accept
Level III	Q8 & 9 • Explanations with algebraic symbols • Explanations with algebraic symbols with examples	Q10. A: Not accept; B: Not accept; C: Accept

MATHEMATICS TEACHERS' VIEWS ABOUT DEALING WITH MISTAKES IN THE CLASSROOM

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Observations in video studies have revealed repeatedly that there are differences in how mistakes are dealt with in different mathematics classrooms. Mistake-handling classroom processes are likely to have a "backing" in the professional knowledge and instruction-related beliefs of mathematics teachers. Accordingly, this study focuses on teachers' views about learning from mistakes and dealing with mistakes in the classroom, as well as on the interrelatedness of these views. Results of a corresponding study with over 130 participating German secondary teachers indicate that most of these teachers have a rather positive view about the importance of learning from mistakes, although there is evidence suggesting more complex interdependencies of mistake-related views.

INTRODUCTION

The role of learning from mistakes for competency development in mathematics can be seen differently according to the learning model referred to. This raises the question of how mathematics teachers see the role of mistakes for learning processes and which ways of dealing with mistakes in classroom interactions they favour. Asserting that a teacher's acting and reacting in the classroom is interdependent with her or his professional knowledge and instruction-related beliefs, the focus of this paper responds to the need for research about instruction-related views of teachers related to learning from mistakes and dealing with mistakes in classroom interactions. The findings of a corresponding quantitative study show partly contrarian views of teachers.

In the following sections, the paper gives an overview on the theoretical background of the study (1), clarifies research questions (2), describes design and sample of the study (3), reports on the results (4), and discusses the evidence (5).

THEORETICAL BACKGROUND

Mistakes in the learning process and dealing with mistakes in the classroom

Mistakes in the learning process are given different roles by different learning theories: On the one hand, from a behaviourist perspective (e.g. Skinner, 1958) mistakes are considered as undesired behaviour that is irrelevant for learning or even inhibiting learning processes. According to this point of view, mistakes should rather be ignored and correct knowledge should be emphasised: in mathematics lessons, for example, a mistake should never appear on the blackboard, because the mistake might be remembered by the learners like correct knowledge.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 449-456. Thessaloniki, Greece: PME.

On the other hand, in constructivist approaches, learning from mistakes plays an essential role for constructing knowledge, which implies that inter-individual discourse related to mistakes provides students with important learning opportunities. In line with a moderate constructivist position, Oser and colleagues (e.g. Oser & Spychiger, 2005) emphasise that only mistakes enable learners to discern the boundaries of "correct" knowledge, i.e. of knowledge conforming to a norm. Accordingly, the development of conceptual knowledge in mathematics classrooms, for example, should be supported by opportunities of discussing mistakes (cf. e.g. suggestions by Garuti, Boero, & Chiappini, 1999).

Hence, almost contrarian implications for ways of dealing with mistakes in the mathematics classroom can be drawn from different learning theories.

This might be one of the reasons for the observation of cultural differences in the way mistakes are dealt with in mathematics classrooms. For example, Stevenson and Stigler (1992) observed differences in classroom routines linked to mistakes between teachers in the U.S. and Japan. Japanese teachers considered mistakes as positive opportunities for discourse and competency development and encouraged the presentation and discussion of solutions containing mistakes in the classroom, whereas the U.S. teachers tended to avoid negative feedback to the learners in order to protect their self efficacy and motivation.

Oser and Spychiger (2005) reported empirical findings from Switzerland showing that relatively few mistakes appear in the classroom interaction. These findings have been replicated in a German video study (Heinze, 2004), stating also that the teachers often played a very dominant role when dealing with mistakes of students. Indeed, the case of Germany is particularly interesting, because according to the underlying goals of the dominant teaching script, central phases of mathematics lessons should consist of a classroom discourse between students and teacher, leading to new contents (Baumert et al., 1997). In this phase, productive discussions of mistakes and a corresponding argumentational exchange could create cognitively activating learning opportunities. However, video studies have repeatedly observed small-step question-answer routines with a dominant teacher role in German classrooms associated with a rather low level of cognitive activation (Baumert et al., 1997; Kuntze, Rechner, & Reiss, 2004). These observations might be explained on the one hand by socio-mathematical norms, according to which the learners know when mistakes are "permitted" and when they are "forbidden" (Heinze, 2005) and on the other hand by beliefs and views of mathematics teachers about learning from mistakes and ways of dealing with mistakes in classroom interactions. As there is still hardly any quantitative research in this second domain, the study focuses on aspects in this area.

Views of mathematics teachers concerning mistakes in the classroom

Professional knowledge of mathematics teachers encompasses not only sub-areas like "content matter knowledge" or "pedagogical knowledge" (cf. Shulman, 1986), but its components may be seen in a spectrum between declarative or procedural knowledge on the one side and prescriptive beliefs on the other side (cf. Pajares, 1992). Moreover, components of professional knowledge may be of a global, of a content-specific (cf. Törner, 2002) or even of a situation-specific nature (Kuntze & Reiss, 2005). Views of mathematics teachers about the role of mistakes for learning are connected to rather global epistemological aspects, whereas views about dealing with mistakes in the classroom tend to be more specific for particular instructional situations.

In a study with 20 participating primary U.S. teachers, Barnett and Sather (1992, p. 11-12) identified three types of teachers according to their views related to students' mistakes: The type label "conceal errors and only acknowledge the 'right' way" included teachers holding the conviction that incorrect solutions should rather be ignored or skipped, that there is no rationality behind mistakes and that mistake-related discussions should be avoided because of possible negative motivational consequences for the learners. Teachers belonging to the type label "expose errors and fix them" were in favour of making students explain their incorrect solutions in order to localise mistakes. Rationality behind mistakes was accepted by these teachers, but teacher-centred short explanations of correct solutions were favoured. The type label "expose errors for inquiry and debate" stood for teachers who saw mistakes as a starting point for reflective discussions providing learning opportunities for students. Such teachers preferred to avoid instantaneous feedback on the correctness of solutions and they even planned mistake-related learning opportunities in their lessons.

Some of the mistake-related components of professional knowledge identified by Barnett and Sather (1992) were included in a quantitative empirical study which gave evidence about relationships among these views and relationships with judgements on mistake-related videotaped instructional situations as well as with global beliefs of the teachers (Kuntze, Heinze & Reiss, 2008). However, the findings called also for a broader quantitative examination of views about learning from mistakes and about ways of dealing with mistakes in classroom interactions.

RESEARCH QUESTIONS

Taking into account also the findings related to mathematics lessons in Germany reported in the section above, the results of Barnett & Sather (1992) can be integrated into the following components of mistake-related views of teachers, which include possible reasons for avoiding argumentational classroom discourse when dealing with mistakes:

- Constructivist or behaviourist views related to learning from mistakes reflect corresponding global epistemological beliefs, as described above.
- Seeing rationality behind mistakes and favouring deepened considerations of mistakes in the classroom discourse are other facets of views linked to the role of learning from mistakes (cf. Barnett & Sather, 1992).

- Moreover, views about ways of dealing with mistakes in the classroom can concern reasons for avoiding discussions of mistakes like the danger of exposure of students and negative motivational effects (cf. Barnett & Sather, 1992), distraction from the chain of thought of the lesson, or lost time caused by mistake-related discourse.
- Possibly as a consequence of the perception of such dangers, the teachers might favour short corrections of mistakes or even a skipping of mistakes in the classroom interaction, like observed in the video study by Heinze (2004).

Focusing on these views of mathematics teachers, the study aims at providing evidence for the following research questions:

- Which views about learning from mistakes and dealing with mistakes in the classroom do mathematics teachers hold?
- Which interdependencies among these views can be identified?

SAMPLE AND METHODS

The research questions were explored in two studies. The sample of the first study consisted of N=137 German in-service secondary mathematics teachers working at academic-track schools (54 female, 76 male, 7 without data; 34 teachers aged up to 35 years, 34 teachers aged 36-45 years, 34 aged 46-55, 30 aged more than 55 years, 5 without data). The teachers were asked to answer a multiple-choice questionnaire (four-point Likert scale) containing nine scales (see Table 1) and referring to the mistake-related views introduced above.

74 of the 137 teachers were included a second, deepening study. The questionnaire of this sample contained additional scales about views of learning from mistakes and dealing with mistakes in the classroom as well as an additional sub-questionnaire about individual judgements on four classroom situations. In this deepening questionnaire, multiple-choice and open items were used. However, due to length limits, this paper focuses on results of the first study. Some of the results of the second study are presented in the graduate thesis of Susanne Schmailzl (2008).

RESULTS

The questionnaire scales and sample items are displayed in Table 1. As can be seen from this Table, the reliability values of the scales are good. In a corresponding factor analysis, 70.9% of the variance is explained by 7 factors which mainly correspond to the scales (two pairs of scales coincide in a common factor, respectively).

In order to examine possible interdependencies between different views, correlations between the scales have been calculated (Table 2). For the behaviourist view related to learning from mistakes, there are considerable correlations with most of the scales concerning reasons for avoiding discussions of mistakes. These scales are also interrelated among themselves. The scale “constructivist view related to learning from mistakes” shows only small correlations with the exception of the perception of rationality behind mistakes, which correlates with r = 0.48. In particular, the evidence in Table 2 suggests that behaviourist and constructivist views concerning the role of mistakes for learning are not necessarily antipodes on a common dimension. This is supported also by a factor analysis for the scales (Figure 1), in which the scales of behaviourist and constructivist views concerning the role of mistakes can be found in two separate factors together with other scales, respectively, which suggests that there is a structure of two almost independent dimensions of mistake-related views.

Scales	Sample item	number of items	α (Cronbach)
Behaviourist view related to learning from mistakes	When mistakes are presented on the projector or on the blackboard, students will remember the incorrect instead of the correct knowledge.	4	0.81
Constructivist view related to learning from mistakes	Learning from mistakes is a prerequisite for constructing mathematics-related knowledge.	4	0.86
Avoiding discussions of mistakes because of danger of exposure	In order not to expose students in class, their mistakes should not be treated intensively.	4	0.90
Avoiding discussions of mistakes because of distraction from chain of	In order not to distract the students from the chain of thought of the lesson, mistakes should not be treated intensively.	4	0.83
Avoiding discussions of mistakes because of fear of lost time	In order not to loose time, I avoid time-consuming discussions of students' mistakes, as far as possible.	4	0.84
Favouring of short corrections of mistakes	A short and clear correction of mistakes is better than analysing mistakes in the classroom.	2	0.88
Favouring of skipping of mistakes	It is not necessary to consider incorrect utterances of students, if other students give the correct answer.	2	0.84
Rationality behind mistakes	Behind a student's mistake, you can generally find rational thoughts.	2	0.87
Favouring of deepened considerations of mistakes	An incorrect utterance of a student should always be considered as long as every student will have understood why it is incorrect.	2	0.71

Table 1: Scales, sample items and reliability values

	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Behaviourist view related to learning from mistakes (1)	-.09	.49***	.54***	.46***	.29***	.20*	-.05	-.09
Constructivist view related to learning from mistakes (2)		-.19*	-.14	-.24**	-.16	-.08	.48***	.17*
Avoiding discussions of mistakes because of danger of exposure (3)			.62***	.45***	.37***	.21*	-.09	-.34***
Avoiding discussions of mistakes because of distraction from chain of thought (4)				.69***	.45***	.47***	-.15	-.28***
Avoiding discussions of mistakes because of fear of lost time (5)					.51***	.47***	-.23**	-.22**
Favouring of short corrections of mistakes (6)						.41***	-.15	-.09
Favouring of skipping of mistakes (7)							-.17	-.28***
Rationality behind mistakes (8)								.22*

Favouring of deepened considerations of mistakes (9)	

*** : correlation significant with p<.001 ** : correl. signif. with p<.01 * : correl. signif. with p<.05

Table 2: correlations between mistake-related scales

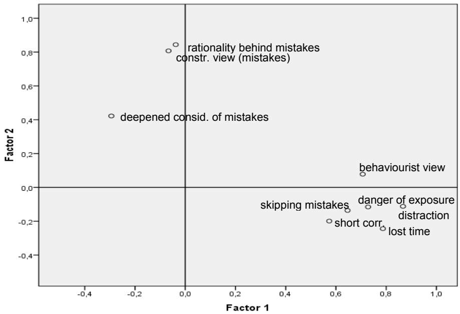

Figure 1: Factor analysis for the mistake-related scales (load diagram) (2 factors, 54.8% of variance explained)

Some key data concerning the research question about what views the mathematics teachers were holding are represented in Figure 2, which shows the mean values and their standard errors for the scales. Moreover, Figure 2 contains the results of a Cluster Analysis (Ward method), which was calculated in order to find out about different patterns of views according to the different scales. The four resulting clusters suggest that there are different views of the corresponding groups of teachers: There is a relatively big group of teachers which is marked by low values of the behaviourist view and corresponding views about dealing with mistakes in the classroom and positive values related to the importance of mistakes for the learning process, rationality behind mistakes and the favouring of a deepened consideration of mistakes. The other extreme case is a smaller group of teachers holding relatively consistent behaviourist (and non-constructivist) views about mistakes and ways of dealing with them in the classroom.

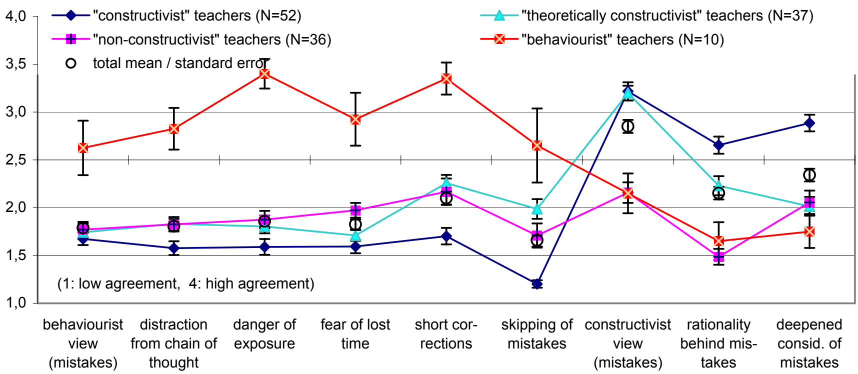

Figure 2: Mean values of scales and results of cluster analysis (Ward method)

Another cluster in Figure 2, named "theoretically constructivist" teachers, is marked by a high value of the constructivist view about the role of mistakes, but low mean values of "rationality behind mistakes" and "deepened consideration of mistakes", which, in a way, contradicts to a consistent constructivist view. The higher values of "short corrections" and "skipping of mistakes" also support the interpretation that this group of teachers holds a rather theoretical and somewhat inconsistent constructivist view. Finally, a group of rather "non-constructivist", but also non-behaviourist teachers shows – when compared to the "constructivist" teachers – a tendency of higher values in refraining from mistake-related discussions in the classroom. These teachers hardly see rationality behind mistakes, but they seem not to hold clear "behaviourist" preferences for dealing with mistakes in the classroom either.

