INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

PME 33

THESSALONIKI 2009

PROCEEDINGS

of the 33rd Conference of the International Group for the Psychology of Mathematics Education

In Search for Theories in Mathematics Education

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Volume 5

Research Reports [Ron - Zod]
Poster Presentations

PME 33, Thessaloniki - Greece
July 19-24, 2009

Cite as:
Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). (2009).
Proceedings of the 33rd Conference of the International Group
for the Psychology of Mathematics Education, Vol. 2. Thessaloniki, Greece: PME.
ISBN: 978-960-243-652-3

Website: **http://www.pme33.eu**

The proceedings are also available on CD-ROM

ISSN 0771-100X

Volume 5 ISBN: 978-960-243-657-8

Cover design & Overall Printing Layout: Dimitris Milosis
Logo: Giorgos Papadopoulos, Adaction S.A.
Production: MOUGOS - Communication in Print

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Research Reports
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ON STUDENTS' SENSITIVITY TO CONTEXT BOUNDARIES

Gila Ron and Tommy Dreyfus — Rina Hershkowitz

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We studied 8th grade students' processes of knowledge constructing while learning probability. We present and exemplify two classes of partially correct constructs which express students' lack of sensitivity to the boundaries of the context in which their constructs are relevant: constructs that are recognized but only in a context that is too narrow for the situation under consideration, and constructs that are implemented in a wider context than warranted.

PARTIALLY CORRECT CONSTRUCTS (PACCS)

In spite of the efforts made by mathematics education researchers, instructional designers, and teachers in the design and implementation of learning situations, the resulting match between the underlying mathematical knowledge elements and students' constructs for these elements is, in many cases, only partial. In previous papers we used the notion of Partially Correct Construct (PaCC) for a student's construct that only partially matches a mathematical knowledge element that underlies the learning context; we showed that contradictory student answers and unexpected student difficulties can often be explained by identifying some student constructs as PaCCs (Ron, Dreyfus and Hershkowitz 2006; Ron, Hershkowitz and Dreyfus 2008). These results have been obtained by using the epistemic actions of the RBC model (Hershkowitz, Schwarz and Dreyfus 2001) as tracers.

The RBC epistemic actions

Processes of knowledge constructing are expressed in the RBC model by means of three observable epistemic actions, *Recognizing, Building-with, and Constructing* (whence *RBC*). Constructing of knowledge is seen as *vertically re-organizing existing knowledge constructs* in order to create a new knowledge construct. Recognizing takes place when learners recognize that a specific construct is relevant to the problem they are dealing with. Building-with is an action comprising the combination of recognized constructs, in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem.

Contextual aspects of Partially Correct Constructs

PaCCs are defined as constructs that match the corresponding mathematical knowledge elements only partially. In this paper we explore PaCCs in which the partial fit between a student’s construct and the corresponding knowledge element lies in student’s failure to recognize the boundaries of the context in which the construct is relevant. Cases in which students recognize knowledge elements only in too narrow a context were studied by Wagner (2006). Smith, diSessa and Rochelle

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 1-8. Thessaloniki, Greece: PME.

(1993) claim that learning involves knowing where and why pieces of knowledge that are conceptually correct work only in a more restricted context. We present, in parallel, cases in which students recognize knowledge elements in too narrow a context and cases in which students implement their constructs in a context that is wider than warranted.

METHODOLOGICAL CONSIDERATIONS

A ten lesson probability unit (Hershkowitz, Hadas, Dreyfus and Schwarz 2007) was designed and experimentally taught to six student pairs (grade 8, age 13) in laboratory conditions and to seven grade 8 classes. The pairs and one or two focus groups in five of the classes were video-taped throughout the unit. Using the epistemic actions of the RBC model as tracers we analysed learning processes to identify expressions of knowledge elements in students' actions and claims.

Mathematical knowledge elements

We consider the partiality of knowledge with respect to the mathematical knowledge elements that underlie the learning context. In this paper, we consider only equiprobable two dimensional sample spaces (for example, the sample space for rolling two dice) and focus on two knowledge elements:

Order: We shall say that students have constructed the order element if they relate to events like (□,Δ) and (Δ,□) (e.g., 'rolling 5 and 2' and 'rolling 2 and 5) as to two distinct events.

Ratio: We shall say that students have constructed the ratio element if they calculate the probability value of a compound event as the ratio between the number of simple events in the compound event and the number of simple events in the sample space.

DATA AND ANALYSIS

We present PaCCs in four learning episodes that are representative for other students' similar PaCCs in the same or other tasks. The first two episodes exemplify situations in which students recognize a previously constructed knowledge element only in a too narrow a context. The other two episodes exemplify situations in which students build-with mathematical knowledge elements in irrelevant contexts.

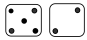
The case of Roni: coloured dice versus identical dice

Roni and Yam worked as a focus group in a class. Answering question 1b of the dice task (see figure 1), Yam speaks first, counting five pairs of consecutive numbers: "One two; two three; three four; four five; five six." Roni dictates their joint written answer: "The game is not fair because out of eleven possible outcomes, six times (on average) Ruti will win, versus five times that Yossi will win"

1a Yossi and Ruti roll two white dice. They decide that Ruti wins if the numbers of points on the two dice are equal, and Yossi wins if the numbers are different. Do you think that the game is fair? Explain!

1b The rule of the game is changed. Yossi wins if the dice show *consecutive* numbers. 2 and 3 are consecutive numbers, or 23, 24 and 25 are consecutive numbers. Do you think the game is fair?

1ci How many possible outcomes are there when rolling two dice?

An outcome of rolling two dice is, for example, .

1cii Reconsider your answers to tasks 1a and 1b: Are the games fair?

1d If Yossi and Ruti play with one red die and one white die, does this change the answers to 1a, 1b and 1c?

Figure 1: The dice task

In answering question 1ci Roni used multiplication: "Every one has six [possibilities]. Thirty six possibilities. Every number six times." Roni found that the sample space has 36 simple events, presumably without considering whether he counted ordered or non-ordered pairs. When answering question 1cii, he repeats his former answer that the game is not fair because out of eleven possible (winning) outcomes, Ruti has six while Yossi has only five.

When considering question 1d Roni says: "… here you have one red die. Here [two white dice] if you get 5, 2, you can't get again: 2, 5; it is considered the same. But here [red and white dice] it's possible. If you get, say, 5 on the red and 2 on the white, maybe it's possible 5 on the white and 2 on the red." The pair's written answer (dictated by Roni) is similar: "There will be a different answer because it can be 5 on the white die and 2 on the red one; and the opposite is considered a different answer."

Roni's oral and written answers supply evidence that he constructed the Order element in the context of playing with coloured dice. However, even after becoming aware of the possibility of reversal, at least in the context of dice with different colours, he does not connect this awareness to the game of Yossi and Ruti playing with two white dice, and does not change his mind regarding the fairness of the game in question 1b. Roni's Order element was constructed for a context that is too narrow for the situation under consideration.

The case of Hagit: different ways to play the game

Hagit, Yana and Shir worked as a focus group in a different class. Their first steps in solving question 1 of the dice task, seem incoherent. Yana is the first who, like Roni and Yam, counts five consecutive pairs as Yossi's winning pairs. Soon thereafter, however, she proposes counting pairs like (1,2) and (2,1) as distinct. Her idea initiates a discussion full of reversals.

72	Yana:	No, no, no… It can be different… each die … it's 10 … He [Yossi] still has the same 10. One two, and two one. [Yana gestures with her hands, putting the two index fingers next to each other, and then crossing them.]

Shir accepts Yana's explanation but Hagit resists:

75	Hagit:	But you can't exchange the die. [Hagit imitates Yana's gestures]. It remains the same thing - constant.

She then checks whether she understood well:

77	Hagit:	Let's say that Ruti… and then Yossi… Say here Ruti has 1 and Yossi 2? The other way 'round, is that what you mean? [Hagit put two pens side by side and then switches their positions.]

Later, Yana returns to her original claim that Yossi has five winning pairs and the discussion continues with the students' roles switched:

112	Hagit:	But it is possible to switch it!
113	Yana;	No! It is not possible to switch it.
115	Hagit:	[I say] it IS possible to switch them: one and two, and two and one. No?
119	Hagit:	[Holding a pen that represents Ruti and a pencil that represents Yossi.] Just a moment. This [the pen] is Ruti, and this [the pencil] is Yossi.
121	Shir:	But they do not play together. He rolls and gets 2 1, and she rolls herself and gets something like this. They don't roll together!
122	Hagit:	Everyone has two dice?
123	Shir:	Yes. Every one has two dice. Not Yossi one and she one.
124	Hagit:	Wrong!
125	Shir:	They don't roll together. They don't roll together, Hagit.
126	Hagit:	They do roll together.

The girls check the wording of the question in their worksheets. Now they argue how to interpret the words "Yossi and Ruti play with two white dice". Shir accepts Yana's claim that this issue does not make any difference and that even if every player rolls his own die, reversed pairs should be counted only once. But Hagit is still in doubt:

140	Hagit:	…What shall I do? Ten out of 12 or five out of 12? [The girls considered 12 as the number of possible outcomes when they answered question 1.]

Hagit originally interpreted the rules of the game so that each player rolls his own die. In this context, the dice are distinguishable and as a consequence, according to Hagit, the order element applies. It is unclear from the discussion whether Hagit would consider the order element as relevant in the case of indistinguishable dice (the same player rolls the two dice) or if she is just uncertain about this question. It is clear, however, from Hagit's insistence in the discussion that rolling the two dice together is for her a new context that needs new considerations.

It may be interesting to note that a week later, after a computer-based activity that was designed to help students' become aware of the need to count reversed pairs like (1,2) and (2,1) as distinct outcomes, the teacher shows the class how to represent the sample space for rolling two dice in a six by six table. In order to help the students to distinguish between reversed pairs, she talks about coloured dice and guided the students to mark each pair with the colour of the die that shows the bigger number. At the end of the discussion, Hagit asks the teacher if she would have to count reversed pairs as two outcomes also in a case of playing with identical dice.

Like Roni, Hagit's construct for the order element was constructed for too narrow a context, namely one in which the dice are distinguishable. Unlike Roni, she wonders about the relevance of order also in the context of identical dice.

The case of Yana: building-with the order element in a context where it is irrelevant

At the end of the computer activity Yana (and the other girls) counted correctly all 36 outcomes of the dice task sample space. She also explained to the researcher that in the previous lesson she was wrong when she counted reversed pairs like (1,2) and (2,1) as the same outcome. That means that she has constructed the order element. In the third episode we shall see that Yana builds-with the order element in a context, in which it is irrelevant, the Spinners task (see figure 2).

Yael and Vered spin the hands of the two spinners in the figure and wait until they stop. (Each circle is divided into equal parts). Yael wins if the product of the two numbers shown by the spinners is positive. Vered wins if product of the two numbers shown by the spinners is negative.
Do you think that the game is fair? Explain!
Calculate the probability of each girl to win?

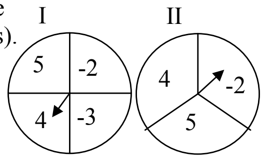


Figure 2: The spinners task

While Hagit and Shir struggle (unsuccessfully) with the Spinners task, Yana makes systematic lists of the winning pairs for Yael and for Vered. She explains to Shir: "Minus two, five – negative; with four – negative; but minus two, minus three is positive; four with five positive. Four and minus two – negative…".

Shir repeats the explanation for Hagit: "Four has four and five positive, and minus two and minus three negative; five has five and four positive, minus two and minus three negative; minus two has minus two and minus three positive, four and five negative. But what if we do it from the other side?" Yana immediately understands Shir's question, and asserts "here it is; I made it" and shows her work to Shir. Yana's work is illustrated in Figure 3. The first row on the right hand side represent, the pairs (-2,5) (-2,4) (-2,-2) (-2,-3). The "- - + +" expresses that two of these products are negative and two are positive. At left, she summarized her conclusions in a table. The second row represents one positive product and two negative one obtained with "-2" on the left spinner, etc. Yana's list includes each outcome *twice*: Once she matched to each number of the left spinner all the numbers of the right spinner and then she matched to each number of the right spinner all the numbers of the left spinner. This is superfluous because all 12 outcomes of the sample space are already represented in the four upper rows on the left side. Yana builds-with the order element in a context in which this element is irrelevant.

	positive	negative
-2	1	2
-3	1	2
4	2	1
5	2	1
-2	2	2
4	2	2
5	2	2

- - + +	-2 = 5, 4, -2, -3
+ + - -	4 = 5, 4, -2, -3
+ + - -	5 = 5, 4, -2, -3
+ - +	5 = 4, -2, 5
+ - +	4 = 4, -2, 5
- + -	-3 = 4, -2, 5
- + -	-2 = 4, -2, 5

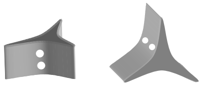
Figure 3: An illustration of Yana's lists and sketches for the spinners task.

The case of Shelly: building-with the ratio element in a sample space that is not equiprobable

The last part of the unit included probability tasks in eight situations that were not equiprobable. Shelly, a student who has learned in a laboratory setting, constructed a strategy based on an area model to successfully deal with most of these tasks. In a post unit interview Shelly was presented with a game using two identical irregular shapes (‘aliens’) with two flat and three bent faces (Figure 4a). Two dots (eyes) are drawn on the bent faces. An alien is considered sleeping if he lies on a flat face, and awake otherwise. Because of the alien's irregular shape, there is no reason to assume that the probabilities of falling on a flat or on a bent face are equal.

The rules of the game are: "We throw two aliens in the air. If both of them fall down awake, the first player wins. If one falls down awake and the other sleeping, the second player wins. If both of them fall down sleeping, there is no winner.” Among other questions, Shelly is asked to evaluate the fairness of the game and the probability of each player to win. Shelly quickly claims that the probability to fall down awake is 3/5 and the probability to fall down sleeping is 2/5, because there are three possibilities to fall down awake and two to fall down sleeping. Later she makes a systematic lists of outcomes (Figure 4b), explaining "I gave to this (the faces with the eyes) the numbers 1, 2, and 3, and to this (the blank faces) the numbers 4 and 5". She concludes that the probability for the first player to win is 9/25 and for the second player 12/25. Shelly calculated the probabilities with the ratio element, which is relevant only in equiprobable sample spaces. She thus built-with the Ratio element in too wide a context, in which using the ratio element is not warranted.

a



b

Both are awake	1:1, 2:2, 3:3, 1:2, 1:3, 2:1, 2:3, 3:2, 3:1
Both sleep	4:4, 5:5, 4:5, 5;4
One is awake/ one sleeps	1:4, 4:1, 1:5, 5:1, 2:4, 4:2, 2:5, 5:2, 3:4, 4:3, 3:5, 5:3

Figure 4: The aliens game. (a) The aliens (b) Shelly's list of events

SUMMARY AND DISCUSSION

Contextual PaCCs

The data presented here exemplify two classes of PaCCs that reflect two ways of students' lack of sensitivity to context boundaries.

Narrow-context PaCC: A student recognizes and builds-with a construct in too narrow a context with respect to the learning situation; the student faces situations where the construct can and should be applied but the student does not see the relevance of the construct for the situation. The limited context in which the student recognizes the relevance of the construct excludes (at least part of) the focus context.

Wide-context PaCC: A student recognizes a construct as relevant and builds-with it where it is relevant, but in addition the student recognizes as relevant and builds-with the construct in a wider context than warranted, where the construct is irrelevant.

The cases of Roni and Hagit exemplify narrow-context PaCCs. They constructed the order element for the narrow context in which the dice are distinguishable. There is a salient difference between Roni's and Hagit's PaCCs: Roni spoke of a difference between the contexts when answering an explicit question that was part of his task, and then moved on without being aware of or bothered by the issue. Hagit raised the issue of order on her own initiative, and this issue bothered her along several lessons.

Yana and Shelly built-with their constructs in contexts, in which the constructs were irrelevant. Yana's building-with the order element, when she had already listed all the possible outcomes, resulted in listing each outcome twice. Shelly built-with the ratio element in a situation where there was no reason to assume that the sample space is equiprobable. Although Yana obtained the correct answer that the game is fair, while Shelly's calculation is baseless in the given context, both Yana's and Shelly's cases exemplify wide-context PaCCs.

Equiprobability

In all the cases that were presented here, sensitivity to the context boundaries is related to equiprobability in two-dimensional sample spaces: The probability of a compound event can be calculated as the ratio between the number of simple events belonging to the compound event and the number of simple events in the sample space only if all simple events in the sample space have the same probability.

Understanding this point includes the need to be aware that outcomes may not be equiprobable. This awareness is especially important in cases where outcomes that visually look alike are actually distinct. This is exactly what happens when rolling two identical dice: With indistinguishable dice, getting 3 on the first die and 5 on the second, looks like getting 5 on the first and 3 on the second, but is actually a different outcome. The tendency to assign the same probability to the outcomes '1 on each die' and '2 on one die and 1 on the other die' is described in the research literature (e.g. Lecoutre, 1992). Lecoutre calls this tendency "the equiprobability bias". Roni and Hagit failed to recognize that ordered pairs need to be considered in an equiprobable sample space, and that this is true, no matter if the dice are distinguishable or not. Yana, on the other hand, built-with the order element, although she already had a list of equiprobable pairs. Shelly was not aware of the need to check whether her sample space is equiprobable in order to build-with the ratio element.

We believe that teachers and instructional designers need to be aware of students' contextual PaCCs in general, and in the context of two-dimensional equiprobable sample space, in particular.

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[1] Research supported by the Israel Science Foundation under grant number 1166-05.

JUSTIFICATION SCHEMES OF MIDDLE SCHOOL STUDENTS IN THE CONTEXT OF GENERALIZATION OF LINEAR PATTERNS*

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This study examined the work of six middle-school students over three years beginning in sixth grade as they justified generalizations for linear patterns given to them primarily in figural form. We present examples of student thinking to illustrate four categories of empirical justification that their work exemplifies.

INTRODUCTION AND RESEARCH QUESTIONS

In previous work investigating the processes of generalization of middle school students (Becker & Rivera, 2006; Rivera & Becker, 2008), we have concentrated on the way students approached forming a generalization and their development of formal methods of generalizing. Equally important in this work has been students' justification of generalizations over the course of three years. We share Lannin's (2005) perspective that students' justifications in the context of generalization "provide a window for viewing the degree to which they see the broad nature of their generalizations and their view of what they deem as a socially accepted justification" (p. 232).

One of the standards promulgated by the National Council of Teachers of Mathematics is Reasoning and Proof (NCTM, 2000). NCTM states that in the middle grades (grades 6-8, ages 10 to 12 years) students should have many and varied experiences using reasoning as they "examine patterns and structures to detect regularities; formulate generalizations and conjectures about observed regularities; evaluate conjectures; and construct and evaluate mathematical arguments" (NCTM, 2000, pg. 262). In this paper, we discuss the ways in which middle school students, immersed in teaching experiments around generalization of linear patterns in which students worked together and were expected to explain their solution methods and to listen to and evaluate others' explanations, developed their abilities to justify their generalizations and evaluate those of others. Here we use the concept of proof discussed by Harel and Sowder (2007), that proof or justification is the process used by an individual to determine the truth of an observation and explain that determination. Justification is the more appropriate term here as the explanations that middle school students are apt to provide are less formal considering the cognitive level of the students. We focus on the following *research questions*: 1) How do middle school students justify conjectures they themselves develop as generalizations for linear patterns? 2) How do they justify generalizations developed by others?

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 9-16. Thessaloniki, Greece: PME.

CONCEPTUAL FRAMEWORK

Harel and Sowder (2007) present a classification of proof schemes that includes: externally based proof schemes; empirical proof schemes; and analytic proof schemes. For Harel and Sowder, a proof scheme consists of methods of ascertaining the truth of an assertion and persuading others of its veracity or lack thereof. Externally based proof schemes reside in some outside authority, such as the teacher or the textbook. Empirical proof schemes may be either inductive or perceptual in nature, but primarily involve justifications made solely on the basis of examples. Finally, analytic proof schemes, transformational or axiomatic, include what most mathematicians would consider formal proof. The salient feature of transformational schemes is that the justification is concerned with general aspects of the conjecture and the reasoning entails settling the conjecture in general. Transformational schemes involve mental operations that are goal-oriented and the ability to anticipate the results of such operations; they are characterized by an understanding that the goal is to justify a conjecture for all values, not isolated cases. Axiomatic justification schemes are transformational ones but in addition, the student demonstrates that any proof must start from accepted principles or axioms (Harel & Sowder, 2007). Considering the cognitive level of our sample, in our work we looked for reasoning that demonstrates understanding and an ability to explain a generalization using both numerical and figural approaches. We did not expect to see any use of axiomatic proof schemes among this middle school sample.

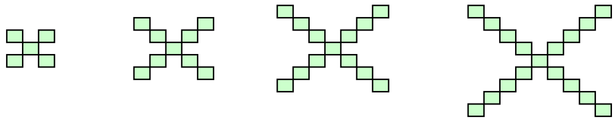
METHODS

This paper reports on results from a longitudinal study that followed a group of 15 students (5 males, 10 females; all students of color) over three years; in each year these students represented a subset of the whole class but other participants varied over the three years so this paper will concentrate on findings from these fifteen students. In each year, students participated in two clinical interviews involving five tasks (conducted by the first author) with analogous questions (e.g. see Figure 1 below), with an intervening teaching experiment related to generalization of linear patterns presented in context (conducted by the second author; see Rivera and Becker, 2008 for more details about the study). Data analysis involved viewing videotapes of the interviews and examining student written work from the interviews by each author separately and identifying patterns in justification schemes. Then identified patterns were compared and discussed to come to consensus. For this paper, we highlight the thinking of selected students that best illustrates a particular scheme. Also, due to the conceptual nature of schemes, we paid particular attention to those cognitive actions (correct or otherwise) that remained stable over some time.

RESULTS

How do middle school students justify conjectures they themselves develop as generalizations for linear patterns? In all three years of the study, we found that the students used empirical justification schemes that were based on examples, either numerical or figural, to justify their own conjectures or those of other students. We classify these justification schemes in relation to patterning activity as follows: *generic case*, and; *formula projection* and *formula appearance match* after a numerical construction of a direct formula.

Tiles are arranged to form pictures like the ones below.



Picture 1 Picture 2 Picture 3 Picture 4

A. Find a direct formula that enables you to calculate the number of square tiles in Picture "n." How did you obtain your formula? If the solution has been obtained numerically, is there a way to explain your formula from the figures?

B. How many squares will there be in Picture 75? Explain.

C. Can you think of another way of finding a direct formula?

D. Two 6th graders came up with the following two formulas:

Kevin's direct formula is: T = (nx2) + (nx2) + 1, where n means Picture number and T means total number of squares. Is his formula correct? Why or why not?

Melanie's direct formula is: T = (nx2) + 1 + (nx2) + 1 –1, where n and T mean the same thing as in Kevin's formula. Is her formula correct? Why or why not?

Which formula is correct: Kevin's formula, Melanie's formula, or your formula? Explain.

Fig. 1 Square Tiles Task from Year 1 in Compressed Form

Generic Case. Some students developed a generic case to illustrate the perceived structural similarity across the given stages in a pattern. In the post-interview in year 1, Dung developed the formula $S = n \times 4 + 1$ visually for the Fig. 1 pattern by seeing four sides with one square in the middle: "there's one in the middle so that could be plus 1 and then on the sides they all have the picture number [pointing to arms], so square tiles equal nx4 plus 1." He then checked his formula for two different values of n.

Formula Projection After a Numerical Construction of a Direct Formula. Some students employed formula projection in which they demonstrated the validity of their numerically-drawn formulas as they saw them on the given figures. Consistently throughout the three-year project, Tamara dealt with every linear pattern by setting up a table of values, then obtaining a common difference, then establishing a direct formula, and then visually verifying that the formula made sense by checking how each term in her formula actually made sense on the given stages. For example, in the case of Fig. 1, she initially set up a two-column table consisting of the ordered pairs

(1, 5), (2, 9), (3, 13), and (4, 17). Then she saw that each dependent term increased by 4, which led her to conclude that her direct formula for the pattern was $s = n \times 4 + 1$. When asked to explain her formula, she referred back to the pattern and used the same reasoning that Dung gave in the preceding section above.

Formula Appearance Match After a Numerical Construction of a Direct Formula. Unlike formula projection, formula appearance match is a numerical-derived justification scheme in which the students simply fit the formula they just established onto a generated table of values that they have drawn from the figural cues. Referring back to the case of Dung above, when asked to find the formula for task 1 in another way, he made a table of values for the first four pictures, found a common difference of 4, and wrote the formula from that, adjusting the constant term to fit the values for numbers of squares and checking it for the first four values. Thus, Dung's first inclination in year 1 was a figural strategy and explanation but when prompted to find the formula in another way, Dung used formula appearance matching. Note that by year 2 of the study, most of the students, including Dung, had switched to a numerical strategy for finding a generalization so formula appearance match became the more dominant justification scheme used, and, in fact, they were less consistent in ability to use formula projection (see more about Dung below).

We note that not all of the reasoning was correct. Emma used an incorrect strategy in dealing with the far generalization task of the Fig. 1 pattern in her Year 1 pre-interview, noting first that "you were adding 4 each time" and then pointing to the configuration of 5 squares inside picture 2 (see Figure 2 below). She then found picture 10 in the following manner reflective of the well-documented incorrect method of direct proportionality. We note that this incorrect method was not evident in the succeeding interviews.

Emma: Five in the beginning [points to Picture 1] and then I got 25 [for Picture 6], so I multiplied that by 5 to get 6 [i.e., Picture 6] so from 1 to 10, instead of 5 it will be a 10, so 50. I think it's 50 tiles.

JRB: Are you pretty sure about the 50? Is there any way to check it?

Emma: I could make it but that would take too long.

JRB: Is there any other way you could count them that would help you get to 10?

Emma: No, I don't see any other way.

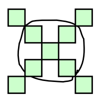
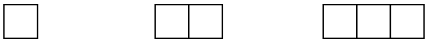


Fig. 2 Depiction of Emma's Gestures on Picture 2

In the Year 2 pre-interview, Dung, in 7th grade, used faulty reasoning to justify a correct direct formula for the Fig. 3 pattern. Dung initially set up a table of values consisting of (1, 4), (2, 7), and (3, 10) and noticed that "the pattern is plus 3 [referring to the dependent terms]." He then concluded by saying, "the formula, it's pattern number x 3 plus 1 equals matchsticks," with the coefficient referring to the common difference and the y-intercept as an adjustment value that he saw as necessary in order to match the dependent terms. When he was then asked to justify his formula, he provided the following explanation in which he projected his formula onto the figures in an inconsistent manner (see Fig. 4):

> For 1 [square], you times it by 3, it's 1, 2, 3 [*referring to three sides of the square*] plus 1 [*referring to the left vertical side of the square*]. For pattern 2, you count the outside sticks and you plus 1 in the middle. For pattern 3, there's one set of 3 [*referring to the last three sticks of the third adjacent square*], two sets of 3 [*referring to the next two adjacent squares*] plus 1 [*referring to the left vertical side of the first square*].

Square Toothpicks Pattern. Consider the sequence of toothpick squares below.



1 2 3

A. How many toothpicks will pattern 5 have? Draw and explain.

B. How many toothpicks will pattern 15 have? Explain.

C. Find a direct formula for the total number of toothpicks T in any pattern number n. Explain how you obtained your answer.

D. If you obtained your formula numerically, what might it mean if you think about it in terms of the above pattern?

E. If the pattern above is extended over several more cases, a certain pattern uses 76 toothpicks all in all. Which pattern number is this? Explain how you obtained your answer.

F. Diana's direct formula is as follows: $T = 4\text{x}n - (n - 1)$. Is her formula correct? Why or why not? If her formula is correct, how might she be thinking about it? Who has the more correct formula, Diana's formula or the formula you obtained in part C above? Explain.

Fig. 3 Square Toothpicks Task from Year 2 in Compressed Form

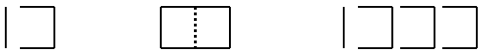


Fig. 4 Dung's Formula Projection in Relation to the Fig. 3 Pattern

Fig. 5 Tamara's Formula Projection in Relation to the Fig. 3 Pattern

Contrast that explanation with the formula projection of Tamara. In the Year 2 pre-interview, she approached the Fig. 3 pattern as follows: She initially made a table up to stage number 10, and filled it in through stage number 5 by adding 3 each time. Then she found the general expression to be Px3 +1 where P is the pattern number. She then used formula appearance match when she checked her formula for the first three pattern numbers and concluded it was correct. She then explained her formula visually in the following manner reflective of formula projection: [In stage 2 of the Fig. 3 pattern, there is] one group of three toothpicks, another group of three toothpicks plus one extra on the end (see Fig. 5).

How do students justify generalizations developed by others? In Years 1 and 2 of the study, when the students were presented with tasks such as the questions in part D of

Fig. 1, they oftentimes employed *intensional/extensional generation*, which is a variant of the formula appearance match scheme. For example, in checking the veracity of Kevin's formula $T = (n \text{ x } 2) + (n \text{ x } 2) + 1$, Anna checked the formula using stage number 4, writing as follows: "T = (4x2) + (4x2) + 1; 8 + 8 + 1; 16 + 1; 17. She also used intensional generation in establishing the correctness of Melanie's formula in part D of Fig. 1 using stage number 3. In some cases, they used a combination of intensional generation and a variant of formula projection. For example, when asked to explain Kevin's formula, Tamara initially used intensional generation (i.e., checked the formula for stage numbers 3 and 4) and then followed it up with formula projection when she reasoned as follows: "two bottom columns and two upper columns plus the one in the middle." In still other cases, some students used formula projection without resorting to intensional/extensional generation. For example, Emma explained Kevin's formula as follows: "Yes it works because in Picture 3 the 3 squares on the side represent (n x 2) and it's the same for the other. The 1 represents the square in the middle." Her explanation mirrored what Tamara saw as well but stated in a less coherent manner.

By year 3 the students had developed their understanding of grouping by way of multiplication of integers, including an algebraic facility for manipulating symbols that enabled them to either justify by simplifying the given expression (such as Kevin's formula in part D of Fig. 1) so that it resembled their own formula (which was simpler since it oftentimes took the standard linear form y = ax + b) or explaining the expression via formula projection. Fig. 6 illustrates the formula projection of Dung in relation to the T Stars pattern. Dung explained Marcia's formula in terms of three groups of the stage number plus the middle star that stayed the same from stage to stage. He also explained Pete's formula in terms of three overlapping groups of (n + 1) stars which necessitated the taking away of two stars in middle that have been initially counted three times.

DISCUSSION

In this report, we extrapolated at least four types of empirical justification schemes as they apply to linear patterning activity. The students in this study used a variety of types of schemes that exemplify empirical justification on the basis of their differing levels of competence. We note that the use of a generic case is a powerful stand-alone method of justification since it is a result of specialization and analogical reasoning. Further, the justification schemes involving formula appearance match, formula projection, and intensional/extensional generation all emerged as effects of students' facility with a numerical method for constructing a direct formula, which made the process of justification more complex for them. In other words, rather than seeing constructing and justifying a direct formula as mutually interdependent, as is the case when a generic case is employed, the three numerically-drawn schemes caused a separation between construction and justification to take place.

In the Year 1 pre-interview, none of the students had facility with use of variables so their generalizations were not algebraic and, to some extent, arithmetical (Radford, 2000). This affected their ability to justify using no more than mere extensional generation.

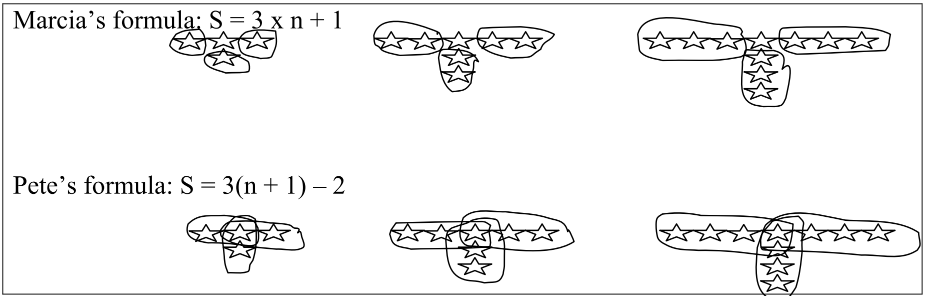


Fig. 6 Dung's Formula Projection in Relation to the T Stars Pattern

As they developed variable fluency, they were more able to develop algebraic generalizations and alternate ways of justifying them. By the end of the Year 1 study and throughout Year 2, the students used all four schemes of generic case, formula projection, formula appearance match, and intensional/extensional generation as a result of their competence for both numerical and visual modes of generalizing. In Year 3, the students' conceptual facility for grouping, which was explored in the context of multiplicative thinking (see, for e.g., Fig. 6), diminished the use of the numerically-drawn empirical schemes in favor of generic case that tend to focus on the structural features of patterns.

We note that even when the students acquired facility with multiplicative thinking in relation to patterning activity, which we have found to be effective and necessary especially in helping them develop equivalent direct formulas on the basis of how they see a particular pattern, it still was not sufficient. This happened when some of them grouped objects or parts in a pattern stage without relating it with the overall structure of the pattern in terms of what stays the same and what changes. An illustrative example is shown in Fig. 7 in which Dwayne, 8th grader in the Year 3 pre-interview, inconsistently grouped objects by threes without considering the structure of the pattern.

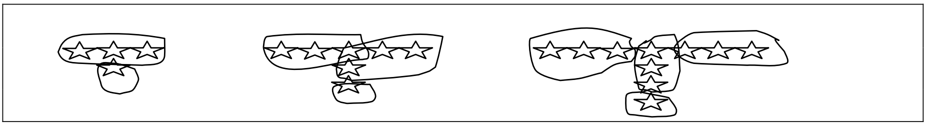


Fig. 7 Dwayne's Formula Projection of S = n x 3 + 1 in Relation to the T Stars Pattern

Finally, it is interesting to note that none of the students exhibited justification schemes that were external-based or analytic. Although at times they asked during clinical interviews for verification of their reasoning from the interviewer (the first author), when that was not forthcoming they relied on their own understanding of the situation. The fact that students did not depend on an external authority to provide a justification is considered a positive outcome from the teaching experiments in which they participated.

CONCLUSION

The middle school students we have studied over the course of three years exhibited cognitive changes in their ability to justify their generalizations in patterning activity. What we are certain at this stage is that the changes in justification schemes are consequences of more learning. One issue that is worth pursuing further involves identifying empirically-grounded and pedagogically-usable cognitive mechanisms that facilitate change in justification schemes that lead to valid and sound mathematical explanations.

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* This work was supported by Grant #DRL 044845 from the National Science Foundation to the second author. The opinions expressed are not necessarily those of NSF and, thus, no endorsement should be inferred.

BELIEFS AND ACTIONS IN UNIVERSITY MATHEMATICS TEACHING

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This paper describes and analyses a lecture in which the Fundamental Theorem of calculus is introduced to a class of undergraduate students at an English university. The four dimensions of the Knowledge Quartet (Rowland, Huckstep and Thwaites, 2004) were used as a framework for analysing the knowledge base that can be seen to 'play out' in the lecturer's exposition. In this paper, the focus is on the beliefs about mathematics and mathematics instruction, and the ways that these are realised in her teaching style and promoted in several 'asides' to the class.

The beliefs of mathematics teachers and students, and the significance of these beliefs in the processes and outcomes of mathematics education, have been a fruitful domain for research (Leder, Pehkonen and Törner, 2002). This has been reflected in many significant contributions to the work of PME, in which methodologies over the last decade have seen a slight shift from Likert-type approaches to observational studies (Leder and Forgasz, 2006). Where teachers' beliefs are concerned, studies have focused almost exclusively on elementary, middle and high school education. Teachers' beliefs about mathematics and its teaching are known to influence their instructional practices (e.g. Thompson, 1992). This paper takes up the classroom-observational trend towards the study of teachers' beliefs, but extends the context to university mathematics teaching. It will be seen that the mathematics lecture which comes under the spotlight here is not presented as 'typical' in that the mode of teaching could be described as 'inquiry-oriented' (Keene, 2008), so that opportunities were created for undergraduate students "to participate in the discourse of the class and routinely explain and justify their reasoning" (*ibid.*, p. 247).

The observational and analytical framework deployed in this study is the known as the 'Knowledge Quartet' (Rowland, Huckstep and Thwaites, 2004). Whereas the grounded theory research from which the Knowledge Quartet emerged was located in compulsory school-age mathematics teaching, this paper began as an investigation into the application of the Knowledge Quartet to a case of undergraduate mathematics teaching. As indicated earlier, the focus of this particular paper is on the beliefs that the lecturer brought to her work, based on observation of a lecture that she gave to second- and third-year students.

METHOD AND METHODOLOGICAL FRAMING

The account of the Knowledge Quartet here must necessarily be brief. The knowledge and beliefs evidenced in mathematics teaching are conceived in four dimensions, or categories, named foundation, transformation, connection and contingency. The application of subject knowledge in the classroom always rests on *foundation* knowledge. This first category consists of knowledge and understanding of mathematics *per se* and of

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mathematics-specific pedagogy. Most significant for this paper, it also includes beliefs concerning the nature of mathematics, the purposes of mathematics education, and the conditions under which students will best learn mathematics. The second category, *transformation*, concerns the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations. The third category, *connection*, includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks. The final category, *contingency*, is the ability to 'think on one's feet', in response to unanticipated and unplanned events. For further details, see Rowland, Huckstep and Thwaites (2005).

One class session towards the end of a course on Analysis taught by the lecturer, Angela, was videotaped by one of her colleagues. With Angela's consent, the videotape was sent to the author for analysis against the Knowledge Quartet framework. Reversing the process by which it was developed, the Knowledge Quartet became the theoretical lens through which the data – the videotape – was analysed and interpreted. In keeping with previous practice (e.g. Rowland et al, 2004) this interpretation is preceded by a 'descriptive synopsis' of the lecture. In previous applications of this kind, participant teachers have been invited to revisit their lesson in a stimulated-recall viewing of the recording, and thereby to comment on, and to inform, the KQ-analysis. A variant of this approach was used in this instance: Angela was invited to comment on a draft of the analysis, some details of which were amended in the light of her feedback. Some of her insights are explicitly recorded in endnotes.

ANGELA'S LECTURE

The lecture takes place in the mathematics department at an English university. It is the penultimate session in a course on Real Analysis, covering continuity, derivatives and Riemann integration, with conventional rigour. As a university mathematics lecturer, Angela is untypical in several respects. She is young (in her thirties) and female. While her Bachelor and Masters degrees are in mathematics, her doctoral thesis was in mathematics education, in the field of advanced mathematical thinking.

The *descriptive synopsis* of the lecture is as follows. The main content-focus of this session is the Fundamental Theorem of Calculus. The class is a mixture of second and third year single-honours or joint-honours mathematics students. There are about 90 students, male and female, in their early twenties, in a raked lecture theatre. There are two screens, each showing an image projected from an overhead projector (OHP). These images include pre-prepared, typed pages from 'gappy' notes supplied to the students. Angela's intention is that the gaps – such as proofs of theorems – will be the focus of activity and writing in the lectures. The lecture is in two main phases.

Phase 1: investigating conjectures about moduli and integrals.

On one of the two OHPs, Angela has displayed the following[1].

> Which of the following would you conjecture are true for <u>every</u> function f that is integrable on an interval $[a, b]$

1. $\left|\int_a^b f(x)dx\right| = \int_a^b |f(x)|dx$ 2. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$ 3. $\left|\int_a^b f(x)dx\right| \geq \int_a^b |f(x)|dx$

4. $\left|\int_a^b f(x)dx\right| = \left|\int_a^b |f(x)|dx\right|$ 5. $\left|\int_a^b f(x)dx\right| \leq \left|\int_a^b |f(x)|dx\right|$

After an introductory sentence, the other OHT continues:

> To develop some intuition for this, make some sketches of these functions on the given intervals, shading them appropriately to show the area represented by the integral.
>
> 1. $f(x)=x^2$ on [0, 2] 2. $f(x)=x^2$ on [-1, 4] 3. $f(x)=x^2-1$ on [-1, 4]
>
> 4. $f(x)=\cos x$ on $[0, 2\pi]$ 5. $f(x)=x + 10$ on [0, 10]

Angela tells the class to spend some time thinking about the conjectures, and indicates that the graphs of the functions are intended to help them. There is a quiet buzz of student talk, though the video shows that this comes from a few pairs: most students are on task individually. Eventually Angela sketches the graphs of functions 3, 4 on an OHT, and refers to the graph of function 4 to show that the first statement is false. Throughout this exposition she invites the class to contribute: they are reluctant and need some goading to commit themselves to calling out answers. Angela goes on to explain why the second statement is true, and the third false. She then offers them some more time to think about statements 4 and 5, because they "involve a bit more thinking … so I'll give you a bit more time". After a while she explains that the point of the two statements is clearer when *a* is *greater* than *b*. She asks "Did you work out what the issue is there?", but gets no response. She goes to the OHP to explain, but hesitates and says "I think I might have given you the wrong one". At this point Angela explains to the class why she had asked them to think about the five statements.

Phase 2: making sense of the Fundamental Theorem of Calculus (FTC)

Angela displays the following on the OHT.

> 3.4.1 The Fundamental Theorem of Calculus (FTC)
>
> Suppose *f* is Riemann integrable on [*a, b*] and let $F(x) = \int_a^x f(x)dx$. Then
>
> 1. *F* is continuous on [*a, b*]
>
> 2. If *f* is continuous on [*a, b*] then *F* is differentiable on [*a, b*] and $F'(x) = f(x)$
>
> 3.4.2 Understanding the FTC
>
> First, think. You have used the FTC a <u>lot</u>. What for?

At this point a student enquires about the unresolved statements from the earlier part of the lecture. Angela explains at some length why she sometimes leaves things for the students to resolve themselves. Returning to the OHT slide, she says that they have used the FTC a lot in treating differentiation and integration as inverse operations. She asks the students whether they understand why the theorem allows this. There is no audible response, although a number of students nod or shake their heads, or look puzzled[2]. Angela says "We're going to take it apart and see why it works".

First she sketches the graph of a 'general' function in the first quadrant, shades the area under the graph between a and x, and labels it $F(x)$. She then looks at a function with a discontinuity, one that the students had met in a recent tutorial:

$f(x) = 1$ on $0 \le x \le 1$; $\quad f(x) = 2$ on $1 < x \le 2$

Angela sketches $F(x)$ for this function, attempting to involve the students as before. Referring back to both parts of the statement of the FTC, F is indeed continuous on [0, 2]. But this f is not itself continuous, and F (in turn) is not differentiable, since it has a "corner" at (1,1). She then goes on to say that she will draw a picture to help them "get some intuition" about why the theorem is true. The picture shows $F(x+h)$-$F(x)$ as the area of a strip of width h under the graph of f, leading to the conclusion

$$F(x) = \lim_{h\to 0} \frac{\text{area of the strip}}{h}$$

$$\approx \frac{hf(x)}{h} \qquad \text{and so } F(x) = f(x)$$

As the students prepare to leave, Angela tells them that the course will conclude with the proof of FTC in the final session next Monday.

ANGELA'S BELIEFS

As indicated in the introduction, several interesting and significant findings in relation to Angela's beliefs, as inferred from her practice, rapidly emerged in the analysis of Angela's lecture. I made the choice, therefore, to make these beliefs the almost exclusive focus of the analysis presented in this particular paper, which is therefore restricted to one aspect of the *Foundation* dimension of the Knowledge Quartet. In fact, I had set out expecting that my attention would home in on pedagogical aspects of Angela's teaching performance. It is important to emphasise, therefore, that my analysis was motivated and framed by Knowledge Quartet, but not with the outcome that I had expected.

The Foundation dimension of the KQ includes the beliefs that mathematics teachers bring to the work of teaching. These beliefs, as conceptualised in the KQ, relate to three different domains:

- beliefs about the nature of mathematics itself;
- beliefs about the purposes of teaching and learning mathematics;
- beliefs about the ways that mathematics is most effectively taught and learned.

Evidence of Angela's beliefs about the first of these is more implicit than explicit, but a formalist view (Ernest, 1998) fits well with the rigour conventionally expected to be present in Analysis courses at university. While the ontology of the core concepts lies in experiences and intuitions about things like functions and graphs, these same concepts are invested in, or even equated with, formal definitions. These same definitions must then underpin the proofs of any results in the topic. Angela's commentary on the "nice picture" that she draws at the end of her lecture might well satisfy many, if not most, of her audience, as a proof of the FTC. Angela herself knows that, by the norms by which she is expected to operate (a) too much of the argument rests on the picture she has drawn, and (b) she can only rigorously accomplish the final step of the proof by invoking the Mean Value Theorem for integrals.

At the same time, Angela's frequent injunctions to "get some intuition" about a result, often by reference to examples, seem to place her practice somewhere within the orbit of Jacques Hadamard's adage that "The purpose of rigour is to legitimate the conquests of the intuition" (Hadamard, cited in Burn, 1982, p. 1). It will become clear later that she sees recourse to examples as a key learning heuristic.

By contrast, Angela makes her beliefs about the second and third domains above very explicit in various 'asides' to the class. It almost seems that she views these meta-lessons, these more general truths about what the students are there for, and how they can help themselves to learn, as more important than the details of the particular mathematics under immediate consideration. Midway through the lecture, she responds to the student enquiry about the unresolved statements from the first phase of the lecture as follows:

> Yes, I'm going to leave you to think about that 'til next week. We're going to cover it, I just want to leave it, you know [*pause*] Um, maybe I'll say something about that now. You remember me saying 'Yeah, I'm going to let you do that', and not answering questions? One or two people remarked about it on the feedback forms, though most don't seem to mind. I'll say what I said at the beginning. One of the things that you should be gaining from a course like this is gaining more confidence in making judgements for yourself, and not needing so much someone to hold your hand and say 'Yes, it's alright'. And that this is [inaudible] in life. When you go out to the real world when you leave here, and you have a job, you will first of all have a manager who will tell you what to do, and will sometimes check your work, but then they will check your work less, and then you will *be* the manager and you will be checking other people's work, and no-one will be checking what you do. And as you get older you've got to get used to this idea that you are responsible for what happened for a lot of things that you do. And I want to help you develop that. Sometimes I misjudge it, maybe, but maybe I'm getting it mostly right.

It is interesting to hear Angela offering this advice to her students at this late stage in the course, although her commitment to her message suggests that she might have said similar things to them earlier. She seems to be drawing out lessons about autonomy and independence. There is a lot in the course that the students could figure out for themselves, if only they had the confidence to do so. What is more, Angela seems to be saying, this is true of life in the "real world" outside the university, in particular in the world of work. Her point about confidence and autonomy (making judgements for yourself) as mature alternatives to dependence on authority seems to be a general one about the purposes of education. However, it has been argued that the case for developing autonomy in and through mathematics is especially compelling because mathematics is not by nature an authoritarian subject (Huckstep, 2001). It is a *democratic* discipline, in that "there is no evidence available to the teacher that is not available to the student" (Sawyer, 1964, p. 82). There is an irony in this: because mathematics is also perceived to be a difficult subject to teach and to learn (DES, 1982), students can be tempted into simply accepting the testimony of the teacher on trust, into letting the teacher do the thinking while they merely imitate the behaviours that evidence that thinking. Angela is at pains to caution against such a regime, urging the students to take more responsibility for their learning.

This last point leads naturally to evidence in the lecture of Angela's beliefs about ways that mathematics is most effectively taught and learned. It is clear from the outset that this is no ordinary university mathematics lecture. University mathematics teaching varies, of course, but the paradigm form must be that in which the lecturer writes a complete mathematical text on a display board of some kind, and utters a spoken version of what s/he writes whilst the students produce a written copy of the text in their own notebooks[3]. There are alternatives, however. In a study of mathematics teaching at two English universities, Burn and Wood (1995) found that the use of lecture time most appreciated by students was "when the printed notes were interspersed with gaps which had to be filled with student work, and where the lecture time included exercise time as an integral part" (p. 29). This, of course, is a close description of Angela's use of her 'gappy' notes. It is not possible to ascertain from the video the extent to which the gaps were "filled with student work" as such, as opposed to a real-time presentation of material by Angela herself. What is clear is Angela's intention that the students are active in exploring and testing the statements of theorems, and other maybe-true propositions like the five listed in Phase 1 of the lecture. The use of examples as a 'methodology of testing' such statements is central to her approach. Another extended meta-statement of Angela's towards the end of Phase 1 clarifies why she is so committed to their being involved in this way.

> The reason we're doing this is we're going to use some inequalities a bit like these in the proof of Taylor's theorem[4]. We're working towards that. Also – the other is a meta-mathematical issue […] faced with things like this, what mathematicians do is have a go, draw some examples … think "I don't know, I'll just get myself a function *f* and see what happens" […] Drawing some examples and getting some insight about things like this can help you feel more confident that these things are what we said.[…] Incidentally, I would probably say that without exception every single one of your lecturers thinks that you are doing that anyway. … they probably don't spend a lot of time generating examples for you because they assume that being mathematically aware people you're doing that anyway. So if you're not, this would be a good time to start.

Angela's meta-lesson to these students about the place of examples in testing conjectures – and in generating them for that matter – is well-founded. It is what Mason, Burton and Stacey (1982) called 'specialising', and it is at the heart of fallibilistic epistemology (Lakatos, 1976). It is indeed how "mathematically aware people" generate and investigate conjectures (Alcock and Inglis, 2008). Fundamentally, Angela is teaching her students one of the key heuristics for learning mathematics and for 'being mathematical' (Jaworski, 1995). She confidently asserts that "every single one of your lecturers thinks that you are doing that", and it would be interesting to know the grounds for her confidence[5]. In any case, there is a potential danger that she might actually be a lone voice in holding and articulating these views, these beliefs about how to learn mathematics, and why it is worth all the effort. If other lecturers act/teach differently, and do not articulate similar beliefs to students with the same conviction – or at all – which version of 'being mathematical' will the students subscribe to?

CONCLUSION

Angela's approach to the subject-matter of this lecture and her meta-mathematical remarks to her students are indicative of a set of beliefs about the nature of mathematics itself, about the purpose of the activity - why she and the students are there at all, and about how she can best assist them to learn and to become effective learners of mathematics. Her teaching style is not unique to her, but it differs from the paradigm exposition and note-copying pattern of the university mathematics lecture. In particular, Angela's approach to teaching the course involves the students in: working on some 'exercises' within the class session; 'specialising' (Mason, Burton and Stacey, 1982) in order to test conjectures, and to make sense of statements of theorems; interacting with the lecturer and (ideally) with each other to share their findings and to contribute to the construction of 'real time' arguments in the lecture; being expected to resolve 'loose ends' outside the lecture.

Angela is explicit about these expectations and her reasons for them. That is not to assume that her beliefs are wholeheartedly embraced, or even properly comprehended by all the students, or even that they are all enthusiastic about this way of working, although Burn and Wood (1995) lead us to expect that they would be. Shared assumptions about necessary and expected behaviours of this kind are the sociomathematical norms (Yackel and Cobb, 1996) which frame, permit and, ultimately, determine the nature of mathematical activity in a class setting. They come about not by imposition, but by building a common culture over time. Judging by Angela's frequent efforts to engage the class in the behaviours listed above, it is reasonable to suppose that her mode of interaction with them is indicative of the way that she has worked with the class throughout the course. Different ways of teaching and their different epistemological assumptions, and the nature of the students' role in lectures, might be a valuable issue for discussion within the team of lecturers, because establishing sociomathematical norms of the kind described is a task for the whole team.

Notes

[1] To save space, the mathematical expressions have been displayed differently here. On the actual OHT display, each equality/inequality/function was on a different line.

[2] The student's non-verbal responses were gleaned from Angela's comments on my account of the lecture. Since the camera is focused on Angela for the whole lecture, apart from one brief episode, I can only gauge the student reaction from what can be heard.

[3] http://www.nd.edu/~jcaine1/pdf/How_to_listen_to_a_Math_Lecture_Korner.pdf (retrieved 07 January 2009)

[4] This was a 'slip of the tongue': the FTC was intended here, not Taylor's theorem (Angela, personal communication).

[5] Angela reflected that "*every single one* … now seems like a rhetorical flourish" (personal communication).

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REDUCING OR OPENING UP FOR COMPLEXITY? THE PARADOX OF HOW TO FACILITATE STUDENT LEARNING

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In what ways could teachers' attempts to make mathematics understandable and easy to learn affect how the topic to be learned is handled in the classroom? From the point of departure of two Learning studies - a collaborative work among teachers aiming at exploring and developing what is made possible to learn in the classroom - I describe how the teachers' good will to facilitate learning by reducing complexity had an un-expected effect. It is also described how the teachers successively got to know that aspects of the object of learning, either taken for granted or deliberately avoided by the teachers, were necessary for learning. Two studies – teaching and learning to convert sentences into algebraic expressions and addition and subtraction with negative numbers respectively - are reported.

It has been reported about students' difficulties to learn how to add and subtract negative numbers (e.g. Gallardo, 1995; Vlassis, 2004) just as students' initial encounter with algebra (e.g. Booth, 1984; Kieran, 2004; Knuth *et al*., 2005). The teachers taking part in the studies reported here, hade the same experience from their practice; these are topics difficult to learn and to teach. Therefore, they paid much attention to the difficulties the mathematical tasks would have for the students when planning the teaching. Hence, they were concerned to facilitate students' learning. The aim of this paper is to demonstrate how their willingness to make mathematics understandable and their attempts to reduce difficulties for the learners affected how the topic to be learned was handled in the classroom.

LEARNING STUDY: A COLLECTIVE ENQUIRY INTO TEACHING AND LEARNING

This paper takes it point of departure in the work of two particular study groups of mathematics teachers in a Swedish compulsory school. They worked collectively in a cyclic process with an approach called Learning study (Runesson, 2008) to enhance their practice. However, the aim of a Learning study is not to improve the lessons in a general way, (c.f. lesson study, Yoshida, 1999) but to improve students' learning of a specific object of learning. Since the variation in pupils' ways of seeing plays a central role in Learning study, the students are tested before and after the lesson to give information about what is problematic for learning a particular concept or skill. The lessons are planned on insights about what is critical for students' learning. After the lessons, the students are tested again. Information from the post-test gives immediate feed-back to the teachers and enables them to analyse and reflect on the

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 25-32. Thessaloniki, Greece: PME.

video recorded lesson from the point of view of students' learning outcomes. They inquire and explore the lesson from the point of view of possibilities for learning; whether the necessary conditions for learning were met in the lesson or not. In a Learning study the capability we want the students to develop, is the focus. So, how the object of learning is handled in the lesson becomes the object of study for the teachers. Having identified critical aspects in the lesson, they revise the lesson plan and in the second cycle a new teacher teaches the lesson to her students (i.e. a new class), a post-test is given and the recorded lesson is observed and – if necessary – the lesson plan is revised again.

The theory of Variation (Marton *et al.*, 2004) forms the basis of the theoretical framework of Learning Study and is used as a guiding principle when investigating students' learning and their learning possibilities in the classroom. According to this theory, in order to learn something, one must be able to discern its critical aspects. And, in order to discern an aspect, one must experience variation in the aspect. The failure of learning can be understood in terms of an inability to discern all the aspects that are necessary to be discerned for a particular way of understanding.

This paper is based on the analysis of data from two Learning studies. One of them at grade 5 (11 years) was in the area of 'pre-algebra' developed by three teachers, the other in 'negative numbers' at grade seven and eight by four teachers (all experienced teachers). Although, mostly in Learning study a researcher from the University serves as a consultant, the teachers always have the ownership of the study. That is, the planning and revision of the lessons are based mainly on the teachers' decisions. The entire cycle took about seven meetings, which were held in the school after school hours over a period of four months. The lessons were taught by the teachers to their own classes in cycles. Permissions were obtained from the parents of students for the lessons to be video recorded.

After the entire cycle the video recorded lessons, the test results and the transcribed audio recordings from the meetings with the teachers were analysed from a variation theory perspective by the author of this paper. The purpose was to get deeper understanding of the process; how the teachers planned the lesson, reflected on the recorded lessons and students' learning outcomes and made decisions about the teaching. In the analysis I carefully followed the cycles with a particular focus; how their ambition to promote and facilitate students' learning affected what was made possible for the students to learn in the lessons.

THE SUCCESSIVELY GROWING INSIGHTS INTO NECESSARY CONDITIONS FOR LEARNING

In the following the two Learning studies will be described in detail. It is reported how the teachers through the Learning study cycles developed their understanding, not just into the student learning difficulties, but into how their ways of handling the topic taught promoted learning.

Learning study 1: Learning to convert sentences into algebraic expressions

The object of learning that is, the capability the teachers wanted the students to develop was to 'converting sentences into algebraic expressions' and vice versa. From previous experience, they knew that an example like: x+5=y, often confuses the students since no calculation is required. When planning the first lesson in the cycle, the teachers were very much concerned about how to facilitate pupils' learning. They assumed that the first encounter with letters would be problematic, particularly the letters 'X' and 'Y'. They said: "X and Y will be frightening to the students". They came up with the idea that, choosing a letter close to the variable would facilitate the understanding. For instance, the example "John is three years older than David", could be represented "John's age - David's age = 3 years" and next, shorten into "J-D=3". Furthermore, it was suggested to present variations of, for example, J-D=3; this could also be written as D+3=J; the relation is still the same.

The lesson was taught according to the plan; *only abbreviations for the variables were used.* (e.g. 'T' for the price of a toffee, 'C' for the price of a chocolate bar and so on) and the same sentence was represented differently as concerns the order of the variables. For instance the example "A chocolate bar costs 5 Crowns more than a toffee":

$$
\begin{aligned}
&C - T = 5\\
&C - 5 = T\\
&T + 5 = C\\
&C = T + 5\\
&T = C - 5\\
&5 + T = C
\end{aligned}
$$

The equations are all permutations of T+5=C, that is, the position of the symbols, and thus the operation, changes.

After having carefully analysed the videorecorded lesson and the results on the post-test, the teachers realised that things could have been done differently to increase the scope for learning. One of the tasks on the post-test was to write *different* expressions for the example 'an apple and a cucumber cost 15 crowns'. Those students, who solved the task, all used the initial letter of the variable ('A' and 'C') but changed the positions similar to the pattern introduced in the lesson. To improve students' learning, the teachers decided that it was necessary to add two more aspects to the original lesson plan that they had taken for granted when planning the first lesson; to introduce *incorrect* algebraic expressions for the example given above (e.g. C+5=T) and to bring up *the idea of variable* (i.e., something that varies) by substituting and varying values for the letters. In this way, they anticipated that the learners would learn that the relation between the variables is the same independent of the values taken. Therefore, in the second lesson, correct *and* incorrect formulas for different examples (e.g. for "My cousin John is 10 years younger than me, Henry" and permutations of J+10=H was given (e.g., H+10=J). It was also demonstrated that the letters could be substituted by numerical values.

However, in lesson 2 something un-planned happened; the students introduced letters other than the initials for the variable, for instance “X’ and Y" “A, B and C" or "square and circle". Although the teacher did not disapprove of this, he gave prominence to the ‘initial letter’ and pointed out that for instance, H and J (comparing Henry’s and John’s age) were the symbols he wanted them to use. In my interpretation this could be traced to the early planning sessions in the learning study; to facilitate learning and not confuse the students with ‘X’ and ‘Y’ which they assumed would be "frightening and strange” to the students. The teachers commented on this in the post-lesson session and now realised that that the students’ suggestion of using other symbols than the initials could create a potential for learning. This insight prompted them to revise the plan for the third lesson; to deliberately introduce a variation of symbols. Thus, in addition to the aspects opened up in the previous lessons yet another one was added.

In the third lesson in the cycle, besides varying the positions of the variables (and thus the operation), they also varied the symbols used (e.g. ‘X’ and ‘Y’ and ‘♥’ and ‘♠'). An assignment - to match an example and different algebraic expressions - was worked with (e.g. “Martha has five more marbles than Colin”). Some algebraic expressions corresponded to the example, others did not. When this example was discussed, something happened that was probably not foreseen by the teacher. When the teacher asked: “Is M=C-8 a correct way to represent the example?” most of the students said it was incorrect. However, one student remarked that the equation could represent the example provided that the number of marbles Martha had was represented by C, and Colin’s marbles by M. This suggestion caused some confusion, but eventually there was an agreement about that it is not only possible to vary the symbols (abbreviations/’X’ and ‘Y’/’♥’and ‘♠'), it is possible to vary the symbols chosen by swapping them also. This implied that more and critical aspects of the object of learning were revealed in lesson 3 just as lesson 2 was "richer" in that respect compared to lesson 1. The idea that the representations chosen are arbitrary was brought out in a more clear and distinct way in lesson 3.

Learning study 2: Learning to add and subtract negative numbers

This Learning study group knew that many students could solve tasks like 5-(3)= , 5-(-3)= and -3-(-5)= with problem solving skills or by using the rule ’two minus signs make plus’; a ‘rule’ often used as a method of a procedure. The teachers wanted to teach for understanding, not just to get the students to come to the correct answer. Initially the teachers anticipated that the different meanings of the operational sign for subtraction and the ‘minus’ sign for a negative number probably was confusing for the students. They planned to systematically vary the meanings by using examples like 8-3 = 5 and 3- 8 = -5 to draw the learners awareness to the different meanings of the sign for a positive or a negative number, but still keep the sign for the operation the same. Furthermore, the first lesson was planned to focus on opposite numbers; to see that adding the opposite numbers, e.g. +5 and -5 equals zero (i.e. to teach negative numbers as ‘patterns’). To overcome learning difficulties they were very much into 'avoiding things'; to make the lesson as simple as possible without any examples that would 'mix up things' for the students. For instance, although they were aware of that subtraction could be seen as a difference between two numbers also, they thought that this would make it more complicated. "Subtraction means 'taking away'. It is stupid to call it a difference. Let us just say 'subtraction'. They know what that is", one of the teachers suggested.

In the first lesson in the cycle, just as was planned; negative numbers in relation to the positive numbers was focused by asking. "What happens when we add the opposite number to five with five, 5+ (-5)?" and "What happens if we go below zero, 5+ (-1)?" in order to the learners themselves discover the rule 'adding (subtracting) a negative number is the same as subtracting (adding) its opposite'. In addition to this, the different meaning of the 'minus-sign', for instance in 5 – (-5)= was elicited by contrasting the same number and varying the sign.

The results on the post-test demonstrated that the progress concerning addition was better than subtraction. On subtraction tasks with two negative numbers, after the lesson the students even showed a lower result on the post-test (29 % correct answers) than on the pre-test (35% correct answers). They realised that there were aspects of subtracting negative numbers that has been taken for granted (when planning and in the lesson) that probably should not have been taken for granted. They decided another approach for the second lesson; to contrast addition to subtraction. They noticed from the video recording that one student brought out something which they had decided to avoid; that "minus could be seen as a difference, not just as take away". This made them realize how their aspiration to help the learners by just using 'take away' might have had the opposite effect. Therefore they decided to teach "5– (-5) is the 'distance' between them [the numbers]". But one problem still remained; when using difference as a metaphor, the students would get both positive and negative differences due to which number was placed first in the expression. Would that be too difficult? In order to make things easier for the students the team decided to just try to avoid examples with a negative difference in lesson 2. The lesson was taught in accordance with the plan, with one exception; one aspect which was present in lesson 1 was missing in lesson 2; the minus sign as both a sign for operation and the number value (c.f. lesson 1 above). This absence was not planned. Probably the teacher just forgot about it.

The post-test showed that the students performed better on tasks concerning subtraction, especially to subtract a negative number from a positive number. Certainly the results increased from 41 % of correct answers on the pre-test to 76 % on the post-test, but results from subtraction tasks with two negative numbers (-5)-(-2) =, did not increase so much (from 41 % to 65 %). So, having planned and conducted two lessons in the cycle without getting the expected outcome, they realised that they had not yet found what was critical for learning to subtract negative numbers. "Could the difference between 5-4= and 4-5= be a critical aspect (i.e., the law of commutativity is not valid for subtraction)?" they asked. A long discussion about whether they should try to avoid or focus on expressions with a negative difference followed. Thus, they reconsidered their view of how to help the learners; from avoiding to confronting. To facilitate learning, however, it was suggested to connect the examples to an everyday context which would be familiar to the learners.

However, due to an unexpected incident (lightening) happened, lesson 3 was interrupted. When lesson 4 was planned they thought that they had found what was critical for their students to learn subtraction with negative numbers. The last lesson in the cycle became a synthesis of the previous lessons and the conclusions drawn from the analysis from them. In this lesson the teacher tried to direct the students' attention to all the critical aspects that she knew about. During this lesson it was possible for the students to experience: *the difference between the two signs for subtraction and for a negative number*, *subtraction could be both seen as 'a take away' and as 'a difference' between numbers* and *commutativity is not valid for subtraction*.

The post-test from the lesson 4 shows the best results compared to the other classes for the learning outcomes. After this lesson there was an increase from 29% correct answers on the pre-test to 81% on the post-test on tasks concerning subtraction of negative numbers. The results on addition tasks with negative numbers are also showing a similar increase. Our conclusion is that there were better possibilities to experience aspects critical for learning provided in this lesson compared to the previous ones.

MAKING LEARNING POSSIBLE BY REDUCTION OF COMPLEXITY?

To summarize, the teachers had an ambition not to make the mathematics too complicated but yet understandable. It is easy to be wise afterwards and classify the first lessons in the cycle as being poor, and since they were not so successful, considering the teachers as unskilled. Remember however, that going so deeply into how to teach and learn a topic was not just a new experience for them, it was also a challenge. However, from the two examples given above I would ask: Did the teachers' good will to facilitate students' learning have the opposite effect thus; did the reduction of complexity prevent the intended learning?

In terms of aspects of the object of learning that were exposed to the learners, thus was made possible to discern, the space of learning was rather restricted in the initial lessons in the cycle. Although the results on the post-test give some indications that when the space of learning was expanding, in terms of more simultaneous aspects present in the lesson, students' performance also changed, it is not possible to say that this was an effect of the wider space of learning alone. What has been showed is that the teachers in both study groups initially held the idea that a reduction of complexity might help learning and make mathematics understandable. Subsequently, aspects, that probably were necessary for learning, were either taken for-granted or deliberately avoided by the teachers in order to facilitate learning.

What significance the opening of a complex of dimensions of variations has for what is made possible to learn, has been discussed previously (Runesson & Mok, 2005; Häggström, 2008). They have studied how the same topic was taught in Asian and Swedish classrooms and point to a cultural difference in how aspects of the object of learning were opened up in the lessons. The Asian teachers seemed to open up for a wider space of learning in terms of complexity compared to their Swedish counterpart, who tended to have a more sequential approach; one aspect at the time spread out over a period of lessons. Häggström also reported that more aspects seemed to be taken for granted by the teachers in the Swedish classrooms; a practice that might reflect the ideas held by the teachers in the studies reported here.

The study has also demonstrated that this initial idea was challenged and changed during the Learning study process. The results on the post test and the analysis of the video recorded lessons informed the teacher groups that sufficient possibilities for learning were not always provided. This prompted them to go more deeply into the nature of the object of learning; to consider what it implies to be able to convert sentences into algebraic expressions and add/subtract negative numbers respectively. In this way, they explored and unfolded the nature of the object of learning. I would suggest that they successively - by an intense and very close analysis of how student learning was related to the lesson - changed their ways of seeing of how to facilitate students' learning. In the process, they successively became aware of that it was necessary to bring out certain aspects simultaneously in order to make it possible for the learners to understand that which was intended. They came to realise that a reduction of complexity did not provide the necessary conditions for learning.

This emerging awareness was sometimes a result of inputs from the students, who at some occasions introduced a variation which was not planned (or deliberately avoided) by the teacher. It seems as if the absence of certain aspects prompted the students to contribute with and elicit those aspects (which they probably had discerned) themselves. For instance, suggesting other letters than abbreviations for the variable, swapping the variables (e.g. the number of marbles Martha had *could* be represented by C as well) and suggesting interpreting 'minus' as 'difference'. Through the Learning study cycles the awareness of aspects critical for learning grew among the teachers. The teacher who implemented the lesson plan in the last lesson had a more developed understanding of what must be brought out in the lesson to provide better learning. Hence, she was more prepared to bring out the identified critical aspects and managed to do so also.

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RELATIONSHIPS AND CONTROL WITHIN SEMIOTIC BUNDLES

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We use a semiotic lens to compare and connect two different case studies, where the subjects have to solve a calculus problem. In both cases they are given a function only through its graph. In the first case they have to find its derivative; in the second case, its anti-derivative. We study the different ways in which subjects use semiotic resources (inscriptions, gestures, speech), focusing in particular on how they link and control them. We analyze how the nature and formulation of the tasks and the subjects' backgrounds may have influenced their use of the semiotic resources. We use the semiotic bundle model, and introduce a new construct, the virtual space of gestures. They allow us to frame cognitively the dynamic evolution of the two situations and to give reasons for the similarities and differences between them.

THEORETICAL FRAME

In recent years many research studies on mathematics teaching-learning processes have pointed out the necessity of taking into account the variety of semiotic resources activated by the students and by the teacher. As semiotic resource (or sign) we consider anything that "stands to somebody for something in some respect or capacity" (Peirce, 1931/1958, vol. 2, paragraph 228), e.g. "words (orally or in written form); extra-linguistic modes of expression (gestures, glances, …); different types of inscriptions (drawings, sketches, graphs, ...); various instruments (from the pencil to the most sophisticated ICT devices), and so on" (Arzarello et al., 2009). To properly frame all such resources within a semiotic perspective, it is necessary to broaden the range of signs that are considered relevant in the teaching-learning process. Over the last decade, various researchers have done this: see for instance Arzarello (2006), Radford (2003), and Roth (2001). Within this stream, we use an enlarged notion of semiotic system, the *semiotic bundle* (Arzarello et al., 2009). A semiotic bundle is a system made of the different semiotic resources and of their mutual relationships that are produced by one or more interacting subjects. It allows us to describe the semiotic processes of subjects in a holistic way as a dynamic production and transformation of various signs and their relationships. The semiotic bundle encompasses the classical semiotic registers (Duval, 2006) as particular cases. In fact, we consider semiotic bundles to model the evolution of learning processes, and the role of a richer variety of semiotic resources (compared to standard semiotic registers), especially gestures.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 33-40. Thessaloniki, Greece: PME.

In this paper we shall deepen the analysis of gestures within the semiotic bundle model by introducing a new construct: the *virtual space of gestures*.

A virtual space of gestures is a space that is created by subjects through a set of gestures and the meanings associated with them. It is related to the gesture space, namely "the limited space in the frontal plane of the body" (McNeill, 1992) in which gestures are usually performed, which in European languages usually extends vertically from the waist to the eyes, and horizontally between the shoulders. For example, a virtual space can be created by a person pointing with her hand to three sequential places and saying "one", "two", "three". The meanings associated with these three deictic gestures create a structure for the gesture space, namely that of the number line: the sequential occurrence of these gestures suggests that "3" occupies a place further along than "2", which is similarly further along with "1". Hence, from a cognitive point of view, the virtual space endows the gesture space with a palpable *structure*. The structure is given by the gestures that have created it and subjects can act within the structure to create further gestures. For example, another subject could easily imagine where "4", "5" might be gestured within the structure just described. As such, the structure can be shared and acted on by different subjects: they can gesture to make (real) explorations for checking conjectures or for concretely simulating some mental experiment (for some examples, see Arzarello, 2008). The idea that the gesture space can be used to provide a structure that has specific meaning in the interaction is present in the literature on gestures: for example, Kita speaks of "representational gestures" as "actions in the virtual environment" (Kita, 2000, p. 165), and McNeill (2003) discusses how deictic (pointing) gestures in conversation structure the shared gesture space in terms of a "deictic field" endowed with specific meanings. For what concerns mathematics learning, a similar construct has been described by Arzarello (2008). In our analysis, the power of a virtual space is particularly evident when the signs produced within its structure are intertwined with other semiotic resources (in particular, speech and inscriptions).

TWO CASE STUDIES

The derivative case

The following task was given to grade 10 students in an Italian school:

> Consider the following graph of a function $y = f(x)$ [Fig. 1]. What can you say about the graph of the function that describes how the slope of the tangent line changes with the change of x? Justify your answers.

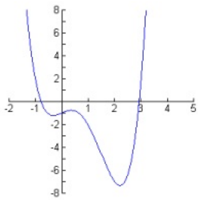

Figure 1

The task is part of a long teaching experiment carried out in an Italian secondary school, with the aim of introducing students to theoretical thinking starting from variation. The lessons prior to the one described here focused on local aspects of functions in different registers: paper and pencil, and CAS software. The "derivative at a point" was introduced as the slope of the tangent to the graph at the point, and the derivative function was introduced to describe its variation.

Students worked in small groups under the supervision of the teacher. We video-recorded the group formed by M, L, and R. In this paper, we analyse only a short portion of their activity, in which they follow the teacher's suggestion and work in the graphical setting.

1. R: We can say that here (pointing to the first minimum of the graph, and miming with the pen a little horizontal segment, Fig. 2) well here there will be a slope equal to zero. Here (pointing to the maximum, and miming an horizontal segment on it) another equal to zero, here (same gesture on the second minimum) another equal to zero.

Figure 2

2. Teacher (to the other students): Is it right what he is saying?
3. M: Yes, because…
4. L: Yes (and he traces quickly with his finger little horizontal segments on the graph)
5. R: That is we can say that in the peaks, in the vertices, we know that there the slope is zero

The students soon identify the stationary points as having slope equal to zero, and use both speech and gesture to express this property. In particular, they use gestures to depict on the graph little horizontal segments, which show in an iconic way how the tangent line is horizontal at those points. In previous lessons, the teacher had introduced the technique of drawing little segments to show the behaviour of the tangent line. This explains why such gestures have a clear meaning even if performed without comment (L in #4). The students used words to express the numerical value of the slope in the point (zero), first correlated with gestures (#1), and then organized in a more complete narrative (#5).

Figure 3

The students appear stuck to the given graph, so the teacher suggests organizing information on the slope function in a Cartesian graph. The students start drawing a diagram (Fig. 5), in which they correctly record the information about the zeroes of the slope function, as well as its sign. Then they try to describe where the slope function increases/decreases:

6. L: […] Let's start from here (pointing on the left part of the graph). Here, isn't it decreasing? That is, it is decreasing less and less (miming the graph moving his hand in the air, Fig. 3). Yes, look!
7. R: The slope…does like so, in theory… (putting the
pen in a slanted position on the sheet, Fig. 4)

Figure 4

8. L: It decreases less and less, the slope function
9. M: It decreases less...
10. L: Less
11. M, R: Yes, right […]
12. M: So, here (completing the diagram shown in Fig. 5) it decreases less and less. Until it reaches zero.

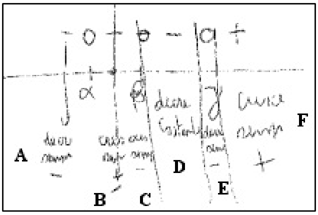

Figure 5 [A: decreasing less and less; B: increasing more and more; C: increasing less and less; D: decreasing constantly; E: decreasing less and less; F: increasing more and more]

L suggests a systematic strategy, namely starting from the left part, "going along" the given graph and imagining how the corresponding new graph looks (#6). He observes that in the left part the slope is decreasing less and less (#6, 8). However, this is not the case: it is the given graph that is decreasing less and less, not the derivative graph. As correctly reported in the diagram, in that interval the slope function is negative and reaches zero at the point α; therefore it has to be increasing. It is possible that L fails to clearly distinguish the two graphs. Other contradictions can be found in the diagram, but the students complete the task without noticing them (#7, 11). Why? We observe that the semiotic resources of the bundle are activated in an inadequate way. The students may be influenced by perceptive facts and are thinking of slope in its everyday meaning, i.e. as an intrinsic characteristic of the steepness of a path (a road has a slope of 5%, independently from the direction one goes), rather than as a number with sign, as in the context of mathematical calculus. Such a misleading interpretation is favoured both by the word "slope" and by the gesture performed by L in the air (Fig. 3). Furthermore, the gesture is quite quick and may be iconic both of the given graph and of its tangent; it does not support an effective exploration of the situation. Nor does L's gesture with the pen shortly afterwards (Fig. 4): it appears as the material instantiation of the previous gestures on the graph (# 1, 4; Fig. 2). All of these gestures refer to the given graph and are not suitably co-ordinated with another semiotic resource in order to relate in a productive way to the graph to be traced. To help students to link the information with the derivative graph, the teacher suggested working in a Cartesian plane. The students do build a diagram (Fig. 5) that is within a Cartesian plane, but quite unrelated to it. They appear not to master the techniques related to the diagram, so the introduction of the new semiotic resource probably has the effect of increasing their cognitive load, instead of helping them to generate a new graph. Thus, their lack of semiotic control hinders their problem solving.

The antiderivative case

Two New Zealand secondary mathematics teachers (Ava and Noa) were presented with a story about a hiking trip and with the gradient graph of the hiking track (Fig. 7, without the circled letters A-H). They were asked to design a method to sketch the distance-height graph of the original track: height as a function of distance, a task which from a mathematical point of view amounts to constructing the antiderivative graphically. They had just successfully carried out a warm-up task of sketching the gradient graph for a given distance-height graph, a task that amounts to constructing the derivative graphically and is therefore closely related to the 'Italian' task (Fig. 1).

Ava and Noa utilised three semiotic resources concurrently while constructing the tramping track in the air. In the excerpt below, Ava and Noa discuss the gradient of the track between 1600 and 2700 m.

1. Noa: And then the point of inflection (points to A in Fig. 7) and then starting to get not so steep (points to B in Fig. 7, gestures shallow positive gradient) up to the summit (points to C in Fig. 7, gestures flat gradient as in Fig. 6a).
2. Ava: Yes (copies Noa's gestures as in Fig. 6a).
3. Noa: And then we are starting to go downhill (points to D in Fig. 7, gestures negative gradient as in Fig. 6b), negative gradient.
4. Ava: Yip, yip. Downhill (gestures negative gradient as in Fig. 6b).
5. Noa: And it's quite gentle gradient (points to D in Fig. 7, gestures negative gradient), getting steeper gradient (points to E in Fig. 7, gestures steeper negative gradient) to the point where it is the hardest gradient (points to F in Fig. 7, gestures very steep negative gradient as in Fig. 6c) and then it starts levelling off (points to G in Fig. 7, gestures less steep negative gradient as in Fig. 6d) and getting easier again until you get to like a bottom (points to H in Fig. 7, gestures a flat gradient).

Figures 6a, 6b, 6c, 6d

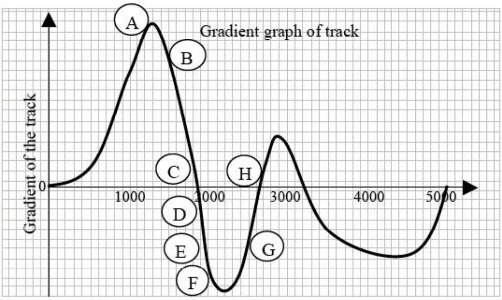

Figure 7: A-H denote the sections of the graph to which Ava and Noa pointed

Ava and Noa use the semiotic resource of speech to verbalise notions such as gradient steepness and sense in order to describe the gradient of the track indicated by the gradient graph. They use a combination of formal mathematical language – e.g., 'point of inflection' (Noa in #1) – and language specific to the context of hiking tracks – e.g., 'getting easier again' (Noa in #5). Sometimes, they use both formal mathematical language and hiking-specific language to describe the same concept – e.g., 'negative gradient' (Noa in #3), and 'downhill' (Noa in #3; Ava in #4).

While verbalising the gradient of the tramping track, Noa and Ava also use deictic gestures (McNeill, 1992) to trace with their index finger and pen (Fig. 6) along the gradient graph, pausing on occasion to point at the segments denoted A – H in Fig. 7. They simultaneously use iconic gestures (McNeill, 1992) in the air to create a *virtual space* in which they act out the slope of the original track (Fig. 8).

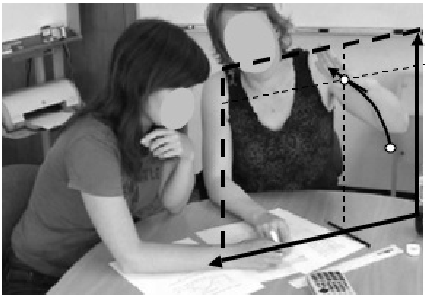

Figure 8: Noa creates a virtual space using iconic gestures. The mathematical properties included in this virtual space are: orthogonal horizontal and vertical axes, gradient steepness and sense (indicated by the shape of the hand), height of the track (indicated by the vertical position of the hand), left-to-right (from Noa's orientation) movement along the track.

Ava and Noa coordinate these three semiotic resources (speech, deictic gestures, iconic gestures) so that they are synchronised with each other. Thus, while pointing to any one portion of the gradient graph, Ava and Noa will simultaneously gesture the slope of the corresponding portion of the track, while verbally describing the features of the track's gradient over that portion.

The synchronous use of these three semiotic resources is likely to have lightened Ava and Noa's cognitive load as they constructed the antiderivative. In using deictic and iconic gestures, Ava and Noa were able to distinguish clearly between two different mathematical concepts: the given gradient graph and the graph of the track (the antiderivative). The right hand performed deictic gestures referring to the gradient graph on the paper, whereas the left hand performed iconic gestures referring to the graph of the track in the virtual space. The semiotic resource of speech enabled Ava and Noa to connect these two concepts together, by referring to both the gradient graph, and its antiderivative (the track's graph).

The iconic gestures that created the virtual space were helpful in reducing the cognitive load in two ways. First, by being a temporary medium, the virtual space lent itself to experimentation and enabled Ava and Noa to correct their errors quickly without leaving a residue of any mistakes they made (in contrast, drawing the graph is a much more permanent action). Second, the iconic gestures in the virtual space afforded more detail than the act of drawing would have, as the flat of the hand itself served as a makeshift tangent line to the graph. Thus, the tangent line was visible and dynamic throughout the enactment of the track in the virtual space.

DISCUSSION

We have used the semiotic bundle to frame the synergies and dynamics among the semiotic resources activated by the subjects in two different cases. In the first case (the students L, M, and R) the coordination was feeble and the performances were poor. In the second case (the teachers Ava and Noa) we observed a high coordination among them, and good mathematical performances. The reasons for such differences are multiple and the short space here allows us only to sketch our findings.

Apart from the differences in the background of the teachers and of the students, we note the type of the task and the text of the task. The students were given a 'forward' task for which some elements toward an algorithm (the tangent line segments) had already been introduced earlier, whereas the teachers were given an inverse task that required them to operate on the unknown origin (antiderivative) of the given gradient graph. In the formulation of the teachers' task, a technical term (gradient) and a realistic situation (track) were used; in the students' task, the term (slope) has also an everyday meaning that diverges partially from the mathematical one, as pointed out above, while the situation is purely mathematical (function).

A further important difference that is at stake in this paper concerns the relationships among the semiotic resources within the semiotic bundles. In the teachers' case the semiotic bundle contains three types of semiotic resources (speech, iconic gestures of the virtual space, deictic gestures while "going along the graph"). The main relationships among these resources are given by the narrative supported by the speech: it links what happens in the structure of the virtual space with what happens in the given graph on the paper. The synchronism between the gestures in the two spaces and the narrative shows the deep intertwining of the three semiotic resources. However, while synchronously linked, the two spaces—the virtual one in the air and the physical one on the paper—are kept physically and cognitively distinct in a very neat way. As such, the virtual space lightens the cognitive load of the teachers.

The situation is completely different for the students: even if they activate the same kind of semiotic resources as the teachers (speech, gestures), the links among them are feeble and do not allow them to activate that synergy of registers that, according to Duval produces a "conceptual comprehension" (Duval, 2006, p. 126). For example, whereas Noa's gesturing (Fig. 6) starts the building of the virtual space that supports the teachers' successful managing of the task, the gesture in the air by L (Fig. 3) pushes the students to confuse the graph of the sought derivative with that of the function itself. Because of the confusion, the students use the representations on the paper to look for the properties of the tangent slope. But within the new semiotic resource (the diagram) they are not able to build techniques apt to manage the passage from the properties of the given function graph to those of its derivative: contrary to the virtual space for the teachers, this fresh resource increases the cognitive load of the task for the students.

It remains to be clarified in what manner the differences in the background, the type of the tasks, and the text of the tasks influenced the subjects' (lack of) success in using the semiotic resources to differentiate and link between the concepts involved. Our analysis highlights that the two stories are different not so much for the variety of the semiotic resources the subjects activated, but for the different degrees of control the subjects exercised over them: the control is interdependently related to the richness of the mutual relationships that the subjects built within the semiotic bundle.

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EXPERIENCES RELATED TO THE PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS FOR THE USE OF TECHNOLOGY IN THEIR PRACTICE

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We are concerned with researching the use that in-service mathematics teachers make of digital technologies, the training they require and the changes in their practices that they need to make in order to harness the potential of such tools. In this paper we present data from several case studies of participating teachers in a three-year development and research project linked to a master's degree program in education for in-service teachers. We divide the paper in two parts: first, we draw attention through a case study of the difficulties that teachers can have when attempting to incorporate DT tools for the first time; second, we present some results from a group of teachers in the program who were asked to reflect on the changes in their practice through both training and classroom implementation of DT.

INTRODUCTION AND THEORETICAL FRAMEWORK

It is clear that a meaningful incorporation of digital technologies in education, requires rethinking and changing the teaching-learning process, leading to possible changes in class structure, time distribution and the teacher's practice. In particular, teachers need to develop new skills – not only technical, but also pedagogical ones— and adapt to the changes brought about by such technologies. This is particularly relevant for us, because in our country, in public schools especially, teaching practices are often still quite traditional (i.e. education is teacher-centred and students have mostly a passive role). Hence, changing the pedagogical structure and dynamics of the classroom – e.g. allowing students to have a more active role when DTs – is one of the challenges teacher education faces in our country today.

We consider, however, that digital technologies (DT) can act as catalysts for change, since they have a great potential for revolutionizing school practices. For instance, through the use of such technologies, students may gain an autonomy that can challenge the restrictions of conservative curricula (Facer et al., 2000). On the other hand, research results indicate that teachers with little experience in the use of DT, have difficulties in harnessing the power of these technologies as *tools for learning,* having as consequence a lack of significant influence of the DT in the school culture (McFarlane, 2001). In fact, studies (e.g Ertmer, 1999; Goldenberg, 2000) indicate that DT can help students learn in a more significant way, only through an *adequate use*. Thus, teachers not only need to be trained in the use of the new technologies but

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also need to understand how to use these tools, and change their practice in order to promote significant learning in their students. However, changing teaching practices is not straightforward: teachers can face struggles when attempting to modify their practices (e.g. Wilson & Goldenberg, 1998). This is consistent with results (e.g. Sacristán et al., 2006) from our own country, where, after a decade of use of DT in mathematics classrooms, its benefits are not so clearly perceived; there are many difficulties in changing a school culture (including teachers' difficulties in adapting their practices when using DT); and it has become clear that the role of the teacher is critical for a successful implementation of DT in the classroom and for fostering meaningful learning in students.

In order to provide strategies to help in the educational transformation and to overcome what Ertmer (1999) calls "second-order barriers", more understanding is needed of all the variables, difficulties and changes that teachers' face when incorporating new technologies into their practice. Documenting those changes, not only from the perspective of researchers, but also from the perspective of the teachers themselves can provide deeper insights into those variables.

A THREE-YEAR DEVELOPMENT AND RESEARCH PROJECT WITH IN-SERVICE TEACHERS

We have been involved in a development and research project, that was linked to a three-year master's degree program in education that began in the autumn of 2005 teachers that had as aim to strengthen the participants' mathematical and pedagogical preparation. Our project seeked to: a) train (and observe) teachers in the use of DT tools for mathematics teaching; and b) research and understand how changes in school practice and culture are brought about by the implementation of those tools into the classroom. More specifically, the project seeked to document the methods, techniques, resources and strategies that teachers use and develop when incorporating new technologies into their practice, and understand the influence and impact of, and on, the broader school community (since we consider teachers part of a symbiotic system that is the school community).

We divide this paper in two parts. In the first we provide sample data from a case study of a teacher (a participant of the above-mentioned master's program) that attempted to incorporate the graphic calculator into his practice; we provide this case study because it is representative of the problems that many of the teachers that we have observed have when they first attempt to incorporate digital tools into their practice. In the second part, we present some results from a professional development program where we asked a group of teachers to reflect on the changes in their practice through both training and classroom implementation of DT, and document their findings.

A TEACHER'S USE OF THE CALCULATOR IN HIS PRACTICE

Here we present results from a case study on the use that Joseph, a middle-school teacher, did of the Voyage 200 TI graphing calculator for teaching first-degree equations to 12-13 year-olds. This teacher had a decade of experience, and was a first-year participant of the continuing education master's program previously mentioned.

For this teacher, we wanted to analyse the knowledge and competencies that he put into practice. For that, we used as methodological and theoretical framework the work of Llinares (2000) on research that attempts to identify characteristics of teachers' knowledge. In the episode discussed here, we used as data sources, videotapes of both a 50-minute class (with approximately 40 students) and that of the planning session (about 3 hours long) for that class.

Joseph chose to work with first-degree equations of the form: ax + b = cx + d, both with paper-and-pencil and with the Voyage 200 calculator. The problem he chose was to "*Find two consecutive whole numbers whose sum is 143*"; this problem had no direct relationship with other contents in his course. Joseph explained that "other problems would require too much previous knowledge and reasoning". It is clear that Joseph had little awareness of what the use of the calculator can bring and require for this and other problems. The problem chosen is a routine problem, meant to be solved with paper-and-pencil, and if a calculator is used, it is most likely as a basic one for trial and error methods.

In his planning session, Joseph established several moments for his class: verbal presentation of the problem; translating the problem to algebraic form (setting up an equation); solving the equation using paper-and-pencil; verifying the results using the Voyage calculator. The problem was given in a worksheet with the suggestion for students to solve it, with teammates, with paper-and-pencil and then use the calculator as a verifying tool (for which he suggests the use of the *Solve*, *Approx* and *propFrac* commands); he then asked (in the worksheet) what method (paper-and-pencil vs. calculator) was easiest, what strategies were used in each case and what calculator commands were used. In his planning, Joseph had considered that first and necessarily, students had to set up an equation describing the problem, then solve it with paper-and-pencil and then verify it with the calculator (even though the problem is not so complex as to require a CAS calculator for verifying the results).

In the actual class, Joseph presented the problem without giving a review of the mathematical content and simply handing out, and reading, the worksheet. However, the use that students did of the calculator was different from the intended one: over 50% of the students used the calculator to find by trial and error the two unknown numbers. Only one team of students set up an equation, but then didn't solve it through paper-and-pencil algebraic manipulation, but simply used that equation to verify their answer with the calculator (see Figure 2). In this sense the actual learning trajectories (Clements & Sarama, 2004) of students differed considerably from the intended one by the teacher.

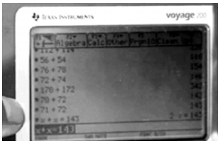

Figure 1: Students solve the problem by trial and error

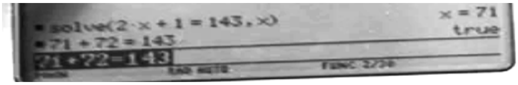

Figure 2: Students' use of the equation as a verification tool

Moreover, even Joseph lost sight of his intended goals: when looking at the answers, he totally disregarded the process by which the students reached their answers, looking only for right ones. There were evident difficulties in students in being able to set up an equation (with only one team achieving it) and in introducing it into the calculator, and several students were unable to verify their results with the calculator (in spite of correctly identifying the answers – i.e. 71 and 72), but the teacher disregarded this. He even ignored all pencil-and-paper work and how students had arrived at the answer or their strategies. And in the follow-up discussions that he had with his students, he simply accepted that most students had used trial-and-error methods and did not engage in a discussion of the problem. In his questions, he focussed on what commands the students typed on the calculator to find the answers, and ignored mathematical processes.

We thus had the following main observations regarding Joseph's class with the calculator: (i) Lack of understanding and knowledge of the potentials of the use of the technological tool for mathematical learning; (ii) Deviation from intended goals; (iii) Absence of use of mathematical language and justifications, and in leading students to reflect on the mathematical processes when using the tool.

The above case-study shows the difficulties that the teacher had in having a clear comprehension (i) of the knowledge being used, and (ii) of the materials and teaching tools. Llinares & Krainer (2006) explain that teachers must have an adequate mathematical preparation in order to propose activities that harness the potentials of instruments and technology. But as Godino (2008, p.12) has emphasised, the professional development of teachers must not limit itself to developing mathematical competencies; it is also necessary to develop competencies of reflection and analysis of their own mathematical activity and of the knowledge that is put into play, so that they are able to select or adapt adequate mathematical problems and reconstruct the configurations of the objects and meanings that are put into play in them. This points to the importance of professional development programmes for teachers where they reflect on their teaching practice, their own mathematical competencies and on the relevance of using certain tools and materials depending on the class-objectives.

It is in this sense that we developed a program where 6 other participants (4 middle-school and two primary in-service teachers) in the above-mentioned master's programme, could reflect on their practice when incorporating digital technologies.

TEACHERS' REFLECTING ON THEIR PRACTICE WHEN INCORPORATING DT TOOLS

Our approach was to have teachers reflect on the changes in their practice through both training and classroom implementation of DT, and document their findings. For this, we follow as a theoretical and methodological framework works such as those of García et al. (2006) and Artzt & Armour-Thomas (1999) which provide models in which teachers reflect on their instructional practice. We also assume that teaching must be a constructive process that requires reorganising and reinterpreting the subject matter and the practice as a result of experience. In this sense, Liu & Huang (2005) emphasize that the knowledge that is derived from social interactions in a (real-life) context is more valuable and significant for the teacher.

During this development and research project, the teachers were involved in three, almost simultaneous, activities: a) in the training, and development of abilities, for the use of digital technologies (DT) in the classroom; b) the design and planning of teaching strategies and activities that integrate DT; and c) engaging in observation and reflection-on-action (Bjuland, 2004) of the changes in their own teaching practice with the new tools. As part of the master's degree courses, the participants also analysed and discussed research papers and results. In addition, some of the teachers became involved in training programs for other teachers, so they have used this opportunity to engage in observation, or training-and-observation, of some of their colleagues (not involved in the development project) when incorporating DTs.

The participants analysed and reflected upon the potentials, limitations and changes brought forth by the incorporation of DT into their practice, and that of their colleagues, from various perspectives: (a) *The perspective of the teacher and didactical use of DT*: Changes in the role of the teacher (e.g. changes of the teacher as lecturer, to that of mediator) and the difficulties in those changes. Changes in teaching methodologies. Changes in their beliefs and conceptions. Design of activities with DT. Articulation of the DT activities with the curricular requirements. Design of assessment techniques for DT activities. Complementarity of different DT tools among themselves, and with non-DT activities (such as those with paper and pencil). New mathematical knowledge and perspectives through DT. (b) *The perspective of the classroom interactions* (e.g. changes in classroom structure*).* (c)- *The possible impact on students* (e.g. in their learning and affect). (d) *The technical perspective* (e.g. technical knowledge for the use of the DT tools). (e) *The social context*.

The participants collected data by using video recording of their –and their colleagues'— classrooms; taking field notes when possible; designing questionnaires for colleagues, authorities and/or students; collecting their activities and assessment designs and sometimes students' work, and most importantly, writing weekly reports. In addition to that, we designed an initial questionnaire for evaluating the participants' conceptions on the use of DT and we held bi-weekly meetings with the participants where they presented oral reports, were informally interviewed, and engaged in reflections and discussions with the other participants. Finally, "independent" researchers (not involved in the development project) carried out some observations and videotapes of the participants during their practice. In this way we were able to carry out case studies of each of the participants, combining data from both their own reflections and reports, and from researchers observations and interviews. Some results of this project have been reported in Sacristán et al. (2007).

Initial Conceptions of the Participants on the Use of Digital Technologies

As with Joseph described above, at the beginning of the three years, the other participants lacked awareness of the different knowledge or practices that DT can bring as well as any disadvantages. The main beliefs that the participants had on the use of the DT was that they are a useful tool for teaching because DT facilitate the construction of graphical representations ("*it is easier to create graphs*"); and that they can save time for some activities in comparison those with paper and pencil. All of the participants also expressed a lack of confidence and concern in the pedagogical and technical knowledge that DT demand from the teacher, and some of them felt uncomfortable in using some of the tools with students; with one of them not believing that the DT tools could help create significant learning in students. As with Joseph, they treated the DT-based activities as separate from other mathematical activities in their practice, used DT to teach the same problems as with paper-and-pencil and many of them focused on technical knowledge of the tools rather than the mathematical content that was put into play.

Results from the Participants' Own Observations during the Project

Since the beginning of the project, the participant teachers' beliefs, attitudes and practices were in constant and profound transformation, and enriched by their long-term exposure and use of the DT in their classrooms, their own reflection of their practice and by their sharing of experiences and by their observations of other colleagues. After three years, in terms of *how their practices, and the classroom dynamics, have changed*, most of the participants recognize that with DT, they need to change the way they teach. One of the participants expressed that although adequate training and continuous support helped her to change her practice, ultimately it is up to the teacher to make the changes. Most participants have expressed how they have now taken a back-seat role in their classrooms, becoming more of observers and guides than lecturers. They have realized more and more, and come to accept, that students can learn on their own, or from collaborative work, with the support of DT, and that they request less and less assistance from the teacher. However, all of them also expressed that it takes time for teachers to get used to this change in the classroom structure.

Most of the project's participants have become aware that the use of technology "is a tool and not an end". Many of these participants engaged in observation of their own colleagues at school; one of the results was that they realised that other teachers seem to not be successful in the implementation of DT into their practices, because they are not clear why they are using them (i.e. they do not have a broader educational goal or plan) and just give students some DT activity for the sake of using the technology (as was the case of Joseph).

Another aspect is that the DT use, has made the participants, and their colleagues, aware of their own limitations in their mathematical knowledge. They observed that many of their colleagues refrain from certain DT activities, because they lack mathematical confidence and feel their deficiencies can be exposed by the DT activities. On the other hand, all the participants realized that the knowledge developed with the DT can be of a more conceptual nature. One of them explained that she has learned to look more at what abilities her students are developing rather than looking for correct answers to problems (which again was one of Joseph's problems). Another aspect was in terms of the choice and design of activities. After three years all of the participants realized the importance of being careful in the choice of the DT-based activities, in solving the activities themselves, and of the difficulties and care needed when designing their own activities.

CONCLUDING REMARKS

What we have observed, is that through training and involvement with the DT tools, teachers' initial difficulties as those illustrated in the case-study of Joseph, and their perceptions of their use has been enriched. Second, the classroom experiences of implementing DT has led them to reflect on the potentialities and limitations of the tools. More importantly, this opportunity to reflect upon, share their personal experiences with the other participants, and observe fellow teachers, has led them to develop a critical and reflective attitude, as well as enabled them to construct didactic strategies for the use of DT that are in accordance with the specific needs of their students. Yet the changes do take time, and only after three years do we feel that the participating teachers have grown confident and wiser in the use of DT in their practices, with half of them having even become advisors to other teachers and programs. Therefore, we keep in mind the words of Goldenberg (2000, p.8): "Provide instruction and *time* for teachers to become creative users of the technology they have."

Part of this research was funded by the CONACYT Research Grant No. 44632.

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LIGHTS AND SHADOWS OF FEEDBACK IN MATHEMATICS LEARNING

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Feedback as an assessment practice for learning, specially in Mathematics, has been studied over the past decades. Based on the evidences gathered from several previous research studies we aimed to study how the "form" and "dimension" of feedback can influence students' learning. Four interpretative research studies were carried out, over a three year period. The participants, students aged between twelve to fourteen years old, show that the trends studied can be determinant to find out how effective feedback is in connection to the students' school achievement and to their conceptions of the tasks given.

INTRODUCTION

Formative assessment is a type of assessment that currently has a key relevance considering the school crises. This relevance is due to four key reasons: (i) a general concern of the OECD countries to end failure and drop out (Field; Kuczera & Pont, 2007), where it is recognized that the modes of assessment centred in retention don't contribute to learning; (ii) the need for assessment to be a tool serving teaching practices in order for them to be closer to the students difficulties; (iii) the fact that formative assessment needs to be recognized as key by the assessment guidelines and the curricular orientations for mathematics teaching (NCTM, 1995; 2000), and (iv) the fact that the teachers are favourable to using a formative assessment but their practices are still poor and little effective (Black & Wiliam, 1998; Torrance & Pryor, 2001). This facts lead to the development of project AREA[i]. This project aimed to understand how formative assessment can become a real learning tool. In this perspective we assume that all learning is complex and created by the individual through its activity and the use of mediator processes of re-interpretation of its own actions. Thus, the formative assessment is a relationship process between the student and the teacher aiming to do a certain task, that is, an assessment for learning. The products of this relationship and of the mediation process in connection to the purposed tasks and objectives are shared between teacher and student. The reflection that the participants do about this information can strongly improve its use. Thus, the key objective of this research study is to study deeply one type of interaction – written feedback. Having as starting point the results of previous research studies about feedback (Black & Wiliam, 1998; Clarke, 1996; Wiliam, 2007), we aimed to understand what the potential of feedback to mathematics learning is and what should be the conditions present in order to maximize its learning potential.

THEORETICAL FRAMEWORK

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 49-56. Thessaloniki, Greece: PME.

A new approach to learning and consequently to teaching lead to new culture and assessment forms (Dierick & Dochy, 2001). With the evolution of the times, we can state that an assessment for learning is not circumscribed to the formal moments of assessment but essentially present in every day moments in the classroom, in the moments were learning tasks are being performed as well as on those were reflection is being done. In this perspective intentionality is another aspect that deserves special attention. It is the intention of understanding and students support that provides assessment with a formative nature (Pinto & Santos, 2006). Assessment goes through gathering evidences, followed by its interpretation and finally in an action based on the hypothesis reached (William & Black, 1996). However, this type of assessment will only be truly a formative assessment if there are implications to the learning.

A key element to an assessment for learning is feedback. Feedback is perceived as the information that shows how apart is the "performed" to the "expected" trying to minimize that difference (Sadler, 1989). The revision of literature done by Black and William (1998) gives some contribution to distinguish feedback regarding its nature. Referring to another revision of literature done by Kluger and DeNisi, they identified three levels of linked processes in the regulation of tasks performance: *meta-task processes*, involving the self; *task-motivation processes*, involving the focal task; and *task learning processes,* involving the details of the focal task. The results gathered from previous studies point out that when feedback is aiming mainly to the task its effect tend to be positive, favouring the improvement of the student's product.

Also Gipps (1999) makes a distinction between two types of feedback: the *evaluative* and the *descriptive feedback*. The first is seen as a judgment of value with an implicit and explicit use of norms. Given its nature it has little effects on learning. This author still divides the descriptive feedback in two other types: the feedback that specifies the progress and the feedback that constructs a way forward. The first sub-type of feedback is the only responsibility of the teacher. It is the teacher that has a control, power and authority to say to the student what he should do in order to improve its production. The second sub-type of feedback is developed in collaboration with the student. Therefore, the responsibility and power over a certain production is shared promoting a deeper understanding about the tasks given and encourages the student to access and to reflect on what they have done.

Tunstall and Gipps (1996) have developed a new type of feedback as a result of a study carried out about oral and written feedback. Assuming that feedback is a socialization tool and as such it is directly connected to values, attitudes and classroom processes, the authors defined two subcategories of evaluative feedback witch opposite ends are, respectively: *Rewarding versus punishing; approving versus disapproving*. As subcategories to descriptive feedback they have considered *specifying attainment versus specifying improvement and constructing achievement versus constructing the way forward*. The authors argue that the use of this categorization suggests that feedback can change according to the style, purpose, meaning and processes. Nevertheless, it is important to stand out that these subcategories, as they are stated, are not necessarily separated. Furthermore, the same teacher can use, in different moments, different types of feedback. Also Jorro (2000) makes a distinction between two types of assessment writing. Making *notes to convey information* that can be translated by vague statements with little contribute to learning; and making *notes as a dialogue* that aims to question, to provide clues and to encourage reflection from the student (Veslin & Veslin, 1992).

In summary, feedback can contribute to improve students' performance when is focused on what is needed to be done in order to improve performance, and when more detailed information on how to proceed is given (William, 1999). We still have to consider how much information should be provided and the appropriate moment to do it. To state that the more feedback the better is not necessarily true (William, 1999). It is the quality and not just the quantity of feedback provided that deserves our attention (Sadler, 1998). The amount of information provided should only be the necessary for the student to move forward and not too much in order to provide the answer (Santos 2004; 2008). Strategies that are favourable to long castings learning include allowing for the student to identify and correct its own mistakes and to reach the correct answers (Nunziati, 1990; Jorro, 2000). To know what is the correct moment to provide written feedback also seems to be crucial. Several studies point out that feedback should never be provided before the student is given the chance to think and work on the given task and a grade given (William, 1999).

The assessment practices that we have mentioned constitute a big challenge to the teachers. To perfect the practices of an assessment for learning does not follow a linear process. There is no facilitating process that can be adapted to existing practices that can guarantee fast results (William *et al.,* 2004). Nevertheless, evidence show that we can obtain gains to student learning through an assessment for learning practices and that the teaching of high level objectives is compatible with success even when this is measured using limited instruments such as external assessment test (Black *et al*., 2003; William, 2007)

METHODOLOGY

This study is based on four research studies about feedback in a mathematics classroom. The studies were carried out over a three year period in Portugal (from the academic year of 2005/2006 to 2007/2008). These research studies were carried out under the scope of project AREA and were developed by two teachers, Sónia e Sílvia, that teach in two different school in different areas of Portugal. One of the teachers carried out one research study per year (a total of three) and the other just one on the final year. The participants of the studies were classes of middle school level with students aged between twelve and fourteen years old.

All of the research studies carried out an interpretative research methodology and used a case study design. The selection of the participant students aimed to gather students with different levels of school achievement in mathematics. The data was gathered through classroom observation, students' interviews (both with audio recording and its total transcription) and analyses of the documents produced by the students. All of the students' productions included two versions. The first version, not yet graded, had received some written feedback by the teacher. The feedback provided always aimed to be focused on the task and not on the students' specific trends (Black & William, 1998) and to be mainly of a descriptive nature (Gipps, 1999).

For the purpose of this research the feedback provided was analysed according to the following categories: form (symbolic/descriptive, this one could be affirmative/interrogative/mixed) and the length of the feedback (short/long).

RESULTS AND ANALYSES

The feedback provided to the student and its effectiveness concerning learning will be analysed considering its form and length.

Form

Analysing the feedback provided to the students throughout this research project we can state that there was progress made regarding the form of the feedback provided by the teacher. The teacher went through a frequent use of symbols to mark a mistake or something incomplete (underline, a cross, a question mark) that gradually started to disappear being then replaced, at the start of the second year of research, by a descriptive feedback as we noticed that the first form of feedback used was only useful to certain students. All of the students perceived feedback through symbols as something that was wrong. However, to the students that had a good achievement in mathematics the symbols were sufficient. They were interpreted as a call for awareness that would make them go over what they had done and use their knowledge to improve the second version of their product. For the students that had an average or weak achievement in mathematics the symbols were not enough for them to change the incorrect information. The students argued that the teacher should say what she wanted: "If the teacher would explain what it was. It could have a note explaining what was wrong, instead of a cross" (student with medium achievement, Sónia, 05/06).

Descriptive feedback can present several statement forms: affirmative, interrogative and mixed. The interrogative form intention is for the student to reflect on their answer, to clarify it and to use their knowledge. In order to do this we associate several clues that will help with the progression of the task such as, "How did you come up with these values?", "What did you conclude?", "How can you convince me that this statement is true?", "Is it always true?". The affirmative form appears to be more often used when it is necessary to explicit a mathematical concept in order to solve a task or to identify a concept that it is clearly not correct, such as "You understood the task very well and your results are presented very well. However, you forgot one thing. The box where you had your little 'cubes' is itself a 'cube' and it was full of your little cubes" (Sónia, 08/09). The mixed form has very similar intentions in connection to the interrogative form to which we add an 'anchor point' in a way that establishes a certainty on the student: "This paragraph does seem to be very clear. Are you sure you understood all the information provided?" (Sónia, 08/09).

When we compare several feedback forms, the interrogative or mixed forms seem to be clearer to the students regardless of their achievement: "I have the questions. It is easier" (student with medium achievement, Sílvia, 08/09), "The overall comments are important but I understand the questions better than the overall comments. When we have questions we try to answer them. It is easier", so the student try to address the questions first: "We think it is best to answer the questions that the teacher poses us" (student with high achievement, Sílvia, 08/09).

An interrogative or mixed feedback that is contextualized by the task and that aims to provide more detail about what it is asked to do can help the student with the task given. It is the case of the following comment: "How can the cube A be formed by 23 little cubes? And B by 18? And C by 10?" or "Where the two brothers born with the same weight? Or was one heavier than the other? Did they maintain the same weight difference forever?" (Sónia, 08/09). Or still by specifying the question connected to the students' contribution to the group work: "Did I help the group with the work? Have I participated and gave my opinion? In what way? Did I listen and respected others opinions?" (Sílvia, 08/09). Nevertheless, not all comments that include questions are effective to all students, specially for those that have a low achievement in mathematics. When the comments given are more abstract, not centred on the task or require the connection between concepts the feedback is not always understood. For example, "Start by saying one result that we learned and that connects elements of the triangle with elements of the square. What does CD^2 represent? And DE^2? Why do we divide the area of square Q by 4? We have represented in the picture triangles and squares..." (Sónia, 08/09). This fact was confirmed by the teacher: "When I used symbols from the task on my comment or a specific mathematical language the student did not understood" (Sónia, 08/09).

In this analyses we still have to consider that certain feedbacks have been ineffective not due to its form as such but due to the students perception of the task itself. That is, for example, the case of a student, with low achievement, that values more reaching a correct answer rather than the explanation of the process use to reach that answer. Faced with the following comment: "You were looking for values that would solve the problem and you were successful because you found them. Now I want you to present other possibilities and to explain why they cannot be solutions to the problem" (Sónia, 08/09), this student added nothing to what he had already done The other teacher found the same problem regarding the final conclusions of the finalized work. Faced with the feedback comment stating: "You mention that the task was difficult but well done. What brings you to say that? Do not forget to justify your comments" (Sílvia, 08/09), the student states that he added nothing on the second stage because "I don't think that the conclusion is very important. If it was the conclusion of the task it would be important. But it wasn't so I don't worry about it. I just do it because the teacher asks me to" (student with medium/high achievement, Sílvia, 08/09). When it is a student with high achievement, and even when the teacher states that on the first stage the task was correctly solved and performed, the student commits to the second stage where he tries to perfect all aspects of the product, even the smallest details.

Length

The comments have a tendency to be short. However its length seems to be connected with the length of the task itself. If the tasks are open the comments tend to be longer. However, this type of comments, as the students themselves admit, are the ones that are more difficult to understand: "I had to gather all the information from the first stage of the task plus the comments to do the second stage. I had to read, look back on the comments, do the second stage. There was a point when I was really confused. The notes should be smaller" (student with a low achievement, Sónia, 08/09); "I think that the teacher should just write 'you need to explain yourself better...' because with all this text we read and start thinking about something more" (student with medium achievement, Sónia, 08/09).

These comments do not seem to help when it comes to re-writing the tasks. The students are lead to several strategies such as: (i) to remove or not what it is wrong; (ii) disregard the comments that they do not understand or ask the teacher for clarification; and (iii) hierarchies the information that they understand giving priority to the aspects related to mathematics and then the ones connected to the formal aspects of the task. Faced with the awareness that it is wrong, but without knowing how to correct it, some students chose to remove the information in order to avoid leaving mistakes on their products. Others chose to leave it as it is as they fear, that by changing it, it will lose coherence or that it would become worst: "We read but we didn't understand what was wrong. We decided to include it again. If we took it out something could be missing. If we change it, it could be worst" (group of two students, one with low achievement the other with medium, Sónia, 05/06). We are then facing different attitudes towards a mistake.

One other aspect stands out in all the studies that relates to other type of behaviour facing a not understood feedback comment. The students with a lower general achievement disregard the comment while students with a high achievement question the teacher regarding the feedback given. This questioning is often related to the fact that they want to present a product directly connected with the requests of the teacher. This fact leads to the presence of oral feedback as an important complement of the written feedback. Oral feedback can be dynamic, constantly adjusted and developed which can be seen as an upgrade of the initial written feedback.

Finally, when the comment is long, and as such withholds a lot of information, students tend to manage their time by delaying other aspects related to the final product, such as the front cover: "We were more concerned with looking for the information. So, in order to be able to do everything, we left the front cover to the end" (group of students, Sónia, 08/09).

SUMMARY AND IMPLICATIONS

The research that served us as a starting point shows that feedback of a descriptive nature (Gipps, 1999), such as notes as dialogue (Jorro, 2000), contextualized by the task and with detailed directions as to how to proceed (William, 1999) is potentially more favourable to learning. However, our study point out that not all feedback with these characteristics has the same positive effects regarding learning. The form of the comments as well its length, the type of student and its perceptions are factors that can influence the effectiveness of this assessment practice. The interrogative or mixed form in comparison to the affirmative form seems to facilitate the students understanding of the feedback and to get them involved in the following stages of the task. Short feedback comments seem to be more effective than the long ones, helping the students to focus in certain specific aspects of the task. Nevertheless, we need to stand out that there seems to be a connection between the length of feedback and the nature of the mathematical task in hand. More open tasks tend to require a longer feedback which can constitute a difficulty for the teacher. Following Tunstall and Gipps (1996) research, feedback as a mediation instrument for the learning and teaching relations, can be appropriate to students in a diversity of ways. The students with a lower achievement in mathematics reveal more difficulties in understanding feedback when it relates or uses mathematical concepts or refers to more abstract ideas. When the understanding of what is requested from the task is not what the teacher intended to be can constitute another difficulty for the effectiveness of feedback. Nevertheless, when the students reveal a high achievement in mathematics, if the feedback is not understood, they tend to question the teacher aiming for some new feedback, this time orally, and thus creating a new learning opportunity. Summarizing, this research study reveals that we cannot mention of totally good or bad feedbacks. Giving feedback can constitute a task of extreme complexity, where the intentionality of the teacher and its ability to reflect on and about its own action are determinate factors to a real assessment for learning practice.

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[i] Project funded by FCT nº PTDC/CED/64970/2006 Project funded by FCT nº PTDC/CED/64970/2006

VALUES OPERATING IN EFFECTIVE MATHEMATICS LESSONS IN AUSTRALIA AND SINGAPORE: REFLECTIONS OF PRE-SERVICE TEACHERS

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110 pre-service teachers' reflections of what were co-valued in effective mathematics lessons in Melbourne (Australia) and Singapore were analysed in a research study reported in this paper. 64 values were recorded, of which 29 were unique to Melbourne. These values were predominantly mathematics educational/pedagogical in nature. A new category of values, (teacher) personality values, was identified. A majority of the most cited values across Melbourne and Singapore were similar, although these might be enacted differently. The research significance and implications for future, related research are discussed.

INTRODUCTION

This paper reports on one method for understanding what enhance the effectiveness of mathematics lessons, drawing on students' reflection and on their presentation of these reflection through drawings. This paper will briefly present what is currently known about effective mathematics lessons and the roles played by values in mathematics pedagogy. The methodology and the research context for the study follow. The presentation of the results will be accompanied by a discussion of our interpretation and learnings. Implications for future research will also be outlined.

EFFECTIVE MATHEMATICS LESSONS

Opdenakker and van Damme's (2006) research provided empirical support for

> the inappropriateness of a narrow conceptualisation of (teacher) effectiveness in terms of testing and assessment of student outcomes in relation to strictly defined curriculum content…. Good teaching involves reflection[,] ….communication and building relationships …. Also, … aspects of learning are influenced by teachers (and other participants in the school) and should be included when conceptualising effectiveness. (p. 16)

The focus on effective mathematics *lessons* (instead of, say, mathematics *teachers*) in this study is an acknowledgement that the school classroom constitutes a learning community with ongoing interaction and negotiation of ideas and their meanings between teachers and their students, and amongst students themselves.

In this light, a teacher's effectiveness is a function of socio-culturally based factors. Any focus on specific classroom practices associated with successful lessons would risk making the mistake of regarding these cognitively-based practices as being

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transferable across space and time (see Chenoweth, 2000; Gowen, 2001). Indeed, while "the structures and meanings of mathematics … and the methods and insights of psychology (especially constructivism) have provided rich theoretical fields for the mathematics education research community[,] they have not, however, enabled us to engage with schooling as reproduction, nor with culture or power, as they are manifest in the mathematics classroom" (Lerman, 2001, p. 87).

Askew, Brown, Rhodes, Johnson, and William's (1997) study of effective teachers of numeracy identified general - rather than specific or explicit - teacher behaviours. Relatively high mean achievement gains were not necessarily related to overall teaching style (see also Hollingsworth, Lokan and McCrae, 2003), but were associated with teachers who had 'connectionist' orientations (vs 'transmission'/'discovery' orientations), focused on students' learning (vs provision of pleasant classroom experiences), provided a challenging curriculum (vs a comforting experience), and held high expectations of initially low-attaining students.

The relationship between effective teaching and teacher values/beliefs is highlighted in Opdenakker and van Damme's (2006) study, which supported "recent research … that a teacher's cognitive and pedagogical behaviours are guided by, and make sense in relation to, a personally held system of beliefs, values and principles" (p. 16).

Given the more internalised and more stable nature of values (compared to beliefs), one way of taking into account the socio-cultural aspects inherent in effective mathematics pedagogy might be to consider what are co-valued by the teacher and his/her students in the mathematics classroom, rather than what is being done by the teacher alone. This perspective has been adopted in the study reported here.

VALUES IN MATHEMATICS EDUCATION

In mathematics education, values are "the deep affective qualities which education fosters through the school subject of mathematics" (Bishop, 1999, p. 2). Values represent "an individual's internalisation, 'cognitisation' and decontextualisation of affective constructs (such as beliefs and attitudes) in her socio-cultural context. Values related to mathematics education are inculcated through the nature of mathematics and through the individual's experience" (Seah, 2004, p. 43). These values might be classified as mathematical, mathematics educational, educational (Bishop, 1996), and institutional/organisational (Seah, 2004).

What get emphasised – or valued – in mathematics lessons can be seen through the decisions teachers and their students make (see Bishop & Seah, 2008). However, there is a complex relationship between form of practice and its underlying value. Different forms might reflect the same value, while any value might be enacted differently in different situations. These have implications for research methodologies concerned with value identification.

That values are so internalised has meant that what one values can be so tacit that personal identification is difficult (Bishop, FitzSimons, Seah & Clarkson, 2001). It appears, then, that research methods using questionnaires and interviews might not be effective in enabling participant identification of values, unless these values are inferred indirectly, using the written and spoken responses (as well as other sources of data, such as lesson observations) as stimuli to further personal reflection.

Thus, in this study, we focused on researching what were co-valued by teachers and their students in effective mathematics lessons, and we noted prior research experience highlighting the difficulties involved in identifying values directly. In this context, the study was framed to help us respond to the following research questions:

(1) What are co-valued by teachers and students in effective mathematics lessons in Australia and Singapore?

(2) How are these values similar and different across Australia and Singapore?

RESEARCH CONTEXT AND METHODOLOGY

In situating this study in Singapore and Australia, specifically, the city of Melbourne and its suburbs, comparisons can be made between two distinct contemporary cultural traditions which are otherwise similar in many other ways (e.g. both with British colonial heritage, comparable populations and GDP, and both are multicultural). Singapore is essentially East Asian with a Confucian heritage culture, whereas Melbourne reflects a Western, liberal culture.

Given the difficulties involved in identifying values, McDonough's (2002) technique was adopted for this study. This technique involves children reflecting on identified situations, and then drawing their mental pictures. (Children's) drawings have been sources of data in prior research, relating to both cognition (e.g. Goodman & Bottoms, 1993) and affect (e.g. Bishop & Clarke, 2005; Norton & Windsor, 2008; Weber & Mitchell, 2000). Children's drawings had also been used by Bishop and Clarke (2005) to infer what young learners valued, and as was similarly suggested in Norton and Windsor's (2008) methodology.

62 first-year pre-service teachers from Melbourne, and 48 from Singapore, participated in the study. They were best poised to draw upon some twelve years of experience learning mathematics in primary and secondary schools. The pre-service teacher participants were first invited to close their eyes, relax, and recall a time in their respective school experience when they felt that "mathematics was being learnt particularly effectively, that is, learnt especially well". Each pre-service teacher participant was encouraged to focus on the first mental picture he/she had, and to visualise what the learning situation looked like. Given that most, if not all, the participants were fresh out of formal schooling, their recall would reflect some last twelve years of schooling in Melbourne or Singapore. The participants were then asked to open their eyes, and to draw on paper their own mental picture, including as much detail as they could recall. They were then invited to write down five words, terms or phrases (that is, context-independent) that were dominant in their reflections.

Ten pieces of drawings from each city were randomly selected for analysis by both researchers independently. What appeared to be valued through the drawings were recorded by each of the researchers. They then met to compare the values named, so as to clarify and standardise how the values identified by each were related to particular features of drawings, and to clarify and agree on common terms for the same value identified (e.g. *authenticity* and *real-life applications*). With this mutual understanding of value identification and labelling, the rest of the drawings from both cities were analysed independently by both researchers.

In view of the subjective nature of interpreting others' values; the analysis of each drawing by both researchers separately should address the possibility of the researchers reading too much into the drawings. To this end, when the 90 pairs of value sets were pooled together (the other 20 pairs had been analysed earlier, as discussed above), only values named by both researchers were selected. Another source of the values data was the list of 5 terms nominated by each teacher participant. In adding these to the common values identified by the researchers, values from these sources which might be in conflict with one another were identified for researcher resolution.

AN OVERVIEW OF THE COLLECTED DATA

A total of 389 counts of 64 different values were identified as being associated with effective mathematics lessons across Melbourne and Singapore primary and secondary schools. Amongst these, 35 values were evident in both cities, while the other 29 were identified in Melbourne only. Table 1 summarises the results, listing the least number of values making up 50% of all the value counts within each city.

Amongst the values identified, there is only 1 mathematical value, no general educational value, and 2 organisational values, while the rest of the 52 values were mathematics educational ones. This dominance of mathematics educational values is both a reflection of the nature of values showing variety of form in mathematics pedagogy, and a testimony to the role that the teacher plays in making professional choices to facilitate effective mathematics learning. The co-valuing of *challenge* and of *authenticity*, say, would not have been possible without the teacher drawing upon his/her content and pedagogical content knowledge.

The key role played by the teacher in effective mathematics lessons can also be discerned through the 9 (teacher) personality values identified. The participants' valuing of aspects of teacher personality is novel, and resonates with Opdenakker and van Damme's (2006) finding that one of the two effectiveness enhancing factors was "teachers with good class management skills" (p. 14).

THE MELBOURNE DATA

The 62 Melbourne drawings of effective mathematics lessons were associated with 207 counts of 64 values. The five most cited values were *fun, whole-class interaction, out-of-class learning, group interaction,* and *challenge.* Cited 62 times, these made up 30.0% of values identified in Melbourne. Together with the following seven values, a total of 12 values (out of 64) accounted for at least half (53.6%) of all the values associated with effective mathematics lessons in Melbourne: *manipulatives, authenticity, quietness*, *hands-on*, *competition*, *visualisation*, and *practice*.

Melbourne (207 counts of 64 values from 62 drawings)	Singapore (182 counts of 35 values from 48 drawings)
fun (21 counts) [ME]	fun (22) [ME]
whole-class interaction (13) [ME]	whole-class interaction (18) [ME]
out-of-class/outdoor learning (10) [ME]	interesting (13) [ME]
challenge (9) [ME]	authenticity (12) [ME]
group interaction (9) [ME]	group interaction (10) [ME]
	manipulatives (10) [ME]
manipulatives (8) [ME]	teacher-centredness (9) [ME]
authenticity (8) [ME]	clarity (9) [P]
quietness (7) [IO]	(teacher) organisation (9) [P]
hands-on (7) [ME]	
competition (7) [ME]	
visualisation (6) [ME]	
practice (6) [ME]	

ME = mathematics educational value
IO = institutional / organisational value
P = (teacher) personality value

Table 1: Values commonly associated with effective mathematics lessons in Melbourne and Singapore.

This allows us to depict what an effective mathematics lesson in Melbourne might look like. With reference to the five most-cited values, an effective mathematics lesson in Melbourne is likely to be one which capitalises on *whole-class interaction* (e.g. drawing ML41), while also tapping on *group interaction*, discussions and collaboration (ML52). The effective mathematics lesson would most likely embrace *fun* (ML35), a value which may be associated with *out-of-class interactions* (ML43). Mathematical games such as ‘Top Dog’ will also infuse an empowering atmosphere of *challenge* (ML35). Of course, this is not to suggest that all these values operated in any one effective mathematics lesson (as the drawings show).

THE SINGAPORE DATA

From the 48 drawings of effective mathematics lessons in Singapore were identified 182 counts of 35 values. The top values identified for the Singapore data were *fun*, *whole-class interaction, interesting*, *authenticity*, *manipulatives*, and *group interaction*. These values constituted 85 of the 182 counts (46.7%) of values identified there. Together with (teacher) *organisation*, *clarity*, and *teacher-centredness*, the 9 values made up 61.5% of all the values associated with effective mathematics lessons in Singapore.

In this light, an effective mathematics lesson in Singapore might be one, say, where real-life knowledge and skills such as the measurement of distance (SL18, valuing *authenticity*) was explored by students in *whole-class* settings (SL11) or in groups (SL23, valuing *group interaction*) with the help of appropriate *manipulatives* such as blocks (SL22). The employment of manipulatives and the harnessing of ICT contributed to the *interestingness* (SL26) of the lesson, and, most importantly of all, the effective lesson was a *fun* experience, such as composing tunes to help in the memorisation of formulae (SL35).

That *teacher-centredness* had been identified so often in Singapore participants' impressions of effective mathematics lessons is noteworthy, given that (Western) research knowledge (e.g. Opdennakker & van Damme, 2006) often relates the learner-centred teaching style with effectiveness. This is understandable, however, considering the influence of Confucian teachings in Singaporeans' value systems, where elders and teachers are accorded respect and authority. As a result, effective lessons in Singapore were often associated with the teacher assuming a central role.

COMPARING THE VALUES IDENTIFIED ACROSS THE TWO CITIES

This study has highlighted several values which were equally significant across Melbourne and Singapore lessons. In fact, there were three common values amongst the most frequently-nominated five or six values within each city, namely *fun* (equal top), *whole-class interaction* (equal second), and *group interaction*. However, socio-cultural realities have meant that any particular value could be enacted in different forms in the two cities. For example, *group interaction* was demonstrated in Melbourne lessons mainly in the way students were seated in class, that is, in groups (e.g. AL21 and AL62). What was absent in any of the Melbourne drawings, however, was the scenario shown in Singapore drawing SL53 where students were normally seated individually in rows, with *group interaction* embodied through the teacher nominating several students to sit with her to complete particular tasks.

Amongst the 12 most cited values in the Melbourne data, one (*visualisation*) was not mentioned in the Singapore data. It is understood, however, that *visualisation* has been highlighted in the 2007 revised Singapore primary mathematics curriculum (CPDD, 2006, p. 6). To what extent does the association of *visualisation* with effective lessons in Melbourne exemplify current institutional focus there on *learning* *for all* (VCAA, 2005) in the context of the last 15 years or so? Perhaps also, how does this reflect culturally different learning styles of students in the two cities?

29 of the values associated with effective mathematics lessons were unique to Melbourne. These included *instant recall*, *mental mathematics*, *repetition* and *proficiency*, all of which were aspects of Singapore mathematics teaching, and reflective of the Confucian heritage culture! That these were not evident in the Singapore data suggests that the Singapore education system and her students are open to looking beyond culturally-embedded ways of pedagogical practice. Singapore's valuing of *teacher-centredness* (as discussed above), however, might imply that any departure from traditional practice would be cautious and thought out.

CONCLUDING REMARKS

This comparative study had sought to investigate what were co-valued by teachers and students in effective mathematics/numeracy lessons in the cities of Melbourne and Singapore, and how these values were similar and different across the two cities. The research design has been unique in that effectiveness in mathematics pedagogy has been interrogated through the perspective of lessons rather than teachers, as well as through participant recall of effective mathematics lessons over an extended period of time rather than over a few lessons in a limited period of time. The use of drawings to articulate mental pictures was also a relatively novel way of data collection.

This study has highlighted how effective mathematics lessons across Melbourne and Singapore value different aspects of pedagogy, education management and teacher attributes, although there were also much similarities amongst the most frequently cited values. These values reveal how the education systems negotiate emerging pedagogical initiatives with traditional practices. There is scope for further research into how these might relate to mathematics performance. These promise to lead to a greater understanding of how values can be harnessed as a form of soft knowledge (Seah, 2008) to complement cognitive and affective knowledge in better facilitating effective mathematics pedagogy and optimal performance in different cultures.

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WHEN LANGUAGE IS TRANSPARENT: SUPPORTING MATHEMATICS LEARNING MULTILINGUAL CONTEXTS

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This paper draws on a wider study conducted in Grade 11 classrooms in South Africa to explore what happens when language is transparent in multilingual mathematics classrooms. Based on an analysis of data collected through lesson observations in a Grade 11 class, we argue the use of language as a transparent resource in multilingual classrooms. Through this learners can gain access to mathematical knowledge while gaining fluency in English, which is presently seen by many parents, teachers and learners as a necessary condition for gaining access to social goods such as higher education and employment.

INTRODUCTION

Teaching mathematics in South Africa to learners who learn in a language that is not their home language is complex. Research shows that teachers and learners in multilingual mathematics classrooms in South Africa prefer that English be used as the LoLT (Setati, 2008). How can the learners' languages be drawn on to support mathematics learning? In this paper we argue for the deliberate, proactive and strategic use of the learners' home languages as a transparent resource in the teaching and learning of mathematics.

We begin the paper with challenging three prevalent dichotomies in research on teaching and learning mathematics in multilingual classrooms. First, is the dichotomy between using English as LoLT as opposed to using the learners' home language(s) as LoLT. Second, is the dichotomy about drawing on socio-political perspectives when analysing interactions in multilingual mathematics classrooms as opposed to drawing on cognitive perspectives. The third dichotomy is about gaining access to mathematical knowledge as opposed to access to English. We then discuss the theory that informed the analysis we present in the paper.

MULTILINGUALISM IN MATHEMATICS EDUCATION

Debates around language and learning in South Africa tend to create a dichotomy between learning in English and learning in the home languages. They create an impression that the use of the learners' home languages for teaching and learning must necessarily exclude and be in opposition to English, and the use of English must necessarily exclude the learners' home languages. In an article published in the Science Africa magazine, Sarah Howie of the University of Pretoria in South Africa,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 65-72. Thessaloniki, Greece: PME.

argued that the most significant factor in learning mathematics is not whether the learners are rich or poor. It's whether they are fluent in English. She insisted, "Let's stop sitting on the fence and make a hard decision. We must either shore up the mother tongue teaching of maths and sciences, or switch completely to English if we want to succeed." (Science in Africa, 2003). This she said drawing on her analysis of South Africa's poor performance in the Third International Mathematics and Science Study of 1995 (see also Howie 2003, 2004). Our argument in this paper is that in a multilingual country such as South Africa the choices are not as simplistic as Howie suggests. Our argument is informed by a holistic view of multilingual learners, which is different from Howie's monolingual view. In our view multilingual learners have a unique and specific language configuration and therefore they should not be considered as the sum of two or more complete or incomplete monolinguals.

The use of the learners' home languages as a transparent resource that we are exploring in this paper is informed by this holistic view of multilingual learners. We accept that the idea of drawing on the learners' home languages during teaching is not necessarily new. The use of code-switching as a learning and teaching resource in bilingual and multilingual mathematics classrooms has been the focus of research in the recent past (e.g. Adler 2001; Barwell 2005; Khisty 1995; Moschkovich 1999, 2002; Parvanehnezhad & Clarkson 2008; Setati, 2005). These studies have argued for the use of the learners' home languages in teaching and learning mathematics, as a support needed while learners continue to develop proficiency in the LoLT at the same time as learning mathematics. All of these studies seem to be in agreement that to facilitate multilingual learners' participation and success in mathematics teachers should recognise their home languages as legitimate languages of mathematical communication. The practical manifestation of the use of the learners' home languages in these studies is through code-switching, mainly to provide explanation to learners in their home languages. In all of these studies code-switching is presented as spontaneous and reactive, the learners' home languages are only used in oral communication and never in written texts. What this paper argues for is the transparent use of the learners' languages in a deliberate, strategic and proactive manner. This is in recognition of the fact that learners want access to English and thus while we draw on their home languages and foreground the quality of the mathematics tasks used during teaching, we also ensure that English is still available to them and they can continue to develop fluency in it.

Past research on multilingualism in mathematics education informed by a cognitive perspective (e.g. Dawe, 1983; Clarkson, 1991) present an implicit argument in support of the maintenance of learners' home languages, and of the potential benefits of learners using their home language(s) as a resource in their mathematics learning. Multilingualism is becoming the norm in many classrooms all over the world, hence the need to consider treating the multilingual learner not only as the norm but also to view his or her facility across languages as a resource rather than a problem (Baker, 1993). Through our work in this study we have come to recognise that separating cognitive matters from the socio-political issues relating to language and power when exploring the use of language(s) for teaching and learning mathematics in multilingual classrooms is not productive. While we accept that cognitively oriented research does not deal with the socio-political issues relating to the context in which teaching and learning takes place, we acknowledge that it is useful in helping us attend to issues relating to the quality of the mathematics and its teaching and learning in multilingual classrooms. In this study we are thus moving against dichotomies, not only of language choices but also of theoretical perspectives.

THEORETICAL UNDERPINNINGS

This study is broadly informed by an understanding of language as “a transparent resource” (Lave and Wenger, 1991). While the notion of transparency as used by Lave and Wenger is not usually applied to language as a resource nor to learning in school, it is illuminating of language use in multilingual classrooms (see Adler, 2001). Lave and Wenger (1991) argue that access to a practice relates to the dual visibility and invisibility of its resources. Invisibility is in the form of unproblematic interpretation and integration into activity, and visibility is in the form of extended access to information. This is not a dichotomous distinction, since these two crucial characteristics are in a complex interplay, their relation being one of both conflict and synergy (Lave and Wenger, 1991: 103). For language in the classroom to be useful it must be both visible and invisible: visible so that it is clearly seen and understood by all; and invisible in that when interacting with written texts and discussing mathematics, this use of language should not distract the learners’ attention from the mathematical task under discussion but facilitate their mathematics learning. This idea is similar to the use of technology in mathematics learning. The technology needs to be visible so that the learners can notice and use it. However it also needs to be simultaneously invisible so that the learners’ attention is not focussed on the technology but on the mathematics problem that they are trying to solve. As Lave and Wenger argue the idea of the visibility and invisibility of a resource is not a dichotomous distinction, it is not about whether to focus on language or mathematics, it is about recognising that the two are intertwined and are constantly in complex interplay.

Lave and Wenger’s concept of transparency was useful in conceptualising language use in multilingual mathematics classrooms. Multilingual mathematics classrooms are characterised by complex multiple teaching demands: the learners’ limited proficiency in the language of learning and teaching (English); the challenge to develop the learners’ mathematical proficiency as well as the presence of multiple languages. The strategy we are exploring is guided by two main principles, which are informed by the theoretical assumptions elaborated in the discussion above. First is the *deliberate* use of the learners’ home languages. We emphasise the word deliberate because with this strategy the use of the learners’ home languages is deliberate, proactive and strategic and not spontaneous and reactive as it happens with code-switching. Second, is that through the selection of real world interesting and challenging mathematical tasks, learners would develop a different orientation towards mathematics than they had and would be more motivated to study and use it (Gutstein, 2003). Many learners in multilingual classrooms in South Africa have what Gutstein (2003: 46) describes as "the typical and well documented disposition with which most mathematics teachers are familiar – mathematics as a rote-learned, decontextualised series of rules and procedures to memorise, regurgitate and not understand". In this study we selected high cognitive demand tasks (Stein, Smith, Henningsen and Silver, 2000), that present real world problems that the learners can find interesting and useful to engage with.

THE STUDY

The study presented in this paper focuses on data collected in Bheki's classroom as part of a wider study on exploring relevant pedagogies for teaching and learning mathematics in multilingual classrooms. In this study Bheki was observed teaching a Grade 11 class in a secondary school in Thembisa, East of Johannesburg. There were 46 learners who were able to communicate in at least four languages and they had the following home languages: Sepedi, Sesotho, IsiZulu and Isixhosa. They were learning English as a subject at second language level as well as their respective home languages as subject at first language level. Data was collected through video recorded lesson observations and individual learner reflective interviews. During the lessons learners were organised into language groups and they were given tasks in two language versions (English and their home language). In this paper we focus only on lesson observation data to explore what happens when language is transparent.

WHEN LANGUAGE WAS VISIBLE AND SIMULTANEOUSLY INVISIBLE

In our analysis we found that when language was transparent learners' interactions were conceptual –focused not only on what the solution is but also why it is correct. In this section we draw on Lesson 2 when the learners engaged with the definition of linear programming, which the teacher introduced in Lesson 1. We do this to illustrate how language typically functioned as a transparent resource during interactions. Below is the task, which learners were working on. It was translated into the four home languages of the learners in the class.

Mandla's cinema hall can accommodate at most 150 people for one show.

a) Rewrite the sentence above without using the words "at most".
b) If there were 39 people who bought tickets for the first show, will the show go on?

c) Peter argued that if there are 39 people with tickets then Mandla should not allow the show to go on because he will make a loss. Do you agree? Why do you agree?
d) What expenses do you think Mandla incurs for one show?
e) Use restrictions to modify the statement above in order to make sure that Mandla does not make a loss.
f) If Mary was number 151 in the queue to buy a ticket for the show, will they accommodate her in the show? Explain your answer.

This task generated a lot of interaction in groups and also during the whole class discussion because the mathematical solution that the learners thought to be correct was not consistent with what they had understood Linear programming to be about. During Lesson 1, Bheki defined Linear programming as the maximization or minimization of a specific performance index, usually of an economic nature like profit, subject to a set of linear constraints. He further indicated that for this exercise to qualify as linear programming the performance index should also be linear. Throughout Lesson 1, Bheki emphasised the fact that linear programming is thus about minimising the losses and maximising the gains. During Lesson 2 learners drew on this information while working on the above task. In the extract below the learners are engaging in a whole class discussion with the teacher on the answer to question b).

242 Maseko: Sithe [We said] the show will not go on because he will not benefit profit from thirty nine people

253 Bheki: Akesizwe isipedi kucala, yes [Let us give a chance to Sepedi group first].

254 Errol: …Hundred and fifty kuya phansi, akanawumela Hundred and fifty kuya phezulu [150 and below, cannot wait for 150 or above].

255 Bheki: Okay, okay, alright okey akebajusifaye ipoint labo okay [Okay, let them justify their argument].

256 Mkhonza: Mawudefina iloku LP ithini idefinition ye LP. [When defining LP, what is the definition of LP?]

260 Errol: Ithi [It says] maximising the profit and minimising the loss, so nami ngiz…[I will also…]

262 Mkhonza: Wena nawubhekile lomuntu lo nakungena abantu abawu thirty-nine kumele ucabange ukuthi le I-one show. Nakungena abantu abayi-thirty-nine akusiyo nehhafo ka hundred and fifty. Mara iprofit izophuma kanjani nakungasiyo nehhafu ka hundred and fifty. Akusiyo hehhafu, nehhafu ka hundred and fifty. [When you look at this person, when 39 people get into the cinema hall, how is he going to make profit from 39 people since 39 is not even half of 150, it is not even a quarter of 150?]

What is emerging in the extract above is the fact that according to the problem Mandla's cinemas can accommodate a maximum of 150 and so what this means mathematically is that the number of people in Mandla's cinema cannot go beyond

150 but it is allowed to be lower than 150. That is if x is the number of people in Mandla's cinema then $1 \leq x \geq 150$. This is the explanation that Maseko and Errol are using for their argument. Mkhonza on the other hand is drawing his argument from the definition of linear programming which Bheki emphasised in Lesson 1. This argument continued through the lesson as Bheki found out what other groups were thinking and also challenging them to think about each other's arguments and not just to focus on why their argument is correct.

276 Sipho: Ibinessman inama risk [A business man has risks] if you are in a business you are going to take the risk. Even when the business… may be kune [there is])…(learners saying yes)

282 Xhakaza: Abantu nakudlala ibhola abantu baya estimate itulu zining. Kunga fika abantu abayi2 ibhola iyachubeka. [When there is a soccer match in a stadium the game continues even when they are two spectators]

What is evident in the discussion above is the fact that learners are trying to reconcile what they understand to be a mathematically correct answer, their understanding of what linear programming is about and when to apply that and also their understanding of how things work in the everyday world. As Xhakaza explains in utterance 282 above, the situation in the problem is similar to what happens in soccer matches held in big stadiums, where even if not many people came to the stadium the game would still go on as scheduled.

Throughout the discussions the learners did not refer to language either as being a resource or a problem. They went on with the mathematics problem as if the language is not there, hence our argument that language functioned as a transparent resource.

CONCLUSION

While language is a resource that can help advance mathematics learning, it can also be a stumbling block for successful learning depending on how it is used. The major challenge in multilingual contexts such as South Africa is the fact that while the power of English is unavoidable, many learners do not have the level of fluency that enables them to engage in mathematical tasks set in English. In this paper we argued for the use of learners' languages as a transparent resource. This argument recognises the political nature of languages. The fact that while the learners' home languages are drawn on they are not presented as being in opposition to English rather as working together with English to make mathematics more accessible to the learners. The question that emerges from this study is whether language is only a resource when it is transparent.

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EXPLORING THE CO-CONSTRUCTED NATURE OF A "GOOD" MATHEMATICS LESSON FROM THE EYES OF LEARNERS

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This paper reports on the analysis of students' views on what constitutes a "good" mathematics lesson. The methodology employed in the current study offered teachers and the students the opportunity in post-lesson video-stimulated interviews to identify those events in the lesson that they felt to be significant and to comment on what is a "good" lesson for them. Post-lesson interviews were conducted and analysed with sixty students in three eighth-grade mathematics classrooms in Tokyo. The analysis reported in this paper reveals that, though there is certain commonality in students' views on what should be happened in a "good" lesson, they tend to construct different meanings associated with the same lesson they have experienced. The co-constructed nature of mathematics lesson has emerged through the eyes of the learners.

INTRODUCTION

The Learner's Perspective Study *(LPS,* Clarke, Keitel & Shimizu, 2006) is an international study of the practices and associated meanings in 'well-taught' eighth-grade mathematics in participating countries. One of the goals of LPS is to document and analyse the data from the mathematics classrooms in participating countries for elucidating findings that eventually complement those of other international studies of teaching practices such as TIMSS Video Study (Stigler & Hiebert, 1999; Hiebert et al., 2003). Given the fact that teaching and learning are interdependent activities within a common setting, classroom practices should be studied as such. The LPS has the potential in that it literally focuses on the perspectives of the *learners* in mathematics classroom as well as those of teachers.

The methodology employed in the LPS offered the teachers and the students the opportunity in post-lesson video-stimulated interviews to identify for the interviewer those events in the lesson they had just experienced that the participant felt to be significant. The teacher and the students were asked to identify and comment upon classroom events of personal importance (Clarke, 2006). The analysis of LPS data has revealed both pattern and variation in the ways in which the teacher and students perceive the lesson (Hino, 2006). The LPS then provides the researchers with the opportunity to explore the commonalities and differences in perceptions of mathematics lessons by teachers and students by means of juxtaposing their reconstructive accounts of the classroom (Shimizu, 2006).

As a part of the larger study, this paper reports on the analysis of post-lesson video-stimulated interviews with the students in three eighth-grade mathematics classrooms in Tokyo that participated in the LPS. The transcriptions of interviews with

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 73-80. Thessaloniki, Greece: PME.

sixty students were analysed with a focus on their views on what constitutes a "good" mathematics lesson. More specifically, the current study aims to address the following questions: (1) how can we characterize the students' views on a lesson counted as "good" one to them? (2) To what extent the co-constructed nature of a "good" mathematics lesson can be identified through the analysis from the learner's perspective?

EXPLORING CONCEPTIONS ABOUT WHAT CONSTITUTES A "GOOD" MATHEMATICS LESSON

Exploring students' views on mathematics lessons will open a window through which we can examine values held by students in the context of teaching (Bishop, Seah & Chin, 2003). If we consider teaching as *cultural activity*, we need to look into what participants in the community of practice, both the teacher and students, value in the classroom and how they perceive lesson events with associated values embedded in cultural activities in classroom. Previous international studies of mathematics classroom have identified coherent sets of actions, and associated attitudes, beliefs and knowledge, that appear to constitute culturally-specific *teacher practices* (Stigler & Hiebert, 1999). The larger international study, of which this study is a part, hypothesizes that there is also a set of actions and associated attitudes, beliefs, and knowledge of students that constitute a culturally-specific coherent body of *learner practices* (Clarke, Keitel & Shimizu, 2006). Given the fact that teaching and learning are interdependent activities within a common setting, classroom practices should be studies as such (Carpenter & Peterson, 1988). The task of accounting for successful instruction is not one of explaining how students take in and process information transmitted by the teacher but explaining how students actively construct knowledge in ways that satisfy constraints inherent in instruction (Cobb, 1988).

The stimulated recall interview is one of the methods for exploring teachers' ideas and beliefs about teaching and learning (Clark & Peterson, 1986). A videotaped lesson as stimulus in the interview has been used in mathematics education research area. The entire videotaped lesson was, for example, used for examining American and Japanese teachers' ideas about what constitutes effective mathematics pedagogy (Jacobs & Morita, 2002). Also, video-stimulated recall interviews were conducted with the teacher and group of students to seek their interpretations of videotape excerpts as the events occurred in a classroom community of inquiry (Goos, 2004).

Although video-stimulated interviews are used in examining teachers' and students' ideas and beliefs, earlier studies have no focus on contrasting conceptions between the teacher and the students of the same lesson they experienced. By contrasting their perceptions of the same lesson, it is possible to identify the discrepancies between the teacher and the students in their perceptions of that practice (Shimizu, 2006). The current study mainly focuses on their views on what constitutes a "good" mathematics lesson to explore the characteristics of mathematics lessons.

DATA AND METHODOLOGY

The technique for undertaking this study involved the development of complex "integrated data sets" that combined split-screen video records of teacher and students with transcripts of post-lesson interviews and copies of relevant printed or written material (Clarke, 2006). Data collection for the current paper was conducted at three public junior high schools in Tokyo. The teachers, one female and two males, roughly represented the population balance of mathematics teachers at the school level in Japan. The topic taught in each school corresponded to three different content areas prescribed in the National Curriculum Guidelines; linear functions, plane geometry, and simultaneous linear equations.

The methodology employed in this study offered both the teachers and the students the opportunity in post-lesson video-stimulated interviews to identify for the interviewer those events in the lesson that the participant felt to be significant. The teacher and the students were given control of the video replay and asked to identify and comment upon classroom events of personal importance. Semi-structured post-lesson video-stimulated interviews, which occurred on the same day as the relevant lesson, included such prompts as follows.

> Prompt Four: Here is the remote control for the video-player. Do you understand how it works? (Allow time for a short familiarization with the control). I would like you to comment on the videotape for me. You do not need to comment on all of the lessons. Fast-forward the videotape until you find sections of the lesson that you think were important. Play these sections at normal speed and describe for me what you were doing, thinking and feeling during each of these videotape sequences. You can comment while the videotape is playing, but pause the tape if there is something that you want to talk about in detail.
>
> Prompt Seven: Would you describe that lesson as a good one for you? What has to happen for you to feel that a lesson was a "good" lesson? Did you achieve your goals? What are the important things you should learn in a mathematics lesson?

Interviewers were supposed to be explicit during the interviews in specifying the point to which the teachers and the students referred. It is clearly possible that students identify significant classroom events quite differently from those intended by the teachers. The analysis that focused on Prompt Four was reported elsewhere (Shimizu, 2006) and this paper focus on the analysis of (the teachers) and the participants' response to Prompt Seven.

The transcriptions of post-lesson interviews with sixty students, twenty students from each three schools, were analysed. For the analysis of the interview data, a coding system was developed. The coding categories emerged from the preliminary analysis of one of the three schools (J1) and then applied to all the data set. Table 1 shows the description of each coding category with an illuminating example of it. It should be noted that these codes do not constitute a mutually exclusive coding system.

Codes	Description	Example (School-Lesson, student)
Understanding/ Thinking	Those responses that refer to understanding and thinking in the classroom	*I can understand the topics to be learned. (J2-L03, M)*
Presentation	Those responses that refer to presenting their ideas in the classroom	*I can present my solution on the blackboard. (J3-L07, I)*
Classmates	Those responses that refer to other students' explanation	*There is an opportunity of listening to classmates (J1-L09, S)*
Whole class discussion	Those responses that refer to whole class discussion	*We all in the classroom exchange ideas actively (J1-L06, U)*
Teacher	Those responses that refer to teacher's explanation	*I listen to teacher's final talk, I always take a note and check a point. (J3-L06, S)*
Other	Other responses	*By preview the topic at home, I attend the lesson with a preparation. (J3-L09, K)*

Table 1: The categories for coding and their illuminating examples

RESULTS

Students' views on a "good" lesson

Table 2 shows the result of the analysis as a whole of students' response to the prompt seven in video-stimulated interviews. It should be noted that the percentages do not add up to 100, because of the nature of coding system.

Codes	Responses	
Understanding/Thinking	27	45.0%
Presentation	10	16.7%
Classmates	4	6.7%
Whole class discussion	16	26.7%
Teacher	10	16.7%
Other	10	16.7%

Table 2: Students' response to the stimulated recall interview

As Table 2 shows, nearly half of the students interviewed (45.0%) described certain learning activities that were related to "understanding/thinking" to be happened in a "good" lesson. As the example in the Table 1 illustrates, those students who described learning activities related to "understanding/thinking" seemed to attach values directly to the importance of their own thinking and understanding in the lesson. The example, "I can understand the topics to be learned", suggests that the students regard a lesson as "good" one, if he can have a clear understand of mathematics topic taught in the lesson. However, many students in this category also referred to other activities in the classroom, as the case of MANA from J2 who mentioned to teacher's explanation as the object of understanding: "*Even if your answer is wrong…to be able to understand what the teacher explained. If that happens, I think//that it was a good lesson*."

Roughly a quarter of the students (26.7%) identified "whole class discussion" as the "component" of a "good" lesson. Then, two categories "presentation" and "teacher" follow it. Only four students (6.7%) explicitly described the activities related to their "classmates" in mathematics classroom. There is a difference between the first four and "Teacher" categories in terms of types of the activities referred by the students. That is, first four categories are directly related to students' own learning activity, while category "teacher" is related to both learning and teacher's activities. The example of "Teacher" in Table 1, for instance, is the one that refers to the teacher's summarizing the main point, taking a note, and checking the key point of lesson.

Table 3 shows the same result by schools. Table 3 reveals that in the interview students in each school referred to activities related to the category "Understanding/Thinking" with higher frequencies, though there are some differences in students' response among three schools. Half of the students interviewed at school J1, for example, described activities related to "whole class discussion" as the component of a "good" lesson, while only one students at school J2 did that.

Code	J1	J2	J3	Total
Understanding/Thinking	8	13	6	27
Presentation	2	4	4	10
Classmates	1	1	2	4
Whole class discussion	10	1	5	16
Teacher	2	6	2	10
Other	6	2	2	10

Table 3: Students' response to the question on a "good" lesson by schools

Relating teacher and learner perspectives

To understand the charcteristics of a "good" mathematics lesson, a detailed analysis was also conducted with an eye of comparing the responses to Prompt Seven between the students and the teacher in the same classroom.

Suzu, a student from the school J3, for example, responded to questions, "When you think it's a good class?" and "What should happen in the class?", as follows.

01. INT: When you think it's a good class,

02. SUZU: Yes.

03. INT: What should happen in the class?

….

04. INT: Do you have anything that you think is a good class?

05. SUZU: I can present my answer, and then listen to my friend's way as well,

06. INT: Yeah?

07. SUZU: The teacher's final comment, or answer,

08. INT: Yeah?

09. SUZU: Listen to it carefully, and to make a good note from it,

The student clearly mentioned to the importance of presenting his answer to the problem to the class and then listing to his friend' way to solve the same problem. He also referred to listening to "*The teacher's final comment, or answer*" carefully and of "*making a good note from it*". These interview data suggest that, students' views on a "good" lesson are shaped through the co-construction of classroom practice. If the teacher keeps summarizing and highlighting the main points of the lesson as a daily routine, the students may become aware of the importance of the particular lesson event which tends to come on the final phase of lesson in the form of teacher's public talk together with time for note-taking. The teacher's summarizing and highlighting, in turn, have to rely upon students' understanding of the mathematical topic taught and to be summarized and highlighted.

Teachers' comments on what constitutes a "good" lesson also suggest the co-constructed nature of a "good" mathematics lesson. Mr. K, the teacher of JP3, mentioned to the importance of "understand" in a "good" lesson as follows: "*practically, what I think is that the students think in many ways...and they understand it well ...The students can ask me or each other where they can't understand*."

Mr. N, the teacher of JP2, on the other hand, referred to a shared goal between the students and himself in a "good" lesson: "*the best lesson is where the teacher and the students can both agree that today's lesson was a good lesson. Um, not just one side but, well this is for the students so of course it's good if the students say it's good. Of course that's good but from a teacher's perspective, if the students assessed something else as good and ignored what the teacher wanted them to understand the most, that's. That sort of class really doesn't have much meaning so I think if the goal of the teacher and the students are the same, that's the best thing".*

As these comments suggest, in the classroom, teacher and student practices can be conceived as being in a mutually supportive relationship. This is not to presume that teacher and students have the same goals or values, or even that they perceive the importance of particular classroom activities in the same way.

DISCUSSION

The analysis of post-lesson video-stimulated interviews with the students reveals that, although there is certain commonality among students' views on what should be happened in a "good" lesson, they tend to construct different meanings associated with the same lesson they have experienced. In fact, the results of the analysis showed that students in each school referred to the activities related to the category "Understanding/Thinking" most frequently, though there were some differences in students' response among three schools. Although those students seemed to attach values directly to the importance of their own thinking and understanding in the lesson, many of them also referred to other activities in the classroom. Thus, "elements" which constitute a "good" lesson from the eyes of learners can be conceived as related together.

The word "good" in the prompt in the post-lesson interview may not be a sufficiently neutral to the teacher and students. The specific term used by them should be chosen to be as neutral as possible in order to obtain data on those outcomes of the lesson which the student values. It is possible that these valued outcomes may have little connection to "knowing", "learning" or "understanding", and that students may have very localized or personal ways to describe lesson outcomes (Clarke, 2006, p.33). This was not the case with most students interviewed in the study. Their descriptions had strong connections to "understanding" and "thinking" and included classroom activities as co-constructed in nature. Also, students interviewed described learning and teaching activities related to the lesson events within "structured problem solving" as the "components" of a "good" lesson. Two categories "presentation" and "whole class discussion" can be regarded as directly related to the part of so-called "structured problem solving" in classrooms (Stigler & Hiebert, 1999). Those students who described activities related to category "teacher", as the case of student SUZU, refers to teacher's highlighting and summarizing the main point as well as their own learning activities.

The results of the current study suggest that values held by Japanese students in the context of teaching mathematics in a "good" lesson directly related to the way in which mathematics lessons are structured and delivered with an emphasis on students' thinking in teaching and learning process as intended by Japanese teachers.

CONCLUSIONS

This paper approached to the co-constructed nature of mathematics lesson from the perspective of learners. Insights into the co-construction of classroom practice can be provided by both similarities and differences in participant perspectives on the same classroom events. In the classroom, teacher and student practices can be conceived as being in a mutually supportive relationship. This is not to presume that teacher and students have the same goals or even that they perceive lesson events in the same way. We need to understand each participant's contribution to the activities of the mathematics classroom. Further studies are needed to explore such participant's contribution to co-constructed meaning in the classroom.

Footnotes

The research reported in this paper was supported by the research grant (grant ID: 13680195) by the Japan Society for the Promotion of Science.

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AN ANALYSIS OF PROCESS OF CONCEPTUAL CHANGE IN MATHEMATICS LESSONS: IN THE CASE OF IRRATIONAL NUMBERS

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This paper reports empirical part of an ongoing research into the conceptual change in the teaching and learning of mathematics, following the theoretical part (Shinno & Iwasaki, 2008). It focuses on the concept and/or learner's conception of irrational numbers. The purpose of this paper is to analyse process of conceptual change in mathematics lessons dealing with irrational numbers. The result suggests the understanding of incommensurability, that is an essential aspect of irrational number, is inevitable for conceptual change in the development from rational to real number concept. Some didactical implications are also discussed.

THEORETICAL BACKGROUNDS

The theory of conceptual change (from now on, TCC) was originally developed by drawing on the philosophy and history of science, in particular Thomas Kuhn's account of theory change (Kuhn, 1962). In mathematics and science education fields, it has been mainly used to explain learner's knowledge acquisition, in particular for characterizing drastic reorganization of prior knowledge. In this paper, first of all, we should like to mention some theoretical backgrounds of TCC as preliminary considerations for the analysis of conceptual change in mathematics lessons.

The prescriptive and descriptive role of theory of conceptual change

The role of theory in mathematics education differs in different research traditions (Bishop, 1992). One of the considerable philosophical or theoretical stances depends on the prescriptive or descriptive roles of theory. As a matter of fact, these two theoretical stances can stand out as being very important on relationship between research and practice in mathematics education (e.g., Bishop, 1992; Silver and Herbst, 2007). Broadly speaking, the descriptive role of theory can be a tool in order to analyze a "phenomenon". On the other hand, the prescriptive role of theory can be a principle in order to guide a "phenomenon". According to Bishop (1992), "both can learn from the other, of course, and the appropriate balance must be struck by every researcher, depending on the their social and political situation" (p. 715). In this way, these two roles can be seen as the different facets of the same theory, though not referring to all theories in mathematics education.

Let us mention research situation on TCC in terms of two different facets of the theory. TCC has been pursued in recent years as the approach to mathematics learning and teaching (e.g., Vosniadou, 2006), however, there is little argument oriented to the

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prescriptive role theory, much less the interplay between two different roles of the theory. Therefore in our previous work (Shinno & Iwasaki, 2008), we emphasized the prescriptive role more explicitly and made an illustration of TCC with the help of preliminary analysis of irrational numbers. Following these attempts, what we wish to show in this research report is the descriptive role of TCC and the interplay between two different roles of TCC.

Domain specificity and conceptual change

TCC is a constructivist approach that rests on certain epistemological assumptions, such that knowledge is acquired in *domain-specific* (Vosniadou, 2007). This is an important feature of TCC, as can be seen in the following quotation:

> Most theories of learning and development, such as piagetian and vygotskian approaches, information processing, or sociocultural theories are *domain general*. They focus on principles, stages, mechanisms, or strategies that are meant to characterize all aspects of development and learning. In contrast, the conceptual change approach is a domain-specific approach. It examines distinct domains of thought and attempts to describe the processes of learning and development within these domains. […] Domain-specific approaches should be seen as complementary rather than contradictory to domain-general approaches. (Vosniadou, 2007, pp. 48-49)

This may suggest that we need to take the specificity of mathematical knowledge into account, for characterizing conceptual change in our field. Our research concern is to inquire domain specificity by focusing on particular mathematical concept: irrational number. In fact, notwithstanding serious issues about understanding the concept of irrational number (e.g., Sirotic & Zazkis, 2007), only few research papers have so far been made at the teaching and learning of irrational numbers.

A priori components for an analytical framework

Based on the epistemological considerations of irrational numbers in authors' previous works (Shinno, 2007; Shinno & Iwasaki, 2008), we could identify three a priori components (two components and one meta-component) of learner's status of knowing in the teaching and learning of irrational numbers as follows[1]:

S: symbolic notation of number
A: attitude towards the notion of infinity
N: nature of knowledge of number

S is the component for demarcating the range of learner's prior knowledge. It is concerned not only with the use of the root sign as a mathematical symbolism, but with the decimal notation and/or infinite decimals. **A** is the component referring to an essential aspect of irrational numbers, that is the incommensurability or irrationality,

[1] The labels of the components have been partly modified. The meanings of these are preserved.

because the incommensurability[2] both conceptually and procedurally involves the notion of infinity. **S** and **A** are the components respectively related to the aspects of form and content of knowledge. **N** is the meta-component about an ontological view of number, for identifying whole characteristic of knowledge of number.

In the following section, with the help of a priori components above, we attempt to analyze a series of lessons and to identify learner's conception.

DESCRIBING AND ANALYZING A SERIES OF LESSONS

Target and method

The lesson observations were conducted in one of the ninth grade mathematics classrooms at a lower secondary school attached to a certain national university in Japan. The target of analysis in this report is the ordinary mathematics lessons of "square root of numbers". A series of lessons, which was planned on 15 lessons, were videotaped. The outline of the teaching unit "square root of numbers" is listed as follows. (L*n* denotes the *n* th lesson)

L1: Quadratic equation
L2: Existence of square root
L3: A number that cannot be expressed as a ratio of integers (fraction)
L4: Root sign (√)
L5: Ordering the square root of numbers
L6: Magnitudes of square root of numbers
L7: Prime number and prime factorization
L8: Multiplication and division of square roots (1)
L9: Simplifying the expressions including square roots
L10: Rationalizing the denominator
L11: Multiplication and division of square roots (2)
L12: Prime factorization and divisor
L13: Addition and subtraction of square roots (1)
L14: Addition and subtraction of square roots (2)
L15: Various calculations including square roots

Phase 1

As phase 1, we describe and analyze the introductory situation (L1, L2), in which learners encounter "square root of 10" as a new kind of number and are asked to reflect their well-known numbers.

In the context of solving the quadratic equation, teacher posed the problem "let's study the solution of $x^2 = 10$", and students tackled the problem by successive approximation

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[2] The "incommensurability" means the relationship between two magnitudes such that there is no common measure (unit). This notion is originated from Ancient Greek: the Euclid's Elements Book states that "*those magnitude are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure*" (*Vol. X, Def. 1*) (Heath, 1956).

with the calculator. A student said "3.1622777" as the solution (because, on the calculator, the square of "3.1622777" displays "10"), and then another student argued, "Since the square of the last digit is not 0 but $7 \times 7 = 49$, it is wrong". Next episode, extracted from L1, implies that (infinite) decimal numbers can play an important role for the "problematic" situation:

1 T: Please stop calculating with your calculator. It seems your calculator has a limitation. Now, can you think how long this number continues?

2 S1: Infinity.

3 T: Does it continue infinitely? Really? Can you explain the reason for that? I want you to think whether it continues infinitely or not if we use a better computer.

4 Ss: Yes.

5 T: Can you explain the reason why it continues infinitely?

6 S2: Even if the same numerals from1 to 9 are multiplied each other, it does not make 0.

7 T: If it stops somewhere, the product should be 0.

The teacher asked, "If it continues endlessly, what is the solution of $x^2 = 10$?", and a student responded, "It cannot be represented as the specific numeral, [???] but it can be found to a certain extent". The next lesson (L2) started with the question "does the solution of $x^2 = 10$ exist?", and then the teacher and students recognized that it exists as the length of a side of the square having the area 10. After introducing the term "square root", the question "what is the difference between the square root of 10 ($\pm$ 3.16227766…) and numbers that we have already learnt?" (Question 1) was posed.

The familiar notation of number, such as place value system of decimal notation or fraction, cannot represent the square root of 10 precisely. The above episode suggests that S2 explained that decimal notations continued indefinitely. Here we can identify the learner's conception as "*comprehending the applicability of prior knowledge*". As long as **S** is considered as representing certain quantity (magnitude), it can be *operational* (cf., Sfard, 1991). In this sense, **S** is seen as the relationship between "number" and "quantity". **A** can be rather *empirical/implicit*, if the notion of infinity has never been dealt with at formal instruction. Namely, for learners, the notion of infinity may reduce to being "indefinite" or "endless". And the knowledge of number, that learners have developed, consists in the concrete or existential. Underlying such ontological view of number, **N** can be characterized as *reality*.

Phase 2

As phase 2, we describe and analyze the challenging situation (L3), in which the teacher shows the proof that indicates that the square root of 10 cannot be expressed as fraction nor repeated decimal numbers.

After the question 1 posed in the L2, in the L3, students explored the reason of the statement "any fractions can be reduced to finite or repeated decimal number". This task could be tackled by focusing on the remainders in the process of long division algorithm. A student took 20/17 as an example and explained as follows.

8 S3: In fraction, in this case, if 20÷17 has a reminder, that should be from 1 to 16. Then, if all of these numbers are found, …

9 T: 1 to 16? Not 0 to 16?

10 S4: Oh, I see. They are limited, so if all of them are found, definitely one of them will appear again. Perhaps in the case of recurring decimal numbers, when we get the same reminder, the quotient will start to repeat.

Another student was aware of one pattern repeated in the process of long division algorithm, and explained the reason of being *repeated*. Then, the teacher and students inductively recognized the fact that repeated decimal number could be reduced to fraction, by finding 0.1=1/9, 0.01=1/99, 0.001=1/999 and using these results.

Following this episode, in the L3, the teacher posed the question "Is 'square root of 10 (±3.16227766…)' repeating?". For answering this question, the teacher showed the proof by contradiction (*reductio ad absurdum*), which gave the conclusion that the square root of 10 cannot be expressed as fraction nor decimal number.

We suppose that the square root of 10 is expressed as fraction b/a. (b/a is irreducible)

$3.1622 ☹ = \frac{b}{a}$

Square both sides.

$$10 = \frac{b^2}{a^2} = \frac{b \times b}{a \times a}$$

Because $\frac{b^2}{a^2}$ is irreducible, $10 = \frac{b^2}{a^2}$ is not true.

Therefore the supposition $3.1622 ☹ = \frac{b}{a}$ is wrong.

Therefore the square root of 10 is not expressed as fraction.

Figure. 1: Teacher's proving on the blackboard

The square root of 10 is a number that cannot be expressed as fraction, and that means a number that students have never learnt. In this way, we can identify the learner's conception as "*becoming aware of the limitation of range of prior knowledge*". What is important here is the context that the teacher shows the proof. Another important point to note is that the teacher has not yet introduced, in the L3, the root sign, the term "rational number" and "irrational number". Hence, in this phase, **S** is still *operational* on the one hand, **A** can be relatively *logical/explicit* on the other hand, because of learner's being aware of the recurrent nature of repeated decimal number and being able to explain the reason. But **N** is still characterized as *reality*; although the teacher showed the irrationality by the formal proof (*reductio ad absurdum*), it is difficult for learners to understand the irrationality or incommensurability in connection with the notion of infinity.

Phase 3

As phase 3, we describe and analyze the process of introducing new terms and root sign (L4, L6), in which number concepts may develop from rational to real number.

The 4th lesson started with the teacher's question "What can be the square root of 10?", which intended to promote student's reflection of last lesson. Students responded in different ways; for example, "3.[…]", "not good number", "cannot be expressed as fraction", "cannot be expressed as decimal number", "(the square root of 10 is) nothing", "endless". After introducing the term "rational number" and "irrational number", the teacher summarized the relationship between different kinds of numbers and introduced root sign.

In order to see the magnitude of square root, in the 6th lesson, the representation of number line was used. By focusing on the construction of number line, the density of rational number was intuitively dealt with, such that the unit can be equally divided by 10, 100. Then the teacher posed the question about the correspondence between numbers and points on the number line:

11 T: Well…, that question is the same as the following question. Can we fill up the number line with fractions? If we keep dividing the number line, we can find another fraction to be filled up with, right?

12 S5: [??]

13 T: If we continue dividing like 1/10 and 1/100, … we can find such as 2/100 and 9/100. In this way, we keep on doing it. Is it possible to fill up the number line with fractions completely? Discuss it with your friend next to you.

A few students answered this question intuitively. Another student questioned as "how about the number that cannot expressed as fraction?". This student's question was chance for the teacher to introduce the term "real number". Here the number line could evolve, though not complete, to the model of real number.

14 T: There are irrational numbers, too. […] So we cannot fill up the number line only with fractions. There are irrational numbers between fractions and we can completely fill up the number line with irrational numbers and fractions. That may sound difficult, but the conclusion is it is impossible. We call rational numbers, that are fractions, and rational numbers real numbers, and the number line can be filled up with them. We have ever dealt with the number line, but irrational numbers exist there in fact.

Through a series of activities for understanding the order and operations of square root, it is expected that new knowledge can be constructed. If new knowledge of number can be integrated into the prior knowledge, we can identify learner's conception as "*reconstructing the prior knowledge*". The root sign as a new mathematical symbolism implies arithmetic operations, rather than representation of certain quantity (magnitude), so it is a computational object that can be processed formally. **S** can be *structural* (Sfard, 1991). Irrational numbers with root sign can be used as a computational object like the use of literal symbol in algebra. Generally speaking, it is likely that the computations with irrationals can be foreground and performed without reflecting on the incommensurability. In this way, since **N** is still characterized as *reality*, this phase suggests an intermediate or quasi status for conceptual change that the **S** can appear to be *superficially* structural.

DISCUSSIONS AND DIDACTICAL IMPLICATIONS

As a result of lesson analysis, we may say that a priori components of learner's status of knowing (**S**, **A**, **N**) can function as a descriptive-analytical framework for conceptual change in the teaching and learning of irrational numbers. In the previous section, we attempted to characterize learner's status of knowing in the process of change (see Figure 2), and to identify learner's conception according to the three phases; 1) comprehending the applicability of prior knowledge, 2) becoming aware of the limitation of range of prior knowledge, 3) reconstructing the prior knowledge.

	S	A	N
Phase 1	*operational*	*empirical*	*reality*
Phase 2	*operational*	*logical*	*reality*
Phase 3	*structural*	*logical*	*reality*

Figure. 2: an analytical framework for conceptual change (characterized)

We are now ready to assess the process of conceptual change from the prescriptive point of view. In this concluding section, with the help of the characterized framework (Figure. 2), the authors discuss the following two points as didactical implications towards designing for conceptual change. In particular, our discussion points here are concerning that the component **N** in Figure 2 has not undergone shift.

Dealing with the infinite decimal numbers

In our interpretations, a series of lessons from the phase 1 to 3 were intended the teaching and learning activities for the evolution of number concepts from rational to real number. In these activities we think that the infinite decimal numbers can be crucial merkmal for capturing student's conception. An important feature of rational numbers is that we can 'see' the symbolizing of these numbers by the decimal notations 'in infinity'. In the case of irrational numbers, although the non-repeated decimal numbers are capable of symbolizing such as rational numbers, we cannot see the written form of all numerals in these decimal notations (Wilder, 1968). The root sign was introduced, in the L4, in line with the teaching situation that the square root of 10 cannot be expressed as fraction. The use of root sign implies arithmetic operations, rather than representation of certain quantity (magnitude). The figure 2 suggests that the relationship between the *operational* and *structural* natures of the symbolic notation of number is not dichotomic rather complementary (Sfard, 1991).

Context for understanding the irrationality

Understanding the irrationality is the issue of essential character of the concept in question. The proof by contradiction (*reductio ad absurdum*) can function as a didactic tool for understanding the irrationality (incommensurability). However it involves a considerable didactic difficulty: many students may encounter a kind of cognitive obstacle because of the formal logic of indirect proof. What is more important is on the shifting on the nature of knowledge of number (**N**): from *realistic* nature of knowledge of number relied on the concrete or existential, to *idealistic* nature of knowledge of number relied on the reasoning or thought. The figure 2 referring to the component **N** suggests the understanding of incommensurability is inevitable for conceptual change. In order to resolve this didactical problem, we focus on the Euclid algorithm as another didactic tool for understanding the irrationality. Because the Euclid algorithm can function as a direct proof of irrationality, and it may enable learners to become aware of irrationality through the operative activity (e.g., Iwasaki, 2004; Shinno, 2007).

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PROSPECTIVE TEACHERS' ENGAGEMENT IN PEER REVIEW AS A MEANS FOR PROFESSIONAL DEVELOPMENT

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Professional development" in the context of prospective mathematics teachers is a comprehensive term that encompasses various types of knowledge, one of which refers to assessment methods. The aim of the present study is to examine the effects of engaging prospective mathematics teachers in peer review on the development of their awareness both to the complexity of the assessment process and to the development of their own evaluation skills. Analysis of the research data reveals that the prospective teachers had difficulties generating categories for evaluating their peers' work. They also realized that assigning grades without providing justification or explanation is ineffective. The evaluation process exposed the prospective teachers to different solutions and consequently helped them improve their own mathematical work.

INTRODUCTION

Unfolding the term "professional development" in the context of mathematics prospective and in-service teachers reveals a wide variety of different aspects, such as development of content, didactic and cognitive knowledge, and knowledge about class management (Shulman, 1987). Evaluation of students' work is one of the main responsibilities of teachers. As such, developing the awareness of prospective teachers (PTs) to the various aspects associated with evaluation and helping them acquire the required skills are essentially an integral part of the process of their professional development. The importance of qualifying PTs to evaluate students' work stems from the fact that the way teachers evaluate their students can either reinforce or undermine the learning process. Instructors are therefore obliged to consider the implemented evaluation approaches carefully and adjust them to their teaching and learning goals. Various assessment methods are customary in the field of education. Nevertheless, observing educational systems reveals that exams are the dominant assessment approach both in schools and in higher education. Being aware of the benefits of the peer review approach, we believe that engaging PTs in peer review activity can support their professional development by exposing them to the complexities of evaluating their future students. In this paper we describe part of a study designed to examine the effects of experiencing peer review on the professional development of PTs.

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THEORETICAL BACKGROUND

In this section we provide a brief literature survey of peer review (PR), its benefits and shortcomings, and the roles of PR in teachers' professional development.

Peer review. PR is a form of innovative assessment (McDowell and Mowl, 1996). It is a process used to examine the work performed by one's equals (termed peers) to ensure it meets specific criteria and to enhance its quality. In the case of PTs, PR is the evaluation of PTs by other PTs from the same group. PR is primarily a formative evaluation process in which participants work collaboratively to strengthen a product (Keig and Waggoner, 1994). The goal of the PR processes is to verify whether the work satisfies the acceptable standards, as well as to provide constructive feedback which includes suggestions for improvements (Herndon, 2006). The use of PR is based on the assumption that peers can recognize each other's errors quickly and easily, and that a larger and more diverse group of people might find more weaknesses and errors in a work. The assessment can therefore either be formative, aimed at providing feedback, or summative, aimed at grading students' works. In their literature review, Falchikov and Goldfinch (2000) point to the increase in students' involvement in assessment across the spectrum of discipline areas. This approach, however, is rarely implemented in higher education (Zevenbergen, 2001).

The benefits and shortcomings of peer review. Gatfield (1999) examined undergraduate students' satisfaction of PR. The data indicated that there was a high level of satisfaction with the process. Students tended to perceive PR as an appropriate form of assessment; they regarded it as a fair method and believed that students should assess their peers. Student PR, both formative and summative, has many benefits in terms of empowering the learners: the process supports the development of autonomy and higher-order thinking skills; improves the quality of learning; develops the ability to evaluate and justify (Topping, 1996); provides opportunities for students to self-monitor, rehearse, practice and receive feedback (Falchikov and Goldfinch, 2000); increases the amounts of feedback that students receive regarding their work; and encourages students to take responsibility for their own learning (Race, 1998). By reviewing works of peers, students gain insight into their own performance, as well as develop their ability to make judgments, which is an essential skill for one's professional lives. At the same time, they learn how to refer to mistakes as an opportunity for learning rather than as a failure. Moreover, PR promotes the exchange of ideas and serves as a basis for guiding academic discourse (Berkencotter, 1995) and the exchange interchange of ideas and methodologies resulting in a more refined product (Reese-Durham, 2005).

PR might expose problems that relate to validity and reliability, and there is no guarantee that the feedback provided will be accurate and valuable. A similar problem concerns peer grading (Falchikov and Goldfinch, 2000). It appears that the subjectivity associated with grading students' works is influenced by students' different standards of what a numerical score should consist of (Conway, Kember, Sivan and Wu, 1993). Even when students are provided with clear criteria for evaluating and grading, there is still a problem of subjectivity: Zevenbergen (2001) examined peer assessment by mathematics PTs of posters created by their peers. The researcher identified an observable trend among high and low achievers and the scores they gave to their peers' works: low achievers assigned higher than average grades and often provided very generic comments whereas high achievers tended to assigned lower than average grades and provided more insightful and critical comments.

The roles of peer review in teachers' professional development. PR is an effective way of ensuring quality teaching. It encourages reflection and analysis of teaching practices and promotes specific feedback over time. PR proponents assert that regular and formal support of struggling new and veteran teachers is merely one of the many factors that play a critical role in enhancing teacher quality. The theory is that teachers who know they are subject to PR are more inclined to identify ways to improve their teaching. PR does not, however, have to replace more traditional methods of evaluation; rather, it can deepen and expand the processes of accountability. PR also allows teachers to take a more active role in their professional development (Rogers and Threatt, 2000).

Copley (2007) suggested a model of focused professional development to enhance a teacher’s practice and student learning in mathematics. One of the model's components related to PR done teachers, where lessons were videotaped and than evaluated.

Our two main aims in implementing the PR approach on PTs were: (i) to increase the PTs' awareness of the complexity of the assessment process and to develop their evaluation skills; (ii) to expose the PTs to their peers' works so as to develop their ability to produce high-standard and high-quality mathematical work and consequently to develop their mathematical knowledge. In this paper, we focus on the first aim.

THE STUDY

In this section we present data concerning the study participants, the nature of the Methods course in which the study took place, and the phases of the study. In addition, we present the methodology used for data analysis.

The study participants. Sixteen prospective mathematics teachers studying at an academic college participated in the study. The PTs group was heterogeneous: Nine were regular PTs, studying in their third year (out of four) towards a B.A. degree in mathematics education and computer sciences or physics for middle and high school. Four were in-service mathematics teachers who chose to earn a teaching certificate in mathematics. Of these four, two had 2 years of teaching experience and the other two had over 10 years of teaching experience. The other three PTs were adults who were considering a second career as mathematics teachers.

The Methods course. The Methods course in the framework of which the PTs experienced PR is a two-semester course and is the second mathematics Methods course the PTs are required to take. In this course, the PTs learn and discuss methods and approaches for teaching various high school mathematics topics and experience various evaluation approaches (one of which is PR).

The course of the study. The study comprised ten phases that together lasted three months, during which no issue related to evaluation was discussed in class. The phases were as follows: (1) The PTs were required to solve an identical given problem and to hand in their solutions; (2) The PTs' solutions were scanned and sent through the course forum mail for evaluation. Each PT received two anonymous solutions for evaluation and grading. The PTs were asked to formulate a list of evaluation categories, to assign each category a weight, and to provide justification for each category and weight. In addition, the PTs had to specifically grade the work according to each category and explain the underlying reasons for each grade. These evaluations were e-mailed to us; (3) Each PT was e-mailed the two anonymous evaluations and grading of his or her work and was asked to address eight questions concerning the evaluation received from their peers and its contribution to his or her future work; (4) Using the "What If Not?" (WIN) strategy (Brown and Walter, 1993) and based on the initial problem, the PTs had to start an inquiry process. The WIN strategy is based on the idea that modifying the attributes of a given problem can yield new and intriguing problems, which eventually may result in some interesting investigations. In this phase, the PTs were asked to produce a list of the initial problem's attributes, suggest alternatives for each of them, and choose one alternative or a combination of several alternatives for further mathematical investigation. Again, the PTs e-mailed us their assignments; (5-6) Phases 2 and 3 were repeated, based on the work produced in Phase 4; (7) The chosen new problem was treated, including a full written description of the conjectures, solutions, indecisions, and so on; (8-9) Phases 2 and 3 were repeated, based on the work produced in Phase 7; (10) Each PT wrote a summative evaluation and final reflection on the entire process. It should be noted that in Phases 6, 8, 9, and 10 the PTs were given additional questions to which they responded. Class discussions related to the 10-phase experience were held only after all PTs completed their final reflections. Due to space limitations, we focus here only on data obtained from Phases 2 and 3.

Data collection and analysis methods. A large amount of data was collected during the various phases of the study. The data received from Phase 2 were analyzed according to the following focal points: (i) Categories: the nature of the criteria; the weight given to each criterion; types of justifications given for each category's weight; (ii) grades: an examination of the grades assigned in each category; types of justifications for the grades; (iii) the nature of the general comments given on the work; (iv) validation: comparison of the categories, weights, grades, and justifications given by the PTs with those we gave. The data received from Phase 3 was analyzed with a focus on the effects of peer evaluations and grading on the individual from cognitive, affective and didactical perspectives. In addition, we held some informal interviews with some of the PTs to clarify statements they had made.

Analysis was ongoing and it continuously informed the data-gathering process. Following analytic induction (Taylor and Bogdan, 1998), we reviewed the entire corpus of data to identify themes and patterns and generate initial assertions regarding the above focal points.