DISCUSSION

As far as the first research question related to the teachers' views is concerned, the over-all mean values indicate that the teachers hold a low behaviourist and a rather constructivist view about the role of mistakes for competency development in mathematics. The teachers claim on average not to prefer to avoid mistake-related discussions and short reactions to mistakes. However, the corresponding cluster analysis enables us to see that there is a small group of teachers holding consistent "behaviourist" views and that the teachers' views seem not always to be completely consistent with bi-polar theoretical expectations. An explanation for this can be that the teachers' mistake-related views might reflect somewhat less a solid background in terms of a consistent professional knowledge but rather individual justifications of possibly unreflected classroom routines.

An additional interpretation focuses on the findings about the interrelatedness of the views examined in this study: teachers often have to decide on ways of dealing with mistakes in situations with conflicting goals. Time limitations, reaching curricular content goals in a lesson or the perception of a conflict between individual learning of a student and collective competency growth might inhibit teachers to have in mind clear guidelines of creating cognitively activating learning opportunities related to mistakes of the learners. Conformingly, reasons for avoiding deepened considerations of mistakes appear interdependent. In further studies, it would be very interesting to include the teachers' instructional practice in the research design, in order to examine the extent of relevance of the teachers' views for practice. In such a study, teachers' views about particular mistake-related instructional situations as considered in the deepening study by Schmailzl (2008) can give additional insight, because situation-specific professional knowledge can be seen as closer to instructional practice.

Summing up, the classification by Barnett and Sather (1992) has been replicated only partly with this larger sample of (German) secondary teachers. Even though a majority of teachers tends to prefer less behaviourist and rather constructivist views, there is evidence showing possibilities of improving mistake-related professional knowledge of mathematics teachers.

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MEANINGS FOR ANGLE THROUGH GEOMETRICAL CONSTRUCTIONS IN 3D SPACE

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Abstract. *The construction of meanings for the notion of angle in 3d geometrical space is studied during the implementation of specially designed geometrical tasks in the classroom. 13-year-old pupils were engaged in exploring the mathematical nature of angles while controlling and measuring the behaviours of geometrical objects in 3d simulations. Pupils worked in pairs using a specially designed computational environment that combines symbolic notation (by means of programming) and dynamic manipulation of graphically represented geometrical objects. The findings reveal a progressive coordination of different facets of angle in 3d space facilitated by the use of the available tools.*

THEORETICAL FRAMEWORK

This contribution reports classroom research [1] aiming to explore 13 year-olds' construction of meanings for the notion of angle in 3d geometrical space during activity involving the construction and manipulation of programmable geometrical figures. The students worked in collaborative groups of two using *MaLT (MachineLab Turtleworlds)*, a microworld environment for geometrical constructions which combines symbolic notation -through a specially designed version of Logo- with dynamic manipulation of graphically represented mathematical objects (Kynigos & Latsi, 2007).

Angles and turns are critical to 3d geometrical knowledge and since they are related to students' everyday experience they can be considered as a rich domain for mathematics meaning-making, not systematically studied up to now. Student's difficulties in the angle domains are well documented in the literature. Research results strongly suggest two major factors influencing students' use of angles: the physical presence or absence of the lines which make up its arms and the conceptualisation of turning as a relationship between two headings of segments (Clements et al., 1996). The situation is aggravated in 3d space by the apparent difficulty of students to recognise the basic elements of an angle. For example, an 'opening-closing door' situation incorporates a more obscure similarity of the standard angle domain since the arms of the embedded 3d angle are rectangles. Moreover, turning in 3d space can be considered as a sophisticated concept since the turning motion itself usually does not leave a trace and the 'heading' must be reconstructed from memory. It is also clear from previous research that students have great difficulty in coordinating the various facets of the angle embedded in various physical angle contexts involving slopes, turns, intersections, corners, bends,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 457-464. Thessaloniki, Greece: PME.

directions and openings (Mitchelmore & White, 1998). Taking this into account Mitchelmore and White (2000) highlight that a mature abstract angle concept depends essentially on learning to link the various physical angle contexts together through "the systematic attempt to investigate our spatial environment mathematically" (p. 233).

In the light of the above data, it makes no sense to perceive angle in 3d space as a mathematical notion on its own but rather it is more useful to consider it in terms of the concepts interrelated with it, the situations in which it may be used and the available representations, which constitute –in the words of Vergnaud (1996)- a *conceptual field*. For instance, a concept tightly related to angle in 3d space is that of a turn, a situation in which it may be used can be a situation evoked by a given task (e.g. an opening-closing door simulation) while the available representations can be based on the use of paper and pencil or on the use of computational tools.

Taking a constructionist perspective (Harel & Papert, 1991) in the present study we have selected the version of Logo developed in MaLT as one context to explore students' ideas around the concept of angle in 3d space based on turning and directionality. Our purpose was (a) to relate children's construction of meanings for the notion of angle in 3d space explicitly to their physical angle experiences and (b) to offer a framework in which to account specifically for their difficulties in coordinating different aspects of the notion of angle as well as to throw light on the paths by which students might come to integrate their various angle concepts in 3d space. More specifically, the main focus of the study concerned the ways by which students conceptualise angle as a spatial visualisation concept representing turn and measure when engaged in activities involving the construction and dynamic manipulation of 2d geometrical objects. Under this perspective our central research aim was to study how students used the available representations in MaLT to construct meanings for angle: (a) as a *geometric shape*, i.e. formed between two geometrical objects which can be either segments (in 2d geometrical figures) or 2d geometrical figures (in the 3d space, e.g. dihedral angles); (b) as a *dynamic amount*, indicating a change of directions both as a turn and as the result of a turn which can also be represented by a variable; and (c) as a *measure* represented by a number.

THE COMPUTER ENVIRONMENT

MaLT is a programmable environment for the creation and exploration of interactive 3d geometrical constructions consisted of the following interconnected components (see Figure 1).

• Turtle Scene (TS). This is a 3d grid-like projection of a simulated 3d space in which a 3d turtle is visualised when a command (or a procedure) is executed. Whenever the turtle moves it leaves behind it a trace which is a selectable 3d cylindrical line.

• Logo Editor (LE). This is the Logo-like programming interface which is linked to the TS. The user is able to write, run and edit Logo procedures to 'drive' the construction of geometrical objects by the turtle using a specially designed version of Logo. A distinctive feature of this specific version is that it provides an extension of Logo commands in 3d space including two new types of turtle turns: 'UPPITCH/DOWNPITCH n degrees' ('*up/dp n'*) which pitches the turtle's nose up and down and 'LEFTROLL/RIGHTROLL n degrees' ('*lr/rr n*') which moves the turtle around its trunk/vertical axis.

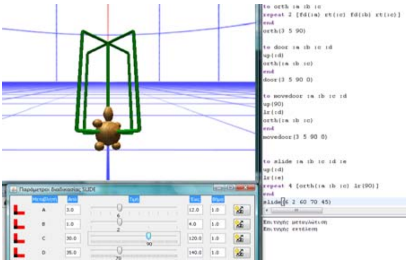


Figure 1: A revolving door simulation in MaLT.

• Uni-dimensional Variation Tool (1dVT). The main part of this component consists of 'number-line'-like sliders (see at the bottom of Figure 1), each corresponding to one of the variables used in a Logo procedure. After a variable procedure is defined and executed with a specific value, clicking the mouse on the turtle trace activates the tool, which provides a slider for each variable. Dragging a slider has the effect of the figure dynamically changing as the value of the variable changes sequentially.

RESEARCH SETTING AND TASKS

The work reported in this paper is part of a design-based experiment (Cobb et al., 2003) concerning the exploration of the mathematical nature of angles in 3d space by groups of 13-year-old pupils who work collaboratively in the classroom setting using MaLT to construct, control and animate 3d geometrical objects often encountered in everyday situations (e.g. doors, revolving doors). The experiment took place in a secondary school with one class of twenty 7 grade pupils (13 year-old) and one experimenting teacher who acted also as a researcher in the classroom. The class had totally 18 teaching sessions with the experimenting teacher over two months.

We divided the activity sequence in two phases and developed for each one of them a strand of two tasks. In designing the tasks for the first phase of the activity we took into account the constructionist theoretical implication that angle as turn (which inevitably involves directionality) in the simulated 3d space of MaLT is closely related to real world experiences as well as to body-syntonic activities such as walking or piloting a flying object. In task 1 students were asked to move the turtle in the right and left corner of the 3d space and then to bring it back at its initial position. It was not specified on purpose what was meant by 'the left and right corner', so as to leave students to explore 3d space and develop their own 'navigation' strategies. In task 2 students were asked to navigate the turtle in such a way so as to simulate the take-off and the landing of an aircraft. In the second phase of the activity angle was approached as a dynamic quantity through the simulation of real 3d objects in the TS.

In particular, in task 3 students were asked to construct rectangles using parametric procedures in at least two different planes of the TS simulating the windows of a virtual room. In task 4 students were asked to develop a parametric procedure so as to simulate the opening and closing of a door while in task 5 they were asked to use the 1dVT to control the four variables of a ready made procedure given to them so as to create the simulation of a revolving door.

METHOD

During the implementation of the activity a second researcher participated in the classroom acting as a co-researcher. The adopted methodological approach was based on participant observation of human activities taking place in real time. The researchers intervened in the children's work by posing questions and encouraging them to clearly explain their ideas and strategies. We used a specially designed screen capture software -called Hypercam- which allowed us to record students' voices and at the same time to capture all their actions on the screen. For the analysis we transcribed verbatim the audio recordings of three groups of students (focus groups) throughout the teaching sequence and we also selected significant learning incidents from the work of all groups in the classroom. Background data were also collected (i.e. observational notes, students' electronic and written works). In the analysis a phenomenological approach (Nemirovsky & Noble, 1997) was adopted according to which researchers' interpretations were focused on the student's interactions with the available representations and the meanings they attached to these from their own points of view while struggling to relate the turtle's move/turn and/or specific 2d representations in forming angular relationships in 3d space. The episodes were selected (a) to have a particular and characteristic bearing on the students' interaction with the available tools and (b) to represent clearly aspects of the students' construction of meanings for particular aspects of angle in 3d space (i.e. geometric shape, dynamic amount and measure) emerging from this use.

ANGLE AS A TURN IN 3D SPACE

During task 1 students have constructed simple crooked lines limiting turtle's motion either on the 'ground plane' or on the 'screen plane' (vertical to the ground plane) while their experimentation concerning the notion of angle was developed around how to find out the measure of turtle turns. Their preference in moving in a 2d plane was tightly linked to their difficulty in conceptualising the role of the rotation commands -especially rightroll/leftroll – while moving the turtle around in the TS. However task 2 seemed to have provided a more fruitful context for the experimentation of students with both new types of turtle turns. It seemed that the simulation of the take-off and landing of an aircraft brought in the foreground the concept of angle as a turn in 3d space with particular measure involving also angle as a slope, an aspect of angle difficultly recognised by students as the one supporting edge is missing (Mitchelmore & White, 2000). As they were engaged in navigating the turtle so as to simulate the take off of an aeroplane, in the initial stages of exploration pupils appeared not to focus on the change of planes or the angles (internal or external) of the crooked line that the turtle had drawn but rather at the angle that was drawn by the turtle in relation to the horizontal ground plane. This is evident in the following excerpt in Group A students' confusion over the way in which the commands up(45) and lt(50) affected/determined the generated graphical outcome visualized in Figure 2. These pupils were reflecting upon the commands given to the turtle so as to explain why the aircraft collided to the ground (Figure 2).

Researcher: Hey, nice take off!! I see you hit the ground!

Student: Look there is a slope up(45) and then a slope of lt(50).

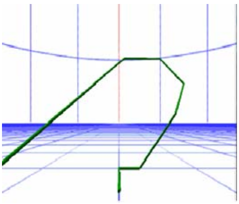


Figure 2: Simulating the taking-off and flying of an aircraft.

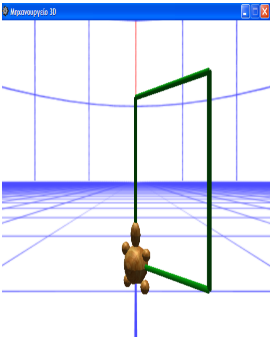
fd(5)
rt(90)
fd(2)
up(45)
fd(5)
lt(50)
fd(6)
rr(30)
up(45)
fd(8)
fd(5)

A closer look at the respective Logo code reveals that students, drawing upon their everyday experiences, considered the horizontal ground plane and the directions of up and down as fixed. This may be a possible explanation of their insistence to use the up command in order to get height regardless of the opposite graphical outcome. Apart from the lack of previous familiarisation with the new kinds of turtle turns, the preference of students to move the turtle only in one plane (during task 1) or their difficulty in coordinating turtle's turns and trace with the notion of angle as a slope could possibly be interpreted in the light of the fact that pupils, accustomed to work with 2d representations of geometrical figures, might have had difficulties in understanding either the conventions used to represent a 3d object in the computer screen (Lowrie, 2002) or the conventions that underlie the move of an entity – here the turtle – in a simulated 3d space. As we will see in the next paragraph simulating animated 3d geometrical objects with the use of variables was critical in helping students coordinate the various facets of angles present and conceptualise angle as a change of direction in 3d space while turning with particular measure.

ANGLE AS A DYNAMIC ENTITY FOR SIMULATING 3D OBJECTS

In task 3 the need to design figures in different planes of the 3d space challenged pupils to move the focus of their attention from directed turns between lines and planes to directed turns between two similar geometrical figures which is related to the conceptualisation of a dihedral angle in 3d space. For example, most groups of pupils recognised that in order to construct windows in two consecutive walls (planes in mathematical terms) the use of the commands 'rr/lr' or 'up/dp' was needed. During this construction process students easily identified the dihedral angle defined by the two consecutive windows (i.e. rectangles) and used the terminology familiar to them from 2d geometry in order to describe it. However, all groups of pupils had difficulties in identifying its measure. For instance, students of group B characterized the dihedral angle drawn by the turtle as an acute and not as a right one as it was the case, although they had commanded the turtle to leftroll 90 before drawing the second window. It seemed that students had focused more on the visual characteristics of the figural representation and were confused by the 'distortion' of the dihedral angle as a result of the use of a vanishing point in the line of horizon of the TS designed to strengthen the sense of depth in the representation.

However, the use of the two new types of turtle turns coupled with pupil's explorations of angle as a dynamic amount that could by dynamically handled and changed sequentially using the functionalities of 1dVT facilitated further the visualization of different planes in 3d space. For instance, in task 4 (Figure 3) students decided to use not a fixed turn measure but a variable after the rightroll command so as to simulate the opening and closing of a door.




```
to rect :a :b :c
up(90)
fd(:a)
rt(90)
fd(:b)
rt(90)
fd(:a)
rt(90)
fd(:b)
rt(:90)
rr(:c)
end
```

Figure 3. The opening-closing door.

The use of 1dVT allowed students to view their constructions from different perspectives which might have minimized the 'distorting' effects of static 3d representation and prompted them to focus on the measure of the turtle's turn in the Logo code. The more the students appeared accustomed to the conventions used in the 3d simulated space the more they were able to coordinate the visual characteristics of the drawn angles with their measure related to turtle's turns. For instance during the dynamic handling of the revolving door simulation (task 5) students were able to overcome the difficulties faced earlier during task 3 and to recognize the four consecutive right dihedral angles created between the four rectangles around the common vertical side of the four rectangles (see Figure 1). Experimenting with the variables of the procedure 'slide' (see below), which was given ready-made to them (task 5), so as to create a revolving door moving around group B students progressively got able to handle different aspects of angle simultaneously. Since for random values of the variable :c four parallelograms appeared around the common side of them, S1 compared the visual outcome with a door and recognized the need to construct initially four rectangles dragging on the slider :c on the 1dVT.

```
to slide :a :b :c :d :e
up(:d)
lr(:e)
repeat 4 [repeat 2 [fd(:a) rt(:c) fd(:b) rt(180-:c)] lr(90)]
end
```

S1: Wait, we should move it here first. It's the angle of the rectangle [*moves the slider :c to the value 90 so as to construct 4 rectangles*], so as to become like this (*i.e. the door*) and then probably turns with this [*moves the slider :e*]. Let's see…

S2: Yes, it definitely turns around with this [*i.e. slider :e*] as it has lr.