RESULTS AND DISCUSSION

Phase 2 -The peer review process. Most PTs admitted that they began by reading the solutions and then formulated categories to match the solutions. Retrospectively, however, they believe that the order should have been reversed: "*The solution should have been judged according to presumed standards and not vice versa*". Many PTs reported difficulties in generating a satisfactory set of categories for evaluating their peers' works, ascribing their difficulties to their lack of prior experience. It should be noted that even the experienced teachers reported encountering difficulties: "*I wondered whether I should generate a set of general categories, one that can be used to evaluate all mathematical assignments, or a set of categories that are specific for this assignment…. After several attempts, I chose the second option. I found the first to be very difficult and confusing*". Some PTs argued that the categories should not be class-independent, that they should be "*realistic*"; specifically, "*one can not set categories and weight them in a way that will cause most of the class students to receive very low grades".* Besides their lack of experience, the PTs' indecisions had other causes:

> "I had difficulties deciding whether the categories should relate to the methods, techniques, line of thought, or perhaps the final solution…How to weight them? Which is more important? Which deserves a higher weight?"

> "One of the things I was concerned with related to my inability to generate categories that would distinguish between constructive and non-constructive evaluation … not to mention the fact that I couldn't decide how to weight them."

> "My most significant dilemma related to the nature of the categories… I wanted to generate categories that would reflect creativity, simplicity, and original observation of the problem. I had no idea how to do that."

From the above statements it appears that the PTs were uncertain as to the nature of the categories - what should they evaluate? The PTs also had hesitations concerning the weight they should attribute to each category, assuming that the greater the category's weight is, the greater is its importance to the process of solving the mathematical problem. The PTs argued that the weight itself "*bears a certain message*"; namely, "*if a teacher wishes to educate his students to work in a certain way, he should weight specific categories in accordance*". The PTs had trouble generating categories that would reflect the consideration of non-measurable aspects, such as creativity and originality. Ultimately, the PTs did not suggest such categories at all.

A total of 32 assessments (two by each PT) were obtained, from which 22 different categories emerged. The most common categories (offered by at least 6 PTs) were: customary proof structure (sequence of logically consecutive claims and justifications) (11 PTs); correct solution (8 PTs); correct final answer (6 PTs). The difference between the first and the second category is that the former refers also to answers that provide incomplete or incorrect proofs but that offer an acceptable structure of geometric proof, whereas the latter refers only to correct proofs. The weight that was given to the first category ranged between 20% and 50% (Average=30%, SD=10.44). The weight of the second category ranged between 20% and 60% (Average=46.1, SD=21.1) and the weight of the third category ranged between 10% and 30% (Average=23.3, SD=6.87). The wide range observed for each category points to the subjective nature of evaluating work. Other examples of categories were: providing the shortest possible proof, clear use of notations, calculations errors, clear drawing, organization and clarity, and more. After generating the categories, the PTs encountered difficulties in evaluating and grading the solutions:

> " I tried not to think of how I would have solved it myself, but rather to slip into the solver mind and suggest ways to improve his work, so that he could retain the 'spirit' of his work."
>
> "I had doubts…What should I evaluate – the correct parts or the wrong parts?"
>
> "It was a difficult decision – should I give points for the correct portions or deduct points from the total for the mistakes?"
>
> "I tried to find the good things in the solution, and provide reinforcements. However, I found myself commenting on the wrong things…I hope I wasn't too harsh…"
>
> "I could see from the work that the solver knew the solution. The final answer was correct. The method, however, was unclear. So I wondered whether I should give points for the solution or deduct points for the method…It was difficult to give a low grade, because I felt that he knew…"

These excerpts reveal that the PTs evaluated and graded their peers' works with care. They were concerned about providing positive feedback and reinforcement, and referred to the evaluation of their peers' mistakes as an opportunity for learning rather than as a failure (Berkencotter, 1995).

Eighteen different types of comments emerge from an examination of the final comments to the works. When the PTs found no errors, they praised the solver ("*Excellent work*", "*It shows that you understand*", and so forth). When errors were discovered, most evaluators chose to open with a positive comment such as "*I believe you can solve the problem, but…*", or "*You have a good start…*". The majority of comments referred to the absence of proof or justification for a specific argument. In some cases the evaluators explicitly added the missing explanation.

***Phase 3 - Feedback on the peer review*s**. Feedback on the peer review process can be divided into two perspectives, namely that of the evaluating PT and that of the evaluated PT. From the evaluators' perspective, PTs stated:

> "When I had to think of relevant categories for the peer review, it made me think about the problem again, what parts of the proof are essential. I realized that my solution was not very good. If I had to evaluate my own work, I would have given myself a lower grade."
>
> "The process of peer review exposed me to different solutions and if I could, I would have changed my original solution."

The engagement in PR exposed the PTs to different solutions of the problem and as a result they gained insights into their own performance and deepened their own mathematical knowledge. This is consistent with Berkencotter (1995) and Reese-Durham (2005) who asserted that in reviewing works of peers, students gain insight into their own work, as well as develop their ability to make judgments, which is an essential skill for one's professional lives. One of the most important insights the students experienced as evaluators was expressed by the PTs as follows: *"When I evaluated the two works, I understood that my solution was not good and what a good solution should look like"; "I saw a solution that is simpler than mine"*. Exposing the PTs, as evaluators, to other solutions made them realize that there are various ways to solve the problem, some of which are simple or "*more elegant*" than their own.

From the evaluatees' perspective, most PTs did not express disagreement, dissatisfaction or reservation about their peers' evaluation and comments; for instance, "*I tried to address the comments with an open mind…although at first glance I didn't agree with all of them. I knew that the evaluator had good intentions, so I tried to evaluate my work using his categories*". The majority of the PTs also indicated the contribution of the evaluators' comments, for example: "*I received constructive comments; ones that enabled me to correct my solution*". A few PTs, however, felt otherwise; for instance: "*The evaluator gave me a low grade without justifying it. I felt insulted… nevertheless, when the other evaluator gave me a low grade on one of the categories, I could accept it since it was justified logically"*. When the solutions were correct, the comment was usually only "*Excellent work*". However, many of those who received such feedback complained: "*I was disappointed. I would like to hear why my work is good, what he thinks made it good, or – is there any way that I can improve my work? I gained nothing from this comment*".

In other words, from the evaluatees' perspective it was very important to receive their peers' written comments. They were disappointed and sometimes even reacted affectively ("*I felt insulted*") when their peers did not justify the grade they gave their work: "*the assessment process can be regarded as formative only if justifications are included*".

In summary, during the early phases of the process, the PTs began developing their ability to evaluate and justify (Topping, 1996) and were provided with opportunities to self-monitor and receive feedback (Falchikov and Goldfinch, 2000). The process enabled them to receive two opinions of their work (Race, 1998) and by reviewing the works of their peers they gained insight into their own performance (Berkencotter, 1995). In addition, they recognized that "*it is much easier to evaluate the works of others than to evaluate my own work. But the process helped me understand how I can be more critical of myself*".

CONCLUDING REMARKS

From the data obtained it is evident that the PTs seriously delved into the PR process. Since developing assessment skills is essential to their professional development as future mathematics teachers, we believe that engaging PTs in PR has the potential to expose them to the complexity of the assessment process and to develop their ability to evaluate their future students in a fair and constructive manner.

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NURTURING TEACHERS FOR GIFTED NURTURING

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This article presents a model of enriching experienced teachers' knowledge and reforming their attitude in nurturing gifted. In this study there are eighteen groups, selected from elementary schools in various areas in the central county in Taiwan, and each group comprises a teacher and two gifted. Teacher is acting both as a co-learner and a mentor in the group so that they have the opportunity to jumping from the book to experience the genuine gifted education by themselves. The philosophy of mathematics taken in this model is naturalism. The materials selected from the sixth to ninth grade natural science course are mostly the authentic Physic phenomenon presented in video. The mathematics knowledge in describing the physics phenomenon is not taught in advance but captured through the investigation of the phenomenon. Teachers' progress is described in detail along Dettmer's view on learning and doing.

INTRODUCTION

The focus of this article concentrates on the in service education model of teachers who are teaching the mathematics gifted. The theme of teacher preparation of gifted in Taiwan is, without exemption, quite similar to other countries in the world. The philosophy taken in establishing the gifted education policy leans toward the mainstream: enrichment, acceleration and ability grouping. Unfortunately, the course structure and content offered in the teacher preparation program do not yield the required disciplines of being a capable teacher of gifted. Unveiling the major of those teachers of gifted, we see that they are coming from different fields such as special education, computer science, Chinese, elementary education etc., but only a few of them graduated from mathematics and science education. With the lack of content knowledge, teachers of gifted can offer only a vast volume of knowledge that is cut and pasted from various sources books. Feldhusen (1997) indicated that the competencies needed by teachers to work with youth talented in mathematics or science might be very different from those needed to work with youth talented in art, music, literature, or computer science. To yield a successful gifted program, teacher education is imperative (Hansen & Feldhusen, 1994). And only the most effective teachers were able to show important growth gains for the top 20% students on pre/post measures (Sanders & Horn, 1998). In order not to squander gifted,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 97-104. Thessaloniki, Greece: PME.

especially in mathematics (Sanders & Rivers, 1996), it is urgent to take actions to raise teachers' proficiencies in nurturing gifted. This article reports a preliminary result of a model of nurturing teachers of gifted in the authentic environment.

LITERATURE REVIEW

In order to develop the ability of the gifted, the material given by the teachers of gifted need to be broader, more depth and more flexible (Heid, 1983). According to Kiesswetter (1992, p. 11), purposeful mathematics thinking is the course of which four steps can be identified: (1) ascertaining constellations (questions) of interest; (2) focusing on the mathematical question under consideration and specifying the steps involved in its solution; (3) creatively and with the help of heuristic procedures arriving at new terms, mathematical formulations, proofs and broadly applied lines of attack; (4) placing the established interconnections within a network. What kind of material can meet the requirement of both Heid and Kiessweterr? In their study, Shy and Tasi (ICME 11) presented some encouraging results that gifted do benefit from the exploration and description of the physics phenomenon. The non-mathematical material provides a wide scope in understanding and applying the mathematics. It also provides ample opportunities for gifted to do mathematics in Kiesswetter's sense. Zimmermann (1992) proposed that the variety of conceptions of mathematics leads not only to varying perceptions of what mathematical gifted is, but also apparently affects the contents of instruction and the way the information is imparted by instructor. The philosophy of mathematics taken by the authors is naturalism. This perspective, accompanying with the use abundant information obtained via internet, influences the management of the material to tackle the mathematics concept and hence yields a new way of teaching gifted. Under these circumstances, what competence is needed for the teacher to organize this kind of teaching material? Furthermore, Davis(1997) asserts that gifted tends to challenges rules, authority, and argumentative. Gifted doesn't seem to easily accept concept which they disagree. Do teachers of gifted have enough content knowledge to handle such situation? Feldhusen(1997) suggests that teachers need special skills and understanding if they are to facilitate the personal, social, and academic development of talented youth. What abilities should teachers of gifted have? Joyce VanTassel-Baska and Susan K. Johnsen (2007) present an in depth introduction about the Teacher Education Standards for the Field of Gifted Education. The standard number nine lists the knowledge and skill that are essential for teachers of gifted. (K1) Personal and cultural frames of reference that affect one's teaching of individuals with gifts and talents, including biases about individuals from diverse backgrounds; (K2) Organizations and publications relevant to the field of gifted and talented education; (S1) Assess personal skills and limitations in teaching individuals with exceptional learning needs; (S2)

Maintain confidential communication about individuals with gifts and talents; (S3) Encourage and model respect for the full range of diversity among individuals with gifts and talents; (S4) Conduct activities in gifted and talented education in compliance with laws, policies, and standards of ethical practice; (S5) Improve practice through continuous research-supported professional development in gifted education and related fields; (S6) Participate in the activities of professional organizations related to gifted and talented education; (S7) Reflect on personal practice to improve teaching and guide professional growth in gifted and talented education. The standard indicates the importance of not only the content knowledge but also the consultant skill. In educational aspect, Dettmer's (2006) expanded educational taxonomies model for learning and doing provides a comprehensive view by adding important dimensions other than the knowledge. The model consists of cognitive, affective, sensorimotor, social and unified domains. With this expanded model we are able to trace teachers' growth in gifted education categorically.

METHDOLOGY

The objects of this study are eighteen experienced teachers of gifted from many elementary schools in Changhua County, a county located in central Taiwan famous for enthusiastic in gifted education. Each teacher has two allowances of gifted of fifth or sixth grade to enroll in this program. The term of this in service teacher education program is six weeks, three hours a week on Sunday. The instructor is a professor with majors in pure mathematics (Topology) and mathematics education. He has more than eight years of experience in nurturing secondary school gifted. The class is organized in the cooperative learning style consisting of eighteen groups. There are two sessions in each week that the first two hours is the open-ending course. In third hour, gifted and teachers are separate for different tasks. Gifted are working problem sheets and teachers are discussing issues whatever they like to share. All the activities are videotaped. And teachers' logs and notes are scanned to capitalize data.

The materials presented in the class are videos showing the phenomenon of physics principles. The subject matters include the pressure, the temperature and weight measurement. The videos, down loaded form the internet and some scientific experiment videos from libraries, are edited so that gifted only have the opportunity to watch the phenomenon with no explanation. The mathematics for gifted to explore are characteristics and the solving skills of linear equations. There is enough time for gifted to discuss the phenomenon. At this moment, the teacher in each group must behave like a co-learner and try to understand how gifted are exploring questions. The teacher may provide his or her opinions but not tries to teach gifted. This is a great opportunity for teacher to learn how to work with gifted to maximize their potential and to work up their creativity. In the first period, first three weeks, gifted are asked to present the key factors in the phenomenon and try to relate these factor in any possible way with their own reasoning. The discourse is central in this model. The instructor's role is to appreciate, to question or to refine the findings and reasoning proposed by gifted. After the discourse, challenging question sheets are provided so that the conception of gifted can be collected. In the second period, last three weeks, the focus is on the representations of the relations between key factors in terms of equations and the graphing. According to the varieties of the properties deduced from the phenomenon, the equations are more meaningful and the strategies for manipulating the equations are versatile.

RESULTS

Most teachers in this model are terrified by the open ended teaching mode in this model. They can't even adapt themselves to the discourse oriented teaching performed by the instructor. All these mental barriers mentioned above have been cleared due to their enthusiastic participating attitudes to behave like an exemplary gifted for their gifted students to follow. At the beginning of the program, teachers do hesitate on responding their opinions or answers to instructor's question in the class. They are worrying that the mistake they made is so embarrassing that will diminish their authority. But the environment of this model do provide a friendly ambiance that teachers are considered as ordinary learners and they won't be blamed by any inappropriate argument so that teachers are getting comfortable in this system to expression their own idea. Although they may not familiar with the very essence of the knowledge behind the phenomenon provided, but what the discourses continued in the class do open the teachers' eyes that teaching is not a best way in nurturing the gifted. This leads to a fundamental retrospect on what they have being doing on nurturing the gifted. And the radical change is undergoing as teachers have realizes that the needs of gifted may be different from that teachers are presumed. To better out line teacher's progress in terms of Dettmer's taxonomy of the developing human potential for learning and doing, the flow charts of the development of a specific teacher are provided. Detailed descriptions, presented in the following four figures, represent samples of teacher's developments. To simplify the diagrams, we give a list of abbreviations of different domains and phases of Dettmer's taxonomy: Cognitive domain: Know(CK); Comprehend(CCo); Apply(CAp); Analyze(CAn) ; Evaluate(CE); Sythesize(CS); Imagine(CI) ; Create(CCr). Affective domain: Receive(ARec); Respond(ARes); Value(AV); Organize(AO); Internalize(AI); Characterize(AC); Wonder(AW); Aspire(AA). Sensorimotor domain: Observe(MO); React(MR); Act(MAc); Adapt(MAd); Authenticate(MAu); Harmorize(MH); Improvise(MIm); Innovate(MIn). Social domain: Aware(SAw); Communicate(SCom); Participate(SP); Negotiate(SN); Adjudicate(SAd); Collaborate(SCol); Initiate(SI);

Convert(SCon). Unified: Perceive(UP); Understand(UUn); Use(UUs); Differentiate(UD); Validate(UVa); Integrate(UI); Venture(UVe); Orginate(UO).

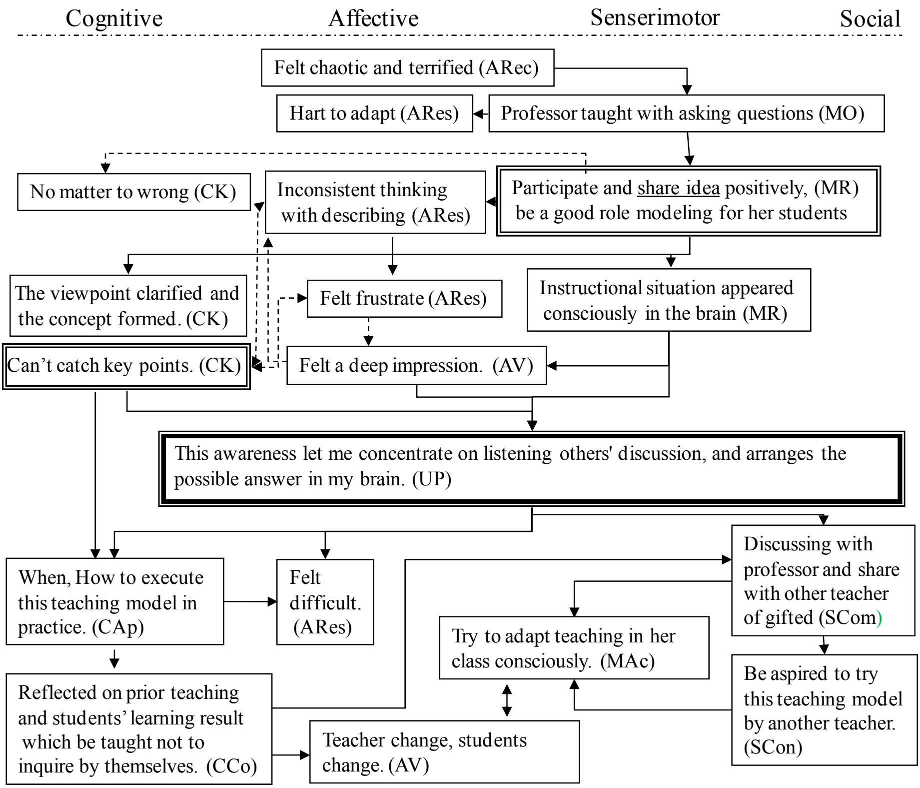

Fig1: Teacher Ru's human potential development network in the first period

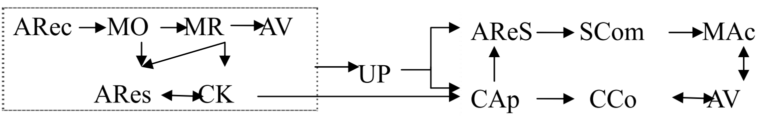

Fig2: Schematic expression of the development of teacher Ru in the first period

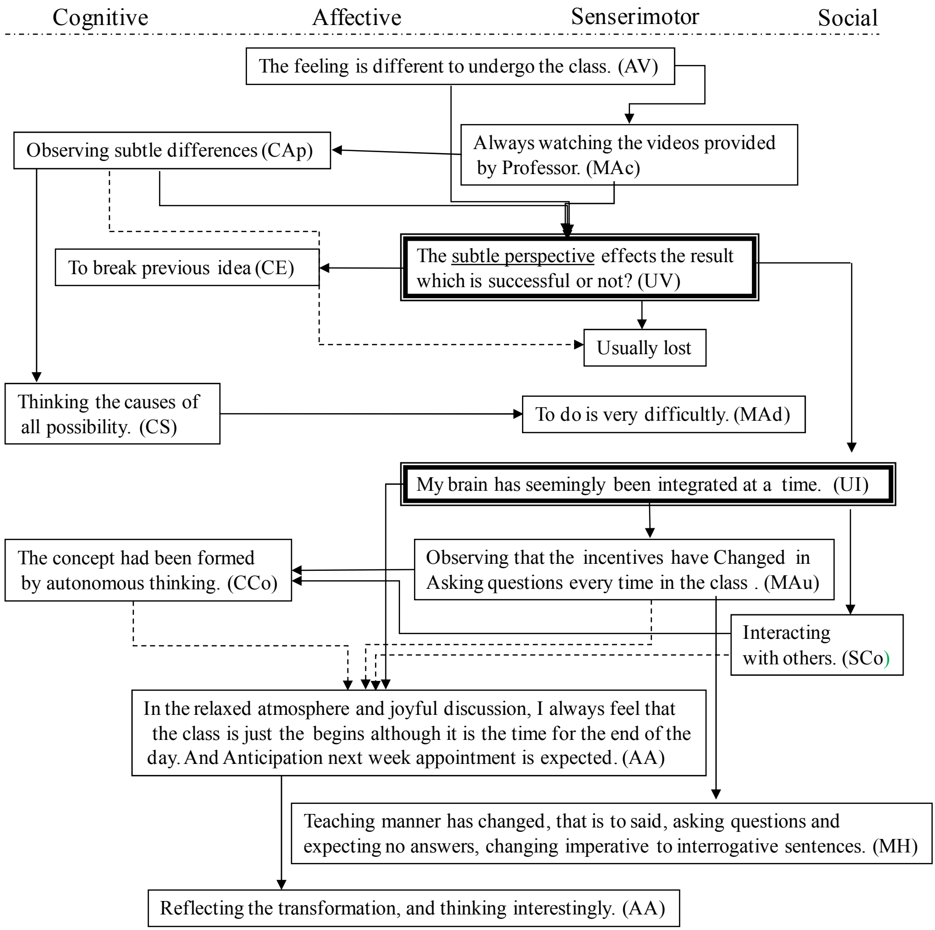

Fig3: Teacher Ru's human potential development network in the second period

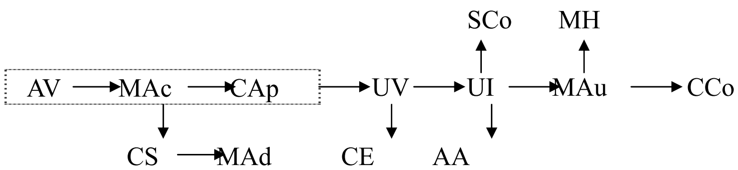

Fig4: Schematic expression of the development of teacher Ru in the second period

CONCLUSIONS

In service teacher education has its all-too-frequent failure in improving teaching performance and teacher attitudes (Sandra, 1986). The results of this study show that teachers are becoming more and more willing to work and to discuss with gifted instead of rushing in stuffing gifted as much knowledge as possible and a lengthy practice on problem solving that will polish their skill. Teachers' major concern on evaluating their quality of teaching switches from the outcome of the performance of gifted to the opportunities for gifted to express their idea and creativities. Thus teachers are more sensitive to the need and the idea of gifted. The issues that are allowed to discuss are almost limitless due to the attitudes of self-image as a co-learning not an authority. As this model provides the authentic environment that they are easily practice their belief in their own class. The implication of conclusion of this model is that teachers of gifted can better learn the knowledge and skill that are essential for teachers of gifted when they are placed in the position of gifted.

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TI-NAVIGATOR IMPLEMENTATION AND TEACHER CONCEPTIONS OF MATH AND MATH TEACHING

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We are currently completing a study on the effects of linked hand-held technology (specifically the TI-Navigator) in early secondary mathematics. We developed a set of criteria specific to TI-Navigator, to evaluate teachers' implementation experiences as revealed in our observation records, interview transcripts, and questionnaire data from the first two years. We found that changes in practice around our criteria were related to teacher conceptions of mathematics teaching.

The TI-Navigator project is a mixed methods study to investigate use of linked hand-held technology in early secondary mathematics. The study was designed to follow a number of "typical" teachers as they implemented the use of the TI-Navigator (a wireless device that supports sharing of information from TI 83 or 84 calculators.) The key questions for the research were: a) What are the effects of the use of TI-Navigator on student achievement in mathematics? b) What are the effects on the attitudes of students towards mathematics? c) What are the effects on teaching practice? and, d) What support do teachers need to use such technology effectively?

While we include background information on student achievement to inform the reader, this paper gives particular attention to our findings on question c). In particular we argue that changes in teacher practice are related to the interplay between technology affordances and teacher conceptions of mathematics.

BACKGROUND

Schools for our study were co-educational, of average size, and full-program (i.e., not specialty schools). All Grade 9 mathematics students in the 2006-2007 year and all Grade 10 mathematics students in the 2007-2008 year were involved. This required that all Grade 9 teachers at the experimental schools agree to use the TI-Navigator. Although we knew that some teachers would be keen to try the new approach, we were aware that some might be tentative or even resistant.

Teacher and Student Background

All participating grade nine students completed a background survey at the beginning of the first year of the study that probed their attitudes towards mathematics and the learning of mathematics and asked them to indicate the typical mathematics teaching activities and practices they had been exposed to in prior years. Responses indicated that for both groups, mathematics had largely been taught in a traditional procedure-focussed manner in the past, with little use being made of computers or other technology.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 105-112. Thessaloniki, Greece: PME.

All participating teachers were surveyed initially to gather background data on their teaching experience and teaching strategies. Overall, study teachers were very experienced, and their responses indicated that they were fairly conservative and traditional in the teaching strategies they employed. Most had used graphing calculators and CBR/CBL, and a majority had used Geometer's Sketchpad, but their use of other technologies—and of manipulatives—was sparse.

Teachers were surveyed at the end of the first year about their use of technologies. Smart Boards, CBL/CBR, Geometer's Sketchpad, and spreadsheets were used in less than 25% of classes by study teachers. While graphing calculators were used in every class by all but one experimental teacher, few control teachers made use of them. Reported use of TI-Navigator by experimental teachers, varied greatly – from less than 25% of classes to every class. (This corroborates our first year observations which showed that technical difficulties slowed implementation of the TI-Navigator). The TI-Navigator was reported as being used primarily for teaching linear relations, solving equations, and graphing.

Student Test Analyses

To determine if there were any significant differences in student post-test scores between the experimental and control groups in each of the two years of the study completed to date, two parallel sets of analyses were employed. Analyses of covariance (ANCOVA) were run on the student post-test scores for each year separately, using the student pre-test scores for that year (grade nine or grade ten) as a covariate to control for pre-existing individual differences in mathematics knowledge and ability directly related to achievement in that year's mathematics curriculum.

Results of the ANCOVA for the grade nine students from year one of the study showed no significant differences between the experimental and control students on the post-test for either the academic stream students ($F_{(1,225)} = .198$; $p = .657$) or the applied stream students ($F_{(1,116)}=3.07$; $p = .082$).

For year two, the analysis of the grade ten test scores revealed a statistically significant difference between the experimental and control students on the post-test (controlling for pre-test differences) for the academic stream students—the experimental students had a significantly higher mean post-test score ($F_{(1,260)} = 9.910$; $p = .002$; $\eta_p^2 = .037$). The treatment effect size for the academic stream of .39 (expressed as Cohen's *d*) is considered moderate in educational research. No statistically significant difference in adjusted post-test means was found between the control and experimental student groups for applied stream students in year two, ($F_{(1,141)} = 0.300$; $p = .585$, $\eta_p^2 = .002$).

LITERATURE AND FRAMEWORK

When we implement use of a technology we challenge systems and individuals to deal with problems, and to adjust their practices. Various frameworks have been developed to help analyse the implementation of innovations. One that is frequently applied in these contexts is the Concerns-Based Adoption Model (CBAM) (Hall & Hord, 1987), which uses a questionnaire to identify concerns of participants, and interviews to discern levels of technology use. Researchers have shown, for example, that in regard to educational innovations, personal concerns need to be addressed before participants are able to focus on consequences for learners (Zbiek & Hollebrands, 2008) – a major contention of CBAM. Another model, PURIA (plays, uses as a personal tool, recommends, incorporates, assesses) developed by Beaudin and Bowers (1997, as cited in Zbiek & Hollebrands, 2008) offers a means to differentiate modes of technology use by teachers. However these models failed to help us analyse our data with respect to teacher use of the TI-Navigator. As a result, in order to answer our question about impact on teacher practice we developed a set of criteria specific to TI-Navigator use.

We believe that the key affordances of TI-Navigator are: 1) the provision of two-way communication between teacher and students, which enables sharing, and checking, and 2) the provision of a (shared) display, which facilitates investigation of the behaviour of mathematical models, heightening the role of visualization in mathematics. We contend that the specific forms of the study teachers' implementation of TI-Navigator around sharing, checking, and modelling were linked to their conceptions of mathematics and or mathematics teaching.

Teacher conceptions

In considering the participating teachers' conceptions of mathematics and its teaching we draw from Hoz and Weizman's (2008) recent work in the area. They developed dichotomous characterizations of mathematics (as either static-stable or dynamic-changeable) and mathematics teaching (as either open-tolerant or closed-strict) (p. 906), and defined extreme conception pairs as *static-closed* and *dynamic-open*. To facilitate our discussion we provide some of the authors' examples of 'official conceptions' – or 'expert characterizations' for each category in Table 1. Hoz and Weizman note that few high school teachers hold all beliefs associated with a category and operate somewhere in the middle; however they found that, "the prevalence of the pair of static-closed and the rarity of dynamic-open among math teachers were implicitly reflected by the teachers' use of textbooks," (p. 910).

Category	Sample notion
Mathematics – static	Mathematics is *a priori* and infallible; mathematics is a clear body of knowledge and techniques.
Mathematics – dynamic	Mathematics is a social construction; the essence of mathematics is heuristics not the outcomes.
Math teaching – open	The student constructs her or his knowledge actively – she or he is doing mathematics; learning is based mainly on personal-social experience and involvement and on discussions that evolved during problem solving.

Math teaching – closed	The teacher is the knowledge authority and she or he is obliged to transfer it to the students; mathematics teaching aims at and depends on the mastery of concepts and procedures;

Table 1: Dichotomous conceptions of mathematics and mathematics teaching with example notions from Hoz and Weizman (pp. 907-908)

Based on our teacher survey results and early observations we characterized study teachers' practice as 'traditional,' or 'very traditional'; this suggests that teachers leaned towards a 'closed' conception of mathematics teaching. For example, a teacher commented that before having students work with Geometer's Sketchpad, she teaches the concepts. And, although the curriculum requires students to carry out investigations, we typically observed instances of direct teaching of the concept(s) before and/or after group work. We also found that, while all study teachers included group work and investigations – and noted the importance of doing so – the discussions observed in most classes at the beginning of the study were very brief (or non-existent); this indicates that most study teachers, at least initially, considered that student input was not essential, i.e., they viewed mathematics itself as "static".

ANALYSIS

Under the categories of sharing, checking, and modelling, we organized a set of three uses to illustrate links between teacher practice with TI-Navigator and a possible continuum – from static-closed to dynamic-open – in teacher conceptions of mathematics, and mathematics teaching. We do not mean to imply however, that the uses we have chosen represent the extremes, in view of the fact that use of TI-Navigator requires that students participate, and that curriculum curtails, to some extent, a completely open-ended approach.

In what follows, we expand on each of these categorizations in turn through a discussion of some general findings, and then examine the experiences of two teachers to probe whether/how TI-Navigator use affected their practice.

Sharing. Teachers can use TI-Navigator to enable a) creation of a joint product, b) analysis of errors, c) deep discussion. In the study, student involvement took a variety of forms. In most observed classes students were asked to contribute a response to a prompt by the teacher. By giving each student a unique value to input as, say, slope, or intercept, teachers could ensure that each student's answer was different. In many classes students were asked to identify mistakes, although teachers didn't necessarily engage the class in analysing those mistakes. In a few observed classes, students were engaged in actively using mathematical ideas and language to share their ideas. For instance, the activities around matching parabolas to curves in photos led to enthusiastic participation and a recent trigonometry lesson around daylight hours involved students in a discussion of real life connections.

Checking. Teachers can use TI-Navigator a) to deliver and mark quizzes, b) to display students' answers for analysis, and c) to modify their lessons as a result of student responses. Initially, most teachers had difficulty with this aspect – both with generating questions and with saving and projecting student answers and results; however, by the end of year two all teachers were using the quizzing and marking capabilities of TI-Navigator with ease. At that time, teachers had various comments about the checking capabilities. Only one was negative – a teacher noted that unless she wants to have a quiz about graphs it is just as easy to do a paper and pencil quiz. Positive comments focused on 1) evaluating – that the tools provide ongoing information to the teacher about student achievement; 2) participation – that they permit all students to participate (instead of just a few), they encourage competition (which keeps students involved), and they allow students to answer anonymously (which helps protect self-esteem); and 3) monitoring student understanding – that they provide instant feedback, which enables the real-time modification of lessons to address student needs. These comments indicate that most study teachers held an open conception on mathematics teaching around the students' role in learning.

Modelling. Teachers can use TI-Navigator, 1) simply to display responses to prompts, or to engage students in 2) guided or 3) open-ended creation and investigation of a model. We observed study teachers using a variety of activities that they had collected at Navigator-focussed PD sessions, or accessed via DVD or the Web. Some of the observed activities simply involved students in following specific procedures and recording the results. Some were examples of "guided investigation." That is, the teacher worked through the development of a model (such as creating a formula to find the surface area of a triangular prism), then students contributed particular values (sometimes based on an experiment), and then discussed the results. What we did not observe was any occasion in which students were invited to develop and answer their own questions about a model's behaviour. We would add that in most observed classes the TI-Navigator image was used as a source for students to see the result of their input in response to a question or prompt, but there was little explicit use of it for "visual reasoning". A few activities used the displayed image to help students make connections to real life – for example, two teachers used the technology to have students develop equations of parabolas to match curves in photos of real-life objects (e.g., the St. Louis arch); however, over the two years, there were only a few observed lessons that we believe encouraged "visualizing" as a way of thinking mathematically. In one case a teacher had students create "stars" by graphing systems of equations to help them visualize why the elimination model for solving equations works. In another, the teacher asked students to send ordered pairs at intervals to help them connect the gradual appearance of points onscreen to the behaviour of the data.

Thus we argue that study teachers did not take full advantage of TI-Navigator's modelling capabilities to help students develop inquiry and visualization skills; that is, though supportive of student involvement, which suggests an open conception of math teaching, teachers may have held a rather "static" conception of mathematics.

Teacher practice

As briefly outlined, we found that most observed lessons did not reflect uses at the dynamic-open end of the continuum of conceptions about math and math teaching. Here we offer glimpses of the practices of two teachers, considering what aspects changed over the two years and what that suggests about the teachers' conceptions.

Dan. At the beginning of the study, Dan had few technical difficulties. He was able to troubleshoot problems (and assist others in doing so) and quickly learned how to use the features of TI-Navigator. Early in year two a research-observer wrote:

> [Teacher] starts off with a quick review –[given a] slope of 2, what slope is parallel? Perpendicular? [He] then reviews equations of lines that are parallel, perpendicular. They then do a Quick Poll. After that they use activity center to send [equations of] lines that are perpendicular or parallel to a given line. Then they have a quiz on parallel and perpendicular lines….teacher reviews the answers - e.g., why would someone choose b)? [Teacher offers] - because they are mixing up vertical and horizontal.

Clearly, at this point Dan was technically strong at incorporating TI-Navigator into his lessons; however as hinted at in the last sentence, he was not yet involving students in discussions; he asked frequent questions but accepted one word answers, or answered himself if students didn't respond quickly. An observer noted that, as a result, students often lost interest in participating. A year later, Dan's interest in technology is still strong. His classes still move along quickly and students are all comfortable with using technology; however, after observing a recent lesson a researcher noted: "The technology is being used to check pencil and paper work. There isn't a sense in which students are thinking about the factors as 'terms' to be multiplied." Thus, we would suggest that this teacher's approach to teaching mathematics has remained largely unaffected, and that it reflects a conception that mathematics is a body of knowledge and techniques, and a conception that math teaching is about transferring concepts and procedures.

Joan. At the beginning of the study Joan (when not using TI-Navigator) spent most of the time at the board, talking as she wrote, while students copied the information into their books; when using the Navigator, Joan never strayed far from the computer, and frequently had difficulty focusing on her teaching while carrying out TI-Navigator tasks. However, despite the technical difficulties Joan was keen to try using the technology, and quite early chose an activity in which students tied knots in ropes, then recorded and plotted the lengths. The lesson was successful in many ways, but the observer wrote: "[The teacher] doesn't engage students in talking ahead of time to situate activity, to predict, to wonder. Questions not complex - one or two word answers."

As the year progressed Joan became more comfortable with the technology and by the end of year one, she was moving around the class to observe student pairs at work. By the middle of year one, she was holding (very) brief discussions about results. After a PD session early in year two, which encouraged teachers to use errors as opportunities, we noticed a change in Joan's practice. An observer wrote:

> [Students are to multiply] (2x-1)(3x-2) and send their answer to match the teacher's graph of y= (2x-1)(3x-2). … many answers are incorrect – several are upside down. Teacher tells them what the answer should be and reads off the names of those who got it right. She points out the parabolas that are upside down and asks what is wrong – and why. [She] sends some who got it right to help the ones who had trouble.

In another lesson later in year two, students were to send quadratic functions whose graph would match the curve on a photo of the St. Louis arch. An observer wrote:

> [Teacher] uncovers the picture and 3 different parabolic graphs show superimposed on the arch. One is correct, one has the right vertex, but is too skinny. One is way off. …[Teacher] goes through the mistakes. …She uses the Smart Board to write the vertex and correct the wrong answer. She basically "corrects" the equation while [the graph] is projected. [Teacher] takes the same picture but moves the x-y axes up. …. She asks students to model this using a parabola.…All students submit the correct parabola. [Teacher] asks – what happened? Did the shape change?

These two clips show Joan: checking student work, involving students in analysing particular errors, engaging students in helping one another, incorporating use of another technology and having students carry out guided investigations. In particular, we note that she explicitly asked students to "model" the curve. Looking back at the end of year one, we see that Joan's explanation of what she liked revealed a clear vision of the need for students to "see" and "do" mathematics:

> What did I like? Probably the fact that I'm a visual learner myself and I think many students are at least partially visual learners. … I think that the fact that they can see on the big screen things that they never did before helped them… this way I could engage all of them and they could see right away the results and I think they really liked that because I could hear them say "Miss, Miss, can we see it"…

At the end of year two Joan acknowledged her changed practice, saying, "To be honest it wouldn't be the same if I suddenly stopped using it. I think that I would feel that I'm missing a big part of my class. I think that class is a lot more interesting." These comments and our observations of Joan's lessons lead us to contend that use of TI-Navigator encouraged Joan to alter her practice – in particular, to move towards a more dynamic-open stance, in which she saw the nurturing of student engagement and discussion as key elements of mathematics teaching.

CONCLUDING REMARKS

Whether TI-Navigator changes teachers' conceptions, or simply supports teachers in following conceptions about mathematics or mathematics teaching that they held but didn't recognize as appropriate or feasible, is a question we cannot answer. But our findings support the contention that successful implementation of TI-Navigator in ways that further sharing, checking, and modelling in mathematics learning is not dependent on teachers' prior comfort with technology but rather, is related to their holding of or development of a dynamic view of mathematics and an open conception of mathematics teaching.

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BEYOND STATIC IMAGERY: HOW MATHEMATICIANS THINK ABOUT CONCEPTS DYNAMICALLY

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Abstract: Researchers have emphasized the role of visualization, and visual thinking, in mathematics, both for mathematicians and for learners, especially in the context of problem solving (see Presmeg, 1992). In this paper, we examine the role that motion and time play in mathematicians' conceptions of mathematical ideas. In order to expand the traditional focus on (and distinction between) visual and analytic thinking (see Zazkis, Dubinsky, and Dautermann, 1996), we employ gesture studies, which have arisen from the more recent theories of embodied cognition. Expanding on Núñez's (2006) work, we show how mathematicians' gestures express dynamic modes of thinking that have been hitherto underrepresented.

INTRODUCTION

There has been a growing body of research on students' ways of thinking in problem solving situations since the 1970s. While researchers initially identified two different modes of thinking (visual and analytic), they later became more concerned with the interrelationships between these modes. Zazkis, Dubinsky, and Dautermann's (1996) study, in particular, argues the two modes of thinking are not dichotomous. They draw specific attention to the visualization of dynamic objects and processes, and argue that perceiving dynamic processes and objects creates more complex mental images than perceiving static objects. Their study motivated us to further probe the interaction between analytic and visual thinking. We also draw on theories of embodied cognition to suggest that dynamic thinking is potentially a bridge between visual and analytic thinking. In this paper, we first present a brief overview of research on the different modes of thinking, and connect this research to emerging theories and methodologies from embodied cognition. We then present the analysis of mathematicians' verbal and non-verbal expressions in describing two concepts: quadratic functions and eigenvectors. Finally, we present a discussion about our findings, and offer suggestions regarding the use of gestures in teaching mathematics.

BACKGROUND

In mathematics education literature on students' mathematical thinking, researchers have proposed distinct modes of thinking. Krutetskii (1976) distinguishes verbal/logical thinking from visual/pictorial thinking. The former is an indicator of level of mathematical abilities whereas the latter indicates a type of mathematical giftedness. Krutetskii's work led mathematics educators to inquire further about the visual/pictorial mode of thinking, and to emphasise its importance in mathematical thinking (see Bishop, 1989, Eisenberg and Dreyfus, 1986 and 1991, Presmeg, 1992; Zimmermann and Cunningham, 1991). In her study of mathematicians' ways of

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 113-120. Thessaloniki, Greece: PME.

coming to know mathematics, Burton (2004) interviews seventy mathematicians working in different fields of mathematics. She identifies three primary modes of thinking: visual/pictorial, analytic/symbolic and conceptual.

While many researchers have distinguished the visual from the analytic, as two modes of thinking (see Clement, 1982), Zazkis et al. (1996) focus on the relationships between the visual and the analytic modes of thinking. They argue that these two modes of thinking are not dichotomous, and propose the Visualization/Analysis (VA) model to describe students' ways of thinking in problem solving. In defining the visual category, Zazkis et al. draw attention to the sometimes dynamic nature of visual imagery, something that Burton (2004) also does, in describing her visual/pictorial category as "often dynamic." According to Zazkis et al., perceiving dynamic processes and objects creates more complex mental images than perceiving static objects. Further, the very act of perceiving static objects involves dynamic actions, as the eye moves across the visual field to build the static objet (Piaget, 1969). Although both Burton and Zazkis et al. recognise the presence of dynamic visual imagery, they do not offer many examples of such imagery, nor do their models of mathematical thinking accord it a primordial role.

More recent research, drawing on theories of embodied cognition (see Lakoff and Núñez, 2000), suggests that dynamic thinking (and not just image-based dynamic thinking) plays an important role in conceptual development. For example, Núñez (2006) argues that mathematical ideas and concepts are ultimately embodied in the nature of human bodies, language and cognition. He has shown that static objects can be unconsciously conceived in dynamic terms through a fundamental embodied cognitive mechanism called 'fictive motion;' he illustrates this mechanism using the concepts of limits, curves and continuity.

In addition to studying mathematicians' linguistic expressions, Núñez broadens the methodological scope by including analyses of mathematicians' metaphors and gestures, which are key to revealing more dynamic thinking processes. As such, Núñez's approach differs from that taken by the researchers cited above, who focus mainly on linguistic expression, and who minimize the role of dynamic thinking in their models of mathematical thinking. We note that the inattention to (and sometimes ignorance of) the role of time and motion in mathematical thinking has strong historic roots: not only does mathematics tend to detemporalise mathematical processes (Balacheff, 1988; Pimm, 2006), but several mathematicians have expressed discomfort at the idea of moving objects (see Frege, 1970).

Núñez writes that gestures have been "a forgotten dimension of thought and language" (p. 174). Recent research, however, has shown that speech and gesture are two facets of the same cognitive linguistic reality. In particular, research claims that gestures provide complementary content to speech content (Kendon, 2000) and that gestures are co-produced with abstract metaphorical thinking (McNeill, 1992). This research supports our methodological approach in this paper, which is to analyse both speech and gesture in describing mathematical thinking. In particular, given the motion aspect of gesturing, we hypothesis that analysing gestures will provide more insight into the dynamical thinking process of mathematicians.

RESEARCH CONTEXT AND PARTICIPANTS

In our larger study, we extend Núñez's work to explore concepts other than limits and continuity. This paper focuses on concepts relating to functions, matrices and eigenvectors. While Núñez studied mathematicians as they gave lectures, we chose to adopt the approach of Burton (2004), who used interviews to examine the nature of mathematical thinking. We designed our interviews using a set of questions aimed at eliciting mathematicians' concept imagery around a variety of mathematical concepts, spanning K-12 and undergraduate mathematics. We interviewed four mathematicians whose interests were in both pure and applied mathematics, and who were all members of a medim-sized mathematics department in Canada. Each interview lasted between 1 and 1.5 hours. Interviews were videotaped and transcribed. We reviewed the video clips and selected to analyse their speech, gestures, analytic and visual thinking about quadratic function and eigenvector.

ANALYSIS OF STUDY

We refer to Núñez's framework, conceptual metaphor and fictive motion to analyse mathematicians' linguistic and non-linguistic expressions. We also use McNeill's gesture classification and transcription to analyse the movements of the mathematicians' hand and arm as they described mathematical concepts. Verbal and gestural excerpts from interviews follow.

ANALYSIS OF SPEECH AND GESTURES: QUADRATIC FUNCTION

In our first analysis, we illustrate the way in which linguistic expression by itself can include evidence of dynamic thinking. In response to our prompt about quadratic function, LG first says: "Well I guess I see, I picture a parabola, right, a parabola which is, um, or a conic section if it's a quadratic function of two variables." In addition to the visual image of a graph of a parabola, he also talks about variables, which point to more analytic/symbolic thinking. It seems that his thinking moves flexibly between visual and analytic thinking which supports the VA model.

LG then continues to say: "I don't think I picture just one. […] I know that there's only one parabola up to scaling. If you took any two parabolas, you can always rotate it, put them side by side, zoom in on one and it will look just like the other." LG not only visualizes the graph of a parabola, but also thinks about the graph in motion, as he translates it, rotates it, and zooms in on it. He conceives a static entity (the parabola, the equation of the parabola) in dynamic terms, as illustrated by the verbs *translate*, *rotate*, *zoom*. In other words, his concept of quadratic function doesn't include just the graph, or the equation, but the parabola in motion: in the language of Sfard (2008), he uses the dynamic aspect of the parabola as a "saming" technique, to make all the parabolas, whatever their shape, size, orientation, be one single object; as he later says "there's only one quadratic function really."

In our next example, we show how the linguistic expression and the non-linguistic expression can illustrate different aspects of mathematical thinking. Once again, in response to our prompt of quadratic function, NN begins by referring to a real object: "Something like a goblet, yeah so both a parabola and a goblet." She uses the goblet metaphorically to describe the shape of a parabola. While both 'parabola' and 'goblet' evoke visual images, instead of dynamic ones, her speech coincidences with a set of gestures. In Figure 1 below, her right hand is cupped under, with fingers pointed upward, as if holding the goblet. Then, she uses her index finger to trace out a parabola starting from left to right and then returning from right to left (see Figure 2). In MacNeill's scheme, this is a *metaphoric gesture*, which 'points' to an abstract object. Note that in this gesture, the finger is moving, as if tracing a curve, or drawing a parabola; it is not a static gesture, as the one used to accompany the word "goblet."

	
Figure 1. shows NN's right hand which depicts a parabola	Figure 2. shows NN's index figure while tracing out a parabolic curve

In our third case, instead of producing the gesture along with the speech, the mathematician replaces speech by gesture. Again, in response to our prompt, JJ says: "initially I thought of algebraically, then I thought of [index figure depicts a concave down parabola] one of these [index figure depicts a concave up parabola] one of these." His gesture resembles that of NN, but differs also in several ways: he draws two different parabola, one concave and one convex, and also, draws them right in front of his body, at chest level. In contrast, NN goes back and forth along one parabola, and draws her in a region above, and to the right or her head. For both NN and JJ, the gestures are metaphorical, referring as they do to abstract objects. However, whereas NN evoked the metaphor of the goblet and the visual imagery of the parabola, JJ speaks first about the algebraic interpretation of the quadratic function, signalling an initial analytic—and very static—conception.

Our fourth and final case PT, combines various aspects of the first three, but in slightly different ways. His thinking is analytic/symbolic, while he says "this would be a function that is ay ex squared plus bee ex plus cee, and then, you could represent that by a parabola." But, he uses a set of gestures (see Figure 3) to actually write out the symbols ax^2+bx+c. He then draws out a very big parabola (see Figure 4), in his upper left spatial field, with his index finger, and says “going like that.” His gesture points to an abstract object. He then says “of course, in that you can include a line, you can imagine a line in there, […] though a line is technically a quadratic function.” In his accompanying gesture, his whole hand moves from left to right, fingers extended, as if cutting out a plane (see Figure 5). That he sees the parabola becoming a line (as the parabola flattens out), it also appears that he sees the parabola moving continuously from a curved line to a straight one—whereas LG saw the parabola move continuously across transformations.

		
Figure 3. PT gestures the quadratic equation.	Figure 4. PT draws a parabola.	Figure 5. PT’s gestures line as parabola.

ANALYSIS OF SPEECH AND GESTURES: EIGENVECTORS

In response to our prompt about eigenvectors, LG first says: “I guess eigenvector might be a resonance so if you are in a big tunnel and you start singing and hit the right note it starts to go really loud resonating your ears.” He uses resonance metaphorically to describe abstract objects, eigenvectors. He evokes a visual image and conceives it in dynamic terms, as he uses the verb *to go.* LG’s linguistic expression alone reveals the presence of dynamic thinking. Unlike the examples above, LG’s dynamic thinking is not necessarily image-based; rather, the dynamism is in the echo, which starts to “go really loud.”

In our next example, we analyse NN’s linguistic and non-linguistic expressions to illustrate dynamic aspects of her mathematical thinking. In response to our prompt about eigenvectors, she says: “stresses, so if I am thinking about a plate being pulled out so it’s gonna move along principles.” She uses ‘stresses’ as a metaphor that refers to eigenvectors. She evokes a visual image of a plate and uses motion, as illustrates by the verb *pull out*, to describe her concept image of eigenvectors. Her speech coincidences with a set of gestures: Figure 6 shows how she embodies a dynamic imagine of eigenvector in the context of a real world example, “a plate being pulled out.” She clenches her hands and moves her arms back and forth, as if holding a horizontal steering wheel, to accompany her verbal expression. This is another *metaphoric gesture*.

Figure 6. Shows NN's hand and arm movements in describing eigenvector.

Our third case, JJ, first says "one idea is, you have the idea of matrix as a linear transformation and, um, so you take a vector and you map it to something else." This seems to describe a visual image of mapping on vector to another, though his description "matrix as a linear transformation" also indicates a more analytic conception of eigenvector. He then continues to say that "you set the matrix up by some inputs, they're gonna come inside and then obviously you say what is the important direction when the two line up of course. So, that is one idea that I use to say that there is something special about that direction." Here, he uses his hands to demonstrate a vector and its transformations: with his index fingers (on both hands) he rotates one finger toward the other (see Figures 7 and 8). His hand movements, which coincidence with the verbal description quoted above, show how he conceives the process of transformation dynamically, as something coming together in the same "direction."

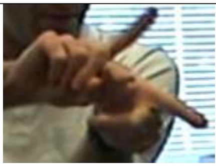

Figure 7. Shows JJ's use of hands to depict a vector and its linear transformation.

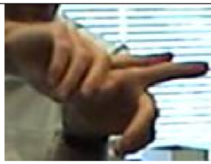

Figure 8. Shows how JJ's line up a vector and its transformation to illustrate the concept of eigenvector.

Our fourth and final case PT, in response to our prompt, says "I think of a matrix, I think of applying the matrix to the vector, and then what you get out is another vector that's in the same direction but either stretched or shrunk." Again, his linguistic expression reveals the presence of fictive motion in his conception of an eigenvector, which he describes as something you "get out," that is "stretched or shrunk." His speech coincidences with arm and hand movements that are similar to NN's gesture: starting with his hands and arms extended (as in Figure 9), he brings them toward each other as he says "same direction" and moves them away again when he says "stretched or shrunk." Unlike NN, who is referring to plates and stresses, PT seems to be thinking about the vectors themselves, and also using metaphorical gestures in describing them.

Figure 9. Shows PT's arm movements in illustrating result of transformation, stretched or shrunk in eigenvector.

DISCUSSION AND REFLECTIONS

The results of our analysis indicate that: first, the mathematicians use gestures and metaphors to express their thinking about concepts. Second, their linguistic and non-linguistic expressions comprise a dynamic component. However, while sometimes this dynamic component is visual in nature, other times it is no. This would suggest that some forms of dynamic thinking are non-visual, and more time-based. In fact, in Thurston's (1994) categorisation of the different "facilities of mind," he includes both a "vision, spatial, kinaesthetic (motion) sense" category and a "Process and time" category, where the latter refers to a facility for thinking about processes or sequences of actions.

As Núñez (2006) points out, the dynamic component of gestures and metaphors promote understanding mathematical concepts (Núñez, 2006). Following Zazkis et al.'s (1996) work, which draws attention to the important interaction between the visual and the analytic, we hypothesise that dynamic thinking is potentially a bridge between visual and analytic thinking: further research on this hypothesis seems warranted.

On a final note, turning now to the teaching and learning of mathematics: we suggest that the instructional use of gestures warrants further study. Cook, Mitchell and Goldin-Meadow (2008) reported that requiring students to gesture while learning a new concept helped to retain the knowledge they had gained during instruction. It seems reasonable to assume that not all gestures will work in this way; however, drawing on the gestures that mathematicians use to think about concepts may well provide guidance to educators looking to identify productive gestures for instruction.

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WHEN THE INFINITE SETS UNCOVER STRUCTURES: AN ANALYSIS OF STUDENTS' REASONING ON INFINITY

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We explore children's strategies in comparing infinite sets of numbers, based on an empirical study. We report four categories of structures that the children identified during this process: geometrical-based structures, topological-based structures, fractal-type structures, and arithmetical-type structures. Using the identified structure, students arrive at different perspectives on problem solving procedures.

INTRODUCTION

A specific literature deals with children's representations about infinity in mathematics. Prior research has suggested that children understand infinity as a property of processes (being endless), rather than as a number-like object that has an order of magnitude (e.g. Fischbein, Tirosh, & Hess, 1979) and has insisted on the role of representations in students' intuitive thinking about infinity (Tirosh & Tsamir, 1996). Because "our mind is essentially adapted to finite realities in space and time" (Fishbein, 2001, p. 309), intuition is decisive in understanding infinity.

On the other side, a large body of recent cognitive science and neuroscience research has revealed that children have stronger predispositions in processing numbers and sets of numbers than it was supposed to have in a Piagetian perspective (e.g. Carey, 2001; Dehaene, 2001, 2007; Hartnett & Gelman, 1998; Karmiloff-Smith, 1992; Mix, Levine, & Huttenlocher, 1999; Wynn, 1992). In particular, an intuition of infinity is strong enough so that students in early grades might be able to construct arguments for the infinity of some sets of numbers (Singer & Voica, 2008).

However, important aspects of infinity contradict intuition and are a source of paradoxes. Mamolo & Zazkis (2008) explore learner's conceptions of infinity through the lens of mathematical paradoxes. They noticed that when students face the comparison of infinite sets, in particular in the ping-pong ball conundrum paradox, intuition seems to be less functional.

In spite of this, many students we interviewed were able to find strategies to compare infinite sets and to express reasons for the result of comparing, even if those reasons were sometimes incorrect. We proceed in this paper to analyze students' approaches in comparing infinite sets of numbers at young ages, in the absence of any formal training regarding cardinal equivalency, as well as at the ages when they already studied calculus. We show that, when arguing about the infinity of a set and when

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 121-128. Thessaloniki, Greece: PME.

comparing the cardinals of two infinite sets, students inevitably arrive at emphasizing structures (structure) of those/ that set.

RESEARCH METHODOLOGY

Samples. The participants in our research were students in elementary education (grades 1 to 4 i.e. 6-7 to 10-11 years old), and students in secondary education (grades 5 to 12 i.e. 11-12 to 18-19 years old). In total, 209 students, among which 111 girls and 98 boys, answered to questionnaires. The children involved in the study were randomly selected from different schools; we avoided a selection based on student's performance in mathematics.

Tools. We have used two ways to collect data: questionnaires and interviews. The questionnaires contained several questions dealing with the concept of infiniteness. In the present paper, we analyze the following two categories of questions: How does one prove the infinity of a given set? (example: *Explain if the following sets are infinite: the set of the even numbers* {0; 2; 4; 6;…}; *the set of the divisors of* 123456789; *the set of the rational numbers between* 1 *and* 2.); How does one compare infinite sets? (example: *Which set of the following pairs of sets has more elements*: {0; 2; 4; 6; 8;…} *and* {0; 1; 2; 3; 4;…}; *the rational numbers between* 1 *and* 2, *and the rational numbers between* 2 *and* 3?). We interviewed 31 students who were selected from the sample who answered the questionnaires. These interviews provided us with the opportunity to rephrase some questions in order to clarify students' meanings and to probe their understanding more fully.

Method. We analyzed the answers given by the students, no matter if they were correct or not, and classified them according to the ways students formulated, explained and illustrated their arguments. We gave special attention to those answers that were accompanied by supplementary explanations. These answers led us to the following hypothesis: *students try to endow the number sets with specific structures in order to provide convincing arguments for the infinity of a set or for finding comparison criteria for infinite sets*. We checked this hypothesis along the interviews. There were three months between the administration of the questionnaires and the interviews. During the interviews, the students looked at their solutions on the questionnaires, and explained what they have been thinking when they were designing their solutions. In doing the interviews we started from a set up schema for the discussions, which was ad-hoc adapted to children's comments. We avoided validating students' judgements, sometimes in spite of their insistence concerning the correctness of their statements. To give a better insight into the children's spontaneous reactions, this study reports a qualitative analysis of the data, focusing mostly on the interviews.

IN SEARCH FOR STRUCTURES

Students refer to various structures to find evidence for the infinity of a set or for the cardinal equivalency of some sets. For example, Tiberiu (grade 9) speaks about the generating rule of the elements of a set seen as a modality to structure the sets:

Tiberiu: The set of natural numbers is infinite, because it is built following a rule: one starts form 0, and adds 1, again and again. The numbers are built following a rule in the set of rational numbers, as well: each of them is a natural number plus a sequence of natural numbers after the point.

We identified four types of structures based on students' comments: arithmetic-based structure, geometric-based structure, fractal-type structure, and density-type structure. We will define these structures in the following sections of the paper. Given the limited number of pages, we tried to select from the long interviews the most relevant quotations that highlight a specific type of structure.

Arithmetic-based structures

In the next fragment of the interview with Dragoş (grade 12), the structure of two real number sets is emphasized as a vehicle to determine a bijection between these sets, seen as an isomorphism.

Interviewer: How do you think we could prove that two infinite sets have the same number of elements?

Dragoş: Err… they should have the same structure… (*He has spoken about structure without any suggestion from the interviewer.*)

Interviewer: What structure?

Dragoş: Meaning that we can be aware, even if it is very difficult … as it is the case here, with these sets [0;1] and [3;4]. I know that after zero follows…I know for sure that it follows zero point zero …something, then zero point zero, zero … we don't know how many zeros, but many – to be very close to zero… then a number. In the same way we could say here too (*he points to* [3;4]) … 3 point zero…zero … and it is the same thing…

Dragoş sees the numbers from the considered sets through their decomposition of the whole part and the fraction part. The reference to „structure" is not only formal: this decomposition takes him, finally, to express the correspondence correctly (Fig. 1).

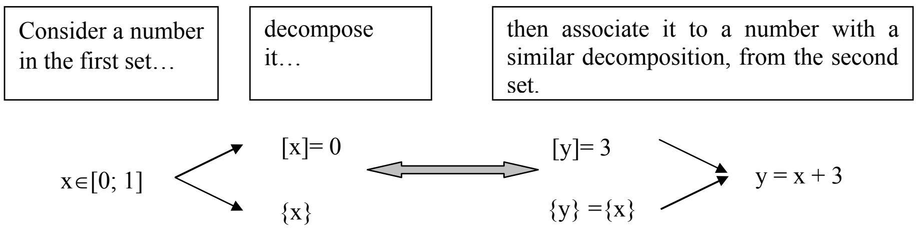

Fig.1. Finding a 1:1 correspondence using the arithmetical structure.

The identification of an algebraic decomposition, which is significant for the given purpose, grants an *a*-structure ("arithmetic-type structure") to sets of numbers. We define the *a*-structure as a means to interrelate the components of a system, according to its algebraic properties. Characteristic for an *a*-structure is the transfer of operations between the sets through generating bijective correspondences (in Dragoş's terms, "*it is the same thing*").

A function is a global concept: when one identifies the correspondence between the sets, the structures of the sets are seen as overall. Consequently, the element-by-element analysis becomes superfluous. Because the students identify functions when they compare two sets through arithmetically characterizing their structures, we might say that the arithmetical structures have a global character. To identify an arithmetical structure of a set supposes to characterize the generic element that extrapolates its properties to all the elements of the set; this does not imply mobility from one element to the other within the set (when the generic element is known the entire set is known). For these reasons, an *a*-structure endows the system (or its subsystems) with an organization that is global and static.

Geometric- based structures

The next excerpt is from the interview with Denise (grade 12):

> Denise: The interval [0;1] and the interval [2;3] look equal (…). They differ by an unit …they have the same length and I have the tendency to say that they are equal… I thought about representing them as a segment…

When Denise refers to the length 1 of the intervals, she uses measure to justify the congruence of two segments. In this way, she construes a geometric argument to prove an algebraic sentence: the measurement that allows comparisons among physical (and geometrical) objects is realized by superposition, and this leads to the idea of correspondence between the points of the two sets. Denise uses the number line representation to move from an amorphous set of numbers to a structured set.

The representation of the sets on the number line and the identification of their geometric properties which are significant for the given purpose, grants a *g*-structure ("geometric-based structure") to the sets of numbers. We define the *g*-structure as a means to interrelate the components of a system which highlight its geometrical properties. These are graphical, visual, iconic. Once a *g*-structure identified, students can use geometrical transformations as a way to prove the cardinal equivalence of two sets of numbers.

Characteristic for a *g*-structure is the use of congruence as a way to show the cardinal equivalency of some infinite sets. Congruence might intuitively appear as superposition through a slide. *G*-structures suppose a transfer between algebra and geometry, during which the initial configuration is mentally modified. *G*-structures also suppose a holistic vision of the set, which is transferred through representation.

For these reasons, a *g*-structure endows the system (or its subsystems) with an organization that is global and dynamic.

Fractal-type structures

Sometimes students identify structures that are of a fractal type. Intuitively, a fractal is a configuration that self-generates through homothety. For example, to argue the infinity of the set of rational numbers between 2 and 3, Alice (grade 6) highlights a tree configuration:

Interviewer: Is the set of rational numbers between 2 and 3 infinite?

Alice: Yes!

Interviewer: Why? Look, I have the smallest number and the biggest… why should this be an infinite set?

Alice: Well, yes, could be 2.1; 2.11 … I mean 2 point … 111 and so on …

Interviewer: And you say they are infinitely many…

Alice: Perhaps they are not quite infinitely many, because finally we still get to number 3, but they can be said as a sequence … it might be … number 2.1., it might be 2.11 to 2.19, and so on … number 2.11 might be 2.111 and 2.119 …and so on…

We notice that, even if Alice does not have a clear idea about an infinite set ("*perhaps they are not quite infinitely many"*), she shows however a spatial-rhythmic perception (Singer & Voica, 2008) about infinity (" *and so on ... and so on…"*). Alice's argument is based on a tree-type graph (see fig. 2).

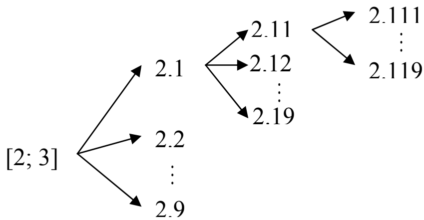

Fig.2. The structure identified by Alice (grade 6) to argue that the rational numbers between 2 and 3 are infinitely many.

When, in order to describe the elements of a set, a procedure is sequentially repeated at different scales, the set was granted an *f*-structure ("fractal-type structure"). We define the *f*-structure as a general means to organize a system which highlights the sequential generating of its subcomponents by repetition at different scales. Because a *f*-structure works with different scales, it has a local character. The *f*-structure is in the same time self-generated in a sequential mode, through the application of a rule. This is why a *f*-structure endows the system (or its subsystems) with an organization that is local and dynamic.

Density-type structures

In interviews and questionnaires we identified many references to the density of sets, seen as a degree of piling up the elements of the set. The next fragment is an excerpt from Denise (grade 12)'s interview.

Interviewer: Which are more: the natural numbers or the rational numbers?

Denise: The rationals.

Interviewer: Why?

Denise: Simply because we have 1.12; 1.13…

Interviewer: So what? In ***N***, we also have 1, 2, 3, 5…

Denise: Yes, but the divisions, in the case of the set of natural numbers are bigger…, on the contrary the divisions, in the case of the set of rational numbers are smaller, and then they are more numerous…

Denise tries to compare the cardinals of ***N*** and ***Q***. She notices that the natural numbers are “rarer”, while the set of rational numbers is more “crowded”. The idea of piling up, crowdedness, the step of succession, or density seen as an intuitive measure of the set – how ”crowded” the set is – leads to associate to a specific set a *d*-structure ("density-based structure").

We define a *d*-structure as a means to interrelate the components of a system which emphasize its local topological properties (concerning vicinity, approximation, frontier). In general, the endowment of a set with a *d*-structure favors extrapolations from local to global. In this way, topological structures in a wider sense are emphasized, i.e. structures that suppose the invariance at changing the shape. Density-based structures have a dual nature. On the one hand, in the construction of a *d*-structure the topological perception is activated, because the child evokes density/ jam/ accumulation of the elements of a set. On the other hand, *d*-structures appear especially in a discrete context, in which the students appeal to the processional recursive perception. This is why a *d*-structure endows the system (or its subsystems) with an organization that is local and static: the child tracks the description of the vicinity of an element of the set.

Because the *d*-structures have a transfer potentiality, allowing extrapolations local-global, the students who intuit *d*-structures use restrictions on certain types of sets (finite) of the given sets, they analyze local density properties on those restrictions, and extrapolate the conclusions to the departure set. This explains a frequent misconception recorded by researchers who studied children’s understanding of the part-whole relationship for infinite sets (e.g. Tsamir, 1999).

CONCLUSIONS: SOME APPLICATIONS

We have presented above the four types of structures identified in students’ comments when they compare infinite sets. We might ask: to what extent are these structures generalizable? Do they only appear in the context of infinite sets or are they present in other mathematical contexts?

The *g*-structures become active within the transfer algebra-geometry, facilitated by the number line (or by the system of Cartesian coordinates), when the problem context allows for a geometrical representation. For example, the graphs of the functions of the type x $-x^2$, x x^2+1, or x $(x+3)^2$, can be drawn starting from the generic function $x \longmapsto x^2$, through the application of some geometrical transformations of the graphs. In this way, one highlights a connection that allows the transfer of the algebraic properties of a numerical function to another numerical function through the geometrical representation. The activation of a *g*-structure facilitates this transfer.

The recursive processes based on changing scales seem to need the activation of a fractal-type structure. For example, a *f*-structure is activated for understanding the transformation from a unit measure to another one. These structures seem to be specific also for the way in which we understand the numerical systems: the numeric magnitude orders of base ten (units, tens, hundreds, etc.), as well as for other bases, are defined through grouping (or dividing) other groups. Fractal-type structures are also activated in doing some algebraic operations. For example, when dividing 11 by 4, we recursively use the grouping of fourths and the division of a unit in ten units of the next lower order of magnitude (fig. 3). The difficulty the students encounter when they learn the division algorithm might be given by the passage among different numerical scales before internalizing a *f*-structure.

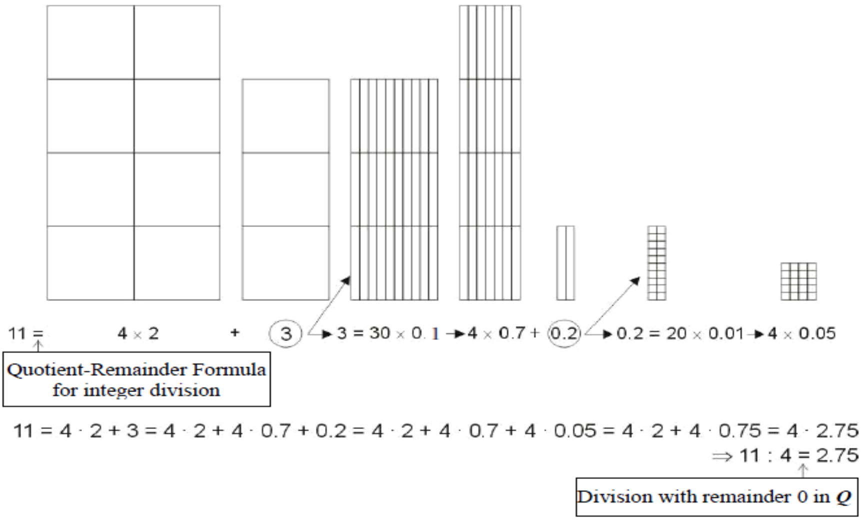

Fig.3. The *f*-structure involved in the division algorithm. The representations show the process of the division for 11: 4

The arithmetic-type structures operate with the decomposition of the elements in their constituent parts. This decomposition uncovers functional transfer between two sets. From this perspective, they are activated in any process that supposes input-output. At an elementary level, a function between two sets, for example $f : A \to B$, $f(x)= x+5$, "transports" the elements x of the input-set A into the elements $x + 5$ of the output-set B.

The phenomena of convergence, limit, or, more generally, the recursive processes in which the scale is preserved suppose the activation of a *d*-structure. Thus, the fact that the students have a local topological perception (expressed through understanding the density properties) can be used (and it is actually) in the design of the functional graphs at the 'endpoints' of the definition intervals, before the in-depth study of calculus. *D*-structures are also typical for statistical methods: we extrapolate conclusions having a local character (obtained on a sample) to an entire population. In conclusion, the four types of structures identified in students' reasoning about the infinite sets are activated in various mathematical contexts and they help or hinder (depending on the students' knowledge) the understanding of some important mathematical concepts and procedures.

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FROM ARITHMETIC TO INFORMAL ALGEBRAIC THINKING OF PRE-SERVICE ELEMENTARY SCHOOL MATHEMATICS TEACHERS

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In this paper we report the study on algebraic thinking of prospective elementary teachers. Students from three teacher education colleges have been asked to solve multi-level problems that had included the main components of algebraic thinking in non-formal manner. Teachers' responses to two of those tasks were closely examined and those analyses are presented in this paper. We found that almost all prospective teachers know to fulfil correct calculations and to recognize arithmetic patterns. However, important misconceptions of study participants is accepting set of observations as generalization itself neglecting the essential justification and thorough prove.

BACKGROUND

Introducing basic forms of algebraic reasoning in elementary school seems an effective tool for enhancing pupils' comprehension of arithmetic procedures (Blanton & Kaput, 2005; Booth, 1984; Kieran, 1992; Lee & Wheeler, 1989; van Dooren, 2003). The transition from arithmetic to algebra might be made easier through the introduction of intuitive concepts from "early algebra" - a systematic use of the ideas of algebraic thinking in the primary school (Carpenter, Franke & Levi, 2003; Schifter, 1999 and others). For example, comparison between two sums 9+8 and 6+5 by means of algebraic considerations that is focused on relation between numbers (Sinitsky & Ilany, 2008) serves as a promising basis for dealing with well-known problem of restricted comprehension of equality sign (Carpenter Franke & Levi, 2003).

Descriptions of early algebra vary in different research sources. However, several features of early algebra are broadly accepted: reasoning about relationships between unknown quantities - including solving word problems with the language of letters and equations; generalizing arithmetic expressions into algebraic patterns and understanding these patterns as shorthand for computations; recognizing identities (including commutative, associative, and distributive laws) as general properties of operations on a given sets of numbers; using a variety of representations, including conventional algebraic notation to derive and to justify generalizations and to articulate them.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 129-136. Thessaloniki, Greece: PME.

In the Israeli syllabus for primary school, several components of algebraic thinking are presented, for example: involving sequences of computations with numerical patterns, primary symbolizing for unknown quantities, study of regularities in numeric tables *etc*.

Accordingly, teachers need non-formal algebra knowledge in algebra in order to stimulate and support such activities in a classroom. Blanton and Kaput (2005) argue that elementary teachers must develop algebraic "eyes and ears" as a new way of looking at the mathematics they are teaching. They must appreciate the power of generalization. They should be able to recognize suitable context when students are approaching this territory. They need to understand mathematical and didactical aspects of justification (CBMS, 2001).

There is an insufficient research on teachers' knowledge in the area of algebraic instruction (Doerr, 2004, Kieran, 2006), especially concerning prospective teachers (Hallagan *et al*, 2006). Recent initiatives call for further study on developing and promoting effective algebraic teaching practices for elementary school teachers (RAND, 2003).

RESEARCH

Although, during their professional education, prospective teachers of elementary school mathematics in Israel learn several courses of arithmetic and algebra subject matter, their educators are doubting the ability of these students to apply theoretical knowledge in future teaching practice (Sinitsky & Guberman, 2006). Thus, we have decided to investigate the familiarity of pre-service teachers with early algebra thinking and comprehension.

Research objectives

The present research concerns two questions:

- What are the aspects of early algebra that prospective teachers of mathematics in elementary school know and understand?
- What are the aspects of early algebra that require additional intervention in the process of professional development of elementary school mathematics teachers?

Population

Participating in this study were forty seven prospective mathematics female teachers in the elementary school from three academic colleges in Israel. These students were in their second or third year of four-year professional development programme - meaning they have already completed the regular algebra and arithmetic courses.

Tool Design

The prospective teachers have been tested with the questionnaire that includes seven tasks on early algebra and check the main components of algebraic thinking in non-formal algebra projection. Four tasks have been posed as multi-level problems to provide data on several components of early algebra thinking (see the list above) of the participants.

Each of these problems starts with numeric examples and goes to generalizations and justifications. In fact, this method based on generic examples (Tall, 1991) is an approach that can be used by elementary school students in solving mathematical problems. Regarding that, participants' approach to proposed questions needs to emulate the early algebra activity of young students. By this way we try to study in which scale pre-service teachers have competence in dealing with such assignments.

In this paper we focus on the analysis of responses to two tasks of the set.

Task 2 Here is a sequence of subtraction exercises:

$\frac{1}{2}-\frac{1}{4}=$ $\qquad$ $\frac{1}{3}-\frac{1}{5}=$ $\qquad$ $\frac{1}{4}-\frac{1}{6}=$

a. Solve the exercises and construct the next exercise of the series.

b. Propose the pattern for generalization and calculation of such differences.

c. Justify the way of calculation

d. Construct additional exercises of the same type and derive the answer without additional calculation.

Task 6

a. Choose an arbitrary two-digit whole number, construct the number from the same digits that are ordered in inverse order and subtract a smaller number from a greater one.

b. Repeat to the same operations with another number.

c. Try to reveal the regularity.

d. Explain/justify the result of previous stage and seek for additional implementation.

RESULTS AND DISCUSSION

The presented problems suppose construction or understanding of trial example(s) followed by generalization and justification, both in arbitrary – but not necessary symbolic – form. According to the structure of the tasks, all the responses of prospective teachers have been grouped. The responses of participants to the question of tasks 2 and 6 are summarized in Table 1 below.

N=47	Arithmetic solution	Generalization in various forms			Justification	Implementation
		numeric	verbal	symbolic		

<table>
<tr><td>Correct</td><td>94
75</td><td>30
-</td><td>13
43</td><td>4
13</td><td>13
15</td><td>9
2</td></tr>
<tr><td>Incorrect</td><td>4
4</td><td colspan="3">17
12</td><td>-</td><td>-</td></tr>
<tr><td>No answer</td><td>2
21</td><td colspan="3">36
32</td><td>87
85</td><td>91
98</td></tr>
</table>

Table 1: Distribution of response (percentage) for task 2 (first number in each cell) and task 6 (second number in each cell)

The amount of students that have successfully accomplished all the stages of the task and derived the complete solution for each task is almost ignorable. The initial stage of specific numerical examples was typically successful completed and reasonable part of participants has made various generalizations.

We have found that only a very small part of participants had tried to justify their guesses in any form. As it turns out from detailed analysis of their answers, pre-service teachers are familiar with the terms and notions of common scheme of problem solving but they make their own comprehension of basic steps of mathematical search. Primary it concerns the stages of generalization and justification.

In our study we have discovered misconceptions concerning both **structure** of **generalizations** and their **role** in justification process. Normally, at the phase of generalization some wide-ranging patterns are elaborated in order to verify, to modify and to apply them at further stages. In such a pattern various relevant data need to be linked and integrated.

Structure and relevance of generalizations

Different views of generalization have been observed through our study. As usual, they might be either correct or incorrect. Notably, each of those generalization types does not connect with any specific manners that express the regularity: they appear in numerical, verbal and symbolic form.

Understanding of the structure and connection inside the specific expression

In Task 2 three exercises on subtraction of fractions have been displayed. Part of the students have calculated the appropriated differences and after that has related just to the regularity in the left sides of obtained equalities. The derived results have been totally ignored through those generalizations. Typical examples of such a pattering were "It is always can be written as $\frac{1}{x+1}-\frac{1}{x-1}$ *(just with omitting of equality sign)*" or "The denominator of the first fraction is always smaller by 2 from the denominator of the second one". In the responses to task 6 similar expressions have been detected, as "If in a given number the unit of digits is smaller than the units of tens, it will be vice versa in the second number".

It is clear that observing such regularity might be a step to construct a pattern for obtaining a result in a general form - but it is not a complete pattern at the moment.

Generalizing of connection between proposed exercises

This sort of generalization may be illustrated with the sentence "In each next row, the denominators of both fractions are greater by 1 than in previous one; $\frac{1}{x}-\frac{1}{y} \Rightarrow \frac{1}{x+1}-\frac{1}{y+1} \Rightarrow \frac{1}{x+2}-\frac{1}{y+2}$." In contrast with the previous reasoning, here the link between the operands of subtraction itself has been not observed. As previously, all the generalizing relates to the given expressions themselves and not to the connection between the exercise structure and obtained result of calculation.

Guessing the answer

Some participants have chosen to guess an answer immediately after few calculations have been fulfilled. They claimed (Task 2) "The difference is always equal to 2 divided by product of denominators" and "The difference is $\frac{1}{2x}$ or $\frac{2}{2x+1}$, it depends on denominators are add or even ones." Similarly, for Task 6 it has been claimed "The difference of pair of such numbers is always equal to 9".

Even though there is nothing wrong with the declaration itself, the lack of argumentation does not allow accepting those statements as really construction of suitable pattern.

Construction of the relevant pattern

Fortunately, part of prospective teachers of mathematics is aware of generalizing of specific derived results as a stage prior to justification and application of the general statement.

> (Task 6) An arbitrary two-digit number with digits *x, y* may be written as $10x+y$, accordingly the second number is $10y+x$. Now we can derive the difference between the larger number and a smaller one.

Neither distribution of generalizations through various forms nor sorting them as formally correct and incorrect provides the point of the main problem of this reasoning. According to the collected data, almost 80% of pre-service teachers catch the single and very partial feature of mathematical object or situation and see it as generalization. It is impossible to classify these statements as false or incorrect ones in mathematical sense, but there are simple **observations** rather than generalizations.

Those participants don't try to look for manifold of links and also are convinced that this reasoning is a justification of their solution for proposed task. As a result, prospective teachers of mathematics miss out proving their guesses and integrating their conjectures in further stages of search.

On justification and conclusions

Accordingly, only quarter of participants has replied the questions concerning justification. Part of them has either started "from the very beginning" with no reference to previous phases: "From $\frac{9}{9}x=\frac{9}{9}y$ it follows that $x=y$ " or failed with algebraic technique: "$(10x+y) - (10y+x) = 10x + y - 10y + x = 11x - 9y$" (both examples are from Task 6).

Nevertheless, typically right generalization has been completed with justification procedure:

> (Task 2) The difference of two fraction may be written in the form $\frac{1}{n}-\frac{1}{n+2}=\frac{(n+2)-n}{n(n+2)}=\frac{2}{n(n+2)}$. When n is odd number the fraction is non-cancellable, other case the numerator of the result will be equal to 1 because of cancellation. In this case the denominator is half-product of denominators of given fractions."
>
> (Task 6) Every two-digit number with digits x, y may be written as $10x+y$, accordingly the second number is $10y+x$, and the difference is equal to $(10x+y)-(10y+x)=10x+y-10y-x=9x-9y=9(x-y)$. Thus, the difference is multiple of 9.

By this or similar way six participants proved that the constructed difference is always divisible by 9. As to application of obtained pattern, only one student noted that the quotient of this division is equal to the difference of digits of chosen (or inverted) whole number.

Since all the justifications have been performed in symbolic form, we still continue to dream about the justification that is generalization of numeric generic example, i.e.: $82-28=(80+2)-(20+8)=(80-20)-(2-8)=10\times(8-2)-(8-2)=9\times(8-2)$. Since such a way of justification might be developed by students that are not familiar with algebraic technique, it is of particular importance for prospective elementary school teachers.

Although problems of non-sufficient implementation of obtained patterns and poor variety in justification styles can not be neglected, but the main finding that needs an intensive intervention is a very low percent of pre-service teachers that are confident with the need of justification at all, and are aware of way to do it.

CONCLUSIONS AND RECOMMENDATIONS

The transition from formal arithmetic and algebraic skills to deep comprehension of algebraic nature of numerical problems and procedures is not trivial. Despite the fact that prospective teachers are familiar with arithmetic algorithms and algebraic symbolism, they lack comprehension of early algebra. For most of them, the fact of using **algebraic notation** or talking about any **regularity** makes **generalization**. Additionally, a guess that has been formulated in general form replaces them the verification and explanation. These pre-service mathematics teachers are not aware of **the need of proof** and necessity of justification of their conjectures. Those who come to justification don't try to do it in non-symbolic form.

As a result, even students with no difficulties in algebraic technique don't succeed in the construction of two-way bridge between arithmetic and algebra. Most future teachers fail to both justify a way to solve arithmetic problem in a generalised form and to interpret derived algebraic result in meaningful terms.

Having such an attitude and background, those teachers of mathematics could not assist their students to develop algebraic-style reasoning within elementary school mathematics. To improve the situation in a framework of professional development of pre-service teachers, special attention must be paid to the development of abilities for proper generalizing. At the same time, it is necessary to clarify and to improve comprehension of the relations between generalization, justification and verification in mathematical reasoning.

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YOUNG CHILDREN'S MATERIAL MANIPULATING STRATEGIES IN DIVISION TASKS

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The aim of the present research, which is a pilot one, is to study kindergartners' ways of manipulating materials in order to divide them in different situations (questionnaire and board game). The main research questions were: What strategies do young children use to manipulate different types of materials in order to divide them? Is there any distinction between children's material manipulating strategies in the questionnaire's division tasks and in the board game's division tasks? Are materials manipulating strategies related somehow to the calculation strategies children use in division tasks? It was found that the young children who participated in the study used several strategies to manipulate materials in order to divide them which differed according to the material's type and the situation. These strategies seemed to be related to the calculation strategies children use in division tasks.

INTRODUCTION

Young children come to kindergarten with a lot of informal mathematical concepts and variable skills and abilities. Several recent studies have shown that kindergartners can solve a variety of division problems before formal instruction on the operation (Baroody, 2004; Huntinh & Sharpley, 1988; Neuman, 1999). For example according to Frydman and Bryant (1988) high proficiency demonstrated by 3-year-old children in equally distributing items may be a repeated drill or repetitive action learned from others. They also reported that children at that age manage quite well in informal tasks in which they have to share out discontinuous quantities among two or more people.

For solving division tasks children use a range of intuitive calculation strategies (Greer, 1992). According to Fischbein et al., (1985) the primitive model which is associated with equivalent groups division is the partitive ('sharing') model. The quotitive ('measurement') model (carried out through repeated subtraction) is acquired with instruction. But other researchers like Mulligan (1992) showed that the young pupils in her study preferred for partitive and for quotitive divisions, an additive building-up model.

However, in these studies it is not mentioned what strategies young children use to manipulate the materials in order to divide them, when dealing with different situations. The aim of this pilot study is to investigate children's ways of manipulating materials in order to divide them in two different situations: in a questionnaire and in a board game.

THEORETICAL FRAMEWORK

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Several researchers have studied, from different perspectives, how young pupils divide (Neuman, 1999: 103). There are studies which investigate the difficulties that appear in partitive and in quotitive divisions as well as studies that investigate the mental processes in calculating division problems. Other studies present the different calculation strategies that children use for solving division tasks. Studies that focused on the factors that influence children's ability to divide constitutes another type of research. Among these factors are mentioned the context and the formality of the situation as well as the type of the materials used.

There are two contrary views about calculation strategies that are used in division tasks. For some researchers (Bell et al., 1984; Bell et al., 1989) mathematically equivalent problems of different semantic structure evoke different calculation strategies and vary widely in difficulty. For others (Kouba, 1989), these problems did not generate different calculation strategies. As Kouba suggests both partitive and quotitive division problems generate strategies such as: 'sharing', 'repeated taking away', 'building up' and 'multiplicative calculation'. The most frequently used strategy for sharing tasks (Frydman & Bryant, 1988; Neuman, 1999) is the 'dealing' competence which is not related directly to counting skill (Pepper & Hunting, 1998). When 'dealing' young children successfully share materials by assigning an item or piece of the material to each recipient in turn, repeating this action until all the parts have been allocated. Besides 'dealing', Neuman (1999) found that children in grade 2-6 used, for partitive division, strategies such as: 'estimate-adjust', 'dividing', 'repeated halving' and 'repeated estimation'. For quotitive division, children used strategies such as 'counting', 'repeated addition', 'chunks' and 'reversed multiplication'. In another study (Mulligan & Mitchelmore, 1997) it was found that students (in grades 2 and 3) used three main calculation strategies of multiplication and division: 'direct counting', 'repeated addition', and 'multiplicative operation'. A fourth model, 'repeated subtraction', only occurred in division problems. Although all the calculation strategies were used with all semantic structures, their frequency varied according to age, size of numbers, language, and semantic structure of the problems.

Children's ability to share is related to the context as well as the formality of the situation. Desforges and Desforges (1980, in Pepper & Hunting, 1998: 165), identified a distinction between social sharing and mathematical sharing. They mention that in social sharing the mathematical importance of the action might be lost. This eliminates the need to use a dealing strategy to establish fair shares. Davis & Hunting (1991) found a total absence of sharing in an informal situation without an adult present. But systematic sharing has been observed in structured interview situations in which discrete items were to be distributed (Pepper & Hunting, 1998: 165).

The type of the material, in the sharing task, as well as the number of the material's parts influences the difficulty of the division. In general three forms of materials are used in sharing tasks. The first one is discrete items which can be distributed by the 'dealing' procedure. The second type is continuous material in which partitions of the appropriate size and number must be generated by the child. The third type is composite material which can be conceptually organized into units or 'chunks' (Wing & Beal, 2004). Studies (Charles & Nason, 2000) have shown that young children's performance in sharing tasks tends to be better with discrete and composite materials than with continuous materials. In one study (Frydman & Bryant, 1988), 4-5 year old children did not have difficulty in equally sharing some hypothetical candies between two dolls when the candies were all in units of one. However, when it was asked to equally share them with one doll wanting her candies in units of two (double units), and the other wanting her candies in units of one (single units), these children gave to the first doll twice as many candies as to the other. Regarding the material's parts, studies (Pothier & Sawada, 1983) indicate that children find partitioning tasks that involve equal numbers of parts easier than those that involve odd numbers of parts because the even number tasks can be solved with successive halving.

The present paper attempts to answer questions related to the above issues using data from an experiment which investigated the way six kindergartners manipulated several types of materials (discrete, continuous and composite) in order to divide them in two different situations—questionnaire and board game. The main research questions were: What strategies do young children use to manipulate different types of materials in order to divide them? Is there any distinction between children's material manipulating strategies in the questionnaire's division tasks and in the board game's division tasks? Are materials manipulating strategies related somehow to the calculation strategies children use in division tasks?

METHOD

Children were asked to judge the relative amounts of material shared between two characters. Two types of research tools were used. The first type was a typical questionnaire and the second type was a board game. Division tasks were involved in both cases and were similar. They have been constructed so that no remainder has been involved. They included partitive and quotitive situations, with three number groups (0-8, 8-14 and 14-20) and with several types of materials (discrete, continuous and composite). With the questionnaire the children were also shown two dolls and were asked (1-9 questions) to equally divide them the given materials. With the board game, there were only two questions (question 8 and 9) that were posed to each child before the game started. They then played the game which had the simple rule: when you reach a purse you must share the contents with your co-player in order to construct a bracelet (each child could open 0-6 purses). Fortunately each child opened 3 purses). The questions were the following (in the brackets materials used in the board game are described):

1. Can you divide this string between these two dolls? (thread)

2. Can you divide these 16 candies between these two dolls? (16 green beads)

3. Can you divide these 8 bananas between these two monkeys? (8 yellow beads)

4. Can you divide these 6 notebooks between these two dolls? (6 rigatonis)

5. Can you divide this strip between these two dolls? (straw)

6. Can you divide these 10 cubes (1-1-1-2-2-3) between these two dolls? (10 purple beads 1-1-1-2-2-3)

7. Aunt Mary wants to divide these 20 lollipops between her two nephews. Can you help her? (20 red candies)

8. The teacher has bought 10 pencils. If she gives 5 pencils to each child, how many children will get pencils? (10 blue candies)

9. The teacher has bought 10 pencils. If she gives 2 pencils to each child, how many children will get pencils? (10 green candies)

The research was conducted in a state Kindergarten on Rhodes in Greece. Initially each child was interviewed individually, according to the structured questionnaire, in a separate room. The interviewer recorded the children's responses as they solved each problem, noting down children's way of manipulating the materials in order to divide them in two. Each interview was lasted for 10-20 minutes and it was audio taped. After two weeks children in teams of two were asked to play the board game. The board game lasted 30-45 minutes and was videotaped.

RESULTS

The data analysis focuses on the material manipulation strategies children used in order to divide the different types of materials in the different situations (questionnaire and board game). Additionally, the relation that these strategies may have with the calculation strategies children use in division tasks is analyzed (in the Tables "ques" means question in the questionnaire and "activ" means activity in the board game).