S1: Yes, but we don't only want it to turn, we also want it to move even further down.

S2: I should change here [*He puts the slider :d to the value 90 so as to have the simulation in a vertical position*].

S1: Yes, 90 is fine.

S2: Now, with this [*points to the slider :e*] it turns around normally.

The above excerpt accompanied by the respective Logo code indicates that students created meanings in relation to angle (a) as a constitutive element of a figure which is defined and stay fixed (variable :c), (b) as a means to move from the horizontal plane to the vertical one in relation to the viewing axis of the user which is again defined and stay fixed (variable :d) and (c) as a means of constantly changing planes in 3d space (variable :e) around the common vertical side of the four rectangles (Figure 1).

CONCLUSIONS

Under the constructionist theoretical perspective the above incidents illustrate the dialectic relationship between the evolution of students' interaction with the available tools and their progressive focusing on angular relationships underlying the current geometrical constructions and representations. As students were engaged in navigating the turtle in MaLT they gain a sense of the mathematical ideas related to the notion of angle through the construction and dynamic manipulation of 2d geometrical objects in 3d space by a process of hypothesising, experimenting and reflecting on the empirical observation of the graphical feedback on the screen. Struggling with the representational convention used and the identification of the angles corresponding to turtle's turning students progressively managed not only to construct dihedral angles defined by parallelograms in different planes but also to approach angle as a dynamic entity for simulating 3d objects through the use of variables. This kind of experience was critical for the coordination of the various facets of the angular relations (Mitchelmore & White, 2000) simultaneously present in the subsequent revolving door simulation where angle was approach either as a geometric shape with a fixed measure (e.g. between 2d geometrical figures) or as a dynamic amount with a varying measure. The above analysis describes angle concept development in terms of interconnected stages of abstraction which can be considered as representing a progressively more refined classification of students' experience based on their interaction with the available tools.

Notes

1. "Representing Mathematics with Digital Media", http://remath.cti.gr, European Community, 6th Framework Programme, Information Society Technologies (IST), IST-4-26751-STP, 2005-2008.

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STUDENTS' ACTIVITIES ABOUT FUNCTIONS AT UPPER SECONDARY LEVEL: A GRID FOR DESIGNING A DIGITAL ENVIRONMENT AND ANALYSING USES

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In this paper, drawing from current literature, we present a grid built to design and experiment one of the digital environments of the ReMath European project. This grid connects various activities about functions at upper secondary level. Analyzing classroom use of the environment, we show how the grid helps to make sense of its potentialities, especially as a tool for functional modeling. Finally, we situate this outcome inside the theoretical ReMath work that aims to progress in connecting and integrating theoretical frames in technology enhanced mathematics learning.

INTRODUCTION

This paper deals with students' activities related to the concept of function at upper secondary level. We are particularly attentive to how in many curricula like in France, activities in the functional world are supposed to engage and support the transition between algebra and calculus and how digital technologies can enrich students' activities in that area and make these more productive.

As pointed out by Kieran (2007) in her recent review of the literature on learning and teaching algebra, particular emphasis has been put on the entrance in the algebraic world at middle school level or even earlier. Nevertheless, a substantial body of research also exists at upper secondary level and pays special attention to digital technologies, some giving more emphasis to activities supported by Computer Algebra Systems (CAS) and the role they can play for supporting the understanding of algebraic equivalence and forms (Kieran, Drijvers 2006), and others stressing more globally the role that activities based upon the use of varied computer representations can play in the understanding of functional dependencies (Arzarello, Robutti 2004, Falcade et al. 2007) or in the introduction to calculus concepts (Maschietto 2008).

Our concern is that, at the moment, these approaches are rather unconnected, and thus difficult to use for analyzing classroom activities, relatively to the global challenge of learning about functions at upper secondary level. Our experience deals with the design and experiment of digital didactic artifacts in the Remath project, one of these, Casyopée, being developed by our team to specifically address this challenge and the fragmentation of frameworks has been pointed out as a difficulty when trying to make sense of their didactical functionalities (Artigue, Cerulli 2008).

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 465-472. Thessaloniki, Greece: PME.

In this paper, we present a grid that we think useful to connect various activities about functions. We show how it helps to make sense of Casyopée’s features especially as a help for functional modeling and for connecting varied activities. Finally, we study an example of students’ activities using Casyopée and we situate this grid inside the theoretical ReMath work.

THE GRID

		Representations and Types of activities				
		Enactive-Iconic		Algebraic		
		Local	Global	Generational	Transformational	GlobalMeta
Objects represented	Covariation and dependenciein a physical system Covariation and dependencie between magnitudes or measures Mathemati-cal Functions of one real variable	Small moves. Obser-ving effect on elements Small moves. Obser-ving effect on values Tracing graphs Browsing Tables	Moving elements Obser-ving transfor-mations Graphs of measure against time or another magni-tude Percei-ving prope-rties of graphs and Tables	Building pre-algebraic “geometrical” formula. Choosing an independent variable. Expressing algebraically a domain and a formula	Computing, recognising equivalent expressions. Choosing an appropriate form	Considering ‘generic’ objects and measures. Interpreting Working on ‘families’ of functions Parameters (animated or formal). Proving

Table 1

Our grid is sketched in table 1. The organization of rows is based on the epistemological ground that the mathematical concept of function cannot be separated from the idea of dependency in physical systems where one can observe mutual variations of objects. This necessary connection has also a cognitive foundation: the idea of function is linked to the sensual experience of dependencies in a physical system (Radford 2005). While research generally deals with one or both of these levels, we consider a third level, the level of dependencies between measures or magnitudes that bridges the physical word and the mathematical word of functions. Falcade et al. (2007) for instance focused on the first level: they choose Dynamic Geometry (DG) as a field to provide students a qualitative experience of covariation and of functional dependency. Arzarello & Robutti (2004) did one of the studies covering the first and third levels, but not the magnitudes. The classroom activities they experimented were about physical movements. Using a motion sensor and a graphic calculator, students produced and interpreted graphs and numeric tables to describe different kinds of motion in term of mathematical functions. The level of magnitudes (distance, time) was actually taken in charge by the calculator: using implicit variables and units for distance and time, it directly transposed the movement into the mathematical world of tables and graphs. Our hypothesis is that activities at this second level can be fruitful for conceptualizing functions: building appropriate variables to quantify observations, distinguishing functional dependencies among more general co-variation, choosing dependant and independent variables… strongly contribute to make functions exist as model of physical dependencies.

The columns in the grid refer to different representations of concepts in calculus adapted from Tall (1996). We choose to separate representations of relationship between two elements that can be thought of enactively or from images or approximations, and those that imply an explicit exact expression and thus an algebraic language. Activities in the columns under the heading "enactive-iconic" involve experience of movements inside physical systems, as well as work on graphical or tabular representatives of these, and 'explorations' (Yerushalmy 1999) on graphs and tables of approximate values. In these activities, a local point of view is related to what happens 'near a value', and therefore 'local' activities tend to explore physical systems by 'small movements", and tables and curves by tracing or browsing near a point, with adequate zooming in. A global point of view considers properties of dependencies and their representatives on whole intervals. In our meaning, distinguishing the local and global points of view in these activities is essential in the transition to calculus (Maschietto 2008). Research in this column insists upon the complexity of semiotic systems involved in these activities. Falcade et al. (2007) for instance insisted that DG tools (Dragging, Trace, Macro...) and particular signs (segments, rays, figures representing either the domain, on which the independent variable varies, or the range, on which the dependent variable varies) offer a common semiotic system that the students and the teacher can elaborate on.

Activities under the heading 'Algebraic' involve signs and rules specific to Algebra. Student activities in algebra have been classified by Kieran (2004) into three categories: generational, transformational, and global / meta-level that correspond to columns in our table: "*The generational activities of algebra involve the forming of the expressions and equations that are the objects of algebra (...). The* *transformational (rule-based) activities includes, for instance, collecting like terms, factoring, expanding, substituting(…) The global / meta-level mathematical activities include problem solving, modelling, noting structure, studying change, justifying, proving, and predicting.*"

We are specially interested by how meaning can develop at the interface between the three categories and enactive-iconic activities. Very significant *generational activities* exist at this interface: finding an algebraic expression (domain and formula) for a function is motivated and takes sense when the function is conceived as a model of an enactive phenomenon. We are especially interested by the connections between generational and enactive-iconic activities at the second level (magnitudes): after independent and dependent variables have been built using a formalisation specific to magnitudes, dependencies can be thought of by reasoning on the laws governing the magnitudes in the system and algebraic work can takes place to express this dependency mathematically.

Rich connections also exist between *transformational* and enactive-iconic activities. For instance students can connect the notion of equivalence, central in the transformational activities with the coincidences of graphs. Yerushalmy (2009) designed the VisualMath curriculum, based on specific graphing software, so that students come to understand what operations on equations are legal ones while performing manipulations as a way to conjecture and understand "on screen" results. Transformational activity is a domain where, for many authors, Computer Algebra Systems (CAS) could alleviate problems related to student weaknesses in computational skills. But, as Yerushalmy note, being 'solution tools' they do not support the construction of a visible map of the point of view of the curriculum.

Global-Meta activities include using algebraic means to express generality in the study of physical systems. Modeling a dependency often involves general objects (for instance arbitrary points in a geometric figure) and thus the model is a family of functions. Algebraically, it is expressed by a function whose domain and formulas depends on parameters. Using the model to solve a problem in the physical system brings together algebraic treatment and enactive-iconic interpretation of parameters.

CASYOPEE

Our choice was to build an open computer environment covering comprehensively the activities in the above grid and allowing easy connection between them. Casyopée has two modules: one is a symbolic window and the other a Dynamic Geometry (DG) window. The symbolic window differs from standard CAS, following Yerushalmy's criticism that standard CAS do not support the curriculum. In Casyopée each object has a clear status with regard to the curriculum. While standard CAS' window is a mere memory of commands and feedback, Casyopée's interface displays dynamically the objects relevant for a problem. The symbolic window supports transformational and global meta activities as well as enactive-iconic activities on functions. An important feature with regard to these activities is that parameters can have two different statuses that a user can switch at any time. One is ‘animated’: the parameter has a value that the user can change by way of a slide bar. This status corresponds to the parameter as a placeholder when the student looks at a particular expression or table or graph, or as a changing quantity when (s)he looks at the evolution of forms by operating the slide bar. The other one is ‘formal’: while keeping a value used for graphs and figures as a placeholder, the parameter is treated formally in algebraic transformations.

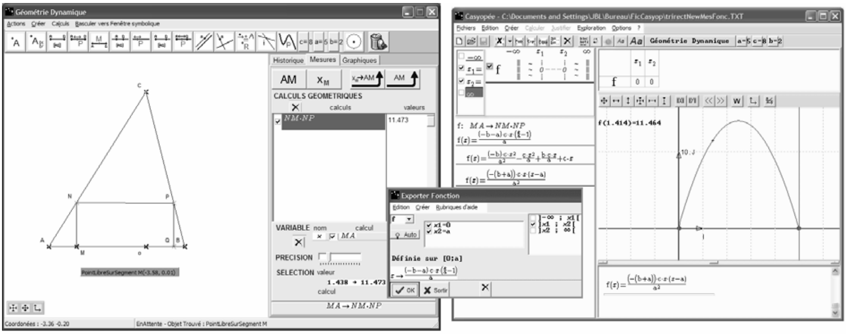

Figure 1: Casyopée’s symbolic and DG windows and the exportation form.

The DG window derives from our choice of geometrical figures as physical systems for enactive activities, consistent with Falcade and al. (2007)’s. It offers the main features of standard DG systems: creation and animation of geometrical objects. It proposes also symbolic facilities for generalisation, parameters entering into the definition of geometrical objects. Finally it offers specific aids for modelling dependencies between magnitudes to respectively create a “geometrical calculation” expressing a measure, select a measure as an independent variable and finally export a dependency between this variable and a calculation, if it exists, as a mathematical function into the symbolic window. These aids ‘reify’ the intermediate level between the figure as a physical system and the mathematical function and allow students to perform a generative activity that would be very difficult without the help of Casyopée.

AN EXPERIMENT

We draw on an experiment in two eleventh grade scientific classes with Casyopée. This experiment is part of the ReMath project. The scenario included first three sessions focusing on capabilities of Casyopée’s symbolic window and on quadratic functions. A second part (two sessions) aimed first to consolidate students’ knowledge on geometrical situations and to introduce them to the geometrical window’s capabilities. Finally, in the third part (one session), students had to take advantage of all features of Casyopée and to activate all their algebraic knowledge for solving an optimization problem. We draw from this last session to show how students developed and connected the activities.

The problem

ABC a triangle, and o the foot of the altitude from C to AB. Among the rectangles MNPQ with M on [oA], N on [AB], P on [BC], Q on [oC] is (are) there (a) rectangle(s) with maximum area?

Figure 1 shows the different features offered by Casyopée to explore and solve this problem. Table 2 shows the main actions that a student can undertake. Casyopée allows going freely forth and back between these actions. For instance a student might explore and model the dependency on a particular figure before using parameters to consider a generic triangle. Although we situated the actions in specific cells related to their main focus, they often involve other representations. For instance the algebraic actions at the level of the figure and magnitudes are deeply linked to the enactive-iconic representations. The aids for modelling dependencies between magnitudes explained above link actions at the level of magnitudes and at the level of mathematical functions: introducing a 'geometrical calculation' for the area (that is to say a formula involving distances, $MN \cdot MQ$ for instance) allows local exploration and conjecturing one maximum, M being the midpoint of [oA]. When a student chooses an independent variable related to M, Casyopée displays possible dependencies between magnitudes in forms like for instance $MA \to MN \cdot MQ$. The values of the variables are updated when moving M allowing further local exploration and understanding of the formalisation. When the student chooses to export the dependency, Casyopée computes the algebraic domain and formula of the corresponding function. Casyopée offers also means for connecting enactive-iconic exploration at the level of magnitudes and of mathematical functions: a cross on the graph of the function is dynamically linked with the position of point M.

	Enactive-Iconic	Generational	Transformational	GlobalMeta
Geometric figure Length and area Mathematical Function	Global exploration: various shapes of the rectangle Local exploration to conjecture one maximum. Local trace of graph of the function, global recognition of a parabola	Introducing 'geometrical calculation' for the area Getting an algebraic formula and a domain	Recognizing a quadratic function, using knowledge to compute an optimal value.	Considering parameters a, b and c to denote oA, oB and oC Working on 'families' of functions Interpreting the generic optimal value

Table 2

Observation

Student's actual paths were very diverse with regard to the time they devoted to each action and the difficulties they had to go from one to another. Some stayed a long time in enactive-iconic exploration. A minority of these students did not really work on mathematical functions, except for graphs. No student went directly to computing an optimal value after exporting a mathematical function: A few students however went more quickly towards this, some doing no or very little exploration. This diversity of paths is for us an indication that students could work with Casyopee at their own pace, developing activities that they could understand. We now report more precisely on specific students' behavior in connected activities.