Discrete materials	**Sharing by …**							
	1	**2 and then by 1**	**2**	**2 and then by 3**	**4**	**3 and then by 1**	**7 and then by 1**	**9 and then by 1**
20 lollipops (ques7)	3	1	0	0	0	1	0	1
20 candies (activ7)	3	0	0	0	0	0	0	0
16 candies (ques2)	2	1	1	1	0	0	1	0
16 beads (activ2)	3	0	0	0	0	0	0	0
8 bananas (ques3)	3	1	0	0	1	1	0	0

8 beads (activ3)	2	0	1	0	0	0	0	0
6 notebooks(ques4)	6	0	0	0	0	0	0	0
6 rigatonis (activ4)	3	0	0	0	0	0	0	0

Table 1: Number of discrete material manipulating strategies children used for partitive problems

Children manipulated the discrete materials in partitive problems using several 'sharing by' strategies (Table 1). The most commonly used strategy was 'sharing the materials by 1' which was used for the smaller as well as for the larger amounts of materials. In the case of the 6 notebooks, this strategy of manipulation was possibly used because of the size of the material (it was the biggest material used). Other strategies that were used were: 'sharing by 2 and then by 1', 'sharing by 2', 'sharing by 2 and then by 3', 'sharing by 4', 'sharing by 3 and then by 1', 'sharing by 7 and then by 1' and 'sharing by 9 and then by 1'.

It is worth mentioning that in the board game all the children shared the materials by 1 whereas in the questionnaire they also shared by other amounts. This may have happened because the materials were in the purses and children did not have sense of the total amount to be shared. This led the children to use simple strategies.

The above material manipulation strategies are related to the counting strategies children often use in partitive problems. For example, the 'sharing by 1' manipulation strategy is in accordance with 'dealing' procedure. In that procedure one item was dealt to each person in every round. The 'sharing by other amounts' material manipulation strategy is in accordance with the 'estimate-adjust' procedure in which more than one item was dealt in the first round and then, in the second round, children shared the remaining amount.

Discrete materials	**Sets of 2**	**Sets of 5**
Pencils (ques8)	-	3
Candies(activ8)	-	1
Pencils(ques9)	3	-
Candies(activ9)	1	-

Table 2: Number of discrete material manipulating strategies children used for quotitive problems

In quotitive problems (Table 2) a typical manipulating strategy was to separate the material into groups of the specified size until the materials were exhausted and then to count the number of groups. This is in accordance with 'repeated subtraction' counting strategy in which the number of items represented by the divisor was repeatedly taken from the whole number and put into separate places. Thus children made sets of 5 and sets of 2 according to the division task.

Continuous materials	**Fold and cut exactly in the middle**	**Cut about in the middle**	**Cut in many pieces**
String(ques1)	2	4	0
Thread(activ1)	1	2	0
Strip (ques5)	2	4	0
Straw (activ5)	1	0	2

Table 3: Number of continuous material manipulating strategies children used

Three different strategies were used when children manipulated continuous materials (Table 3) in order to divide them to two: 'fold and cut exactly in the middle', 'cut about in the middle' and 'cut in many pieces'.

There is a distinction between the strategies that were used in the questionnaire and these that were used in the board game. All children who participated in the questionnaire cut the continuous material given (exactly or about in the middle) in two pieces giving one piece to each doll. In the board game children chose the strategy they used according to the purpose. Specifically when they had to share the thread they cut it (exactly or about in the middle) in two pieces giving one piece to each of them in order to use it as a basis for their bracelet. But when they had to share a straw the two children cut it in many pieces and share them by one in order to pass them through their thread. Whatever strategy they used to manipulate the continuous material, the counting strategy they used was the 'dealing' procedure.

Composite materials (1-1-1-2-2-3)	**Separate and share by 1**	**Connect and share by 5**	**2-1-1-1 & 2-3**	**3-1-1 &2-2-1**
Unifix cubes (ques6)	4	1	1	0
Connected beads (activ6)	0	0	1	2

Table 4: Number of composite material manipulating strategies children used

The manipulation of composite materials drives children to treat the different materials in different ways (Table 4). Regarding the unifix cubes which could be separated and connected, children used three different strategies: 'separate them and share by 1', 'connect them and share by 5' and 'take 2-1-1-1 & give 2-3 to the co player'. Two of the children, in spite of separating and divide per one, like some did, the first one connect them per 5 so that they would have the same height and then share by 5 and the second share them in combinations.

The manipulation of connected beads which could not be separated was different. Children shared them in combinations. They made two different combinations: 'took 2-1-1-1 and gave 2-3 to the co player' and 'took 3-1-1 and gave 2-2-1 to the co player' which are in accordance with the counting strategy of 'estimate-adjust'.

DISCUSSION-CONCLUSIONS

This study aimed to investigate children's way of manipulating three types of materials—continuous, discrete, and composite—in order to divide them in two different situations: a questionnaire and a board game. It was found that the kindergartners used different strategies to manipulate the different types of materials in each situation. Performance with discrete materials was significantly better than the other types of materials and this is consistent with the work of Wing and Beal (2004). On the contrary to this work, in the present study, it was found that the continuous materials were manipulated easiest from the children in relation to composite materials. Composite materials were of particular interest since they needed different manipulation. In the questionnaire, the children had the chance to separate or/and combine the cubes and share them, as they wanted. In the board game, they had to find a way to manipulate the connected beads fairly without separating them. Although this procedure disturbed children it did not lead them to wrong answers, as to Frydman and Bryant (1988) study in which they made unequal sharing giving twice as many items to the one than to the other.

Material manipulation differed in the two different situations and this may be in accordance with the studies which identified a distinction between social and mathematical sharing. In the board game children used simple strategies for manipulating discrete materials whereas the questionnaire drove them to use more compound ones. The strategies they used for manipulating the continuous and the composite materials were different in the board game in relation to the questionnaire and they were in accordance with the purpose of the game (make a construction).

This study extends previous research by comparing children's way of manipulating different types of materials with the counting strategies they use for division tasks. It was found that young children' materials manipulating strategies are in accordance with 'dealing', 'estimate-adjust' as well as with 'repeated subtraction' counting strategies.

Previous findings that problem difficulty varies with semantic structure have been confirmed. Manipulating materials in quotitive problems seemed to be more difficult for children compared with partitive problems.

Some limitations of this study were: a) The use of dividing only in two. This is significant because in studies of children's partitioning behaviour, halving emerges as a very early strategy. Thus, the concept of halving may be qualitatively different from other partitions. b) The very small sample, because this study was a pilot one, so the results can not be generated. c) The chance that is included in a board game which influence the amount of activities which each player can be involved in the board game.

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CROSS-CULTURAL COMPARISON OF THE EFFECT OF CAUSAL NARRATIVES IN SOLVING MATHEMATICAL WORD PROBLEMS

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The purpose of this study was to investigate whether combining causal and outcome related elements with mathematical content improves mathematical word problem performance, thus providing a more meaningful context to construct better situation models within an international context. Participants were 178 undergraduate students from the USA and Turkey majoring in elementary education. They were given four story problems in three formats: standard (minimal verbiage), potential causation (causal and mathematics content overlap), or climax resolution (potential causation with a discernable outcome). There are significant differences among the formats, as well as between genders and countries. Mixed results suggest solving word problems might involve the use of both schema and situation models equally.

INTRODUCTION

There are many factors affecting students' performance in mathematical problem solving. One of them is the wording of the problem. There is evidence that shows it has an effect on students' problem solving (e.g., Coquin-Viennot & Moreau, 2007). Wording also affects the strategy choice of the students (Thevenot & Oakhill, 2005) which in turn affects performance in problem solving. The strategy and thereby performance might be dependent on how the students represented or modeled the problem in their mind.

Whether solving mathematical word problems involves primarily schemas, invariant recipes for problem structures (Kintsch & Greeno, 1985; Riley et al, 1983; Devidal et al., 1997), or situation models, episodic models of the unique elements of the story (Thevenot et al., 2007; Coquin-Viennot & Moreau, 2003), has not been resolved. This query has important implications for how the word problems are designed systematically so that the students will be better problem solvers. The authors suggest a new method for addressing the schema versus situation model conundrum (Smith,

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Gerretson & Olkun, 2009). Borrowing from theories that readers create *situation models* while reading text narratives (Van Dijk & Kintsch, 1983; Zwaan, Langston, & Graesser, 1995), the authors investigated whether combining *causal* elements, with *outcome related* elements into mathematical content boosted students' performance; that is, does putting the story back into "story problems" improve performance on word problems.

THEORETICAL FRAMEWORK

It is possible to explain students' different representations in two competing theories on the solving of word problems: According to *schema* model, students solve word problems by plugging the particulars of a word problem into a schema, while the situation model suggests that students create a model of the problem *situation*. The schema-based explanation of word problems suggests that students successfully solve a number of similar type word problems; then, when encountering another problem of the similar type they access that schema from long term memory (LTM) and use it to compute a solution (Rumelhart, 1980; Schank, 1975). According to the schema approach, students create a "problem model" containing the minimal of information necessary to solve the problem. In contrast, the situation model posits that students approach word problems similarly to reading any other text narrative by first creating a situation model of the unique elements of that story and that any use of schemas comes secondarily.

One implication for this difference might be that if the situation model theory of how readers read text narratives is correct then further contextualizing mathematical word problems could improve student learning. As people read text narratives, they read at a surface level, parsing words and sentences; but also at a deeper level creating a cognitive situation model of the story and updating that situation model as events unfold. Readers' situation models include five dimensions (Zwaan, Langston, & Graesser, 1995): protagonist (who the main characters are), goal (of the characters), causal (how events cause changes in the situation), temporal (flashbacks and flash-forwards, etc), and spatial (where in the setting events take place). Jahn (2004) posits that combining *potential causal* with *spatial* elements increases reader's monitoring of spatial elements. Further, the lead author of the current study expanded the theoretical model to include a second form of causation, i.e., causation resolution, or climax resolution. In other words, the current authors speculated that combining *potential causal* elements with mathematical elements in word problems might boost monitoring of the mathematical situation in the story. We further hypothesized that the climax resolution format would result in better word problem performance than potential causation.

If the schema explanation of word problems is correct, one would predict that a minimalistic presentation of the problem, with little more than the mathematical elements, would result in more correct answers. However, if the situation model explanation of solving word problems is correct, one would expect that providing additional meaningful contextual information to the word problem (in other words, making it a better story) might motivate students to construct a more detailed situation model, resulting in more correct answers. We also investigated how these differences play out in two different cultures in comparatively similar populations. Consistently there are large differences between USA and Turkish students in large scale mathematics comparisons (TIMSS, 1999; PISA, 2003) favoring the USA students around 8^{th} grade.

METHODS

Participants

The participants were 178 third year undergraduate elementary education students from the USA (n=109) and Turkey (n=69), including 154 females and 24 males, enrolled in a mathematical methods course in the fall semester. Since each word problem has been analyzed as a separate unit the number of subjects will vary across comparisons.

Materials

The testing material consists of four very simple word or story problems developed by the investigators. Two of the narratives, story problems 2 and 3, have some spatial content, of which problem 3 has the most. The fourth problem was mainly verbal arithmetic. All four problems were also presented in three contextualized versions: standard, potential causation, and climax resolution. There are some differences among the three versions, in terms of the level of information about the context as such:

The standard format includes almost the minimal amount of information needed to solve the problem. Standard format: "*At seven o'clock in the evening, Pete the Frog fell into a damp, slippery well. As Pete ascends the nine-foot well, each hour he climbs three feet up, but slides one foot back down again. Between which hours will Pete reach the top of the well?*" The potential causation format includes, in addition to the sentences in the standard version, several sentences which link the mathematical content to potential causation in the story. In the following, the additional potential causation sentences are bolded. Potential causation format: *"At seven …., slippery well.* ***He immediately began to climb up the wall to escape. If he doesn't get to the top of the well before the temperature reaches freezing, Pete will die.*** *As Pete ascends … top of the well?"* In the climax resolution format, mathematical content is connected to potential causation in the story; additionally, sufficient information is added to the potential causation format so that the student can use the mathematical content to compute the outcome of the story. In the following, the key additional climax resolution phrase is bolded. The addition of the phrase "by midnight" allows the reader to compute the outcome of the story. Climax resolution format: *"At seven … slippery well. He immediately …top of the well* ***by midnight****, when the … freezing, Pete will die. As Pete ascends … top of the well?"*

Items were presented to students as written on paper. Each student answered four different problems with the three formats randomly assigned to each student based on Latin Square approximation. In other words, each of the four narratives was written in one of the three different formats (standard, potential causation, and climax resolution) so that the total number of problems answered by each student was four.

Procedure

During a one-time session (non-graded portion of their course), students sat down in their regular classes and were presented with the four different story problems in one of three different formats. The fourth narrative was randomly formatted so that the presentation of materials was counter-balanced in terms of problem, format, and order. Therefore, one third of the students read one particular word problem in standard format, one third of students read that same problem in potential causation format, and one third read it in climax resolution format. Every student encountered all four of the story problems. The students read each narrative as many times as they needed. There was also sufficient space around each of the problems to solve (collected as data), should they need to write or draw anything associated with the problem. They were instructed to write down their solutions, not to erase anything they had written on the paper, and to put their final answer in a designated area. It took approximately 25 minutes to answer the four items.

RESULTS

Since the four problems were not standardized, each of the narratives was effectively a separate between-subject experiment. Four of the investigators (two of whom are professors in mathematics education) graded the students' answers to the word problems assigning a value of 1.0 to correct answers, 0.5 to partially correct (with some valid logic and plausible), and 0.0 to totally incorrect answers with fallacious logic. The inter-rater reliability was sufficiently high (94.6%).

	USA		Turkey		Combined	
Problem	N	Mean	N	Mean	N	Mean
1 Frog	112	.746	69	.703	181	.729
2 Garden	108	.579	69	.598	177	.586
3 Newlywed	107	.449	69	.540	175	.484
4 School	109	.544	70	.825	179	.654

Table 1. Mean scores obtained from each narrative based on country

As seen in Table 1, for both groups (USA and Turkish) the most difficult item was problem 3, with the most spatial content. The easiest item was problem 1 for the USA students but problem 4 for the Turkish students, which is mainly verbal arithmetic.

	N	Mean	SD	t	p
US	436	.581	.44	-2.63	.009**
Turkey	276	.668	.41		

Table 2. T-tests between US and Turkish students total scores

As presented in Table 2, on average the Turkish students did significantly better on the total scores of the four narratives in three formats than did the USA students. However, the only significant difference between the USA and Turkish students occurred on problem 4, favoring Turkish students (Refer to Table 3). There was no significant difference between males and females on average scores (Females, N=616, M=.618, SD=.43 and Males, N=96, M=.594, SD=.45, p=.613).

	N	Mean	SD	t	p
US	109	.544	.47	4.238	.000***
Turkey	70	.825	.37		

Table 3. T-tests between US and Turkish students' scores on narrative four

For the whole group analysis based on narrative and format, students performed statistically significantly better on problem 1 (frog) format 1 (standard) than on format 3 (climax resolution), [$F(2, 178) = 4.121$, $p<.018$]. For all other problems, there were no statistically significant differences among formats.

NARRATIVES		FEMALES			MALES				
Problem	Format	N	Mean	SD	N	Mean	SD	t	p
2 Garden	3 Climax resolution	52	.553	.48	9	.111	.33	2.65	.010**
3 Newlywed	1 Standard	51	.471	.37	9	.222	.36	1.85	.070
3 Newlywed	3 Climax resolution	48	.479	.45	9	.778	.23	1.95	.056
4 School	3 Climax resolution	51	.613	.48	8	1.00	.00	2.25	.028*

Table 4. Scores on narrative and format based on gender

As depicted in Table 4, scores on each problem based on format were analyzed using independent t-tests. According to results, there were significant differences between females and males on two of the narratives. While female students did significantly better on problem 2 (garden) format 3 (climax resolution), males did better on problem 4 (School), format 3 (climax resolution). The results approached significance on two of the other narratives (see Table 4, rows 2 and 3). While females did slightly better on problem 3 (newlywed) format 1 (standard), males did slightly better on problem 3, format 3 (climax resolution).

NARRATIVE		USA			TURKEY				
Problem	Format	N	Mean	SD	N	Mean	SD	t	p
3 Newlywed	1 Standard	37	.338	.35	23	.587	.40	2.59	.012*
3 Newlywed	3 Climax resolution	37	.480	.45	23	.793	.39	2.74	.008**
4 School	2 Potential causation	37	.574	.48	23	.891	.30	2.86	.006**
4 School	3 Climax resolution	35	.579	.49	24	.792	.42	1.75	.086

Table 5. Scores on narrative and format based on Country

As shown in Table 5, the Turkish students did significantly better on three of the formats than did the USA students. These were problem 3 (newlywed) format 1 (standard), problem 3 (newlywed) format 3 (climax resolution), and problem 4 (school) format 2 (potential causation). The difference between USA and Turkish students also approached statistical significance on problem 4 (school) format 3 (climax resolution) favouring Turkish students.

DISCUSSION AND CONCLUSIONS

Without considering the other factors, for both USA and Turkish students the problem with the most spatial content was found as the most difficult of the four problems. Although the solution and the answer were very simple, the students had the lowest average score on this problem, most probably due to the difficulty they had in drawing a diagram for the city plan.

Generally, Turkish students did significantly better than did the USA students. This finding is in contrast to both previous comparisons (Smith, Gerretson, Olkun, Yuan, Dogbey & Erdem, 2008; Olkun, Smith, Gerretson, Yuan & Joutsenlahti, 2009) as well as the large scale international comparative studies such as TIMSS (1999) and PISA (2003). The difference is especially high on problem 4, which is mainly verbal and arithmetic.

Considering the different formats for the same four problems, students did better on problem 1 (frog) format 1 (standard) than on format 3 (climax resolution). This finding provides evidence on the use of schema based models. When the scores were analyzed based on gender, however, the results are mixed. For some of the problems, females did better on some of the formats while males did better on others. Similar mixed findings were obtained when the analysis was done based on country. These findings indicate that both models are equally viable for explaining students' problem models or there are other interfering factors, which could be investigated further.

IMPLICATIONS

Educationally, even though the mathematical content was identical in all formats, *students' performance was dramatically changed across formats*. If only one model, either schema or situation model, is more primary than the other in word problems, then perhaps the current way of teaching word problems is misguided or at least insufficient. This suggests that, at least for elementary education majors solving story problems, quality of story is equally important as minimizing cognitive load. Involving the mathematical content with causal elements of the story might motivate some students to better focus on the task to correctly work problems. Efforts to redesign story problems so that mathematics content overlaps with causal elements of the storyline might result in increased performance on word problems.

The current results suggest that some students treat word problems fundamentally as stories so that context should be added in a way that makes the mathematics content more fundamental to the story; hence, the mathematical elements should be part of the plot. Furthermore, since there is mounting evidence that situation models are at least as primary as schemas in solving word problems, further research on using situation model theory to articulate strategies for learning and teaching word problems is warranted. As well, the complex results of the current study may also indicate other subtle factors affecting students' modeling of problems. Students' learning or thinking styles may be a factor in this choice. Another factor might be different early experiences with mathematical story problems. Further research should address these issues.

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CREATIVE ABILITY AND CRITERIA IN RECOGNISING GEOMETRIC FIGURES

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The aim of this study was to investigate the relationship between the type of criteria that students use to recognise geometric figures and their creative abilities to complete figures using various geometric shapes. A group of fifth and sixth grade students (N=201) completed a three-part test. The results show that the students who used different type of criteria in recognising geometric shapes differed in solutions of creativity tasks. More specifically, the mean performance in the creativity tasks of students who gave responses based on critical attributes of shapes was much better than the performance of students who gave responses based on visual properties of shapes.

INTRODUCTION

Creative performance is an essential part of doing mathematics (Pehkonen, 1997). A number of researchers proposed conceptual framework to define creativity in mathematics and tried to determine factors for creativity in different areas of mathematics, for instance problem solving, problem posing (Haylock, 1987; Runco & Okuda, 1988; Pehkonen, 1997; Sriraman, 2004; Leikin & Lev, 2007). However, although much work has been done in this area, little attention has been given to the elementary school students' creativity in geometry and to the aspects of their geometrical thinking that enhance their mathematical creativity. This study aims to investigate the type of criteria that students use to recognise geometrical shapes and the relationship to their creative abilities to complete various geometric figures.

THEORETICAL BACKGROUND

Mathematical creativity

A number of different definitions for creativity have been used in research. Creativity in general is a notion that embraces a wide range of cognitive styles, categories of performance and kinds of outcomes (Haylock, 1997). Singh (1987) describes the creativity as "the process of generating significant ideas, making theoretical ideas practical, converting innovative ideas from other fields into the new field" (p.186). In this line, creativity in mathematics implies creation of original, inventive, novel and unpredictable ideas in the field of mathematics (Singh, 1987). A number of researchers characterised creative responses in mathematics mainly by fluency (the number of acceptable responses), flexibility (the number of different ideas or

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 153-160. Thessaloniki, Greece: PME.

categories of responses used) and originality/novelty (the relative infrequency of the responses) (Haylock, 1985, 1987; Singh, 1987; Leikin & Lev, 2007).

There are two main approaches, according to Haylock (1987), to the recognition of creative thinking in mathematics. The first is the overcoming of fixation, the breaking of a mental set and the second is to have divergent thinking. A study made by Imai (2000) found that students who can overcome fixation in mathematics can contribute varied and original ideas in open-ended problems.

Runco (1986) suggested that gifted and non-gifted children approach open-ended problems differently, with the former utilizing components that facilitate originality. Furthermore, Leikin and Lev (2007) found that gifted and non-gifted proficient students differed meaningfully in fluency, flexibility and originality of their solutions in conventional and non conventional tasks from regular students. It is hypothesised that the above differences between the groups of students in their creative solutions in mathematical tasks are caused probably by the type of thinking that students use. More specifically, Haylock (1985) found that highly creative mathematically students tend to think in broad rather than narrow categories.

Criteria in recognising geometric figures

Criteria that students use to recognise geometric figures have been classified in many different ways throughout the years. Using the van Hiele levels of geometrical thought, researchers such as Clements, Swaminathan, Hannibal, and Sarama (1999) and Tsamir, Dirosh, and Levenson, (2008) coded the criteria students used to respond to the questions about their selections (e.g. “How did you know that was a rectangle?”). The first level of van Hiele thought contains visual responses that refer to the form of an object and descriptions such as “pointy”, “round”, or “skinny”. The second level of van Hiele thought consists responses based on critical attributes of the geometric shapes. In mathematics, critical attributes stem from the concept definition (Tsamir et al., 2008). For example, for the definition of a triangle there are four critical attributes: closed figure, three, vertices and straight sides. But there were responses in which the children referred to non-critical attributes such as thin, fat, long, sharp. Non critical attributes are “usually attributes of a prototypical example only” (Hershkowitz, 1989, p. 69) and it may be viewed as responses of the first level of van Hiele which is visual (Burger & Shaughnessy, 1986; Tsamir et al., 2008). The third level of van Hiele geometric thought involves responses that contain only necessary and sufficient conditions required to identify an example of the concept, i.e. definitions (Tsamir et al., 2008).

THE PRESENT STUDY

The purpose of this study

The purpose of this study was to explore the relationship between the type of criteria used by elementary school students to recognise geometrical shapes and their creative abilities to complete figures using various geometric shapes. More specifically, the present study addresses the following questions: (a) What is the relationship between the type of criteria used by students to recognise circles, squares, triangles and rectangles and their creative abilities to complete figures using various geometric shapes? (b) Are there differences in the creativity abilities to complete figures among students who use different type of criteria?

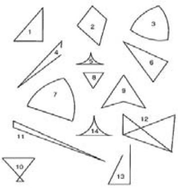
Participants and Test

In the present study data were collected from 201 students (105 females and 96 males), 137 fifth-graders and 64 sixth-graders students, ranging from 10 to 11 years of age. These students were from 5 primary schools in Cyprus from rural and urban areas.

All participants completed a three-part test which was developed on the basis of previous studies (e.g. Burger & Shaughnessy, 1986; Razel & Eylon, 1991; Clements & Battista, 1992). The first part of test examined students' mental images of geometric figures, the second part measured students' performance in shape-selection tasks (see figure 1) and the third part investigated students' ability to complete figures using various geometric figures (see figure 1).

Shape-selection tasks

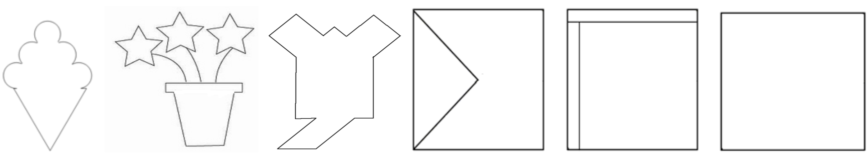
Write the number of the figures below that are triangles. (Burger & Shaughnessy, 1986; Clements & Battista, 1992)



Why did you select these shapes as triangles? (RT)

Creativity tasks

Complete the figures below by drawing different geometric shapes in these, for instance square, rectangle, circle, triangle etc.



(Cr1a) (Cr1b) (Cr1c) (Cr2a) (Cr2b) (Cr2c)

Figure 1: Sample of the shape-selection tasks and of the creativity tasks.

For the purpose of this paper, we used only students' responses to the questions about the justification of their selections from the second part of the test and their performance in the creativity tasks from the third part of the test. More specifically, we used the criteria that students use to justify their shape selections (circles - RC, squares - RS, triangles – RT and rectangles - RR) and classified them according to the coding of children responses after selection of shape made in previous studies (Clements et al., 1999; Tsamir et al., 2008). For every shape-selection task, we classified criteria students used to recognise geometric figures into five categories based on the figure's attributes. The first category includes responses based only on angles of the shapes (RSa, RTa, RRa). The second category refers to responses based only on sides of the shapes (RSc, RTc, RRc). The responses of the third category based on non-critical attributes (it appears only in task refer to circles - RCb, for example "These shapes are circles because they have no angles and sides"). The forth category includes responses based on two critical attributes of shapes (RCd, RSd, RTd, RRd) and the fifth refers to responses that include all critical attributes of shapes (RSe, RTe, RRe). The responses of the first three categories are characterised mainly by visual properties of shapes; these properties are easy to emerge from a prototype example of the shape. We also analysed students' drawings in "specific" figures (these figures have a concrete shape e.g. an ice-cream etc) and in "free" figures (these figures give to students more free space to complete them) in terms of fluency, flexibility and originality (characteristics of creative responses). Fluency was measured by student's ability to use various geometric shapes to complete the figures. Flexibility was measured by student's ability to fit together different geometric shapes in the figures using many different ways. Originality was measured by student's ability to combine the geometric shapes in the figures in an original way which is not common, or frequent amongst students' responses.

Data Analysis

To answer the research questions of this study, two different analyses were conducted: a similarity statistical analysis using a computer software called C.H.I.C. (Classification, Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier, & Gras, 2000) and an Analysis of Variance one way using SPSS. The similarity statistical analysis is a method of analysis that determines the similarity connections of the variables.

RESULTS

To explore the relationship between the type of criteria that students' use to recognise geometrical shapes and their creative abilities to complete "specific" and "free" figures using various geometric shapes, we employed the statistical similarity analysis for the data of this study and gave us the similarity diagram (see figure 2). This diagram allowed for the grouping of the creativity tasks and the type of criteria students use based on the homogeneity by which they were handled by students.

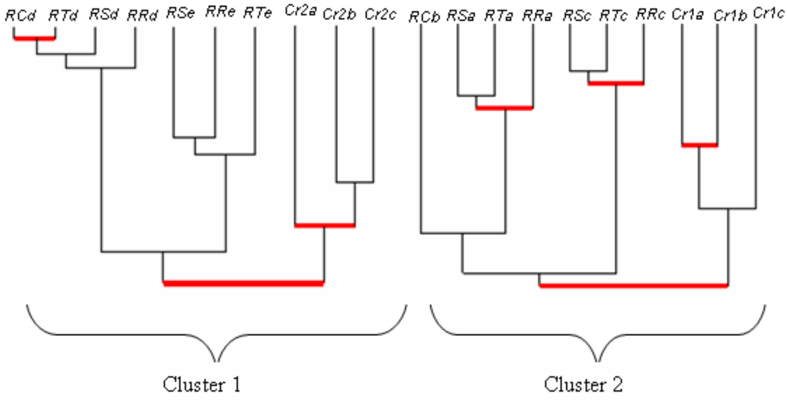


Figure 2: Similarity diagram of students' recognising and creative abilities.

Note: The similarities in bold color are important at level of significance 99%.

In figure 2, two distinct clusters of variables were formed. The first cluster consists of students' responses based on critical attributes of shapes and their creative abilities to complete "free" figures using various geometric shapes. While the second cluster consists students' responses based on visual properties of shapes and their creative abilities to complete "specific" figures using various geometric shapes. More specifically, the first cluster involved three similarity groups. The first two groups included the criteria that students use to recognise circles, triangles, squares and rectangles based on two critical attributes and on all critical attributes of shapes respectively. The third group involved the creativity tasks with "free" figures. All above groups of similarity of the first cluster show that students who used criteria based on critical attributes of the shapes in recognising geometric figures solved the creativity tasks with "free" figures in a similar way. The second cluster involved three similarity groups too. The first two groups consisted from the criteria that students use to recognise circles, triangles, squares and rectangles based on visual properties of the shapes (angles, sides, non critical attributes), while the third group included the creativity tasks with "specific" figures. The similarity groups of the second cluster show that students who used criteria based on visual properties of the shapes solved the creativity tasks with "specific" figures in a similar way.

The sample of this study was clustered into six groups according to the criteria students were using in recognising geometrical shapes in every shape-selection task. To examine the differences between these groups in respect to their creative abilities to complete figures using various geometric shapes used the Analysis of Variance one way. Table 1 presents the mean performance of the above student groups on creativity tasks in total. In addition, Table 1 presents the results of the Analysis of

Variance one way by specifying the F and p values for creativity tasks. The dependent variables in the Analysis of Variance one way were the performance of students in the creativity tasks, and the independent variables were the classification of students according their criteria used to recognise geometric figures. The means of students' performance on creativity tasks shown in Table 1 are all smaller than one since the "creative" answers by terms of fluency, flexibility and originality were summed up and then divided by the total number of creativity tasks (the means performance on creativity tasks are not very high, because these tasks are very complicate and difficult to solve them in the paper.)

		Types of criteria							F	p
		Visual properties			Critical attributes		No reason			
		Sides	Angles	Non critical attributes	2 critical attributes	All critical attributes				
Creativity tasks	Circle	-	-	0.15	0.22	-	0.09		4.317	0.002*
	Square	0.14	0.16	-	0.18	0.04	0.08		2.432	0.036*
	Triangle	0.12	0.15	-	0.18	0.38	0.10		2.327	0.034*
	Rectangle	0.14	0.15	-	0.19	0.21	0.10		2.097	0.067

*Indicate statistical significance at $p<0.05$.

Table 1: Comparing performance of students' groups according to the type of criteria used in creativity tasks.

From Table 1, it can be seen that there were statistical significant differences among students who used different type of criteria to recognise circles, squares and triangles and on the total scores in the creativity tasks ($F_{(4,196)}=4.317$, $p=0.002$; $F_{(5,195)}=2.432$, $p=0.036$; $F_{(6,194)}=2.327$, $p=0.034$ respectively). However, using Bonferroni procedure, there were no statistical significant differences in creative abilities between the students who used criteria based on critical attributes and those who used criteria based on visual properties. Furthermore, the comparison of the scores in the creativity tasks for students who used different type of criteria to recognise rectangles revealed no significant differences ($F_{(5,195)}=2.097$, $p=0.067$). Additionally, the mean performance in the creativity tasks of students who used criteria based on critical attributes of shapes was much better than the performance of students who used criteria based on visual properties of shapes (see Table 1). Students who used criteria based on critical attributes of shapes seemed to have been more flexible in the completion of the geometric figures, than students who used visual based criteria.

DISCUSSION

The purpose of the present study was to investigate the relationship between the type of criteria used by elementary school students to recognise geometrical shapes and their creative abilities to complete figures using various geometric shapes. We found that students who used different type of criteria in recognising geometric shapes differed in solutions of creativity tasks. More specifically, students who gave responses based on critical attributes of shapes solved the creativity tasks with "free" figures in a similar way. While students who gave responses based on visual properties of the shapes solved the creativity tasks with "specific" figures in a similar way. The creativity tasks with "free" figures are more complicated than the creativity tasks with "specific" figures. The above finding, suggests that the knowledge of critical attributes of shapes is necessary to complete complicated figures using various geometric shapes, while the knowledge of visual properties of shapes is adequate only for the completion of the simple figures.

The results of this study showed that the mean performance in the creativity tasks of students who gave responses based on the critical attributes of shapes was much better than the performance of students who gave responses based on the visual properties of shapes. It is hypothesised that students who used critical attributes criteria can combine geometric figures in a more sufficient and flexible way than students who used visual properties criteria. These findings are confirm the results by Haylock (1985) who found that highly creative mathematically students tend to think in broad rather than narrow categories.

Overall, the above findings are very important in mathematics teaching and learning. Creative performance is an essential part of doing mathematics (Pehkonen, 1997) and thus teachers need to develop ways to enhance students' creative abilities. The results of this study suggest that it may be possible by enhancing students' ability to use critical based attributes criteria to become more creative in the composition of geometrical figures and in mathematics in general.

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A SENSE OF MEASUREMENT: WHAT DO CHILDREN KNOW ABOUT THE INVARIANT PRINCIPLES OF DIFFERENT TYPES OF MESUREMENT?

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This study investigates children's knowledge about measurements based on the idea of invariants, and of number sense. Forty children (6 and 8 years old) made judgments about situations involving measurements of volume, time, mass, distance and length. Task 1 examined the ability to identify the appropriate unit to measure different objects, and Task 2 the understanding of the inverse relation between the size of the unit and the number of units needed to measure something. Results showed that children have a sense of measurement and that this sense does not manifest itself equally in relation to different types of measurements. Also, children showed a better understanding about the inverse relation between the size of the unit and the number of units than of the relation between unit and object to be measured.

INTRODUCTION

This study investigates children's mathematical knowledge about measurements based on two theoretical perspectives: one associated with Vergnaud's (1997) idea of invariants, and the other related to the idea that has been called number sense in the literature (e.g., Greeno, 1991; Sowder & Shapelle, 1989). Considering that a given concept involves, among other aspects, a set of invariants (related to the properties that define it and to the schemes used by individuals to deal with situations that make the concept meaningful), it becomes important to identify which invariants related to the concept of measurements would need to be understood by children. Considering, on the other hand, that number sense permeates any and all mathematical concepts, it becomes equally relevant to examine what intuitive knowledge children present about measurements in different situations.

Curry, Mitchelmore and Outhred (2006) present a series of principles which involve the understanding of measurements. According to our analysis, it is possible to interpret these principles as being the invariants of the concept of measurements. Two of these principles are considered in this investigation:

(1) The relation between unit of measurement and the object to be measured. This principle deals with the understanding that there is an appropriate unit to measure a given magnitude; for instance, understanding that an object's height is measured in meters, while the weight of an object is measured in kilograms. (2) The inverse relation between the size of a unit of measurement and the number of units necessary to measure something. This principle deals with the understanding that the smaller

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 161-168. Thessaloniki, Greece: PME.

the unit of measurement, the larger the number of units and vice-versa. For instance, when measuring a person's height in feet and the same person's height in inches, since the foot is larger than the inch, a smaller number of feet will be needed than of inches.

Traditionally, the research requires actions of measuring something in order to obtain a precise value based on actions implemented by the children (e.g.; Bragg & Outhred, 2001; Piaget, Inhelder & Szeminska, 1960). This study proposes to examine simultaneously different types of measurements without requesting measuring actions. Knowledge of measurements is investigated based on the methodological paradigm usually adopted in research on number sense, in which children do not need to do precise numerical calculations, but rather make judgments about numerical situations (e.g. Ribeiro & Spinillo, 2006; Spinillo, 2006). In this study, the children are asked to make judgments about situations which involve measurements of volume, time, mass, distance and length. The aim is to investigate children's knowledge of different types of measurements in the perspective of number sense. Does a sense of measurement manifest itself equally in relation to different types of measurements or not? Is the understanding of the invariant principles (relation between unit and the object to be measured; and inverse relation between the size of a unit and the number of units necessary to measure an object) something general? Or is this understanding somehow related to a specific type of measurement? Are these principles of the same level of difficulty or one is more difficult than the other?

METHOD

Participants and Procedure

Forty low-income children aged 6 and 8 years old, attending the first and third grade of elementary schools in Recife, Brazil were individually asked questions related to the two principles mentioned above. Two tasks were given, involving five types of measurement: volume, mass, time, distance, and length. The children were asked to justify each response. Audio recordings were made of the interviews.

The tasks

Task 1 (first principle) examined if the child could identify the appropriate unit to measure different objects. The child was told that it was a guessing game in which s/he had to discover what a boy had measured. For the open-ended questions, it was told that "John measured something, and said that this thing measured 5 litres (3 kilos/4 hours/12 kilometres/2 meters). What do you think he measured?" Some of the multiple choice questions were: "John measured something, and said this thing measured 20 litres. What do you think he measured: the time he took to do his homework, or how much petrol is in his father's car?"; "John measured something, and said this thing measured 60 kilos. What do you think he measured: how much a person weighs or the time he took to get to school from home?"; and "John measured something, and said this thing measured 5 litres. What do you think he measured: how much milk there was in a pan, or how tall a door was?"

Task 2 (second principle) examined if the child understood the inverse relation between the size of the unit and the number of units needed to measure something. The child was told that Ann and John used different units to measure things. For instance, "John measured the length of a table with matchsticks. Ann measured this same table with popsicle sticks. Who will need more sticks to measure the length of the table: Ann or John?" Other examples were: "John weighed a pack of rice on a scale with small weights. Ann weighed this same pack of rice on a scale with large weights. Who will need more weights to weigh the pack: John or Ann?"; and "John measured the time of a football match in minutes. Ann measured the time of this same football match, but in seconds. Which number will be greater: the number of minutes or the number of seconds?"

RESULTS AND DATA ANALYSIS

Based on the correctness of the responses and justifications given, four types of responses were identified:

Type 1: incorrect response accompanied by vague, subjective, or inappropriate justification, or no justification at all. Examples:

Interviewer: (Task 1) John measured something, and said that this thing measured 12 kilometres. What do you think he measured?

Child: A large table

Interviewer: How did you figure that?

Child: Because there's one at my sister's house.

Interviewer: (Task 2) John weighed a pack of rice on a scale with small weights. Ann weighed this same pack of rice on a scale with large weights. Who will need more weights to weigh the pack: John or Ann?

Child: Ann.

Interviewer: Why?

Child: I don't know.

Type 2: correct response accompanied by a vague, subjective, or inappropriate justification, or no justification at all. Examples:

Interviewer: (Task 1) John measured something, and said this thing measured 20 litres. What do you think he measured: the time he took to do his homework, or how much petrol is in his father's car?

Child: How much petrol there was in the car.

Interviewer: Why?

Child: Because petrol is really heavy.

Interviewer: (Task 2) John measured the time of a football match in minutes. Ann measured the time of this same football match, but in seconds. Which number will be greater: the number of minutes or the number of seconds?

Child: Seconds, because seconds are much faster.

Type 3: correct response accompanied by appropriate justification. Examples:

Interviewer: (Task 1) John measured something, and said this thing measured 60 kilos. What do you think he measured: how much a person weighs or the time he took to get to school from home?

Child: How much a person weighs.

Interviewer: Why?

Child: Because a person can be weighed in kilos, I think.

Interviewer: (Task 1) John measured something, and said this thing measured 5 litres. What do you think he measured: how much milk there was in a pan, or how tall a door was?

Child: How much milk there is in a pan.

Interviewer: Why?

Child: Because a pan can have 5 litres or more.

Interviewer: (Task 2) John measured the length of a table with matchsticks. Ann measured this same table with popsicle sticks. Who will need more sticks to measure the length of the table: Ann or John?

Child: John.

Interviewer: Why?

Child: Because Ann has the bigger one and his matchstick is smaller than a popsicle stick so he'll need more.

The 6-year-old children gave more Type 1 responses than Type 3, both in relation to each type of measurement separately, as well as in relation to the total percent of each response (Mann-Whitney: $p<.02$). The same pattern of results was found at age 8 ($p<.05$), except in relation to the measure of mass, which presented a lower percentage of Type 2 responses than the other types (Table 1).

			6 years			
Responses	Volume	Mass	Time	Distance	Length	Total
Type 1	51	50	48	64	54	53,4
Type 2	41	34	33	28	34	34
Type 3	8	16	19	8	12	12,6
			8 years			
Type 1	39	44	41	34	41	39,8
Type 2	36	22	34	47	36	35
Type 3	25	34	25	19	23	25,2

Table 1: Percentage of types of responses in Task 1.

Among 6-year-old children, Friedman detected significant differences between the types of measurements in relation to Type 1 (X^2 =10,777, p=.030) and Type 3 (X^2=10,774, p=.029) responses. Wilcoxon showed that Type 1 responses were more frequent in measurements of distance than in other measurements ($p<.05$), while Type 3 were rarely found in measurements of distance (8%) and of volume (8%) ($p<.05$).

At age 8 there were significant differences between the types of measurements in relation to Type 2 (X^2=11,912, p=.018) and Type 3 (X^2=11,278, p=.024) responses. There were more Type 2 responses in measurements of distance (47%) than in those of mass (22%), while Type 3 responses were more frequent in measurements of mass (34%) than in those of distance (19%) (Wilcoxon: $p<.05$). The 6-year-olds rarely gave Type 3 responses (12,5%), the percentage of which is significantly higher at age 8 (25,2) (Mann-Whitney: $p<.05$). On the other hand, Type 1 responses were more frequent at age 6 (53,4%) than at age 8 (39,8%) ($p<.05$). Furthermore, younger children gave few Type 3 responses when making judgements about distance and volume, while among 8-year-olds this only occurred in relation to measurements of distance. Actually, the 8-year-old children performed better (66%) than the 6-year-olds (36%).

On Task 2, children of both ages (Mann-Whitney: $p<.05$) gave Type 3 responses significantly more frequently than Types 1 and 2, except in measurements of time, in which Type 3 responses were absent (Table 3).

			6 years			
Responses	**Volume**	**Mass**	**Time**	**Distance**	**Length**	**Total**
Type 1	20	35	37,5	22,5	32,5	29,5
Type 2	12,5	7,5	62,5	7,5	15	21
Type 3	67,5	57,5	0	70	52,5	49,5
			8 years			
Type 1	20	12,5	62,5	10	12,5	23,5
Type 2	7,5	15	37,5	5	10	15
Type 3	72,5	72,5	0	85	77,5	61,5

Table 3: Percentage of types of responses in Task 2.

Significant differences were found (Friedman) between the types of measurements in relation to Type 2 (X^2=32,938, p=.000) and Type 3 (X^2=41,965, p=.000) responses at age 6. According to Wilcoxon, Type 2 responses were observed more frequently in measurements of time than in the other measurements (p<.002), while Type 3 (more elaborate) responses were more frequent in measurements of distance and volume, and absent in measurements of time. Among the 8-year-olds, significant differences between the types of measurements (Friedman) occurred in relation to Type 1 (X2=11,467, p=.022), Type 2 (X2=19,300, p=.001) and Type 3 (X2=49,202, p=.000) responses. Wilcoxon revealed that Type 1 and 2 responses were more frequently given in measurements of time than in other measurements (p<.01), and that Type 3 responses, although widely observed in other types of measurements, were absent in measurements of time (p<.02).

Comparisons between age groups showed that the 8-year-olds gave more Type 3 responses (61,5%) than the 6-year-olds (49,5%) (Mann-Whitney: p<.05). This fact was also found in relation to measurements of mass, distance and length (p<.02). On the other hand, Type 1 responses were significantly more frequent at the age of 6 than at the age of 8 years both in general as well as in relation to most of the types of measurements investigated.

DISCUSSION AND CONCLUSIONS

Children have a sense of measurement and this develops from 6 to 8 years old. This sense does not manifest itself equally in relation to different types of measurements. The results in Task 1 suggest that distance and volume are the most difficult types of measurements for both age groups to understand, since these measurements rarely received correct responses accompanied by appropriate justification (Type 3). One possible explanation for this result is that measuring the distance between objects and the volume of recipients is an unfamiliar mathematical activity in the everyday lives of the 6 to 8 year old children, which explains the difficulty in identifying the appropriate unit to measure different objects. On the other hand, children had no difficulty in relation to measurements of length and mass, for instance. Actually, measuring the size of objects and weighing objects are mathematical activities commonly carried out by children from a very early age at home. Thus one may conclude that this sense of measurement is more developed in relation to some types of measurement than to others.

In Task 2, Type 3 responses were widely used, indicating that the children of both ages have an understanding of the inverse relation between the size of the unit and the number of units necessary to measure something. However, the measuring of time is difficult for both ages since they gave no Type 3 response when making judgments about time in this task.

In general, Task 2 seems to be easier than Task 1, this was due to elaborate responses (Type 3) being more frequent in Task 2 than in Task 1; while elementary responses (Type 1) were more often given in Task 1 than in Task 2. Thus (except for measurement of time), one may say that the principles are not of the same level of difficulty and that children presented more elaborate knowledge of the inverse relation between the size of the unit and the number of units needed to measure a given object than of the relation between unit of measure and object to be measured.

One may conclude that it is important to identify the invariant principles related to the concept of measurements children do understand and those they do not. This may help to understand how a sense of measurement develops and how to develop a sense of measurement in children. The relationship between number sense and invariants can be considered a new approach to the investigation of measurement in particular, and to the investigation of mathematical knowledge in general

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UNDERSTANDING LINEAR ALGEBRA CONCEPTS THROUGH APOS AND THE THREE WORLDS OF MATHEMATICAL THINKING THEORIES

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Every year around the world a vast number of undergraduate students study linear algebra at university. Despite their lack of full understanding and experiencing difficulties with the basic concepts the traditional ways of teaching is still been carried out. Furthermore majority of students successfully pass the final examination. In her PhD thesis (Stewart, 2008) the author created and applied a theoretical framework combining the strengths of two major mathematics education theories in order to investigate the learning and teaching of linear algebra at the university level. She produced and analysed a large data set concerning students' embodied and symbolic ways of thinking about elementary linear algebra concepts. It was anticipated that this study may provide suggestions with the potential for widespread positive consequences for learning mathematics. This paper highlights some of her findings and in particular discusses the effectiveness of the framework.

INTRODUCTION

Over the past three decades, some researchers have been concerned with the difficulties related to the first year linear algebra courses. They believe that teaching and learning this "high cognitively demanding" course (Dorier & Sierpinska, 2001), is a frustrating experience for both teachers and students (Hillel & Sierpinska, 1993) and despite all the efforts to improve the curriculum, "linear algebra remains a cognitively and conceptually difficult subject" (Dorier & Sierpinska, 2001, p. 255). According to Hillel (1995), unlike calculus, linear algebra is generally the first course that students encounter which is based on mathematical theory, built systematically from the ground up. Harel (1997) supports this contention describing, linear algebra as very different from calculus, in the sense that there are more theorems and equivalent theorems introduced in a linear algebra course in the period of a semester, than for calculus. Moreover, unlike the formal definitions of concepts in calculus such as function, limit and continuity, which may resonate with students' previous experiences, students have little intuition for linear algebra terms such as span or eigenvector. Many university lecturers and researchers around the world are trying to suggest alternative ways of teaching linear algebra. Some are trying to change the curriculum by deciding which parts should or should not be part of stage one university linear algebra courses and some believe that we should introduce linear algebra in high school so students have a better background in the subject. The reality is that no one really knows what is the best way of teaching the subject and it even seems that there may be no such thing as a right way of teaching linear algebra (Day & Kalman, 1999). This research proposed applying APOS theory, in conjunction

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 169-176. Thessaloniki, Greece: PME.

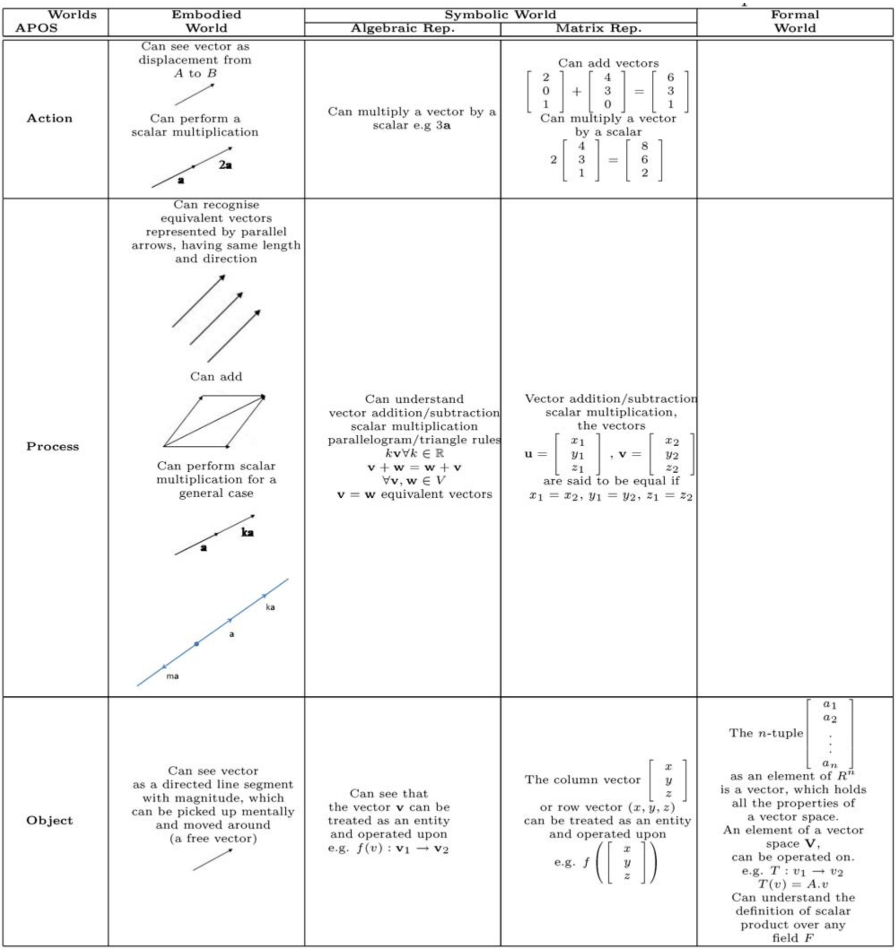
with Tall's three worlds of embodied, symbolic and formal mathematics, to create a framework in order to examine the learning of a variety of linear algebra concepts by groups of first and second year university students. The aim was to investigate the difficulties in understanding some linear algebra concepts and to propose potential paths for preventing them.

THEORETICAL FRAMEWORK

It is believed that a rich theory combined with a well-thought-out framework based on the theory, and sensible gathering and analysing of the data are the essence of high quality and beneficial research. Lester (2005) argues that theory should play an indispensable role in our research. In his view the common saying that "The data speak for themselves" has no value since in reality, data actually have nothing to say. He claims that "were the researcher guided by theory, a natural question would be to ask WHY? Having some theoretical perspective guiding the research provides a framework within which to attempt to answer Why questions. Without a theoretical orientation, the researcher can speculate at best or offer no explanation at all" (p. 174). Presently mathematics education owns many theories which are directly related to aspects of mathematics. This however was not always the case, since until not so long ago most theories came from general educators and psychologists such as Piaget (1965), Dienes (1960) and Bruner (1966), who only had something to say that was particularly related to mathematics (Tall, 2004d). Although a theory is not a statement of truth it should be an attempt to understand how mathematics can be learnt and what a programme of research can do in order to assist learning (Dubinsky & McDonald, 2001). While there is no perfect learning theory to describe how students learn and think, it was the aspiration of this research to consider a theoretical framework to suggest possible ways of advancing students' conceptual understanding and to relate them to learning of some basic linear algebra concepts. Hence for the purpose of this thesis I employed two advanced mathematical learning theories that suggest alternative, but related, ways of teaching and learning. These theories were Tall's theory of learning, employing the constructions of embodied, symbolic and formal worlds of mathematics, and the action, process and object of Dubinsky's APOS theory. The theory of APOS has been applied to many areas of mathematics at the university level. Since many linear algebra concepts have embodied, symbolic and formal representations, and they involve ways of thinking that are outlined by APOS theory's actions, processes and objects, the theories were considered an ideal combination for research on student learning and understanding of linear algebra, as well as the teaching of the concepts. Thus, they provided a platform for building this framework (see Figure 1 for a section of the framework on the concept of vector and scalar multiplication). The framework was constructed by creating a grid with 12 cells to examine a learner's action, process, and object thought processes of each chosen concept (the left-hand column) in each of the three mathematical worlds of embodied, symbolic and formal (the top cells). This formulation was achievable since it is possible to have action, process, and object type of thinking in each of the embodied and symbolic and formal worlds of mathematics. For example it is possible to have an action view in the embodied world or in the symbolic world, or both, or think about mental processes in the symbolic world. Thus, the theoretical standpoint for this framework implies that students can do actions, think about processes, and encapsulate their processes to form objects in each of the embodied, symbolic and formal worlds of mathematical thinking. Despite the fact that Dubinsky's APOS theory refers to learners' mental views and Tall's worlds are mathematical thinking, the theories seem to blend naturally together. Although this framework was clearly not perfect it was aimed at providing a starting point for analysing student thinking by providing evidence of students' level of thinking based on the specific cells or regions of the framework. For example by analysing students' work the researcher could observe which region or cell of the framework student thinking was in. It was also possible to trace which parts of the framework describe thinking students were not able to produce. So the framework may be used to discover students' thinking abilities. Furthermore, by finding out weak points in a student's understanding and thinking, the instructor would see the areas that need improvement, and how to address them. Moreover, from a teaching point of view the framework was designed to make sure that every aspect of the concept mentioned in the cells of the framework is covered in the lectures. This should help to give students an overall view of each of the concepts and help them to build linear algebra knowledge that might assist them to reach formal world thinking.

METHOD

This research comprised several qualitative case studies to study students' thinking about some basic linear algebra concepts, namely vector, scalar multiple, linear combinations, linear dependence/independence, span of vectors, subspace, basis, and eigenvectors and eigenvalues, where they were taught by the researcher in the context of the proposed framework. The participants were first and second year general mathematics students from the University of Auckland who had volunteered to take part in this study. Based on the aim and design of each case study a total of four linear algebra tests were made. The tests were designed to analyse students' conceptual understanding of the linear algebra concepts described above. The questions included describing certain definitions and assessing students' geometrical understanding of the concepts. Some questions were also constructed to examine students' ability to recognise the concepts in different representations, and their ability to move between them. In some ways the questions were different from what the students had seen before, as these questions were not asking for procedural answers, and required more thinking and sound grasp of the concepts (see Figure 2 for questions). As part of the case studies interviews and concept maps were also employed. For comparison reasons a recent PhD in mathematics student also participated in this study.

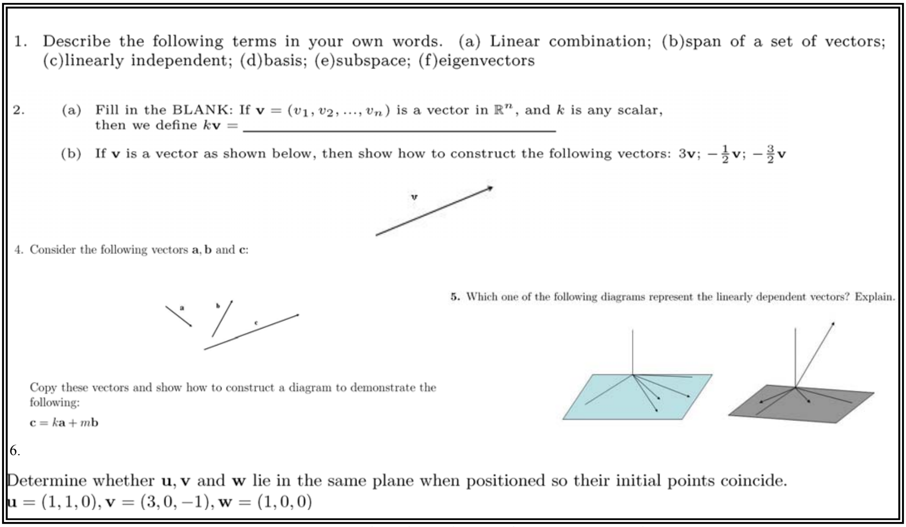


Worlds / APOS	Embodied World	Symbolic World: Algebraic Rep.	Symbolic World: Matrix Rep.	Formal World
Action	Can see vector as displacement from A to B Can perform a scalar multiplication $\mathbf{a}$ $\mathbf{2a}$	Can multiply a vector by a scalar e.g $3\mathbf{a}$	Can add vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$ Can multiply a vector by a scalar $2\begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 2 \end{bmatrix}$	
Process	Can recognise equivalent vectors represented by parallel arrows, having same length and direction Can add Can perform scalar multiplication for a general case $\mathbf{a}$ $\mathbf{ka}$ ka a ma	Can understand vector addition/subtraction scalar multiplication parallelogram/triangle rules $k\mathbf{v}\forall k \in \mathbb{R}$ $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ $\forall \mathbf{v}, \mathbf{w} \in V$ $\mathbf{v}=\mathbf{w}$ equivalent vectors	Vector addition/subtraction scalar multiplication, the vectors $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ are said to be equal if $x_1 = x_2,\ y_1 = y_2,\ z_1 = z_2$	
Object	Can see vector as a directed line segment with magnitude, which can be picked up mentally and moved around (a free vector)	Can see that the vector $\mathbf{v}$ can be treated as an entity and operated upon e.g. $f(v) : \mathbf{v_1} \to \mathbf{v_2}$	The column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or row vector (x, y, z) can be treated as an entity and operated upon e.g. $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$	The n-tuple $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ as an element of R^n is a vector, which holds all the properties of a vector space. An element of a vector space $\mathbf{V}$, can be operated on. e.g. $T : v_1 \to v_2$ $T(v) = A.v$ Can understand the definition of scalar product over any field F

Figure 1. The APOS-Three Worlds Framework for Vector and Scalar Multiplication

RESULTS AND DISCUSSIONS

The analysis of the data comprised examining students' written responses to questions on each concept, the transcribed interviews and their written concept maps. Although over 100 pages of results were produced in researchers' thesis, for the purpose of this paper only a small amount will be discussed. The results of this research highlighted many areas of difficulties that students have with linear algebra concepts. The extensive evidence revealed that the majority of students had major problems understanding the concepts that are the essence and foundation of a linear algebra course. However, evidence showed that including embodied ideas and using multiple representations had an effect on students' understanding. In other words, employing a visual, embodied approach to the teaching of linear algebra concepts, which are often treated symbolically or formally, may enrich students' understanding (Stewart & Thomas, 2007).



1. Describe the following terms in your own words. (a) Linear combination; (b)span of a set of vectors; (c)linearly independent; (d)basis; (e)subspace; (f)eigenvectors

2. (a) Fill in the BLANK: If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\mathbb{R}^n$, and k is any scalar, then we define $k\mathbf{v} =$ ________________

(b) If $\mathbf{v}$ is a vector as shown below, then show how to construct the following vectors: $3\mathbf{v}$; $-\frac{1}{2}\mathbf{v}$; $-\frac{3}{2}\mathbf{v}$

4. Consider the following vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$:

Copy these vectors and show how to construct a diagram to demonstrate the following:

$\mathbf{c} = k\mathbf{a} + m\mathbf{b}$

5. Which one of the following diagrams represent the linearly dependent vectors? Explain.

6.

Determine whether $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ lie in the same plane when positioned so their initial points coincide.
$\mathbf{u} = (1, 1, 0), \mathbf{v} = (3, 0, -1), \mathbf{w} = (1, 0, 0)$

Figure 2. A selection from the test questions.

Traditionally almost all of the concepts that are considered in this study are primarily introduced through their formal definitions. Lecturers often argue that they would like to show pictures or describe the concepts in more details, but a lack of time forces them to omit what in many cases are the building blocks of mathematics concepts. However, investigating these concepts from the standpoint of the theoretical framework in this study intended to offer a broader view by employing a variety of representations that may deepen learners' conceptual understanding.

Action process object in embodied symbolic and formal world thinking

The findings from the seven case studies (including the pilot study), suggest that most first and second year students had difficulties understanding basic linear algebra concepts. Evidence showed that students were mainly thinking and representing their understanding of the concepts in a manner described by the action-symbolic-matrix or/and process-symbolic-matrix cells of the framework (the 4 cells in the centre section). Very few students demonstrated embodied or object views, and the majority of students tended to show an action/process view in the symbolic world. As Harel (1997, p. 109) points out:

> In the absence of a concept image that sustains the concept definition, these students are unable to retain the concept definitions for a long period of time. Once the concept definition is forgotten, they are unable to retrieve or rebuild it on their own.

However, majority of researcher's students showed a good grasp of a variety of ideas in linear algebra, and were able to make connections between them, and gave a richer concept maps for the requested concepts. It was interesting that they answered most questions and there were no lack of explanations when it came to the interviews. It seems that having a grasp of the concepts in the embodied world helped them to move more confidently toward the symbolic and the formal worlds. Students' formal understanding of concepts was mainly examined through the definitions. Here I found students in two separate groups. The first group were students who struggled to give a clear definition of the required term. As many students could not remember the definition and were confused about the concepts, they simply described an action related to the concept in the symbolic-matrix world. The second group consisted of students who mainly came from the researcher's class and were happy to give a definition and their definitions contained the key elements of the concept (a process-formal view). However, the interviews gave more insight on what these students really thought about the definition, and as it turned out most students gave the impression that while solving problems they did not think about the definitions. It was also noted that the PhD graduate who mostly presented his thinking in the symbolic-formal world and produced perfect formal world definitions, expressed appropriate embodied thinking in questions where they were examined.

The effectiveness of the framework

It is difficult to evaluate fully the effectiveness of this framework, considering that there has been limited experience with teaching based on the framework, and limited data to analyse. Based on the results of this research the majority of students were in the symbolic world, so the question would be, if the three worlds of the mathematical thinking are hierarchical, how did the students reach the symbolic world without passing through the embodied? In other words if student thinking is based in the symbolic world surely they would have had embodied ideas too, since they are relatively easier than the symbolic ones. The answer to this question is not trivial. In Tall's description of the three worlds, he often refers to the entire mathematics from school mathematics to calculus and more advanced algebra and right through to the definitions and axioms in the formal world. There has been no study examining the development of a single concept of advanced mathematics through these three worlds. How could we apply his theory to a single concept? In other words, to construct conceptual understanding does one have to start from the embodied, travel through the symbolic, and finally arrive at the formal world? As Tall (2007) claims in an ideal world this is likely the case. Most students need to symbolise the embodiment and embody the symbolism first and only after fully integrating them they will reach at the formal world. However, in the real world it is possible to be solely in the symbolic world of thinking by following the steps of the instructor in the class. For example students can dwell in the world of matrices, where they find a basis for a Null space and row-reduce matrices to see whether a set of vectors are independent or not, and ultimately pass the course and satisfy the expectations of the examiners. However, in all of this they may not know what the concepts actually mean and where they come from and not have embodied conceptions. Apparently this is not only quite likely but it is a common scenario. As Tall (2004a, p. 30) reveals:

> There are many occasions when individuals do not encapsulate a given process into a thinkable object and instead carry out the procedures in a routinised way based on repetition and interiorisation of learned operations.

On the other hand, a mathematician can comfortably live in the symbolic and embodied worlds since he/she is able to reverse and construct embodied views, as well as going forward to the formal world. The possession of a rich schema allows him/her to tie all the pieces of his knowledge in a way that the student may not be able to. Thus, the claim is that it is the embodied view that gives deep meaning to the concept allowing us move toward the formal world. In the case of the students in this research, it appears that since they often lacked embodied aspects of the concepts and thus could not move away from the symbolic world to the world of formal mathematical thinking.

CONCLUDING REMARKS

The construction of the framework was one of the most challenging and exciting parts of this thesis, and represents a useful contribution to the literature. It proved difficult to incorporate a combination of two fairly powerful and significant learning theories, that in some ways differ, and unite them to form a unique new frame that can operate as a basis for research on examining students' understanding, and as a teaching tool. One aspect of this was that it was not easy to cover all the cells of the framework, for example, finding actions that were external yet could be performed in the formal world, or think of an object in the embodied world. Thus, some parts of the framework still remain empty. In a more theoretical approach in further research, it would be beneficial to consider the hidden assumptions made by the two theories and examine their (in-)compatibility. It is also recommended to extend the ideas presented here to include further linear algebra concepts in more advanced mathematics courses. Further research applying a similar framework to other mathematical areas of study, such as analysis would also be beneficial. I consider both theories that were employed to build this framework equally important and useful, however it was the desire of this research to highlight the importance of teaching and learning using embodied ideas. So in a way the first column of the framework was emphasised more. Deeper investigation of other columns or rows of the framework, for instance studying the formal column and evaluating how students reach formal thinking could produce valuable results.

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BUILDING COLLECTIVE MATHEMATICAL KNOWLEDGE IN THE ELEMENTARY GRADES USING PROBLEM BASED LEARNING AND PEDAGOGICAL CONTENT TOOLS

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This design research explored strategies for enhancing mathematics instruction for students from a low socio-economic diverse population by developing collective mathematical knowledge using problem based learning and pedagogical content tools. Results showed that teachers' opportunities for reflection and discussion focused on developing collective knowledge influenced many pedagogical decisions. The most prominent pedagogical decisions influenced were a) selecting tasks and sequence of problems, b) integrating pedagogical content tools, and c) orchestrating classroom discourse through pedagogical moves and questioning. As a result the teacher researchers developed a working framework to guide their efforts in building collective knowledge to enhance students' learning.

THEORETICAL FRAMEWORK

Sociocultural approaches emphasize the interdependence of social and individual processes in the co-construction of knowledge (Vygotsky,1986). Through participation in activities that require cognitive and communicative functions, children are drawn into the use of these functions in ways that nurture and 'scaffold' them" (pp. 6-7). Vygotsky (1986) described learning as being embedded within social events and occurring as a child interacts with people, objects, and events in the environment. Through socially shared activities learners also internalized processes. Following this approach, researchers have explored sociomathematical norms and how teachers actively guide the development of classroom mathematical practices and individual learning through capitalizing on opportunities that emerge through students' activities and explanations (Ball, 1993; Cobb, Wood & Yackel, 1993; Lampert, 1990; McClain & Cobb, 2001).

However, recent research offers some important insight on participation gaps which exist among students from diverse social, cultural, and racial backgrounds in mathematics classrooms and how classrooms can be structured to better afford opportunities to participate in mathematics by a wider range of students (DIME, 2007). Typically, school systems that serve economically disadvantaged or minority student struggle to meet academic achievement and traditionally reform movements have aimed at remedial models to improve students' achievement. According to this model, students from diverse populations get more of the basic skill learning without the opportunity to participate in learning opportunities that develop unique talents,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 177-184. Thessaloniki, Greece: PME.

creative thinking and problem solving strategies. For students to be prepared for the 21st century they need to be critical thinkers, problem solvers, and effective communicators who are proficient in core subjects, technology literacy skills, and life skills.

In reform oriented approaches like problem based learning, students work in teams to explore real-world problems and create presentations to share what they have learned. Compared with learning primarily from textbooks, this approach has many benefits for students, including deeper knowledge of subject matter, increased self-direction and motivation, improved research and problem-solving skills, and understanding how academics connect to real-life and careers. The study by Boaler (1999) found that students at the problem-based school did better than those at the more traditional school both on math problems requiring analytical or conceptual thought and on those considered rote, that is, those requiring memory of a rule or formula.

This following collaborative design research focused on building collective mathematical knowledge in the classroom by integrating students' reasoning and proof and designing meaningful 'pedagogical content tools'. Pedagogical content tools have been defined as "devices such as graphs, diagrams, equations, or verbal statements that teachers intentionally use to connect students thinking while moving the mathematical agenda forward" (Rasmussen & Marrongelle, 2006, p.388).

The research questions that guided this design research were:

- What influences do teachers' deliberate reflections and discussions focused on developing collective knowledge in the classroom and making connections and generalizations have on pedagogical decisions?
- What design features of the project and pedagogical content tools promote development of collective knowledge, algebraic connections and generalizations methods among elementary students?

METHODOLOGY

The participants in this study were sixteen fourth through sixth grade students who participated in the summer program focused on problem based learning. These students attended a Title One designated elementary school in a major metropolitan area with a diverse population of 600 students at the school: 51% Hispanics, 24% Asians, 16% Caucasians, 3% African Americans and 6% others, with over 50% receiving free and reduced lunch. Many of these students' former classroom teachers recommended them based on their exhibition of mathematically promising traits during the academic year. The focus of the summer program was to immerse students in authentic mathematics problem solving while utilizing local community resources such as invited community speakers. The idea was to expose students from diverse backgrounds to challenging mathematics while fostering their algebraic thinking and positive attitude towards mathematics. The project called *MATH 4-1-1: Young*

Mathematicians on Call had students solving rich engaging, meaningful and mathematically complex problems presented by the community.

The study used qualitative methods, specifically, the design research approach and research memos. The design-based research method is aimed at improving educational practices through systematic, flexible, and iterative review, based upon collaboration among researchers and practitioners in real-world settings, which leads to design principles or theories (Brown, 1992; The Design-Based Research Collective, 2003). Using design research which emphasizes the processes of iteration, feedback loops and narrative reports, we refined the key components of our shared learning activities. As teacher researchers, we took active notes and memos as we proceeded with the research exploring the interplay between the individual and collective knowledge in the mathematics classroom through discourse and the development of new ideas through records of students' thinking. We collected video recordings of each class session, retained copies of students' work and work displayed in the "collective workspace" and recorded observations and reflections throughout the planning, teaching, and debriefing phases of the study. Daily debriefing meetings allowed for formative analysis which focused on what was revealed during the class session and how to plan for subsequent classes by modifying the task, tools and teaching methods based on the feedback. In addition, these memos and artifacts (i.e. students' written work contained drawings, solution procedures, numeric notations and explanations) were analyzed for emerging themes at the end of the project for a summative analysis.

RESULTS

Qualitative analysis from the teacher-researchers' memos indicated that the process of collaborative planning, teaching and debriefing focused on developing collective knowledge impacted many pedagogical decisions. The most prominent pedagogical decisions influenced were a) selecting tasks and sequences of problems, b) integrating pedagogical content tools, and c) orchestrating classroom discourse through pedagogical moves and questioning.

a) Selecting tasks and sequences of problems. In selecting the problems, the teacher researchers were deliberate in developing and posing problems that were related yet increasingly more complex. This allowed for students to naturally make connections to previous problems solved in class and to build upon the knowledge they had acquired. By having the previous problems displayed on the *Generalization Posters* and readily accessible, students had entry to problems and to solution strategies and built new knowledge based on previous knowledge. The problems were embedded in a real-life service project for the school to raise money for a natural habitat in the school courtyard. Through this, we used a business theme to work on many classes of real life math problems such as budgeting, analysing cost and revenue, maximizing profits, figuring out combination problems, using discounts and comparing prices using unit pricing. Another design feature that allowed for students to extend their thinking was through *Thinking Connection Cards*, which presented related problems with increasing complexity. Due to the nature of the multi-age and grades of the students, *Thinking Connection Cards* allowed for differentiation and extension for students who were ready.

b) Integrating pedagogical content tools to extend students' reasoning. In order to promote students' mathematical reasoning in the classroom, we implemented a design feature called *Collective Workspace* and *Generalization Posters*, sometimes referred to as *Poster Proofs*. *Collective Workspace* was a method for students to bring their individual work to their group and discuss different solution strategies and compare each others' strategies specifically looking for connections, efficiency, multiple representations and generality. This collective space and time allowed for students to connect their way of knowing to other strategies. In addition, it allowed students a chance to debate on which strategy was most efficient and effective to broader classes of problems.

Figure 1. "Collective Workspace" to build collective mathematical knowledge

Generalization Posters were created as a class to summarize the essential mathematical learning. Pictures of individual strategies were attached to these posters so that students could use them as a reference for future problems. It also allowed for students to build on each other's ideas so that every student has ownership of the collective thinking. Just as mathematicians over centuries built on conjectures and theorems, these young mathematicians were given the same opportunity to engage in building collective knowledge

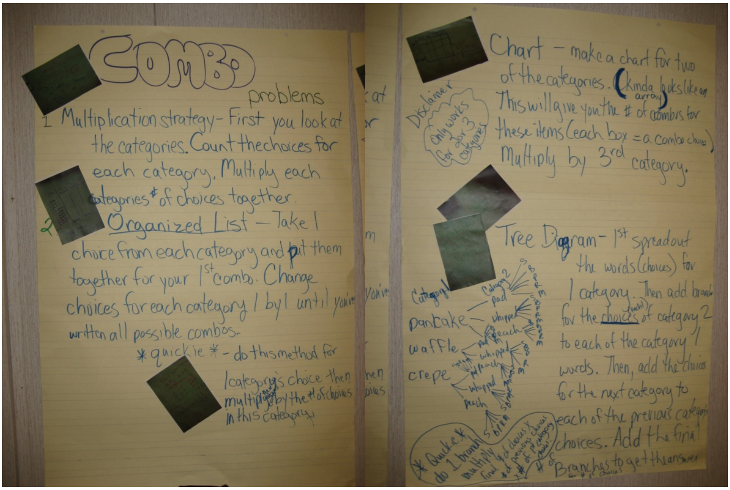

Figure 2. Generalization Posters with strategy photos

c) Orchestrating classroom discourse through pedagogical moves and questioning. The teacher's role in extending students' thinking during this task was in engaging students to share strategies and to look for an efficient way to solve problems and to generate a rule. To analyse the mathematical discourse, teacher researchers used these codes to make sense of pedagogical moves and questioning strategies.

PEDAGOGICAL MOVES and QUESTIONING

Zooming in and zooming out: making generalization
Connecting: making connections among representations or algebraic concepts
Marking: marking critical features which the students should pay attention to.
Directing: keeps the students on task and encouraged to persist;
Extending: Pressing on for justification
Scaffolding: simplifying or clarifying

In one of the combination problems, students were asked to determine the number of sundae choices an ice cream shop owner could offer her customers. Some of the students began by drawing the ice cream with different flavors and toppings, but quickly found that drawing a picture was not an efficient strategy. Below is an excerpt from the discussion that took place during the collective workspace.

Teacher: Let's look closely at how your classmates solved this problem. *(PMQ: Zooming In)*

Lana: I drew a picture of the ice cream with its topping and syrup, but it was not easy so I decided to list all the different combinations. Then I noticed my partner was using the first letter of the flavor, topping and syrup and it seemed like a short cut then writing out the whole word, like strawberry.

Teacher: So you decided to use S to stand for strawberry and C for chocolate and V for vanilla. I see that you have listed the possible sundaes. How did you know you had all the possible combinations? *(PMQ: Connecting and Marking)*

Jose: I decided to create a chart with the flavors going down and the topping going across and had a 3 by 4 table. But then I realized for each I also had to decide it I wanted caramel or chocolate syrup. So I had to take the 12 types of ice creams and double them for the syrup and got 24 different combinations.

Mariam: I used a tree. I started with the 3 flavors and each flavor had 4 topping choices and then from there I had 2 syrup choices, so I knew that it would be 3 x 4x 2 = 24 different kinds.

Teacher: I see that Mariam used multiplication to help her see how many combinations she had. Do others see how this equation may appear in your solution? *(PMQ: Connecting*) So how are your different strategies similar or different from each other? Take a few minute to look at your own and turn to a partner and share. *(PMQ: Zooming in and zooming out)*

Frances: I noticed that Lana's list was done in a similar fashion as Mariam's tree. She seemed to start with one flavor and go to the next topping and then to the syrup. She wrote it each time making sure she did not double it up.

Teacher: Frances mentioned flavors, toppings, and syrups. What were in each category? *(PMQ: Marking and extending)*

Brandon: There were many choices, for example, there were four topping Choco chips(CC), oreo cookies(OC), rainbow sprinkles(RS) and fresh berries(FB).

Teacher: So what can we write on our Generalization Poster about combination problems? *(PMQ: Zooming out)*

At the end of class, the Generalization Poster read,

IN GENERAL, when solving a combination problem with categories and choices, you can find the number of possible combination by multiplying the choices in each category, for example: Number of flavors x number of toppings x number of syrups=Number of possible combinations. F x T x S= total But multiplying will only tell you the total number, not the different types of combinations. For a list of combinations, the tree method works quite well and keeps the list organized. A table is easy if you have two categories but when you have more, you might have to make another table. A smart way to save time is to use a shorten form or just the first letter of the choice so that you are not wasting time writing it all out.

It was during the conversation that took place in the collective workspace that students negotiated the meaning of solving combination problems and concretized the learning for the individual and for the collective group. As evidenced by the excerpt, the advancement of ideas that resulted from students' reasoning became a collective record through the *Generalization Poster*. In addition, student generated representations, such as, the table, tree diagram, equation and verbal explanation became important pedagogical content tools for scaffolding questions for algebraic connections, explanations and generalizations and for students to compare, connect and extend their thinking.

CONCLUSION

Through this research, we developed a working framework called Building Collective Knowledge to Enhance Students' Learning. Principles to this framework included, a) *adhering to the authenticity of problems*, which proved to be motivating for students. The teacher researchers ensured that the task required students to use higher ordered thinking skills, to consider alternate solutions, and to think like a mathematician; b) *making connections and generalizations* to important mathematical ideas that go beyond application of algorithms by elaborating on definitions and making connections to other mathematical concepts, which led to; c) *navigating through guided reinvention,* (Gravemeijer & Galen, 2003) where students go through similar processes as mathematicians so that they see the mathematical knowledge as a product of their own mathematical activity (p. 117); d) *elaborating and communication through justification*, where students demonstrate a concise, logical, and well-articulated explanation or argument that justifies mathematical work; e) *participating in shared learning* and the interdependence of social and individual processes in the co-construction of knowledge.

In this study, we benefited from the opportunity to plan and debrief together which allowed us to determine when, what kind and how to use tools such as graphs, diagrams, equations, spreadsheets, or verbal statement to connect students thinking and to build collective mathematical knowledge in the classroom. This process required the combination of pedagogical and mathematical knowledge. This study suggests that integrating reflective planning with effective mathematical tools such as representations, notations and explanations and the use of critical pedagogical moves and questioning can help build collective mathematical knowledge in the classroom.

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EXPLORING THE RELATIONSHIP BETWEEN TASKS, TEACHER ACTIONS, AND STUDENT LEARNING

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We are examining actions that teachers take to convert tasks into learning opportunities. In this paper, we contrast ways that three teachers convert the same task into lessons, and the way that their lessons reflect their intent. We found that the teachers did what they intended to do, that this was connected to their appreciation of the mathematics involved, and directly influenced the learning opportunities of the students. To the extent that the potential of the task was reduced, this seemed due to the lack of mathematical confidence in the case of two of the teachers.

TASKS AND TEACHER ACTIONS IN IMPLEMENTING THEM

We are investigating ways that particular types of mathematics classroom tasks create opportunities for students and challenges for teachers. Various authors have argued that classroom tasks are the medium through which teachers and students communicate, and that the type of task influences the nature of the learning (e.g., Christiansen & Walther, 1986; Hiebert & Wearne, 1997).

The data presented below are from the *Task Type and Mathematics Learning*[1] (TTML) project which focuses on four types of mathematical tasks as follows:

- Type 1: Involves a model, example, or explanation that elaborates or exemplifies the mathematics.
- Type 2: Situates mathematics within a contextualised practical problem to engage the students, but the motive is explicitly mathematics.
- Type 3: Involves open-ended tasks that allow students to investigate specific mathematical content.
- Type 4: Involves interdisciplinary investigations in which it is possible to assess learning in both mathematical and non mathematical domains.

The focus of our overall research is to describe how such tasks respectively contribute to mathematics learning, the features of successful exemplars of each type, constraints which might be experienced by teachers, and teacher actions which can best support students' learning.

The focus here is on actions that teachers take in implementing tasks in their class. We draw on the Stein, Grover, and Henningsen (1996) model of task use, in which they described how the features of the mathematical task as set up in the classroom,

[1] TTML is an Australian Research Council funded research partnership between the Victorian Department of Education and Early Childhood Development, the Catholic Education Office (Melbourne), Monash University, and Australian Catholic University.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 185-192. Thessaloniki, Greece: PME.

and the cognitive demands it makes of students, are informed by the mathematical task as represented in curriculum materials, and influenced by the teacher's goals, subject-matter knowledge, and knowledge of students. One of the interesting results from Stein et al. (1996) was the tendency of teachers to reduce the level of potential demand of tasks. Doyle (1986) and Desforges and Cockburn (1987) attribute this phenomenon to complicity between teacher and students to reduce their risk of making errors. Tzur (2008) argued that there are substantial deviations between the ways that developers intend tasks to be used and the actions that teachers take. Tzur argued that there are two key ways that teachers modify tasks: at the planning stage if they anticipate that the task cannot accomplish their goals; and once they see student responses if they are not as intended. Charalambous (2008) argued that the mathematical knowledge of teachers is one factor determining whether they reduce the mathematical demand of tasks based on their expectations for the students. A related issue is the extent to which students are allowed to create their own solutions, as compared with following a method proposed by the teacher. It has been argued that students choosing their own approaches, and their awareness of those choices, are key elements of mathematics learning (Watson & Sullivan, 2008).

The following comparison of three lessons based on the same task is intended to offer insights into the relationship between teachers' intentions, their actions, and the effect on the students, and especially on the relationship between the teachers' intentions, actions, and the task's potential.

THE OVERALL PROJECT AND METHODS USED FOR THIS PHASE

In a prior phase of the overall project, we worked with teachers to ensure that teachers have access to high-quality task exemplars. We led teacher development meetings focusing on the nature of the respective task types, the associated pedagogies, ways of addressing key constraints, such as diversity in culture, language background and readiness to learn, and student assessment.

At this current phase of the project, we worked with groups of teachers on coherent sequences of lessons, termed *teaching units*, drawing on a mix of the task types. The lessons reported below were from a teaching unit developed by a group of three combined Grade 5/6 (ages 11/12) teachers from the same school serving a middle-class community in Melbourne, Australia. The first step was for the teachers, termed Ms A, B and C (although not all were women), to identify the focus, which they proposed to be ratio and rates. The teachers met to plan the teaching unit, after which the researchers joined with the teachers to brainstorm possible activities from each of the task types. The teachers prepared a pre-test, including items such as "Write everything you know about fractions", and some specific content items. Each of the three teachers was observed in 7 lessons, many of which were 90 minutes long. The observation schedule was developed from Sullivan, Mousley, and Zevenbergen (2005), and records details of classroom events, including the timing, teacher actions, some quotes, and the reactions of the observer. There were audio-recorded interviews with the teachers before and after the lessons, and the teachers completed a planning pro-forma before each lesson. We developed a content test in collaboration with the teachers for the conclusion of the unit, and we supervised its administration and its scoring.

The teaching unit was taught over a three-week period. The teachers had developed a somewhat unusual format (unrelated to our project) in that they had arranged the class into like-achievement groups and created a set of up to nine tasks for each of the groups, although many of the tasks were similar across the four ability groups. The students could choose the order in which they worked, and this choice was emphasised by the teachers as having a pedagogical purpose. In the teachers' plan, one of the tasks for each of the groups was recorded, simply, as follows:

> Usain Bolt ran the 100 m. in 9.7 sec. How fast is that in km/hr? How fast you can run in km/hr?

The first part satisfies the definition of a Type 2 task: it is set in the context of the contemporary Olympic Games with potential to be interesting for the students; and it has an explicit mathematical purpose of conversion between comparable rates. The second part of the task could also be considered Type 3, with the openness being in the choice of the method, the choice of the mode of recording, the variety of correct answers, the possibility of interrogating the answers, and through the personal result.

Our specific questions in this phase were: *How do teachers' actions relate to the task potential and to their intentions?* and *What is the impact of the teacher actions on student learning?*

THREE DIFFERENT IMPLEMENTATIONS OF THE ONE LESSON

All three teachers taught a lesson based on this task. The following are summaries of the lessons as derived from the teacher interviews and observations, with some interpretative comments. In the summaries we also draw on the responses to the following question on the student summative assessment that directly addressed the content involved in this task:

> Usain Bolt's brother, Lightning Bolt, ran for one minute around the (school) track and covered 550 m. How fast did he run on average in kilometres per hour?

We also presented students with a list of the 20 possible tasks they may have completed, and asked them to identify which one they liked the most, and from which one they had learned most.

Teacher A

The written plan prior to the lesson indicated that Ms A intended to have an initial discussion linked to previous lessons, and a whole class discussion on km/hr, after which the students would work outside in pairs on the task, then a whole class debrief adding to an overall map of the concepts involved in the unit that the class was progressively and collaboratively developing.

As part of the 26 minute introductory discussion, Ms A, an early career teacher with confidence in her ability and mathematical knowledge, posed the following problem:

> (The class turtle) escaped. He covered 10 metre in 30 seconds. How fast is this in km/hr?

Note that the *Turtle* question is of a different form from the *Usain Bolt* question. After working on this problem, one student wondered whether he could walk that fast. Ms A adjusted her plan to facilitate this incidental opportunity. Then, the students were asked to work out how fast they could run. There were detailed directions on organisational matters (e.g., use the stop watches), but no instruction on how to do the running task.

The students then spent 30 minutes outside. The students worked in groups, with some choosing to measure how far they could run in a particular time. When asked, the students in those groups said that they chose their method deliberately since it would be easier to calculate. For example, one student said:

> [student name] and me chose to do ten seconds because if you do ten seconds, it needs to add to one hour but the distance doesn't really have to add to anything…

Other groups measured how long it took them to run a particular distance.

When the students returned to class, they continued to work in groups. Most of the eight or so students who had chosen the easier method calculated their answer readily. Many other students who had chosen the more difficult method struggled with the calculation. The teacher was extremely busy trying to help the students working on the difficult method, while the better students completed the work quickly (but pretended they had not yet finished). This phase took 25 minutes. There was no concluding review, and therefore no discussion of the differing methods.

In the post lesson interview, Ms A recognised what had happened.

> So those that had thought about time and a unit of time prior to it were able to do it more readily than those that had thought about a unit of distance. So if I was to do it again … I would try to make the specific ratio idea clearer… I always try to put it back on them.

Sixteen (or 73%) of the 22 students in this class correctly answered the *Lightning Bolt* question on the test. In the survey of task preferences, five students chose the *how fast can you run* task as the one they most liked (none chose the *Usain Bolt* question), giving comments like "It was fun and hard" and "We got to go outside I liked running around". A different five students chose the same task as the one from which they learned most, giving comments like 'I learnt things I didn't know before".

We interpret this experience overall to suggest that Ms A had thought about the task and its pedagogical purpose, and gave the students ample opportunity to devise their solution path for themselves. The task was well introduced, with the *Turtle* question being meaningful to the students, and at a lower level of difficulty. Ms A had not anticipated the way that the form of the calculation chosen determines the level of difficulty, although she realised this during the lesson. The task, and this lesson, clearly created opportunities for students and most students were able to respond to the assessment task. Nearly half of the class chose this as the task they either most liked or felt they learned most learned. The constraint was the lack of success by some other students, and the organisational difficulties created by having some students finished while others were struggling. This highlights that such contextualised and open-ended tasks are complex to implement. Even so the implementation of the task was as intended, and even when she realised the student difficulties, Ms A neither moderated the level of challenge for the students nor reduced the potential of the task, and certainly maintained a commitment to students choosing their own methods.

Teacher B

The lesson of Ms B was in two parts. In the pre-lesson interview, Ms B, a confident early career teacher, who was uncertain about aspects of her mathematics knowledge, described part of the lesson:

> …then we're going outside to calculate their kilometres per hour and that would be quite hard for some of them. So I'll just have to see how it goes with how far we get. The timing will be easy because we'll just time 100 metres and then convert it.

In the introduction, which took 10 minutes, Ms B posed this task as "how fast can you run a kilometre?" She invited the students to suggest how they might do this. Various considerations such as the ability to maintain running speed were proposed. One student suggested "we could do 100 metres and times by 10" and this idea was adopted. Interestingly, other methods proposed by students were rejected quickly, as they appeared not to conform to Ms B's plan. One student asked whether they could do 3 sprints and find the average, and this was confirmed as a good idea by Ms B:

> How you work it out is up to you, but you'll need to share the trundle wheel and stopwatches. Work out as a group how you're going to record results. When you've worked that out you can come and get a stopwatch and a trundle wheel off me. So groups of three or four would be best.

The students then spent 25 minutes outside on the sports field working on their data collection in small groups. The last five minutes of this part of the lesson was back in the classroom, with the students together. There was a discussion on how they could find an average, with the teacher giving the instruction, "go back through your maths book and see if you can find how to do it". This part of the lesson concluded with the instruction:

> Now think about how to change your time to how many km per hour you can run. Take it home and talk to your parents about it. See if you can work out how to do it.

In the post-lesson interview, Ms B was asked about the outcome of the lesson:

> I think they'll all need a little bit of help but I think some of them will be able to work it out with a bit of help. A couple of them have already done the *Bolt* question where you convert his speed of how it takes him 9.7 seconds to run a hundred metres. I've already

> worked with a copy of students to help them convert that into kilometres per hour. So they'll be able to use that information to help them.

In the following lesson, after an unrelated introduction, Ms B did a six minute review of the "how fast can you run" lesson. After discussing strategies for calculating their average speed in km/hr, the students then worked individually or in pairs for five minutes on possible methods for calculating the speed. There was limited success. Ms B led a discussion about Usain Bolt's speed with questions like "how many times do you have to multiply the 9.7 to get to an hour?" She wrote on the board "9.7 × ? = 60" leading to a procedural presentation of a solution. She then repeated this with some of the students' times (e.g., 21 seconds) modelling the procedure and then asking them to work on their own answer. They worked on this for 25 minutes.

Ms B's students were less successful than Ms A's on the assessment item, with only six (or 35%) of the 17 students in this class correctly answering the *Lightning Bolt* question on the test. None of Ms B's students chose either task as the one the liked, but five reported learning most from the running task and another three said they had learned most from the *Usain Bolt* task. They wrote comments like "really didn't know how before" and "how to convert from … seconds into kilometres per hour".

We infer that Ms B posed the task (how long would it take you to run a kilometre?) this way to make the calculation easier, but it did not do this. The task she implemented was the task she intended. The orientation of the teacher towards allowing students to make their own choices was evident in her posing the home based continuation to the first part of the lesson. In contrast she did not allow students to choose their own method of solution to the task. In class, Ms B's attempt to simplify the task and the direct and the formal way that she presented a solution method was both planned and perhaps limited by her lack of confidence with the mathematics underlying the task. In other words, she attempted to reduce the potential challenge for the students, having anticipated student difficulties, and this seems connected to her own lack of confidence with the task itself. As it happens, her attempt to make the problem simpler for the students actually made it more procedural and more complex. Her students did not do well on the assessment item, but nearly half felt they had learned something from the experience.

Teacher C

The lesson of Ms C, in short, was similar to that of Ms B, but different in three major ways. She showed a video of the actual race, she spent time in the introduction and conclusion on calculating time differences (which was irrelevant to the ratio aspect of the lesson), and she drew skilfully on students' suggestions. The observer's noted:

> (Ms C) invited that same student to explain his method to the class, said: "I got 100 m. and divided by 9.71, which gets me how much metres you got in a second, and then I multiplied by 3600." Ms C asked where the 3600 came from, and the student replied "60 seconds by 60 minutes," and gave the answer as 37075.18. Ms C then restated the method suggested by the student, followed by a discussion of the need to divide by 1000.