Enactive-iconic and generational activities. Many students did mistakes when creating a geometrical calculation for the area of the rectangle. They moved M and observed inconsistencies between numerical values or variations of the area and the shape of rectangle. This enactive feedback allowed them to correct the geometrical calculation. The expression resulting of the exportation of this dependency was another important feedback for the students in the generational activity. Students often changed their choice of an independent variable until they got a sufficiently simple expression. We think that this use of feedbacks, although not always reflective, was an important step in understanding the notion of independent and dependant variable.

Enactive-iconic, transformational and global meta activities. To compute the maximum's x-coordinate, students needed an expanded form of the function and used Casyopée to get it. A team got a non parametric expanded expression, because they did not switch the parameters from 'animated' to 'formal'. Then they computed a numerical value for this particular case. While recognizing that they only partially solved the problem, they could not prove the property for a general triangle. Other teams did, but had some difficulty to apply a formula to the parametric quadratic expression. We think that, overcoming these difficulties, they got an extended understanding of the method, linked to meanings given to the parameters from the geometrical situation.

CONCLUSION

As shown in the paper, the grid was an operational tool inside the ReMath project, for guiding the design of Casyopée and of learning situations. An ambition of ReMath was to progress in connecting and integrating theoretical frames in technology enhanced mathematics learning. Most of the framework involved in this connecting activity such as the theory of didactical situations, the anthropological theory of didactics, activity theory, the theory of semiotic mediation, and the instrumental approach are situated at a rather general level. These general theories were influential but too general for piloting in a precise way the design of artefacts or the use of these. For fulfilling our design needs, we felt the necessity of building local frames, like the grid in this paper. We tried to build coherence among the diversity of approaches to this area, selecting some key perspectives whose complementarities seemed to us potentially productive and organizing these into a structured landscape.

More remains to be done in order to better connect this local level with the global level of macro-theoretical approaches. For that purpose, the system of cross-experimentations which has been developed in ReMath seems encouraging. Casyopée for instance, has been experimented by teams in Italy and in France referring to different macro-theoretical perspectives, respectively the theory of semiotic mediations and the theory of didactic situations. It is thus possible to explore how they combine in didactic action with the local frame we have presented here. We will be able to present more about this at the conference.

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PARTICIPATORY STAGES TOWARD COUNTING-ON: A CONCEPTUAL CAUSE FOR 'REGRESS' TO COUNTING-ALL

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This study addressed a 3-decade unexplained inconsistency in children's progress from counting-all to counting-on. Mixed-methods analyses of 37 (USA) first-graders' solutions to addition problems, particularly using Tzur and Simon's (2004) participatory/anticipatory stage distinction, allowed identifying both stages within two renowned invariants that constitute counting-on, Start Counting Singletons From n and Keep Track via Double Counting. The prompt-dependent participatory stages provide a lens for explaining children's peculiar regress to counting-all.

INTRODUCTION

This study revisited children's difficult-to-promote transition from counting-all to counting-on—particularly the puzzling inconsistencies in their use of these schemes to add items in two separately counted quantities (Carpenter & Moser, 1984; Fuson, 1992; Steffe et al., 1983). For example, a child may find that one quantity consists of 4 items and another of 3 items, and then be asked how many items are in both. She may start over from 1 to Count All items in the composed quantity (e.g., 1-2-3-4-5-6-7) or start from the number of the first quantity and Count On the items in the second (e.g., 4 [pause]; 5-6-7). To determine when to stop counting, the child has to keep track of how many items are being added. Steffe and von Glasersfeld (1985) considered counting-on to be a good indicator for a child's initial concept of number. Yet, why children use counting-on and then solve other problems, presented later even within the same session, by 'resorting' to the less advanced counting-all was not explained (cf. Carpenter, in Steffe et al, 1983).

In particular, we addressed the questions: (a) What intermediate stages can be inferred in children's transition to counting-on, and (b) how these stages explain why children who use counting-on revert to counting-all? We argue for using fine-grain analysis (Tzur, 2007) of two stages (see below) in a child's construction of each of two conceptual invariants (see below) that constitute counting-on at a quality indicative of a concept of number. We also argue that a child's prompt-dependent and hence provisional access to her conception of counting-on provides an additional lens for explaining 'regress' to counting-all.

CONCEPTUAL FRAMEWORK

The general constructs that guided this study draw on Piaget's (1985) core notions of assimilation, anticipation, and reflective abstraction. Key to this study is a distinction between two stages in learners' conceptualization—participatory and anticipatory—that Tzur and Simon (2004) distinguished to address two separate issues: (a) how the

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three parts of a scheme that von Glasesfeld (1995) postulated might be linked and (b) peculiar student 'regresses' when solving fraction problems, from more advanced reasoning they already appeared to use (cf. Tzur, 2007). These explanations converged into the notion of reflection on activity-effect relationship (AER), which further specified reflective abstraction (Simon et al., 2004). It articulates learning as changes in a learner's anticipation of the invariant AER they construct.

A novel anticipation of AER is first formed at a participatory (provisional) stage. The participatory stage is provisional not in terms of the content of the anticipated invariant, which is essentially the same in both stages, but in the learner's inability to access the AER compound unless somehow being prompted for the activity. The learner, once prompted and initiating the activity via assimilation into recognized situation and goal of extant conceptions, re-notices its effect(s), regenerates the new AER compound, and uses it to solve problems. Critically, however, the learner does not yet have access to the AER compound. Thus, she spontaneously uses AERs that are constituents of previously established anticipatory schemes.

A participatory stage is transformed into the more robust anticipatory stage. Relative to the task, the learner can spontaneously call up and use the novel AER compound. The anticipatory stage entails that a learner has transformed the previous scheme into a new one via coordinating the participatory AER compound with situations in which it is relevant for the learner, precisely the link that was not yet abstracted at the participatory stage. The new AER is ordinarily also linked to a sign that serves as a mental 'pointer' assimilated into the new scheme (Steffe & von Glasersfeld, 1985).

The stage distinction implies that a learner's solution reflects the interplay between two variables: (a) her assimilatory schemes and hence her interpretation of the task (Steffe, et al., 1983) and (b) features of the problem situation as designed by an adult for triggering certain learner's solutions (Carpenter, Hiebert, & Moser, 1983). To articulate the former, fine-grain assessment of the conceptual stage at which a learner has constructed the novel conception is needed (Tzur, 2007). To claim conceptualization at the anticipatory stage, assessment tasks must initially avoid prompting for the invariant AER compound, that is, follow a 'hard' (prompt-less) to 'easy' (prompt-inclusive) introduction of problems.

The content-specific constructs for this study draw on Steffe and von Glasersfeld's (1985) fundamental distinction between pre-numerical and numerical ways of operating. This is the transition from counting-all to an advanced form of counting-on. To provide a cognitive basis for explaining counting-all and two qualitatively distinct forms of counting-on, Steffe et al. (1983) introduced the notion of a counting extension (e.g., 4+3). Extension is observed when the child gives some indication (e.g., a short pause) of a completed sequence of counting acts and then continues counting, where the immediate successor of the last number word of the completed sequence serves as the first number word of the extension. The child counts each item in a single sequence of counting within which she introduces a temporal partition (e.g., “1,2,3,4; [pause] 5,6,7”). Later, the pre-numerical child constructs a primary form of counting-on as an extension that reorganizes counting of the first quantity into its anticipated effect (e.g., “4; [pause] 5,6,7”), to which Steffe and Cobb (1988) referred as the Initially Nested Number Sequence (INS). Eventually, the primary form of counting-on is transformed into its advanced, numerical extension form. Counting-on numerically (CON) is a scheme rooted in the child’s anticipation, prior to adding, that the extension of counting acts (to items in the second quantity) must be monitored via double counting. For example, the child may explicitly coordinate the two sequences (e.g., “4; [pause], 5-is-one, 6-is-two, 7-is-three”). Steffe and Cobb (1988) referred to this scheme as the Explicitly Nested Number Sequence (ENS). CON consists of coordinating two new invariants. One involves anticipating at which number word to begin counting and at which number to proceed-through-completion the extended count of the second quantity, termed “Start Counting Singletons From *n*” (SCSF-*n*). The other involves anticipation of the need to keep track of items in the second quantity, termed keep track via double counting (KTDC).

METHODOLOGY

A 20-minute, task-based, clinical interview was conducted in October of 2007 with every child in two Grade-1 classrooms (n=37) in a public elementary school in the USA mid-west. Additional interviews with a sample of 16 children who differed in their use of counting-on for different tasks took place about 6-8 weeks later. Each child’s behaviors as she solved three tasks (see below) designed according to Tzur’s (2007) fine-grain assessment framework were videotaped and transcribed. The first two tasks were similar to those used in studies that reported on children’s regress to counting-all. The third was used to better distinguish between participatory and anticipatory stages in the construction of SCSF-*n* and KTDC.

Every interview began with the Cup task. Matt presented a cup and told the child that *n* Unifix® cubes were in it. He then presented *m* more cubes outside the cup, and asked the child how many cubes would be in the cup if those outside were put inside. Next, Matt posed the Pile task, where all items were visible. The child placed *n* cubes in one location, *m* cubes in another location, and solved how many cubes would be if both piles were combined. Past research indicated that children might solve it via counting-all after solving the Cup task via counting-on. Finally, the child solved the Line task. Along the floor, Matt placed a cloth marked by thin hash lines (but no numbers) that divided it into 20 adjacent spaces, and asked the child to walk or hop over *n* spaces, then over *m* more spaces. Standing at the ending space ($n+m$), the child was then asked how far from the start she was standing.

Data analysis consisted of mixed methods. Quantitative analysis consisted of comparing proportions of student responses to the three tasks via a 2-sided Fisher’s exact test (Agresti & Finlay, 1997). Qualitative data analysis consisted of two phases. First, logs of major events in each interview were created to note segments in which children’s behaviors indicated a participatory and/or anticipatory stage of either invariant. Second, transcribed videotapes, representative of each stage, were observed and analyzed through line-by-line reading of the transcript coupled with repeatedly observing critical video segments. A major attention was given to comparing and contrasting between the child's responses to the Cup/Pile tasks and the Line task.

ANALYSIS

Using the sequence of tasks (Cup-Pile-Line) to assess all 37 first graders' conceptions of counting-on corroborated previous findings: After using this scheme to solve the first and/or the second task, most children resorted to using counting-all to solve the third task. Table 1 summarizes these results. Comparing each task separately (Columns 1-3) shows a regress to counting-all: whereas 30 students used counting-on for the Cup task, only 17 used it for the Pile task and a mere 5 for the Line task. To examine the extent to which item availability could explain this difference, we combined the responses to the Pile and Line tasks (Items Available, Column 4) and compared them with the Cup task (some items not available). A 2-sided Fisher exact test shows significant interaction ($p<.0001$) between item availability and the scheme the child used spontaneously. Yet, item availability did not explain the interaction ($p<.005$) between task solved and scheme used when all items were available (Pile vs. Line). Indeed, a conceptual source seemed involved, which led to the qualitative analysis (below) of a participatory stage in the transition to each of the two invariants that constitute numerical counting-on (CON).

	1. Cup	2. Pile	3. Line	4. Items Available (Pile & Line, n=74)
Count On	30 (81%)	17 (46%)	5 (14%)	22 (30%)
Count All	7 (19%)	20 (54%)	32 (86%)	52 (70%)

Table 1. Schemes Children Used for the Tasks

Participatory Stage: Start Counting Singletons From *n* (SCSF-*n*)

To solve the Cup task (8 cubes inside, additional 8 cubes outside), Aaron began re-counting (all) from one, but spontaneously stopped at 4 and changed to counting-on: "8; 9-10-11-12-13-14-15-16." That is, he almost immediately regenerated the anticipation that counting will yield the effect "8," a number he realized and then explained to Matt, would necessarily be identical with the number of given cubes. Proceeding to the Pile task (8+5), Aaron could then extend the count without a prompt: "8 [pause]; 9-10-11-12-13." When asked why he started at 8, Aaron replied: "Because I remembered you taught us to start at the first number and then keep on going." Indeed, after regenerating for himself his participatory SCSF-*n*, Aaron could spontaneously solve the Pile task via anticipating the effect of potentially counting the first quantity and thus extending it to complete the count of the second quantity.

After Aaron successfully used counting-on to solve the Pile task (items available), Matt proceeded to the Line task (7 spaces + 6 more spaces). He spontaneously solved it via counting-all: "1-2-3-4-5-6-7-8-9-10-11-12-13." For the second task (8 spaces + 7 spaces), Aaron again attempted to use counting-all, but Matt interrupted him.

Excerpt 1: Prompted SCSF-*n* During the Line Task

M: (Moves to the 8th space to block Aaron's view) Is there another way to figure it out?

A: (Utterly confused): Fingers? With Seven? Eight …. (Aaron puts up eight fingers to represent the first quantity and tries to count-on seven more fingers, eventually abandoning this process after 15 seconds as he says): No, fingers don't help... (He then begins counting at 8 without keeping track on his fingers, stops again, and finally uses counting-on from 7): 'Seven [pause]; 8-9-10-11-12-13-14-15-16."

M: Okay. Are you standing on the 16th space? How do you know?

A: (Aaron does not answer the question; instead, he begins counting from the space on which Matt stands, nodding at each space, then says) Eighteen.

M: You're standing on the eighteenth space. How do you know?

A: (Does not respond. He looks behind him at the extra spaces on the line, and sweeps his hand back and forth as though he is counting them) Yeah, I'm on sixteen.

Excerpt 1 provides evidence for a critical feature of a child's transition to counting-on in general and to SCSF-*n* in particular, which seems compatible with the progress from the INS to the TNS (Steffe and Cobb, 1988). Aaron initially needed a self-generated prompt for the Cup task, which brought forth re-processing of SCSF-<u>n</u> also for the Pile task, where items were available <u>and the two quantities spatially separated</u>. Yet, when items were available and the quantities were <u>not</u> separated he did not spontaneously use the same scheme although just a few minutes earlier he explicitly articulated its advantage. Aaron used a scheme (counting-all) to which he did have access, but he could not access the evolving SCSF-*n*.

Aaron is a typical case of how a participatory stage of SCSF-*n* impacts children's regress to counting-all, hence of a conceptual source for this 3-decade old unexplained phenomenon. One should expect that children whose conceptions include SCSF-*n* only at the participatory stage would 'resort' from counting-on to counting-all when task features (e.g., spatial break between quantities) do not provide the needed prompt for accessing that invariant. This was indicated by Aaron's "awakened" ability to access his SCSF-*n* once prompted by Matt's stepping onto the 8th space and creating the potential for Aaron's imposition of such a break.

Participatory Stage: Keep Track via Double Counting (KTDC)

Karen spontaneously used counting-on for the Cup and Pile tasks. Yet, when asked to hop 6 spaces and 5 spaces over the empty number line, she first resorted to counting-all (1-2-3-4-5-6-7-8-9-10-11). She was then asked to move 7 spaces and 6 more spaces and began counting-all (1-2-3-4…) again, at which point Matt interrupted.

Excerpt 2: Participatory Stage of KTDC

M: How many spaces did you move the first time?

K: Um… Seven.

M: You moved 7. Not starting at one—how can you tell me what square you're standing on?

K: (Think for approximately 5 seconds; then she closes her fist and begins extending her fingers while silently mouthing the number-words to herself similarly to how she was taught by Matt back in October, finally announcing): Twelve.

M: How do you know that you're standing on 12?

K: Because I started at 7 and went "7 [pause]; 8-9-10-11-12-13" (As she counts to explain the answer, she again begins with a closed fist and extends one finger at a time, then self-corrects herself). I'm on thirteen.

Unlike Aaron, who was taught the same keeping track method but could not access it even when his SCSF-*n* was activated via a prompt, Karen could use an almost identical first prompt to recall her KTDC invariant, hence execute numerical counting-on. Her access to KTDC seemed indirect: Being prompted not to start at one triggered her SCSF-*n*, which in turn triggered her KTDC anticipation. The key point is that this anticipation was immediately and independently accessible to her in the Cup and Pile tasks, but not in the Line task, where items were available <u>and</u> no spatial separation between the quantities was apparent. The prompt did <u>not</u> eliminate item availability, but indicated the possible mental separation between quantities that she could impose in the abstract. Such a solution seems consistent with Steffe and Cobb's (1988) analysis of children who develop the Explicitly Nested Number Sequence (ENS). It was particularly demonstrated in her use of finger monitoring as she intentionally embedded the count of first seven items within the combined quantity—the numerosity of which she attempted to make definite.

DISCUSSION

This study provided empirically grounded evidence for two claims: (a) the transition to each of the two invariants (SCSF-*n* and KTDC) that, <u>coordinated</u>, constitute numerical counting-on (CON) and the concept of number, may involve a participatory stage in which the students can count-on only if being prompted and (b) a child at the participatory stage of each invariant may solve some tasks via counting-on while resorting to counting-all for later tasks. Most importantly, this study demonstrated that a plausible explanation to the 3-decade unresolved puzzlement concerning this regress is conceptual. That is, this regress may be rooted in the stage at which a learner has formed each or both invariants, not mainly or entirely in the structure of a task.

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CONSTRUCTING CONTEXTUALIZED QUESTIONS TO ASSESS PROBABILISTIC THINKING: AN EXPERIMENTAL INVESTIGATION IN PRIMARY SCHOOLS

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The use of contextualized questions is widespread because it makes the assessment more authentic. Even more so, the contextualization of questions is very important in the assessment of probabilities curriculum where teachers expect their pupils to transfer their probabilistic thinking skills for decision-making in the real world. This research extends previous work to experimentally investigate the effect of different contextualization conditions on the responses of pupils to probability questions. It was found that different contextualization conditions may dramatically alter the responses of pupils to the same probability question. Based on the results, the authors make specific question-construction suggestions to teachers who would like to use contextualized questions to assess the probabilistic thinking of their pupils.

INTRODUCTION

A possible definition of probabilistic thinking is that it is a mode of reasoning attempting to quantify uncertainty, as a tool for decision making (Tversky & Kahnemann, 1974). Nevertheless, pupils' probabilistic thinking has been shown to be influenced by culture and is sensitive to cultural experience (Amir & Williams, 1994, 1999). In a previous study (Canizares et al, 1997) pupils' strategies, when comparing probabilities in tasks, were analysed and pupils' arguments were sometimes found to be based on irrelevant aspects such as the context of the questions. Inspired by these results, Lamprianou and Afantiti Lamprianou (2002) conducted a study to investigate the extent to which the probabilistic thinking of primary school pupils in Cyprus was affected by irrelevant or subjective information. Their results agreed with the results of Canizares et al (1997).

However, it is frequently suggested by the optimists that pupils' errors and misconceptions can be a starting point for effective diagnostically designed mathematics teaching (Williams & Ryan, 2000). It has been argued that if the teachers are aware of the most common errors and misconceptions that pupils have in probabilities, they will try to develop classroom strategies for helping students to confront them (Fischbein & Gazit, 1984; O'Connell, 1999). However, teachers need tools (i.e., tests with appropriate questions) with which to investigate pupils' probabilistic thinking in their own class. Kapadia (2008) suggests that the end-of-year national tests in England show that misconceptions on key probability ideas remain –

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surely the teachers could benefit by instruments that would allow them to diagnose these misconceptions. How should the questions of tests be optimally constructed?

Contextualized questions

The use of contextualized questions (e.g. with wording and artwork referring to various situations such as from the social or physical world) is widespread in many subjects because it makes the assessment more authentic and more relevant to the real world. We agree with Ahmed and Pollit (2007) that contextualization is very popular for low and high stakes assessment, and also in international comparative studies such as the TIMSS (2004). The context of questions is very important and APU (1988) has measured it to affect success rate on a question from a few percentage points up to 20%.

In the area of the research on the probabilistic thinking of the primary school pupils, most researches simply conclude that the context of a question may affect pupils' responses without going the necessary extra step to investigate how and why this happens. For example, in a typical study, Lamprianou and Afantiti Lamprianou (2002) investigated the responses of 426 primary school pupils to questions like: "*In a zoo there are 2 elephants and 4 monkeys. Today the staff of the zoo wants to randomly select an animal to wash. What is the most likely animal to select first, an elephant or a monkey? Why?*" The responses of many pupils contained subjective elements like "*they will select an elephant because they cannot catch monkeys; they climb on the trees*".

In this research, we introduce the definition of the *general context* of the question as the wider environment in which the question is put: in this case it is the physical world and more specifically, a zoo. We define the *particular context* as the immediate environment of the question which, in this case, is the desire of the staff of the zoo to randomly select an animal. The particular context defines the *stimulus contrast* which we define as the distance between the *vector of contrast dimensions* (e.g. external characteristics) of a monkey and an elephant. The qualitative analysis of pupil responses in the Lamprianou and Afantiti Lamprianou (2002) research showed that almost 20% of the pupils explained their response to the zoo question by making references to various characteristics of the animals such as their size, their ability to run or climb etc: the stimulus contrast was too large, for the pupils, to miss.

However, how would the responses of the pupils change if the particular context changed to tigers and lions? In this case, the general context would be the same, but the particular context would be much different. The stimulus contrast would arguably be smaller between a lion and a tiger (than the elephant/monkey contrast). We would therefore, expect the responses of the pupils to be less affected by this new context.

Ahmed and Pollit (2007) suggested that a question in a given context is focused to the extent that "*it addresses the aspects of the context that will be most salient in real life for the students being tested. A more focused context will then help activate relevant concepts, rather than interfering with comprehension and reasoning*" p.1.

They also suggested that contextualizing questions may sometimes make them more difficult to answer because the context (a) could make the language (the wording of the question) more demanding, (b) may be more or less familiar to the respondent, and (c) could contain too much irrelevant information so that the respondent would need to select only the relevant information. So the contextualization may affect the language and the familiarity of the question, and distract the attention of the respondents. Through an experiment with science questions, they found that altering the context of a question may alter the way somebody responds to it.

Mevarech and Stern (1997) have shown that the context may not only divert students' attention from the mathematical task (in their case: interpretation of graphs) but can sometimes activate simplistic mental models rather than the abstract mathematical thinking required. Similarly, Boaler (1993) found that students sometimes choose the procedure for a mathematics problem according to the context.

We could cite many more references to show that it is not just in the probability questions that the context affects the responses of the pupils, but this is a wider issue in education. In the past, this phenomenon has been cited as a result of the errors and the misconceptions of the pupils in probabilities – as if this was a characteristic of individuals; our new approach is that this is a phenomenon related to the context of the questions and may be reduced by proper question-construction approaches.

This research extends previous work by Lamprianou and Afantiti Lamprianou (2002) to experimentally investigate the effect of different contextualization conditions on the responses of pupils to probability questions. We hope that teachers will keep using contextualized questions to assess the probabilistic thinking of pupils, and this research aims to offer them specific guidance on how to construct these questions so that the context will not interact too much with the measurement process invalidating the assessment results.

METHODOLOGY

Method and instrument

For the purposes of this research, we conducted an experiment using probability questions from a past research by Lamprianou and Afantiti Lamprianou (2002) who also described the original test. Four questions were chosen from the test:

Q3: *I have two bags with marbles. Bag A has 4 marbles, 2 blue and 2 green. Bag B has 6 marbles, 3 blue and 3 green. (a) From which bag do I have the largest probability of picking randomly, without looking in the bag, a blue marble? (b)Why?*
Q6: *Andrew wants to decorate his Christmas tree. On the carpet, there are 4 big and 6 small golden balls. Since he does not care if he starts the decoration with a small or with a big ball, he selects randomly a ball from the carpet. (a)What is more likely to select firstly, a small or a big ball? (b)Why?*
Q8: [the zoo item is described above]

Q9: *In a box, we put 4 cards with the number '5', 2 cards with the number '50' and 3 cards with the number '100' in a box. I randomly pick, without looking, a card. What is the most likely number to read on the card? Why?*

We formulated an experimental design where the original questions, and variants of these questions, will be presented in the same test.

In the case of Q6, we replaced the words 'big' and 'small' golden balls with 'green' and 'red' balls. Lamprianou and Afantiti Lamprianou (2002) found that the size of the ball distracted the pupils. Our hypothesis is that Q6_colour will be easier than Q6_original because the stimulus contrast will be much smaller and the probabilistic thinking of the pupils will be affected less.

The variants of Q9 are presented in Figure 1.

Q9_nonumbers: *In a box, we put 4 red cards, 2 blue cards and 3 green cards. I randomly pick, without looking, a card. What is the most likely colour of the card? Why?*

Q9_picture: I have 4 white, 2 grey and 3 black balloons. What is the most likely colour of balloon to blow out first? Why?

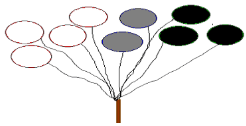

Figure 1: The variants of Question 9

Our hypothesis was that the stimulus contrast would be the smallest for Q9_nonumbers, larger for the Q9_picture and the largest for the Q9_original. In this case the picture would probably help some pupils to visualize the sample space but the fact that they would not want their balloons to blow out would probably increase the perceived stimulus contrast (see Ahmed and Pollit (2007) for a similar case study in the context of science).

In the case of Q8, we built two variants: we replaced 'elephants' and 'monkeys' with 'lions' and 'tigers' (Q8_tiger) and also with 'goldfish' and 'fighters' (Q8_fish). Our hypothesis was that the stimulus contrast would be smaller for the variants and substantially larger for the Q8_original. The Q8_fish might prove to be slightly more difficult than the Q8_tiger because of the contrast between goldfish and fighters.

The variants of question 3 are presented in Figure 2.

Q3_cars: *I have two boxes with cars. Box A has 2 blue and 2 green cars. Box B has 3 blue and 3 green cars. (a) From which box do I have the largest probability of picking randomly, without looking in the box, a blue car? (b)Why?*

Q3_picture: *I have two bags with marbles. Bag A has 2 blue and 2 green marbles. Bag B has 3 blue and 3 green marbles. (a) From which bag do I have the largest probability of picking randomly, without looking in the bag, a blue marble? (b)Why?*

Figure 2: The variants of Question 3

Lamprianou and Afantiti Lamprianou (2002) had found that the reference to the total number of marbles in the bags (i.e. the wording) in the original question was confusing the pupils. We simplified the wording and built the Q3_cars version (and changed 'marbles' to 'cars' to conceal from the pupils the fact that we used the same question). In the Q3_picture variant, we included a visualization of the two bags (of the sample space). Our hypothesis was that Q3_cars might be easier than Q3_original (if the change in the wording was significant) but we certainly expected that the Q3_picture would be the easiest.

Test administration and the sample

The final instrument was administered to 424 pupils in 12 different district schools in all districts of Cyprus in November-December 2008. The schools were selected in such a way so that we would have a satisfactory geographical coverage of the island, but was non-probabilistic. The sample includes complete tests from 211 boys and 213 girls. Half of the pupils were 11 and half were 12 years old (5^{th} & 6^{th} grade).

The researchers (or their assistants) visited the schools and supervised the administration of the test under standardised conditions. The pupils were reassured about the anonymity of the test, were encouraged to attempt it in their full potential and were given ample time to respond. They were explained that, on each question, they would get 1 mark if they gave a correct response and they could get a second mark if they explained how they worked to reach their answer.

Methods for data analysis

In order to keep the reporting of the results accessible to people not familiar with statistics, we will mainly report percentages of pupils' correct responses. We will also use the Sign test and the Friedman test (which are one-sample, non-parametric equivalents of the t-test and the repeated-measures ANOVA) to investigate if the difference in the performance of pupils on various questions is statistically significant. The performance of the pupils to any question is measured on a scale from 0 to 2 (0: incorrect response, 1: correct response, 2: correct response and explanation of thinking). We will also present results of the qualitative analysis of the exact responses of 30 random scripts of pupils to the ‘explain why’ section of the questions (all questions asked: “*Explain, with as much detail you can, your way of thinking to give this response*”).

RESULTS

Our hypothesis that Q6_colour would be easier than Q6_original was correct. Overall, 424 pupils answered both questions. 79% of the pupils gave a correct response to Q6_colour but only 66% of them gave a correct response to the original question. Similarly, 57% of the pupils gave a correct explanation to the original question while the percentage for the variant is 67%. The Sign test suggests that the pupil performance on the two questions is significantly different ($z=4.643$, $p<0.001$).

Our hypothesis that Q8_original would be more difficult than Q8_fish and Q8_tiger also proved to be true (percentages of corresponding correct responses are: 69.3%, 73.3%, 72.6%, N=424). The corresponding percentages of correct explanations are 57.4%, 59.7%, 62.7%. The Friedman test was statistically significant (x2=6.012, df=2, p=0.048) (the Sign test showed that the Q8_original is more difficult than any of its variants).

Our hypothesis that Q9_nonumbers would be the easiest, Q9_picture would be of average difficulty and Q9_original would be the most difficult question also proved to be correct. The corresponding percentages of correct response were 80.0%, 74.5%, 56.7% (N=424). The corresponding ‘explain why’ percentages correct were 71.9%, 56.8%, 46.0%. The Friedman test was statistically significant (x2=42.056, df=2, p=<0.0001). A series of Sign tests showed that the differences in the pupil performance (as we predicted them) were statistically significant at the 0.001 level.

Finally, our hypothesis that Q3_picture would be the easiest question and that Q3_cars would be easier than Q3_original proved to be correct. The corresponding percentages correct are 63.2%, 53.1% and 47.2%. The corresponding ‘explain why’ percentages correct are 51.2%, 41.5% and 38.9%. The Friedman test was statistically significant (x2=22.183, df=2, p=<0.0001). A Sign test showed that Q3_picture was the easiest, but there was no significant difference between Q3_cars and Q3_original.

The content analysis showed that the probabilistic thinking of many pupils was affected by the size of the balls on the original question but only one child said that “green is a bright and beautiful colour” on Q6_colour. Also, pupils’ thinking was affected by information like the colour of the fish, the strength of the tiger, the speed of the monkey etc. Overall, the responses of the pupils were more affected in the cases of the questions were the stimulus contrast was larger.