Twelve (or 52%) of the 25 students in this class correctly answered the *Lightning Bolt* question on the test. This success rate is in between those of the other two teachers. Five students reported that they most liked the running task and a further two said they learned most from it.

In summary, Ms C had the intention of being explicit about the method the students should use, but drew the method from a student, before restating this method. After the lesson, she was aware that the method she chose was complex, and expressed a view that she would describe a simpler method another time. Ms C intended to restrict the student choice of method, and so reduced the potential of the task, and this seemed directly connected to her own lack of confidence in the mathematics needed to solve the task.

SUMMARY AND CONCLUSION

The first part of the task that was the basis of these lessons is complex because of its real world nature (had Bolt run 100 m. in 10 secs it would have been easier), and the other part of the task (the running) was complex because of its openness and the student choice involved. Between a third and a half of the students in each of the three classes claimed to most like or most learn from one or other of the parts of the task. This is significant given that there were a number of interesting tasks from which they could have chosen. Ms A did not moderate the demands of the task, although she did not use the *Usain Bolt* part. Both Ms B and Ms C intended to present a particular method of solution, apparently motivated by their lack of confidence with how to do the task themselves. At this level, knowing a formal method for rate conversions is of limited value, and the real potential of the task is the opportunity for students to work out a method for themselves. The reduction in the potential of the task by teachers B and C was mainly in the restriction of the students' choices of the methods of solution.

The three lessons confirm the applicability of the Stein et al. (1996) model, which asserts that the classroom implementation of a task is influenced by the teacher's goals and subject-matter knowledge. In each case, what the teachers intended was what they did. Anticipating student difficulties, two of the teachers (B and C) moderated the demands of the task before the lesson, and each of these teachers was explicit in the method they expected the students to use. Ms A's students were more successful on the assessment item and, paradoxically, some of her students discovered the easier method of solving the task in class. This highlights the complexity of converting tasks to lessons in that some challenges are difficult to anticipate, and must be dealt with as they arise. In this case, the confidence of the teacher who allowed her students more freedom to explore was rewarded with more interesting responses from the students and apparently better learning. All three teachers were willing and able to draw on the student ideas and were prepared to spend time and energy to facilitate this. All three teachers had designed the *learning unit* with an emphasis on student choice of task, but the choice of method for this task was only part of the lesson of one of the teachers. To the extent that the potential of the task was reduced by two of the teachers, it can be attributed, in a similar way to the teacher studied by Charalambous (2008), to their lack of mathematical confidence in solving the task themselves, and not to any lack of familiarity with, or confidence in, student enquiry or problem-based methods.

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"PROBLEM VARIATIONS" IN THE CHINESE TEXT: COMPARING THE VARIATION IN PROBLEMS IN EXAMPLES OF THE DIVISION OF FRACTIONS IN AMERICAN AND CHINESE MATHEMATICS TEXTBOOKS

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This paper illustrates the features and roles of "problem variation" in the Chinese textbook by comparing of examples of division of fractions in American and Chinese mathematics textbooks. The analysis reveals striking differences in concept organization and solution organization in the textbook examples. Specifically, 85.7% of the examples in the Chinese textbook were found to include problems with variation of concepts connection, and 57.1% of the examples were problems variation of solutions connection. Such an approach to problem variation is known to build a mathematics structure for learners by connecting concepts and tends to build a method system by connecting methods. Yet neither concept variation nor variation with solutions connection appeared in the American textbook. Together with problems variations, the Chinese textbook has distinctive features and rationales to organize and clarify the mathematics knowledge it covers.

INTRODUCTION

Efforts to explore possible contributing factors to observed cross-national differences in students' mathematical achievement have led to the discovery that the form of the curriculum is one of the key factors (see, for example, Schmidt, McKnight, & Raizen, 1997). In particular, researchers have analyzed textbooks to understand their potential effect on students' mathematical achievement in the United States and other countries (e.g., Schmidt et al., 1997). Existing textbook studies, however, have usually focused on content analyses, including content-topic coverage and problems presented in the textbooks (Li, 1999; Sugiyama, 1987). Less attention has been placed on the analysis of problem variation in the examples, which is a kind of problems design composed in terms of connections between concepts and solutions to different problems, – where an example may be made up of origin problem and its variation problems. The reason for the lack of attention to problem variation may be because the definition of problem variation is difficult to achieve with a clear boundary. However, the natures of the problem variation tend to enhance the understanding of concept and solution hidden behind examples. Moreover, the aims of the problem design tend to enhance the discernment of concept organization and relations among different solutions underlying problem, which incline to build a clear mathematical structure and a solution system at last. Concepts /solutions connection feature behind problem variation reflects the mathematical structure /solutions underlying among the problems. Without concepts connection behind design which illustrates object itself.

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Although it is well known that problems are central to mathematics, and that problems are central to the mathematics curriculum (Halmos, 1980), to date studies of problem variations seldom appear in the field of teaching and learning of problem solving. Recently, the importance of examples is not only getting more and more recognized (see, for example, Watson, Zaslavsky & Mason, 2006), there is also the related study of variation that is forming a “hot” topic in the education field (Qingpu Educational Reform Group, 1991; Marton & Booth, 1997; Gu, Huang & Marton, 2004; Sun, Wong & Lam 2005; Sun , 2007; Wong, 2007). Among these studies there is a growing agreement that variation (“bianshi” teaching, or teaching with variation) study could be regarded as a kind of Chinese wisdom of mathematics teaching. Variation study is gaining more and more attention in the mathematics education field. The study reported in this paper is designed to provide an example of an exploration from an international comparative study perspective in order to understand how to construct curriculum for mathematical understanding and its particular execution in China by problem variation. From an international comparative study perspective, Ma（1999） investigated US and Chinese teachers subject matter knowledge across the four basic operations of arithmetic. Remarkably, Ma found striking differences on the advanced arithmetic topic of division of fractions through asking teachers to calculate, and illustrate the meaning of, items such as:

$$1\frac{3}{4} \div \frac{1}{2} = ?$$

Ma’s analysis showed there were gaps between American teachers and Chinese teachers on the concept understanding and solution multiplicity on the topic.

It is the case that the textbook is an important, stable and visible mediator between teaching and learning, directly embodying the ways in which the textbook writers view both mathematics and the teaching and learning of mathematics. Especially in terms of the examples in the textbook, these directly demonstrate what concepts and solutions the textbook designers want the students to learn. The important question that arises is whether the design of the concepts and solutions of division of fractions in US and Chinese textbooks are related the gaps mentioned above between US and Chinese teachers. As such, the issue then arises what are examples differences between US and Chinese textbooks and what are differences of problem variations in the example to reveal concepts and solutions on the topic of division by fraction between US and Chinese textbooks.

The research questions: what are the features and roles of "problem variation” in the Chinese textbook by comparing of examples of division of fractions in American and Chinese mathematics textbooks?

THEORETICAL FRAMEWORK

The theoretical framework used to address this issue of comparing examples in US and Chinese textbooks is the theory of variation (Qingpu Educational Reform Group, 1991; Marton & Booth, 1997; Gu, Huang & Marton, 2004; Sun, Wong & Lam 2005; Sun，2007; Wong, 2007). The natures of the examples tend to enhance the understanding of concept and solution hidden behind examples. Moreover, the aims of the problem design in the examples tend to enhance the discernment of concept organization and relations among different solutions underlying problems variation.Students may learn recapitulating connection of concepts and connection of solutions from problems variations (i.e. variable or invariable) after solving many problems (i.e. abstracting mathematics relations from the relations among problems again), similar to learn capitulating concepts and solutions from a single problem. The textbook deficiency in connection of solutions and connection of concepts determined their limitations to the teaching and learning of mathematics structure and its method system. Even a mathematics / method structure could not make up for their ignorance connection of concepts/ method (Sun, 2007).

METHOD

Materials

Given that Chinese textbooks are highly centralized in the Chinese education system, the textbook published by People Education Press, and written by the math textbook developer group for elementary school (2005), is often regarded a representation of Chinese national curriculum by most scholars of textbook comparative study (for example, Li, 2000) due to the textbook having been used by the majority of Chinese students with diverse populations for thirty years. This textbook is often regarded a representation of Chinese national curriculum. For this study, the US textbook selected was Bolster, Boyer, Butts & Cavanagh (1996). This textbook was identified as one of the most popular and widely used mathematics textbooks by the 2000 Mathematics and Science Education Survey conducted by Horizon Research (2001).

While some care was taken in selecting the textbooks to analyze, it is not being claimed that these textbooks represent all the textbooks that might be used in the two countries. However it is interesting to note that there were considerable similarities among other versions of American textbooks considered for this study.

Analysis Framework

The analysis framework was informed by the categories developed in the work of Sun (2007). It is well known that the tasks in the texts, such as examples or exercises, are not always made up of a single problem alone, but often a group of problems. Such a group of problems naturally contain the original problem and with its variation problems with or without concepts connection and with or without methods connection, when examined under the variation perspective.

RESULTS

1. Differences between US and Chinese textbooks

The Chinese textbook presented whole content of division by fraction across grades K-8 in its 6th volume alone. The textbook included seven examples which illustrated concept of reciprocal, concept of fraction, concept of addition of fraction, concept of multiplication of fraction, concept of equation, concept of partition, concept of measurement, concept of products and factors.

In contrast, the American textbook presented content of division by fraction at 6th volume, 7th volume, and 8th volume across grades K-8. It includes five examples which illustrated reciprocal concept, measure concept with division of fraction algorithm and area concept. All the five examples required a single computation procedure only.

2. Differences of problems variation with concept organization in the examples

Under the variation of concept feature window, the examples to introduce the new concept at the beginning in the Chinese textbook: Original problem: *Each box of fruit candy weighs 100g, how much does 3 boxes weigh? Variation problem 1: 3 boxes of fruit candy weigh 300g, how much does each box weigh?* Variation problem 2: *Fruit candy weigh totally 300g, each box contain 100g fruit candy, how many boxes need it?*

The example to introduce the new concept at the beginning in the American textbook: Original problem: *For an art project, Maia cuts pieces from several ribbons. How many ½ inch pieces can she cut from this 5-inch red ribbon?* Variation problem 1: *How many 3/8-inch pieces can be cut from a 3/4-inch piece?*

In Chinese textbook, there is problems variation with concepts connection (the problem design targeting different concepts in a set of problems) between multiply concept and division concept underlying original problem and variation problems, which may be conductive the concept clarification among multiplication concept students are familiar with, with division concept and equation concept students are unfamiliar with. There is a “unit conversion” from 100g to 1/10Kg which switched the division of whole number to the division of fraction. However, in US counterpart, we can see there is a concept of division by fraction alone, i.e. problems variation without concept connection.

The whole data also indicated that Chinese text not only introduced new concept by the problems variation with concepts connection but also enforced new concepts by abbreviated problems variation with concepts connection （the problem design targeting different concepts in a single problem）in 85.7% of the examples in Chinese textbook (These 6 examples with these kinds of problems variation or abbreviated problems variation with concepts connection , in total, account for about 85.7% of all 7 examples). But none of them appeared in the American textbook.

3. Differences of problems variation with solution organization

In fact, Chinese textbook took three steps to extend the knowledge of division by fraction by problems variation with solution connection (also called one problem multiple solutions). first, the algorithm of division by *whole* number was clarified by using the problem variation with multiplication and division methods connection, second, the algorithm of division by *fraction* was further illustrated by using the problem variation with multiplication and division methods connection, At last, the concept of division by fraction was further clarified by using the problems variation with equation and arithmetic methods connection. About 57.1% of examples with problems variation with solutions connection present problem-solving solution structure with more than one solution in the example or component of the example in Chinese textbook (These 4 examples problems variation with solutions connection , in total, account for about 57.1 % of all 7 examples). But none appeared in the American textbook.

DISCUSSION

The results above are relevant to documented cross-national differences in American and Chinese teachers' mathematical performances (Ma, 1999) and curriculum difference (Li, 1999). Since every State in US uses quite different textbooks, the generalisation of curriculum difference must be limited. However, the study indicates how the Chinese intended-curriculum elicit the mathematical structure and the rationales behind algorithms by problem variations under the mirror of the American textbooks.

The "problem variations" play a crucial role to shape a characteristic of the Chinese text with *curriculum organization, focus of curriculum and its intended problem-solving process,* which are different from the US text.

The "problem variation" designed in terms of connections between concepts and solutions, by using variant concepts and variant methods, may provide a setting to prevent students only memorize the concepts overarched in the origin problem and the methods applied the origin problem, may help students to compare the similarities and differences between the origin problem and the variation problems, may go further to reflect the similarities and differences of the concepts and methods to be used. It is possible that the identified variation may, from the variation theory of view, enhance the possibility to learn. For instance, if multiplication and division co-vary within the same problem set, it may make it possible to see connections between the two concepts. Thus, it is very important that the identified differences matters for students' learning or possibilities to learn. Also, problems variations are conductive to shape the process of problem-solving into a reflection system, which provide a setting in which students with their teachers can reflect, inter-relate and generalize.

By problem variation with concepts connections, the Chinese text elicited a knowledge package of division by fraction with connections among concept of reciprocal , concept of the fraction, concept of multiply, concept of division, concept of division by fraction , concept of subtraction of fraction, concept of partitive division of fraction , concept of multiply of fraction , concept of equation by using 7 examples above. Chinese textbook also presented each a concept with its relations in order to derive the structure of division fraction.

By problem variation with solutions connections, the Chinese text try to permeate the rationale underlying the algorithm of division by fraction, i.e. division is equal to multiplication its reciprocal of divisor, by explaining the division method and the alternative approach of multiplication its reciprocal of divisor at same time, which promote "rationale" thinking and present how the "rationale" and the "procedure" are inter-related in the process of explanation *why* division is equal to multiplication its reciprocal of divisor.

Another role problem variation played in the Chinese text is longitudinal coherence in weaving all concepts into a whole knowledge tree, from the beginning to introduce a new concept (i.e. the division concept) from an old concept (i.e. multiplication concept), to the ending to extend the relation between division and multiplication concept into the idea of equation, the basic and powerful rudimentary for algebra curriculum students will learn latter. The whole organization of methods in problem design of all 7 examples just promoted approaches to revisit and enforce basic concepts which students learned already in arithmetic: the fraction concept, division concept, partitive concept, equation concept. In particular, it addresses the invariant meaning of division between division by fraction and division by whole number in all the examples i.e. asking for the amount of every a section through dividing the sum into several sections.

Comparing with other textbooks in two countries, I found that there were considerable similarities within countries. For example, the textbook (Rose, Tourneau, McDonnell, Burrows & Ford, 1996) presented all 15 examples of problems variation without concepts and solutions connection. The exercise section is almost same as the example section of division by fraction. Problem variation with concepts connection and multiple methods connection seemed to be a salient idea secretly permeated through at some teaching materials (formal ones, such as textbook or informal ones such as teaching plan) at school and learning material (such as students` exercise or worksheets) after school in china. The data further suggest different significances related to problems variation in Chinese curriculum:

Problems variations have played an important role to make *any a* concept to connect with others and *any a* method to connect with others, which tend to build the whole mathematics structure by connecting concepts and tend to build the whole method system by connecting methods .

CONCLUDING COMMENT

Besides the above, it should be realized, on the one hand, that the examples in Chinese text are more challenging and more abstract than the American counterpart because of problems variations. This may lead to much more learning difficulty for Chinese students than US counterpart. Moreover, there exist distinctive features that Chinese text seldom stresses on the intuitive knowledge, such as measure experience, visual pictures, than American text, which may be disadvantages, which it is not enough for the intuitive knowledge development.

On the other hand, the results not only show the value of studying problems variations in the curriculum design field as a teaching scaffolding tool, but also show the value of comparing problems variations in the curriculum comparison field, which provide new lenses for comparing textbooks. Problems variations, a different window, holds more promise for revealing cross-national study than comparing problems and examples alone. Since this study is limited in its scope of topics selected as well as in its focus on comparing examples, Therefore, the findings has its limitation. I recommend the findings call for an extended fine-grained comparison of examples across content topics or grade levels on problems variations in the textbooks or other curriculum materials.

Notes

The latest curriculum, called as spiral variation curriculum, based on Chinese problem variations carried out successfully in Hong Kong. The model, an improvement on variation (bianshi) teaching developed by Gu - and in line with Marton's theory of variation (Qingpu Educational Reform Group, 1991;Gu, Marton & Huang, 2004) - was tried out in 21 classes in primary schools in Hong Kong, The effect of curriculum are significant (Sun, 2007; Wong, 2007).

"Bianshi" is written as "變式" in Chinese, with "Bian" meaning "changing" and "shi" meaning "form". "Bianshi" can be translated liberally as "changing form".

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MAKING SENSE OF ACCUMULATION FUNCTIONS IN A TECHNOLOGICAL ENVIRONMENT

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The present case study was designed to analyze the process of making sense of the accumulation function when the integral concept is being studied in the geometry track. This study is guided by the socio-cultural learning theory and the assumption that mathematical concepts are learned by means of constructions in three worlds: embodied, symbolic, and formal. We concentrate on the first. The learning environment designed for this study includes activities focusing on the use of computer applications that support direct manipulations of graphs and parameters related to the visual representations of accumulation and integrals. The case study focuses on two 17 year old students who have studied differentiation but not integration. In the course of the discourse micro-analysis we identified four elaborations of the meaning of the accumulation function: (1) noticing the presence of a lower limit, (2) awareness of the negative and positive areas of bounded rectangles, (3) paying attention to zeros in the integral as indications of accumulation of negative and positive areas, and (4) explaining the global effect of the change of the lower limit as vertically transforming the integral function.

INTRODUCTION

The objective of this study is to understand the process of making sense of the integral graph when the integral concept is first studied in the geometry track (De Lang, 1987). In the geometry track the relation between the function and the anti-derivative graph is considered to be the central object of calculus learning. Following Bruner (1966), Tall (2003) identifies three modes of mental representation, sensory–motor, iconic and symbolic, categorized into three distinct modes: the embodied world, the symbolic world, and formal world, each world having a different warrants for truth (*ibid*, p. 6). Technological tools have the potential to support settings in which learners can experiment with interactive changes of parameters, visualize graphic feedback, directly transform graphs, and view dynamic multiple-linked representations of the derivative and the integral. The current study was designed within a setting that promote experimentation in what Tall (*ibid*.) defined as "the embodied world." The technological tools we used in this experiment are considered to be external tools or signs that have cultural meaning. Our aim was to analyze how the personal meaning that learners bring to the learning of integration developed to be consistent with the cultural meaning that is part of the semiotic systems upon which the tools are designed. The case study explored over 20 hours of learning by Hadel

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 201-208. Thessaloniki, Greece: PME.

and Shada, two 17 year old students who have already studied the concepts of function and derivative but not that of integral.

THE CULTURAL AND PERSONAL MEANINGS OF SIGNS

According to the socio-cultural theory of learning, mediators that are external artifacts are internalized through active learning processes to tools that facilitate perceptual activity and sense making. Signs are often described as an instrument of psychological activity, analogous to the role of tools in labor (Vygotsky 1978). The basic analogy between signs and tools lies in the mediating function that characterizes both. According to Vygotsky, externally oriented tools may be transformed into internally oriented ones through a semiotic process of internalization, in particular by the use of a semiotic system in social interaction. Signs in general and mathematical signs in particular play two roles. Radford, Bardini, Sabena, Diallo, & Simbagoye (2005) define these roles as "social objects in that they are bearers of culturally objective facts in the world that transcend the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions" (*ibid.*, 117). Berger (2004), who studied the integral sign, used the concept of meaning in two ways: personal meaning, "to refer to a state in which a learner believes/ feels/ thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not)," and cultural meaning, " to the extent that its usage is congruent with its usage by the mathematical community" (*ibid.*, p.83). Computerized mediators are designed along the culturally accepted signs, and we consider learning as a development towards bringing personal and cultural meanings in line.

COGNITIVE CHALLENGES IN UNDERSTANDING THE LOWER LIMIT IN THE ACCUMULATION FUNCTION

The integral is a complex sign that can be interpreted in different ways: an anti-derivative that then appears as an indefinite integral; the area between the graph and the x-axis; the Riemann sum representing length, area, or volume (this view leads to the definite integral); and the accumulation function $F(x) = \int_a^x f(u)du$, where the upper limit, x, is a variable and the lower limit is a fixed parameter. Thompson & Silverman (2007) claim that understanding the various variables participating in the accumulation function is a challenge for students. The lower limit representing as the parameter a in the above formula indicates the initial point of the accumulation area, which is bounded between the function and the x-axis. Graphically, the lower limit value is the value of an intersection point of the F(x) graph with the x-axis. The relative zero and the area sign are challenging. For example, consider a positive function defined in the domain $x>0$ and $a=2$. The area bounded between the graph function and the x-axis is positive for $x>2$ and negative for $0<x<2$. (See Figure 2, right of the origin)

The students participating in this research received a short introduction on the use of the Calculus Sketcher Integral tool (Figure 1) (Shternberg, Yerushalmy & Zilber 2004) and of the "Calculus UnLimited" integral tools (Figures 2 and 3) (Schwartz & Yerushalmy 1996). Both environments dynamically and visually link between f(x) and F(x), and students were asked to explore and explain possible connections between each two graphs. We video-recorded the students and captured their computer screens, then analyzed the data according to the Radford's (2003) categories of attention, awareness, and objectification. According to Radford, to learn something is to notice it in the culture and be aware for its existence in culture, which then leads to objectification. Below we analyze part of the data in order to present the four-step development of objectification of the accumulation function.

AWARENESS OF THE PRESENCE OF A LOWER LIMIT

The Integral Sketcher provides 7 icons that can be used as a drawing tool for simple or compound graphs on the upper f(x) system. Each icon is drawn as a function in an interval and produces an integral function (accumulation function graph) in the lower system. The default initial value of the integral is (x, 0), where the x value coincide with the x-coordinate in the upper f(x) graph (Figures 1 and 2). Graphs and axes can be dragged. If f(x) is dragged vertically, the integral graph F(x) is distorted but remains anchored at (x, 0). If f(x) is dragged horizontally, F(x) is relocated to coinciding x values. In the course of dragging, students attended to these dynamic phenomena, described them, and attempted to explain them (Figure 1).

Hadel translates the function graph vertically, then drags it pulling horizontally the end points of the function in the interval. She notices that no matter how she distorts f(x), F(x) changes but always remains zero at the left point of the interval. This constraint, whereby the lower limit of accumulation is also the lower value of the interval, has been incorporated into the tool by the software designers. Apparently, it had a strong visual effect that led Hadel to the next step.

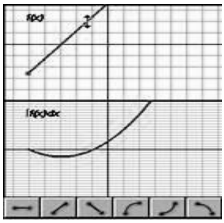


Figure 1

EXPLORING THE ZERO IN THE INTEGRAL GRAPH

Hadel and Shada deliberately explored the zero that remained invariant under dynamic changes when they later worked with the Calculus UnLimited integral tool. The 7-icon tool supported continuous dragging, but using the Calculus UnLimited tool students were able to fix the initial accumulation value by a numeric user input. As a result, visual changes can be viewed in a discrete mode. Any change in the lower limit value changes the location of the first rectangle. Hadel and Shada drew the function f(x) = x and asked for the accumulation of rectangles starting at 0. Then (at a1, a2) they changed the lower limit 0 to 3, and as a result the positive integral points graph shifted down and included both positive and negative values, as shown in Figure 3. The students tried to make sense of this visual change.

a1. Hadel: Go change the "lower" limit [the term lower appears on the tool button].

a2. Shada: [Enter a lower limit of 3] What is that? [Figure 2 appears on the screen]

a3. Hadel: I don't know [silence for four seconds] [Shada changes the lower limit to -4]. If I put that zero [the lower limit] that is between that [bold point in integral graph] then the origin must be negative. Try to make it positive again [changes the lower limit to 5]. Look at that, between five and zero it is negative, and there negative product of negatives [triangle A in Figure 2] is positive [indicates with the mouse the area left to the origin: triangle A]. But here [indicates the area between the origin and the rectangles] it must be negative because you fixed it to [lower limit] zero.

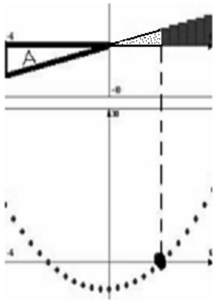


Figure 2

a4. Shada: No, it is negative between five and minus five. Not between the origin and five.

They tried out other negative and positive values for the lower value. Observing these visual changes, the students realized that at the value of the lower limit the integral value is zero (a3). At this point, the terms "negative" and "positive" appeared in the conversation. They first considered what we term "relative zero:" the integral graph is positive to the right of this point and must be negative to the left (between the origin and the zero point of the integral graph) "Look at that, between five and zero it is negative." Hadel then turned to the upper system and explained her idea about the area sign and said "and there negative… product of negatives is positive." She did not consider the negative values of the integral graph shown to the left of the origin. Shada noticed that and therefore rejected Hadel's conclusion (a3). In sum, at this stage the lower limit in the function graph receives the meaning of a zero point in the integral graph, which always indicates positive values to its right and negative values to its left.

MULTIPLE ZEROS AND AREA ACCUMULATION

A relative zero is not the only zero, however. The integral graph intersects the x-axis at any point that the accumulation of the area is zero. The students chose to produce a symmetric cubic function (Figure)4 and fix the lower limit at -2. According to the "relative zero" conjecture, they assumed negative values to the left and positive values to the right of it (b1).

b1. Hadel: Before this [indicates the first rectangle in the above system (A), Figure 3] it must be negative [left side] and after it [right side] must be positive, but here it's the opposite!

b2. Shada: After this [point C] it is positive and after that it is positive, [point B] [she indicates the areas where x>2 and x<-2] but why this? If we take -2 why is it negative between 2 and -2?

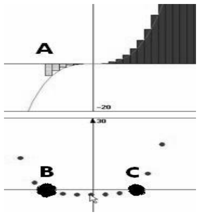


Figure 3

Observing the integral system, Shada noticed that to the right of the zero value the integral graph obtains negative values all the way to the second zero and positive values beyond the second zero. This contradicted their previous conjecture and served as a springboard to move from a local observation of signs of rectangle areas on the function graph to the accumulation of areas.

c1. Hadel: it is maybe the area in 2 and in -2 cancels each other. When you start to measure from -2, right? The area from 0 to -2 is the same as the area from 0 to 2. As a result the areas cancel each other [figure 4]. Is that right?

c2. Shada: But why do you get negative?

c3. Hadel: Because the area here is bigger than the area here [indicates to computer screen].

c4. Shada: I am not convinced.

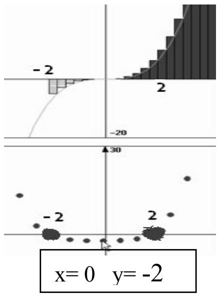


Figure 4

c5. Hadel: Look at the area between -2 and 1. The area from -2 and 0 is bigger than the area from 0 to one. That is, the area must be negative. Try to take the area from -2 to 0 as -2 and the area from 0 to 1 as 1 and a half. -2 plus 1.5 is -0.5. That is, we got a negative area. And as a result negative between -2 to 2.

Hadel used the balance metaphor. She noticed that there are two groups of rectangles with identical areas and opposite signs that cancel each other out. Shada was not convinced by Hadel's explanation (c4). Hadel provided a numeric example to explain her idea and for the first time used the idea of accumulation to explain the appearance of the second zero (c5). Hadel chose two areas from the right side to the lower limit.

She assumed that the first area, between -2 and 0, had a value of -2, and that the second area, between 0 and 2, had a value of 2. She claimed that summing these areas would produce the result zero. They examined another numeric example, this time the area between -2 and 0, reducing the area from 0 to 1.5. Again, Hadel claimed that summing these areas would produce the result -0.5. Attention to multiple appearances of zeros created awareness to a new aspect: accumulation.

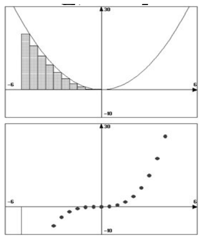
LOWER LIMIT CAUSES THE RELOCATION OF THE INTEGRAL GRAPH

The possibility of changing the lower limit discretely with the Calculus UnLimited tool allows choosing the lower limit value anywhere on the displayed interval. This option was important in the fourth step, leading to objectifying the lower limit as the cause for the vertical transformation of the integral graph.

d1. Swidan: If we change the lower limit, what will happen? [Figure 5 showing In the background]

d2. Shada: It will go up and down [moving her hands up and down].

d3. Swidan: Up and down, why up and down?



d4. Hadel: When we enter zero [in the lower limit], the area is zero there [changes the lower limit to -4, (Figure 6)]. Now here the area is zero [indicates -4] and there it is more than zero [in the origin].

d5. Shada: At -4 it is zero [the area], to the right of -4 it is positive, that is more than zero.

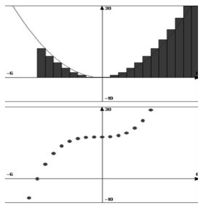


Figure 6

Swidan (the interviewer and first author) asked a question about the change that will occur in the integral function when the lower limit changes. Shada was aware that changing the lower limit caused the vertical transformation of the integral graph (d2). Hadel explained this behavior by comparing area values in two different situations of lower limits values. Initially, they fixed the lower limit at zero. Hadel and Shada knew that the area value was zero also in this point. Next, they changed the lower limit to -4 and compared the area at -4, which is zero, with the area at the origin, which it is more than zero. At this stage they were no longer dealing with specific point values but were explaining the connection between the lower limit, as an initial point of accumulation, and the intersection point with the x-axis, as well as the connection between the area of the rectangles and discrete points in the integral graph. They constructed the global view for the dynamic taking place in the integral graph as a result of changing the lower limit.

DISCUSSION

This case study suggests a four-stage development of understanding of the accumulation function as students are learning within two environments. One environment offers tools to sketch a function graph in intervals, producing an integral graph that is forced to be zero on the left value of the integral, and enables dragging and transforming both graphs. The second environment enables users to graph any function and choose numeric values for the initial value of accumulation, which is graphically presented as positive and negative area rectangles under the function curve, and an integral point-wise graph on which each point shows visually the accumulated value at that point. Both environments are semiotic systems that represent culturally accepted meanings and representations of integral and accumulation function. We consider learning to be a process in which the personal meaning of signs and representations is consistent with the culturally accepted ones. In this case study, we followed Shada and Hadel in their attempt to understand the role of the lower limit in determining the connection between the function graph (with which they were familiar) and the newly appearing graph in the system of linked graphs. Using the sketching tool, the students learned that the integral graph is always zero at the beginning of the function's interval. This fact was noticeable because sketching and dragging allowed students to change anything but the left x value of zero. The other visual changes were too sketchy and complex, and apparently were not useful at this stage. Moving to the second environment, the students were able to visualize other components connected to that same "zero." They determined a value to the "lower" parameter and became aware of the representation of rectangles that changed whenever the lower limit changed. At first they connect the positive and negative values of the integral with those of the function. Trying out different function and parameter values has led to a conflict: the zero that was no longer the beginning of an interval or an origin of the graph system became a "relative zero," indicating positive values to the right and negative ones to the left. This description was appropriate as long as they were observing a single zero and x values were positive. When they observed functions where the integral had multiple zeros, the students were forced to revise their image. At this stage they concluded that zero was not a "switch" between positive and negative values of the area of rectangles but rather a point bounding an area from the left to the point the sum of which was zero. In this study we did not explore the development and use of the symbolic formula but we assume that at this point the students were ready to understand Riemann sums and were getting closer to understanding the connection between a function and the accumulation function. Indeed, Shada and Hadel continued their visual exploration, and when prompted noticed that a change in the value of the lower limit shifts the integral graph vertically. It is possible that this reminded them of the first dynamic impression they had while dragging with the sketching tool, where they could not relocate the zero point whereas here they could. To explain this behavior, they produced the idea of accumulation of positive and negative areas. But they went beyond calculations by observing the cause and effect of the operation with the lower limit and stating that which the culturally accepted meaning offers: the accumulation of areas bounded by the curve and the x-axis can begin anywhere to produce a family of vertically transformed graphs. The design of the environments was driven by the two culturally accepted meanings of the integral, related concepts, and design constraints. The four steps were therefore deeply situated in the work with the given environments. The integral-related mathematical objects, the operations with them and on them, and representations that at first were mere artifacts, were eventually interiorized and acquired personal meaning that in the course of learning became consistent with the accepted meaning.

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CONCEPTUALIZING PROFESSIONAL DEVELOPMENT IN MATHEMATICS: ELEMENTS OF A MODEL

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This theoretical paper discusses the concept of models for mathematics professional development. After examining the mathematics professional development literature, we propose a definition of models for mathematics professional development that includes four elements: goals, theories, contexts and structure. We present aspects of professional development that comprise each element.

INTRODUCTION

In the United States, the current focus on accountability has increased attention to the effectiveness of mathematics professional development (MPD), and many recent studies have examined MPD initiatives to determine which ones were effective (e.g., Heck et all, 2008; Yoon et all, 2007). Despite differences in the definition of effectiveness used across studies (Darling Hammond & Youngs, 2002; Guskey, 2003), the search for effectiveness has generated interest in finding a common set of features that are present in various MPD programs deemed successful. Creating such lists of features of effective MPD can lead, some believe, to a consensus in the field about what constitute best practices in educating practicing mathematics teachers.

In this theoretical paper we claim that, by focusing on discrete features, these efforts to define effectiveness have supported the perspective of MPD as a research field that lacks a more systematic approach (Ball & Cohen, 1999), continuing to shy the field away from considering the conceptual and theoretical frameworks that support (or not!) the combination of various features of successful MPD into coherent MPD programs. To address this problem, we propose a set of elements that should be included in the design and description of MPD initiatives, creating what we are calling *models* for MPD. We contend that a more systematic approach to examining elements within a model of effective MPD (instead of discrete features) can strengthen the emerging, and still under-theorized, field of research and development in MPD. We also argue that by defining elements that comprise models of MPD, the field can begin to more consistently examine what is being done and learned in MPD through the development of shared language and frameworks.

EXAMINING THE MPD LITERATURE

In her review of the literature about MPD, Sowder (2007) claimed that the fast growth of the field since the early 1990s was due to the realization that the improvement of instructional practices required better-prepared teachers who could change the way mathematics was taught. This need to reform, Sowder noted, was true across many countries. She summarized various synthesis studies that examined

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successful MPD initiatives (e.g., Borasi & Fonzi, 2002; Clarke, 1994; Hargreaves, 1995; Hawley & Valli, 1999). The commonalities she found across these studies indicated that, to be effective, MPD should include: teacher participation in deciding the purpose of the intervention, support from various stakeholders, engagement in collaborative problem solving, modeling of appropriate instruction, continuation over time, and use of formative assessment.

Although such lists of features that make professional development effective can be helpful for researchers and developers in MPD, they are also vague. Many terms used in the MPD literature can be interpreted in various ways. For an example of this vagueness in MPD language, consider the term workshop. Loucks-Horsley et all (1998) defined workshops as "structured opportunities for educators to learn from facilitators or leaders with specialized expertise as well as from peers" (p. 86). They noted that workshops allowed participants to focus intensely on an issue of interest. When one examines the MPD literature, however, the term workshop, although frequent, is used in many different ways. In one description of an MPD program, workshops were one feature within a constructivist program that included a two-week summer institute, weekly meetings throughout the school year and four workshops to support collegial sharing among participants. In a different MPD, professional developers called workshop a two-week, university-level mathematics education course in the summer, with a one-day follow up after the school year started. In a third example, professional developers implemented a two-year intervention designed from a situated perspective on learning and developed around a sequence of iterations comprised of three workshops each, all focusing on the same mathematical task. In their more systematic review of the literature on effective MPD, all interventions Yoon et all (2007) examined used workshops. These interventions, however, differed in terms of content, duration, contact hours, follow-up activities, and level of teacher engagement in learning opportunities.

Far from being exhaustive, these descriptions are illustrative of the variance that is present when one talks about offering workshops for mathematics teachers. They show that little meaning exists in the statement that a teacher participated in a workshop. Borrowing from the same examples, all three studies also used the term model to refer to their programs. As in the case of the workshop, model was also used with varying meanings and there were varying amounts of explanation in each article about what these models were. Further, there was no consistency across articles about how to describe a model, which makes these examples characteristic of the MPD literature: key terms are often used but rarely defined.

CONSIDERING THE IDEA OF MODELS

In the general literature about staff development, a few efforts to define general professional development models can be found. Sparks and Loucks-Horsley (1989) discussed five models for staff development. Their description of each model included three elements: assumptions that guided the design of the professional development, research underpinnings, and phases that constituted the work with the teachers. For example, one of their models, named individually-guided staff development, was designed based on the underlying assumption that individuals are the best judges of their learning needs and are capable of self-initiated and directed learning. The research work of Rogers (1969), Knowles (1980), and Levine (1989) supported this model due to their foci on adults' search for growth, self-directedness, and needs at various professional stages, respectively. Professional development under this model included several phases such as: need identification, development of a plan, learning activities, and assessment of whether learning fulfilled the need. Training, another model proposed by Sparks and Loucks-Horsley (1989), rested on the assumption that teachers can learn to replicate behaviors and techniques that are new to their repertoire. Joyce and Showers' (1988) research findings about components for skills development supported the training model.

A training model was also considered by Little (1993) in her critical piece about the lack of fit between professional development configurations and reform teaching. She proposed four alternatives to the training model that rested on the common claim that "the most promising forms of professional development engage teachers in the pursuit of genuine questions, problems, and curiosities over time, … and communicate a view of teachers not only as classroom experts, but also as productive and responsible members of a broader professional community" (p. 133). The alternatives Little proposed to the training model were teacher networks, subject matter professional associations, collaborations for school reform, and institutes or centers. Common across these alternative models were principles such as meaningful intellectual engagement, pursuit of knowledge, and explicit accounting of the contexts of teaching and schooling. Elmore (2002) built on Little's criticisms to define a "consensus model" for teachers' professional development. His model included a focus on student learning, a clearly articulated theory of adult learning, active participation of administrators, use of data, and alignment between practice and message.

In the United States, the recent government regulation No Child Left Behind Act (U.S. DOE, 2002) defined professional development (section 9101.34) through fifteen features of the activities included in interventions. These features ranged from targets for the professional development to the broader picture surrounding the initiative to what the intervention should address. Although NCLB did not discuss professional development models, its definition of professional development included features related to the goals, context, and content of interventions. Context and content, together with processes, were also included in the National Staff Development Council (NSDC) Standards (2001).

Also regularly used in mathematics education is Loucks-Horsley et all's (1998) discussion of strategies. In the opening section of their book, these authors proposed a framework to guide the design of MPD. This framework highlighted the ways programs were designed to target particular contexts and goals, as well as guided by beliefs and knowledge about learners, teaching, change, and the nature of mathematics. Critical issues influencing the MPD design included features that occurred across various contexts such as issues of equity, leadership and cultures. In their framework, contexts, goals, and knowledge and beliefs influenced the plan that was implemented and then assessed before being revised.

PROPOSING A MODEL FOR MPD

Building on Loucks-Horsley and colleagues' goal, context, knowledge, and plans, we propose a definition of a model for MPD that includes four elements: goals, contexts, theories, and structure. In this definition, the structure of an MPD intervention is what mathematics teachers experience as participants. Structure is at the center of the model; it is shaped by and un-detachable from the goals, contexts, and theories that guide the intervention. Goals specify what is to be accomplished through a particular intervention; they define what needs are being addressed. Contexts are features from the environment that surrounds the intervention. Contexts shape the conceptualization of the intervention and help explain why an MPD is set up in a particular way to address a particular need. Theories are the larger assumptions about teaching and learning that guide all aspects of the MPD.

The continuation of this paper briefly presents what particular features of MPD should be included in each of the elements proposed to define a model for MPD. These features are based on previous findings from the emerging MPD research literature.

DEFINING ELEMENTS OF A MODEL FOR MPD

Goals

In her review of existing interventions, Sowder (2007) organized MPD around the goals to develop: (1) a shared vision for mathematics teaching and learning; (b) a sound understanding of mathematics for the level taught; (3) an understanding of how students learn mathematics; (4) deep pedagogical content knowledge; (5) understanding the role of equity in school mathematics; and (6) sense of self as a mathematics teacher. These goals reflect some of the progress made in different areas of research within mathematics education. For example, research has begun to associate mathematics-related goals of MPD to student learning. Concepts such as mathematical knowledge for teaching (Ball, Hill, & Bass, 2005) and its connections to student achievement (Hill, Rowan & Ball, 2005) increased the need to assure teachers have appropriate and teacher-specific content knowledge of mathematics. Similarly, research showing that when teachers attend to student reasoning there are gains in student achievement made research results from studies about student development of understanding in areas such as word problems (Carpenter et all, 1999), rational numbers (Lamon, 1999), geometry (Battista, 2007), and proofs (Harel, 2006), to mention just a few, essential for teachers. We propose that these six goals be included in the definition goals for an MPD model. However, all these goals represent perceived needs of teachers. Thus, we also propose that in examining the goals of an MPD intervention, researchers and developers also take into account other possible goals emerging from needs of the designers, administrators, policymakers, or others.

Contexts

The importance of context cannot be underestimated and is highlighted in most attempts to summarize what is known about MPD. In a recent discussion about design issues regarding the study of impacts of professional development, Wayne et all (2008) noted two important aspects of the school environment that shaped decisions about professional development: curricular context (adopted curricula at the schools) and ambient context (other professional development opportunities that co-exists with the intervention under focus). Beside considering curricular and ambient contexts in our definition of contexts, other features included under context of an MPD are descriptive information about participants and providers (background demographics, involvement of stakeholders other than teachers, teaching assignments, etc.), teacher engagement in decision-making processes related to the intervention, compulsory versus voluntary participation, and role of accountability. Participation of stakeholders other than teachers and teacher engagement in decision-making processes are two contextual features discussed not only within the general professional development literature, but also within the literature about effective school organizations and school change.

Theories

Sowder (2007) concluded that "however professional development is designed, it will be ineffective unless it is grounded in sound theory of learning" (p. 171). In their review, Borasi and Fonzi (2002) also noted the importance of theories on "how people learn best" to the design of PD. Wayne et al (2008) considered two important learning theories for designing professional development: the theories that guide what providers do when interacting with teachers (theory of teacher change) and theories about K-12 instruction espoused by designers and providers of the intervention (theory of instruction). Whereas the former is connected to theories about adult development and learning, the latter is connected to theories about children learning. When an overarching learning theory guides the MPD, coherence between these two theories is to be expected. It is reasonable to hypothesize that the more congruency between these two sets of theories, the more effective an MPD is likely to be. Theory of teacher change and theory of instruction are both included in our definition of MPD models.

Structure

Structure is probably the aspect of the MPD model that has gained most attention from researchers and developers. How to design the content and format of learning experiences for teachers? How many hours should teachers meet? Where? When? To do what? For example, consider number of contact hours as an important aspect of MPD. Yoon et al. (2007) found that interventions with contact hours ranging from 30 to 100 hours showed positive and significant effects on student learning, while interventions with fewer hours (ranging from 5 to 14 hours) had no effects. Garet et all (2001) examined both the number of contact hours and the span of time of MPD interventions. They found that both dimensions were important for the impact of MPD interventions and had independent effects on teachers' self-reported outcomes such as improved practice. Garet and colleagues identified additional features of high-quality MPD such as active learning, coherence, collective participation, and content focus. Kennedy (1998) also found that among MPD programs that examined their impact on student learning, programs with a stronger content focus had a bigger impact, despite differences in organizational features. Her work highlighted the importance of attending to the content of the intervention beyond recommendations on how the intervention should be organized (format). Thus, the structure of an MPD intervention needs to include both content and format. Content may consist of the mathematics topics covered, a focus on student learning of particular topics, or a focus on mathematics curriculum. Format describes how opportunities for learning are organized, and presented. It includes number of contact hours, span, location, type of contact (in person, distance learning, mixed), the activities carried out and the artifacts used.

SUMMARY

Examining the current literature about MPD, we noted that there is no consistent use of language or framework for describing MPD initiatives. We also noted that studies have focused on discrete features of professional development when trying to examine what makes an MPD effective or successful. We believe there is a need to clarify language and frameworks in the emerging field of MPD, and in this theoretical paper we proposed that researchers and developers of MPD should take into consideration the notion that to conceptualize an MPD initiative one should attend to models. Further, we proposed that such models are composed of four elements (goals, theories, contexts, and structure), and we suggested various features of MPD that should be included within each of these elements. We claim that an appropriate design or description of MPD should attend to all the four elements that constitute a model for MPD. More important, we propose that MPD is a multi-dimensional construct with (at least) four dimensions that interact with each other in a potentially complex way.

LOOKING FORWARD

We see our definition of MPD models as an initial definition to be discussed and revised by researchers and developers in the emerging field of MPD. After a wide debate among researchers about what should constitute a model for MPD, we suggest it would be advantageous to the field to come to an agreement about the various elements that comprise a model for MPD. We believe this focus on models can help the field move the discussion of effective MPD away from the listing of discrete features and toward a system approach to examining MPD. Not only that, the discussion of model and of elements such as goals, theories, and contexts can support a move toward a more careful examination of fundamental, but hard to measure, aspects of MPD. We believe that such changes in language and in the approach to provide and study MPD are necessary to support the growth of MPD as field of research and development within mathematics education.

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A MULTIFACETED APPROACH TO TEACHING ALGEBRA: STUDENTS' UNDERSTANDING OF VARIABLE

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The concept of a variable can be a major cause of student difficulty in learning algebra. We propose studying different aspects of variables (unknown, function and generalized number) in parallel with each other using real contexts to associate meaning with the variable involved. We call this a multifaceted approach to teaching algebra. We are investigating whether studying the application of variables using multi-representational environments before moving on to symbol manipulations can reduce student misconceptions and improve algebra performance. Here we report preliminary results of a two-year longitudinal case study of a teaching intervention in one school. Results show that experimental classes demonstrated fewer misconceptions regarding variables as compared to the comparison classes.

THEORETICAL BACKGROUND

Researchers have indicated student difficulties in the interpretation of letters in algebra (Booth, 1995; Küchemann, 1981; MacGregor & Stacey, 1997). Booth (1995) indicated that many student errors in algebra arise as a consequence of procedures that children use to solve arithmetic problems of a similar kind. Her results reinforced Küchemann's (1981) report that the majority of 13-15 year-olds find it difficult to interpret letters as generalized numbers. MacGregor and Stacey (1997) obtained data from a sample of approximately 2000 students in Grades 7-10 (ages 11-15) in 24 Australian schools. They found several misconceptions of algebraic letters by Grade 7 students: letter ignored, arbitrary numerical value, alphabetical value, abbreviated word, label, letter equals 1, and letter as a general referent (e.g., $h = h + 10$, where h represents Dane's height and Con's height) (MacGregor & Stacey, 1997, pp.10-14).

Letters are used in different contexts in secondary school, each with a slightly different meaning, and all of the distinct meanings must be coordinated if a sound conceptual understanding is to develop. These contexts include letters as specific unknowns ($y = 2x + 5$, find y if $x = 3$), letters as generalized numbers (2, 4, 6, 8, ..., $2n$), and letters as variable quantities ($A = l \times b$, where A represents the area of a rectangle with length l and breadth b). In order to create a cohesive and comprehensive picture of mathematics it is necessary that algebra pedagogy makes connections between different uses of variables, between algebra and other branches of mathematics, and between mathematics and other curriculum areas and the real world (Goos, Stillman, & Vale, 2007).

Bednarz (2001) proposed a teaching approach based on generalization to problem solving. Various meanings of letters (generalized numbers, unknown quantity,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 217-224. Thessaloniki, Greece: PME.

functions) were used in three phased teaching sequence for 24 secondary school students of low mathematical ability. Her results suggested that different representations employed by students to represent a problem helped in constructing meaning for algebraic symbolism and notations. Stacey, Chick and Kendal (2004) believe that algebra can be used to model real world situations and modelling, and proving activities can provide meaning and purpose to algebra.

Trigueros and Ursini (2001) presented a theoretical model called the 3UV model to approach the study of algebra. This model involved integrating three aspects of variable, namely unknown quantity, variable quantity, and generalized number. They proposed a two-phased algebra teaching plan in which the three uses of variable are studied separately, then introducing teaching situations in which students can integrate these three aspects. We propose studying these three aspects simultaneously using real contexts to associate meaning with the variable involved, combining patterns and linear functions and using multiple representations such as tables of values, algebraic expressions and graphs in Grade 7; algebraic manipulations of expressions and equations would be studied later in Grade 8. We call this a *multifaceted approach* to teaching algebra.

The purpose of the present study is to investigate the effect of a teaching sequence based on a multifaceted approach to the variable concept on students' conceptual understanding of variables and their general algebraic competence.

METHOD

Patterns and Algebra is one of the five content strands of the mathematics syllabus for Grades 7 and 8 used in New South Wales, where we conducted our research. In Grade 7, the first year of secondary school, students typically study algebra in two teaching blocks, each of two to three weeks' duration. They briefly study patterns and then move quickly to substitution and simplification of algebraic expressions. In Grade 8, approximately the same amount of time is devoted to algebra instruction; students learn to factorize algebraic expressions, substitute into algebraic formulas, solve linear equations, and represent relations graphically. The teaching program used in our study covers all of the syllabus content in a similar timeframe, but we have changed the order of topics. In our program, the Grade 7 students learn about algebraic, numeric and graphical representations of linear functions, making connections between variables and applying them in a variety of realistic contexts. In Grade 8, they study manipulation of variables and the solution of linear equations using real life contexts.

To give meaning to the variable and link it to the real life situations, a functional approach is used and the role of variables in linear relationships is highlighted using multiple representations such as tables, graphs and algebraic relationships. Students work on real life problems designed to trigger discussions about mathematical situations, to develop strategies to explore and solve problems, to develop appropriate language to express mathematical ideas, and to explore mathematical relationships by making generalizations and giving reasons to support their conclusions.

Phase One of the study, which we report here, took place during 2008 in four Grade 7 classes from a secondary girls' school in Sydney. The classes (called Set1, Set2, Set3, and Set4) were streamed according to their mathematical ability level. The experimental group consisted of the 49 students in Set2 (medium ability, 27 students) and Set4 (low ability, 22 students), and the comparison group consisted of 54 students in Set1 (high ability, 27 students) and Set3 (medium ability, 27 students). Note that we have adopted a conservative approach by choosing the comparison classes to be of higher general mathematical ability than the experimental classes.

Prior to the teaching intervention, a professional development workshop for teachers of the experimental classes was conducted to inform the participants of the aims and objectives of study and discuss the proposed teaching sequence. We provided teaching resources for the teachers (McMaster & Mitchelmore, 2006, 2007) which they were free to use as they saw fit in conjunction with their textbook and their own resources. The teachers planned some lessons and obtained feedback from the researchers.

The first author observed and videotaped one algebra lesson for each class every week during the algebra teaching periods. The topics taught in each group during the four weeks of the algebra blocks are shown in Table 1. All classes spent 13 or 14 lessons on these topics.

Experimental classes	*Comparison classes*
Patterns, linear sequences, generalizing patterns, input-output functions, generalized number, linear relationships, dependent-independent variables	Number patterns, finding a rule, method of finite differences, number plane, ordered pairs, evaluation of algebraic expressions, addition and subtraction of like terms, multiplication and division of algebraic expressions

Table 1: Topics studied in experimental and comparison classes

After completion of the algebra teaching blocks, a test was administered to assess students' perceptions of variables and to assess their ability to move between words, algebra and tables of values. The test was administered by the teachers, marked by the first author, and checked by another researcher. There was a total of five test items. The last one concerned pattern extension and is not presented here due to limited space; the other four items are shown in Figure 1.

Item 1: Sarah's mother gave her 2 times as many chocolates as Hannah. If Hannah has x chocolates then Sarah will have ....chocolates. When her father came home, he gave each of the girls 5 more chocolates. Describe the number of chocolates each girl has using x. Sarah has .... chocolates, Hannah has ....chocolates.

Item 2: Look at the number pattern and then answer the questions.

Input (n)	1	3	5	7	9
Output (m)	3	9	15	21	27

Describe the relationship between the first number 'n' and second number 'm'.

State the pattern rule algebraically using n and m. Explain how you got your answer.

Find m when $n = 23$.

Item 3 [adapted from Goos et al., p. 242, 2007]: There are 20 passengers for every bus. Write this relationship algebraically using x for the number of buses and y for the number of passengers.

Item 4 [adapted from MacGregor & Stacey, 1997]: Complete the table using the following rule.

Rule: Add 3 to the input number and then multiply by 5.

Input (x)	2	3	5	12	43
Output (y)					

State the relationship between x and y algebraically. Explain how you got your answer.

If y is 15, what is x? Show your working.

Figure 1: Items 1-4 of algebra posttest.

RESULTS AND DISCUSSION

The results of the algebra posttest, along with classroom observations and student work samples, were analysed to investigate students' perceptions and knowledge of variables. Table 2 records student achievement on the posttest for a number of outcomes.

Despite their lower overall mathematical ability, the experimental classes were considerably more successful than the comparison classes in writing algebraic expressions to model given problem situations—both from situations described in words and from tables of values. Particularly striking is the comparison between the lowest ability class (Set4) taught by the experimental method and the middle ability class (Set3) taught by the traditional method. This finding is probably at least in part due to the greater emphasis on modelling using algebra in the experimental classes (see Table 1).

Task	Experimental group		Comparison group	
	Set2	Set4	Set1	Set3
Write an algebraic expression from a verbal statement	81	40	70	23
Write an algebraic expression from verbal statement accompanied by a table of values	91	73	90	26
Finding the unknown variables in a linear algebraic expression	78	61	84	86
Extending a pattern	93	80	99	84

Table 2: Percentages of students giving correct responses, by class.

Evaluation of algebraic expressions was part of the algebra syllabus followed by the comparison classes, but they had no previous experience of solving a linear equation to find the independent or dependent variable. The experimental classes had studied patterns and linear expressions, but they also not familiar with an equation and its solution. The difference in content covered would explain the slight superiority of the comparison group over the experimental group in finding unknown variables.

As Table 2 shows, all classes performed very well in pattern recognition. Again the lowest ability group Set4 achieved well above expectations—their previous mean mark in mathematics was 57%.

Error analysis

The results in Table 2 are partially explained by the difference in the content of the experimental and comparison classes. To explore in greater depth the perception of variables that students in the two groups had formed, we conducted an error analysis. In particular, we attempted to identify the errors reported by MacGregor and Stacey (1997) noted earlier. The results are shown in Table 3.

The most common error was to consider x as an object. For example, for Item 1, Cathy stated that Sarah initially had xx chocolates and that, after her father gave her five more, Hannah had $2x + 5x = 7x = x^7$ and Sarah had $4x + 5x = 9x = x^9$ chocolates. Gina's answers were $xx = x^2$, $x + 5x = 6x$ and $x^2 + 5x = 6x^2$ respectively. Anita gave two sets of answers: $x = 1$, $x + 5x = 6x$, $2x + 5x = 7x$ and $x = 2$, $2x + 5x = 7x$, $4x + 5x = 9x$, respectively. As can be seen in Table 3, this error was far less common in the experimental group than the comparison group.

A number of students assigned numerical values to variables unnecessarily, giving in Item 1 the answers 2, 7 and 9 instead of $2x$, $2x + 5$ and $x + 5$, respectively. This error was uncommon, but less prevalent in the experimental group.

Error	Experimental group		Comparison group	
	Set2	Set4	Set1	Set3
Variable considered as an object	0	27	33	73
Numerical values assigned to variables unnecessarily	4	9	11	15
Letters as abbreviated words or labels	0	0	11	15
Expressions conjoined incorrectly	4	5	19	22
Incorrect exponential notation	0	9	4	30

Table 3: Percentages of students showing various errors, by class.

Several students used letters as abbreviations. For example, in Item 3, Stephanie gave separate expressions for the number of buses and the number of passengers: $20p \times 2b = 40p$ and $20p \times 5b = 100p$, using b for buses and p for passengers instead of x for the number of buses and y for the number of passengers. This error occurred about as frequently as the previous error in the comparison group, but was not found in either of the experimental classes.

Students often conjoined algebraic expressions incorrectly, for example reasoning that $2x + 5 = 7x$, $x + 5 = 6x$ or $x^2 + 5x = 6x^2$. Others used exponential notation incorrectly, for example, $x + x = x^2$ and $x + 5x = 7x = x^7$. These errors often occurred when variables were considered as objects (see above). Again, they occurred frequently in the comparison group but rarely in the experimental group.

In summary, although student achievement in the two groups showed no marked advantage of one teaching sequence over the other, there appeared to be a considerable difference in the concept of variable that students in the two groups had formed. Students in the experimental group showed far fewer of the traditional misconceptions than the comparison group, the difference being particularly noticeable in the lowest ability group.

CONCLUSIONS

The four weeks of the experimental multifaceted approach to teaching algebra appear to have resulted in students acquiring a much more viable concept of variable than results in the traditional method. We conjecture that this outcome is the result of the considerable emphasis on the meaning of a variable, and the situation of the algebra instruction in familiar, interesting, and simple (i.e., linear) contexts.

The test results confirmed that a majority of the experimental students had learnt how to model linear situations using algebra, a major focus of the content. This finding and the finding that they tended to avoid such common beginners' errors as $2x + 5 =$

$7x = x^7$ indicates that they had acquired a far sounder understanding of what a symbol like x represents than the comparison group.

Particularly encouraging are the results for the lowest-achieving group. Teachers often despair of teaching algebra to such students. The transition from computation with fixed numbers in arithmetic to algebra, where numbers are variable, is frequently seen as far too demanding for mathematically weak students. The preliminary results of the present study suggest that this conclusion may not be appropriate. Although there was still some tendency to regard variables as objects, students in the lowest-achieving class showed few conceptual errors. They were also more successful in modelling linear situations than the class "above" them that had followed a more traditional course. Perhaps algebra is not so difficult after all.

It was observed that the four participating teachers all used different teaching styles. The teachers of the experimental classes closely followed the teaching resources provided (instead of their text book), without putting a great deal of emphasis on the various different aspects of the variable concept. Apart from this change, they seemed to be keeping to their normal teaching style. All four teachers were experienced and appeared to teach effectively. As a result, we did not feel that the success of the experimental program could be assigned to a teacher effect. The crucial factor seems to have been the content rather than a particular teaching style. This finding is also most encouraging, because it suggests that simply changing the order of the algebra topics in Grades 7 and 8 may be sufficient to make a considerable change in effectiveness, without the need for extensive teacher professional development.

The main rationale for the design of the experimental program was to build up a sound concept of variable in Grade 7 that would enable students to learn algebraic manipulation in Grade 8 more successfully than is done in the current algebra program. Phase Two of the study, following the experimental and comparison students into Grade 8, is currently under way. It will be most interesting to see whether the students in the experimental group, most of whom have acquired a sound concept of variable, will indeed learn to manipulate algebraic expressions and solve equations more successfully than the comparison classes who are already showing serious misconceptions. The results will be reported next year.

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LEARNING SYSTEMS OF LINEAR EQUATIONS THROUGH MODELLING

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We report on the effectiveness of the use of models as a teaching strategy to develop university students' ideas about systems of linear equations and their solution. Students' productions throughout the modelling process were analysed, together with observation guides and tests. In particular, students' use of variable was investigated in relation to their problem solving strategies.

INTRODUCTION

We are currently involved in a three-year project whose main purpose is to develop ways of teaching linear algebra through the use of models to university students coming from different degree programs. Research shows that students' difficulties in linear algebra are widely spread and highly persistent; even when different teaching approaches are used in the classroom (Sierpinska, 2000). Students are said to have problems interpreting the solution of systems of equations both graphically and analytically. Additionally, it has been found that students have problems in dealing with systems with infinite number of solutions, with different representations, and with the concept of variable (Trigueros *et al*., 2007; Afamasaga-Fuata'I, 2006; Nickerson, 2006).

For this project, ideas coming from the models and modelling perspective (Lesh and Doerr, 2003) are being used in order to select, analyse and often transform realistic mathematical problems so that they can be used in the classroom. Mathematical activities are also being designed with the purpose of providing students with experiences that can deepen their conceptual understanding of systems of equations. In particular, relationships between different representations are being emphasised through those activities.

The purpose of this paper is to report on the effectiveness of the use of models as a teaching strategy to develop students' ideas about systems of linear equations and their solution. We are also interested in exploring the relationship between students' modelling strategies and their use and understanding of variables.

THEORETICAL IDEAS AND RESEARCH QUESTIONS

Two complementary frameworks were used in the analysis of the data reported in this paper. On the one hand, as mentioned before, the didactical approach used was based upon the models and modeling perspective, where it is considered that model-eliciting activities help students develop powerful mathematical ideas in a meaningful realistic context. On the other hand, since work with systems of equations involves algebraic knowledge, the 3UV model (3 Uses of Variable model) (Trigueros and

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 225-232. Thessaloniki, Greece: PME.

Ursini, 2003) is used to analyse students' understanding of variable through the interpretation of their work.

The modelling perspective focuses on the development of conceptual tools which are useful in decision making. Researchers working on this perspective (Lesh and English, 2005) have developed criteria to verify if selected problems have the potential to be successfully applied in the classroom to contribute to students' learning. The main idea consists in introducing realistic situations where students engage in mathematical thinking and where complex products and conceptual tools are generated to accomplish the intended goal. These products are constructed during cycles of work and reflection and can be self-evaluated by students on each cycle.

The 3UV model has been proposed as a means to analyse students' responses to algebraic problems, to compare their performance at different school levels in terms of their difficulties with this concept, and to develop activities for teaching the concept of variable (Trigueros and Ursini, 2003). The 3UV model considers the three uses of variable that appear more frequently in elementary algebra: specific unknown, general number and variables in functional relationship. For each one of these uses of variable, aspects corresponding to different levels of abstraction at which it can be handled are stressed. These requirements can be summarized as follows:

- The understanding of variable as unknown requires to: recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem (U1); interpret the symbols that appear in equation, as representing specific values (U2); substitute to the variable the value or values that make the equation a true statement (U3); determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations (U4); symbolize the unknown quantities identified in a specific situation and use them to pose equations (U5).
- The understanding of variable as a general number implies to be able to: recognize patterns, perceive rules and methods in sequences and in families of problems (G1); interpret a symbol as representing a general, indeterminate entity that can assume any value (G2); deduce general rules and general methods in sequences and families of problems (G3); manipulate (simplify, develop) the symbolic variable (G4); symbolize general statements, rules or methods (G5).
- The understanding of variables in functional relationships (related variables) implies to be able to: recognize the correspondence between related variables independently of the representation used (F1); determine the values of the dependent variable given the value of the independent one (F2); determine the values of the independent variable given the value of the dependent one (F3); recognize the joint variation of the variables involved in a relation independently of the representation used (F4); determine the interval of variation of one variable given the interval of variation of the other one (F5); symbolize a functional relationship based on the analysis of the data of a problem (F6).

An understanding of variable implies the comprehension of all these aspects and the possibility to shift flexibility between them depending on the problem to be solved.

The research questions that guided our work are:

- How does the use of models influence the development of students' understanding of systems of linear equations?
- What is the relationship between students' modelling strategies and their use of the different aspects of the concept of variable?

METHODOLOGY

Research was conducted within a course on Linear Algebra for students in Social and Administrative programs, at university level, at a small private university in Mexico. A modelling situation that had been successfully used in Linear Algebra courses for Economics and Mathematics students, and designed according to the modelling perspective, was used:

The following diagram (Figure 1) represents a street plan in the busiest first two blocks in the financial district of a city. The traffic control center has installed electronic sensors that count the amount of vehicles passing through specific points in the city. The arrows represent the direction of each street and the numbers the amount of vehicles per hour that pass through that point as accounted by the electronic sensors. At each crossing point there are roundabouts that direct traffic and allow for a continuous flow of traffic through the entire system. Cars are not allowed to park on the streets.

The traffic flow should be allowed to follow its usual course at the sensor points. However the Traffic Control Center is interesting in analysis possible traffic diversion policies. These policies are necessary when road works take place or other special traffic disruption events occur. The students are presented with the following specific questions:

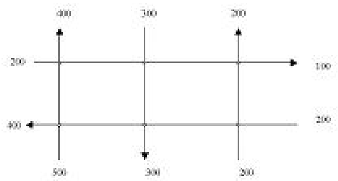

Figure 1. Street plan

1. If we were able to set minimum quantities of cars to circulate in a particular road (stretch between roundabout), what would this amount be for each stretch to maintain the normal flow of traffic in the system? Is it possible to close of one of the roads? If so, which ones can or cannot be closed?
2. The Traffic Control Centre can divert traffic by closing of some of the roads. This is done by installing diverting signs at the begging of each road. How many of such signs are needed? Is it possible to use them at the beginning of any road? Is there a particular selection of road signs that would make it easier to perform the flow evaluation?
3. Is your model well adapted to consider a restriction of no more than 200 cars each hour in a particular street? How would you modify it?

Work on the problem required of several modelling cycles. Students worked collaboratively in groups consisting of three students. Observation guides were designed to keep trace of students' work in class and of whole class discussion. Students had to hand in their advance at the end of each cycle. This information was completed with the analysis of responses of all students to two tests questions regarding solution and interpretation of linear systems of equations. All these productions were analyzed by the researchers and the teacher separately and results of the analysis were negotiated between them.

ANALYSIS OF RESULTS

All students involved in the course had experience in solving systems of linear equations, at least those where the number of equations is the same as the number of unknowns. In the case of two variables they were also able to represent the system and to find and interpret the solution both analytically and graphically. They showed however, difficulties when working with larger and more complex systems.

Students' understanding of variable and their modelling strategies. As the course progressed, important differences were found in students' modelling and problem solving strategies. We found that these differences could partly be explained by analysing students' understanding and use of the concept of variable.

At the beginning of the course, while working in groups during the first cycle of the modelling experience, it was observed that students discussed the problem and were able to identify and symbolize the variables, interpreting them in the context of the problem. Groups posed systems of linear equations to model the situation at hand, as shown in Figure 2:

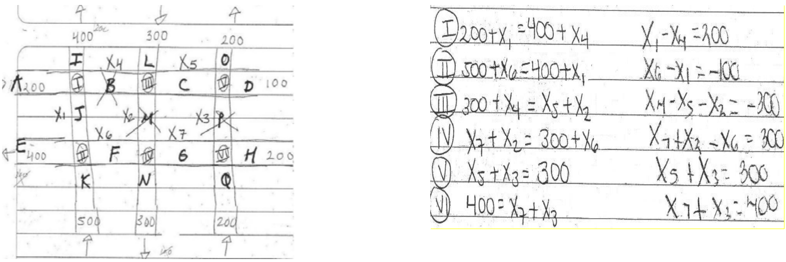

Figure 2. Representing street problem with system of equations

Students appeared to be competent in the use of variables as unknowns and as a general numbers. A closer look, however, revealed differences in students' problem solving strategies. Throughout the duration of the course we observed that, when students worked independently, not all of them were able to solve problems which were similar to the one they had solved within their small groups. The following are two examples of students who were not able to pose a system of equations for a simpler version of the street problem:

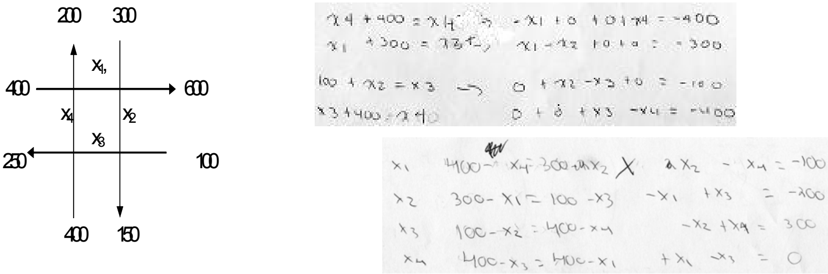

Figure 3. Special case of streets problem

We found that differences in modelling and problem solving strategies were related to students' use of variables. Some students showed persisting difficulties in identifying and interpreting unknowns, while others who were proficient in identifying and interpreting them showed difficulties with the manipulation and symbolization of variables. Other students did not show any difficulty when working with variables.

The effectiveness of the modelling approach was strongly related to students' use of variable. Modelling as a teaching strategy showed to be effective for students who, from the beginning of the course, were able to work flexibly the different uses of variable, in particular its use as an unknown and general number. Results show that these students benefited, to a great extent, from the modelling experience. They deepened their understanding of variable; generalized the methods they had learned before in order to handle more complex situations; were able to learn new methods to solve system of linear equations, and deepened their understanding of variables in functional relationships, together with their graphical representation.

Among those students who had difficulties with the uses of variable, two groups could be identified. A first group of students benefited from the modelling experience and demonstrated, by the end of the course, an improved understanding of the three uses of variable. These students enriched their solution procedures, were able to learn new ones and consolidated their notion of solution set and their abilities to use different representation registers. However, they did not develop enough flexibility to use variables in order to approach any problem that involved the use of systems of linear equations. We observed that a second group of students benefited less from the modelling experience. Throughout the course they consistently showed difficulties in identifying, interpreting, manipulating as well as symbolizing the unknowns. They learnt new algebraic methods, but apparently not solidly enough to understand why and how they worked.

The interpretation of the solution of the system proved particularly difficult for those students with a weak understanding of the concept of variable. Even when all students learned solution methods that were new to them and which were introduced through complex problems where simpler methods did not work, they often used the algorithms blindly, without knowing why or how they worked. Interpreting the solution of larger systems became an obstacle that was difficult to overcome. Students expected to find a unique solution for all systems and had difficulties in interpreting the meaning of the solution set when this condition did not hold. In this case, interpretation of the solution set requires moving from interpreting variables as unknowns only, to interpreting them as in a functional relationship, which needs the students to be able to understand joint variation of variables.

Even when students were able to find the solution set for specific problems, our results show that interpreting it in terms of the problem context represented by the system was also a difficult task. We consider that this difficulty can be explained in terms of interpretation of joint variation of variables in the solution set.

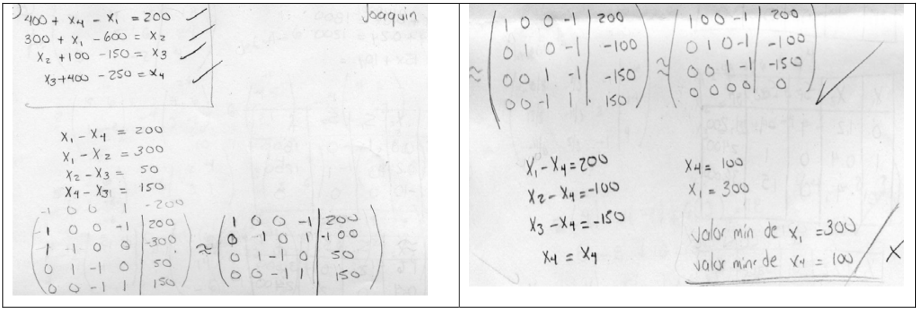

Figure 4. Interpreting the solution set

Additionally, many students were unable to relate the conditions imposed by the solution of the problem to the restrictions that came from the modelling situation proposed. As a result of this interrelation it is sometimes necessary to consider either a finite number of solutions or to bound the domain of the variables in the solution set. Frequently, students seemed to forget the context of the problem and focus only on the solution process. When they considered the context, they were not able to find a way to work simultaneously with both the solution set and the restrictions.

Modelling as a teaching strategy for the learning of systems of equations. We observed that modelling contributes to motivate students' involvement in mathematics lessons. Moreover, the use of modelling was useful in bringing into light both students' difficulties and their level of understanding of certain mathematical concepts. This allowed the teacher to identify specific needs, which would be harder to discover without the use of modelling. We observed however, as has previously been commented, that the use of a modelling approach by itself is not enough for all students to develop new mathematical concepts based on their previous knowledge. Even though some activities were developed specifically for the students to work with new concepts, their use was not enough for all students, in particular for those who had difficulties with the concept of variable. We believe that these students might have benefited from the use of activities specifically designed for them to reflect on their previous algebraic knowledge. This might have opened the possibility for the development of a more flexible use of the different aspects of variable.

Students who showed difficulties with the concept of variable were able to follow the modelling process with the help of other students in small groups and with whole class discussions. However, when they had to work independently they showed difficulties to work both with problems stated verbally and with modelling situations. They were not able to extrapolate, to similar situations, the techniques used in modelling situations developed during class. These students tended to revert to more basic knowledge and procedures when faced with new problems to model, rather than using the newly acquired mathematical concepts and modelling techniques.

It might be claimed that the modelling situation used to introduce systems of equations did not satisfy the construct-generalization principle of the models and modelling approach, which deals with the extrapolation of the model to other similar situations. However, we had already used this problem with other groups of students where it proved to satisfy it. Students in our research group, however, were weaker in terms of their previous algebraic knowledge. We observed that high-achieving students were able to apply the new concepts introduced in the course to different contexts and problems. However, less advanced students were not even able to use a model they had previously worked with when it appeared in simpler or more specific situations. We found out that this was due, in most cases, to students' inability to use the variables adequately, the even when they were specified in the problem both verbally and in a diagram.

Another interesting result found was that students had fewer difficulties with the identification, interpretation and symbolization of variables, and with finding a model to represent the situation when the wording of the problem was such that they could easily translate the sentences into equations, or deduce the system from them. Their difficulties were more apparent when situations involved the need to make use of an additional condition that had to be deduced from an analysis of the problem situation. This was the case, for example, for the traffic problem where they needed to recognize an equilibrium conditions for the roundabouts to be able to write the equations. This ability is also strongly related to the understanding of the concept of variable. We observed how the less advanced students tried setting the equations unsuccessfully, following techniques of the first class of problems, and how they could not make sense of the results or the interpretation of their solutions.

CONCLUSIONS

Using a modelling approach as a didactic strategy to teach system of equations was useful to guide the acquisition of new concepts and modelling techniques. But we can conclude that in order for all students to develop a deep understanding of this topic, modelling needs to be complement with additional activities to help student reflect on their previous knowledge, and if necessary develop it. Without these activities some students seem to revert to memorized and arithmetic based procedures which are inadequate and useless for more complicated modelling situations.

The use of modelling in class showed two advantages: a) it helps motivate students and b) is a useful diagnostic-tool to evaluate students' understanding of concepts related to the solution of systems of linear equations. But it is important to stress that the modelling approach proved to be a more powerful tool for high-achieving students, than for those not as advanced.

This work shows that students' modelling strategies are strongly related to their flexibility in moving between the different uses of variable. Those students who showed proficiency working with variables and those who developed this flexibility during the course developed richer strategies and were able to use them to model and work with different kinds of problems. They also developed their capability to interpret and analyze both real situations and models' solutions.

In spite of students having taken several algebra courses, plus two calculus courses (one and several variables), many of them showed a lack of flexibility in the use of variables. Many of them were not able to move between them, and showed also difficulties with the different aspects involved in each use. Difficulties related to the concept of variable seem to be difficult to overcome, as has been widely reported in the literature. This was also apparent in this course where variable interpretation is a vital component in modelling.

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DEVELOPING MATHEMATICAL CONTENT KNOWLEDGE: THE ABILITY TO RESPOND TO THE UNEXPECTED

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In this paper I present some findings from a four-year study into the development of content knowledge in beginning teachers using the Knowledge Quartet as a framework for reflection and discussion on the mathematical content of teaching. Findings which relate to the participants' ability to react to pupils' unexpected responses are discussed. Data from three case studies suggest that the framework helped participants to consider their unplanned actions when teaching mathematics. There was also evidence that over the course of the study the participants become more able to react helpfully to children's responses due to development in their pedagogical content knowledge.

INTRODUCTION

In the discussion group led by Neubrand, Chick and Leikin (2008) at PME in 2008, a number of different perspectives on research into the mathematical content knowledge of teachers was discussed. One of these perspectives was that of Deborah Ball and her colleagues at the University of Michigan This involves a "practice-based theory of knowledge for teaching" (Ball & Bass, 2003) based on the premise that mathematical content knowledge for teaching can only be identified in the context of actual practice. At the same conference I reported some findings from my study also based on a 'practice-based' perspective (Turner, 2008). The basis of this study was the use by beginning teachers of a framework for the identification and development of mathematical content knowledge. This Knowledge Quartet (KQ) framework had been developed by a group at the University of Cambridge (Rowland, Huckstep and Thwaites, 2004). Whereas the Michigan work focused on identifying different kinds of mathematical knowledge involved in teaching (Ball, Thames & Phelps, 2008), the work of the Cambridge group focused on the classification of *situations* in which mathematical knowledge surfaces in teaching. The KQ offers this classification of the situations through which mathematical content knowledge of teachers is 'made visible' as a framework for the analysis of teaching.

The KQ framework was developed from observation and videotaping of mathematics teaching. Analysis of this teaching produced 18 'emergent' codes (Glaser and Strauss, 1967) of situations in which mathematical content knowledge of teachers was 'made visible' e.g. concentration on procedures, making connections between concepts. These were later classified into four 'superordinate' categories based on associations between the original codes. These categories make up the four dimensions of the Knowledge Quartet; *foundation, transformation, connection* and *contingency*. For a detailed account of the development of the KQ framework see Rowland (2008).

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 233-240. Thessaloniki, Greece: PME.

In this paper I focus on the development of teachers' content knowledge as 'made visible' through the lens of one of these categories or dimensions, that of situations in which teachers act contingently. The *contingency* dimension of the quartet may be considered to be about the ability to react to unplanned situations or to 'think on one's feet'. There are three codes which emerged from the empirical research subsumed under this category; deviation from agenda; responding to children's ideas and use of unplanned opportunities. Most mathematics lessons are planned before the act of teaching takes place and teachers bring their curriculum knowledge, subject matter knowledge (SMK) and pedagogical content knowledge (PCK) (Shulman, 1986) to the planning of a *text* for the lesson (Shulman, 1987). Teachers predict how pupils will respond to their planned teaching based on *knowledge of content and students* (Ball, Thames & Phelps, 2008) as well as on their previous experience of teaching, and amend their *text* accordingly. However, not all pupil responses can be predicted. Etienne Wenger (1998) saw the way in which teachers are able to act contingently within the context of planned teaching as central to effective pedagogy.

> Pedagogical debates traditionally focus on such choices as authority versus freedom, instruction versus discovery, individual versus collaborative learning, or lecturing versus hands-on experience. But the real issue underlying all these debates is the interaction of the planned with the emergent. Teaching must be opportunistic because it cannot control its own effects. (Wenger, 1998, p. 267)

A teacher's ability to react appropriately to unplanned-for responses depends, at least in part, on their bank of SMK and PCK. Bishop (2001, pp. 95-96) offered an example of a teaching incident which illustrates the role of teacher's content knowledge in reacting to pupils' responses. In this example 9- 10-year-olds were asked to give a fraction between ½ and ¾. A response given was $^2/_3$, "because 2 is between the 1 and the 3, and on the bottom the 3 lies between the 2 and the 4". The way in which a teacher might react to this response would depend on their SMK – were they aware of Farey sequences and mediants, and on their PCK – did they know how to disprove the generalisation inherent in the pupil's justification. Mathematical content knowledge, both SMK and PCK, plays a part in determining the way in which teachers react to unexpected pupil responses. The *contingency* dimension of the Knowledge Quartet therefore offers a lens through which to identify teachers' mathematical content knowledge.

THE STUDY

The aim of this study was to investigate the way in which beginning teachers' mathematical content knowledge for teaching might be developed through focused reflection using the KQ framework. Throughout the study this framework was used as a tool for analysis, evaluation and development of the teachers' mathematics content knowledge. The study began with 12 student teachers from the 2004/5 cohort of primary (5-11 years) postgraduate pre-service teacher education course at the University of Cambridge reducing, as anticipated, to 4 in the fourth and last year of the study. Data came from observation and analysis of teaching using the KQ as well as from post-lesson reflective interviews, group and individual interviews and participant written accounts. Transcripts of interviews and written reflective accounts were all systematically coded using the qualitative data analysis software NVivo. A grounded theory approach (Glaser and Strauss, 1967) was used which led to the emergence of a hierarchical organisation of codes into a number of themes. For a more detailed account of the research methodology see Turner (2008).

Case studies were built from the KQ analysis of observed teaching as well as from analysis of the data coded using NVivo. The analysis of observed teaching, using the 18 codes and four dimensions of the KQ, provided a 'spine' for presenting findings in relation to the development of participants' mathematical content knowledge. Data from the NVivo coding of interview and written data support, supplement and enrich these findings. Six themes in the development of the participants' mathematics teaching emerged from the NVivo coding. These were*, beliefs, confidence, subject knowledge, experience, reflection* and *working with others*. Discussion of findings about participants' mathematical content knowledge, including that revealed through their ability to act contingently, drew mainly on data from the NVivo themes of *subject knowledge* and *confidence*. Data from the theme of *beliefs* formed the basis for an analysis of participants' conceptions about mathematics teaching. Data from the themes of *experience, reflection* and *working with others* gave insight into the influences on developments in participants' mathematical content knowledge and into influences on changes in their conceptions of mathematics teaching.

FINDINGS IN RELATION TO THE ABILITY TO ACT CONTINGENTLY

Knowledge of errors and what they suggest about children's understanding of mathematical ideas is part of a teacher's PCK. The way teachers respond contingently to mathematical errors therefore gives some insight into their PCK. At the beginning of their teaching careers the participants did not always make good use of opportunities for teaching offered by children's errors. For example, during Amy's lesson observed during her training year in 2004/5, the class of 4-5 year old children were asked to write some 'teen' numbers. Amy focused on correcting children's reversals of digits but did not address their errors which involved writing the numerals in the wrong order e.g. '01' for ten and '21' for twelve. However, when reflecting on this lesson using the KQ framework Amy acknowledged that the ordering of the numerals was a more significant error and suggested that she should have used this to discuss place value rather than focusing on digit reversals.

The *contingency* dimension of the KQ framework helped Amy to think about a more useful way of responding to the children's errors. Such reflection may be described as reflection *on action* (Shön, 1983). The following year, Amy demonstrated the ability to respond helpfully to children's errors *in action.* At the conclusion of a lesson on counting some children were having difficulty counting the number of times Amy hit a chime bar, and continued counting after the last chime. Amy responded to this difficulty by asking the children to close their eyes, count the number of chimes in their heads and only give the answer once she had finished. Amy's knowledge of the cardinal principle enabled her to 'think on her feet' and suggested this effective strategy.

Kate responded helpfully to children's errors during a lesson observed in 2005/6. The children were asked to show an even number using cards showing the digits 0-9. When children reversed the order of digits on their cards i.e. '801' for '108', Kate used place value cards showing '100' and '8', to demonstrate how these numbers are written in the contracted form.[1] During the post lesson interview Kate confirmed that her use of the place value cards had not been planned, "I didn't really plan to talk about that but that was a bit of a last minute thing".

In a lesson observed later in 2005/6, Kate made good use of a child's error. Kate displayed a measuring cylinder on the interactive whiteboard and asked a volunteer to indicate the level to which 100ml of liquid would come. A child pointed to the interval marked '1000 ml'. Kate asked the class how they knew this did not show 100 millilitres and the children responded that it had an extra zero and was a thousand. Kate was clearly aware that children's errors can be used to advantage when teaching.

> I took advantage of Lily confusing 100 with 1000 on the interactive measuring cylinder to discuss place value. (*Kate, reflective account of observed lesson, 2005/6*)

Kate's use of the *contingency* dimension of the KQ helped her to think about how she responded to children's errors. She became increasingly confident in using children's errors to inform her teaching and appeared to relish such opportunities.

> When estimating how many cubes long a book was Harriet-Mae said "eighty" and then corrected herself to say "eighteen". I used this as an example to question the children about which of these was a sensible estimate and we discussed why 80 was not. (*Kate, reflective account, 2006/7*)

Some of the instances in which Kate made use of children's errors involved the effective use of resources, e.g. her use of place value cards to demonstrate how numbers are written in the condensed form. The use of appropriate resources in order to address unexpected difficulties is another aspect of acting contingently which was found to have developed over the course of the study, particularly in the case of Kate. An instance in which Kate made use of a resource i.e.100 grids[2], that she had not planned for was recorded in a reflective account of her mathematics teaching and demonstrates her concern with the flexible and appropriate use of resources.

> When we were suggesting different ways to count, the children wanted to count in 100s. Someone said 0, 100, 1000. I used 100 grids to represent units of 100 and then counted in 100s to 900. I asked the children if they knew another word for 10 hundred and someone

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[1] Place value cards consist of sets of cards showing the digits 1-9, 'tens numbers' 10-90 and 'hundreds numbers' 100-900. Cards are placed on top of one another to show the contracted form e.g. [100] and [8] → [10[8].

[2] 10 x 10 grids with the numbers 1-100 arranged in rows 1-10, 11-20, 21-30 etc.

> said '110' so I had to demonstrate the difference using 10 hundred grids compared with 1 hundred grid and 10 cubes. (*Kate, reflective account, 2006/7*)

Jess was also aware of the need to be able to use resources contingently. Under the heading of 'Contingency', Jess recounted an incident in which she had been unable to use a particular resource to demonstrate why a child's response was incorrect.

> We were looking at lines of symmetry on the interactive whiteboard. The children were shown a variety of shapes and had to identify where the lines were and how many lines. When a child identified a line which didn't exist, I found it hard to prove they were wrong without actually folding paper. (*Jess, reflective account, 2006/7*)

Jess felt able to demonstrate that this answer was incorrect by using a different resource. However, more secure pedagogical content knowledge might have enabled her to make use of the interactive whiteboard.

Kate demonstrated secure PCK through her contingent use of a resource during a lesson observed in 2007/8. The lesson had been planned by another teacher as a Powerpoint presentation and Kate had not had the opportunity to make any amendments before teaching. Kate acted contingently when a slide was displayed showing 23 + 12 as (20 +3) + (3 + 2). This modelled a 10-10 strategy which she thought inappropriate. Kate 'deviated from the agenda' and made a new slide to model the N10 strategy i.e. 23 + 10 + 2. For a discussion of the 10-10 and N10 strategies see Beishuizen (2001). Kate's knowledge of the two methods enabled her to make an 'informed' choice about which to use. Kate further demonstrated her ability to use unplanned resources by producing a 100 grid and modelling the procedure for addition of ten by moving down one row.

Discussion of children's unexpected methods for solving problems was another form of contingent action observed during the study. Participants became more likely to carry out such discussion in their teaching over the course of the study. In the lesson I observed in her training year, Jess revised how to interpret pie charts with her class of 9-11year olds. Jess displayed a pie chart showing preferred flavours of ice cream and asked a question which involved calculating ¼ of 32. The class had already found that 3/8 of 32 = 12 and 1/8 of 32 = 4. One child explained that he had calculated ¼ of 32 by adding 12 and 4 and dividing by two. There were rich opportunities for discussion of equivalent fractions in this but Jess simply responded "that works well". Our discussion in the post-lesson interview suggested that Jess did not see an opportunity for discussing equivalent fractions because she did not understand the child's method.

In the lesson observed early in 2006/7, Jess appeared more willing to explore a child's calculation method. Jess asked the children to record their methods for solving '20 ÷ 2' on individual whiteboards. Most children drew pictorial representations modelling the partative method that Jess had previously demonstrated. One child however, had simply written '20 ÷ 2 = 10'. When Jess asked how he had arrived at the answer he explained that he knew "ten add ten is twenty" so had "put ten on one side and ten on the other". Jess said "knowing ten add ten is twenty to find twenty divided by two is like using the opposite". Jess understood and related his method to division as the inverse of multiplication. She might have gone on to explore how this could be used to solve other division problems, especially since later in the lesson some children worked on using multiplication facts to solve division problems.

Over the course of the study, the participants became more likely to discuss children's methods of calculation. They were also more likely to ask for and accept children's ideas as starting points for their teaching. Amy recognised that she had missed an opportunity to work from children's ideas during the lesson observed in early 2005/6.

> The children in my 'treasure counting' group had some good ideas for how we could count all the coins more quickly. It would have been good to try out the children's ideas, despite asking for their suggestions I went ahead with what I had planned to teach them (*Amy, reflective account, 2005/6*).

Amy wrote this comment under the heading of 'Contingency', indicating that this dimension of the KQ had helped her to focus on how her teaching might encompass children's ideas. Comments in Jess' reflective accounts under the heading of 'Contingency' suggest that the KQ framework encouraged her to think about exploring children's thinking in order to make her teaching more meaningful.

> They often catch me out when discussing subtractions – "Why can't you do 4 – 8?" "You can, it's a minus number!" I have started to get these children to explain in more detail what they have said so I understand where they are coming from and also so some of the other children start to realise some of these things too. (*Jess, reflective account, 2006/7*)

During the study the participants became more willing to discuss children's methods and to explore their ideas. This was underpinned by a growing confidence that their mathematical content knowledge would enable them to understand the children's thinking and develop this in their teaching. An extension of this willingness to explore children's methods and ideas was a growing confidence to allow children to investigate mathematical ideas for themselves. In a lesson on measurement that I observed early in 2006/7, Amy gave the children a selection of objects and containers which they could use in order to investigate ideas about capacity. Amy observed what the children were doing and asked questions, made suggestions or gave them further resources to support their learning.

> Callum and Joshua filled bigger boxes with small toys and found they couldn't count that number. I don't think it mattered too much though. They were enjoying the practical experience of filling a container, they were practicing judging when a container is 'full' and they saw how they could fit more small toys in a box and less big toys.(*Amy, reflective account of observed lesson, 2006/7*)

Kate also become increasingly confident about letting children take greater ownership of their mathematics.

> I was really pleased – my upper group have finally started to work through a problem systematically on their own initiative. I didn't mention it this week as I had not really thought of that approach and they did it anyway! So we shared their systematic approach as a class. (*Kate, reflective account, 2006/7*)

In this instance, Kate had not intended that the children should investigate the problem in their own way. However, she was clearly happy that they did so and felt confident to discuss their strategies with the class.

Observations of the participants' teaching, and their reflective accounts showed that they became more able to respond contingently during their mathematics teaching. There was also convincing evidence from reflective accounts that the participants saw the ability to act contingently as a factor in effective teaching and that they believed they had developed in this respect over the course of the project.

> I am more experienced, so I am aware of children's common misconceptions, and can therefore adapt in response contingently, or plan for these. Generally I think there is more contingent teaching going on and I am more confident to be flexible. I can respond quickly to a child by setting up an activity I know will extend from what they are doing. (*Amy, group interview, 2006/7*)

Kate also thought that she had become more responsive to how children reacted to her teaching.

> Quite a lot of the things that I remember talking about arose out of what the children did. One of the children who came up to write something on the board got something wrong and if he hadn't I possibly wouldn't have made that a focus. (*Kate, interview, 2007/8*)

By the end of her second year of teaching Jess felt that responding contingently in her mathematics teaching was something she was able to do automatically.