DISCUSSION

This research introduced the concept of "particular context" (versus "general context") to investigate how the context of a question affects the probabilistic thinking of pupils. The degree of the effect depends on the stimulus contrast: if it is larger, then the effect of the contextualization will be larger. The magnitude of the stimulus contrast is defined by the "vector of contrast dimensions": one may imagine this to be an n-dimension vector where each dimension may be one perceived characteristic of the stimulus.

Let us discuss an empirical example of this experiment. The general context of Q9_original is the familiar "*I have a container X (e.g. a bag, a box, an urn etc) where I put objects Y (e.g. marbles, cards etc)*". The pupils are very familiar with this general context which probably puts them into a mode of class-style probabilistic reasoning. However, the particular context is realised as cards with integer numbers written on them. Although the stimulus (for the pupils) should be the count of the cards (e.g. **4** cards with the number X), in Q9_original the stimulus vector has another dimension: the numbers on the cards are strikingly different in value: "5", "50" and "100". The stimulus contrast is very large, thus the great effect on the probabilistic reasoning of the pupils (only 57% answered correctly and 46% explained their answer properly). That is why one one of the pupils responded that "*it is more likely to pick a card with the number "100" because this is the biggest number of all*".

When the contrast is reduced (the numbers on the cards are turned to colours) as in Q9_nonumbers, the pupils shift to a class-style probabilistic reasoning. When we introduced the balloons picture (Q9_picture), the 'balloon' dimension of the stimulus contrast carried their thinking away again. In another case, when the general context and the particular context had the same 'class-style' characteristic such as in Q3_picture (i.e. picture of bags containing marbles), the question became easier. It is interesting to mention that during the content analysis of their responses, we identified some pupils who drew pictures to represent the sample spaces. In none of the cases did they try to put the sample spaces of their drawings into a context of their own: the general pattern was that they used the class-style paradigm by drawing marbles into bags or circles (presumably marbles) into urns or squares etc.

Our research results encourage us to make the following suggestions to the teachers that would like to use contextualised questions to assess the probabilistic thinking of their pupils: (a) the contextualization will not necessarily affect the probabilistic thinking of your pupils, (b) try to avoid particular contexts with high stimulus contrasts, (c) the visualization of the question will help provided it does not introduce new dimensions in the vector of contrast dimensions.

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I ACTUALLY STARTED TO SCREAM: DOING SCHOOL MATHEMATICS HOMEWORK

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Homework is an activity done by large numbers of students all over the world. Many concerns have been raised including, especially in primary schools, whether any academic benefit is gained and whether parents have appropriate resources to actively support/teach their children. In this paper, we explore the stories that two ten year old girls tell about doing their mathematics homework with family help and the pressures that it puts on them to take control of their own learning. We discuss the opportunities and constraints to students doing homework as a consequence of the social and institutional relations that they operate within.

HOMEWORK AS TRAUMA

Although mathematics homework is done by students all over the world, it is not without controversy. As well as showing limited effects on primary students' academic performances (Inglis, 2005), homework can be the source of great trauma. Recently, an Egyptian mathematics teacher was convicted of the manslaughter of an 11 year old boy after disciplining him for not completing his homework (http://www.abc.net.au/news/stories/2008/12/26/2455355.htm). This level of physical violence is rare, but homework is often a source of frustration for many families (Kralovec & Buell, 2000). In this paper, we contend that homework can be emotionally traumatic, especially for children who are in difficulties with mathematics. We use the stories told by two children to explore the impact of homework on their lives and the opportunities and constraints for reducing the trauma associated with it.

Using a socio-cultural framework, Street, Baker and Tomlin (2008) described homework as an example of school numeracy practices that is done in the home domain. There is an expectation that parents or care givers have, at the very least, a monitoring role, but the interplay of children, parents and teachers' expectations about homework is not a simple one. Street, et al. (2008) also described how teachers often expressed disappointment with the support that they believed the parents were providing. However, teachers can have an inaccurate view of what was actually being provided (Lange, 2008a; Street, et al., 2008).

Nevertheless, parents who had bad experiences of doing school mathematics themselves may not have the confidence or requisite knowledge to provide help (Anthony and Walshaw, 2007). Many immigrant parents struggle with seeing what the school set as homework as being appropriate mathematics and with knowing how to provide support to their children (Abreu & Cline, 2005; Bratton, Civil, & Quintos,

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2005; Hawighorst, 2005).

When parents' expectations differed to those of the school, often the children were the ones who had to determine an approach that resolved the differences (Street, et al., 2008). For example, in Abreu and Cline's (2005) study, some students expressed a devaluing of their parents' support as their methods of doing mathematics were different to those of the school. For other students, often the only option when the frustration level from working with their parents became too great was to resist. This could raise the ire both of teachers and parents. Homework thus has the potential to be traumatic.

There has been little research into students' perceptions of doing mathematics homework, but what has been done suggests that students who are in difficulties with mathematics are the ones who are most likely to suffer frustration. Street et al. (2008) describe how one parent reported that her daughter who was a low achiever in mathematics had requested that her mother stopped trying to help. Abreu and Cline (2005) quoted a parent of another low achieving student who stated that when she tried to help her daughter "she'd get frustrated and I'd get frustrated, we'd … just get … at loggerheads" (p. 714). High achievers seemed more able to explain the school methods of doing mathematics so that parents were able to understand the differences in approaches. This did not seem to be a possibility for low achieving students.

Given that "homework also is exertion of teacher power over students, a way of controlling students' behaviour in and out of school time, and a way of asserting school norms and values" (Lange, 2008a), the possibilities for disputes and emotional trauma is great. Children are the ones who are most likely to suffer from this trauma and also the ones who often sort out how to deal with the different pressures. In this paper, we investigate the experiences of doing homework of two girls, who both had difficulties with mathematics. We focus on the features of the situations that hinder or facilitate the likelihood of trauma when doing homework.

METHODOLOGY AND THEORETICAL FRAMEWORK

The interview data in this paper comes from a larger study exploring children's perceptions about their mathematics education. Hence, the interviews were semi-structured life world interviews, i.e. interviews that "seek to obtain descriptions of the interviewees' lived world with respect to interpretation of the meaning of the described phenomena" (Kvale & Brinkmann, 2009, p. 27), in this case mathematics education. Children aged 10-11 years in a Danish Year 4 class were interviewed (Lange, 2008b). The extracts come from an interview with two girls that covered a number of topics. They are analysed using Street et al.'s (2008) *ideological model of numeracy*. Table 1 describes the four inter-related dimensions of the model. We concentrate on how clashes between home and school in relationship to values/beliefs and social/institutional relations arise and then are resolved.

Dimensions	Description

Content	The mathematical concepts, such as times tables.
Context	The situation in which a numeracy practice takes place.
Values and Beliefs	The participants' beliefs about how numeracy practices should progress and how new skills and knowledge are taught within them.
Social and Institutional Relations	The overarching factors that channel what are seen as appropriate choices in the other three dimensions.

Table 1: Dimensions from the ideological model of numeracy (Street et al., 2005)

The children's narratives were rarely rounded, monologic stories. Usually they unfolded as dialogues involving active listening and questions (Kvale & Brinkmann, 2009). Consequently, we have included a long transcript with the original Danish and an English translation. They have been tidied a little to make them easier to read. Line numbers indicate where these extracts came from in the interview.

INTERVIEW EXTRACTS

Isabella's homework experience

The first extract begins discussion on homework in the interview. In it, Isabella describes learning her six times tables with her mother. It does not seem to be stressful for Isabella and may even have been an enjoyable time with her mother.

244	Isabella	Nogle gange når jeg skal også lave lektier, for eksempel hvis vi nu har fået sekstabellen for, ik å, så laver min mor sådan for eksempel hun skriver seks gange en og så seks gange to. Så vender hun dem alle sammen om og så drejer hun dem rundt og så skal jeg så tage en op og så skal jeg så sige det og så tager hun dem alle sammen i en bunke til sidst og så viser dem og så skal jeg sige dem, så til når jeg kan dem. Så stopper vi.	Some times when I do my homework, for example if we have been set the six times table then my mother, like for example, she writes six times one and then six times two [on pieces of paper]. Then she turns all of them over and then she turns them around and then I must pick one up and then I must say it and then she takes all of them together in a pile at last and then show them and then I must say them then until when I know them. Then we stop

The following provides a description using Street et al., four dimensions.

Content: This was times tables and in particular the six times table.

Context: The practice of times tables was done at home, but so that mastery of them could be displayed at school. From other interviews (see Lange, 2008b), the children told about being tested at school on times tables learnt at home. The teacher would then record each child's achievement with ticks on a chart.

Values and Beliefs: Knowing times tables was important at both home and school and this can be seen through the teacher's recording of achievement and through the time that Isabella and her mother spent on learning them at home. Learning of times tables was through memorisation, not understanding of for instance number relations. The method for learning times tables at home resembled the testing done at school, except that Isabella had the opportunity to redo any incorrect ones till she had correctly memorised them all. She could take as long as she needed and there were no consequences if she got one wrong, except that it would be returned to the pile. This would most likely be different to how times tables were done in the school setting where the teacher would not have the time to allow the child to keep repeating the questions until they were all correct. Therefore, although the approaches at home and school were similar, the way that interactions occurred between the child and the adult were different and worked to the advantage of the child at home.

Social and Institutional Relations: In other research, learning of times tables was considered important by many parents (Street et al., 2008; Abreu & Cline, 2005) as it had been heavily stressed in the parents' own schooling. Isabella's mother's also seemed to hold this view as she provided support for her daughter's learning of them. The relations between home and school are strengthened through the shared understanding and support for the learning of times tables knowledge.

Maria's homework experience

The next extract shows that Maria's experiences of receiving family support was traumatic. It is a longer extract as Maria described what happened and how she resolved the issue by requesting that her parents no longer help her, even though she knew that she struggled with doing mathematics.

247	Maria	Det var fordi engang da var mine forældre sådan at de gerne ville hjælpe mig. Hele tiden. Jeg synes bare det var alt for meget. De ville hjælpe mig med matematikken fordi jeg ikke var særlig god til det. (T: ok) Og så endte det så med til sidst at min far fordi jeg blev sådan, jeg begyndte faktisk at skrige. (T: ok) Og det skete næsten hver gang og så begyndte jeg at græde og så rendte jeg ind på mit værelse. Og det var sådan det var. Men nu, nu gør jeg det jo ikke mere. Nu laver jeg dem bare selv fordi nu kan jeg godt finde ud af det. …	It was because once then my parents was so that they wanted to help me. All the time. I just thought it was far too much. They wanted to help me with maths because I was not particularly good at it. (T: ok) And then it ended with at last that my dad because I got like, I actually started to scream (T: ok). And that happened almost every time and then I started to cry and then I ran into my room. And that was how it was. But now, now I don't do that any more, you see. Now I just do them [the sums] myself because

			now I can work it out …
251	Troels	Ja. Kan du fortælle lidt mere om det? Hvordan var det?	Yes. Could you tell a little more about it. How was it?
252	Maria	Mm. Det var jo ikke rart fordi at – Og så bagefter så kom min mor ind og så. ”Ahm, ok” sagde hun ”vi skal nok prøve at lade være”. Så skete det så igen, hvor jeg så begyndte at græde. Så satte jeg mig i en stol og så sad jeg der i vores stol. Så sad jeg bare _ Det var helt vildt. Og så _ ind på mit værelse. Og så smækkede med døren. Og så blev min mor sur for det gider hun ikke have jeg gør. …	Mm. It was not pleasant, you see, because – And then afterwards then my mum came in and then “Ahm, ok” she said “we will try not to do it”. Then it happened again where I then started to cry. Then I sat down in a chair and then I sat there in our chair. Then I just sat _. It was crazy. And then _ [I went/ran] to my room. And then slammed the door. And then my mum got cross because she won’t have me doing that …
257	Troels	Hvordan fandt du ud af at du ikke var så god til matematik?	How did you realise that you were not so good at mathematics?
258	Maria	Jeg synes, jeg følte det sådan helt, det kunne jeg altså bare ikke overskue, så jeg havde bare lyst til at smide hele ud. (T: ja) Jeg havde bare lyst til at bare gøre sådan “Nej nu smider jeg matematikken væk. Nu vil jeg slet ikke have det mere” (T: nej) Jeg synes det er dumt /	I think, I felt it like completely, I just could not cope with it [or: take it in] so I just felt like throwing all of it out (T: yes). I just felt like doing so “No now I throw maths away. Now I will not at all have it more” (T: no). I think it is stupid /
259	Isabella	Nogle gange hvis der kommer sådan nogle problemer så har jeg bare lyst til sådan smide, væk med det. Alt. Og så sætte sig i stolen og så bare sidde og slappe af	Sometimes if such problems come then I just feel like, like throwing, away with it. All of it. And then sit down in the chair and just sit and relax
260	Troels	Ja, ja	Yes, yes
261	Maria	Men det kunne man jo ikke og så derfor så begyndte jeg at blive ked af det. Og min far: ”Og så skal du jo gøre sådan. Og hvad er det? (T: mm) HVAD ER DET?” (T: mm) Og så hvis man ikke kunne finde ud af det så, så (sagde han) sådan	But you could not do that, you see, and then therefore I started getting upset. And my dad: “And then you must do so. And what is that? (T: mm) WHAT IS THAT?” (T: mm) And then if you could not work it out

		her ”Du skal ikke tælle på fingre. Hvad er det? (stemmeføring der antyder at M er dum)” (T: ja) Yyhhh …	then, then (he said) like this: “You must not count on fingers. What is that?” [in a voice that suggests that M is stupid] (T: yes) Eeehhh …
264	Troels	Men hvordan gik det så over? …	But how did it go away? …
267	Maria	Nej, ok det var sidste år det begyndte at stoppe fordi nu lod de mig være med det for jeg har sagt til min mor “Mor jeg synes altså ikke at - I behøves altså ikke at hjælpe mig så meget.” (T: ok) Så fandt jeg så ud af det selv og det var rigtig svært og jeg kæmpede helt vildt hårdt for det (uf)	No, ok it was last year that it started to stop because now they let me alone with it because I have said to my mum “Mum I really don’t think that – you really need not help me that much.” (T: ok) Then I worked it out myself and it was really difficult and I fought for it really very hard
268	Troels	Hvordan kæmpede du?	How did you fight?
269	Maria	22:56 Altså jeg sad timer med det der matematik. Så sad jeg bare der. (T: mm) Så fik jeg så skrevet tallene ned (T: ja). Skrevet tallene ned (T: ja I: ja) – Nu kan jeg godt.	I sat for hours with that maths, you see. I just sat there. (T: mm) Then I got the numbers written down (T: yes) The numbers written down (T: yes I: yes) – Now I can

Street al.’s (2008) four dimensions shows significantly differences between Maria’s and Isabella’s experiences.

Content: Maria was working on sums and in particular was finding effective ways to determine the answers. In other parts of the interview, she described subtraction as being the part of mathematics with which she had the most difficulty.

Context: Like Isabella, homework was initially done with family help. Maria described her parents as wanting to help her because she was not good at mathematics. Her parents made several (unsuccessful) attempts at trying to help her.

Values and Beliefs: In this extract, there was a mismatch between how Maria and her father felt the sums should be worked out. Maria’s father did not value using fingers. Although Maria was aware there were other methods, she did not seem confident in using them until she knew how they worked. The learning required more than memorisation. It seemed that her father did not have the skills to explain the method that he wanted her to use.

Social and Institutional Relations: The relations between Maria and her parents seemed to survive the trauma of doing mathematics homework, in the same way that Maria’s relationship with mathematics seems to have continued. It could be that because Maria believed that these relationships had to continued that she forced herself to understand the mathematics. Both she and Isabella expressed a wish to throw mathematics away when the problems became too hard. However, her need to find a way forward that did not include help from her parents meant that she persevered. The social relations at home were different to those at school. Maria's reaction to her family's help of screaming and slamming the door may not have been acceptable at home but was tolerated. It is unlikely that Maria would have exhibited the same behaviour at school when she was unable to do the mathematics (see Lange, 2008b). Therefore, in the home environment she can act out her frustration and also request that her parents not help her any further. At school, it is not acceptable for a child to refuse the help of a teacher, yet at home the different type of social relations provides Maria with an opportunity to find her own solution for overcoming the difficulties she was experiencing.

OPPORTUNITIES AND CONSTRAINTS

If mathematics is to be done as homework, there is potential for students to experience trauma, especially if they are in difficulties with mathematics. This trauma arises when students are unable to do the homework and is compounded when the support that the parents provide is in conflict with what the child has learnt at school or beyond what the child is capable of doing. However, as well as placing constraints on the way that help is provided because of the parents' own understandings about mathematics, doing mathematics at home also offers opportunities that are not available in school.

Isabella's story suggested that when her mother's beliefs about valuable mathematics and the ways that it should be learnt at home matched those of the school then Isabella was not pulled between two different approaches. Learning through memorisation has less potential for differences of approach than when the learning requires understanding. Isabella may still experience trauma if she was unable to be successful in doing the mathematics and this can be seen in the second extract where she talks about throwing mathematics away. However, this would be a different kind of trauma than if she had to navigate a path between her parents and her school.

Maria's experience was different as in order to do the sums she needed to understand the method she was to use. When the method that she adopted was not the one expected by her father, Maria experienced great distress. Not only could she not do the maths but her interactions with her father contributed to her feelings of being inadequate. However, again the way that social relations operated at home provided opportunities to find her own solution that were not possible in school. She could reject her parents help but only if she persevered with finding a solution. If she had not been able to find a solution then the result may not have been successful.

Doing mathematics homework can be traumatic. For some children, there is significant pressure to successful determine their own solution to doing the mathematics, so that they can negotiate a pathway between home and school expectations about what this entails. If these children are unable to find a solution, then homework will continue to be a conflict zone at home and at school where there are likely to be consequences for not completing the homework.

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HIGH ACHIEVING FEMALES IN MATHEMATICS: HERE TODAY AND GONE TOMORROW?[1]

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The continuing lower representation of mathematically capable females among those entering careers in mathematics and related areas is continuing to attract research attention. In this paper I draw on data gathered, some years after their school success, from the highest achievers in the annually conducted Australian Mathematics Competition. Using purposeful sampling, insight is gained into the lives - after they have left school - of these high achieving mathematics students, and in particular on their attitudes to mathematics, their motivations, self descriptions of aspects of their lives, their career choices and factors influencing those choices. Although inevitably a small group, in this paper particular emphasis is placed on the female medallists.

INTRODUCTION

Gender differences in mathematics performance and participation in post compulsory mathematics courses have attracted considerable research attention over the past four decades. A careful reading of the literature has consistently revealed a substantial overlap in the performance of males and females; gender differences in performance – when found – are small. Yet gender differences in performance, most often in favour of males, have continued to be reported when above average performance is considered, for students in advanced post compulsory mathematics courses, and on selected mathematical tasks assessed through standardized or large scale testings. For example, based on their secondary analysis of TIMSS data, Mullis and Stemler (2002) concluded that males were typically over represented among high achieving students. This pattern was again observed for grade 8 Australian students in the TIMSS 2007 data: “Around three per cent more males than females achieved the advanced benchmark, and eight per cent more males than females achieved at least the high benchmark” (Thomson, Wernert, Underwood, & Nicholas, 2008, p. 62).

In recent years, the continuing lower representation of females among those entering careers in mathematics and related areas: science, technology, engineering and mathematics [STEM] has attracted particular attention. Hyde, Lindberg, Linn, Ellis, and Williams (2008) examined data from some seven million school students in 10 American states on tests assessing cognitive performance and concluded: "Gender differences in math performance, even among high scorers, are insufficient to explain

[1] The support of Professor Peter Taylor, and particularly in contacting the medallists, is gratefully acknowledged.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 497-504. Thessaloniki, Greece: PME.

lopsided gender patterns in participation in some STEM [science, technology, engineering, and mathematics] fields" (p. 495). Drawing on data from various mathematics competitions Andreescu, Gallian, Kane and Mertz (2008) pointed to the influence of socio-cultural and environmental factors on students', and particularly females', willingness to continue studying mathematics beyond high school level, and thus to have the option of pursuing careers in mathematics and related fields. More generally, Lacampagne, Campbell, Herzig, Damarin, and Vogt (2007) noted in their review of issues affecting gender equity in mathematics that "particularly missing in the research are reasons for women's lack of persistence in continuing in mathematics from the undergraduate to the graduate program" (p. 250).

AIMS

In this paper I draw on data gathered, some years after their school success, from the highest achievers in the annually conducted Australian Mathematics Competition [AMC] and thus to gain insight into their mathematics related memories of school as well as their lives since then, their attitudes to mathematics, their motivations, and factors influencing their career choice.

Particular emphasis is placed in this paper on the female medallists, inevitably a small group. Though limited by space constraints imposed on a PME research report, relevant contextual background information is provided for both male and female respondents to enable some comparisons also to be made between the two groups.

THE AUSTRALIAN MATHEMATICS COMPETITION [AMC]

Aspects of the AMC, and gender differences in performance on the competition papers, have been reported in Leder, Forgasz, and Taylor (2006). In brief, the first AMC was conducted in 1978, with 60,000 students from some 700 schools entering the competition. Since then the competition has grown considerably.

The AMC, conducted under the auspices of the Australian Mathematics Trust [AMT] is open to students of all standards and in recent years about one-quarter of Australia's secondary school students have participated in the competition. The consistently high student participation rates are clear testimony of the value assigned by schools to the AMC.

The AMC papers are composed by a specially convened committee, drawn from teachers and university academics. Three separate Competition papers are set for students in grades 7 and 8, grades 9 and 10, and grades 11 and 12 respectively. The papers contain questions in arithmetic, algebra, and geometry. The questions are graded, with the early questions accessible to students of all standards while problems placed later in the papers are challenging to the most elite students. Each year a small number of entrants, about 1 for every 10,000 students entered, receive a medal. "These are awarded on the judgement of the committee to students who are outstanding within their region … (and) their year group" (AMT, 2007).

Some medallists, including some of the earliest recipients, maintain contact with the AMT through personal communications or via the AMT website. For others, older address details are still available. Although roughly equal numbers of females and males enter the AMC, especially in the early secondary school years (Leder et al., 2006), there are fewer female than male medallists in any one year.

The Sample

Purposeful sampling was used to select information-rich cases in this study. This approach has been used by others who have studied the development of gifted individuals, for example, Bloom (1985) and Lubinski, Webb, Morelock, and Benbow (2001). AMC medallists, i.e., the highest achievers in the AMC, and specifically those students who were awarded a medal between 1978 and 2006, comprised the pool of participants in the current study.

Between 1978 and 2006, 690 medals were awarded to students attending a secondary school in Australia. The actual number of recipients was in fact less, 420 students, since some students have won a medal in more than one year. Thus the *potential* sample consisted of 420 students. However, as described below, the actual sample obtained comprised 90 individuals (80 males and 10 females).

METHOD

Previous work on the development of exceptionally high achievers in mathematics – particularly Csikszentmihalyi, Rathunde, and Whalen's (1993) longitudinal study of talented teenagers, Burton's (2004) study of contemporary mathematicians, and Eccles' (2005) model of academic choice – informed the design of the study and the scope of the main survey used for data gathering. These authors highlighted personal values and qualities, as well as environmental factors as significant predictors of success and future behaviours. Details of the survey are provided in a later section.

To comply with ethics requirements, previous medallists were not contacted directly but were alerted by a letter from the AMC organizers to the URL of an online survey specifically aimed at former medallists. In the letter, which was sent to their last known address, they were encouraged to complete the survey within a reasonable time frame. No reminder messages or letters were sent.

The instrument

The survey covered five broad areas: background; school and university; career/vocation (actual or intended); work habits; and some general issues about self. Some items, which typically yielded rich data, were open-ended: for example, “What did winning a medal mean to you?”; “Who influenced your choice of career? List more than one if applicable.” Others were in 5-point Likert response format to which students responded SA (strongly agree) to SD (strongly disagree): for example, “Once I undertake a task, I persist”; “If at all possible, I’d rather work alone than with others to complete a task”. A second survey, which enabled more detailed explorations of issues already opened up, and also contained several items used by Liljedahl (2004), was sent to a small subgroup of the respondents.

Data gathering and synthesis

SurveyMonkey (http://www.surveymonkey.com) was used to create the online survey and to validate, collect, and summarize responses.

RESULTS AND DISCUSSION

Response rate

Of the 420 letters sent out, 52 were returned as undeliverable, leaving a potential pool of 368 respondents. In addition, any completed survey in which the items relating to an AMC medal were not answered was discarded. (Although only medallists were specifically directed to the survey site by letter, others who perused the AMT website also had access to the survey and may have explored it out of curiosity.) The total of 90 useable surveys received within the time frame set represents a response rate of at least 25%. This is within the limits reported by McBurney and White (2004) in their comparison of response rates for different methods of survey administration. "Surveys printed in magazines", they wrote, "may have a 1% or 2% response rate. Mail surveys often have return rates between 10% and 50%" (p. 247).

The sample was considered representative as respondents included medallists from 1979 to 2005 - whose dates of birth ranged from 1960 to 1994 - mirrored the ratio of single and multiple medallists found in the larger sample, and reflected a male dominance among the AMC medallists consistent with the gender ratio for high performers reported by Leder et al. (2006).

Background and group data

As mentioned above, 90 useable surveys were received from medallists: 80 from males and 10 from females. Four of the latter were born outside Australia. For the males the figure was 23%, virtually identical to the proportion of 22.2% given in the Census figures for Australia's population as a whole (Australian Bureau of Statistics, 2008). It is noteworthy that Andreescu et al. (2008) reported a high proportion of non-American born females in their group of young talented female mathematicians in the USA.

For the vast majority of the medallists, both female and male, both parents had tertiary education qualifications.

School and university

Mathematics, English, and science were each listed by three female medallists as their favourite subject at school. In contrast, mathematics was nominated as their favourite subject by two-thirds of the males. The different preferences were also reflected in the mathematics subjects the medallists studied at university. Four of the females continued with mathematics or statistics until third year, three took no mathematics at all, and the remainder took some – mainly first year – mathematics subjects. Again, by contrast, the majority of the males took multiple mathematics courses as can be inferred from the careers they chose. Very few of the males (under 10%) did not study mathematics at university. Though the small number of female medallists precludes firm comparisons being made, these differences are noteworthy.

None of the AMC medallists, neither females nor males, mentioned any negative aspects of winning a medal. However, that teachers and schools, as well as the medallists themselves, reacted variously to the recognition of high mathematical achievement is captured evocatively by the following two excerpts, both from female medallists:

> The AMT sent some extra challenging problems, but it wasn't really followed up. I did do some of them. If my school had given me any encouragement or some time off the incredibly boring school maths classes to do them, I would probably have done a lot more. So actual benefits - negligible… My best experience of maths education was the one year I spent in Japan as an 8 year old in 1976, which gave me a huge head start, followed by two enlightened maths teachers in 4th & 5th grade primary school back in Australia who allowed me to work through every text book they had at my own pace. I finished grade 10 maths half way through 5th grade. Unfortunately, none of my subsequent teachers were as inspired. In 6th grade they couldn't find anything for me to do. High School was much worse - I had to start on year 7 maths like everyone else, and was bored to tears for the rest of my school days. I like a challenge, but there wasn't one.
>
> (Leonie[2] who became a freelance orchestral musician)
>
> I think (it) got me an invitation to participate in the Tournament of the Towns - which in turn meant regular exposure to (a) more challenging mathematics, and (b) other extremely talented students. I gained a great deal from this program. At school, I got somewhat embarrassed by the fuss and teacher pride, but on the other hand my teachers were happy to let me do other things in class once I finished class work.
>
> (Barbara who is studying to become a "statistician or physicist")

Many of the males indicated that winning a medal opened mathematically exciting doors. Martin's comment, reproduced below, captured the sentiments expressed by many:

> Selection into the Mathematical Olympiad training programme, with many flow on benefits, including: learn much more mathematics and at a higher level, meet like-minded people many of whom are now good friends, encouragement to continue with mathematics.
>
> (Martin, doing statistical research at a highly prestigious university outside Australia)

Work habits and motivation

There was much overlap in the motivations and working habits of the female and male medallists. They thrived when doing difficult, challenging, and highly skilled

[2] All names are pseudonyms

work. Once started, they persisted with a task. Their motivation and task commitment was high. Working competitively, but also working cooperatively, was liked by many. They strongly rejected the suggestion that they might sometimes “work at less than my best so others won’t resent me”. They were divided whether they worried “that my success may cause others to dislike me”, though the males disagreed more strongly than the females with this statement.

Career choice

Medicine has proved a popular career choice for the female medallists. Four are now medical practitioners. Another four were working in, or towards, mathematically related fields (medical scientist, meteorologist, statistician/physicist, and software engineer), while the remaining two worked in fields which, they emphatically indicated, did not involve mathematics. None of the females described herself as a mathematician, while in contrast, some 15% of the male medallists did so. Many of the males (some 40%) are working in mathematics-related professions, including actuary, engineer, hedge fund trader, software developer and computer scientist. Medical practitioner was given by 15% of the males as their occupation.

As a group, the women indicated that their choice of career was influenced by parents, teachers at school and university, and personal "calling".

Many of the medallists had won, at school and university, a variety of other honours and prizes. Thus many excelled in multiple areas and ultimately had to select from multiple possible career pathways.

> I wanted to do scientific research of some sort since early high school. The only difficulty was deciding on a field, since I was interested in too many areas of science – mathematics, physics, chemistry, biology, eventually also computer science… In my undergraduate degree, I tried to choose majors that would cover as many fields as possible, so I could put off the decision. Eventually I realised that I liked computer science best of all, and did my honours in computer science. (Karen, software engineer)

Various factors were recalled that influenced them, i.e., the females, to move away from mathematics. They included

- personal interests: “I knew I wanted to be a mathematician/scientist at a young age, perhaps around 13, however I only made the choice of meteorology in uni, age 20. ...At uni I was interested in a lot of things, but eventually narrowed it down to applied maths and later meteorology.”
- The way mathematics had been taught: "... because it (mathematics) was badly taught and there was almost no encouragement", "A perception that ... mathematics wasn’t as interesting or enjoyable as the problem-solving, competition maths I was heavily involved in and enjoyed".
- Parental influence: "Being female – my father believed that it was disadvantageous to be female in the science/engineering fields, and so dissuaded

me. I chose medicine … I was interested in the science of the human body, and was also encouraged by my parents."