> I just think about contingency as a question that I hadn't thought of, that's just what you automatically do in anything, like thinking on your feet. (*Jess, reflective account, 2006/7*)

During her third year of teaching Jess mentored two student teachers working in her class. She clearly perceived the ability to act contingently as an aspect of their teaching on which she should focus and for which she was able to offer support.

> When it comes to thinking on their feet they could do this quite well. They often asked the child to explain what they meant a second time, which gave them more thinking time and they used me to answer difficult questions. (*Jess, interview, 2007/8*)

Conclusion

Analysis of teaching and of reflection on teaching, through the lens of the *contingency* dimension of the KQ, revealed aspects of mathematical content knowledge of the participants. The participants' use of the *contingency* dimension of the KQ framework focused their thinking on the way in which they responded to unplanned-for events in their mathematics teaching. There was convincing evidence that the participants recognised the ability to act contingently during mathematics lessons as a factor in effective teaching. Over the four years of the study the participants became more able to respond helpfully to children's errors and make better 'unplanned-for' use of resources. They became more proficient at understanding, discussing and basing their teaching on children's methods and ideas. Participants' also began to adopt a more enquiry-based approach to their mathematics teaching which was more likely to require them to act contingently. These developments in participants' ability to act contingently were underpinned by development in their mathematical content knowledge. In directing participants' reflection towards their contingent actions, the KQ played a role in the development of this aspect of mathematical content knowledge.

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INVESTIGATING SPATIAL REPRESENTATIONS IN EARLY CHILDHOOD

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This study explores the development of spatial representations in pre-school children. A sample of 30 children from 4.5 to 5.5 year–old were selected from different kindergarten classes and examined individually in originally designed spatial tasks. The children were invited to observe, one by one, two dimensional Lego configurations and retain their characteristics in order to reconstruct them from memory. The analyses of the children's reconstructions demonstrate a continuous improvement of their spatial thinking and provide interesting information about the spatial characteristics they retain mentally when they attempt to copy a spatial situation.

INTRODUCTION

The development of spatial thinking attracts the interest of many early childhood mathematics programs. The spatial oriented activities are considered, not only as an important source of conceptual development, but also as a necessary support for the improvement of the children's mathematical thinking in general (Diezmann & Watters, 2000). The significance but also the particularity of the spatial sense is demonstrated by its identification as a special dimension of intelligence (spatial intelligence), which defines an ability to perceive the spatial world and mentally represent it accurately.

Based on these considerations, the current study attempts to examine the development of spatial sense in early childhood, by analyzing children's performances in tasks that involve arrangements of spatial elements. The research dealing with this issue has generally a psychological orientation (Case & Okamoto, 1996; Siegler, 1998; Newcombe & Huttenlocher, 2000), while our study attempts to approach the ways young children process the spatial information focusing mainly to its mathematical characteristics.

The data reported here derive from a wider research concerning the development of spatial sense in early childhood which consists of two parts. The first part studies this development and some of its evidence will be presented in this paper, while the second part focused on experimentation with spatial activities for the improvement of spatial representations in early childhood (Ikonomou & Tzekaki, 2005).

THEORETICAL FRAMEWORK

The spatial representations are mental construction related to spatial information: objects, shapes, orientation, location and spatial properties, relations and

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 241-248. Thessaloniki, Greece: PME.

transformations (Owens, 2002). This mental construction, as a process and as a result, depicts the world (objects and facts) in the individual's mind facilitating its functionality. Most of the children's interaction with the natural and social environment requires the use of appropriate spatial representations necessary for handling and facing spatial situations (Fuys, & Liebov, 1992). .

In the relative literature, there are several approaches concerning the development of spatial sense and spatial representations. Following Piaget's legacy with its dominant topological primary thesis, several psychological researches investigated the development of young children's conceptions of space in different ways (using children's drawings, figures or constructions) suggesting that the different aspects of spatial knowledge are developed with age. More specifically, some research supports that the children before 6 years are unable to successfully coordinate spatial information related to two different reference systems (Case & Okamoto, 1996).

However, other researchers, like Siegler (1998), disagree with this uniformity and argue that, depending on the situation, children adopt more than one spatial approaches. The research of Newcombe & Huttenlocher (2000) reports on many aspects of the spatial development in early childhood investigating a variety of spatial situations like: position in space related to reference systems, classification of spatial information, memorization of spatial information, etc. According to these studies 5 to 6 year old children are capable of perceiving and handling many aspects of two-dimensional spatial situations. This early development of spatial skills is also confirmed by following research (reported in Clements, 2004; Kersh et als., 2008).

Based on these finding, our study attempts to examine the development of spatial sense and spatial representations of 5 to 6 year old children. More specifically, we investigated whether this development is related to the mathematical or geometrical characteristics of the assigned spatial tasks, an approach that could help us highlight existing differences in the abilities and performances. The findings of this research confirmed our initial hypotheses and formed the basis for the design of a teaching experiment that aimed at improving the development of spatial sense in early childhood (Ikonomou & Tzekaki, 2005).

RESEARCH SETTING AND METHOD

A sample of 30 children from 4.5 to 6.5 years old were selected from different kindergarten classes in the area of Thessaloniki. The children belonged in two different age groups (4.5 to 5.5 and 5.5 to 6.5) having 15 pupils in each one. Individual interviews with them were conducted in the middle of the school year.

During the interview sessions, pre-schoolers were called to reconstruct Lego configurations. These spatial situations consisted of single, double or triple Lego bricks laid in different positions on a base plate of 10X15 cm that were given to the children by the researcher. The children looking at the prototype either, copied the given configuration on their own base plate, or perceived and then mentally represented the configuration to reconstruct it from memory. In the second case, the children's constructions and their success or failure in the different spatial characteristics provided evidence about the spatial information each child could locate and retain in memory. In this paper we will present only the part of the results that concerns these age groups and refers to the case "reconstruct from memory".

For this research twenty original tasks with Lego bricks arrangements were designed. The selection of this common toy material facilitated children's work, while at the same time Lego bricks and base plates could be combined in ways that presented different spatial characteristics.

In fact, the research tasks involved five different variables (fig. 1) that describes a spatial situation with bricks arranged on a Lego base plate: (1) number of pieces, (2) shape of bricks, (3) direction of bricks (horizontal/vertical), (4) relative positions of bricks (distance and alignment) and position of bricks specified by (4) origin, (5) orientation (up/down – right/left).

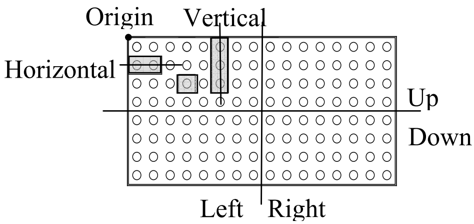

Fig. 1. Variables of a configuration with 3 Lego bricks on a base plate

The variety of these spatial characteristics enables the design of tasks with escalating difficulty and, despite their simple structure- they allow the demonstration of many aspects related to children's development of spatial sense. The involvement of preschoolers in this kind of reproductions raises interesting questions: How do children analyze and reproduce these situations? Which spatial characteristics do the children retain and reproduce in their constructions? How each characteristic is combined with the others in the children's mind? Can these data provide evidence about the development of children's spatial representations?

For example, there are some tasks of this study that involve only few variables (position of a single piece, close to the edges or along the sides of the base plate). Do they demand less analysis and thus produce simple holistic representations that most of the children can mentally retain?

On the other hand, do the spatial arrangements that involve more characteristics like direction and relative positions (distances) of the bricks (see fig.2, tasks LB5, LB6, LB7) or even more complicated situations (see fig.2, tasks LB8, LB16, LB18,LB19) demand more advanced mental representations that children gradually develop? We believe that this kind of detailed analysis is necessary in understanding how the children process spatial information. Moreover, it is essential for the design of appropriate spatial activities that aim at improving spatial abilities in early childhood.

In this paper only seven tasks will be presented because they provide important evidence concerning the significant differences between the age groups and the spatial characteristics of the tasks. These tasks are summarized in figure 2.

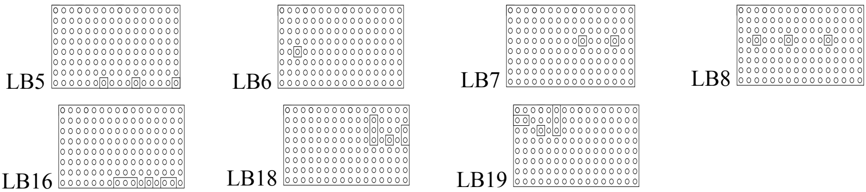

Fig. 2. Seven out of twenty research tasks

RESULTS

Tasks and age groups

The differences in the spatial abilities between the two age groups appear clearly in the following table (table 1) that presents the means of the successful performances at the seven tasks. A performance was declared as successful when the child could accurately reproduce bricks placement in his/her own base plate in respect to the five variables previously presented.

Tasks	Group 1	Group 2
LB5	13.33%	46.67%
LB6	26.67%	60.00%
LB7	20.00%	46.67%
LB8	13.33%	40.00%
LB16	20.00%	66.67%
LB18	0.00%	20.0%
LB19	0.00%	33.3%3

Table 1. Means of the children’s successful performances at the 7 tasks

As it is apparent in this table, the ratio of the younger children (group 1) who managed to reproduce the bricks’ configuration from memory, taking into consideration position, orientation and relative distances between pieces are low. This ratio becomes zero when it comes to more complicated tasks. On the contrary, older children (group 2) show higher scores performing the same tasks. Lower performances in this group appear only in the two final tasks where all spatial variables are involved. However it is important to underline that a percentage of 20%

to 33% of these students manage to reproduce these quite complicated tasks respecting all the spatial characteristics. In this table, we can also easily examine the differences between children's performances related to the specificity of each task. An important reduction of the students' achievement (that zeroes younger children's results) appears in the management of more complicated tasks, in which, not only many pieces and spatial characteristics are involved but also all these elements must be combined (position, orientation, directions and relative distances). This result suggests that children are probably able to handle each of these characteristics separately but encounter increasing difficulties when they attempt to associate them in more complex arrangements. An overview of the differences between performances of the two groups at seven tasks is better illustrated in the following diagram (fig. 3).

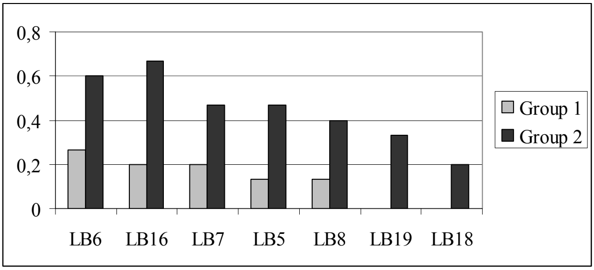

Fig.3. Differences between groups and tasks

This overall presentation provides only general information about the spatial characteristics that children can retain in their mind and recall for their reconstruction. An attempt to analyze students' performances, with respect to the different variables of each task can contribute in a better understanding of the way they perceive these characteristics.

Tasks and variables

In fact, the systematic analysis of what children achieved during the reconstructions of the proposed Lego arrangements from memory provides interesting evidence about the spatial characteristics they approach and the way they handle them, thus reveling to us which representation they form in their minds. The quantity of data deriving from these analyses is quite extended; therefore we will present and comment only a part of it.

The first comments concern the spatial characteristics that children approach successfully. Most children (80-90%) identify successfully and retain the *number* and *shape* of the bricks in all tasks. Similarly, almost the same percentage of them places the bricks on the *right* or on the *left* side of the base plate easily. This ease to place the bricks correctly does not mean that preschoolers are able to recognize right from left side in a spatial situation, but that they follow their own right/left orientation. Results are also similar to the ones regarding the *horizontal/vertical* direction in the tasks with relevant arrangement. However, the complexity of task LB19 leads younger children to lower achievements concerning horizontal/vertical direction.

Opposite to these variables, the children present a variety of performances dealing with *up/down* orientation and identification of an *origin*. The following table summarized (table 2) these performances.

Tasks	Group 1		Group 2	
	Up/down	*Origin*	*Up/down*	*Origin*
LB5	46.67%	80.00%	66.67%	80.00%
LB6	40.00 %	46.67%	86.67%	66.77%
LB7	86.67%	20.00%	80.00%	66.67%
LB8	93.33%	26.67%	73.33%	66.67%
LB16	53.33%	33.33%	86.67%	86.67%
LB18	60.00%	33.33%	60.00%	60.00%
LB19	73.33%	73.33%	73.33%	60.00%

Table 2. Means of successful performances concerning up/down orientation and identification of an origin

The children encounter certain difficulties dealing with '*up/down*' orientation, particularly at tasks where the decision about whether a brick is located up or down is not the result of a simple placement but demands a more conscious identification of what is up or down in the base plate. Some children of group 1 demonstrate apparent difficulties to identify what is up and what is down in tasks LB5 and LB6, where bricks are located almost in the middle of the base plate.

Low scores of younger children in the task LB16 (where the bricks are laid along the bottom side of the plate) are related to a situation we are referring at as 'mirror placement': as children were sit opposite to the researcher, they tend to consider the position of the bricks in the original model as being in the upper part of the researcher's base plate so they put their bricks in the same "symmetrical" place, in the upper part of their own base plate. The lower success in LB18 for both groups could also be owed to the complexity of the spatial arrangement where a simple displacement can easily change up to down.

Interesting evidence concerning the tasks present the way children identify *an origin.* The data support that the decision about the origin, namely the edge of the base plate in relation to where one or more bricks should be placed is not simple. For the task LB5 in which a single brick is placed at one of the plate's edge, this decision becomes simple and successful even for the younger children. The same explanation applies for older children in the task LB16. The choice of an origin becomes less obvious for younger children in the task LB6 and the tasks LB7 and LB8, because of the position of the bricks near the middle of the base plate. This location requires an organized and purposeful choice of an edge to be the origin in relation to where other bricks will be placed. Younger children fail to do this choice whilst older students can accomplish. This is also confirmed by their scores in tasks LB 18 and LB 19.

Finally, as far as the relative positions are concerned results provide interesting evidence regarding mostly the distance between pieces. These results are summarized in the following table (in the tasks with three bricks, two distance are presented).

Task	Group 1		Group 2	
	1st distance	2nd distance	1st distance	2nd distance
LB5	73.3%	73.30%	80.00%	93.30%
LB7	60.00%	-	80.00%	-
LB8	86.70%	46.70%	73.30%	60.00%
LB16	66.70%	53.30%	86.70%	86.70%
LB18	46.70%	20.00%	73.30%	40.00%
LB19	13.30%	20.00%	40.00%	40.00%

Table 4. Means of successful performances concerning relative distances

High performances of both age groups in task LB5 show that children are able to retain relative distances between bricks. Older children maintain this ability in the tasks LB7, LB8 and LB16, in which younger children also produce an interesting performance. Low scores in the rest of tasks could be attributed to the involvement of more variables (orientation and origins) that make these spatial situations (LB18, LB19) rather complicated. Children do not face difficulties in locating the distances between pieces but they are not able to combine them with other spatial information.

DISCUSSION

This study confirms in general that the preschoolers (5 – 6 years old) demonstrate a continuous improvement of their spatial thinking related to spatial characteristics they are able to deal with reproducing spatial situations. Despite important individual differences, the ways in which they represent and handle this kind of spatial tasks reveals their easiness and ability to locate and retain information concerning number and shape of pieces, as well as their relative rectilinear placement. They also easily follow the 'right/left' of this placement as far as it corresponds to their own left/right orientation. However they present a variety of performances when they start dealing with other spatial characteristics like origin, up/down orientation and relative positions (alignment and distances) and even more when they attempt to combine this information. Their spatial abilities keep developing with age and thus most of 6 years old children improve their performance in more complicated spatial situations.

An overall consideration suggests that children from early years dispose an important spatial background related to two dimensional situations, which they develop gradually. Preschoolers develop their ability to analyze spatial situations progressively, attain one by one their spatial characteristics and later learn how to combine them. This development is not determined and it is obviously closely related to young children's involvement with appropriate spatial oriented activities.

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PROMOTING TRANSITION FROM PARTICIPATORY TO ANTICIPATORY STAGE: CHAD'S CASE OF MULTIPLICATIVE MIXED-UNIT COORDINATION (MMUC)

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We analyzed 2 episodes from a teaching experiment with 14 fifth-graders in the USA to examine how a student with learning disability in mathematics (SLD) advances from a participatory to an anticipatory stage of a multiplicative conception. Data reveal Chad's struggle to: differentiate units of One (1's) from composite units (CU), convert 1's->CU, and add the CU. Instruction that orients Chad's reflection on relationships between his activities and their effects via task re-negotiation promotes his reorganization of a structural similarity between problem situations, which he independently calls upon a week later. We discuss this significant transition to what Steffe and Cobb (1988) called the Explicitly Nested Number Sequence (ENS) and its implications for explaining construction of the more robust, desired stage.

INTRODUCTION

This study examined learning (cognitive) processes as a student progresses from the provisional, prompt-dependent participatory stage to the more conceptually robust, and mathematically empowering, anticipatory stage. Tzur and Simon (2004) distinguished these two stages, and a few recent studies (Tzur, 2007a) have examined their implications. Yet, this foundational transition has never been studied. This study is important particularly because it addressed this transition as a SLD constructed the anticipatory stage of a key multiplicative conception that underlies understanding of the distributive property. Deprived of such a conception at a stage that allows independently, consistently, and properly solving multiplicative problem situations is likely to hinder students' development of ratio, proportional, and algebraic reasoning (Hiebert & Behr, 1991). Our central thesis in this paper is threefold: (a) transition to the anticipatory stage may require two different types of reflection (see below), (b) a researcher (or teacher) can intentionally and successfully orient such reflections via re-negotiation of tasks and establishment of norms and practices, and (c) SLDs' difficulties may be rooted in conceptual or procedural short-term memory limitations.

CONCEPTUAL FRAMEWORK

Cobb et al.'s (cf. Cobb & Yackel, 1996) emergent perspective, which coordinates, reflexively, social and cognitive approaches, served as the overarching framework. Due to space limits, we present constructs pertaining to this study, while referring to articulated accounts found elsewhere. The three main constructs used as social lenses are social norms, socio-mathematical norms, and mathematical practices. In a community of learners, such norms and practices govern, but do not determine,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 249-256. Thessaloniki, Greece: PME.

expectations and behaviors of its members. They are constantly negotiated among all members, to establish common goals and taken-as-shared meanings. As such, norms and practices constitute the milieu in which learners' cognitive processes take place.

As cognitive lenses, we used Tzur and Simon's (2004) recent distinction of two stages—participatory and anticipatory—in the construction of new mathematical conceptions via the mechanism of reflection on activity-effect relationship (Simon et al., 2004). This mechanism commences via a learner's assimilation of a task into her available schemes, which furnish an anticipated global goal, and possibly sub-goals, toward which she carries out mental activity sequences with/on objects. Two types of comparison the mind continually executes constitute the reflective process by which learning—transformation in one's anticipation—occurs. Type-1 involves comparison between the learner's goal and the actual effect(s) of her activity; Type-2 involves comparison across her re-presented activity-effect records. Common to both stages is the invariant anticipation of why particular effects follow an activity; they differ by the learner's access to that anticipation. At the participatory stage a learner can only access an evolving anticipation if she is prompted for the activity that generates its effect(s). At the anticipatory stage, she can spontaneously access the anticipated relationship to consistently and properly employ it for solving similar tasks.

Steffe and Cobb's (1988) content-specific notion of Explicitly Nested Number Sequence (ENS) guided the study. It refers to a number scheme a child constructs in which abstract composite units (CU), integrated from units of one (1's), are clearly (for the learner) embedded within and linked to one another. For example, 6 is nested within 7, is nested within 8, etc., and such CU can themselves be embedded within larger CU (e.g., 2 CU of 4 and 7 CU of 4 are embedded within 9 CU of 4).

METHODOLOGY

This study was conducted within the larger context of the NSF-funded, *Nurturing Multiplicative Reasoning in Students with Learning Disabilities* project[1]. The first author (Ron) led a teaching experiment to promote and study how seven pairs of 4th and 5th grade SLDs construct multiplicative conceptions. This paper focuses on the case of Chad who worked with a partner (Tara, both are pseudonyms).

Data were extracted from two consecutive episodes, as the children were engaged in variations of a turn-taking, 'platform' game we call, "Please Go and Bring for Me …" (PGBM). Its base-format involves sending a student to a box with Unifix Cubes to produce, and then bring back, a tower of a given number of cubes. After 2-9 'trips' for bringing same-size towers, the 'bringer' is asked how many towers (i.e., CU) she brought, how many cubes are in each tower (i.e., unit rate, UR), and how many cubes (1's) there are in all (*N* towers of *M* cubes are symbolized as NT_M; $5T_4$ = 5 towers of 4 cubes each). Two variations of PGBM, central to this study, promoted construction

[1] *This research is supported by the US National Science Foundation under grant DRL 0822296. The opinions expressed do not necessarily reflect the views of the Foundation.*

of two key conceptions: (a) multiplicative same unit coordination (MSUC) and (b) multiplicative mixed unit coordination (MMUC). The former might ask: "I placed $6T_4$ under the cover. How many towers will we have if we brought additional $3T_4$?" The latter, which Steffe and Cobb (1988) considered as a strong indication of the ENS, might ask: "I covered $6T_4$ here and 12 cubes there. If you put all 12 cubes into T_4 and moved them under the other cover, how many towers will you have in all?"

Analysis proceeded in 3 iterations. First, after a teaching episode the team discussed major events and recorded their significance. These notes served for planning next episode tasks and for retrospective analysis. Second, individual team members read the transcripts of each episode and highlighted segments with critical events (changes in anticipation, teaching moves). This included hypotheses of why children solved or failed to solve a problem the way they did. Third, highlighted segments were discussed, line-by-line, to identify powerful, explanatory segments. All segments were organized in a story line (presented next) that conveys the transition process.

ANALYSIS

We begin this section with two 'snap-shots' of Chad's MMUC—before (participatory) and after (anticipatory) we first detected the transition—which allow characterizing this advance. We proceed with articulating the constructive process involved via presenting 4 critical excerpts and then analyzing all four at once.

Participatory Stage

On 2008-12-02, Ron first asked Chad and Tara to solve a 'What If' task: "Imagine you'd bring a T_3, and another T_3, etc., until you brought $7T_3$; Figure out the total number of cubes (1's)." Both properly figured out "21." As they watched, Ron then produced and covered the $7T_3$, and asked: "If $2T_3$ are added, how many T_3 would there be? Chad immediately and spontaneously used counting-on to add six 1's to those previously found: "21; 22-23-24; 25-26-27." Ron asked if he meant $27T_3$ and Chad nodded yes, which indicated he could not yet anticipate the part of the activity sequence of differentiating and selecting 1's from CU. Thus, in spite of Ron's strong encouragement to keep with the socio-math norm of answering in terms of towers, Chad shifted to the unit he operated on via multiplicative double counting of 1's. This unit shift took place although the task did not yet mix units! That is, Chad was asked, plainly, about 7 towers and 2 more towers; he replied by operating on 1's.

To re-test our pre-episode hypothesis that Chad had constructed unit differentiation and selection at the participatory stage, Ron prompted for a Type-1 reflection: "So you got 27… Did you count towers or cubes? I'm not saying you did either one; I'm asking what did you count?" Chad immediately responded, "27 cubes." When asked: "Can you figure out how many towers?" Chad immediately corrected himself by counting-on CU: "Um … started with 7, then you added 2 more, so 7; 8-9." This indicated that re-negotiation of the goal (towers) prompted Chad's reflection on the unit he operated on. It led to his adjustment of the unit and proper completion of the activity. Indeed, Chad could not independently access this differentiation.

Anticipatory Stage

Ron began the next week's episode (2008-12-09) with a mixed-unit task. Chad and Tara produced $5T_4$; then Ron covered the towers, counted 8 individual cubes, and placed them under another cover. Chad immediately exclaimed with excitement, "Ooh, I know!" Ron continued introducing the task by communicating the socio-mathematical norm to guide the children's solution: "Whenever you say anything with numbers you always count [say] 2-something, [as in] 2 towers, 2 cubes, 7 cubes in a tower, the number always … refers to something". Chad first said there would be 6 towers, explaining: "Um, so there were 8 [cubes] so then I went, 'Four!' [shows a hand with 4 fingers] that was one full tower; then another one [shows 4 fingers of the other hand], so that's 2 towers, and then 4 [towers] in there [covered] so equal 2 more." Ron asked how many towers were covered and Chad said, "1, 2, 3, 4, 5, oh … no." Ron suggested that Chad take his time and try again. Chad said immediately: "3, 4, then 5 … Umm … 7."

A week after Chad needed a prompt for a task that merely involved addition of the same unit (towers), he did not need a prompt for the more demanding, MMUC task. Rather, in the context of PGBM he clearly anticipated and could spontaneously carry out the entire activity sequence. His finger motions showed he intended to (a) figure out how many towers could be produced from the invisible eight 1's, and (b) add that number to the number of invisible towers (CU). That is, Chad assimilated the MMUC task into his evolving, global goal of adding towers (not cubes), which triggered what for him became an intentional, proper activity for the situation: differentiate 1's (cubes) from CU (towers), select the former as the object for the integration operation and complete it (compose 8 cubes into 2 towers), select the CU (towers) of the other collection as the unit for addition, and complete the addition of CU. Symbolically, Chad's invariant (anticipated) mental activity sequence to accomplish that goal can be expressed as: DIFF(1's<->CU); SEL(1's); COMP(1's->CU); SEL(CU<->1's or Unit Rate); ADD(CU); SAY(total number of CU and reason for it).

Holding the number of towers (5) in his short-term memory while converting the 8 cubes into towers still proved difficult for Chad, so he miscalculated the sum (2+4=6). We stress that in the previous week Chad needed a prompt to access a necessary part of his activity sequence – a conceptual limitation. A week later, Chad anticipated and intentionally carried out the entire activity sequence, merely lacking access to a given fact in the problem (a procedural limitation). The conceptual difference lies in Chad's global goal—to add CU (towers)—that seemed to regulate his entire activity. We make this inference for two reasons. First, Chad never gave any sign of confusing between cubes and towers. Second, his note ("1, 2, 3, 4, 5, oh … no") indicated he independently both realized his error and adjusted his entire process to operate on the correct number of towers (5+2=7).

Chad's intentional solution to the MMUC task, including his independent self-correction, indicated an initial anticipatory stage of his ENS. That is, for him 5 CU of 4 and 2 more CU of 4 were nested (embedded) within 7 CU of 4 even though the given eight 1's required him to impose organization into 2 CU. We ask: How can Chad's advance to the anticipatory stage of MMUC be explained?

Promoting Construction of the Anticipatory Stage

To explain Chad's progress, we returned to the previous week's episode. His solution to the initial task of adding $2T_3$ to the (hidden) $7T_3$ indicated that he already anticipated the differentiation of 1's from CU. He mistakenly selected and operated on the 1's, but did not add different units like other children would (e.g., Tara solved it as 21 cubes+$2T_3$=24 towers). Ron's task re-negotiation and Chad's self-correction emphasized for Chad the need to keep track of the unit. Excerpts 1-4 below present how these events underlied Chad's transition to an anticipatory stage of MMUC.

Excerpt 1 (2008-12-02)

R (Refers to Tara's $9T_5$ and Chad's $7T_5$): How many more towers does Tara have?
C (No finger motion; excitedly raises his hand while R still asks the question;): I got it!!
R (After Tara responded the difference is 9 towers, asks Chad): What did you get?
C (Immediately, no hesitation): Two towers.
R: Why two?
C: Um, … She got 9 then I got 7 so … so I added 2 more and got 9, so I have 2 more.
[Later, Chad explained that to equalize their numbers, Tara could put away two towers.]

Excerpt 2 (2008-12-02, a bit later)

R (Places the equalized groups of 7 towers of each child under cover, then asks): So we had 14 towers, then added [remaining] two. How many towers of 5 will we have?
C (Immediately, no hesitation): Sixteen (16)!!

Excerpt 3 (2008-12-02, first MMUC task)

R: [We had] six towers, four cubes in each; 24 altogether; that's what you found. (Covers the 6 towers, then counts 12 individual cubes and places them in front of Chad and Tara while saying): Five and Five are ten, two more is 12. (He repeats the question about how many towers of 4 are under the cover and both children clearly recalled, "Six.") So if I put these [12 cubes] in towers of four and moved them all together [under the cover], how many towers will I have?
C and T: (Think quietly for about 15 seconds. None indicates having an answer.)
R (Lifts and holds the cover between the six towers and the 12 individual cubes): $6T_4$; 12 [cubes] here. If you mushed all into towers, how many towers will we have?
C (First, he recounts the revealed array and finds there are 6 towers; then he rearranges the 12 cubes into 3 groups of 4 each, finally says with confidence): Nine [towers].
R: How many?
C: Nine. 'Cause it's, like, you have 4 here (builds $1T_4$ out of cubes); so that [array] was 6 [towers]; So 7; then another [$1T_4$] – that's 8 [towers]; then the last one is 9.

Excerpt 4 (2008-12-02, second MMUC task)

R (Produces and covers $7T_5$; places 10 more cubes under another cover; re-asks about number of towers and cubes under each cover to re-establish given quantities): So $7T_5$; we have 10 cubes here. If we put those 10 into T_5 and mushed them altogether, how many towers will we have?

[An exchange between R and T about her solution, "21 towers" took place here.]

R: Chad, what do you have?

C: (Thinks for about 40 seconds.)

R: You had your hand up earlier; do you have an answer how many towers?

C: (Thinks for about 20 more seconds, silently attempting to count on his fingers.)

R: [Chad] Do you wanna' help [Tara] understand what you're trying to do?

C: (Continues thinking and attempting to count fingers for 75 seconds. Clearly, he cannot hold in his mind the entire activity needed while both addends are covered.)

R: This is a very challenging question.

C: (After 7 seconds, saying as a matter of fact, not defeated): I don't know what to say.

R: Okay, so you have something in mind but you don't know what to say. Let's see if we can move one step like we did earlier (lifts cover from $7T_5$).

C: (Immediately raises his hand): Nine [towers].

R: [A little later] You seemed to have struggled before I put this [cover] up and then it seemed to click. What were you struggling with?

C: Umm … that [array] was 7 [towers of 5]. I knew that. Than, um, 10, and I was trying to add it up; I thought it was something, but I still didn't get it (shakes head, "No").

R: So not seeing [the $7T_5$] kind of confused you?

C (nods "Yes"): Yeah, umm, I had trouble because I tried to look and see how many towers, because I forgot how many towers are there.

[A little later, when Ron asked Chad to explain his solution, "Nine" to Tara]

C: There was 7 [towers of 5]; so there were 10 [cubes], right (quickly flashes open/close his both hands to indicate 10 fingers), so I broke it in half by 5 to 10, and there were 10, so 5 (lays left hand out with 5 open fingers); 10 (lays the right hand out); so, there are TWO (moves his two hands to indicate the 2 CU he just counted). So we got 5 (flashes out left hand quickly) and 10 (flashes out right hand quickly). So 5 right here (left hand laid on table); and 5 right here (right hand laid on table); so that is 2 towers so that [array] is 1-2-3-4-5-6-7 (counting towers), and we put 2 more <u>like we did last time</u>, and got 8, 9.

R: What do you mean "like last time?"

C: Like last time, Umm, we had 7 [towers total].

R: When you counted those things into towers of [4]?

C: (Nods head "Yes" affirmatively!)

Excerpts 1 and 2 demonstrate that, in a MSUC task, Chad knew exactly which unit (CU) he was supposed to operate on, for figuring out the difference between $9T_5$ and $7T_5$, or for adding $7T_5+7T_5+2T_5 = 16T_5$. That is, he developed anticipation of the proper differentiation, selection, and operation on CU—in the absence of those units and with numbers larger than 10! Being engaged in the two tasks seemed to promote Chad's coordination of those 3 initial portions into an internalized activity sequence.

Bringing forth this activity sequence led, in Excerpt 3, to the first time Chad manifested carrying out the entire activity sequence also for the anticipatory MMUC (explained above). Indeed, he could complete the sequence only after being prompted by Ron's lifting of the cover. At that point, Chad could accomplish his global goal ("find how many towers") via reflecting on and interiorizing records of properly differentiating, selecting, and operating on CU to accomplish the sub-goal of converting 1's into CU, which allowed adding the CU (converted + visible). We infer that his anticipatory access to the 3 initial portions of his activity sequence allowed his amazing (for SLD) tenacity during the next task—he knew how to start!

Facing the second MMUC task ($7T_5$, 10 cubes), Excerpt 4 provides a glimpse into the transition from a participatory to an anticipatory stage. Chad's explanation of his difficulty was corroborated with his motions as he struggled to keep in his mind the number of converted ($2T_5$) and given ($7T_5$) CU. The mental work needed to convert 10 cubes into $2T_5$ seemed to block his access to the known number of invisible CU. Once visible, he could again complete the activity sequence. At this point came a critical intervention, in the form of a typical social norm in our community—asking Chad to explain his solution to Tara. The data clearly showed how he had to repeat his open-hand motion to re-present, for himself as well as for Tara, the required conversion of 1's into CU. We contend that each repetition became an input in his type-2 reflection (comparing records of experience). In turn, those type-2 reflections on relationships between his activity sequence and their effect (knowing how many CU) led to another type-2 reflection Chad spontaneously imposed on the situation, namely, comparing across his solutions to both MMUC tasks ("like we did last time"). Ron's request to articulate what he meant seemed to have brought forth Chad's 'cementing' of the new invariant anticipation—the tasks seemed the same for him in terms of the entire activity sequence he carried out, including the crucial sub-goal of converting 1's into CU with which he struggled. It was this spontaneous contribution that led us to predict Chad would solve a similar task in the next week's episode in the way presented above (within the **Anticipatory Stage** sub-section).

DISCUSSION

This study provided an empirically grounded, first-of-its-kind manifestation of the transition to the anticipatory stage—the desired, robust form of a mathematical conception. Via a study of a child with learning disabilities in mathematics, we articulated this transition in the fundamental but difficult-to-grasp MMUC and the related Explicitly Nested Number Sequence (Steffe and Cobb, 1988). Our analysis showed how two reflection types, identified by Simon et al. (2004), engendered the desired advance and the corresponding, predictable 'transfer' of the child's novel understanding. It also highlighted an essential distinction between conceptually- and procedurally- rooted difficulties that may hamper SLDs' learning and outcomes. Much more work is needed before researchers can fully explain the diverse aspects of such a transition. Yet, this study provides an 'existence proof' and a first glimpse into its working. By adding grounded analysis to studies about the stage distinction in the domains of fractions (Tzur, 2004) and teaching (Tzur, 2007b), the study supports the vigour of the reflection on activity-effect relationship conceptual framework.

This study also demonstrated the possibility of using the emergent perspective (Cobb and Yackel, 1996) to explain links between conceptual teaching and learning of mathematics. The researcher-teacher's purposeful use of norms and practices (e.g., naming units a child operates on/with) for orienting particular types of reflection (e.g., between Chad's global goal of finding the number of towers and his actual effect/solution given as number of 1's), indicated how teaching could specifically, though indirectly, promote intended conceptual advancements in students. This contribution seems consistent with Cortina's (2006) approach in his recent study on instructional design and students' progress in the domain of ratio.

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MENTAL MODELS AND THE DEVELOPMENT OF GEOMETRIC PROOF COMPETENCY

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We propose a cognitive model of geometric proof competency based on ideas from psychological research on deductive reasoning and the idea of figural concepts (Fischbein, 1993). We describe two possible mechanisms contributing to the development of geometric proof competency: deductive reasoning ability and perceptual chunks. Data from a longitudinal study indicates the particular influence of perceptual chunks.

INTRODUCTION

Learning to construct mathematical proofs is regarded an important objective of mathematics instruction in many countries (e.g., NCTM, 2000). Moreover, in most countries, proofs are presented within elementary Euclidian Geometry. Accordingly, this contribution considers geometric proof competency from a cognitive point of view, aiming at a description of its development in lower secondary school. Following Weinert (2001), competencies are defined as cognitive abilities and skills, which individuals have or which can be learned by them. These abilities and skills enable them to solve particular problems and encompass the motivational, volitional, and social readiness and capacity to utilize the solutions successfully and responsibly in variable situations.

In this sense, our interest is not to explicitly describe the students' *understanding* of mathematical proof, but to model their competency to construct geometry proofs. We hold a normative point of view on mathematical proof which is influenced by curricula for secondary school regarding the common basis of accepted mathematical statements and also includes criteria for the acceptance of students' proofs as mathematically correct.

MODELING GEOMETRIC PROOF COMPETENCY

There are approaches to model students' competence to construct geometry proofs from a cognitive perspective. Duval (1991) distinguishes two levels of proof construction: The organisation of propositions into a deductive step and the arrangement of deductive steps into a proof. This distinction is reflected in a three-level model of geometric proof competence proposed and empirically validated by (Heinze, Reiss, & Rudolph, 2005): Level I encompasses geometric calculation problems which do not require any general deductive arguments, level II corresponds

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 257-264. Thessaloniki, Greece: PME.

to solving problems which require the construction of single-step proofs, and level III includes proofs requiring more than one step.

Duval's (1991) ideas build upon a view of deductive reasoning as manipulation of propositions with regard to certain formal rules. There are psychological theories that describe deductive reasoning in this sense. Alternative theories exist that have a similar or better empirical basis (see Bara, Bucciarelli, & Lombardo, 2001, for an overview). In this contribution, we do not address the question which theory is more adequate to describe deductive reasoning in general, but try to apply an alternative theory – the Mental Model Theory (Johnson-Laird, Byrne, & Schaeken, 1992) – to geometry proofs. Our aim is to provide a framework for describing the development of geometric proof competency as well as several of students' problems that can be observed.

The Mental Model Theory and geometric proof competency

According to the Mental Models Theory (MMT), deductive reasoning is achieved by building mental models of the premises, i.e. mental representations – which usually have a figural component – that represent a set of "states of affairs" in the real world, which are compatible with the premises. Depending on the kind of reasoning, these models are conceptualized in different ways. Bara, Bucciarelli, and Lombardo (2001) propose a conceptualization that applies to a wide variety of reasoning types. For geometry proof, we will propose a different conceptualization.

Deductive reasoning according to the MMT consists of three phases (Johnson-Laird, Byrne, & Schaeken, 1992). Given a set of premises, the individual builds a mental model representing a set of states of affairs that are compatible with the premises (e.g. a representation of a set of examples). Secondly, the individual draws a provisional conclusion based on the model. This conclusion reflects properties of the model which are not directly accessible from the premises. Finally, the provisional conclusion is validated by checking it against alternative models of the premises. If no model can be found that is incompatible with the provisional conclusion, it is accepted and integrated into the model, otherwise it is rejected (and the process starts again).

With regard to mathematical reasoning we propose an extension of the MMT by a fourth phase. The specific nature of deductive arguments in mathematics is to ensure that indeed no alternative models can exist which render the conclusion false. This is usually done by searching for some (accepted) theorem that eliminates the possibility of such incompatible alternative models. The central idea of MMT is that deductive arguments are found using inductive processes (based on a single model) which are afterwards checked against alternative models and only then secured by searching for an appropriate theorem.

For single-step proof problems, one may get the impression that the first three phases of the reasoning process are not necessary. If the student performs only the one step necessary to prove the hypothesis (i.e. recalls the necessary theorem), this is true.

Otherwise the problem-solving process of searching a chain of deductive arguments starts in the same way as for multi-step proofs – with possible dead ends. Nevertheless, this lower complexity is indeed one of the specifics of single-step proof problems.

Mental models for geometry proof

To make predictions based on MMT, it is necessary to conceptualize the term "Mental Model" for the context of geometry proof. Mental models can be formed by encoding verbal information or as a result of perception (Johnson-Laird, Byrne, & Schaeken, 1992). In most geometry proof problems, the premises are usually given as verbal or symbolic information together with a geometric figure[1]. Thus the mental model has to integrate two kinds of information: visual information and conceptual information reflecting the premises given explicitly. Fischbein (1993) discusses mental representations of this kind under the term "figural concepts". Figural concepts are understood as mental representations having figural and conceptual character, but which are not reducible to one the two aspects without loss of information. Fischbein refers to figures intrinsically controlled by conceptual constraints. For example, the mental visual image of an isosceles triangle contains less information than the figural concept that has the information of equal sides attached to it. It is possible to operate mentally with the figural representation, keeping track of the conceptual constraints. According to Fischbein (1993), an ideal case would be a figural representation totally controlled by the conceptual constraints. Nevertheless, he points out that the figural representation is subject to Gestalt forces that can support or impede problem solving processes.

When considering multi-step proof problems, reasoning is a problem-solving process which is directed towards the hypothesis given in the problem formulation. This additional information is relevant during the reasoning process because it values certain possible intermediate conclusions from a given model higher than others. The conclusions that are considered "closer" to the hypothesis will be preferred. Thus the hypothesis should be considered as a part of the mental model. Here we follow Duval (1991) with the idea that the *status* of an information must be considered, distinguishing the hypothesis from other conceptual information in the model.

DEVELOPMENT OF GEOMETRIC PROOF COMPETENCY

When analysing the development of geometric proof competence, different perspectives can be taken into account. Küchemann and Hoyles (2006) for example gave evidence that individual development need not result in a monotonous increase of performance in solving the same proof problem over several years. From a perspective on the system level using appropriate tests consisting of several proof items which meet the requirements of psychometric models, we can expect an

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[1]If no figure is given, constructing a figure compatible with the premises is usually an important strategy.

increase in students' performance. Nevertheless, it is not possible to describe specific individual learning trajectories. Our approach here is to analyse development on the system level using quantitative methods.

Proof competency is a complex construct, and its development can occur in several ways. An increase can be due to a better ability to reason deductively in the sense of the four phases described above, or due to an increased ability to generate, manipulate and check mental models. Moreover, an increase in geometric content knowledge may facilitate an improvement in both fields. We will give two examples.

Firstly, an increase in deductive reasoning ability may occur from an improved capability of coordinating the problem-solving process of searching and selecting possible arguments. This includes, for example, the ability to generate and keep track of intermediate hypotheses in the mental model that are proven in a subordinate proof problem, before the intermediate hypothesis is integrated into the model. In order to facilitate this, the student must be able to deal with information of different status (Duval, 1991) within the same mental model. Moreover, general problem-solving skills and meta-cognitive competencies have an influence on this ability to coordinate the reasoning process.

On the other hand, improvement in geometric proof competency may occur by a reduction of the complexity of the proving process: Koedinger and Anderson (1990) describe how *perceptual chunks*, i.e. mental associations between certain prototypical figural configurations and mathematical concepts that apply to them can support the proving process. If a student matches a sub-figure of his/her mental model with a configuration corresponding to one of his/her individual perceptual chunks (e.g. a pair of vertical angles), the result of a corresponding proof step (i.e. the conclusion that the angles are congruent) can be integrated into the mental model immediately without generating and checking alternative models first. If the corresponding theorem (the vertical angles theorem) is also part of the chunk, the corresponding proof step does not have to be constructed within the problem-solving process. This reduction of complexity is of particular importance, if the number of proof steps to be constructed is reduced to one – making the item change the from competence level II to level III in the model of Heinze, Reiss, and Rudolph (2005) for this single student. We can assume that students learn perceptual chunks during problem solving in their geometry instruction.

We will focus on these two aspects of development, keeping in mind that different processes may be at work additionally, e.g. improved availability of problem schemata (Koedinger & Anderson, 1990).

RESULTS FROM A LONGITUDINAL STUDY

To contribute to the question how the two processes in the theoretical model described above interact, we will present data from a longitudinal study on geometric proof competency we conducted with students from grade 7, 8, and 9 (13–15 years old)[2]. The sample consisted of N=196 students that took part in all three measurements are taken into account in our analysis. All students visit the high-attaining school track "Gymnasium" in Munich (Germany). Achievement tests with 10–12 open-ended items were administered each year within one 45-minute lesson by trained assistants.

Test instruments and psychometric model

The tests were constructed with respect to the competence model proposed by Heinze, Reiss, and Rudolph (2005), using items from all three levels. We started with this simple model considering the number of deductive steps necessary for a proof, but we also took into account that the complexity of a proof item may be smaller due to the availability of perceptual chunks. This means that the allocation of items to the three levels may differ between the different grades. At this point we benefit from the fact, that our sample was recruited from German gymnasiums, i.e. the high attaining school track that is visited by about 30% of the students. We can assume that this sample is quite homogenous with regard to mathematical knowledge, so that the analysis of perceptual chunks can be expected at a certain grade. One example of an item assigned to level III in grade 7 and to level II in grade 9 is given in figure 1. It was solved by about 10% of the students in grade 7 which corresponds well to the assignment to level III. In grade 9 about a third of the students succeeded on this item. We assume that for 9th graders the sub-figure corresponding to a pair of vertical angles is part of a perceptual chunk which leads to an integration of the information about the equal angles as described above even before the proving process starts. In this case, only one deductive step remains: The new angle and the other two lie at a straight line, thus add up to 180°.

Prove $\alpha + \beta + \gamma = 180°$

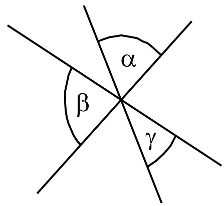

Figure 1: Item in grade 7/9

Our statistical approach to describe the development of geometric proof competency in spite of different tests used in different years is to model the empirical data using the dichotomous unidimensional Rasch Model (Rasch, 1960)[3]. Checks of item fit values for each item provided satisfying results, indicating that the model is appropriate. The use of probabilistic test theory makes it possible to model individual regressive performance on single items as reported by Küchemann and Hoyles (2006). On the other hand, probabilistic modelling makes it difficult to interpret the competency of a single student. Hence, all results reported here are on the system level. The three tests were connected by nine anchor items. Four of these anchor items were assigned to different competency levels in different grades, like the item in figure 1.

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[2] By July 2009, data from a fourth point of measurement in grade 10 will be available.

[3] The scaling was performed with the Software *ConQest* (Wu, Adams, & Wilson, 2006) using WLE estimates.

Quantitative development from grade 7 to grade 9

Since ability parameters from the three measurements are aligned on a common scale using the Rasch Model, a repeated measures ANOVA was used to analyse the development. The effect for time of measurement was significant ($F(2,390) = 35.06$, $p < 0.001$, $\eta^2 = 0.154$). This indicates that geometric proof competency changed between grade 7 and grade 9. Mean values, standards deviations, effect sizes, and data of t-tests are given in table 1, showing an increase in both periods of time.

The effect sizes are comparable to the growth of mathematics achievement within one school year in Germany (e.g., Prenzel et al., 2006). The increase is larger in the first period which can be ascribed to instruction in geometry proof at the beginning of grade 8 and the fact that the tests were based on contents from grades 7 and 8.

Splitting the sample into thirds with respect to their proof score in grade 7, the influence of prior knowledge on the development can be studied. Results of t-Tests showed significant increases in the first period for the lower and middle third and in the second period only for the middle third. The group with highest prior knowledge could not increase its performance significantly between grade 7 and 9[4]. As found in previous studies, only students from this group were able to solve a considerable number of multi-step items in grade 7 (Heinze, Reiss, & Rudolph, 2005). On the average, they also solved about 90% of the single-step items correctly. The fact that these students are not able to improve their performance indicates that the ability to find multi-step proofs does not increase for these students.

	M	SD	d	p (t-test)
grade 7	93.7	7.09	0.37	<0.001
grade 8	96.3	7.22		
grade 9	98.3	7.67	0.27	<0.001

Table 1: Quantitative results

competence level I (%)	competence level II (%)	competence level III (%)
76.6	55.8	23.4
74.7	50.1	18.5
78.7	42.9	15.8

Table 2: Qualitative results

Qualitative development from grade 7 to grade 9

The data described above shows that students are able to solve more complex proof problems in higher grades. We tried to analyse which of the two development processes derived from the model above is more plausible to cause this effect. In any case, a student in grade 9 is expected to solve more, and more complex proof problems than a student in grade 7. But if this better performance is mostly due to an individual mental reduction of complexity by the use of perceptual chunks, it can be questioned if this is a desirable increase of proof competency. From a mathematics education point of view the increase of deductive reasoning ability should also be an

[4] We emphasize that no ceiling effects occurred in the tests at all times of measurement.

outcome of proof instruction. Table 2 contains mean solution rates for the sub-tests corresponding to the three competency levels in each grade. Here items were assigned to different levels in different grades, if a reduction of complexity by perceptual chunking processes could be expected from our knowledge of very common geometry concepts in the corresponding grades. Here, hardly any improvement in the performance on those multi-step proofs, which cannot be reduced mentally in their complexity, can be observed.

These data support the assumption that the observed quantitative increase in geometric proof competency is primarily *not* due to improvement in deductive reasoning capability, but rather to an improvement in building mental models efficiently, reducing the complexity of the proof problems.

DISCUSSION

The presented data indicate that improvement in geometric proof competency may not be primarily caused by better reasoning skills, but rather by a better quality of geometry knowledge, particularly by the availability of perceptual chunks. Our interpretation of these findings supports Duval's (1991) idea that the number of proof steps is an important predictor of the difficulty of a proof problem. Nevertheless, we assume that additional effects may lead to a reduction of complexity on an individual level, rendering multi-step proof problems into single-step problems due to their cognitive representation.

On the theoretical side, an explanation is provided by the proposed adaptation of the MMT, if the idea of figural concepts is used as a basis to conceptualize mental models for geometry proof. This makes it possible to integrate the work done by Koedinger and Anderson (1990) about perceptual chunks into a common framework with a psychological theory of deductive reasoning. Syntactic theories based on the manipulation of verbal or symbolic propositions would not be adequate for this integration. Particularly regarding geometry proof, the model proposed here provides additional explanative power regarding some effects regularly observed in research on proof: Inductive proof schemes (Harel & Sowder, 1998), for example, can be understood as omitting the validation phase three (and phase four) in the reasoning model. Circular arguments can occur because of an invalid integration of the hypothesis into the model without considering its special status.

Implications of MMT in connection with mathematical proof in general have been discussed by Stylianides and Stylianides (2007). With respect to geometry proof, the idea of figural concepts as mental models deserves special attention and provides new ideas to foster geometric proof competency by supporting students in generating, manipulating and checking mental models efficiently. In view of our results, the ability to deal with the specific complexity of multi-step proofs deserves special attention. One first approach that can be interpreted as providing a technique to foster the construction of mental models is the reading-and-coloring strategy studied by Cheng and Lin (2005).

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IN SEARCH OF CHARACTERISTICS OF SUCCESSFUL SOLUTION STRATEGIES WHEN DEALING WITH INEQUALITIES

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Analysis of how students solve inequalities provides a fruitful arena to explore most of the aspects that can reveal students' understanding of algebra. Research has shown that the solution of inequalities is difficult for students. Several authors consider that the understanding of variables, variation and function is required for success. In this study, two students' strategies while solving complex rational inequalities (a routine task and a non-routine problem involving parameters) are analyzed. The comparison of students′ work shows that the above mentioned capabilities are necessary for students' success, but not sufficient. Structure sense and the capability to make sense of solution actions and to organize them are essential. These capabilities are fundamental in students' mathematical development.

INTRODUCTION

Several researchers have focused on the difficulties students face when working with complex algebraic tasks, in particular, inequalities (Bazzini, 2007; Tsamir, Tirosh and Tiano, 2004; Bazzini and Tsamir, 2001; Boero, Bazzini and Garruti, 2001; Boero and Bazzini, 2004; Barbosa, 2003; Tsamir and Bazzini, 2001). Students' tendency to make invalid connections between quadratic and rational inequalities and the tendency to extrapolate their knowledge about equations to solve inequalities have been found (e.g. Gallo and Battú, 1997). The importance to be able to work with functions, variation and variables when facing inequalities has been stressed, considering that they involve these concepts in situations which need complex treatment (Sokolowski, 2000; Assude, 2000). However, Sackur (2004) indicates that the functional approach does not automatically lead to errorless solutions.

Being inequality complex algebraic problems it has been stressed that students need to have developed as well "structure sense" (Hoch and Dreyfus, 2004, 2006; Linchevski and Livneh, 1999), "manipulation algebra" and "axiomatic algebra" (Tall, 2004). Trigueros and Ursini (2008) have pointed to some aspects they consider fundamental for approaching complex algebraic problems. They stress the necessity to be able: to work with variable using it in a flexible way; to differentiate and integrate its different uses throughout the process of solution; to analyze problems from the start instead of starting to do blind manipulation. Focusing on students'

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 265-272. Thessaloniki, Greece: PME.

structure sense, these researchers consider, as Hoch and Dreyfus (2004, 2006) do, the necessity to be able to see algebraic expressions as global entities; to uncover some structure in the expressions; to use definitions and the restrictions they impose on the solution of a problem and to foresee their implications; to give priority to the different actions that have to be performed in the solution of a problem. Additionally, they stress the necessity to keep in mind the different issues involved in a problem and to keep track of the role of each of them in the solution.

The present study aims to contribute in this same direction by exploring students' understanding of algebra through analysing algebraic capabilities they need when solving algebraic rational inequalities. In particular, we focus on the following questions: How important is for students to have a good understanding of variable, variation, function and structure sense in order to work successfully with inequalities? What characterizes successful solution strategies when dealing with inequalities?

THEORETICAL FRAMEWORK

To approach these research questions we use the 3UV Model as theoretical framework (Trigueros and Ursini, 2003) which has been proved to be useful to analyse different aspects of students' work with algebra (Trigueros and Ursini, 2008). The 3UV Model considers the three uses of variable that appear more frequently in elementary algebra: specific unknown, general number and variables in functional relationship. For each one of these uses of variable, aspects corresponding to different levels of abstraction at which it can be handled are stressed. Presented in a schematic way these requirements are:

- The understanding of *variable as unknown* requires to: recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem (U1); interpret the symbols that appear in equation, as representing specific values (U2); substitute to the variable the value or values that make the equation a true statement(U3); determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations (U4); symbolize the unknown quantities identified in a specific situation and use them to pose equations (U5).

- The understanding of *variable as a general number* implies to be able to: recognize patterns, perceive rules and methods in sequences and in families of problems (G1); interpret a symbol as representing a general, indeterminate entity that can assume any value (G2); deduce general rules and general methods in sequences and families of problems (G3); manipulate (simplify, develop) the symbolic variable (G4); symbolize general statements, rules or methods (G5).

- The understanding of *variables in functional relationships* (related variables) implies to be able to: recognize the correspondence between related variables independently of the representation used (F1); determine the values of the dependent variable given the value of the independent one (F2); determine the values of the independent variable given the value of the dependent one (F3); recognize the joint variation of the variables involved in a relation independently of the representation used (F4); determine the interval of variation of one variable given the interval of variation of the other one (F5); symbolize a functional relationship based on the analysis of the data of a problem (F6).

An understanding of variable implies the comprehension of all these aspects and the possibility to shift between them depending on the problem to be solved. Moreover, we believe that to solve complex algebraic problems it is necessary as well to have structure sense (Hoch and Dreyfus, 2004) and the capability to keep in mind the different issues involved in a problem and to keep track of the role of each of them in the solution (Trigueros and Ursini, 2008).

METHODOLOGY

Two tasks were designed to provide information about students' structural sense and their capability to flexibly use variables in the solution of rational inequalities:

Task 1: Solve the following inequality: $(-2x^2+5x-6)/(x-1) \geq 4$

Task 2: Find the value of k in order for the solution set of the inequality

$(3x - 13)/(x+k) \leq 2$ to be $x \in [-3, 8]$ (Hint: find specifically the negative values for k).

The two tasks were analyzed in terms of the 3UV model to highlight the uses of variable and the corresponding aspects involved. Special attention was paid to the aspects revealing structure sense, and the necessity to keep in mind the different issues involved in the task and to keep track of the role of each of them in the solution. This initial analysis guided the analysis of the responses given to the tasks by 42 university students attending the first semester of an introductory mathematics' course for actuarial science, economy, engineering and economy. The two tasks were included in a test they had to answer at the end of a pre-calculus course.

We differentiated first of all correct from incorrect answers. After that, the only one successful student and one unsuccessful student, representative of students using arithmetic approach to determine the unknown in Task 2, were selected for interview. The analysis of responses and the interview (based on students' written responses) aimed to clarify the factors contributing to students' success when working independently. It was not our aim to use the interview to investigate if students could solve the problems with help, but to follow their reasoning when they work alone.

RESULTS

Even though the 42 students were enrolled in a pre-calculus course, where some algebraic topics were reviewed and solution of inequalities was emphasized, and they had taken several algebra courses at high school, where usually inequalities are not discussed, only one of them, Daniel, was able to solve correctly the two inequalities. No other student could solve Task 2 (rational inequality involving a parameter), although 25% of them could solve Task 1 (a routine task not involving parameters).

Analysis of the two selected students' written responses and their explanations during the interview, were compared to identify elements related to the research questions.

Daniel, the only student who successfully solved both tasks, approached Task 1 by, first of all, manipulating the inequality (G2, G4) in order to simplify it until he got: $(2x^2-x+2)/(x-1) \leq 0$. Asked to explain how he proceeded after that, he said:

"I compared the sign of the quotient to zero. This (pointing to $(2x^2-x+2)/(x-1)$) *has to be less than zero... I can not simplify it anymore ...This expression* (referring to $2x^2-x+2$) *does not have real roots, as I wrote here* (pointing to his written answer "no real roots"). *I tried to find the roots with the formula and it has no roots, so… this means that this expression* (signaling $2x^2-x+2$) *is always positive ... for any value of x"*

Analyzing his written answer we observed that taking into account the sign of the inequality he differentiated two possible cases (positive denominator and negative numerator; negative numerator and positive denominator) using so the properties of inequalities and looking at the expression as an entity. He explained:

"Since this is greater than zero (referring to the first case with $2x^2-x+2\geq0$)*...this* (pointing to x-1) *must be less than zero, strictly less because x must be different than 1"*

He solved this last inequality to get x ε $(-\infty, 1)$ as solution set (U1, U4, F5), considering the unknown as a moving variable (Ursini & Trigueros, 1997) and keeping track of the restrictions of the problem and the role of each of them in the solution. He proceeded in a similar way when solving Task 2. He described it so:

"They tell me that the solution is for x is in this interval, closed interval (pointing to [-3, 8])*... is the solution of the inequality ... I wrote this here where it says "solution"* (pointing to "solution $-3\leq x \leq 8$" in his written exam) *... the solution for x ... so I know that x is between -3 and 8 ... I need to use that information to find k ... I first simplified the inequality to get this* (points to his writing "$(x-2k-13)/(x+k) \leq 0$") …

As we have mentioned above, Daniel considered two possible cases, as he explained:

"Then I considered two cases… one of the factors has to be positive and the other negative, and that gives me these two possibilities (pointing to where he wrote "$x - 2k - 13 \geq 0$ and $x + k < 0$ or $x - 2k - 13\leq 0$ and $x + k > 0$"). *Than I used the information they give me for x, to try to find what happens to k ... that is why I drew this line and marked -3 and 8 on it* (pointing to the drawing of a straight line where he had marked -3 and 8)*… to remember it… and to use it with this other information, the one I had found* (pointing to "$x\geq 2k+13$ and $x< -k$") *for the first case ... If x is -3, the lower value, than k is less or equal to -8* (he obtained it from $-3\geq 2k+13$)*, but we have also this* (pointing to "$x< -k$") *and for x equal 8 ... so, k is less than -8."*

Asked about the second case he said:

" … I did the same in the other case, using the 8 … and found k greater than 3, so the values for k are these (pointing to where he wrote "k < -8 or k > 3")*."*

It is clear from his description that Daniel identifies x as the unknown of the inequality although he realizes that the unknown of the problem is the parameter k (U1, G2) and he can use the information given (the interval solution of the inequality) to find its values (G4, U4, F2). He considers the variation of x (F5) and the joint variation of x and k (F4). Daniel can shift flexibly between the different uses of variable. He is able to look at algebraic expressions as global entities, to use definitions and the given solution of the problem as a data and go back to look for the conditions on k that determine that solution. He is able to organize the different actions that have to be performed to solve the problem. He can switch from interpretation to notation and he makes sense of the rules. These characteristics seem to be essential to be successful in solving the problem.

Carlos, representative of a sub-group of unsuccessful students, was also able to simplify the inequality until he had $(-2x^2+x-2)/(x-1) \geq 0$ (G2, G4), but he could not continue and he started all over again. When interviewed he explained:

"First I tried to simplify it, but when I arrived here (pointing to "$(-2x^2+x-2)/(x-1) \geq 0$") *I did not know what to do, because, here, this numerator is quadratic … I tried to do something else* (pointing to his writing where he multiplied the original expression by (x-1), using so a property valid for equations but not for inequalities). *I multiplied by x minus one, and then here* (pointing to the right hand side of the given inequality) *I multiplied it by 4* (indicating the multiplication "4(x-1)")*… but again, I had this quadratic equation* (pointing to the left side of the given inequality $-2x^2 +5x-6$) *… I tried to find its solutions with the formula, but it does not have solutions and I did not know what el se I could do."*

When working with Task 2, Carlos, as other students did, substituted a negative value for k and worked with the inequality. He was happy with his procedure:

"I substituted -2 for k and simplified the inequality … I arrived to this (signaling on his writing "$x-9/x-2 \leq 0$"). *I drew a number line and marked 2 and 9 in it…then I knew that if x is greater than 9 the quotient is positive, and that it is also positive if x is less than 2 … if x is in the middle* (meaning $2<x<9$)*, it is negative, so that interval is the solution, the interval from 2 to 9…I know x has to be between 3 and* 8 (referring to the given interval)*, but so x is also between 2 and 9"*

Asked why he chose -2 for k, he said:

" … It just happened … I chose the right value for k!"

The analysis of Carlos´ responses shows that he was able to manipulate variable as general number (G2, G4), he could look at k as unknown (U1) but he could not use the data and the relation between x and k to determine its values (lack of structure sense). He got stuck in the first task because he could not interpret the quadratic equation without real solutions (U2, U4, G2) and he could not look at expressions as entities. He used procedures that are not valid for inequalities. In Task 2 he could not work with the parameter and he used arithmetic procedures to determine its value. He could determine intervals (F5) but he could not interpret correctly the variation of the unknown x. From his explanation it is clear that he did not understand the meaning of solution set. He considered that if the interval he found contains the solution set, it guarantees a correct solution. It is worth noticing as well that he misread the given interval, [-3, 8] as [3, 8]. He did not consider the possibility of using other values for k showing so a difficulty to conceive the unknown as a moving variable (Ursini & Trigueros, 1997).

DISCUSSION AND CONCLUDING REMARKS

The comparison between successful and unsuccessful students makes it possible to underline some factors that can explain success in the solution of complex algebraic inequalities. Daniel's approach shows that to be able to succeed in solving complex rational inequalities it is necessary to have a good understanding of the concept of variable. This implies understanding each of the aspects involved in each of the uses of variable, as well as to be able to flexibility move between them. A simple inequality, without parameters, involves at least the use of unknown and general numbers. But when a parameter is involved it is necessary to work with related variables as well, so the three uses of variable and their aspects intervene. Carlos had a basic understanding of variable, he could manipulate, seeing variables as general numbers, and he could interpret the parameter k as unknown. But he could not move flexibly between different uses, in fact, he used arithmetic methods trying to determine the value of k avoiding further manipulation. He was able to consider variation, but he could not make sense of the solution set.

Daniel was able to consider expressions as entities and recognize which rules and definitions were useful to solve the inequality, that is, he had structure sense (Hoch and Dreyfus, 2004). Carlos lacked it. Moreover, solving the given inequalities required handling several conditions, distinguishing between different cases and taking decisions along the solution process. To do this required organizing and re-organizing the given information and the information produced during the solution process and, in consequence, to prioritize the actions to be taken making sense of their meaning. Daniel could cope with this but Carlos did not.

This study corroborates what other authors have already found about the role of variable, variation, flexibility with the uses of variable, and structure sense in the solution of inequalities, in particular, rational inequalities. But, we could observe that the solution of inequalities does not require necessarily a good understanding of the concept of function. There are many examples of inequalities that can be successfully solved by using only unknowns and general numbers, as in Task 1. Functions can be also used in the solution strategy but are not indispensable. What seems to be absolutely necessary is the possibility to consider the variable as a dynamic entity, a “moving entity”.

Results show as well that some of the characteristics that have been enumerated to describe structure sense are important for students’ strategies to be successful. But aspects related to students’ capability of making sense of what they are doing, organizing their procedure, taking decisions along the procedure and the capability to work back from a given solution to finding the conditions of the problem, that is, to be able to revert the solution process, are a must in order to be successful.

Our results show that the characteristics related to successful strategies in the solution of inequalities are the possibility of moving between formalism and making sense of one’s actions, together with the possibility to follow a procedure and revert it when necessary. This constitutes a starting point to develop the capability to work with problems requiring advanced mathematics and for a long term students’ mathematical development. We agree as well with Niss (2006) who said in another context: “It seems to be essential for successful learning of mathematics that these facets of symbolism and formalism are put explicitly on the agenda of teaching instead of being relegated to implicit tacit learning between the lines”. Additionally, we would say that the development of these capabilities that seem to characterize successful students should be promoted from early grades and continuously monitored through schooling.

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BRIDGING THE GAP BETWEEN DISCRETENESS AND DENSITY

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We conducted a short intervention study using expository texts with the purpose of fostering secondary students' understanding of the density of numbers. The participants were 46 8th and 52 10th graders, who were administered a pre-test, then an expository text, and last a post-test. The experimental group was exposed to information about the infinity of numbers in an interval, and also to a 'bridging analogy' between students' initial conceptions of the segment, and the segment as a dense array of points. The experimental group outperformed the control group, who was exposed only to information about the infinity of numbers, and also provided better explanations of their answers.

INTRODUCTION

In this paper we report a short intervention study that investigated the potential instructional value of the number line in secondary school students' understanding of the density of numbers.

The property of density of rational and real numbers is described as the possibility to always find at least one number between any two numbers. This implies that, within a dense set of numbers, between any two elements there are infinitely many others and that no element has a (unique) successor. It is amply documented that understanding density is difficult for students at various levels of instruction (e.g. Lehtinen, Merenluoto, & Kasanen, 1997; Neumann, 1998; Tirosh, et al., 1999). In line with prior research, in previous work (Vamvakoussi & Vosniadou, 2004, 2007) we found that secondary students answered frequently that there is a finite number of numbers in intervals defined by rational numbers. In addition, our findings showed that the symbolic representation of the numbers defining the interval had a significant effect on students' responses. More specifically, students were very reluctant to accept that there can be decimals between fractions and vice versa; they also treated integers, decimals and fractions differently with respect to the number of intermediate numbers, for instance a student might answer that there are infinitely many intermediates between decimals, but a finite number of intermediates between fractions. These findings suggest that, in this context, students treated different symbolic representations of numbers as if they were different numbers, indicating a view of the rational numbers set as consisting of different, unrelated 'sets' (i.e. integers, decimals, fractions).

The number line is grounded on the analogy "numbers are points on the line". As such, it calls for a re-conceptualization for numbers, which might arguably help students conceive of rational numbers as individual entities, and also facilitate their

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understanding that, for instance, 0.5 and 1/2 are interchangeable representations of the same number, rather than different numbers, since they correspond to the same point. This could promote students' understanding of rational (and real) numbers as a unified number system (Kilpatrick et al., 2001). In addition, being a continuous representation itself, the number line could (in principle) be used to confront students' belief that numbers are discrete in nature. From a historical point of view, various descriptions of density emerged in mathematicians' attempts to capture the characteristic properties of the geometrical line, far before the emergence of any notion of the arithmetic continuum (e.g. Bell, 2005; Klein, 1968). In a pilot study (Vamvakoussi & Chatzimanolis, 2008) we found that secondary school students (grades 7^{th} -11^{th}) were more apt to accept the infinity of points on a segment, than the infinity of intermediate numbers in an interval.

However, the number line is a highly abstract analog itself, which requires from students to coordinate their understandings coming of two different domains, namely the domain of number and that of geometrical magnitudes, and in addition introduces a number of new conventions to be learned (English, 1993).

Lakoff and Núñez (2000), make a sharp distinction between the notion of the 'holistic line', which can be conceptualized as the trace of a moving object (e.g. the pencil, when it does not leave the paper) and is continuous in an everyday sense, and the notion of the line as a set of points, which they characterize as a mathematically elaborated metaphor of the line. They argue that these are two conflicting images of the line, and that the fact that students are not aware of this conflict contributes to their difficulty in interpreting the number line. Moreover, they argue that the number-points correspondence is far from being transparent to students.

Dealing with the geometrical line per se presents students with difficulties, since they may not distinguish between the abstract, geometrical line and its physical representations, such as a line drawn by a pencil on a piece of paper bearing features such as width, which is not supposed to belong to its idealized counterpart (Fischbein, 1987). Such considerations may also interfere with students' understanding about the number of points of a segment. More specifically, conceptions of points as material spots are at odds with the infinite amount of points on a line segment. Such conceptions may underlie students' belief that longer segments have more points (Fischbein, 1987). Reviewing the related literature, Sbaragli (2006) points to the conception of a segment as "a necklace of beads", put the one immediately next to the other, which seems to underlie many secondary students' descriptions of the structure of the segment.

Students' early experiences with the number line at the elementary school should also be taken into consideration. Quoting Dufour-Janvier, Bednarz, and Belanger (1987), English (1993) points out that students tend to see the number line as a series of 'stepping stones' with an empty space in between, commenting that *"this may explain why so many secondary students say that there is no numbers, or at the most* *one, between two whole numbers"* (p. 24). Another common metaphor for the number line, namely that of the ruler (e.g. Doritou & Gray, 2007) may also convey the idea that the number of numbers in an interval is finite.

Findings from our pilot study were compatible with the above considerations. More specifically, our participants described the geometrical line as a real-world object that gets thicker if magnified; or, given the possibility of unlimited magnification, they described the segment as consisting of points, the one immediately next to the other. Interestingly, the latter description was offered also by students who had consistently answered that there are infinitely many points on a segment. We took this to be an indication that, from the student' point of view, the 'infinity many intermediates' aspect of density does not necessarily imply its 'no successor' aspect. This assumption is in line with findings showing that students believe that, for instance, 2.9999… is immediately before 3, i.e. such number in principle exists, albeit it can not be precisely defined (e.g. Lehtinen et al., 1997).

The question rises, how can the gap between the segment as conceived by students and the segment as a dense array of points be bridged? We drew on the 'bridging analogy' approach, developed by Clement and his colleagues (see, for example, Clement, 1993). This approach involves the interpolation, between students' initial understanding of a situation and the intended scientific idea, of one or more intermediate anchoring situations, expected to trigger a correct intuition, i.e. one that can be developed toward understanding the target situation. We devised the "rubber line" anchor: "The line is like an imaginary rubber band that never breaks, no matter how much it may be stretched". This analogy is (partially) grounded on students' experience with a real world object and it aims at conveying the idea that no matter how close two points seem to be, there are always more points to be found in between, by stretching the rubber line. The 'rubber line' is compatible with students' conceptions of the geometrical line, which lead them to believe that there will eventually be two successive points, but explicitly contradicts this expectation.

We designed a short, text -based intervention with the purpose of investigating the added value of this approach in students' understanding of the denseness of points and numbers on a segment and in an interval, respectively. We assumed that students exposed to explicit information about the infinity of numbers in an interval would improve their performance in similar tasks. However, we hypothesized that students exposed to the 'rubber line' analogy would perform better in items related to the 'no successor' aspect of density and would provide better explanations for their answers.

METHOD

Participants.

The participants were 46 8^{th} and 52 10^{th} graders, from 4 classes of the same school in the suburbs of Athens.

Materials.

We constructed two texts (T_{INF}, T_{RL}). T_{INF} reminded students that all numbers can be placed on the number line. Then it referred to 0 and 1 on the number line and evoked the notion of 'space' between them. It provided the correct answer ("there are infinitely many numbers between 0 and 1"), accompanied with several examples. The first part of T_{RL} was identical to T_{INF}. In its second part, the 'rubber line' anchor was employed to explain how it is possible for two points to 'look' as if they were successive, and yet have infinitely many points in between. T_{RL} concluded by emphasizing the numbers-points correspondence and the implication that there are infinitely many numbers in any interval.

We designed two questionnaires as pre- and post-tests. They had 9 forced-choice items in common, focusing on the infinity of numbers in an interval and the infinity of points on a straight segment ('infinity items'). The post-test included 5 additional items, asking students to evaluate a statement about the existence of two successive numbers, and to justify their answer ('no successor' items).

Procedure.

The students were administered a) the pre-test, b) the expository text, and c) the post-test. Students in the same class received the same type of text. The procedure lasted 45 minutes.

RESULTS

Students' mean performance in the common items of the pre- and the post-test was computed by scoring the "Finite number" answer as 1 and the "Infinitely many" answer as 2 (see Table 1).

	Text type	N	Pre-test		Post-test	
			Mean	S.D.	Mean	S.D.
8th grade	T_{INF}	25	1.227	.188	1.640	.291
	T_{RL}	21	1.302	.298	1.651	.332
10th grade	T_{INF}	25	1.351	.337	1.782	.237
	T_{RL}	27	1.391	.394	1.848	.239

Table 1: Mean scores in the pre- and post-test, as a function of grade and text-type.

No significant performance differences between the two age groups, or between the two text type conditions within grade, were found before the intervention (tested with Mann-Whitney U test).

Students' performance in the 'infinity' items increased after the intervention (Table 1). A Wilcoxon signed ranks test, comparing students' mean performance between the pre- and the post-test, within grade and text-type condition, showed that this difference was significant, for all groups. More specifically, 8th graders' performed significantly better under the T_{INF} condition, z= -3.920, p<.001, and also the T_{RL} condition, z=-3.861, p<.001. Similarly for 10th graders, under the T_{INF} condition, z= -3.374, p<.001, and the T_{RL} condition, z=-3.895, p<.0001.

No significant differences in students' performance in these items' were found between the two text-type conditions, for any of the grades (tested with Mann-Whitney U test).

Finally, a Mann-Whitney U test comparing 8th and 10th graders' performance in the post-test showed a significant difference, z=-3.756, p<.001, in favour of 10th graders.

Students' responses in the additional, 'no successor' items of the post-test were categorized as incorrect, partially correct and correct and were scored as 1, 2, and 3, respectively, by two independent scorers (mean scores are presented in Table 2). For a student to be credited with a correct response, she had to make a correct choice, i.e. deny the possibility of the two given numbers (or points) to be successive and also present a principle-based explanation. A response was categorized as 'partially correct' when the student made the correct choice, but offered no explanation (most typical case), or came up with a counter-example without referring to a more general principle. So for example, a correct choice accompanied with the explanation "*It is not necessary for 2.002 to be immediately after 2.001. Take for example 2.0015 or 2.0012.*" was scored by 2, whereas a correct choice accompanied with the explanation "*Between 3/7 and 4/7 there are infinitely many numbers, so we can always find a number closer to 3/7*" was scored by 3.

	Text type	N	'No successor' items	
			Mean	S.D.
8th grade	T_{INF}	25	1.544	.743
	T_{RL}	21	1.968	.852
10th grade	T_{INF}	25	2.088	.698
	T_{RL}	27	2.467	.716

Table 2: Mean scores in the additional items of the post-test.

A Mann-Whitney U test comparing students' mean performance between the T_{INF} and T_{RL} conditions, within grade, showed significant difference in the case of 10th graders, z= -2.159, p<.05.

The added value of the T_{RL} became clearer when we looked at the number of incorrect responses in the 'successor items' of the post-test. As can be seen in Table 3, students not exposed to the 'rubber line' text were far more often found to deem at least one pair of numbers (or points) successive, than the T_{RL} students. It is interesting to note that, in this respect, the T_{RL} 8^{th} graders outperformed the older students who were not exposed to the rubber-line text.

Grade	Incorrect answers in the 'successor items'	T_{INF}	T_{RL}	Total
8^{th}	None	6 (24%)	12 (57.1%)	18 (100%)
	At least one	19 (76%)	9 (42.9%)	28 (100%)
	Total	25 (100%)	21(100%)	46 (100%)
10^{th}	None	8 (32%)	18 (66.7%)	26 (100%)
	At least one	17 (68%)	9 (33.3%)	26 (100%)
	Total	25 (100%)	27 (100%)	52 (100%)

Table 3: Frequencies and percents of students who did or did not make a mistake in the 'no successor items', as a function of grade and text-type.

Table 4 presents a categorization of students based on the number of correct answers they gave in the 5 'no successor' items. It can be noticed that students exposed to the 'rubber line' text provided more often elaborated explanations than their T_{INF} fellow students, within both age groups.

		Number of correct answers in the 5 'no successor' items			
Grade	Text type	0 or 1	2 or 3	4 or 5	Total
8^{th}	T_{INF}	17 (68%)	5 (20%)	3 (12%)	25 (100%)
	T_{RL}	9 (42.9%)	5 (23.8%)	7 (33.3%)	21 (100%)
	Total	37 (56.5%)	15 (22.7%)	14 (21.2%)	46 (100%)
10^{th}	T_{INF}	12 (48%)	3 (12%)	10 (40%)	25 (100%)
	T_{RL}	6 (22.2%)	5 (18.5%)	16 (59.2%)	27 (100%)
	Total	25 (33.8%)	18 (24.3%)	31 (41.8%)	52 (100%)

Table 4: Categorization of students based on the number of correct answers in the 'no successor' items of the post-test, as a function of grade and text-type.

We note that students exposed to the 'rubber line' text employed this information, and the numbers-points correspondence in general, in justifying their answers (see Examples 1 and 2).

Example 1. (Two points cannot be found the one immediately next to the other) "*Because you keep stretching the line and you find that there are more points - this process does not end.*"

Example 2. (2.002 cannot be the successor of 2.001) "*Because if you place them on the number line, there are infinitely many points between these numbers, and therefore infinitely many other numbers*".

CONCLUSIONS- DISCUSSION

Our findings showed that providing information via an expository text about the infinity of numbers in a specific interval (i.e. the one defined by 0 and 1) helped students improve their performance in similar tasks, under all conditions. Tenth graders profited in general more from this short intervention than the younger students, since their performance did not differ significantly before the intervention, but they performed significantly better afterwards. This finding can be attributed to the fact that older students were more apt to extract information provided by the texts and connect it to their existing knowledge, than the younger ones.

The added value of the 'rubber line' anchor manifested in students' responses in the additional items of the post-test, which focused on the 'no successor' aspect of density. Based on findings from our pilot study (Vamvakoussi & Chatzimanolis, 2008), we took this aspect to be more difficult for students, than the 'infinitely many intermediates' aspect of density and we addressed it in geometrical context, bridging students' conceptions of the segment and its relation with points and the notion of a segment as a dense array of points via the 'rubber line' anchor. We also emphasized the numbers-points correspondence, with the purpose to facilitate students to transfer the information about the denseness of points to the domain of numbers. Our results showed that both 8th and 10th graders profited of this approach, since they were more consistent in denying the possibility of two numbers or points to be successive, and were also more apt to provide explanations, than their fellow students who were exposed only to information about the infinity of numbers in an interval.

Keeping in mind that, besides being short, the intervention was rather conservative in the sense that it was text-based and did not allow for any interaction in the classroom, our findings suggest that purposeful, long-term use of the number line in instruction maybe valuable, provided that it is accompanied with adequate explanations that bridge the gap between students' conceptions of the number line and the intended mathematical meanings.

Acknowledgement

This work is supported by the project ANALOGY: Human-The Analogy Making Species, financed by the FP6 NEST Program of the European Commission. (STREP Contr. 029088)

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ADD? OR MULTIPLY?
A STUDY ON THE DEVELOPMENT OF PRIMARY SCHOOL STUDENTS' PROPORTIONAL REASONING SKILLS

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This paper focuses on the development of two phenomena that were so far studied separately: the use of additive strategies in proportional situations and the use of proportional strategies in additive situations. We observed a decrease in the over-use of additive strategies from 3rd to 6th grade and a simultaneous increase in the over-use of proportional strategies. In the intermediate stage between both types of over-use, students did not reason correctly but rather switched between additive and proportional reasoning based on the numbers that appear in the problems.

INTRODUCTION

Since the 1980s, a lot of research has focused on the development of proportional reasoning skills (Hart, 1981; Tourniaire & Pulos, 1985). Particular attention has been paid to the tendency in students to approach proportional situations additively instead of multiplicatively (e.g., Hart, 1981; Lin, 1991). More recent research has shown that while students acquire proportional reasoning skills, they increasingly tend to use them also beyond their applicability range (e.g., Fernandez, Llinares, & Valls, 2008; Modestou & Gagatsis, 2007; for a review, see Van Dooren, De Bock, Janssens, & Verschaffel, 2008). Of particular interest in that line of research is the tendency in students to approach additive situations as if they were proportional (e.g., Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). In the current paper, we confront both – seemingly opposite – lines of research with each other.

THEORETICAL AND EMPIRICAL BACKGROUND

Proportional reasoning and the over-reliance on additivity

Because of its wide applicability, proportionality takes a pivotal role in primary and secondary mathematics education. Although full competence in proportional reasoning is not achieved easily, children already at a young age can master relatively simple proportional situations (e.g., Spinillo & Bryant, 1999), for instance by relying on repeated addition strategies: "If 1 pineapple costs 2 euro, 3 pineapples cost 2+2+2 = 6 euro". The actual teaching of proportionality generally only starts in the upper elementary (or lower secondary) grades, where pupils intensively practice proportional reasoning skills with missing-value proportionality problems such as:

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 281-288. Thessaloniki, Greece: PME.

"Grandma adds 2 spoonfuls of sugar to juice of 10 lemons to make lemonade. How many lemons are needed if 6 spoonfuls of sugar are used?" (Kaput & West, 1994).

Given the pivotal role of proportional reasoning in mathematics education, a lot of research has focused on how students acquire proportional reasoning skills, which difficulties they experience, and how instruction can enhance reasoning. One of the most often observed errors is that students apply an additive (or "constant difference") strategy to proportional situations (e.g., Hart, 1981; Kaput & West, 1994; Lin, 1991; Tourniaire & Pulos, 1985). For example, for the lemonade problem above, students then reason that for 2 spoonfuls of sugar you need 10 lemons, so for 2+4=6 spoonfuls you need 10+4=14 lemons. This kind of error seems most typical for younger children without instructional experience with the multiplicative relations in proportional situations. But also after instruction, additive errors still occur in students, particularly on more difficult problems. A major factor contributing to difficulty is when the numbers given in the problem do not form integer ratios (Hart, 1981; Kaput & West, 1994; Lin, 1991; Tourniaire & Pulos, 1985). For instance, when the lemonade problem mentioned before is transformed into "Grandma adds 2 spoonfuls of sugar to juice of 5 lemons to make lemonade. How many lemons are needed if 3 spoonfuls of sugar are used?" it is more difficult to execute the multiplicative operations because the ratios are not integer (5/2 and 3/2). Before extensive instruction in proportional reasoning, students will fall back to additive reasoning (2+1 spoonful of sugar in the lemonade, so 5+1 lemons).

Research on the over-use of proportionality

Recent studies have indicated that students are often inclined to apply proportional methods outside their applicability range. Research has shown that particularly for missing-value problems, students strongly tend to apply proportional methods despite the non-proportional character of the problem situation. For instance, more than 80% of 12-16-year old students answer "24 hours" to the problem "Farmer Gus needs 8 hours to fertilize a square pasture with sides of 200m. How much time will he approximately need for a square pasture with sides of 600m?" (for a review of this research, see De Bock, Van Dooren, Janssens, & Verschaffel, 2007).

Particularly relevant for this paper is the application of proportional methods to problems that have an additive structure. Van Dooren et al. (2005) gave the following problem to 3rd to 8th graders: "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run?" Correct additive answers to this problem ("16 laps") decreased from 60% in 3rd grade to 30% in 6th grade (and then increased again to 45% in 8th grade), while the percentage of wrong proportional responses ("24 laps") increased from 10% in 3rd grade to more than 50% in 6th grade.

A recent study by Van Dooren, De Bock, Evers, and Verschaffel (in press) showed that the specific numbers that appear in word problems also play a role in students' tendency to use proportional strategies to solve non-proportional problems, in a similar way as numbers cause students *not* to apply proportional strategies to proportional problems. Solutions to word problems where the numbers formed integer ratios (like the runner problem mentioned above) were compared with cases where the numbers formed non-integer ratios, such as "Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 4 laps, Kim has run 6 laps. When Ellen has run 10 laps, how many has Kim run?" It was found that in the latter case 4th to 6th graders were less inclined to over-use proportional strategies, and therefore performed better than on the version with integer ratios between numbers. This was particularly the case in 4th graders. In 5th, and especially in 6th grade, students became skilled in doing proportional calculations involving non-integer ratios, and therefore also over-used these skills.

RESEARCH QUESTIONS

The theoretical framework showed two lines of research that so far developed largely separately. To the best of our knowledge, the tendency to over-use additive strategies in proportional situations and the over-use of proportionality in additive situations have never been studied in combination in the same students. In the current study, we investigated the development of both kinds of reasoning in their close interaction.

Key questions that led the current study were therefore: Can we observe a development in students from an additive approach to various kinds of problems (including the over-use in proportional situations) towards a proportional approach (including the over-use in additive situations)? If so, how does this development take place, and, more particularly, how does the intermediate stage look like? Is there an intermediate phase in which students appropriately apply additive and proportional solution methods, or is this phase rather characterised by randomness, or by reliance on irrelevant problem characteristics? And, finally, can students in this intermediate stage at the same time over-use proportional methods in additive situations *and* over-use additive methods in proportional situations?

METHOD

Participants were 325 3rd, 4th, 5th and 6th graders (for exact numbers in each grade, see Table 2) from 2 randomly chosen middle-sized Flemish primary schools. These students solved 4 experimental problems.

The design and examples of word problems are shown in Table 1. There were two major types of problems: Two of the word problems were *proportional* problems, for which proportional calculations (i.e. finding the value of x in $b/a = x/c$) lead to the correct answer. The other two were *additive* word problems, for which additive calculations (i.e. finding x in $b - a = x - c$) are required. As can be seen in Table 1, proportional and additive problems were formulated very similarly. The crucial difference between proportional and additive situations lies in the second sentence. For example, for the integer additive problem in Table 1, the additive character of the situation lies in the fact that both girls run at the *same* speed, but one started *later*, so

Problem type	Number type	Example
Proportional	Integer (I)	Evelien and Tom are ropeskipping. They started together, but Tom jumps slower. When Tom has jumped 4 times, Evelien has jumped 20 times. When Tom has jumped 12 times, how many times has Evelien jumped?
	Non-integer (N)	A motor boat and a steam ship are sailing from Ostend to Dover. They departed at the same moment, but the motor boat sails faster. When the steam ship has sailed 8 km, the motor boat has sailed 12 km. When the steam ship has sailed 20 km, how many km has the motor boat sailed?
Additive	Integer (I)	Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run?
	Non-integer (N)	Lien and Peter are reading the same book. They read at the same speed, but Peter started earlier. When Lien has read 4 pages, Peter has read 10 pages. When Lien has read 6 pages, how many has Peter read?

Table 1: Design and examples of experimental items

that the difference in laps between both girls remains constant. The problem can be turned into a proportional one by changing it into:

> Ellen and Kim are running around a track. They started together, but Kim runs faster. When Ellen has run 4 laps, Kim has run 8 laps. When Ellen has run 12 laps, how many has Kim run?

Another element of the design was that we experimentally manipulated the type of numbers that appeared in the word problems, in a similar way as happened in Van Dooren et al. (in press): Given numbers were chosen so that when focussing on the ratios between the given numbers, one ends up working with either integer ratios (I-version) or non-integer ratios (N-version). Care was taken, however, that even when working with non-integer ratios, the outcome was always integer.

For our research questions, it was important that each student would solve all four experimental items. At the same time, we needed to avoid that students' responses on one item would influence their behaviour on other items. The four experimental items were therefore distributed over two tests that each student had to solve with one week in between. The first test contained problems on various mathematical topics, including two of the experimental items: a proportional word problem (I- or N-version) and an additive word problem (I- or N-version). The second test contained the other two experimental items, and again various buffer items.

Potential differences among problem contexts were controlled for by counterbalancing these in the design of eight different test variants: Some students got the experimental items as shown in Table 1, while others got the additive problems reformulated as proportional variants and vice versa, and the non-integer variants reformulated as integer variants and vice versa, so that any uncontrolled variance would be cancelled out.

RESULTS

General results

First, students' answers were coded as either *proportional* (P, proportional calculations as defined above were done with the given numbers), *additive* (A, additive calculations were done with the given numbers), or *other* (O, other calculations were done). Evidently, this coding also allows to analyse performance, since P-answers are correct for proportional problems (and incorrect for additive ones), and A-answers are correct for additive problems (and incorrect for proportional ones).

Table 2 shows the solutions. First of all, these results confirm earlier findings. For *proportional problems*, there is an overall increase in performance with age. Also, the I-version elicits more correct responses than the N-version, while the N-version elicits more additive errors. This influence of numbers diminishes with age, without entirely disappearing however: 3rd graders do not give correct proportional answers to non-integer proportional problems (and rather commit additive errors), but more and more correct answers are given towards 6th grade.

These trends are mirrored in the results for the *additive problems*. The N-version elicits more correct responses than the I-version, while students commit more proportional errors to the latter one. Again, an influence of age is found: 3rd graders hardly commit proportional errors to the N-version, but more and more proportional errors occur towards 6th grade.

	Grade	Proportional problems I-version	Proportional problems N-version	Additive problems I-version	Additive problems N-version
Proportional solutions	3 (*n*=88)	**19**	**0**	*17*	*1*
	4 (*n*=78)	**29**	**3**	*36*	*1*
	5 (*n*=81)	**80**	**18**	*55*	*10*
	6 (*n*=78)	**85**	**41**	*68*	*33*
	Total (*n*=325)	**50**	**14**	*42*	*11*
Additive solutions	3 (*n*=88)	*49*	*95*	**55**	**80**
	4 (*n*=78)	*44*	*71*	**46**	**81**
	5 (*n*=81)	*23*	*69*	**39**	**92**
	6 (*n*=78)	*10*	*35*	**19**	**58**
	Total (*n*=325)	*31*	*68*	**39**	**75**

Table 2: Overview of solutions given by students (in %) (Correct solutions are indicated in bold, erroneous solutions in italic)

Answer profiles

More important for our research questions are students' individual solution profiles. We analysed the combinations of students' solutions to the integer and non-integer versions of the proportional and additive problems, using the following categories:

- **Correct**: Proportional problems were solved proportionally and additive problems solved additively
- **Additive**: Proportional and additive problems were solved additively
- **Proportional**: Proportional and additive problems were solved proportionally
- **Number-sensitive**: Integer problems were solved proportionally and non-integer problems additively
- **Other**: All cases that did not fit into these profiles

The categorisation strictly followed the above rules. For example, for additive reasoners, no proportional responses on any of the four items were tolerated, for proportional reasoners no additive responses were allowed, and for number-sensitive reasoners, no additive responses to integer problems and no proportional responses to non-integer problems were tolerated. The categories were also applied, however, when one "other answer" was given while the other three answers perfectly fitted to the profile.

Table 3 shows the results of the categorisation. A first observation from this table is that the category "Other" is relatively large in 3rd grade, but gradually decreases towards 6th grade, indicating that students' solutions became more systematic. There is, however, hardly any progress in the category "Correct". It remains quite small. Even in 6th grade only 8% of the students succeeds in not committing a proportional error to an additive problem nor an additive error to a proportional problem. So, the increase in systematicity lies elsewhere. It is also not situated in the "Additive" category. In 3rd grade, almost half of the students are systematic additive reasoners, and this gradually decreases to 6% in 6th grade. Apparently, with increasing educational experience, students are less inclined to reason additively – that is, to apply additive strategies to all four items, regardless of item characteristics.

	Grade				
	3 (*n*=88)	4 (*n*=78)	5 (*n*=81)	6 (*n*=78)	Total (*n*=325)
Correct	1	5	6	8	5
Additive	47	37	17	6	27
Proportional	0	3	9	32	10
Number-sensitive	15	28	51	42	34
Other	38	27	17	12	24

Table 3: Overview of solution profiles of individual students (in %)

The decrease in "Other" and "Additive" profiles goes along with an increase in "Proportional" and "Number-sensitive" profiles. In 3rd grade, there are no "Proportional" students, but this number gradually increases so that in 6th grade almost one third falls into this category. In 3rd grade, there are already 15% students with a "Number-sensitive" profile (i.e. responding additively to problems with non-integer numbers and proportionally to problems with integer numbers), this number increases up to 5th grade where more than half of the students fall into this category, and a small decrease afterwards.

The observation that so many students adapt their solution strategy to the number characteristics of problems (rather than to the underlying mathematical model) suggests that students can commit additive errors (to non-integer proportional problems) at the same time as they commit proportional errors (to integer additive problems). A further analysis of the profiles showed that this was indeed the case in 9% of the 3rd graders, 27% of 4th graders, 23% of 5th graders and 26% of 6th graders.

CONCLUSIONS AND DISCUSSION

This study focused on two phenomena that were so far studied totally separately: the over-use of additive strategies in proportional situations and the over-use of proportional strategies in additive situations. We investigated how both kinds of errors developed throughout primary education.

First of all, students behaved similarly as already reported in the literature: Additive reasoning decreased with age and proportional reasoning increased. More additive errors occurred on non-integer versions of proportional problems, and more proportional errors occurred on integer versions of additive problems. This effect became smaller with age, as students became more inclined to over-use proportional methods, regardless of the number structure.

More importantly, our study enabled to investigate individual solution profiles of students. This analysis revealed that many 3rd graders reasoned additively by default to the experimental problems, but this tendency decreased rapidly and was almost gone in 6th grade. It was replaced by two other types of reasoning. First, there was a strong increase in the tendency to apply proportional methods to all the experimental items. This was absent in 3rd graders but very prominent in 6th graders. The second trend was an increase of number-sensitive reasoning, meaning that students consistently reason proportionally when the numbers in the word problem form integer ratios and additively when this is not the case. Remarkably, there were very few students reasoning correctly across all problems, even in the oldest age group.

These results suggest that there indeed is a developmental trend in many students from applying additive methods "anywhere" in the early years of primary school to applying proportional methods "anywhere" in the later years. As such, this could be expected based on what was reported in the two lines of literature. More importantly, our study has shown that in the intermediate stage students do not pass through a phase where they correctly apply additive and proportional methods. Rather, they switch between methods, but based on a superficial problem characteristic, namely the numbers that are given in a word problem. This made it also possible that students at the same time over-used proportional *and* additive methods.

Further research might focus on how the development continues after 6th grade, and whether and when students develop towards a correct application of both methods. Also a true longitudinal (or even microgenetic) study in this field might be useful.

Acknowledgement

This research was partially supported by grant GOA 2006/01 "Developing Adaptive Expertise in Mathematics Education" from the Research Fund K.U.Leuven, Belgium.

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WHAT DO STUDENTS SAY ABOUT THE ROLE OF THE MINUS SIGN IN POLYNOMIALS?

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A lot of researches have drawn attention to students' difficulties with polynomial reductions involving negative numbers. The aim of this study is to examine the origin of these difficulties in the light of the use of the minus sign. Twelve eighth grade students were interviewed about the role they give to the minus sign in polynomials. The results show that the students remained in an arithmetical discourse characterised by a subtraction minus, meaning operating on entities without signs. They also defined the role of the minus as depending on its place in the polynomial. It was noticeable that none of the students were able to express a multiple role for the minus sign. This lack of generalisation and flexibility prevents the students from reifying the terms of the polynomial and the polynomial itself, and so from making sense of the procedures they use.

CONTEXT AND THEORETICAL BACKGROUND

Our line of questioning derives from previous work that had been carried out on learning to solve equations (Vlassis, 2002). We experimented with situations in which eighth grade students (13-14 year olds) learned to solve equations. This research showed an obvious increase in errors as soon as negative numbers are introduced in equations. Based on the literature (Gallardo, 2002), we expected that negative whole numbers would create some difficulties. Nevertheless, we were surprised by the extent of the problem, particularly as the students participating in the research had been learning algebra for 18 months. I addition to the well-known problems related to the negative solution, we identified several types of difficulties in polynomial reduction operations, when students had to simplify the equation members or to apply the solving procedures. However, the required reduction operations did not involve any particular difficulties such as brackets or two signs following each other. The terms were either a single number or a number with a one-letter coefficient, with a maximum of four polynomial terms such as $2 + 7x - 3x + 8$. Some years previously, Kieran (1984) and Linchevski and Herscovics (1996) had already observed these same results in the context of learning to solve equations, but had gone no further than simply recording their observations. Initial investigations in order to gain a better understanding of these difficulties were carried out by Vlassis (2004). Interviews with 17 eighth grade students showed that for the most part they based their procedure on arithmetical rules, or referred to algebraic rules that they had recently studied, such as the signs rule.

The objective of this article is to pursue these initial investigations further by analysing students' statements about the role of the minus sign in polynomials. We begin by describing the results of a test consisting of polynomials constructed on the same basis as that described below and containing at least one minus sign. We then

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 289-296. Thessaloniki, Greece: PME.

present the statements made by students on the meaning they attribute to the minus sign in polynomials. This kind of analysis is crucial to gaining a better understanding of the students' strategies in these operations, which are basic algebraic operations used not only to solve equations, but also in other kinds of algebraic activities such as the application of operational properties, factorisation, and the study of functions.

FROM NEGATIVE NUMBERS TO THE MINUS SIGN

Why did we focus our analysis on the minus sign, and not on negative numbers? During our initial investigations, we had also asked the 17 students to identify the negative numbers and positive numbers in the polynomial $-18a - 2y + 5a - y$. In mathematical terms, it is clear that this question is not really orthodox, since it gives rise to the idea that one can instantly conclude that a term such as $-18a$ is negative. However, this question brought to light some interesting phenomena. It revealed that the students differed in the way that they dealt with the different terms that make up a polynomial. Out of the 17 students questioned, 9 gave the 'expected' response and identified each term with its sign, and out of these, 3 pointed out the ambiguity of the question. Out of the remaining 8 students, 7 systematically circled the first term with its sign, while the terms within the polynomial were considered either completely independently of the signs preceding them (4 students), or inconsistently – sometimes together with the sign and sometimes without it (3 students). When interviewed on this subject, these students explained that they had not circled the plus or minus sign with the term *'since the sign is not part of the number'*. However, all of them had circled *-18a* with its sign *'since it is a negative number, while the signs inside the polynomial mean you have to subtract or add'*.

These initial explorations and explanations given by students regarding the signs led us to shift the focus of our questioning from the concept of negative numbers to the use of the minus sign, and to move towards socio-cultural approaches. These were of interest to us because of their primary focus on *'*signs*'* rather than concepts according to Vygotsky's principles. The implication is that signs and meaning co-emerge. According to Cobb (2000), 'the ways symbols are used and the meaning they come to have are mutually constitutive and emerge together' (p. 18). If we examine the minus sign from this point of view, we can say that the way this sign will be used by learners will be interrelated to the meaning they attribute to it.

To demonstrate the multidimensionality of the minus sign, we have coined the term 'negativity' (Vlassis, 2008). This represents a model of the different uses of the minus sign where its different meanings have been classified according to two criteria. The first relates to the three main functions of the sign as established by Gallardo and Rojano (1994): unary, binary and symmetric. The unary function corresponds to the role of the minus sign as a sign attached to the number to form the negative numbers representing relative number, solution number, result number and formal negative number (Gallardo (2002). The binary function relates to the minus sign as an operational sign. In this category, we distinguished arithmetical subtraction and algebraic subtraction. In the symmetric function, the minus sign is also regarded operationally, but with a different function. This time, it consists of the action of taking the opposite of a number or of a sum. The second criterion refers to the work of Sfard (2000), who classifies symbols as structural signifiers (→ unary function), representing mathematical objects such as number, function, set or group and operational signifiers (→ binary and symmetric function), relating to operations.

METHOD

In order to gain a better understanding of the difficulties experienced by students when using the minus sign in polynomials reductions, we gave 131 8th grade students a test consisting of 28 polynomials. These students came from 6 classes in 3 schools which differed in terms of the students' socio-cultural backgrounds: 2 classes came from a school with students from a privileged socio-cultural background, 2 classes from a school with students from an underprivileged socio-cultural background, and 2 classes from a school with students from an average socio-cultural background.

The 28 polynomials, presented in Table 1, had the same structure as that described in the introduction. Twelve students were interviewed during the last two months of the school year. The students were selected according to the results they achieved in the test. We selected 4 low-ability students (with a test average of 50% or less) (L); 4 medium-ability students (with a test average of between 60 and 70%) (M); and 4 high-ability students (with a test average of 80% or more) (H). The interviews took place during the students' lunchtime two to five weeks after the test. Each interview lasted about half an hour. The author was responsible for conducting the interviews.

In this article we have chosen to focus on two research questions, with particular emphasis on the second question.

1. What are the characteristics of the polynomials that were most frequently answered incorrectly in the test?
2. What meaning do students attribute to the minus sign in polynomials?

From the point of view of the teaching received by these students, they were introduced to algebra at the beginning of the 7th grade, through the generalisation of the properties of operations. During this school year, they learned operations in N and Z. In the 8th grade, the focus of the mathematical curriculum was on operations in R, more complex operations in Z, and solving linear equations with one unknown element.

Results

1. What are the characteristics of the polynomials that were most frequently answered incorrectly in the test?

N°	Item	% C	N°	Item	% C	N°	Item	% C

1	**-8 + 5 - 2n – n**	61	**10**	7 - 2n - 4 + 10n	72	**19**	8a - 7 - 3 + 10a	79
2	**-12 - 7 - 7n + 4n**	66	**11**	-15 - 9y – 4y + 10	73	**22**	2 + 7x – 3x + 8	82
3	**3n - 4 + 7**	66	**12**	**4 - 6n - 4n**	73	**23**	7y - 4 - 9 + 3y	82
4	**6 - 5a - 3 - 4a**	66	**13**	-17 - 3x – 5x + 6	74	**24**	5a – a	83
5	**9 - 3a - 3 - 7a**	66	**14**	20 + 8 + 7n - 5n	75	**25**	-8x - 4 + 10x	86
6	**2x - 7 - 6x – 4**	70	**15**	29x - 9 - 3	75	**26**	7 – 6n + 13	86
7	**5 - 4y + 7 – y**	70	**16**	-2a - 4 - 3 - 8a	76	**27**	**-4n - 3n**	89
8	**-24b + 4b - 6a - 2a**	70	**17**	-18a - 5y + 5a - 3y	76	**28**	-7x - 8x	90
9	-17 + 3 - 14x - 25x	72	**18**	6y - 20 + 3y - 12	77			

Table 1: Percentage of students who correctly answered the polynomials (% C)

The table shows that slightly over one-third of the polynomials (8 out of 28) were only answered correctly by 70% or fewer of the students. It should be remembered that the students in this study were at the end of the eighth grade, and that these operations are particularly elementary for students at this level. The eight polynomials which had the fewest correct answers are highlighted in grey in the table. By analysing their structure, one can identify three factors which seem to contribute to increasing the difficulty of the polynomials.

1) A trinomial or a quadrinomial including a pair of like terms preceded by a minus sign and placed in the second part of the expression. Half of the polynomials were characterised by this structure (polynomials 1, 2, 3, 8).

2) A quadrinomial in which the first term is positive and all the other terms are negative (polynomials 4, 5 and 6).

3) The use of the convention of omitting the coefficient 1 (items 1 and 7).

Note that the first polynomial, which combines two of these difficulty factors, was the most frequently incorrectly answered of all the 28 polynomials. The analysis of the students' statements presented in the next section sheds interesting light on these observations.

2. What meaning do students attribute to the minus sign in polynomials?

To analyse the meaning attributed to the minus sign by the students, three situations have been identified:

- The minus sign is placed at the start of the expression (for example in $\boxed{-}15 - 9y - 4y + 10$)
- The minus sign is found between two like terms (for example in $20 + 8 - 7n \boxed{-} 5n$).
- The minus sign is found between two unlike terms. Two situations are covered here, that where the minus sign is found in front of a pair of like terms, such as in $4 \boxed{-} 6n - 4n$ **(1)**, and that where it appears in another context, such as in $6 - 5a \boxed{-} 3 - 4a$ **(2)**. (*For clarity, the minus sign in the question has a box around it.*)

Table 2 shows the different roles attributed to the minus sign by students according to the place that the sign occupies in the expression, and according to the student's level in the test.

	The sign is used to	**At the start of an expression**	**Between two like terms**	**Between two unlike terms (1)**	**Between two unlike terms (2)**
Binary	***subtract***		3L - 3M - 3H	1H	1L - 1H
	go below zero		1M - 1H	1H	1H
	separate			1L - 2M	
	apply the sign rule			1L	1L
Unary	***form a negative***	2L - 4M - 4H		2H	2M - 2H
Sub Total *Don't know* ***Total***		***10*** *2L* ***12***	***11*** *1L* ***12***	***8*** *2L - 2M* ***12***	***8*** *2L - 2M* ***12***

Table 2: Main meanings attributed to the minus sign according to its position in a polynomial

The second column in Table 2 shows the different meanings of the minus sign referred to by the students. In the first column these are categorised as binary or unary. The other columns show, for each position of the minus sign in a polynomial, the number of students who identified each function, and their performance in the test. For example, 2H means two high-ability students.

In general terms, Table 2 shows that all of the high-ability students and some of the medium-level ones attributed a mathematical role to the minus sign or referred to the number line. The low-ability students invented roles for it according to the procedures that they applied in order to reduce the polynomial (separating, applying the signs rule), or proved unable to assign any function to it at all. Table 2 also shows that in the students' eyes, the meaning of the negative sign clearly depends on its position in the polynomial.

When the minus sign is found at the start of an expression, it is almost always defined as the sign of a negative number. Ten of the 12 students claimed this without hesitation. They explained that it means that the term which follows is negative, or that it is to make the term negative. The two students who did not express this view were low-ability students who were unable to explain the role of the minus sign other than as a subtraction sign when found between two like terms. The first term of the expression seems to be considered the prototypical negative number. This observation is consistent with our other analyses mentioned earlier.

When the minus sign is placed between two like terms, Table 2 shows that 9 out of the 12 students agree that the minus sign is used to subtract. They seem to refer this situation to an arithmetical context where the minus sign is used to take away entities without sign. One low-ability student made a comment which is fairly characteristic of all the students. This is how the student explained the second minus sign in $20 + 8 - 7n \boxed{-} 5n$: '*So 5n is positive, but the minus is there to make - 5n since if it was negative it would have been put in brackets*'. This idea that if the term was negative it would have been put in brackets was found several times in the students' statements. Two students, one medium-ability and one high-ability, mentioned the number line model. This seemed to make sense to them in the context of subtracting whole numbers, even when these are terms with letters, as in the case of the polynomial they had been given.

<u>When the minus sign is placed between two unlike terms</u>, Table 2 shows that opinions are much less homogenous than in the other two contexts. Students seem to be puzzled about how to interpret the minus sign. Two cases are considered:

- In the case of the polynomial $4 \boxed{-} 6n - 4n$, where the minus sign precedes a pair of like terms, 3 students thought that the role of the minus sign was to separate the terms. According to one of these students (medium-ability), it was only there to separate the terms. '*First I wrote 4 and there were no other [terms] without letters. So I put 4 minus, I kept the minus, then I did 6n - 4n and got 2n, then I wrote 4 - 2n*'. This incorrect strategy, called 'bracket reasoning' (Vlassis, 2004), is very frequent among students in this context. One low-ability student thought that the minus was used to apply the signs rule. This difficulty in being able to describe the minus sign's function in this context reveals the ambiguity of the sign, and could explain why this kind of polynomial is so often answered incorrectly. The difference in the number of correct answers between $-4n - 3n$ (89%), item 27 in Table 1, and item 12, $4 - 6n - 4n$ (73%) could be explained in the same way. In the binomial, the meaning of the two minus signs is clear, and refers to the first two situations described above (columns 2 and 3 in Table 2). In the trinomial $4 - 6n - 4n$, the first minus sign in front of 6n is again a source of ambiguity, and leads to errors.

- In the case of $6 - 5a \boxed{-} 3 - 4a$, 2 students, one high-ability and one low-ability, believed that the minus sign meant that one should subtract 3 from 6. Again, the terms of the polynomial are considered as entities without signs. Four students were incapable of attributing any role to the minus sign, and one student again linked its function to the signs rule. Students' comments in this latter case show that the presence of three minus signs within a polynomial, with ambiguous status (from the students' point of view), increases the degree of difficulty, as the results in Table 1 show where this type of polynomial is one of the most frequently incorrectly answered.

Finally it can also be seen from Table 2 that none of the students brought up the idea of algebraic subtraction (which consists of adding the opposite), although the students interviewed had started learning about this rule over 18 months previously. In addition, none of the students mentioned that the same minus sign can have several functions (subtraction and formation of a negative number). Even the high-ability students did not seem aware of the conflict created by their comments, since when they had to move the term they correctly put the sign in front of it.

Here is an extract from the interview with H2, who is a very high-ability student.

1 INT: What's the 2nd minus sign used for in - *7x* - *8x*? (This student had been questioned about an additional binomial in order to take a closer look at their method of reasoning).

2 H2: The minus sign is used for showing you go down on the 'scale', you have to subtract another number and, ummm,... this number is positive, you have to go down again.

3 INT: Where is there a positive number?

4 H2: If you look at the numbers like that, if you don't consider the signs, you say that the *8x* is positive, and you already have a negative (-*7x*), you subtract a positive one, you go down, if *8x* had been -*8x*, you would have gone up to *1x*.

5 INT: So, is the minus sign separated from the *8x*?

6 H2: It depends on the way you solve the difference. For me, since it is a difference between like terms, I would put it separately. If you put it together then there is no more sign between the two of them and you will have to write a plus sign, which will make a difference.

Line 6 shows clearly that the role of the minus sign depends on the procedure being used ('*It depends on the way you solve the difference'*) and brings out, in a student of very high ability, the idea of subtraction between entities that have no sign.

DISCUSSION AND CONCLUSIONS

These analyses show that the students have not genuinely entered into an algebraic discourse characterised by flexibility in the use of the minus sign and independent of its place in the polynomial. Algebraic use of the minus sign consists of moving in an appropriate manner from a unary point of view (structural) to a binary point of view (operational). This difficulty in using the minus sign appropriately prevents students from recognising mathematical objects. At a micro level, students fail to reify the terms of the polynomial, which are regarded as entities without sign, with the exception of the first term, while at a macro level they also failed to reify the polynomial as the sum of its signed components. Take, for example, *-8 + 5 - 2n - n*. Here, the polynomial needs to be understood as follows:

$$\boxed{\boxed{-8} + \boxed{+\ 5} + \boxed{-\ 2n} + \boxed{-\ n}}$$

Sfard (2000) argues that the fact that the same signifier has to play two apparently incompatible roles, operational and structural, most certainly makes reification harder. These results also reveal teaching that is centred on procedures and overlooks the need to make explicit the multiple meanings of algebraic symbols. Such teaching prevents students from recognising the mathematical objects they are dealing with and from imparting meaning to the procedure they use. Yet, Duval (1998) argues that it is objects that are studied in mathematics. Sfard (1991) suggests that a procedural stage should be passed through before reaching the structural stage of mathematical objects. For the reification of polynomial terms, the procedural phase of an expression such as *-2n*, for example, could be followed by solving equations where the expression *-2n* in *-2n = 32* has to be considered as the product of *-2* and *n*. In the case of the students interviewed, the curriculum features a progression in the opposite direction: first, polynomial reductions during the 7th grade, then solving equations at the end of the 8th grade. This failure to reify polynomial terms results in a failure to reify the polynomial itself. It should be remembered that these basic operations form the foundation for other more complex operations; failure to reify them makes the whole structure of algebraic learning more fragile.

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RE-MYTHOLOGIZING MATHEMATICS BY POSITIONING

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Using spatial metaphors, we analyse the way positioning is conceptualized in current mathematics education literature, and the way it may be alternatively conceptualized. Our theorization favours immanent experience over transcendent discourses. We claim that changing the way mathematics is talked about and changing the stories (or myths) told about mathematics is necessary for changing practices.

INTRODUCTION

With growing awareness of the significance of social interaction in the development of mathematical understanding, and the related attention to interpersonal positioning, we ask whether mathematics teaching should be oriented around equipping students for action, or building a particular identity. The difference between action and stability is central to Harré and van Langenhove's (1999) conceptualization of positioning, from which we, in this paper, articulate a theoretical lens for evaluating accounts of classroom positioning in mathematics education research. In developing a relatively radical positioning theory that focuses on moments of action rather than on apparently stable characteristics of individuals and the discipline, we claim that changing the way mathematics is talked about and changing the stories (or myths) told about mathematics is necessary for changing the way mathematics is done and the way it is taught.

POSITIONING THEORY – LOCATING THE SUBJECT

Rom Harré and Luk van Langenhove's positioning theory, which is articulated in their edited book, describes the "dynamic stability between actors' positions, the social force of what they say and do, and the storylines that are instantiated in the sayings and doings of each episode" (van Langenhove & Harré, 1999, p. 10). 'Positioning' refers to the ways in which people use action and speech to arrange social structures. As outlined in their introduction to the book, in any utterance clues in word choice or associated actions evoke images of known storylines and positions within that story. The storylines can stem from culturally shared repertoires or can be invented. For example, a teacher may say something that positions herself as a coach and the student as a motivated athlete. Neither require experience as a coach or an athlete but they would have to know stories about coaches and athletes.

In any conversation, an initial utterance would be called first order positioning as it introduces the positioning within a certain storyline. In a subsequent utterance, if someone moves to change the positioning within the storyline or to change the storyline, it is called second order positioning. We use the following conversation

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from a middle school mathematics class to illustrate these different types of positioning. Italicization, in this case, represents reading from the textbook.

Teacher: Let's go ahead, read on.

Cory: *The class then made a graph of the data. They thought the pattern looked somewhat linear, so they drew a line to show this trend. This line is a good model for the relationship because, for the thicknesses the class tested, the points on the line are close to points from the experiment.*

Teacher: Okay, now, let's look at that line again: *This line is a good model for the relationship because for the thicknesses the class tested, the points on the line are close to the points from the experiment*. Take a look at what they did. Now, their data was a little bit scattered, a little more scattered than ours was. But, they still were able to draw a line that seemed to fit the data pretty well. … That's sometimes called a line of best fit. We're gonna use that term an awful lot. Cory read on.

In this episode, there are multiple storylines because there are multiple relationships, involving the teacher, Cory, other students in the immediate classroom as well as the mythical class mentioned in the textbook, the textbook, its authors, and others. The teacher initiates a typical teacher-student storyline, telling Cory to read from the textbook. This is first order positioning. Cory is complicit, which is either a low-impact form of second order positioning, or is a substantiation of the teacher's first order positioning – together, in agreement, they establish a storyline.

In another storyline, the textbook authors take the initiative with first order positioning. By writing about a particular mathematical situation and giving instructions for action, they tacitly suggested that they have provided all the necessary information. The teacher resists somewhat by interpreting the graph of the data in the textbook and comparing it to the data that his class has collected (lines 08-10). When the teacher makes it clear that he is aware of the local situation, and that the textbook authors are not, he takes some authority away from them. At the same time, the teacher positions them with the authority to tell how to draw a line that is a "good model for the relationship" (lines 06-08). Third order positioning is explicitly metadiscursive: it is reflective with explicit conversation about positioning. If, for example, the teacher in the excerpted situation would have told the students, "When we read a textbook, we should remember that the authors don't know about our classroom as well as we do," it would have been third order positioning.

Positioning theory concentrates on the moment of interaction and thus recognizes that multiple storylines can be enacted simultaneously. This focus on what Davies and Harré (1999) referred to as the immanent includes attention to the moment in time and to the people present in this moment. This is juxtaposed with interpretations that privilege the transcendent, and which attend to factors outside of the current interaction. Davies and Harré (1999) use Saussure's distinction between discourse practice and the discursive systems in which they are situated: "La langue is an intellectualizing myth—only la parole is psychologically and socially real" (p. 32).

With their attention to relationships in the moment, van Langenhove and Harré (1999) argued that all positioning is reciprocal. Thus, in every act or utterance, a person simultaneously positions him- or herself, and the other people with whom he or she is relating. As a result, expressions of identity are contextual and enact polarizations of character within the storylines at play in the context. Also relating to immanence, positioning is dynamic. We characterize this dynamism by saying that storylines are contestable and contingent in the enactment of any particular conversation. First, as described above, storylines are contestable because whenever one person enacts a certain storyline the others in the interaction may choose to be complicit with that storyline and the way they are positioned in it or they may resist and enact a competing storyline. Second, storylines are contingent in that different people may see different storylines being enacted in any given situation. As stated by Davies and Harré, "two people can be living quite different narratives without realizing they are doing so" (pp. 47-8).

For us, the most radical aspect of Harré and van Langenhove's (1999) positioning theory is their claim that la langue (sometimes called 'the discourse,' 'the discipline,' 'the Discourse,' or 'the discursive system', albeit with different nuances) is a myth. This would suggest, for example, that there is no such thing as 'mathematics' as a discipline. Rather 'mathematics' is unique in any interaction. Whether la langue is real or not is not a question for us. Instead, we are interested in the interpretive value of considering classroom practices with the assumption that there is no exterior structure that forces particular interactions. This view illuminates discourse participants' freedom to conceive alternative practices. No one can enforce a particular storyline or positioning in a conversation. Any participant is free to make moves (with speech or action) to establish a particular positioning.

We recognize that myths are powerful: they often feel more real than anything. For instance, though race distinctions are a myth (constructed, not inherent), these distinctions are often the most powerful reality in the lives of people suffering the effects of racism. The word 'myth' refers to stories that are well known in a culture. With this sense of the word, calling a story a myth makes no claim about its veracity. Rather, it makes a claim that the story is very well known and formative in the way people think. Myths are stories people live by, so we claim it is possible for people to position themselves in relation to a discipline whether 'the discipline' is something real or not. Positioning in relation to the discipline is commonplace because there are powerful mythologies relating to mathematics in academic cultures – for example, 'mathematics is useful', and 'mathematics is independent from values'. Thus we argue that even attention to a transcendent discipline can have its place in consideration of immanent experience. People take their storylines from their myths.

POSITIONING STUDENTS IN MATHEMATICS EDUCATION RESEARCH

To illustrate some of the characteristics of our view of positioning in juxtaposition with alternative views on positioning, we use a series of spatial images as metaphors. Our images draw attention to issues related to immanence, reciprocity, contingency, and contestability in the defining and applying of positioning. In the conference presentation, we will illustrate the significance of each feature of positioning using mathematics education literature in which positioning is central.

Immanence

We have developed our own interpretation of Harré and van Langenhove's (1999) radical focus on the immanent: we share their view that focusing on the immanent is preferable but we understand how one could use an immanent lens to reconcile scholarship that focuses on the transcendent. To illustrate the difference between positioning that foregrounds the transcendent and positioning that foregrounds the immanent, we visualize a mathematics student as a point, A. One could locate the position of the point with Cartesian coordinates – point A might be at (2,1). However, we could avoid analytic geometry and locate point A without a coordinate system by describing its location in terms of other points to which it relates – the point A may form a triangle with B and C. Figure 1 illustrates these two ways of seeing point A. We emphasize how different the same point A looks in each way of seeing the point's position.

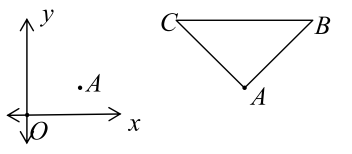

Figure 1

Locating points in relation to other points is like locating students in relation to other people in their mathematics learning. Student A relates to student B and teacher C, for example. By contrast, locating analytically is like theorizing that positions students in relation to mathematics. In analytic geometry the representation of the point's position mentions no other points, just as some scholarship considers the positioning of students in relation to mathematics without mentioning how this positioning relates to other individuals. Instead, they are positioned within a system.

We might argue that the origin is a significant point in the Cartesian system, but it is a point that is taken differently than other points in the system. Similarly, when interpreting scholarship that characterizes student positioning in relation to mathematics (the system), we can recognize that the discipline may be taken as an entity but it is mediated through a person (e.g., a mathematics teacher), or multiple persons (e.g., students perceived as "good" at mathematics). Thus there are one or more unrecognized persons central to the discipline.

The way one chooses to focus significantly impacts the portrayal of the student. For example, our interpretation of the transcript above took an immanent focus, with attention to interpersonal dynamics. With a transcendent focus one might foreground the attention to the apparent technical necessities of modeling and establishing 'good fit'. In the presentation, we will use Evans's (2000) work in which he proposed "a notion of the context of mathematical thinking that can be captured by the idea of positioning in practices” (p. 8) and Lerman’s (2001) illumination of repressive aspects of the practice of mathematics teaching to illustrate the significant complexities of immanent versus transcendent foci.

Reciprocity

The reciprocity of positioning relates closely to immanence in positioning theory, because immanence requires referring to a person in relation to others and the relationship goes both ways. In the illustration in Figure 1, locating point A in terms of points B and C implies, even requires, that B and C are also in relation to A. Together they form a triangle. For example, from our transcript, the teacher, enacting a leader–follower storyline by telling Cory what to do, typecasts Cory as a follower. Cory seems to comply. In the presentation, we will point out the paucity of attention to such reciprocity in the mathematics education literature, and we will use Evans (2000) to show how reciprocity can be obscured by using the words ‘position’ and ‘positioning’ as nouns instead of verbs.

Contingency

We now draw attention to two issues that relate to the contingent nature of positioning. Firstly, to illustrate that one can interpret the same situation using different storylines, we show in Figure 2 that while person A can be in relation to student B and teacher C there are other co-incidental possibilities for the positioning of A in relation to B. One could focus instead on the relation to D and E, with which A and B form a square in a different plane. Teachers may interpret situations thinking only of their own perspective. There are more dimensions – even more than three. It is valuable for teachers and even students to attend to the various possible points of view in mathematics classrooms. In our transcript, though Cory seems to be complicit with the teacher, we do not know why he is. Significantly different storylines would have him complying for different reasons – for example, to garner teacher approval, or because he sees the teacher’s guidance as helpful in his pursuit of understanding.

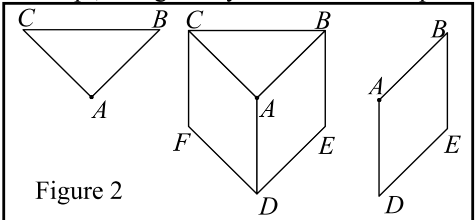

Figure 2

In the presentation, we will draw attention to Nasir and Saxe’s (2003) identification of different sources of cultural capital that students can draw upon when considering their ‘place’ in a classroom interaction. Their different funds of knowledge have implications on participants’ senses of each other’s capacity. We will also use Ainley’s (1988) investigation of students’ perceptions of the questions teachers asked to show an example of students and teachers having different interpetations of the same situation. We will argue that it is even possible for different positionings to co-exist in a complex weave.

Secondly, we said above that A, B, E, and D together form a square in our illustration. Looking at the shape without added context, however, we see a rhombus, not a square. The perspective of the person visualizing the positioning is significant. Thus, it may be true in a way to say "positioning is [a certain way]" in research reporting, but it would also be true to recognize that this is only one perspective on the positioning. Analyzing positioning from a vantage point that feels exterior to a situation is different from analyzing it from the perspective of a participant. A significant question is: who decides what the positioning is? In the context of interaction, the participants' decisions on this are most significant, but such a participatory position is relatively unavailable for researchers. Thus, with our transcript, we offered accounts of positioning but we want to make clear that there are other viable interpretations. Attending to more of these by drawing on various participants' perspectives would be helpful. In the presentation, we will consider Evans's (2000) recognition of the multiplicity of available positionings.

Contestability

Relating to the complexity due to the multiple possibilities for visualizing positioning in any given moment, there is further complexity due to the ephemeral and dynamic nature of positioning. All the illustrative images we used above are static images. It is difficult in a print medium to show them moving and changing shape or to make them as fuzzy as they should be to capture our view of positioning. Second and third order positioning, described earlier, remind us that even when one vision of positioning is initiated, it is contestable. The participants in the relationship can make moves to change that positioning, with either tacit moves (second order positioning) or explicit moves (third order). For example in the transcript, we could see the teacher first establishing the textbook and its authors as authoritative (by using it to structure the lesson), and then undermining this authority (by saying that 'they' do not know the situation in the real classroom).

Not only are the relationships between participants contestable, but their relationships to 'the' external power (the mythological discipline) are also. To illustrate, keeping point A in the same position, we could move the other points with which A associates to form different triangles and other shapes, not necessarily polygons. And in the analytic system, we could freeze A and move the coordinate system's origin. When visualizing a student's positioning in relation to mathematics, it is important to remember that different people (including students) may have very different senses of what (or where) mathematics is, and of how a person can relate to it. In the presentation, we will use examples from Sensevy et al. (2005) to show how using 'positioning' or 'positions' as verbs instead of nouns gives a different sense of the available options for classroom participants. We will also use Gates (2006) to demonstrate the rigidity of the discipline suggested with a focus on dispositions (this word has 'position' as its root).

DISCUSSION

Our take on Harré and van Langenhove's (1999) positioning theory favours a focus on immanent practice, instead of attention to transcendent discourses, and highlights the reciprocal, contingent and contestable nature of positioning. We see benefits in theorizing for particular purposes and we suggest that more attention to immanence and its related features is warranted in the analysis of mathematics learning.

We described above how myths are stories people live by. No matter how real one thinks mathematics as a discipline is, it is possible to recognize that students position themselves in relation to the ‘mythological’ discipline, and it is misleading to write about the discipline as if it is uniformly experienced by all people. Students experience the discipline through their teachers as mediums of the discipline, but they also may experience the discipline as a presence. The repression associated with mathematics expresses itself in interpersonal utterances, which are experienced in unique contexts. In the presence of such a powerful myth as ‘mathematics’ it is worth considering how educators could demythologize the discipline and thus render it powerless, or perhaps less powerful. More appropriately, we suggest, is the possibility to re-mythologize such a powerful discipline by reconceptualising it with human stories that invite identification with storylines that are not traditionally a part of mathematics classroom discourse.

We are recommending a relatively radical approach to positioning in mathematics education, less radical than Harré’s and van Langenhove’s (1999). Instead of de-mythologizing mathematics and rendering it impotent as a discipline, we advocate re-mythologizing it by drawing attention to the narratives at play in classrooms and outside classrooms. We suggest the following questions as potent for research and for use by mathematics teachers to invite narrative into the classroom. The first two questions are Morgan’s (2006, p. 229) and the others are adapted from her work and generalized to extend outside of written texts, which was her focus: 1) Who does mathematics (Is a human agent present)? 2) What processes are human agents engaged in? 3) Who are these human agents doing these things for and why? 4) Who is looked at as an authority? 5) What roles are available to the primary human agent and the other human agents in the interaction? 6) How does the interaction connect with human relationships outside the classroom context?

Morgan showed (and we would corroborate) that the field of linguistics offers useful tools for identifying answers to these questions. We would also point at two other fields of mathematics education research to help identify possible storylines and positioning within them. Ethnomathematics takes the view that all mathematics is cultural, and so claims that any mathematics is set in human story. It can add to one’s repertoire of ways to participate in mathematics. Identity work also has potential for this end as it draws attention to various ways students might see themselves.

Perhaps the best way to deal with the power of a weighty discipline like mathematics is not to fight it, but rather to ignore its weight by simply engaging students in the doing of mathematics. Let students position themselves in various ways and help them recognize that positioning themselves within various storylines in various ways can only strengthen their mathematics.

Acknowledgment: This research was supported by the National Science Foundation (Grant No. 0347906) and the Social Sciences and Humanities Research Council (Grant title: "Positioning and Authority in Mathematics Classrooms"). Opinions, findings, and conclusions or recommendations expressed here are the authors' and do not necessarily reflect the views of the granting bodies.

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MATHEMATICS, MIND AND OCCUPATIONAL SUBJECTIVITY

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Occupational subjectivity is a term that is used to describe the way individuals orient themselves in the world both recreationally and vocationally, that is, how they behave as occupational beings, justify occupational choices and imagine occupational futures. This paper exposes the ways in which the study of mathematics at school is powerfully implicated in occupational subjectivity. A group of upper secondary school children reflect on making choices about whether to continue studying mathematics once it is no longer compulsory, what kinds of mathematics to study when faced with several options, and the place of mathematics in occupation beyond school. The children could be seen to select occupational subject positions in reference to discourses linking self, mind and mathematics.

'OCCUPATION' AS A THEORETICAL FOCUS

Occupation is usually considered to be something we take up after leaving school, but recent theories present occupation as much more than one's vocation or career. According to Kielhofner's (2008) model, human occupation encompasses the entire range of activities in which an individual engages and by which s/he self-identifies. Occupation in this view takes into account not only the paid work that we do, but also the many other ways we spend our time including our ventures, pursuits and recreation. These activities constitute our lives, and at the same time produce us as occupational beings, offering us a means by which we can know and describe ourselves. Occupation then, reflects those things that we personally regard as worthwhile, constrained and shaped by the discursive practices in which we engage. Occupation is not static, but an active, ongoing practice in which an individual is made visible as an 'occupied' being in a continuously unfolding occupational narrative. In adopting the term 'pupil career' to describe children's strategic engagement in the business of schooling, Pollard and Filer (1996) emphasised the role of schooling as occupational, thus schooling itself can be regarded as occupation, and occupation and schooling treated as inseparable.

Occupational subjectivity has been used as a concept by a number of sociologists interested in the applications of postmodern theory in their studies of *work* and *workers* (e.g. Bryant, 2006; Winecki, 2007). These researchers seek to describe how individuals actively take up subject positions through the choices and behaviours that constitute their 'occupation', and show that this process is necessarily political since 'doing occupation' is at the same time doing gender, class, disability, ethnicity and so on. As Bryant pointed out, subjectivity cannot be regarded as something one *has* as a fixed quality, but as something one *does* or chooses, and is therefore conflicted,

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 305-312. Thessaloniki, Greece: PME.

unstable and subject to the socially determined ‘availability’ of particular occupational positions.

> Subjectivity is contradictory and involves political choice between different subject positions and the discourses in which they have their meanings. Agency is not voluntary, involving free choice between subject positions (e.g. the choice of labourer or manager, farmer or farmer's wife). However, such choice is shaped by experience, access and interpretation of various discourses which have their basis in social and cultural institutions. (p. 63)

Occupational subjectivity begins early in life, and becomes a volatile dynamic for students as they near the time to leave school. This process of subjectification can be seen in children’s selection of school subjects and their justifications for such choices. Gottfredson (2005) who described occupational choice as an individual’s process of compromise between what is ideal and what is possible within socially constructed and occupational space suggested that the primary oppositional differentiators in occupational choice are the binaries of masculinity/femininity and high/low social status. Occupational choice then is not simply about what an individual does in life, but how the individual *becomes* an occupational self.

In her research demonstrating that mathematics education is both gendered and gendering, Walkerdine (1997) alerted us to the ways in which occupational choice is not ‘free’ but heavily circumscribed, girls and boys offered differential subject positions through their learning of mathematics that are implicated in and reflective of their social positioning in society at large. Mendick (2006) studied upper secondary school students’ selection of mathematics, and offered the idea that “‘identity work’ positions our choices as producing us, rather than being produced by us” (p. 23). In this view children are psychically active and distinctive ‘selves’ as well as socially interactive and connected beings in communities of practice; children choose to act in particular ways as learners not because of who they are, but in a continuous process of self-production.

RESEARCH METHOD

This paper uses data collected during an eleven-year ethnographic study that followed the mathematical careers of ten children from the beginning of their third year at primary school as seven-year-olds, to the end of their schooling and in two cases into the world of work. The purpose of the study was to investigate the loss of interest and concomitant decline achievement in mathematics that had been documented in large-scale quantitative research. The children were chosen at random from schools randomly selected from all schools in a large urban area in New Zealand. The group consisted of four girls and six boys. They were engaged in semi-structured conversational interviews over Years 3, 4 and 5 of their primary schooling, and again during Years 11, 12 and 13. During their primary years the children were observed in their mathematics classrooms, and their teachers interviewed. Their parents also participated in the study, offering important insights into how children’s learning of mathematics is circumscribed not only by schooling but also by family. By Years 11, 12 and 13, the children were making choices about subjects to study and were considering life beyond school, including tertiary study and careers. One of the children in the group left school early in Year 12, and another two at the end of Year 12. Three of the children attended schools in Australia, the UK, and Europe during the eleven years of the project.

DOING MATHEMATICS, DOING OCCUPATIONAL SUBJECTIVITY

The children in this study demonstrated that they viewed mathematics as a determinant, one way or another, of occupational subjectivity. This was reflected in the way they ranked subjects according to their occupational importance. When asked towards the end of Year 13 which subjects they believed were most important to study at school, maths and English figured highly for most of the children.

Liam: Definitely English. Probably maths as well, not so much now as being important generally unless I wanted to do something with maths in it, but up to Year 11 I think it's pretty important for general use. (Mid Year 13)

Rochelle: I think the most important would be maths and English, I don't know why, you do need to learn English and maths, 'cause otherwise you know, you'd be dumb wouldn't you? Well not dumb, but … (Mid Year 13)

In these statements, the importance of mathematics seemed to be related less to its practical use in everyday life than its role in creating occupational opportunity. The children could be seen as engaged in strategic choice-making in their learning of mathematics, including the ways in which they exercised learner agency during mathematics lessons. This is illustrated in Jared's comments.

Jared: I choose not to listen (*laughs*) because I don't like maths. (Early Year 12)

Jared: My friends were sitting right next to me. We'd distract each other all the time, flick paper, just like, mucked around quite a bit. We weren't really interested in the lesson so we just talked to each other the whole time. (Mid Year 13)

Jared did not enjoy or achieve highly in mathematics as a primary school student and by upper secondary school had adopted a position of withdrawal from the subject. His choice was occupational in the sense that it was tied to his future beyond school as a further comment revealed:

Jared: I don't think [the mathematics teacher] paid much attention to me. He paid more attention to the smarter kids, probably because they've got a future. (Mid Year 13)

In rejecting mathematics for its failure to interest him, Jared counted himself out of the opportunities that success in Year 12 mathematics afforded, such as entry into university. His choice was made not only within the circumscription of the 'smartness' he believed that he lacked, but also the socially sanctioned and endorsed complex of social interactions between friendship group, home, school and classroom. Jared and his similarly disaffected friends legitimized – for each other at least – the strategic action of disengagement from mathematics. The teacher endorsed this occupational positioning since he paid Jared little attention.

Equally disaffected with mathematics, Georgina talked of her choice to drop the study of what she termed ‘normal’ and ‘real’ mathematics.

> Georgina: In Year 10 when we were writing out the options we wanted to do in Year 11, I said I wanted to do Maths Numeracy instead of normal maths, and then my Dean and my maths teachers in the department of Maths had to talk about it between all of them to confirm me, because apparently it’s a lot easier than real maths, but for the kids that are actually in real maths it’s just average, it’s not hard and it’s not easy.
>
> Researcher: How do you think you would have gone if you’d chosen normal maths?
>
> Georgina: I’d have failed it, because it was different teachers and bigger classes, and harder stuff like geometry and algebra and Pythagoras. (Early Year 12)

Georgina’s differentiation between Maths Numeracy and Mathematics, in which she referred to the Mathematics option as ‘normal maths’ and ‘real maths’ positioned her in a group that was seen to be taking a much easier option that was by implication ‘not normal’ and ‘not real’ mathematics. Her choice appeared to be based on the self-prediction of failure in ‘normal’ maths. Georgina left school at the end of Year 12 and failed the numeracy test for admission to university. Instead she studied basic word processing skills at polytechnic and gained work in a call centre. She did not see her lack of mathematical proficiency as a disadvantage.

> Georgina: At college and at school I was always really struggling with maths and I was trying to work real hard … but [dropping maths] hasn’t disadvantaged me in any way at all and I don’t think [lack of maths] will ever stop me from doing anything. (17 years)

Planning for their occupational futures was the chief consideration in the children’s choices about studying mathematics. This was determined to a large extent by perceptions of their abilities in mathematics. Where mathematics was concerned, their choices were not simply confined to whether to continue their study of mathematics beyond the compulsory years, but which of the mathematics options to choose. Mathematics was the only secondary school subject split into two or more options in the final years of schooling. This was rationalized as catering for what was perceived as a much more significant gap between students’ abilities than occurred in other subjects.

Early in Year 3 the children’s parents talked about their hopes for their children’s learning of mathematics including how long they anticipated their children would take the subject. This was often linked to their thoughts about their children’s educational futures. Jessica’s mother hoped her daughter would take mathematics to Year 13, but had already begun to imagine her seven-year-old as an art student.

Jessica's mother: I hope she takes it right through secondary school. I hope it's going to be one of her main subjects … I mean I don't know her … she may decide to become an accountant or whatever. It's hard to…She'll probably be an art student or (*laughs*) … I can imagine her doing something like that. (Early Year 3)

When Jessica, a private school student, reflected on her choice of subjects, the same perceptual rift between the art student and the mathematics student could be seen.

Jessica: I think maths is definitely one of the more academic subjects. There are the more academic subjects and what people call the 'bum' subjects, but the thing about drama and P.E. and say, computer studies that don't take like, the 'smarts', it's more about, um, it doesn't take the stuff like maths does, like the intelligence and the quick thinking and things, it's more about your creativeness, it's kinda like art and things, not that that's a bum subject, but it's a different sort of learning, so it might be that they take those roads, it might be that they are more of a creative person than a logic person. (Mid Year 13)

Jessica's alignment of mathematics with logic, intelligence and quick thinking, and other subjects with creativity, illustrated that choosing mathematics (or not) was an act of occupational subjectivity. To choose or reject mathematics was to make a statement about the self. For all of the children, the choice about whether to continue studying mathematics, and what kind of mathematics, was tied to their perceptions of the value of mathematics in their lives. As soon as Fleur discovered that it was not necessary for the career path she was likely to choose, she dropped mathematics.

Fleur: I'm not doing [Year 12 mathematics], I'd rather do classics and history and geography … I wanna do psychology or sociology [at university]. I like the people [aspect] … Miss Highly said that you need maths in bio [tertiary biology] but the others you don't have to. (Early Year 12)

Fleur's teacher reinforced the widely-held perception that only science subjects required advanced mathematics. Fleur did not see herself as one of the best students at mathematics since it required a brain that could think 'the right way', and was relieved to drop it.

Fleur: I think my abilities are ok, I think they might be better than I think they are, but I still am not happy with it … my teachers always tell me and my parents, "Could do better, has the ability, needs to put in more effort, needs more confidence." … This will sound stupid, I think some people's brains are trained to think in the right way. (Early Year 12)

Fleur: I dreaded maths so much, like going to the class, so this year it was nice not to have to do it. There weren't many of us. They didn't realise that you could not take it and still get into university. (Late Year 12)

Within institutional structures of schooling, teachers often played a significant part in children's choices, their advice reflecting commonly-held beliefs that children are naturally ‘good’ at mathematics or otherwise. Dominic who had wanted to be a pilot since he was seven years old had to abandon this dream, as his mother described.

> Dominic is 16 in December & has just had to make a decision about whether or not to take mathematical methods for his year 11. His maths teacher advised that he needed to think about it very carefully as he would need to work extremely hard to do well. What he lacks intuitively needs to be supported by application apparently (& he's a minimalist). It's been a soul-searching week for him, because he's always wanted to be a pilot & without this particular maths he can't do physics & they're prerequisites for flight training. He's compromised & is now doing General maths A (whatever that is), because he still enjoys it & this one is easier apparently. Not enough to be able to do physics as well though. (Email, Late Year 11)

Dominic describes this incident in his own words.

> Dominic: My maths teacher thought [Maths Methods] might be hard for me, ‘cause I’ve never been, well you know, flash hot at algebra and that kind of thing, so yeah, she kind of advised me to do some maths which is the general kind of thing and to sort of go down the humanities path ‘cause I’m much better at that. If I was going to fight an uphill battle, you know, keep on the same tier as everybody else who can do [maths] sort of naturally, that was what I wasn’t really sure about, whether I could do something I’m good at and which I can get much better marks at, but I’m not really sure if I wanna do it, if you know what I mean…(Mid Year 12)

Dominic used the metaphor of ‘path’ to indicate the way subjects were arranged at his school to channel children in their ‘natural’ occupational directions.

The parents of both Peter and Toby had gained university qualifications and were working in highly-paid professions. From early in the study they said that they expected their sons to do well at mathematics.

> Peter’s mother: My husband thinks maths is the most important subject at school. And when it comes to choice of careers, if you’re good at maths, you’ve got a huge choice ahead of you ... (Early Year 3)

> Toby’s mother: Yes. I would like to think he would carry on doing quite well in maths. Take it right through [secondary school]. That’s what I can see at the moment. (Early Year 3)

Within family these expectations, Peter’s choice about mathematics was also tied to choices about science subjects, and weighing up where his strengths lay.

> Peter: I wanted to keep my options open so I took Stats … I was just hearing from other people, people who had done calculus; they just say it’s really difficult. (Mid Year 13)

Toby’s decision to continue with maths, and the choice of the statistics option rather than calculus was also influenced by other students’ choices and his perception that he was not ‘natural’ at maths.

Toby: It'll be a good thing to have for later in life and all that, yeah, it might keep some doors open…Amongst my mates [maths has] been like the sort of thing that they maintain even if they might not like it a lot, but, yeah, it's considered one of the important subjects so they seem to just continue it … you need it later in life … I heard there was easy credits in Stats, from older, [students] from last year, and, ah, yeah, I just sort of decided, didn't really think about dropping it. Most of the people in [the calculus class] are pretty good at maths, because to take Calculus you've got to be pretty good at maths … it seemed like too much work. Too hard. I'm not very natural at maths, so … (Mid year 13)

At Liam's school, where most students came from working class families, the choice to study mathematics was linked to perceptions of 'braininess' and only those who were considering tertiary study were likely to continue mathematics to Year 13.

Liam: I talked to my last year's maths teacher 'cause I got on with him real well. He said, like, if I put more time [into it] I should be able to do it, Level 3 … I think at our school everyone thinks it's only brainy people [who] try and do it, especially at Year 13, it's like calculus and stats, and calculus is the harder one, and when you look at the people in there and even the people in my class, it's mostly prefects and people that are hard-out like, striving to go university. (Mid Year 13)

In his statement, Liam identified perceptual links between intellect and mathematics, which in turn characterised students already occupationally marked through their selection as school prefects (leaders), their futures as university students both assumed and assured. Since Liam did not fall into these categories, the choice to study mathematics into his final year at school was conflicted.

Jessica and Liam noted strategic advantages that might be realized some time in the future in choosing to study mathematics:

Jessica: Maths is a good thing to do even from the point you should do things sometimes that you don't want to do even if they are not enjoyable or whatever, 'cause they're character-building or whatever, and sometimes, one day that lesson might come back and you know, be beneficial. (Mid Year 13)

Liam: At least I can say I've done [maths] to Level 3. Like in the future if I ever do come across any of that stuff at least I'll recognize that a little bit. I won't be totally like (*mimes looking very puzzled*) "Um, um …"

Rochelle, an indigenous student who saw herself as a capable student of mathematics, left school early in Year 12. The decision was linked to her social life, her disaffection with school in general, and the difficulty of the subjects she had chosen, including mathematics.

Rochelle: I don't think you really need it all, I mean, where are you going to use algebra? ... I had had enough of sitting down in the classroom listening to teachers … I think I chose the wrong subjects … I would have taken easy

> ones like tourism … One of my mates just dropped out and joined the army. But I knew that if I left school I'd have to get a job. (17 ½ years)

Launching herself into the workforce was something Rochelle saw as particularly challenging, her choice of school subjects creating an occupational dilemma.

CONCLUSIONS

The children's accounts illustrate that their choices about studying mathematics and continuing at school were primarily occupational, that they were social since they were made with reference to or in collaboration with teachers, friends and family, and that they were multi-faceted and conflicted. The children's occupational subjectivities were shaped by notions of themselves determined by the degree to which they exhibited a kind of 'mind' they had come to associate with mathematical ability, and how they positioned themselves, and were positioned by others, with reference to this apparently naturally occurring quality. The discourse supporting subject choice constituted the children not only as students of mathematics but as occupational beings at the same time. This accords with the views of Hardy (2004) and Walshaw (2007) who argued that children are made as subjects within the regulatory practices of mathematics education that determine the subject positions that students may take up. As the children actively counted themselves in or out of mathematics based on (self)judgements about mind, mathematics and subjectivity constructed within school and family, they produced themselves as gendered and classed social beings in their constructions of occupation, occupational self and occupational pathways.

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ENHANCING MATHEMATICAL IDENTITIES AT THE EXPENSE OF MATHEMATICAL PROFICIENCY? LESSONS FROM A NEW ZEALAND CLASSROOM

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Issues relating to student diversity are among the most complex and challenging issues facing mathematics education today. We wanted to know what teachers do when charged with developing the mathematical competencies of diverse students. In reporting on an investigation of teaching and learning within a Year 9 (age 13) classroom, we document the efforts and compromises made by the teacher in his attempts to take into account his students' sociopolitical realities whilst endeavouring to enhance their mathematical identities and proficiencies.

DIVERSITY WITHIN MATHEMATICS CLASSROOMS

Reform initiatives of recent years press for fundamental and complex changes to mathematics pedagogy and challenge deeply rooted beliefs about teaching, learning and curriculum content. These reforms respond to the realisation that specific groups of students continually register low proficiency levels in mathematics. They also recognise the challenge of student diversity inherent in mathematics classrooms today. We wondered what kinds of dilemmas teachers confront when working at developing the mathematical identities and proficiencies of disadvantaged students. Marshalling evidence about how teachers work at enhancing students' access to powerful mathematical ideas—irrespective of socio-economic background and out-of-school affiliations—is a primary necessity for mathematics education (Nasir & Cobb, 2002).

Like students in a number of other countries, many students in New Zealand, particularly from disadvantaged backgrounds, do not succeed with mathematics; they are disaffected and continually confront obstacles to engage with the subject (Anthony & Walshaw, 2007). Recent analyses of international test data have revealed patterns of social inequity that confirm a trend of systemic underachievement established over past decades (see Anthony & Walshaw). Findings set out by the Ministry of Education (2004) reveal that New Zealand results, compared with the results of the 32 OECD countries participating in the Programme for International Student Assessment (PISA), are widely dispersed. While 15-year-old New Zealand students performed significantly above the OECD average and are placed within the second highest group of countries, at the same time a high proportion of students are situated at the lower levels of proficiency.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 313-320. Thessaloniki, Greece: PME.

Cobb and Hodge (2007) have argued that issues relating to student diversity are among the most complex and challenging issues facing mathematics education today. However, diversity is part of the New Zealand way of life and changing demographics over the next decades will require mathematics teachers to cater for increasingly diverse groups of learners. We ask how does a teacher reconcile the identities students are invited to construct in the mathematics classroom with their identities within the practices of home communities, local groups and wider communities within society? Specifically, in what ways do students' 'diverse sociopolitical realities' (Ministry of Education, 2004), impact on the types of mathematical identities and the level of mathematical proficiency offered them in the mathematics classroom? As Cobb and Hodge have noted, finding out how such realities impact on mathematics teaching is substantially more urgent than at any previous time.

METHODOLOGICAL CONSIDERATIONS

Teaching is enacted with effect in various ways by different teachers and in different classroom settings with different students (Watson, 2002). Sfard (2005), along with many other researchers, has argued that classroom development is best researched with an emphasis on the social context of learning. The idea that teaching and learning are located within a complex social web draws its inspiration from Vygotskian (1978) ideas and the work of activity theorists. This body of work proposes a close relationship between social processes and conceptual development and is given a clear expression in Lave and Wenger's (1991) well-known social practice theory, in which the notions of 'a community of practice' and 'the connectedness of knowing' are central features. Thus, students' mathematical identities and proficiencies are developed within a complex web of relationships surrounding the organisation and facilitation of knowledge production.

We explored how identities are created and how proficiencies are developed in one classroom. The research is part of a larger study of three mathematics classrooms with three different schools. It represents the New Zealand component of the Learner's Perspective Study—an international research project involving researchers within Year 9 classrooms, providing in-depth coverage of teachers' and students' perspectives over a series of 10 consecutive lessons in the same classroom. The teacher in the study reported on here, like the other two in our project, had been identified by the local educational community as an effective mathematics teacher. His co-educational school caters for around 800 students from Year 9 (13 years of age) to Year 13 (17 years of age), from a range of backgrounds and diverse ethnic affiliations.

The principal data collection method for the research involved video recordings from cameras located as unobtrusively as possible within the classroom. The method produced a split-screen record of teacher and student actions. We used the record for video-stimulated recall interviews conducted immediately after the lessons with individual students. Data which we also draw upon specifically for this report from our large data set include interview material recorded on three occasions with the teacher. In those interviews, in response to watching the videos of the lessons, the teacher shared his beliefs about mathematics teaching and learning and discussed his specific pedagogical strategies.

The classroom video clips, supplemented by the interview material, provided a rich data base sufficiently complex to support an analysis of the dilemmas the teacher confronted in the classroom and the responses he made to resolve those dilemmas. Specifically, it deepened our sensitivity to how the teacher, whom we name here as Mr Polson, dealt with the diverse sociopolitical realities he confronted in his mathematics classroom comprised of low achievers. We explore his efforts through two aspects of his practice and characterise those primary organisers of the analysis as (a) developing mathematical identities; and (b) developing mathematical proficiency.

Key to abbreviations in analysis:

I	Interview
L1	Lesson 1
I, L1	Interview following lesson 1.

DEVELOPING MATHEMATICAL IDENTITIES

Mr Polson's practice is founded on an ethic of care: "You kind of take care of the kids in your class" [I, L5]. He wants his students to develop a sense of belonging and well-being and influences this by establishing a classroom space that is imminently hospitable. Researchers (e.g., Goos, 2004) have found that teachers who care tend to identify, recognise, respect, and value the mathematics of diverse cultural groups. As he points out, "I don't give praise individually because Māori kids don't like it" [I, L8]. In a similar way to the teacher in a study undertaken by Angier and Povey (1999), he works hard at developing a web of relationships that will allow every student to develop the sense of belonging that is essential if they are to engage with mathematics. In lesson 9, he encourages Terry:

Terry	Mr Polson, do I have to do this?
T(eacher)	Yes, because you are smart.
Terry	No I am not.
T	I think you are Terry. I think you are good man, a good man, a good man.

Watson (2002), in her landmark study with low-attaining students, found that teachers believed that students want to learn in a 'togetherness' environment. This is the kind of environment Mr Polson works hard at creating. As one student says, "we have got the coolest class ever." Mr Polson takes pains to ensure that he nurtures every student, irrespective of mathematical proficiency: "Phoebe is a step behind the others and I have to be careful what I ask her because I don't want to embarrass her" [I, L2]. When a new girl joins the class on day 3 of the research he asks her at the end of the lesson to read out the answers to the day's mathematics problems. "I guess she sees that as being part of the class—that she has contributed something…The class has seen her, and they have heard her voice" (I, L8).

The caring and social nurturing that takes place in this classroom implies respect. As Michelle explains that it's "a two way thing because to get respect you have to respect back, like you have to give them respect before you get respect back." Respectful practice involves listening carefully to students. Michelle makes the point:

> He actually listens to what we are saying because most teachers don't. They don't really care what we are saying and they think they are always right all of the time and they won't listen to our side of it. Mr Polson takes it on and thinks about it instead of just letting it go.

Careful listening demonstrates a respect and intellectual support for students on the part of the teacher (Hackenberg, 2005). However, lack of consistent student attendance in mathematics class presents a dilemma for the teacher focused on providing the affective support that will promote cognitive development.

> I think we had 26 or 27 students today. The other day we were about 15…It makes it tricky. One boy had 50% attendances a term. Kylie at the back of the room left school two weeks before the end of term one and didn't attend any school then came back at the end of the second term. So they come and go. Another girl Danielle was away a whole week last week and Hemi was also away a whole week…So it's an interesting class because of that. [I, L3]

Uneven daily engagement is also an issue: "The first day of the week they are pretty hyper[active]; after a long weekend particularly. As the week goes on they get much more settled" [I, L3]. The economic and material disadvantage that goes hand in hand with non-attendance and differential lesson engagement operates to structure the overall learning opportunities that are made available to the class.

DEVELOPING MATHEMATICAL PROFICIENCY

Teachers build their practice on specific beliefs about learning and knowing mathematics. Mr Polson is confident in the reasoning powers of his students. He says: "Part of it is to do with their confidence in their own knowledge. Part of it is to do with them trusting me as a teacher" [I, L10]. Like the teachers in Watson's (2002) study, the value he places on students' thinking and reasoning influences the way in which the students view their relationship with mathematics. Within the supportive community Mr Polson has established, students are confident in the soundness of their mathematical identities. All ten students interviewed believed they were good at mathematics. When asked what makes a good mathematics student, they responded variously, "listening and taking everything in and talking at the appropriate times," "being cooperative," "responsible when you are putting your hand up to talk," "trying your hardest one hundred percent," "paying attention and doing everything right," "knowing your times tables," and "doing the work." However, the kind of mathematical identities the students had been encouraged to create, tend to be inflated, developed at the expense of higher level thinking and deep conceptual understanding. In the following excerpt taken from lesson 9, the class has been working out how many tenths there are in 3.28.

T	What is three times ten, Rowena?
Rowena	Thirty.
T	So what do you think the answer might be if three times ten was thirty, what do you think the answer might be?
Rowena	I don't know sorry.
T	What is three times ten?
Rowena	Three times ten is thirty.
T	What do you think three point two times ten might be?
Rowena	I don't know.
Michelle	32.8.
T	That's right. How many tenths are in that altogether? Thirty-two point eight. (writes down a rule for multiplying decimals by 10). Rowena, just for you, because I know you love them. We have got a math law. A rule, but you could write 'law' if you like.

After the lesson Michelle (M) was interviewed about her answer:

I:	You volunteered an answer. So to multiple 3.28 by ten, what do you do?
M:	You put the decimal point back one.
I:	Why do you think you do that?
M:	I don't know.
I:	Is that something you learned today?
M:	Yes.
I:	So you would be fairly confident in 4.37 by 10?
M:	43.7
I:	So you have learnt something today but trying to explain it is a bit difficult, about why it would work.
M:	Yes, it's something I would remember forever. It's like the same thing with 3.2. I know it would be 32.0

The interviewer moved to inquire about her understanding of decimal ordering.

I:	0.021 and 0.87. How could you know which is the smallest or largest using Mr Polson's rule? [The rule: "When we want to compare decimals we can 'make them up' to the same number of decimal places."]
M:	0.021 is higher because it has more digits behind the decimal point so that is how I know. And if they have the same number of digits you put them in the order of numbers.
I:	If they have got different digits you look for the one that has the most digits?
M:	Yes.
I:	Do you understand his rules? Did you understand this or did you just copy this down and not think too much about it?
M:	He said we can do our strategies or his strategies.
I:	So it is a rule to fall back on if you are not sure?
M:	Yes if you don't know what you are doing.
I:	What would you do for this problem? [order from largest to smallest: 3.210; 3.87]
M:	That one is bigger because it is 210 because I have got the same for the first number. I would take the first number off and then I take away the decimal point and it's like 210 and 87.
I:	So the 210 is the bigger one?
M:	Yes.

Too often in mathematics classrooms, as Thompson (2008) has noted, students are provided with limited opportunities to experience significant mathematical ideas. When confronted with the task of promoting mathematical reasoning, and given the reality that many students in his class register poor attendance levels, Mr Polson makes much use of rules, repetition of tasks, and small procedural steps. He explains his actions in this way: “You have to make a judgment sometimes when you are going to push it and where you are not….I have always felt that people have got different capabilities and you don’t have one size fits all” [I, L10]. Earlier he had noted: “You make decisions as you go along and you make decisions according to feedback that you are getting from the class. Sometimes you will interpret the right way and sometimes you won’t” [I, L5].

His tendency to simplify a task into small procedures is, however, received with favour by students. John says: “It’s easy because he does it step by step.” As Michelle, who is absent from many classes, points out, “I am able to keep up—he doesn’t move that fast.” And as Kelly volunteers: “the way he tells us the questions—he gives them in an easier form, an easier way to put it so we can work it out easier.” In direct opposition to this view, the teachers in Watson’s (2002) study believed that tasks should challenge and that support should be provided for students to task risks. Conceptual development comes from the opportunities provided for them to engage in higher levels of thinking. In Mr Polson’s classroom such opportunities were profoundly limited.

CONCLUSION

Instructional practice is a complex and multilayered practice (Anthony & Walshaw, 2007). The reforms of recent years have pressed for fundamental and complex changes to pedagogy that acknowledge the major issue of student diversity. Such reforms map out how the core dimensions of mathematics teaching might enhance the proficiencies of students from diverse communities. Examining how teachers respond to these challenges provides insight about the dilemmas teachers face and, additionally, contributes to our overall understanding of teaching and learning.

The teacher in the study developed an overall goal structure that prioritised affective support. Foremost in his mind was his care of his students. His objective was to use the resources of a caring community to develop his students’ sense of self as mathematical learners. In this respect he achieved his goal. Through the participation structures he had developed within the class, and through his careful listening and attentive feedback, he enhanced students’ confidence in their own mathematical abilities, enabling them to identify strongly as sound mathematical learners.

Like all teachers, he also hoped to enhance their mathematical proficiency. However, although he had developed an acute sensitively to his students’ affective needs, the pedagogical strategies he employed at times constrained the possibility of deep conceptual knowing. Whilst he firmly believed in his students’ mathematical reasoning powers, that belief did not have the full measure of his practice. When confronted with the realities of poor student attendance and attention, his instructional practice became oriented more towards "teaching for mechanistic answer finding" (Even, 2008, p. 63) than towards providing opportunities for his students to think, reason, communicate, reflect upon and critique what they did and said in class.

If the primary aim of mathematics education is provide students with the means to participate in a microcosm of mathematical practice, by which they learn how to appropriate mathematical ideas, language and methods, then research that taps into classroom life has the potential to illuminate how teachers might best achieve that aim. Shaping students' higher level thinking involves significantly more than developing a respectful, trusting and non-threatening climate. It involves socialising them into a larger mathematical world that honours standards of reasoning and particular ways of mathematical speaking and thinking. Given the sociopolitical realities that are present within everyday classrooms, finding out how teachers achieve that task is more urgent today than ever before.

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SCORING STUDENT RESPONSES TO MATHEMATICS PERFORMANCE ASSESSMENT TASKS: DOES THE NUMBER OF SCORE LEVELS MATTER?

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This study investigates whether using different numbers of score levels to score student responses to mathematics performance assessment tasks would result in different score rankings of student responses. The validity of score interpretation and use depends on the fidelity between the constructs to be measured and the scores obtained from the assessment (Messick, 1989). This study finds that the ranking orders of obtained scores can be changed significantly by altering the number of score levels used in task-specific scoring rubrics that are developed based on the same conceptual framework of the performance assessment program.

STATEMENT OF THE PROBELM

Performance assessments have been used as a valuable tool to evaluate both the process that a student employs in problem solving and the product that the student presents at the end of the problem-solving process. To ensure the validity of performance assessment results, Messick (1994) suggested "where possible, a construct-driven rather than task-driven approach to performance assessment should be adopted" (p.22). In a construct-driven approach to the design of scoring rubrics, a generic or general rubric is typically developed in the early stages of a performance assessment program to judge the knowledge and skills required for student performance at each score level. This generic or general rubric provides guidance to the subsequent development of a specific scoring rubric for each assessment task and "helps ensure consistency across the specific rubrics and it is aligned with the construct-centred approach to test design" (Lane & Stone, 2006, p. 395).

Usually, the number of score levels is specified in the generic or general rubric and criteria at each score level are delineated according to the construct of the assessment domain. It is very common that this specified number of score levels (e.g., 3- or 4-points) is kept the same across task-specific rubrics. Lane and Stone (2006) provide a number of examples of the use of a generic rubric along with task-specific scoring rubrics to develop a construct-driven scoring rubric.

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In practice, the complexity of performance assessment tasks varies, even within the same assessment instrument. For a task requiring less complex thinking skills or involving less steps to solve, it may not be easy to find an intermediate process that corresponds to criteria specified in the generic rubric for a middle score(s). On the other hand, for a task involving more complex problem-solving processes, disagreements often arise in scoring a partially correct solution when applying the criteria from the generic rubric to the task. As a result, applying the same score criteria across different tasks in developing task-specific rubrics may not result in the same interpretation of the scoring levels that are specified in the general rubric.

Messick (1994) recommended that scoring rubrics should not be "specific to the task nor generic to the construct but in some middle ground reflective of the classes of tasks that the construct empirically generalizes or transfers to" (p. 17). Achieving a balance in the "middle ground" while using the same number of score levels across tasks is truly a challenge in practice. Lane and Stone (2006) pointed out that "[T]he number of score levels used depends on the extent to which the criteria across the score levels can distinguish among various levels of performance. The performance reflected at each score level should differ distinctively from those at other score levels" (p.395). Therefore, it is instructive to investigate whether using different numbers of score levels to score student responses to the same tasks would result in the same construct being measured.

PURPOSE OF THE STUDY

This study attempts to determine whether using different numbers of score levels to score student responses to mathematics performance assessment tasks results in different score rankings of student responses. Two assessment tasks were selected from a mathematics performance assessment program that is designed to measure student learning of algebra in middle school. After a generic holistic scoring rubric was developed, two task-specific scoring rubrics were used to score student responses to each of the tasks. The two scoring rubrics differed in the number of score levels. One used the number of score levels specified in the generic rubric and the other altered the numbers of score levels, depending on the complexity of the task's problem-solving process. A comparison was made to examine whether there was a significant difference between the obtained student scores when different numbers of score levels were used to score each of the two performance assessment tasks. In particular, this study answers the following research questions:

1) Do scoring rubrics with different numbers of score levels result in the same ranking of students' obtained scores?
2) Do inter-rater agreements differ when using scoring rubrics with different numbers of score levels?

METHODS

Mathematics Performance Assessment Tasks

Two tasks were chosen from among the tasks in a performance assessment program, each task represented a unique aspect of knowledge in algebra and each required a different level of cognitive demand in the problem-solving process. In particular, Task A involved a relatively small number of steps to complete, but Task B required a greater number of steps to complete. The solution to each task required students to apply their knowledge of middle school algebra or pre-algebra.

Task A assessed a student's problem-solving skills in a map-reading context involving ratio and proportion. Task A requires students to demonstrate their understanding of a proportional relationship involving a map scale and a real distance scale by obtaining a missing value.

Task B was presented in a spatial-geometric context using dots arranged in trapezoids. It requires students to identify regularities to extend a geometric pattern and to effectively communicate these regularities by explaining how they got their answers.

Scoring Rubrics

Before developing task-specific scoring rubrics, the generic holistic scoring rubric developed for a middle-school mathematics performance assessment was adopted (Lane, 1993; Silver & Lane, 1992). The generic rubric specifies five levels of scores (i.e., 0 to 4 points). Subject matter experts, including college professors in mathematics education and educational measurement programs, mathematics teaching specialists, and graduate students in mathematics education programs worked together to develop 5-level task-specific rubrics based on the generic rubric for each task in the assessment instrument. The subject matter experts also developed a brief description of each of the score levels. This helped ensure that the task-specific rubrics could be used consistently across all the tasks in the assessment instrument. The following is a brief description of the 4-point score levels specified in the holistic scoring rubrics:

<u>4 points</u>: Answer and explanation must show a correct and complete understanding.

<u>3 points</u>: Answer and explanation must be basically correct and complete, except for a minor error, omission, or ambiguity.

<u>2 points</u>: Answer and explanation should show some understanding, but it is otherwise incomplete.

<u>1 point</u>: Answer and explanation show a limited understanding.

<u>0 points</u>: Answer and explanation show no understanding of the problem or without any explanation.

Another task-specific scoring rubric with an altered number of score levels was also developed for each of the two tasks chosen for the study. The altered number of score levels was tailored for each task by considering the cognitive demand of the task, and the number of steps involved in solving it, as well as the scoring criteria specified in the generic rubric. Appendix A provides the task-specific rubric with the altered number of score levels for Task A. Because of the relatively small number of steps involved in solving Task A, the second task-specific rubric used score levels ranging from 0 to 3 for this task. For Task B, the second task-specific rubric used 0- to 8-point score levels since more steps were involved in the task solution.

Participants

The mathematics performance assessment program used in this study was designed for use in the U.S.A. National Science Foundation funded project titled Longitudinal Investigation of the Effect of Curriculum on Algebra Learning (LieCal) (Cai & Moyer, 2006). The LieCal project investigates the effects of using the Connected Mathematics Program (CMP) on algebraic learning in middle school as compared to more traditional middle-school mathematics curricula. The LieCal Project is being conducted in 16 middle schools of an urban school district serving a diverse student population. In the school district, 27 of the 51 middle schools have adopted the CMP curriculum while the remaining 24 middle schools are using other curricula. Eight CMP schools were randomly selected to participate in the LieCal Project from among the 27 schools that have adopted the CMP curriculum. After the eight CMP schools were selected, eight Non-CMP schools were chosen based on comparable ethnicity, family incomes, accessibility of resources, and state and district test results.

The mathematics performance assessment was administered in the fall of 2005, the spring of 2006, spring of 2007, and spring of 2008. The assessment in the fall of 2005 serves as the baseline data. Each spring, three-performance assessment forms were administered with five tasks requiring an array of algebra knowledge and problem-solving skills. Of the five tasks, two were common across the assessment forms and three were unique to each form.

Student responses to Tasks A and B in the spring of 2007 administration were used in this study. These includes 1242 students responses to Task A and 414 responses to Task B. The different number of responses obtained is due to the fact that Task A appeared on all three forms and Task B was only used in one form.

Raters and Rater Training

During the summer of 2007, eight middle school math teachers from geographic areas far from the LieCal schools participated in scoring sessions for the LieCal project. The eight teachers were grouped into four pairs to score student responses to each of the assessment tasks administered in the spring of 2007. The four pairs of raters scored all student responses to the assessment tasks using both scoring rubrics. A training session was held before each task was scored. During the training sessions, the teachers (1) worked on the task and discussed the knowledge involved, possible solution strategies, and possible errors; (2) reviewed the scoring rubric along with a set of student responses with scores; (3) independently scored a set of student responses; and (4) discussed in groups the results of the independent scoring and possible reasons for disagreements.

RESULTS

Rater Agreements

Reliability of test scores is a necessary condition for the establishment of the score validity. Inter-rater reliability is one source of evidence for score reliability. Table 1 shows the inter-rater reliability coefficients for each of the tasks corresponding to each type of scoring rubric. For each task, there is a high inter-rater agreement for both scoring methods.

Table 1 Inter-rater Reliability Coefficients

Task	0 to 4-point Score Levels	Altered Number of Score Levels
A	.943	.982
B	.938	.964

Tables 2 and 3 show cross-tabulated frequency distributions of score differences from pairs of raters using the two scoring rubrics for Tasks A and B, respectively. Chi-square tests of the frequency distributions did not show significant differences for either Tasks A or B. This indicated that inter-rater agreements did not differ significantly between the two scoring methods.

Table 2 Cross-tabulated Frequency Distributions of Score Differences from Pair of Raters—Task A

		Frequency of score differences using the 0 to 4-point scoring rubric					Total
		0 points	1 point	2 points	3 points	4 points	
Frequency of score differences using the altered number of score levels (0 to 3 points)	0 points	916	239	19	5	3	1182
	1 point	29	23	3	0	0	55
	2 points	2	2	0	0	0	4
	3 points	0	0	0	0	1	1
	Total	948	264	22	5	3	1242

Table 3 Cross-tabulated Frequency Distributions of Score Differences from Pair of Raters—Task B

		Frequency of score differences using the 0 to 4-point scoring rubric			Total
		0 points	1 point	2 points	
Frequency of score differences using the altered number of score levels (0 to 8 points)	0 points	251	46	1	298
	1 point	77	17	0	94
	2 point	14	7	0	21
	3 point	0	0	1	1
	Total	343	70	1	414

Differences in Obtained Test Scores

Table 4 presents the descriptive statistics of the students' obtained scores using each of the task-specific scoring rubrics for Tasks A and B, respectively.

Table 4 Descriptive Statistics of Obtained Scores

Task	N	0 to 4-point Score Levels			Altered Number of Score Levels		
		Mean	SD	Score Range	Mean	SD	Score Range
A	1242	1.70	1.75	0 – 4	1.09	1.34	0 - 3
B	414	1.53	1.16	0 – 4	2.81	2.44	0 - 8

To examine whether the use of the scoring rubrics having different numbers of score levels resulted in the same ranking of students' obtained scores, nonparametric statistical tests, including both the Wilcoxon signed ranks test and the exact sign test, were conducted. The statistical analyses revealed that significantly different rankings of the obtained scores were found for each of the two tasks. For Task A, Wilcoxon signed ranks Z-statistic was 14.802 ($p<.001$) with a similar statistic from the exact sign test ($z = 15.395$, $p< .001$). For task B, Wilcoxon signed ranks Z-statistic was 6.228 ($p<.001$) with a similar statistic from the exact sign test ($z = 5.487$, $p< .001$).

SUMMARY

The validity of score interpretation and use depends on the fidelity between the constructs to be measured and the scores obtained from the assessment (Messick, 1989). The validity of a performance assessment involves both the nature of the tasks presented to students and the scores used to evaluate student performance (Taylor, 1998). By altering the number of score levels used in task-specific scoring rubrics that are developed based on the same conceptual framework of the performance assessment program, raters can reliably score student responses regardless of the scoring methods. However, this study found significantly different ranking orders of obtained scores. Nevertheless, this study is significant because it provides a forum to investigate further the influence of scoring rubrics on the validity of performance assessment results, e.g. whether the maintenance of consistent score levels across tasks also presents challenges to the validity of scores obtained from a construct-driven approach to the development of the task-specific scoring rubrics. In addition, findings from this study have potential theoretical and practical implications for scoring student responses to open-ended tasks.

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Appendix A Task-Specific Scoring Rubric for Task A

Mathematical elements of this task:

- Proportional Reasoning
- Scaling
- Unit conversion

	Points Earned	Section Points
What is the actual distance between Martinsburg and Rivertown?		
Correct Answer to actual total distance (*216 miles*)	1	
No answer or incorrect answer	0	**1**
Show how you found your answer.		
Shows sound, logical reasoning process through the use of words, diagrams, pictures, number sentences and includes proper calculations. Answer must show all steps involved in solving (the work may include minor computation errors). • Example 1- 54+54+54+54=216 with a graphical division of the distance from Martinsburg to Rivertown into 4 sections. The four segments should be clearly shown. • Example 2- 54×4=216 with the segment from Martinsburg to Rivertown is 4 times as long as the segment from Martinsburg to Grantsville. The distance from Grantsville to Martinsburg is 54 miles, so if you go between Grantsville and Martinsburg 4 times around the distance will be 216 miles. • Example 3- 12÷3=4 and 54×4=216. • Example 4- 54÷3=18 and 18×12=216. • Example 5- $\frac{54mi}{3cm}=\frac{xmi}{12cm}$, x=216.	2	
Shows proper and correct calculations but omits steps • Example - *54×4=216* (with no other explanation). • Example - *54+54+54+54=216* (with no other explanation).	1	
Little or no explanation of work. OR Incorrect explanation of work – explanation lacks or has very weak logic. Example- *If you multiply 54 and 12 you get 564 which is your answer.*	0	**2**
Total Points		**3**

THE SOCIOMATHEMATICAL NORMS IN THE ELEMENTARY GIFTED MATHEMATICS CLASSROOM

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The major goal of this research is to investigate the establishment and content of the sociomathematical norms constructed in the mathematics classroom for elementary mathematics-gifted 5th graders. The research is based on the theory of sociomathematical norms proposed by Yackel & Cobb (1996).

Through the method of classroom observation, the research discovered three sociomathematical norms established in these mathematics-gifted students and they are "acceptable mathematical explanation", "mathematical efficiency" and "mathematical difference". In addition, these three sociomathematical norms are distinctively different from those constructed in the normal students. The research offers possible explanation for this disparity between the mathematics-gifted students and normal students and proposes some possible significance and application of how these sociomathematical norms may have influence on the learning of the mathematics-gifted students.

INTRODUCTION

One of the important goals of current mathematics curriculum reform is to shift away from teacher-centered teaching to student-centered teaching. And the role of teacher is to become a facilitator in supporting students to learn mathematics (NCTM, 2000).One of the advocated teaching method is the discussion method. Teachers are expected to actively conduct the mathematics instruction and guide students' learning activities in the classroom (Lampert, 1990; McClain & Cobb, 2001).

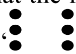
Under such premise, the mathematics meanings generated from the mathematics activities in the classroom are constructed as a process of the social interaction between the teacher and students. There are some norms that might be constructed to promote the discussion method to run smoothly in the classroom. They can be divided into two categories: (1) social norms, in which they can be applied to all subjects; (2) sociomathematical norms, in which they are characteristics only in the mathematics class (Yackel & Cobb, 1996). Some examples of the sociomathematical norms include what the definitions of "different" mathematics solution, "sophisticated" solution, "efficient" solution, and what can be stated as acceptable "explanation" and "justification" (Yackel & Cobb, 1996; McClain & Cobb, 2001). The researches from Kazemi & Stipek (2001), McClain & Cobb (2001), Pang (2001) all agreed that the construction of sociomathematics norms can enhance students to develop their mathematical problem-solving ability, encourage them to investigate question and

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 329-336. Thessaloniki, Greece: PME.

seek for different solutions actively, as well as facilitate students' intellectual autonomy.

The research on promoting the gifted students' intellectual autonomy carries substantial significance since they have higher intelligence, faster development and usually have greater contribution to the humankind. The major goal of present research is to investigate the sociomathematical norms constructed in the elementary mathematics-gifted students in their mathematics class using discussion teaching method. Furthermore, a dialogue between our results and related literature is concluded to compare the similarities/differences of sociomathematical norms between normal students and the mathematics-gifted students.

LITERATURE REVIEW

The present research is an attempt to understand the sociomathematical norms constructed in the elementary mathematics class for the mathematics-gifted students. The theoretical foundation is based on the constructivism, symbolic interactionism and ethnomethodology from the sociomathematics norms research done by Yackel & Cobb (1996). They found some sociomathematical norms including "acceptable mathematical explanation and justification", "mathematical difference", and "mathematical sophistication". For an instance, when students calculate the problem of "16+14+8=?", students think that "10+10=20, 6+4=10, 20+10+8=38" and "8+4=12, 12+10=22, 22+10=32, 32+6=38" are different mathematical solutions. Furthermore, as McClain & Cobb (2001) carried on Yackel & Cobb (1996) research, they found that as students tried to solve the graphic problem of "", there are at least two solutions from students: (1) counting solution; (2) grouping solution, i.e. "1+4=5". Students think that counting solution and grouping solution are two different solutions and they think that the later one is the more sophisticated solution. And for the graphic problem of "", as students asked to compare the solution of "2+2+2=6" (adding two each time) and "4+2=6" (using a group of 4 and adding two), they think that the later one is a more efficient solution.

Yackel & Cobb (1996) argued that the construction of sociomathematical norms can facilitate the development of intellectual autonomy in students. In the process of constructing the sociomathematical norms between the teacher and students, students would carry out the practice of discussion, explanation, dialectics and choosing. They would gradually evaluate mathematics problems based on their knowledge and intelligence and would not necessarily follow their teacher's perspective. During the process of negotiating with other students, their ability of using mathematical language to communicate, analyze and solve problem is developing progressively and consequently their intellectual autonomy in learning mathematics is also improved.

Since the mathematical ability of the mathematics-gifted students is noticeably advanced of the students of same age, they usually can offer unique and extraordinary explanations and solve the problems from many different perspectives (Colangelo, Assouline & Cross, 2004). The research done by Yackel & Cobb (1996) and McClain & Cobb (2001) was aimed for elementary 1st and 2nd graders. The present research is targeted at elementary mathematics-gifted 5th graders and it is attempted to discover some different content and quality of sociomathematical norms as not being reported in the literature.

RESEARCH METHODS

The research method is periodical classroom observation to gather the specifics on how sociomathematical norms are constructed in the elementary class for mathematics-gifted students.

A. Research Subjects and Place

The research is based on 42 mathematics-gifted 5th graders. These students took lessons in a university of education on weekends and the setting of the classroom is similar to their classroom.

B. Data Gathering

The research lasts from Sept. 8th, 2007 to June 21st, 2008. During this period, these elementary mathematics-gifted students and instructor had 24 lessons. The teacher-student and student-student interactions in these lessons were analyzed and served as the evidence of how some sociomathematical norms are constructed. The data included notes during classroom observation and video-tapes of interactive teacher-student discussion. And the data included verbal and non-verbal information from teacher and students during the discussion.

C. Data Analysis

The present research is based on the sociomathematical norms reported in the research by Yackel & Cobb (1996). In the research results, the symbol T is designated as the instructing teacher, the symbol S1, S2, etc. is designated as individual students and the acronym of Ss represents the whole students.

RESEARCH RESUTLS

The following is an excerpt from one of the teacher-students interactions as they were solving a mathematic problem. It is used to explain the content of the sociomathematical norms constructed in elementary mathematics-gifted students. The mathematical problem is stated as the following, “From the numbers of (2,3,4,5,6,7), if one is to pick any three different numbers, how many ways to make their sum an even number?”

three even numbers $C_3^3 = 1$

two odd and one even $C_2^3 \times C_1^3 = 9$

1+9=10

A: 10 ways

S1’s solution

S1: For "three even numbers", there is only one way (i.e. 2,4,6). For "two odd numbers and one even number", there are three ways to pick two odd numbers from the three odd numbers and there are also three ways to pick an even number from the three even number, so <u>3×3=9.</u> So the answer is 10.

S1 used the combination symbol (i.e. C^3_3) to represent his calculation, but he didn't connect his explanation with the combination symbol. He didn't explain why there are three way to pick two odd number out from the three odd numbers, either. The topic of combination is taught in senior high school mathematics in Taiwan, so for those students who never learned combination might not understand the meaning of this calculation presented by S1.

Then, S2 presented another solution, as shown below.

$$\frac{\cancel{6}\times5\times4}{\cancel{3}\times\cancel{2}\times1}=20 \text{ ways}$$

(1) three odd numbers
(2) two even numbers
and one odd numbers
} sum is odd

$$20-1-\frac{3\times\cancel{2}}{\cancel{2}}\times3$$
$$=20-1-9=10$$

A: 10 ways

S2: There is a total of 20 ways to add up three numbers. To have the sum to being even number, one needs to minus the odd-numbered sum. ... Let's try two even numbers plus one odd number. If one is to fix the odd number to being 3, then there are 6 (3×2) ways that the sum would be even number. But why divides by 2? Because for example, if the two even numbers are 2 and 4, but (2,4) and (4,2) are the same so I divide by 2. And because there are three odd numbers, one needs to multiply by 3. So there are the answer is 20–1–9, there are 10 ways to make the sum an even number.

From the abovementioned data, it showed that S1 and S2 have different solution and different path of thought on the same problem. S1 solved the problem according to what is asked for in the question and S2 solved the problem by eliminating what is contradicting with the problem.

The teacher attempted to stimulate clearer explanation from S2. So the teacher asked S2 some further questions.

T: S2, how do you come up with this calculation, □ $\underline{\frac{6\times5\times4}{3\times2\times1}=20}$ □*?*

S2: Because there six numbers, so <u>6×5×4</u>.

⋮

S3: Why divide by <u>3×2×1</u>?

S2: For example, <u>ignore this (using hand to cover 6), 5 and 4, 5×4 is a combination, means two numbers. However, (2,3) and (3,2) are the same so you need to divide by 2.</u> Why divide by 3? Let's take (2,3,4) as an example. 2 can placed here (2□□□□), and it can be placed here (□□2□□), *and it can also be placed here (□□□□2). So you have three combinations: (2,3,4), (3,2,4) and (3,4,2). And then you need to multiply by 2 as they are counting twice (using hand to point at the other two □), so it is 3×2×1.*

T: Understand?

Ss: Yes. (Not many students are responding, and S4 raised his hand for further explanation.)

T: S4 is volunteering to help S2 out.

S2 had learned combination before, so he uses "6×5×4" to explain the all possible ways to take three numbers out from six number. However, S3 haven't learned combination so he didn't understand the solution offered by S2.

S2 simplified the question by using two numbers, covering up "6" first and demonstrating how to find all possible ways for choosing two numbers out of five numbers (5x4 ways of combination). And then he used (2,3) as an example to explain that the sum of "2□3" and "3□2" are the same so one needs to divide by 2. And then he used (2,3,4) as an example and drew an analogy from two numbers to three numbers. However, there were not good responses from students and S4 raised his hand for volunteering to offer supplementary explanation. Researchers believe that S4's active behaviors demonstrate that he already understood and accepted the explanation of S1 and S2. S4 supposed that other students might not get the solution offered by S1 and S2 and he believed that he could explain more clearly and he hoped to share his thoughts.

$$\boxed{C_3^6 = \frac{6\times5\times4}{3\times2\times1} = 20}$$

S4: When you are choosing numbers, if numbers are the same but only being placed differently, they are considered identical. The symbol "C" is denoted as combination. The calculation on the top,"<u>6×5×4", means the total possible ways of combination, and the calculation on the bottom means the numbers we are choosing.</u>

T: If you understand this symbol (C), please raise your hand. (Most students raised their hands but some didn't). There are some people didn't see this symbol before, S4, please explain.