- Longer term career implications: "A perception that it would be difficult to find an interesting and rewarding job as a career mathematician".
- Other considerations: Feeling that their work would make best use of their talents, did not involve close supervision, and "leaves room for other things in my life" were highly valued. Prestige of career and financial reward were relevant but were not listed as being of prime importance.

But, as forcibly put by one of the female medallists:

> the traditional model of identity in which people realise 'what they want to be' is not the only way that people experience career choice ... I didn't make a decision 'not to become a mathematician'; I made a series of decisions to study other things and work in other fields…. During my 10 years at university I had various breaks from part-time work either while I was living at home with my parents, or while I was receiving a PhD stipend. Since completing my PhD I have had 3 separate work contracts with gaps of up to three weeks in between while seeking the next job.

CONCLUDING COMMENT

The survey data provided useful glimpses of the factors influencing the longer term career choices of students identified at school as exceptionally high achievers. The way mathematics and other subjects were taught, teacher influence, personal interests, openings available at critical times, and the perceived opportunity to use personal skills were apparently more important factors than outstanding mathematics achievement *per se* in shaping their career paths. The following excerpts may be indicative of the pressures and perceptions influencing the more constrained career choices (compared to the males) of the female medallists:

> Unfortunately, Australia (like many other countries) does not value academic achievement, and although there has been talk of a maths/science teacher crisis for decades (one of the year 8 maths teachers at my high school was actually primarily a metalwork teacher) little has been done to promote maths and maths teaching as a career option. More promotion is needed for maths generally, as leading to a wide range of careers. When I was at school it was assumed that a uni maths degree would lead only to a teaching career or academia. (The careers counsellor was totally useless.) Other maths graduates I know have had similar experiences. (Madeline)

> Australia (like other Western nations) continues to sustain disparities in men's and women's achievements in the workplace, public life and the economy (amongst other things). (Caroline)

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ANALOGICAL REASONING BY THE GIFTED

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The powerful role of analogical reasoning in discovering mathematics is well substantiated in the history of mathematics. Mathematically gifted students, thus, are encouraged to learn via in-depth exploration on their own based on analogical reasoning. In this study, 4 gifted students (two in the 6^{th} grade and two in the 8^{th} grade) were asked to formulate or solve mathematical problems through analogical reasoning. All 4 students produced fruitful constructs led by analogical reasoning; however, findings showed there to be different tendencies in students' analogy and mathematical construction. In addition, findings suggested different educational needs and support are necessary for the gifted.

INTRODUCTION

Polya (1954, 1962) proclaimed analogy an essential mathematical reasoning ability, and studies on the meaning and development of analogical reasoning (English, 2004; Alexander et al., 1997) have shown how important analogy is in the development of logical thinking. The powerful role of analogical reasoning in the construction of new mathematics is also well documented in the history of mathematics. Mathematically gifted students, thus, are encouraged to go further in their explorations and discoveries based on analogical reasoning.

Sriraman (2003) observed that gifted student thinking behaviours including creative problem solving, formalization, and so on corresponded to those of mathematicians. Lee (2005) found that gifted students eagerly seek advance to higher level of reasoning through reflective thinking or reflection on their earlier thoughts and reasoning. Similarly, mathematically gifted student behaviour has been characterized as being faster; i.e., they are able to efficiently visualize, generalize, simplify, abstract, and grasp the meaning and structure of a problem in a much shorter time period than their peer groups (Heid, 1983; Sriraman, 2004). However, there have been relatively few studies on the analogical reasoning of the gifted.

One of the major challenges in gifted education is to develop an educational program that conforms to the characteristics and needs of gifted students. To date related studies have revealed, at least three requirements that must be met: access to advanced mathematical content (Johnson & Sher, 1997); exposure to challenging mathematics problems (Johnson, 1993); and opportunity to develop creative thinking (Sheffield, 1999). These three requirements were considered in the development of an educational program for the gifted that incorporates analogical reasoning. The objective of this study was to obtain detailed

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 505-512. Thessaloniki, Greece: PME.

information on the way mathematically gifted students utilize analogical reasoning.

THEORETICAL FRAMEWORK

Rota and Palombi (1997) describe mathematician behaviour as the systematic concealing of analogical trains of thought, the authentic life of mathematics, by enrapturing discoveries. Poincaré points out that mathematical facts worthy of study are those that, by their analogy with other facts, are capable of leading one to knowledge of a mathematical law, just as experimental facts lead to knowledge of a physical law. More explicitly, Atiyah mentions that mathematics is the science of analogy, and he argues that finding analogies between different phenomena and developing techniques to exploit these analogies is the basic mathematical approach to the physical world (cited in Corfield, 2003: 81-82). Analogies, thus, can be seen as cognitive aids to the discovery and learning of mathematics.

Polya (1962) explains analogy as a kind of similarity. He claims that the essential difference between analogy and other kinds of similarity lies in the intention of the thinker. If one intends to reduce several aspects shared by objects in order to definite the concepts that make the objects similar, one clarifies the analogy. Polya explains his rational by reasoning that a triangle can be analogous to a pyramid: a triangle is obtained if all points of the segment are connected to a point outside the line of the segment, and a pyramid is obtained if all points of the polygon are connected to a point outside the plane of the polygon. Using this same reasoning, he shows why a prism can be regarded an analogue of a parallelogram.

Piaget (1952) claims that there exists spontaneous functioning of the schemata of displacement by analogy and this analogy entails imagination of new combinations. Analogy is, thus, an instrument that differentiates initial schema by assimilation. According to Piaget, there are two levels of relations that form the solution of analogy problems or analogical reasoning. Lower order relations are simpler ones that are generated between near or closely paired concepts. Higher order relations, in contrast, are those produced between more distant or removed concepts.

METHOD

To investigate the use of analogical reasoning by mathematically gifted students, this research paper intentionally samples appropriate cases, collects data via observation, and performs an in-depth analysis of the data as suggested by Strauss and Corbin (1990). The subjects of this research are two 12-year-old elementary 6th graders (E1, E2) and two 14-year-old junior-high 8th graders – all of whom are receiving education at an attached academy for the gifted at a university. E1 and M1 excel in algebra while the other two show greater aptitude in geometry, as determined from their 12-month-long education at the attached academy prior to this study.

A 9-hour long educational program (3 units, each composed of 3 hours) is provided to the gifted students. Each session with the elementary school students is conducted separately from the middle school students, and for each student one research assistant is assigned to conduct concentrative observation on and an interview with the student. This approach facilitates observation of the ways in which gifted students might differ from each other regarding: (1) the kinds and aspects of objects featured in gifted students' analogical reasoning, (2) the ways in which gifted students compare objects and find correspondences, and (3) the depth of gifted students' mathematical understanding as it emerges in the formulation and justification of an assumption by analogy. To ensure reliability, all utterances by students and interactions with interviewers are audio-/video-taped.

The tasks used in the nine-hour educational program are detailed below.

- [Task 1] Which geometric figure can be regarded an analogue of a triangle? Explain your answer.
  [Task 2] A triangle and a tetrahedron can be regarded analogous. How is this possible? Explain your answer.
- [Task 3] Make conjectures on a tetrahedron based on your knowledge of a triangle and justify your answers.

The above detailed tasks were developed with the objective of requiring students, individually, to discover possible mathematical conjectures on a tetrahedron and justify the conjectures using analogy. Hence, the classical form of an analogy problem, i.e., "A:B::C:?" in which three components of the problem are given and the fourth is to be determined, was modified to meet study objectives. The first task asks gifted students to find analogous objects without any given information about object relations. In other words, students attempted the problem "A:?::?:?". This modification was made to investigate whether gifted students possess innate analogy reasoning. The second task requires students to postulate relations in order to clarify the given analogy "A:?::C:?". By attempting this problem, individual divergent abilities surface as they compare two figures and make correspondent properties, which complete analogy. The final task differs from the classical analogy problem form in that it requires gifted students to formulate whole conjectures by analogy based on knowledge rather than simply determining the fourth component in an analogy problem. These modifications meet recommendations stipulated by prior research on gifted education (Johnson, 1993; Sheffield, 1999).

RESULTS

All four students actively participated in each of the task solving processes. Each student revealed quite different analogical reasoning, and table 1 highlights the kinds and characteristic aspects of each gifted students' analogical reasoning.

Task	E1	E2	M1	M2
1 (A:?::?:?)	Polygon Polyhedron Tetrahedron (elements)	Polygon Tetrahedron (feature)	Polygon Pyramid Tetrahedron (elements)	Tetrahedron (feature)
2 (A:?::C:?)	Role Feature	Property Relation	Feature	Property Relation
3 (A:?::C:?)	The sum of interior angles Number of sides, faces	The sum of interior angles Inscribed Face angle	The sum of interior angles Inscribed Circumscribed	The sum of interior angles Inscribed Face angle Solid angle

Table 1: Emerged analogical reasoning

Interestingly, all students, except M2, initiated the first task by drawing and analysing a triangle and various kinds of polygons. E1 and M1, both of whom are strong in algebra, investigated triangles and polygons by focusing on elements such as angles, sides, and faces. E2 and M2 who were identified as being strong in geometry focused on features of a triangle. They describe a triangle as "a fundamental figure by which all kinds of polygon can be made" (E2) and "a figure with the smallest number of sides in a plane" (M2). When postulating and justifying conjectures on a tetrahedron, each of the four students spent a considerable amount of time looking at the correspondent property to "the sum of the measures of an arbitrary triangle's interior angles is π."

Evaluation of Analogies

At the onset of the study programme, the instructor briefly explained the relationship "dog : bark :: cat : meow" (Alexander et al., 1997) to ensure students understood analogy. E1 found that a triangle, irrespective of its shape, has several unique elements such as vertices, angles, and edges. Quickly realizing that a polygon also has these same elements, E1 concluded that any polygon is analogous to a triangle. E1 quickly adopted his generalization to a polyhedron with the same reasoning. E2 and M1 seemed to proceed along the same route as E1. They explained that any polygon can be analogous to a triangle because of the similar expression for the sum of the measures of its interior angles and the same summation for the sum of the measures of its exterior angles.

After a while, all students deemed those analogies unproductive. The following is a part of a conversation between E2 and the interviewer.

E2: This is so boring! What can I do make it more exciting?

Interviewer: Are you not satisfied with your constructs?

E2: Not at all. But there must be something more important, something I am overlooking. My analogy just isn't that impressive.

Interviewer: You want to find something notable and interesting?

E2: Yes, I ought to find another example for the analogy.

While three of the students eventually turned to polyhedrons, which led to the consideration of a tetrahedron, M2 focused on tetrahedrons from the very beginning. The strong interest in tetrahedrons finally resulted in students making an analogy between a triangle and a tetrahedron. This observation provides evidence that gifted students are aware of the use of meta-cognition. That is to say, they can question and evaluate their own analogies, which leads to useful mathematical discoveries (Kramarski & Mevarech, 2003). The students are unlikely to have considered solid geometry or a tetrahedron as a correspondent object of a plane geometry or a triangle if they had been satisfied with initial analogies. They most likely would have stopped after making preliminary analogies.

M2, in particular, did not ponder long the analogy of a polygon or a polyhedron to a triangle. He quickly began focusing on a tetrahedron and spent most of his time pondering the relation or structure of the analogy he needed to make without producing much output. In the interview, he was asked why he decided to proceed immediately with an analogy attempt to a tetrahedron unlike other students. His response is detailed below.

M2: It just came to mind! A tetrahedron is very similar to a triangle indeed.

Interviewer: Why not consider a quadrangle? It's similar to a triangle, isn't it?

M2: Yes, in a sense. However, I think I need to change this (pointing the word "triangle" written by him) to something very similar and very different.

Interviewer: Why does it have to be different?

M2: I don't know but I think it should be; otherwise, nobody would welcome my idea.

According to his answer on the necessity of being different, he is seen evaluating his potential analogies of a triangle mentally. Therefore, even without a single explicit trial, he came to the idea of a tetrahedron. This finding suggests gifted students are willing to review their approaches to and results of finding analogous figures of a triangle, which easily leads to discoveries of useful analogy.

Once a similarity between a triangle and a tetrahedron had been established, students centred on describing the roles or features of the two figures in plane geometry and solid geometry. E1 and M1 wrote: "a triangle is the basis of all polygons and a tetrahedron is the basis of all polyhedrons. Both figures can be considered a starting point and the simplest among figures of each geometry." In other words, once E2 and M2 completed a list of all the properties of a triangle and a tetrahedron, they tried to find a correspondent property for each of the listed properties.

Space of Knowledge and Thinking Strategies

According to Alexander et al. (1997), each turn of an analogy problem should involve encoding, and inferring takes place when a relationship is constructed between the first pairing in a stated problem to make analogy. During the second and third task, the four students' space of knowledge on a triangle was significantly different as shown in table 2.

Space of Knowledge	E1	E2	M1	M2
The sum of the measures of an arbitrary triangle's interior or exterior angles is π.	○	○	○	○
The bisectors of the angles of a triangle intersect at a point that is equidistant from the three sides of the triangle.	○	X	○	○
The perpendicular bisectors of the sides of a triangle intersect at a point that is equidistant from the vertices of the triangle.	○	X	○	○
The medians of a triangle are concurrent	X	X	○	○

Table 2: Difference of Space of Knowledge

The preliminary assumption, before the onset of this study, was that students' space of knowledge is the main contributor to their analogical reasoning, especially at the encoding and inferring stages. E2, however, performed very well despite a relatively smaller space of knowledge on a triangle than E1. Also, M1 and M2 showed considerable difference in analogy creation even though their spaces of knowledge appeared very similar. E1 and M1 did not clarify analogy using knowledge of incentre and circumcentre or centroid, whereas E2 and M2 did and as a result reached a rich analogy.

The most impressive difference occurred when E2 and M2 focused on face angles and the sum of the measures of those angles in a tetrahedron. They discovered that the sum of the measures of an arbitrary tetrahedron's face angles is invariant identical to the sum of the measures of a triangle's interior angles. M2 realizing the need to define the angle made by three faces in the corner of a tetrahedron concentrated on the relationship between a solid angle and the sum of the measures of face angles. He made the conjecture: "the sum of solid angles of a tetrahedron is constant akin to the sum of the measures of face angles of a triangle." In contrast, E1 and M1 did not look at face angles. They focused their attention on the measure of solid angles of a tetrahedron. Both ultimately asked the instructor for solution hints and in the end gave up on the third task.

CONCLUSION

As Sheffield (1999) points out, talented students should be inspired to think like mathematicians. Participants in this study appeared to experience the deep thinking that is necessary to solve problems made with analogies, a process equivalent to the one that mathematicians undertake. The subjects had to reflect on prior knowledge and develop new concepts such as face angle and solid angle based on analogical reasoning. Students with excellence in algebra (E1, M1) had relatively more difficulty clarifying higher-order analogy than students deemed good at geometry (E2, M2). Quality in the justification of the conjectures by analogies also differed between the two groups. This finding suggests a possibility that analogical reasoning is connected to student learning preference or capacity in content areas. All subjects, however, were found adept at making meaningful analogues of a triangle since they all made use of meta-cognition when searching relations for analogies. In the future, methodologies including the development of tasks and teaching settings, measures to evaluate the depth of mathematic exploration through analogy, and research on how to promote education related to analogy for gifted students will enhance gifted student mathematics education.

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