S4: For example, for (2,3,4) and (2,4,3), if one is to count ways of permutation, they are regarded as two different permutations. But now (2,3,4) and (2,4,3) are considered as same because they have the same sum. If one puts the number "2" first, then there are two way to place "3" and "4". And "2" can be placed in the position of "3" or "4". So there is a total of "3×2×1". (S4 wrote on the board: 234, 243, 324, 342, 423, 432).

T: Understand?

Ss: Yes. (Good responses from students).

S4 thinks actively and independently and tries to explain using words like "if numbers are the same but only being placed differently, they are considered identical" and "(2,3,4) and (2,4,3) are considered as same because they have the same sum" to explain the difference between permutation and combination. And using words like "If one puts the number "2" first, then there are two way to place "3" and "4". And "2" can be placed in the position of "3" or "4"" to amend the original explanation. He also wrote the six possible ways to permute (2,3,4) as an effort to explain the mathematical meaning of the denominator "3×2×1". And his explanation got much better responses from other students, meaning that his explanation is more acceptable by his fellow classmates. And S4 further explains:

> *S4: If one is to take two even numbers and one odd number, there are three ways of choosing an odd number. To take two even numbers, let's consider to take all even numbers and throw out one by one, ruling out 2, then 4 and then 6. So there are also three way. So 3×3=9....*

S4 and S2 offered different explanation on how to pick "two even numbers and one odd number". S4 explained, "To take two even numbers, let's consider to take all even numbers and throw out one by one, ruling out 2, then 4 and then 6"; and S2 explained, "If one is to fix the odd number to being 3, then there are 6 (3×2) ways that the sum would be even number." S4 might consider that the explanation offered by S2 is difficult to understand and believe that he had more efficient way to explain the combination of picking two even numbers and one odd number. Therefore, researchers conclude that not only may students have different solutions but also for a same solution, they may have different path of thoughts and interpret differently.

> *T: Who writes out all the possible ways of combinations, please raise your hands. (Some students raise their hands).*
>
> *S7: Writing all them out?!*
>
> *S8: It's fast.*
>
> *T: If the question is "<u>100×99", when will you be done writing?</u>*
>
> *Ss: Haha!*

The reason why the teacher asked the number of students who wrote out all possible ways of combination is to facilitate students to distinguish different ways of solutions and to stimulate them to think how to solve a problem using a more efficient way. From the response of S7, "Writing all them out?!", and the final laughter from all students, researchers assume that most students think that writing out all possible ways of combinations one by one is a less efficient solution.

CONCLUSION & DISCUSSION

There are three sociomathematical norms demonstrated in the mathematics classroom for these mathematics-gifted students and they are "acceptable mathematical explanation", "mathematical efficiency" and "mathematical difference". For the present research, the sociomathematical norm of mathematical difference is embedded in mathematical efficiency so only the first two sociomathematical norms will be discussed. And they are also being compared with the sociomathematical norms constructed in the mathematics class for normal students.

(1) Acceptable Mathematical Explanation

The establishment of “acceptable mathematical explanation” is more difficult for the gifted students than the normal students. There are three possible phenomena when normal students, who have similar intelligence and mathematics learning experiences, are solving mathematics problems: (1) understand the required knowledge to solve the problem and can retrieve the knowledge successfully; (2) understand the required knowledge but fail to retrieve the knowledge successfully; (3) have some experiences of the required knowledge but do not fully understand and hence cannot retrieve the knowledge successfully. For that reason, the major part of establishing the sociomathematical norm of “acceptable mathematical explanation” in normal students is to “recall” fellow classmates’ related experiences and/or knowledge to solve the problem. On the other hand, there is a greater difference of intelligence and mathematics learning experiences among mathematics-gifted students and consequently their mathematical ability ranges significantly. Other than to “recall” related experiences and/or knowledge like in normal students, the mathematics-gifted students may also need to take on the task of teaching to “generate” new knowledge among their fellow classmates. For an instance, for students who presenting their solution using the combination symbol (C^6_3), they need to teach other students who haven’t learned combination yet before they can understand their solution. In other words, a more and wider range of cognition and knowledge is required in the mathematic-gifted students to construct the sociomathematical norm of “acceptable mathematical explanation”. Comparatively, through this process of establishing this sociomathematical norm as they have more input and stimulation, the mathematics-gifted students enjoy better development in intellectual autonomy and independent learning.

(2) Mathematical Efficiency

Due to the substantial difference in mathematical ability among mathematics-gifted students, the efficiency of their solutions also diverges vastly. For instance, while some students are capable of using combination to solve the problem, some students still use the least efficient way of writing out all the possible ways to solve the problem. Furthermore, even for same solution, there is also some differences in how to explain the solution efficiently. For example, even though both S2 and S4 use combination to solve the problem but they have different approaches on explaining how to choose “two even numbers and one odd number” and the explanation of S4 is more efficient than the one provided by S2, as validated by their fellow classmates’ responses. This kind of phenomenon, in which different explanations generate different mathematical efficiency, seem to be absent in the mathematics class for normal students. One of the reasons may be due to the difference of the complexity in solving problems between the gifted and normal students. The construction of the norm of mathematical efficiency, including the efficiency of using different solutions as well as the efficiency of explaining the same solution, can facilitate the mathematics learning in the gifted students and enhance their mathematical ability. And hence it is valuable to understand how to promote the construction of this kind of sociomathematical norm to improve mathematics learning in the mathematics-gifted students.

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IMAGINARY DIALOGUES WRITTEN BY LOW-ACHIEVING STUDENTS ABOUT ORIGAMI: A CASE STUDY

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Within the framework of commognition by Sfard (2008) thinking is seen as self-communication and in commognitive research discourses become the unit of analysis. A case study is presented here about a learning environment, where low-achieving students built three-dimensional origami models and reasoned about the mathematics behind them. They wrote imaginary dialogues in which they let protagonists talk about their mathematical thoughts. This kind of mathematical discourse of a single 16 year old low-achieving student is analysed against the background of Sfard's framework.

INTRODUCTION

Thinking as communicating

Anna Sfard defines thinking as an "individualized form of (interpersonal) communication" (Sfard, 2008, p. 91) or shorter as "self-communication" (ibid., p. 128). Furthermore, she uses the term *commognition* for both, the processes of thinking and communicating, to reveal their developmental unity. In the framework of commognition, mathematics is seen as a discourse (ibid., p. 129), which for example can be colloquial or literate. The mathematical discourses become the "principal object of inquiry" (ibid. p. 276), which can be analysed under various questions like how and when discourses change or what the roles of the interlocutors are. Sfard also states that "mathematical self-communication may be difficult to observe" (p.276). *Thinking* which is thought as a personal channel is invisible, whereas the interpersonal channel as in a dialogue can be perceived. When vocal or written human talk is used as data, in commognitive research the verbatim use of words is regarded along with the interaction which are conducted.

Mathematical writing

Mathematical writing has been used and explored in different forms in mathematics education. Borasi & Rose (1989) describe advantages journal writing has for students, their interaction, and for their teaching. Gallin & Ruf (1998) investigated how journal writing becomes a written dialogue between teachers and students. Shield and Galbraith (1998) distinguish between two categories of mathematical writing: journal writing and expository writing. They also point out that mathematical writing can enhance understanding. Three modes of mathematical writing were investigated by Clarke, Waywood & Stephens (1993): Recount, Summary and Dialogue, where Dialogue means an internal dialogue. They stress the value of the

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 337-344. Thessaloniki, Greece: PME.

latter, where students name difficulties and "reasons why there are thinking in a certain way" (ibid.). A different form of written internal dialogue was studied by the first author (Wille, 2008). Students wrote imaginary dialogues between protagonists who talk about different mathematical tasks or questions.

Origami in mathematics education

> The whole point of having the students construct origami polyhedra is for the hands-on experience to build conceptual understanding of the objects that they build. (Hull, 2006, p. 135)

There are three kinds of literature about origami. There are publications that contain paper folding diagrams, others regard the mathematics behind origami and some discuss also the use of origami in mathematics education. In a book of Simon, Arnstein & Gurkewitz (1999) one can find several folding diagrams in order to build polyhedra. Row (1966) describes various kinds of paper folding of polygons and discusses the underlying mathematics. Haga (2008) presents in his book a variety of mathematical explorations that can be made with origami paper folding. He also formed the concept *origamics* which means scientific origami in contrast to the pure paper folding. Franco (1999) and Hull (2006) developed several learning environments for the use of origami in mathematics education.

LEARNING ENVIRONMENT

Origami task

In the centre of the learning environment stands a modular origami unit called *Sonobe unit* named after the Japanese origamist Mitsonobu Sonobe (see Figure 1).

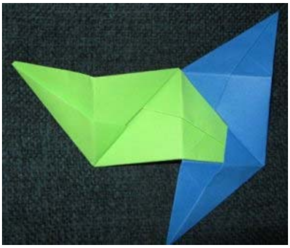

Figure 1: two Sonobe units

By plugging the units together different polyhedra can be built (see Figure 2).

Figure 2: four Sonobe models

In the beginning the students learn to fold some easy origami models and the Sonobe unit. They get no folding diagrams but complete models that they unfold and analyse by themselves. After this the students get the following *Sonobe exercise*:

> By folding many equal two-dimensional units and putting them together a three-dimensional model can arise.
>
> 1. You get 12 Sonobe units. Try to put them together to a three-dimensional model.
>
> 2. Count the edges, vertices and flat faces of your model. Can you calculate the number of Sonobe units out of them? Is your calculation also valid for the other models that you see on the table and can you find out how many Sonobe units were used to build them?

Sonobe units are also explained by Franco (1999) where she calls Sonobe units *Star-Building Units*. Different to the learning environment above, Franco does not discuss the relation between the number of edges, vertices and faces of a Sonobe model and the number of Sonobe units. Folding diagrams for Sonobe units can also be found in Simon, Arnstein & Gurkewitz (1999).

The mathematics behind the task

At this point we will only discuss the relation between flat faces and Sonobe units, since in the case study here presented only this relation is mentioned by the student.

The flat faces of a Sonobe model have the shape of a triangle or a square, which can be separated into two congruent triangles. For example if one counts the number of flat faces of a Sonobe cube, one gets six square faces or twelve triangular faces. In the following we only regard the triangular faces. Each Sonobe unit has two pockets and two points. A point is later put into a pocket of another Sonobe unit (as in Figure 1) in order to build a Sonobe model. Therefore each triangular face of a Sonobe model consists of two different Sonobe units. On the other hand one Sonobe unit is part of four triangular faces. Therefore if u denotes the number of Sonobe units and f the number of triangular faces, we get the relation

$$2u = f.$$

METHOD

The study was carried out in 2008 and 2009 with low-achieving students of age 14 to 16. Some of them went to a grammar school (Gymnasium) and some to a secondary modern school (Realschule) in Niedersachsen, Germany. They met four to five times in small groups. In this case study we will only focus on one 16 year old student out of this group, who we will call Lara in the following.

Since in the commognitive framework of Sfard the mathematical discourse of students is the unit of analysis and since *thinking* is seen as *self-communication* which is difficult to monitor, the data for this study mainly consists of the students' *written self-communication*. The students wrote *imaginary dialogues* in which two or more protagonists talk about different tasks. Using imaginary dialogues in mathematics education is also part of a long term study of the first author.

In each learning unit the students were asked to write an imaginary dialogue between two protagonists talking about how to fold the Sonobe units or how to build a Sonobe model out of them. The Sonobe exercise mentioned above contained also the following task:

> Write an imaginary dialogue while you are thinking, in which you try to answer to the questions above.

While the students were writing and folding, they were videotaped at the same time. The transcripts were also part of the data for the qualitative analysis. That way the interaction as well as the written texts could be regarded.

Although of course that the students let their protagonists communicate with each other, we cannot expect that we see how the students think. But we can explore what kind of inner dynamics can be seen in their imaginary dialogues and which mathematical ideas are presented. In a long term study the changes in the imaginary dialogues also can be investigated. But the latter will not be the focus of the case study here presented.

The research questions behind this case study are:

- Are imaginary dialogues suitable at all for low-achieving students?
- Are mathematical thoughts which are mentioned by low-achieving students comprehensible in their imaginary dialogues?
- Which forms of imaginary dialogues are used?
- Which learning environment of imaginary dialogues might lead to a productive mathematical process?

FINDINGS

In Figure 3 one can see the Sonobe model that Lara built out of the given twelve units. It has both square and triangular faces.

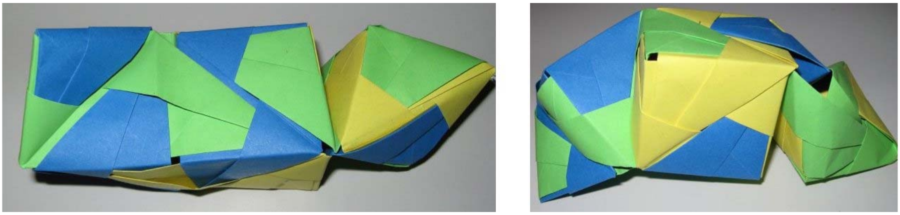

Figure 3: Lara’s Sonobe model from two sides

After Lara gets the exercise she builds her Sonobe model for 19 minutes. Then this she gives it to the teacher. Having it back Lara turns her model in her hands with a concentrated face. A few times she takes another model from the table. After 6 minutes she has a Sonobe model built out of 30 units in her hands and asks the following:

Lara:	Is this the one with twelve units?
teacher:	No, that is this one. (points at another Sonobe unit)

After this dialogue Lara only takes her model and another model that is built out of twelve units in her hands turning them constantly around. After another 6 minutes she looks for 21 seconds on her blank sheet of paper and suddenly takes her pen. She smiles. First she writes “15 edges”, “24:2” and “24 faces” with a pencil on the margin of the paper. Then she takes a blue pen and with a concentrated face writes again her imaginary dialogue down with only a 7 seconds break between the first and the second sentence.

Lara uses in her imaginary dialogue the name of her older sister for one protagonist and her own name for the other. We call her sister Katy here. Lara’s imaginary dialogue originally is in German. The translation is the following dialogue.

1	Katy:	How do I find out how many Sonobe units I have built within a three-dimensional solid?
2	Lara:	Well, you must calculate all faces of your three-dimensional solid.
3	Katy:	But how do I do that, if I have partly triangular faces and partly square faces?
4	Lara:	Then you count nevertheless only the triangular faces. Therefore, if you have a cube for example, then one face counts as two faces.
5	Katy:	Okay, I understand. And now?
6	Lara:	After this you calculate[1] the number of your face by 2, because always 2 units stick together.
7	Katy:	Wait a moment, on this solid (my own) I count 24 faces.

1 Lara uses the German expression “rechnest” which means “calculate”, but here it can also be translated by “divide”.

8	Lara:	Yes, that's is right. Then you must just divide the number of the faces by 2.
9	Katy:	Okay, that is 12.
10	Lara:	Exactly. That's already your solution. In your three-dimensional solid 12 Sonobe units were used.

Under the dialogue she writes with a pencil:

"Number of faces = 24

→ divide by 2, because always 2 are connected with each other.

24 : 2 = 12 units"

Form of the imaginary dialogue

Three characteristics can be noticed in Lara's imaginary dialogue:

- One protagonist asks several questions while the other one answers.
- Lara names the protagonists after herself and her older sister, and the one who explains has her own name. (On a different occasion Lara explained to the teacher that her sister usually is the one who is smarter.)
- At one point the protagonists seem to intermix. In (7) Katy says "my own", although the Sonobe model was built by Lara.

In imaginary dialogues of other students one often finds a form, where there is an initial question, then a long answer and at the end a sentence like "thank you, that helped a lot". There are also imaginary dialogues in which the protagonists communicate symmetrically, i.e. where both have ideas, explain and ask. Lara uses an alternating dialogue which gives room for questions in between. That Lara uses her own name for the one who explains might show that she feels herself competent to elucidate what she has found out. Only in (3) Katy names the key to the solution in her question, since she refers to the difference of square and triangular faces. Finally the intermix "my own" in (7) could indicate that the imaginary dialogue is part of an internal discourse of herself. Another possibility is that she and her sister usually share things with each other.

Lara's explanations

In Lara's explanations the following can be noticed:

- The explanations are set up in small steps. In (5) for example Katy is satisfied that she has understood the last step and asks for the next.
- Examples are mentioned as in (4) and in (7) to (10) where the whole exercise is repeated.
- Lara uses in (2) and (6) the German words "berechnen" and "rechnest" which means calculate, even that the words for counting and dividing, respectively, seem more adequate.

- Not only an algorithm is mentioned to calculate the number of Sonobe units but also an explanation is given in (6) ("because always 2 units stick together") and another in the notes after the imaginary dialogue.

We can assume that Lara thinks that explanations in small steps are promising or maybe others often explain mathematics to her in small steps. She might also understand tasks well, if they are explained with examples.

The whole repetition in (7) to (10) starting with "Wait a moment" points out that in (6) a difficult statement, namely "After this you calculate the number of faces by 2, because always 2 units stick together", was uttered. It is probable that Lara means that two Sonobe units stick at each triangular face. That might have been the focal point of her thoughts, since although the protagonist Lara already said how to calculate, the protagonist Katy wants to repeat the calculation at this point.

The use of the verb "calculating" can indicate that Lara thinks of mathematical tasks in an operational way, where one needs to calculate something. But Lara also goes beyond the pure calculation and gives a reason for the division by 2. Already in (3) she mentions through the protagonist Katy the difference between triangular and square faces. Moreover, after the little break of 7 seconds after writing her sentence (1) she wrote continuously her imaginary dialogue until the end. Therefore this could be a sign that at this point, at least before writing sentence (2), she had her explanations already in her mind.

Mathematical thoughts within the imaginary dialogue

In Lara's explanations, that is why one divides the number of triangular faces by two, there is a little gap. Lara names the key idea to count only the triangular faces and that two Sonobe units are part of each triangular face as in "because always 2 units stick together" in (6) and in "because always 2 are connected with each other" in her notes at the end. By the use of only these arguments the calculation resulting in the number of units would have been: number of triangular faces *times* two, not *divided* by 2. Lara does not name that each Sonobe unit is part of four triangular faces, why we really need to divide by 2 at the end. It seems probable that she did a turnaround mistake in her reasoning triggered by the numbers 24 and 12 of her example. Nevertheless, her algorithm is correct and she names two of the main ideas for the explanation.

For Lara the operational way to receive the number of Sonobe units is in the center of her imaginary dialogue. We do not find a structural correspondence between the number of triangular faces and the number of Sonobe units as in an equation.

SUMMARY

Finally, we can resume the research questions from above and answer to them regarding Lara's mathematical discourse:

- For the low-achieving student Lara the mathematical writing in form of an imaginary dialogue was suitable to express and explain her algorithm and her mathematical reasoning behind it.
- Her writing is comprehensible and clear, apart from the fact that we do not know, if she really did a turnaround mistake in her reasoning or if she actually thought about the Sonobe units that are part of four different triangular faces.
- Lara's imaginary dialogue has an alternating and asymmetrical form, where one protagonist explains and the other one asks several questions. That way she was able to point out a difficulty in understanding the mathematical reasoning, namely the explanation in (6) which is followed by "Wait a moment" in (7).
- Finally, regarding Lara, the learning environment with origami Sonobe units led to a productive mathematical process, which could be detected in her imaginary dialogue. How the imaginary dialogue itself might have contributed to the mathematical process cannot be answered at this point.

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SPONTANEOUS STUDENT QUESTIONS: INFORMING PEDAGOGY TO PROMOTE CREATIVE MATHEMATICAL THINKING

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The questions students asked themselves, or their group, whilst collaboratively constructing new mathematical knowledge are examined in data drawn from three studies (upper elementary, lower secondary, and upper secondary students). Lesson video of group interactions and post-lesson video-stimulated interviews captured student talk in groups, and group discussions in whole class settings. It was found that students tended to structure further exploratory activity by spontaneously formulating questions that were not specific to the mathematics within the task at hand but that helped to explore this mathematics. It is argued that these types of questions could be adapted for use by teachers to support the development of creative mathematical thinking where students or groups are not yet asking such questions.

INTRODUCTION

'Spontaneous' questions students ask during group or individual problem solving (e.g., Dreyfus & Tsamir, 2004; Dreyfus & Kidron, 2006; Williams, 2003; 2004; 2005; Wood, Williams, & McNeal, 2006) have been studied using clinical interviews (e.g., Dreyfus, Hershkowitz, & Schwarz, 2001; Dreyfus & Tsamir, 2004), classroom discourse (e.g., Groves & Doig, 2004; Wood, Williams, & McNeal, 2006), small group discourse (e.g., Barnes, 2000; Wood, Hjalmarson, & Williams, 2008), student post-test interviews (e.g., Cifarelli, 1999), student post-lesson interviews (Hershkowitz, 2004), video-stimulated post-lesson interviews (Williams, 2003; 2005) and reflections on written notes and computer printouts produced whilst developing new knowledge (Dreyfus & Kidron, 2006). Spontaneous, for the purpose of this paper refers to activity that is not teacher directed and does not arise from teacher 'hinting' or 'telling'. As spontaneity is associated with creativity (Csikszentmihalyi & Csikszentmihalyi, 1992), and creative mathematical thinking is integral to developing deep mathematical understandings (Burton, 1999), spontaneous student questions that do not provide direct mathematical input could inform pedagogy that promotes this process. Although some of the student questions identified do provide specific mathematical input (e.g., see Cifarelli, 1999), the three studies selected for analysis in this paper include many questions that do not provide such input. These questions are the focus of this study. Research Question: Are there spontaneous student questions that teachers could ask to stimulate further thinking without providing mathematical input. If so, how could they be adapted to eliminate teaching directing or controlling of student exploration?

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THEORETICAL FRAMEWORK

Williams (2005) identified six activities in the 'Space to Think', which were student activities associated with the creative development of new mathematical knowledge. These students were in classes where such student activity was not the specific intention of the teacher. Even so, they manoeuvred their own space to think because the problems they were given provided opportunities to discover mathematical complexities peripheral to, beyond, or deep within the problem set. The six activities in the Space to Think are:

1. Inclining to explore or enacting optimism (Seligman, 1995)
2. Spontaneously identifying a mathematical complexity and formulating a question to explore it (Williams, 2000)
3. Manoeuvring cognitive autonomy to enable control of own thinking
4. Autonomously accessing mathematics not being told what to use (Williams, 2002b)
5. Spontaneously pursuing an exploration rather than following the directions of another (Williams, 2002b)
6. Asking questions to structure further exploration (adapted: Cifarelli, 1999).

This analysis focuses predominantly on student questioning or Activity 2 and 6 in the Space to Think. It extends beyond these questions to examining the types of thinking the questions promote using Williams' (2005) thinking framework in that analysis (of Activity 5). This integrates Krutetskii's (1976) problem solving processes of 'analysis', 'synthetic-analysis', 'evaluative-analysis', 'synthesis', and 'evaluation' with Dreyfus, Hershkowitz, and Schwarz's (2001) 'observable cognitive elements' associated with the process of abstraction in context: 'recognising', 'building-with', and 'constructing'. Recognising is the process of realising what mathematics is appropriate to use in a certain context, or recognising a context in which it is appropriate to use certain mathematics. In ascending order of complexity, categories of complex thinking in Williams' thinking framework are: Recognising, Building-with (including Analysis, Synthetic-analysis, Evaluative-analysis), and Constructing (including Synthesis, and Evaluation). When recognising occurs as part of evaluation, the next cycle of complex thinking associated with this is more complex than the previous process. The categories within this framework are illustrated in Table 2, and also in Williams (2007a). Activities 1, 3, and 4, although equally important are not the focus of this study. See Williams (2006) for more information.

RESEARCH DESIGN

The context, research subjects, and data collection instruments are described and links between data collection instruments and types of data required are discussed.

Upper-elementary school class: A group of students in a Grade 5/6 class in a government primary school in Australia (pseudonyms: Eliza, Patrick, Eriz) worked together on three problem solving tasks across a school year. This group (which sometimes included another member) had creatively developed new knowledge in the previous two tasks. In the task under study, they were asked to make as many of the numbers from 1-20 as possible using only four of the digit four and a combination of operations. Fours and operations could be reused to make additional numbers. See Williams (2008) for more information about the task and student interactions.

Junior secondary school: One Year 8 student pair (Leon, Pepe) within the international Learners' Perspective Study (LPS) creatively developed new knowledge even though such activity was not the explicit intention of their teacher. They extended the teacher task 'find the number of rectangles with perimeter 38 cm'.

Upper Secondary: William and Talei, a pair of students in a higher-level calculus class in their final year of secondary schooling, worked with a task intended to develop an understanding of second derivatives. These students had not previously been exposed to second derivatives or sign diagrams. The graph of y = f(x) provided was approximately a cubic in appearance. An equation was not given. A sign diagram of the function was drawn and explained by the teacher and sufficient notation provided for students to draw further sign diagrams. Notation for second derivatives was provided f''(x) and described as representing the gradient of f'(x). See Williams (2002a) for more information about the task and the student interactions.

Study: School Type; Student Pseudonyms	Video of Student Groups in Class	Whole Class Reporting At Intervals From Each Group	Student post-lesson interviews
1. Government Grade 5/6 Class	Yes	Yes	Video stimulated
2. Government Junior Secondary Class	Yes	Not formal part of approach. Some across class sharing in the instance reported.	Video-stimulated
3. Government Secondary School Year 12 Class	Yes	Yes	Informal: Usual practice; casual after class discussions

Table 1: Studies and data collection instruments

The data collection instruments employed in each study are summarised in Table 1. In each study, pairs or groups were videotaped as they undertook a challenging problem set by the teacher or teacher-researcher or developed by the students using the teacher task as stimuli. The studies are listed in chronological order from most recent (1) to least recent (3). The researcher-teacher in Study 1 was the teacher in Study 3 and the teacher in Barnes (2000) and Groves and Doig (2004). Her approach

(Class Collaboration, see Williams, 2000) was used in Study 1 and 3 with refinements to improve the pedagogy in Study 1.

In particular, dynamic visual images were developed by students as they worked with the problem solving task in the classroom in Study 1. This refinement resulted from identifying Activity 4 in the Space to Think where students who did not possess appropriate cognitive artefacts to progress with their exploration sometimes developed relevant mathematics as they focused idiosyncratically on dynamic visual displays in the classroom or on the workbooks of others. For further description of Class Collaboration, see Williams (2000, 2007b).

RESULTS AND ANALYSIS (SEE TABLE 2)

Groups selected to illustrate the nature of spontaneous questions were representative of other groups undertaking creative thinking. Spontaneous student questions asked (see Table 2, Column 3) were often appropriate for other explorations rather than just the problem at hand. Table 2 shows the group and who posed the question (underlined in Column 1), the context (Column 2), the complexity of thinking before the spontaneous question and the complexity of thinking stimulated by this question (Column 4). These more general questions stimulated activity like considering more than one representation or solution pathway at the same time without indicating which ones might be useful. They included eliciting the making of judgements about correctness, reasonableness, or elegance, which is a process of 'evaluative-analysis'.

Table 2 also shows these spontaneous questions each stimulated an increase in the subcategory of complex thinking (e.g., synthetic-analysis complexifying to evaluative-analysis) [Table 2, Row 3]. The rows of Table 2 are arranged in ascending order from least to most complex thinking to illustrate the complexifying of thinking and the increased proximity of ideas as the process becomes more complex. The nature of the thought processes and the ways they progress to more complex processes, along with a greater transparency in the progressive connecting of ideas is evidenced as the ideas develop.

DISCUSSION AND CONCLUSIONS

There are two aspects of Table 2 that are crucial to understanding how spontaneous questions can stimulate creative mathematical thinking.

Firstly, the questions are general rather than of the more specific type posed by Cifarelli's (1999) students in test situations. They do not focus directly on the mathematics within the task so do not tend to eliminate spontaneous student thinking by telling and hinting through questions. Elegance has been associated with curtailment (Krutetskii, 1976), which can involve realising the equivalence of some attributes (see Williams, 2004). The student/s can then use the representation or solution pathway they decide is most appropriate and 'unpack' other mathematics as needed.

Year, Students	Context	Spontaneous Question	Initial and Stimulated Complexity of Thinking;
1. Grade 5/6 Patrick, Eliza, Eriz	Students sharing three minutes initial individual work on the task with their groups.	What could I start with that I would find a little more difficult?	Initial: **Analysis**. Stimulated: **Building-with, Synthetic-analysis;** simultaneous use of several 'difficult' operations.
2. Grade 5/6 Eliza, Patrick, Eriz	Students try to find dimensions of box made of 32 small cubes with only 24 cubes to build with	How can we represent this when we do not have enough resources?	Initial: **Building-with, Analysis**; used trial and error. Stimulated: **Building-with, Synthetic-analysis**; used 3D models and 2D diagram simultaneously to solve.
3. Year 8 Leon, Pepe	Teacher asked for other dimension of a rectangle when one is known.	Can I find a faster way?	Initial: **Building-with**, **Synthetic-analysis,** examining several approaches. Stimulated: **Evaluative-analysis,** judgement made by considering physical and numerical simultaneously
4. Year 12 Class William, Talei	Moved pen along two graphs simultaneously. Each student attended to points and graphs of interest to them until both focused on the absence of a critical feature for the turning point (in f''(x)).	All the rest seems fine, but what is happening here? Why does this happen?	Initial: **Evaluative-analysis**, check consistency of fit with other mathematical knowledge. Stimulated: **Constructing, synthesis**, realised why the particular pattern related turning points on the original graph and f"(x): 'It's always turning the same way!"
5. Grade 5/6 Eriz, Patrick, Eliza ** Questioner underlined	Draws upon group finding about boxes being made up of layers of cubes. Adds his own understanding of how to find the number of blocks in a given box.	Is there a rule that always works and will eliminate the need for a lot of counting?	Initial: **Building-with, synthetic-analysis**, counts number of blocks in layer; multiplies by number of layers. Stimulated: **Building-with**, **Evaluative-analysis**, linked pattern and number of blocks in cube. His interview suggested he knews why this pattern works.

Table 2: Spontaneous questions and how they can stimulate more complex thinking.

Secondly, Table 2 shows that thinking tends to shift from less to more complexity in progressive steps as a result of responding to spontaneous questions. In addition, ideas become increasingly intertwined. For example, students progresses from analysis (separating components to examine them), to synthetic-analysis (two or more aspect considered simultaneously), to evaluative-analysis where judgements are made as a result of synthetic analysis and the equivalence of attributes begin to become apparent.

If teachers could find ways to ask questions similar to these spontaneous student questions, but ask them in ways that do not control or direct the avenues students explore, there is potential to increase creative student thinking. The teacher questions in Barnes (2000), Groves & Doig (2004), and Williams (2007b) could assist in modifying these spontaneous student questions to fit with pedagogy intended to elicit creative thinking. As the teacher in each of the three aforementioned papers who has demonstrated the ability to pose questions to elicit creative thinking (see Barnes, Groves & Doig, and Williams), I will adapt some of the spontaneous student questions reported herein to elaborate my meaning. When asked by a teacher, these student questions are more like 'wonderings' than direct questions: [adapting Question Row 1, Table 2 adapted] "I wonder what types of things you will try first … will you stretch to try ideas that are almost out of your reach or do questions you can easily do … I wonder what you will find most useful?" Responses to these questions from students are not expected although various types of responses may be made. If the teacher responds to these in ways that do not indicate an affirmative or negative, then opportunities for creative thinking remain. With the adapted questions for teacher use, there is a hesitancy and sometimes a indication that the teacher may not presently be aware of solution processes: [adapting Question Row 2, Table 2]: "If you find you do not have enough resources, can you think about other ways to proceed because I am not providing more resources. I am interested in what ways you decide to think about this?" Again, the teacher responds in similar ways to student responses. It will be interesting to see what happens when these types of questions are trialed by more classroom teachers. This will provide opportunities to further test the generalisability of these findings, and opportunities to study the complexifying of thinking, and to find more spontaneous student questions to adapt.

ACKNOWLEDGMENTS

The International Centre for Classroom Research (ICCR) supported data collection and analysis for Studies 1 and 2. Financial support for data collection was provided through a Deakin Central Research Grant (Study 1), and the Learners' Perspective Study (Study 2). The University of Melbourne, and the MAV also provided scholarships to support my studies . My thanks to all concerned.

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ALIGNMENT BETWEEN TEACHERS' PRACTICES AND PUPILS' ATTENTION IN A SPREADSHEET ENVIRONMENT

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The research reported here builds upon a body of literature that involves exploring the potential of technology to support the learning and teaching of algebra. More specifically, it focuses upon the interactions between teachers and 11-12 year old pupils during spreadsheet-based activity and identifies themes which account for variation in pupils' longitudinal trajectories. This paper develops one of these themes, the construct of alignment, which describes the relationship between teachers' practices and pupils' focus of attention. A chain of episodes from a case study is used to illustrate the construct, which remains relatively stable over time.

BACKGROUND

Researchers studying the use of spreadsheets typically point to the use of algebra-like notation and the activities of writing formulae and graphing as offering access to the meaning of symbolic notation and to algebraic activity. As a generic tool, spreadsheets offer the potential of developing mathematical meanings for such notation and activity. This study was informed by the social constructivist paradigm and took the perspective that mathematical meanings are guided by teachers and negotiated in social interaction. It sought insight into the ways in which spreadsheet experience and teachers' pedagogic practices shape pupils' construction and evolution of meanings for algebra.

Technology and teachers' practices

Hoyles and Noss (2003) identify a common research trajectory for studying the potential of digital technologies in mathematics education:

> 'one that starts by documenting potentials and obstacles in software use and then gradually shifts to discussions of tool mediation, tasks and activities, and the role of the teacher' (p.16).

Relatively few studies take teachers as an explicit and central focus. Whilst a number of researchers reflect upon pedagogy, studies tend to focus primarily upon the technology and learners rather than on the role of teachers, the interaction between learners and teachers, and sociocultural practices.

Reflection on the affordances and limitations of technology such as spreadsheets points to the need to analyse transparency in light of how the technology is actually used. The design of tasks and the practices of teachers play vital roles in shaping pupils' activity and thinking, and hence the kinds of algebra that the technology affords. Referring to a recent systematic review of the role of technology in the

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learning of algebra, Goulding and Kyriacou (2007) note that technological tools 'could encourage each individual student to develop their own very idiosyncratic knowledge which may or may not accord with the common knowledge the teacher was intending to develop' (p.38), and conclude that 'the mathematics teacher has a pivotal role in structuring and supporting the learning' (p.39). Several researchers have called for more attention to teachers' practices in using technology (for example, Ferrara, Pratt and Robutti, 2006) and Haspekian (2005) has recently signalled the complexity for teachers of integrating spreadsheets into their practices.

Overview of the study

This study has drawn upon data collected for the *Purposeful Algebraic Activity* project[1], which involved the design of purposeful spreadsheet-based tasks (see Ainley, Bills and Wilson, 2005). These formed the basis of a teaching programme for pupils in the first year of secondary school (aged 11-12). Pupils across the attainment range were represented in five classes, who participated in approximately twelve hours of spreadsheet-based activity over the year. The pupils' usual teachers collaborated with the project team throughout and taught all of the lessons. The experienced teachers had the freedom to use the teaching materials as they wished, and made decisions about introducing tasks and offering interventions, for example. Additional data collection involved tracing one of the classes through follow up work in their second year of secondary school (aged 12-13).

This paper reports on one aspect of the longitudinal study, the interaction between teachers and pupils. Focusing on the social construction of meaning, and the relationships between pupils' meanings and teachers' practices, it attempts to address the question 'How do pupils' meanings evolve over time?' The practice of what teachers *actually do* and the way that this shapes (and is shaped by) pupils' activity, has received relatively little attention in the literature, both in relation to technology and more broadly.

DATA COLLECTION AND ANALYSIS

Two main sets of data were collected in the project: observation data from the teaching programme lessons; and interview data providing 'snapshots' of pupils' achievement over three years. Data from over seventy teaching programme lessons, which is the focus of analysis here, includes audio recordings from a radio microphone of teachers' interactions with pupils. These interactions included work with the whole class, often using an interactive whiteboard, and discussions with individuals, pairs and small groups when circulating the room. Field notes, including chronological jottings of non-verbal activity, were also made. In each lesson video and screen recordings were taken of a targeted pair of pupils.

In addressing the question of how pupils' meanings evolve over time, the aim was to understand the 'big picture' through synthesising commonalities in pupils' trajectories and accounting for longitudinal patterns, whilst embracing the micro-level analysis of rich classroom interaction. The analysis of the data in the whole study involved four phases using NVivo software: coding pupils' meanings; coding teachers' practices (see Wilson and Ainley, 2007); case analyses; and building a conceptual framework. These phases of analysis reflect the theoretical framework in moving towards studying the evolution of pupils' meanings within the social cultural context of the classroom. The latter phases, which are more relevant to the focus of this paper, involved analysing changes over time and developing a conceptual framework that accounts for pupils' evolving meanings within a social context.

The development of case analyses involved drawing upon Cobb and Whitenack's (1996) methodological approach for conducting longitudinal analyses of large sets of qualitative data such as classroom video recordings and transcripts. The method considers individual pupils' mathematical learning within a social context, and involves identifying regularities in social aspects that remain stable across chains of episodes. This phase involved analysing episodes chronologically for four themes: the pupils' meanings (as previously coded); the social relationship between the pairs of pupils; the classroom context, including the learning opportunities available; and shifts in the pupils' thinking. Drawing on Mason's (2002) notion of account of and account for, the aim was to offer an *account of* pupils' evolving meanings through chains of episodes and to develop conjectures that *account for* the trajectories. Similarities and differences between episodes were used to refine initial conjectures and to work towards explanatory constructs that accounted for variation.

It was soon evident that pupils experienced different kinds of trajectories, and within classes of similar attainment there were distinct differences between pupils' evolving meanings. Nonetheless, through analysing what remains stable over time, the refinement of conjectures led towards general assertions about important themes in all of the pupils' trajectories. The three themes, which were identified as important, are *ownership*, *connections* and *alignment*. The strength of each theme accounted for variation, so where *ownership* (of the task, spreadsheet and social context), understanding of *connections* (between aspects of their activity) and *alignment* were strong, the pupils made strong gains in their evolving meanings for algebra. The theme of *alignment*, identified as a significant construct, is the focus of this paper.

ALIGNMENT IN CASE STUDIES

The term *alignment* is used to refer to the relationship between teachers' practices and pupils' focus of attention.[2] It embraces the idea that teachers' practices may be timely in meeting pupils' needs if they are strongly aligned to what pupils are attending to. The construct, together with *ownership* and *connections*, is fine-grained enough to account for variation in pupils' trajectories, even as they participate in the same collective activity. While a class of pupils are working on the same task, whole class interventions may be more strongly aligned to some pupils' focus of attention than others. The strength of *alignment* over various chains of episodes was relatively stable over time. Although space precludes the inclusion of rich evidence from many cases, I offer here a chain of episodes involving Caroline and Alice to illustrate.

Caroline and Alice: a chain of episodes

The chain of episodes presented here involves two lower attaining pupils, Caroline and Alice, working on a task called *Mobile Phones*. The pupils were presented with information about two different tariffs offered by a mobile phone company, together with the call time someone uses each month for half a year.

Tariff	Line rental	Calls
A	£12.95	20p per min.
B	£14.50	15p per min.

November	15 mins.	February	44 mins.
December	48 mins.	March	113 mins.
January	80 mins.	April	63 mins.

The pupils were invited to set up a spreadsheet and scatter graph to help them investigate which is the best tariff. They then asked whether they would advise everyone to choose the same tariff. Caroline and Alice were taught by Judith.

Lesson 1: 'doing a sum'

Judith began the first lesson by explaining how a mobile phone contract worked. She established the routine that the pupils would use the spreadsheet efficiently to construct a formula, and pointed to the value of using the spreadsheet to calculate.[3]

Judith	Are you gonna do all the sums, or are you gonna <u>get the "computer to do the sums?</u>
Pupil	Computer …
Judith	You're gonna have to first of all think how do I put that formula in to make it work out the answer for me, okay?

Caroline and Alice read the worksheet but were unsure about what they had to do. Judith clarified the task to them, explaining how the tariff worked for fifteen minutes of calls, and reinforced the norm of using the spreadsheet efficiently ('you've got to find out a way of <u>getting the computer to work out</u> that answer for you'). They entered the headings as suggested, calculated 'fifteen twenties' mentally, and recognised that the cost for the month would be £15.95.

Caroline	So put fifteen pounds ninety-five (points to the spreadsheet), put fifteen pounds … (clicks on cell) Oh (facial expression of realisation) we're meant to <u>do it on the com"puter</u> (looks at Alice)

The act of clicking on the spreadsheet cell seemed to remind Caroline of the routine of writing a formula, particularly Judith's instruction to 'get the computer' to do the calculation. Caroline explained to Alice how a contract worked and they started to construct a formula. They began to engage in the discourse that Judith had used.

Alice	So how do you do a sum?
Caroline	Yeah

Alice	Click on this, here we go here we go here we go, three pounds
Caroline	No 'cause the computer's meant to work it out properly
Alice	Twenty p times fifteen minutes … (enters '20' and then deletes)

Caroline and Alice put up their hands for help, but Judith stopped the class and emphasised the need to calculate in pounds rather than pence. Judith asked a pupil to demonstrate writing and filling down a formula for tariff A. She emphasised that the formula works for any number in the cell, and that it is efficient to fill down.

Judith	Equals. (.) Those many minutes, however many they happen to be, what do you want me to type? … Right, so once you've got it in once, you don't have to type it in lots of times, you can do it, drag it down

Caroline and Alice entered the spreadsheet formula, partly while Judith was talking, and then went on to write a formula for tariff B. Although they were very unsure about constructing a formula, and the intervention curtailed their initial attempt, they did achieve success and the routine of 'doing a sum' appeared to shape their activity.

Lesson 2: ' a real chatterbox on the phone'

In the second lesson Judith demonstrated how to construct a scatter graph and pointed out that the cheapest tariff is indicated by the lower line and that the tariffs cost the same where the lines cross. She demonstrated identifying the approximate crossover point by reading from the graph and then entering numbers in the spreadsheet.

Judith	Let's try thirty-one. Do I get the same answer? (.) Yes. Now I could have been there "ages trying to work out which of these numbers it is. The graph gives you an idea of where to start thinking. Okay? … I'm a <u>real chatterbox on the phone</u> … If I make lots and lots of phone calls, which company is the cheapest?

Prior to constructing a scatter graph Caroline and Alice had to enter formulae for new tariffs. Caroline mobilised the idea of filling down a formula ('all you have to do is <u>bring it down</u>'), showing her increasing participation in the developing practices of the class, but she was unable to fill down because she had not entered a formula.

Caroline	It won't go look, I want all them down there (points finger down column) and look, it won't do it
Alice	That's 'cause you haven't <u>typed a sum in</u> yet there

Although she made an error with the syntax, Alice recognised the need to enter a formula and appeared to mobilise a previously constructed *connection* between using a cell reference and being able to fill down. Having eventually dragged down the formula Alice and Caroline constructed a graph showing the tariffs.

Researcher	So which tariff is the best would you say? Which would you choose?
Alice	I would say, probably the pink again [tariff B] there 'cause <u>it's lower</u> (points) …
Caroline	Oh yeah for a <u>really chatterbox on the phone</u>

Their engagement with the context and their language use (‘chatterbox’) suggests that Caroline and Alice’s construction of meanings was shaped by Judith’s earlier practices. For the particular values graphed, Alice made an appropriate choice about which tariff would be cheapest, explaining that ‘it’s lower’ as Judith had done earlier in the lesson. Caroline and Alice seemed to appreciate the need to consider a range of values and relate these to the kind of user, suggesting a developing sense of functional relationships and a keenness to engage in socio-mathematical practices.

Alignment in the case of Caroline and Alice

The relatively novel approach of tracing aspects of activity, such as language use, offered real insight into the way that teachers’ practices can shape pupils’ unfolding meanings. Caroline and Alice benefited from personalised support, which was closely *aligned* to their focus of attention. They remembered Judith’s instructions (‘we’re meant to do it on the computer,’ ‘type a sum’) and understood that a formula with a cell reference can be filled down to generate data. They also used Judith’s language (‘a real chatterbox on the phone,’ ‘it’s lower’) and recognised that graphs are useful to show functional relationships. This chain of episodes, in which Caroline and Alice showed some *ownership* of the task and routines but made relatively few *connections* themselves, is typical of their activity in the teaching programme. Their successes appear to follow from strong *alignment* between Judith’s practices and their own focus of attention. In the first lesson, for example, Judith responded quickly to their early request for help, emphasising the need to write a formula.

The nature of *alignment*, referring to the relationship between teachers’ practices and pupils’ focus of attention, is such that there is an element of chance regarding how strongly aligned a particular interaction may be. Nonetheless, some characteristics appear to contribute to the relative stability of alignment over a period of time. In this case, Judith appears to be highly attuned to Caroline and Alice’s needs. The pupils sat at the front of the room and she actively monitored their progress while circulating the room. However, alignment involves more than a ‘one-way’ awareness of what pupils are attending to. It refers to the *relationship* or *match* between teachers’ practices and pupils’ attention. There is evidence across a range of episodes that Caroline and Alice actively sought help and managed their own learning.

Variation in other cases

Although space precludes the inclusion of evidence from other cases, it is important to acknowledge that there was variation in the strength of alignment across the different case study pairs, though in each case it was relatively stable over time. Examples of strong alignment include timely interventions, such as demonstrating dragging down a formula to Ashley and Cameron (low attainers in the same class as Caroline and Alice), which helped Ashley to recognise the value of dragging down to generate data (‘so that’s gonna do it for us now, the next one Cameron’). This particular intervention was closely aligned to the pupils’ current needs, unlike previous class discussions about the value of dragging down a formula, which were difficult for Ashley and Cameron to follow as they had not successfully filled down a formula themselves. Over time, Ashley and Cameron received a high level of personalised responsive support, which was strongly aligned to their needs. In contrast, the case of Mark and Japinder (middle attainers also taught by Judith) illustrates weaker alignment. While most personalised interventions were aligned, analysis indicated that most whole class interventions were often untimely for these pupils who experienced some difficulties writing formulas and did not seek help.

DISCUSSION

I suggest that the construct of alignment is relatively novel and timely. Recent theoretical and empirical contributions to the literature have emphasised the role of the teacher in a technological environment. Within the instrumental framework, Trouche (2004) has introduced the term 'instrumental orchestration' to describe the 'external steering of students' instrumental genesis' (p.296), in which an artefact such as a spreadsheet is appropriated by a learner. Using this analytical frame Tabach et al (2008) recognise the strengths and complexities of learning algebra with spreadsheets in a computer intensive environment. Within the framework of semiotic mediation Mariotti (2002) also identifies the central role of the teacher:

> 'the *mathematical* meaning incorporated in the artifact may remain inaccessible to the user … evolution is achieved by means of social construction in the classroom, under the guidance of the teacher' (Mariotti, 2002, p.708)

The notion of alignment develops this research agenda by highlighting the importance of teachers 'reading' pupils' attention and responding to pupils' unfolding activity. The idea of responsiveness has been identified in the broader education literature. Tharp and Gallimore (1988) refer to responsive assistance, and Rogoff (1990) describes the sensitive adjustment of support to meet the needs of learners.

The construct of alignment, however, differs from responsiveness as it refers to the *relationship*, or *match*, between teachers' practices and pupils' focus of attention. Teachers' practices may be timely in meeting pupils' current needs, but not planned as responsive. Alignment captures the essence of this match, including the element of chance, and embraces the idea that learners actively guide their own participation (Rogoff, 1990). The case studies offer evidence of strong alignment when teachers' practices are highly attuned with pupils' focus of attention and when pupils actively seek help and ask questions, which alerts teachers to their needs and promotes a strong match. Importantly, alignment was relatively stable over time in the case studies. The strength of *alignment*, alongside *ownership* and *connections*, accounts for variation in pupils' trajectories, and warrants further research in other contexts.

Notes

1 The Purposeful Algebraic Activity project and this study were both funded by the Economic and Social Research Council

2 With thanks to Professor Janet Ainley for her very helpful comments on this focus

3 The following conventions are used in the transcripts
“ precedes emphatically-stressed syllable … omission of part of transcript
__ highlights language used by teacher and pupils (.) pause

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MATHEMATICAL KNOWLEDG FOR TEACHING AT THE SECODARY LEVEL FROM MULTIPLE PERSPECTIVES

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Mathematical situations from secondary mathematics classes were analysed and used to develop a framework describing mathematical knowledge for teaching at the secondary level. The framework involves broad categories of Proficiency with Mathematical Content, Proficiency in Mathematical Activity, and Proficiency in Using Mathematics in Problems of Practice. A second, in-progress phase of research uses the mathematical situations to gather perspectives on mathematical knowledge for teaching from students preparing to be secondary mathematics teachers and teachers who are currently teaching mathematics at the secondary level. Areas of alignment as well as differences among the perspectives will be discussed.

FOCUS OF THE RESEARCH

The mathematical preparation of mathematics teachers is a matter of international concern to mathematics educators and policymakers. Often requirements are stated in terms of mathematics courses at the collegiate level or specific topics that need to be included in collegiate mathematics courses (CBMS, 2001; Krauss, Baumert, & Blum, in press). A strong case has been made for improving the mathematical knowledge of teachers (Ma, 1999), but there is also evidence that completion of mathematics courses is not sufficient for developing the mathematical knowledge that is needed for teaching mathematics (Monk, 1994). Researchers are trying to identify the mathematical knowledge that is useful to teachers as they help others to learn mathematics (Adler & Davis, 2006; Ball, Lubienski, & Mewborn, 2001; Even, 1993). Knowing and using mathematics for yourself is not the same as knowing mathematics in a way that helps you understand the mathematical thinking of someone else, make connections between what is known and needs to be learned, or make explicit ideas that are often implicit in mathematical work. These are a few of the unique demands that teaching places on one's mathematical knowledge.

At the elementary level, significant work has been done in identifying mathematical knowledge for teaching (Ball & Bass, 2000), and a correlation between teachers' mathematical knowledge for teaching and students' mathematics achievement has been shown by Hill, Rowan and Ball (2005). Researchers at Penn State University and the University of Georgia have been working to identify mathematical knowledge for teaching at the secondary level and to build a framework that would explain the knowledge that is useful for teaching secondary mathematics. This research report will discuss two phases of that research. The first phase included the collection of data from secondary mathematics classes and the construction of a

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framework based on mathematics educators' analyses of the data. The second and in-progress phase of the research is investigating the perspectives of students preparing to be secondary mathematics teachers (preservice teachers), and teachers who are currently teaching secondary mathematics (practicing teachers). We are interested in the mathematical knowledge for teaching that preservice and practicing teachers identify and how their constructions of mathematical knowledge for teaching compare with the frameworks being developed by mathematics educators.

THEORETICAL FOUNDATIONS

Shulman (1986) identified the notion of pedagogical content knowledge, which included pedagogical knowledge that was specific to a content area. Ball and colleagues have claimed that Shulman's pedagogical content knowledge only captured part of the construct of mathematical knowledge for teaching. They identified four components: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2004). Later, they added two additional kinds of knowledge: knowledge of curriculum and knowledge at the mathematical horizon (Ball & Sleep, 2007). From another perspective, McEwan and Bull (1991) suggest that all knowledge may be pedagogical in some way. Our work has also been informed by several frameworks developed to describe the nature of mathematics teaching and associated knowledge. Adler & Davis (2006) described differences in teaching when one is unpacking mathematical ideas in contrast to compressing ideas. Cuoco argued that mathematics teaching should help students attain habits of mind such as looking for patterns, experimenting, and describing (Cuoco, 2001; Cuoco, Goldenberg, & Mark, 1996), which helped us think about knowledge related to mathematical processes that teachers use as they help learners develop mathematical habits of mind. Even's (1993) work on functions explored the connections between content knowledge and pedagogical content knowledge. Employing a situative perspective, Peressini, Borko, Romagnano, Knuth, & Willis-Yorker (2004) developed a conceptual framework that considered content knowledge, pedagogical content knowledge and teachers' professional identity. Collaborative work in Australia and the US produced a conceptual framework to help researchers describe teacher education and development within practice (Tatto et al., 2008). Learning from these previous efforts, we have attempted to describe mathematical knowledge for teaching at the secondary level with attention to the dynamic and structural nature of such knowledge.

METHODOLOGY OF BUILDING A FRAMEWORK

In seeking to understand the mathematical knowledge that is useful for teaching at the secondary level, we took a situative perspective by beginning in actual mathematics classrooms (Peressini et al., 2004). We wanted to draw from interactive mathematics lessons and discussions. We began by visiting mathematics classes and looking for situations that seemed to be critical points in the lesson where teachers had to make decisions about what mathematics to explore (or to ignore). Often, the situation was created by student questions or errors, but interesting student claims and insights also provided rich opportunities. Researchers were either professors or doctoral students and all had secondary teaching experience. Most situations occurred during mathematics lessons in high schools or conversations among teachers discussing their lessons. Some situations were drawn from collegiate classes for preservice teachers where they were discussing secondary mathematics lessons.

Mathematics educators analysed each situation in the collection of more than 50 situations, inferring the mathematical knowledge that would be useful to teachers as they made decisions related to each particular situation. This analysis focused on mathematical knowledge that the teacher could use, but we intentionally did not address what the teacher should have done. Decisions on what to *do* depend on context as well as knowledge, and we were interested in the *mathematical* knowledge that could help teachers make the important decisions within mathematics teaching. In particular, we wanted to identify the knowledge that would allow the teacher to make the most informed decision in that situation. Although it was difficult, we tried to separate the pedagogical knowledge from the mathematical knowledge and focus on mathematical knowledge. This analysis resulted in creating individual case reports for each situation. Each case report contained: (1) a prompt that explained the situation from the mathematics class, (2) a set of 3-5 foci that each addressed a specific mathematical concept we thought was relevant to the given situation, and (3) commentary that provided a rationale for the foci that were chosen, or elaborated on related mathematics that was not developed in any selected focus. The range of topics in the full set of case reports includes topics from algebra, geometry, statistics, number, trigonometry, and calculus. The prompts and foci became our data that informed the building of a framework.

Figure 1 shows an abbreviated form of a case report built from a student's question in a mathematics classroom (the prompt). In the complete case report, Focus 1 develops the idea of linear interpolation, emphasizing that this process produces an estimate. It shows that the value of $2^{2.5}$ is not half way between 2^2 and 2^3. It includes both a tabular and graphical representation. Focus 2 includes a graphical representation showing the continuous function when the exponent is a rational number and the base is positive. Focus 3 discusses the properties of integral exponents and extends them to rational exponents explaining that $2^{2.5}$ can be interpreted as $(2^{½})^5$ or $(2^5)^{½}$. A post commentary notes that the function $f(x) = b^x$ would behave differently if $b < 0$. Although the foci and commentaries do not exhaust all important knowledge related to the situation, they do offer key mathematical ideas that are helpful for teachers to know when they discuss $f(x) = b^x$ or respond to the situation.

Prompt

During an Algebra I lesson on exponents, the teacher asked the students to calculate positive integer powers of 2. A student asked the teacher, "We've found 2^2 and 2^3. What about $2^{2.5}$?"

Commentary

The prompt centers on the extension of the domain of the exponent to numbers beyond integers. The foci explore the nature of exponents numerically, graphically, and analytically. The table with integral values in Focus 1 suggests a pattern for a curve and an extended domain that is illustrated in the graphical representation in Focus 2. Although Foci 1 and 2 provide an estimate, the analytical treatment in Focus 3 generates an exact value. These foci help expand the concept of exponentiation beyond repeated multiplication to accommodate the use of some non-integer exponents.

. . .

Mathematical Focus 2

A more accurate estimation of 2^x where $x \notin Z$ can be obtained through graphical analysis of the function f with rule $f(x) = 2^x$, $x \in R$.

It should be recognized at this point that one is assuming that the domain of the function f with rule $f(x) = 2^x$ can be extended from the set of integers to the set of real numbers. In particular, the resulting graph of the function with the new domain will be represented by a continuous curve. This graph allows one to obtain an estimate for f(2.5) with varying degrees of accuracy depending on the technology or method employed.

For example, one can estimate the value of f(2.5) from a calculator-generated graph of f and the trace option. Alternatively, one can look at the intersection of the function graph with the vertical line x = 2.5.

(Graphical representation displayed.)

. . .

Figure 1: Situation on Exponents

EMERGING FRAMEWORK

The entire set of case reports was analysed in order to identify categories of mathematical knowledge for teaching. A rough framework emerged from the data suggesting three broad categories of mathematical knowledge for teaching. The first category was Proficiency with Mathematical Content. It included conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition and historical/cultural knowledge. These terms were similar to the components defined by Kilpatrick, Swafford and Findell (2001). The second category was Proficiency in Mathematical Activity. It included ideas such as representing, exemplifying, generalizing, defining, justifying and exploring. The third category was Proficiency in Using Mathematics in Problems of Practice. Although this category is closely related to pedagogy, we focused on the mathematical ideas of unpacking mathematics, recognizing conventions, making connections within mathematics, identifying structures in mathematics, and analysing students' mathematics. After identifying these broad areas and specific characteristics, we returned to the case reports to see how well these components of the framework matched the work of teachers.

PHASE TWO: PERSPECTIVES FROM PRESERVICE AND INSERVICE TEACHERS

Although the framework will continue to evolve, we would like to know how the framework, built by mathematics educators, compares to ideas generated by preservice and practicing teachers. We are currently beginning the second phase of the work, and have begun collecting data from preservice teachers. Data will be gathered from practicing teachers throughout the spring semester of the school year.

Methodology

The development of the framework by mathematics educators evolved over two years of extensive work on case reports. We are not expecting preservice or practicing teachers to do such work, but we are interested in how they think about the situations that were drawn from actual mathematics lessons and discussions. We are using focus group discussions to generate ideas and make public the thoughts of our participants after they have read a prompt from a case report. Initially, each participant writes down his or her ideas about the mathematical knowledge that would be useful to the teacher in the provided prompt. The moderator engages the focus group in discussing the mathematical knowledge that was identified, encouraging participants to build on each other's ideas. After the discussion of mathematical knowledge, participants are given a copy of each focus, one at a time. Participants are asked, individually, to rate each focus (created by mathematics educators) based on how useful the mathematical knowledge described in the focus would be to a teacher. The focus group then discusses individual ratings and works toward a consensus of the value of various mathematical ideas. All focus group discussions are video taped and analysed. Our goal is to identify the mathematical knowledge that is considered useful by the group.

Eight preservice teachers are participating in a focus group, which meets three times during the semester. They are students in a class that focuses on methods of teaching secondary mathematics, one of the last classes prior to their student teaching experience. The focus group participants react to the prompts and discuss their ideas, and the moderator (professor of the course) helps to facilitate participation by all members of the group, but does not comment on the discussion. The moderator does help the participants focus on mathematical knowledge when they begin to stray towards pedagogical concerns. The data collection from practicing teachers will proceed in a similar manner with a focus group of practicing teachers reacting to the same three prompts as the preservice teachers. The videotapes of each focus group will be analysed to identify salient mathematical knowledge that emerged from the group discussion. Then the videos will be analysed comparing the teachers' perspectives to the categories of the framework developed by mathematics educators. Comparisons will be drawn among all groups.

Preliminary findings for Preservice Teachers

Initial findings suggest that the areas of mathematical knowledge that the focus group of preservice teachers identified as relevant to the situation were consistent with those identified by the mathematics educators. However, while the broad mathematical concepts were similar, there were significant differences in the amount of detail, vocabulary, and depth of explanation generated by the two groups. Although we had anticipated such a difference, the discussions offered insights into the priorities placed on the mathematical knowledge by the preservice teachers. For example, one of the prompts drew attention to potential confusion between the inverses and reciprocals of trigonometric functions. Both mathematics educators and preservice teachers identified a need for the following areas of mathematical knowledge: (1) defining and understanding inverse functions, (2) recognizing notational issues, and (3) representing functions and their inverses graphically. However, we found no evidence that preservice teachers valued the rigor or structure expressed in the descriptions of the mathematics educators. For instance, where the educators showed attention to defining the operation and identity element involved in finding an inverse, the preservice teachers referred to "doing" and "undoing" without ever defining the operation as composition of functions or defining an identity with respect to multiplication or composition.

The preservice teachers discussed many of the characteristics found in the framework developed by our research group. Their discussion included elements of category 2, Proficiency in Mathematical Activity, such as defining and representing mathematical ideas. They also focused on some elements of category 3, Using Mathematics in Problems of Practice, including understanding students' thinking, connecting ideas, and attending to notational conventions. They did mention one aspect of category 1, Proficiency with Mathematical Content (understanding conceptually), but they seemed to be more comfortable discussing ideas related to Mathematical Activities and Using Mathematics in Problems of Practice than ideas related to Mathematical Content. Some individuals exhibited difficulty with the mathematical content, but as a group they were able to clarify the general ideas, sometimes building on what the mathematics educators had written in the foci. We hypothesize that difficulty with the particular mathematical content may lead to a less rigorous description of mathematical knowledge for teaching related to Proficiency in Mathematical Content.

When asked to rate the foci as to the relevance of the mathematical knowledge described by each, the prospective teachers found them to be highly relevant, which is not surprising since they had previously identified the main ideas described therein as relevant to the situation. However, six of the eight members of the focus group rated the focus dealing with the definition of inverse as less important than the one dealing with notational issues and the one addressing a graphical approach. This reveals that the prospective teachers were perhaps less attuned to the need for structure in mathematics

Currently, we offer an evolving framework for mathematical knowledge for teaching at the secondary level, based on secondary classroom situations. After the completion of the second phase of our research, we will be able to describe in what ways mathematical knowledge for teaching, as defined by mathematics educators, is aligned with the thinking of preservice and practicing teachers. By looking at multiple perspectives, we are laying the foundation for describing the mathematical knowledge that needs to be emphasized in programs for both preservice and practicing teachers.

Note: This research was part of the work of the Mid Atlantic Center for Mathematics Teaching and Learning (MAC-MTL) at Penn State University and the Center for Proficiency in Teaching Mathematics (CPTM) at the University of Georgia and reflects the work of many researchers. Both Centers are funded by the National Science Foundation in the U.S. (Grants: 0119790, 0083429, 0426253). Any opinions, findings, and conclusions or recommendations expressed in this presentation are those of the presenters and do not necessarily reflect the views of the National Science Foundation.

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TOWARDS A SIX-DIMENSIONAL MODEL ON MATHEMATICS TEACHERS' BELIEFS

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Drawing upon the assumption that mathematics teachers' beliefs influence their practice, research has extensively examined the following dimensions: beliefs about the nature of mathematics, beliefs about mathematics teaching, and beliefs about mathematics learning. Many studies highlight inconsistencies between teachers' beliefs and practices. A number of opinions are offered for explaining this mismatch. In this theoretical paper, I talk about three dimensions of teachers' beliefs that are often neglected. These are their self-efficacy about mathematics, their self-efficacy about mathematics teaching, and the social context within which beliefs are developed and expressed. Therefore, six dimensions towards a new model are suggested.

INTRODUCTION

In mathematics education, teachers' beliefs have been the subject of extensive research, based on the assumption that what teachers believe is a significant determiner of what gets taught, how it gets taught and what gets learned in the classroom (Wilson & Cooney, 2002; Chapman, 2002; Lerman, 2002; Middleton, 1999). According to Aguirre and Speer (2000), "being able to identify and describe the mechanisms underlying the influence of beliefs on instructional interactions would deepen and enrich our understanding of the teaching process" (p. 327-328). Older and recent studies highlight the importance of examining, analysing and changing teachers' beliefs in order to successfully apply mathematics curricula reforms (i.e. Ernest, 1989; Handal, 2003; Shahvarani & Savizi, 2007). Without shifting teachers' beliefs, changes can "be cosmetic, that is, a teacher can be using new resources, or modify teaching practices, without accepting internally the beliefs and principles underlying the reform" (Handal & Herrington, 2003, p. 62).

However, the terminology regarding mathematics-related beliefs (of both students and teachers) is 'messy' (Pajares, 1992). This has created confusion in the field (McLeod, 1988; Pehkonen & Pietila, 2003), as "there is still no consensus on a unique definition of the term *belief*" (Törner, 2002, p. 75). Many authors seem to be aware of this deficiency and thus establish their own vocabulary, such as conceptions, philosophy, ideology, perception, world view, image, disposition and so forth, leading to a 'definitional vicious circle'. Beliefs are thought to be connected in order to form belief systems (Ernest, 1989; Andrews & Hatch, 2000; Aguirre & Speer, 2000), which are clusters within the individual (Törner, 2002). These systems are not necessarily logically structured (Richardson, 2003); therefore, an individual may hold beliefs that are incompatible or inconsistent.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 369-376. Thessaloniki, Greece: PME.

THE THREE-DIMENSIONAL MODEL OF TEACHERS' BELIEFS

Despite the apparent ambiguity regarding definitions and the nature of the term, opinions upon the dimensions of mathematics teachers' beliefs appear to converge. Current thinking in the mathematics education literature, according to Aguirre and Speer (2000), focuses primarily on how teachers think about (a) the nature of mathematics, (b) its teaching and (c) its learning. In this context, beliefs are defined as conceptions, personal ideologies, world views and values that shape practice and orient knowledge.

Ernest (1989) offers a detailed discussion upon the above three dimensions. The last two (beliefs about mathematics teaching and learning) are described by him as teachers' *espoused models about mathematics teaching* and *learning*. Beliefs about the nature of mathematics can be summed up into three views: the instrumentalist, the Platonist, and the problem solving. Correspondently, three views regarding *espoused models of mathematics teaching* are proposed. The teacher can be an instructor, an explainer, or a facilitator. With respect to teachers' *espoused models about mathematics learning*, Ernest distinguishes between two key constructs: learning as active construction and learning as passive reception of knowledge. Subject to certain conditions, the *espoused models of teaching* and *learning mathematics* are transformed into classroom practices, leading to the *enacted model of teaching mathematics*, the use of mathematics texts or materials, and the *enacted model of learning mathematics* (Ernest, 1989).

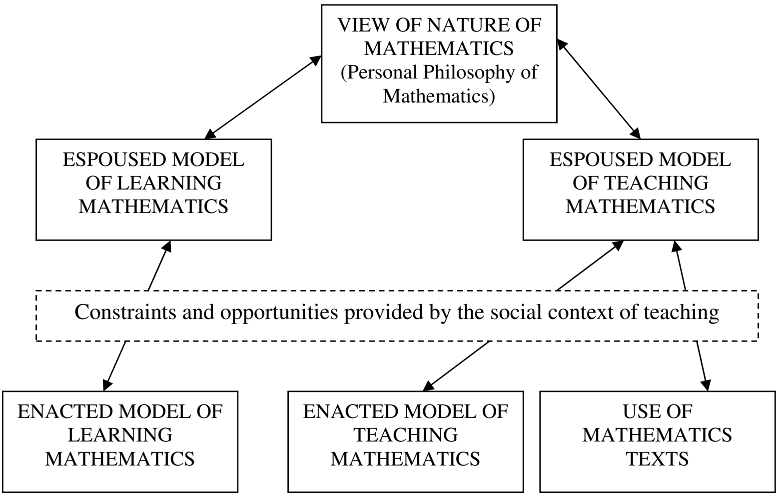

Figure 1: Ernest's model about the three dimensions of teachers' beliefs and their connection to the enacted models of teaching and learning mathematics.

The relation between teachers' beliefs and instructional practices is considered complex and cannot be described in terms of cause-and-effect. Nonetheless, several studies highlight inconsistencies between teachers' espoused models and what is actually observed during instruction (Thompson, 1984; Raymond, 1997; Beswick, 2005). Different opinions have been offered to explain this mismatch. Approaching this issue from teachers' perspective, Ernest (1989) suggests two key reasons. The first is the powerful influence of the social context. Teachers work under certain constraints, such as school and parent expectations, the use of specific material, fellow teachers' beliefs/practices and so on, which affect the enactment of their models about mathematics teaching/learning. The second is the teacher's level of consciousness of her/his own beliefs, and the extent to which the teacher reflects on her/his instructional practice. From another perspective, Leatham (2006) argues that one prevalent pitfall of research in teachers' beliefs is "to take a positivistic approach to belief structure, assuming that teachers can easily articulate their beliefs and that there is a one-to-one correspondence between what teachers state and what researchers think those statements mean" (p. 91).

Most studies follow Ernest's triadic categorisation, by taking into account beliefs about the nature of mathematics, about mathematics teaching and about mathematics learning. The last two components (teaching and learning) are often viewed either as teachers' espoused models on what the *ideal* is or what teachers claim to do in practice. In my opinion, however, a very important dimension is neglected: teachers' self-efficacy beliefs. One could argue that Ernest's dimension about the espoused models of teaching includes a teacher's self-efficacy beliefs. However, this is not explicitly expressed. As a consequence, researchers use Ernest's three-dimensional model without taking into consideration teachers' beliefs about their own competence. From my perspective, this might be another reason why inconsistencies between teachers' beliefs and instructional practices are often observed. A teacher could, for example, express constructivist beliefs with respect to the ideal mathematics teaching. Nevertheless, the same person might feel insecure regarding constructivist and problem-solving oriented approaches in teaching mathematics, therefore adopting a more traditional teaching style.

TEACHERS' SELF-EFFICACY BELIEFS

Bandura is considered by many as the inaugurator of the research in the field of self-efficacy, which is defined as "…beliefs in one's capabilities to organize and execute the course of action required to produce given attainments" (Bandura, 1997, p. 3). According to him,

> …perceived self-efficacy influences choice of behavioural settings. People fear and tend to avoid threatening situations they believe exceed their coping skills, whereas they get involved in activities and behave assuredly when they judge themselves capable of handling situations that would otherwise be intimidating (Bandura, 1977, p. 193-193).

Efficacy is future-oriented judgment that has to do with perceptions of competence rather than an actual level of competence (Hoy & Spero, 2005). People regularly overestimate or underestimate their actual abilities. Nonetheless, "these estimations may have consequences for the course of action they choose to pursue and the effort they exert in those pursuits" (Hoy & Spero, 2005, p. 344), and can mediate performance (Bandura, 1997; Charalambous, Philippou, & Kyriakides, 2008).

In mathematics education research, studies have focused primarily on teachers' self-efficacy beliefs (TEB) with respect to mathematics teaching. For example, drawn upon Bandura's definition, Charalambous et al. (2008) define TEB as the "sense of ability to organize and execute teaching that promotes learning" (p.126). Their research results suggest two subcategories of TEB, (a) beliefs about mathematics instruction (instructional skills in mathematics) and (b) beliefs about management in the mathematics classroom.

Gassert, Shroyer, and Staver (1996) investigated the factors that influence the science teaching self-efficacy of elementary teachers. Their results yielded three categories of factors: antecedent, internal, and external. Antecedent factors included science-related experiences in and out of school, teacher preparation, and science teaching experiences. Internal factors included attitudes toward science and interest in science. External factors affecting science teaching self-efficacy included the school workplace environment, and student and community variables. In mathematics education research, though, most studies about TEB examined beliefs regarding teaching competence, and not beliefs about interest and competence in mathematics. An exception to this constitutes a study by Stipek, Givvin, Salmon, and MacGyvers (2001). The researchers examined, amongst others, elementary teachers' self-confidence and enjoyment of mathematics and mathematics teaching. Their study shows that teachers who embraced more traditional beliefs about mathematics and learning had lower self-confidence and enjoyed mathematics less than teachers who held more inquiry-oriented views. Moreover, teachers' self confidence as mathematics teachers was significantly correlated with students' perception of their own competence as mathematics learners.

SOCIAL CONTEXT: THE HIDDEN DIMENSION

As Ernest (1989) points out, the social context within which teachers work has a great impact on the enactment of their beliefs. Regularly, though, researchers neglect its important role when conducting research on mathematics teachers' beliefs. Acknowledging and examining the social context within which teachers' beliefs are developed and expressed could offer insights into a better understanding of the complex relation between beliefs and practices.

Social context can, in my opinion, be viewed from at least two perspectives. The first, which I shall call the 'micro-context', has to do with the individual's working environment (school), experiences (as a student, trainee, and from prior teaching) and personality. Approaching social context from this perspective, Raymond (1997) talks about a number of factors that influence the relation between teachers' beliefs and practices. The degree of influence each factor has, claims Raymond, can vary from slight to strong. Schematically, her ideas are presented in the figure and table below.

"Although the model cannot be applied universally without amendment, it suggests complex relationships between beliefs and practice and builds toward an understanding of factors that contribute to the inconsistency between them" (Raymond, 1997, p. 570).

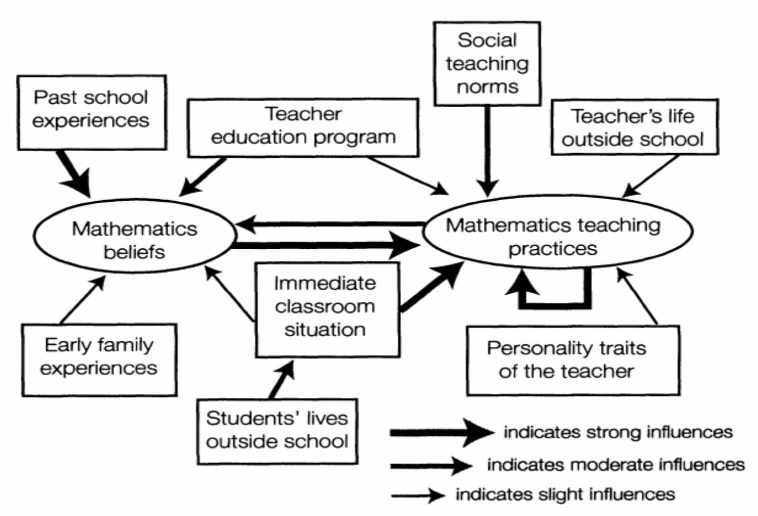

Figure 2: Raymond's model about the factors that influence the relation between mathematics teachers' beliefs and practices

The following table explains each of the model's components, as described by Raymond (1997).

Mathematics beliefs:	About the nature of mathematics and mathematics pedagogy
Mathematics teaching practices:	Mathematical tasks, discourse, environment, and evaluation
Immediate classroom situation:	Students (abilities, attitudes, and behaviour), time constrains, the mathematical topic at hand
Social teaching norms:	School philosophy, administrators, standardised tests, curriculum, textbooks, other teachers, resources
Teachers' life:	Day-to-day occurrences, other sources of stress
Students' lives:	Home, environment, parents' beliefs (about children, school, and mathematics)
Teacher education	Mathematics content courses, methods courses,

program:	field experiences, student teaching
Past school experiences:	Success in mathematics as a student, past teachers
Early family experiences:	Parents' view of mathematics, parents' educational background, interaction with parents (particularly regarding mathematics)
Personality traits:	Confidence, creativity, humour, openness to change

Table 1: Raymond's model-brief explanations

The second perspective from which social context can be approached is what I shall call the 'macro-context'. Macro-context has to do with issues related to culture and the educational system within which teachers are educated and work, as a whole. Mathematics teaching and learning are culturally embedded. Results from international/comparative studies suggest that there are more significant similarities between teachers' mathematics related beliefs (Correa et al., 2008; Santagata, 2004; Andrews & Hatch, 2000) and practices (Leung, 1995; Andrews, 2007; Givvin et al., 2005) within single countries than they are across countries.

TOWARDS A SIX-DIMENSIONAL MODEL

Future research on mathematics teachers' beliefs could focus on six distinct, but also overlapping dimensions: (a) beliefs about the nature of mathematics, (b) beliefs about mathematics teaching in general (how mathematics teaching should be), (c) beliefs about mathematics learning, (d) self-efficacy beliefs about mathematics teaching, (e) self-efficacy beliefs about mathematical competence and (f) the social context within which beliefs are developed and expressed (micro- and macro-context). Both qualitative and quantitative methods could be employed for in-depth understanding of the above and the development of a comprehensive model that takes the relation between these dimensions and their impact on instructional practices into account.

Another issue that has to be taken into consideration is the transferability of such a model when research is conducted in different countries. Researchers in the field of international/comparative mathematics education could examine the connections between the above six dimensions and their relation to teaching practices. This would contribute to a better understanding of the cultural differences related to mathematics instruction.

Acknowledgments

I would like to thank my supervisor, Dr Paul Andrews, for his helpful suggestions and comments.

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DESIGNING INNOVATIVE WORKSHEETS FOR IMPROVING READING COMPREHENSION OF GEOMETRY PROOF

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This study explored the effects of innovative worksheets of geometry proofs on students' reading comprehension of geometry proof. The worksheets treatment was designed with reading comprehension strategies. One mathematics teacher and two classes of ninth-grade students who were taught by her participated in this quasi-experimental classroom study. While the experimental group was instructed with the innovative worksheets in two forty-five-minute sessions, the control group was instructed with their regular materials about "inequality" during winter vacation courses. ANCOVA was conducted on posttest and delayed posttest with pretest as a covariate. Results showed that the score of the delayed posttest of the experimental group was significantly higher than the control group although there was no significant difference in posttest. The study supports the value of a mathematics proof curriculum with reading perspective following similar instructional strategies.

INTRODUCTION

For learning content, visualization and dynamic construction are suggested (Hanna, 2000). For learning to construct proofs, investigation of propositions or conjecturing is included to inspire the need for proof, and validating proofs is helpful for understanding proofs and further constructing a valid proof (Koedinger, 1998). For example, conjecturing activities try to engage students into finding some patterns or properties from several numerical examples, geometric figures or situational phenomena. These kinds of activities are based on the practice of mathematicians, and they require students to formulate their own proposition.

How to coordinate the divergent essence of conjecturing and the convergent essence of proving is still questionable. Transforming the verbal representation of dialogue into the literal and symbolic representation of proof is an obstacle which students must overcome for understanding the nature of proof (Sfard, 2000). Listening, speaking, and doing proofs are considered necessary activities for learning proofs, and reading should not be left out. Solow (2002) suggested useful strategies to read and do proofs to college students. How junior high school students learn and accommodate these strategies and convert the strategic knowledge into action are still critical issues of learning mathematical proofs. Especially, how well junior high school students comprehend mathematics proof is required for analysis as the quality of teaching proofs with the perspective of reading literacy is evaluated.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 377-384. Thessaloniki, Greece: PME.

Yang and Lin (2008) viewed reading comprehension of geometry proof (RCGP) as an object of study in addition to a processing component of producing, validating or interpreting arguments. They had conceptualized RCGP; that is, understanding proofs from the essential elements of knowing how a proof operates and why a proof is accepted besides knowing proof methods or ideas and what it proves. Six facets -- basic knowledge, logical status, integration or summarization, generality, application or extension, and appreciation or evaluation-- were formulated.

Reciprocal teaching method has been one of the most outstanding strategy instructions (Rosenshine & Meister, 1994). It is designed to improve reading comprehension by teaching cognitive strategies such as question generation, clarification, summarization, and prediction. Students attempt to gain meaning from text using theses strategies (Palincsar & Brown, 1984). Both reciprocal teaching only and explicit teaching before reciprocal teaching focus on the instruction in the cognitive strategies and students' practice of these strategies. It at least takes more than three lessons in most of the reciprocal teaching studies (Rosenshine & Meister, 1994). It is worth trying to use cognitive strategies to design worksheets instead of training students to use these strategies if improving students' RCGP with less teaching time is required, especially when the number of researches on students' reading strategies of comprehending geometry proofs is still few.

Based on the understanding of RCGP and transactional perspective on reading comprehension, we tried to design innovative worksheets for improving students' RCGP. The purpose of this study was to evaluate the effects of the innovative worksheets designed with reading strategies on RCGP for ninth-grade students who had learnt geometry proofs in school. For the experimental instruction, students were asked to answer questions individually, and then discuss in groups or present before the whole class. The performance of experimental students on RCGP was compared to that of students who did not receive any instruction of reading geometry proofs but took a test of RCGP where their mathematics teacher explained answers to them.

METHOD

Design of Worksheets

According to schema theory (Anderson & Pearson, 1984), we can derive meaning of texts based on our preexisting knowledge, and the better understanding can be realized while proper schemas can be triggered. Information must be processed in working memory before modified schemas are stored in long-term memory (Nassaji, 2002). On the other hand, instructional designs that increase germane cognitive load are beneficial to students' learning because students are guided to focus on cognitive processes that are necessary for accommodating schemas (Sweller, van Merrie¨nboer, & Paas, 1998). Thus, the innovative worksheets are set to trigger or to structure students' schemas, and to reveal and to acquire cognitive processes by peer and class discussion.

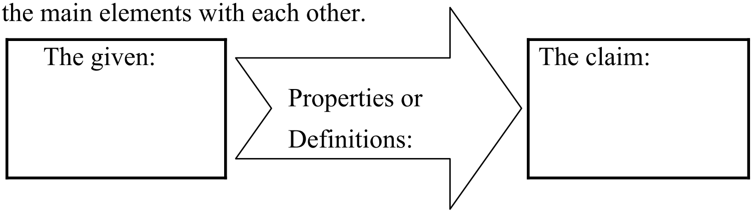
There are two worksheets. The worksheet ONE provided a proof and its corresponding propositions. The design of the following tasks were based on reading strategies of questioning, predicting, summarizing and clarifying (Brown & Palincsar, 1987; Palincsar & Brown, 1984) which are commonly used by good and active readers. The worksheet TWO provided another proof and its corresponding, and the following tasks are the same as the worksheet ONE. The ideas of using these strategies to design the worksheets are described in the following.

Questioning is to ask questions related to the text for monitoring and regulation comprehension of written materials. Students might have difficulties in creating their own questions while reading proofs. In this study we adopt this strategy to provide genetic questions, Q1-1:What is “the given” in the above proof?, and Q1-2:What is “the claim” in the above proof?, to prompt students to recognize the logical statuses of related statements and to monitor if they can identify them.

Predicting is to predict what will come next in the text you are reading. While reading a proposition and its proof, one should predict what can be inferred based on the given (forward, Q1-3:Thinking in the beginning of “the given” in the above proof, what can you infer in the next step?) or on the conclusion (backward, e.g. Q1-4:Which result can you derive from “$\overline{BC}=\overline{AD}$”directly in the above proof?). This might trigger relevant knowledge.

Asking for *clarification* if needed is one approach to modifying schemas. For examples, readers should point out what they do not understand or clarify whether their understanding is coherent. Q2-1:In the above proof, which properties are written unclearly in the given?, Q2-2:In the above proof, which proof steps do you not understand? Please circle them., and Q2-3:In the above proof, can you think which proof steps are redundant or unnecessary? Please delete them with straight lines in the above proof. are designed to check if students understand the meaning of this proof through self-evaluation after questioning and predicting guidance.

Summarization is to summarize several statements or paragraphs in simplified substance. For helping students understand a proof structure and synthesize the proposition and its proof steps, we provide a proof mapping as a visual display. Students are initiated to identify important proof steps and chain proof steps logically by attending to structure the critical elements of a proof through this proof mapping by Q3-1:Now, please use brief sentences to write down the main elements of the proof in the following mapping, and Q3-2: Please discuss in group. Write down the difference of the main elements with each other.



In this study, we also design tasks to ask students to think the predicting questions again and to compare their answers with the initial answers, to describe the underlying relationship of a proof mapping and to reflect their reading strategies based on their metacognition of what they understand and how they read by Q4-1:If you were a teacher, which problem do you formulate to let students write down the proof?, Q4-2: Do you have the same answer between the question you devise in (4-1) and (1-1) (1-2)? What is the correct answer?., Q4-3:Please go back to check (1-3) and (1-4), and you can revise the conjecture of (1-3) and (1-4) if you need., Q4-4: Referring to (3-1), please describe the relationship between the given, properties or definitions, and the claim., Q4-5:While applying the properties or definitions, what should you notice?, and Q4-6: While you read mathematical proofs hereafter, how will you read to easily comprehend proofs?. Theses questions are designed to reveal and acquire cognitive processes of reading geometry proofs.

Instructors and Instruction

The participating teacher had taught in junior high school for six years. She had taught ninth-graders' geometry proofs twice. She joined the first author to discuss the worksheets and the way to implement the worksheets. One of the two classes was randomly chosen to instruct with the worksheets. The other class was instructed with their regular materials about "inequality".

In the experimental group, students were asked to answer questions 1-1 to 1-4. The instructor asked one or two students to show their answer, and then discussed if the answers were plausible with all students; Questions 2-1 to 2-3 were adopted to check if students have sufficient pre-knowledge to understand this proof. The instructor can explain some properties, e.g. ASA, while students don't know it. Questions 3-1 to 3-2 provided a framework to summarize proof steps and asked students to discuss with peers. After peer discussion and presentation, the instructor explained how to summarize proof steps with the given, the applied properties and the conclusions. Questions 4-1 to 4-4 asked students to rethink what this proof proves and clarify their initial understanding and their summarization. To reflect their strategies of comprehending proofs, questions 4-5 to 4-6 asked students to point out the conditions of applying a property and to describe how to understand proofs.

In the control group, the mathematics teacher introduced the concept of linear inequality with one variable and the skills to find the solution of linear inequality. Students were asked to solve problems of inequality, discussed with peers and then explained to the whole classmates.

Subjects

One mathematics teacher and her two classes of 66 ninth-graders (14 to15 years old) participated in this quasi-experimental classroom study. These students who had leant formal geometry proofs in school were the subjects of this study, because their RCGP may not be advanced by the traditional geometry proof instruction (Lin & Yang, 2007).

One class of 9th graders (N=32) was instructed with the innovative worksheets while the other (N=34) was instructed with their regular textbook during winter vacation courses. However, only 24 and 26 students of the two classes respectively completed both posttest and delayed posttest. Based on the analysis of pretest, the absent students in the experimental group performed a little better than in the control group while compared to their class peers, and the absent students in the two groups performed a little poorer than their class peers.

Dependent Measures

Five quantitative measures were derived from this instrument of RCGP (Yang & Lin, 2008): basic knowledge comprehension, logical status comprehension, summarization comprehension, generality comprehension and application comprehension. The sum of the five measures represented the performance of RCGP. It took about thirty minutes to complete this test in both the two groups. Moreover, the instructors spent about fifteen minutes explaining the answers of this test for the students.

The posttest instrument as well as the delayed posttest for measuring RCGP were developed according to the operational definitions of the first five facets. The proof steps read in the posttest and delayed posttest were more than that in the pretest. The Cronbach's alpha reliability coefficients of the pretest and (delayed) posttest instruments were .84 and 0.73 for the ninth graders.

Procedure

This study consisted of four different phases: *Phase 1 Pretest.* Prior to the instruction, the questionnaire of Reading Comprehension of Geometry Proof (Yang & Lin, 2008) was administered to both experimental and control groups. *Phase 2 Intervention.* The experimental group is instructed with the worksheets in two forty-five-minute sessions. The control group was instructed with their regular materials about “inequality“ in two forty-five-minute sessions by the same mathematics teacher. The two sessions were part of winter vacation courses. *Phase 3 Posttest.* Two weeks later of the winter vacation courses, all students answered the posttest questionnaire. *Phase 4 Delayed Posttest.* About three months later of the winter vacation courses, all students answered the questionnaire which is the same as the posttest questionnaire.

RESULTS

The means and standard deviation on the total scores of reading comprehension of geometry proof (RCGP) from pretest, posttest and delayed posttest for the experimental and control groups were presented in Table 1.

	Experimental		Control	
Test (Full Score)	M	SD	M	SD
Pretest (29)	12.92	4.24	12.46	3.25
Posttest (29)	14.29	4.19	13.07	5.11

Delayed Posttest (29)	14.00	4.67	10.73	4.26

Table 1:Means and standard deviations on the total scores of RCGP for the experimental and control groups.

Regarding pretest, the mean scores for the control groups were a little lower than the mean score for the experimental group. However, the two independent samples t test revealed no significant difference among the two groups (t(48)=.428, p=.671). Analysis of pretest results provided further support that the groups are comparable. Students' performance on RCGP were really improved even if the raw scores on pretest, posttest or delayed posttest did not show trivial increase for the two groups, because the posttest and delayed posttest questionnaires were harder than the pretest questionnaire.

Regarding posttest and delayed posttest, the distributions of the scores on RCGP for each group followed a normal distribution, according to the Kolmogorov–Smirnov one-sample test. Before using the analysis of covariance, posttest scores were transformed with the power of square for fitting the assumption of equal variances of the scores between the two groups. Test of homogeneous regression coefficients also confirmed equal regression slopes between the two groups while either posttest or delayed posttest scores were dependent variables. ANCOVA was conducted on posttest and delayed posttest with pretest as a covariate.

There was no significant main effect on posttest for the two groups (F=.38, p=.541); however, there was a significant main effect on delayed posttest for the two groups (F=6.53, p=0.014). Results showed that the delayed posttest of the experimental group was significantly higher than the control group. The progress from pretest to delayed posttest in RCGP could be attributed to the treatments of innovative worksheets while the two groups were taught in the same school by the same mathematics teacher during the winter vacation and the following three months.

The low observed power of posttest (d=0.093) indicated that a larger sample size for testing the effect of the experimental instruction on RCGP is required. Although not significantly different, the changes following explaining a test of RCGP and teaching reading proofs were somewhat larger than the changes following explaining a test of RCGP and teaching inequality. Delayed posttest intended to identify whether the performance on RCGP changes were retained or lost, suggest that these changes were retained just in the experimental group (d=0.706). The control group showed significant decrease from posttest to delayed posttest.

CONCLUSION AND REFLECTION

Significant difference in the total scores of RCGP was observed between the two groups regarding delayed posttest with pretest controlled, though no significant difference in the total scores of RCGP was observed between the two groups regarding posttest with pretest controlled. This proved that after two forty-five-minute sessions of teaching, the significant effect can be observed after 12 weeks. Moreover, the retained progress in RCGP is related to the instruction of reading geometry proofs and not merely due to retesting conditions or spontaneous development.

There are some explanations for the absence of short-term effects. First, the test-retest condition might make statistically equal progress in the two groups for a short period of time, because reviewing a test is also a kind of learning. Thus, the delayed posttest is always thought to show the effectiveness of instruction. But, why cannot the progress of the control group last for a longer period of time to the delayed posttest? Based on schema theory (Norman, Genter, & Stevens, 1976), knowledge representation consists of nodes and networks, and meaningful learning is to interconnect the nodes by links. Our control group might have not made nodes or networks store in long-term memory because test-retest condition did not really facilitate semantic linkage. However, the instruction for the experimental group might benefit students' cognitive operations, e.g. chunking (Batting & Bellezza, 1979) or inference making (Pressley, 2000), which might further affect students' learning after instruction. Thus, tracking cognitive operations which could be effective over time is our next step for modifying the innovative worksheets and integrating different models of text comprehension.

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EXPERIENCING EXAMPLES WITH INTERACTIVE DIAGRAMS: TWO TYPES OF GENERALIZATIONS

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We examined the learning of mathematical concepts using examples displayed by means of relatively simple interactive diagrams (ID). We studied changes in concept images by qualitative discourse analysis of 5 pairs of 15 year old students. Each pair was interviewed while working on two algebra and functions activities, one that required learning a new concept and another that required relearning a previously studied concept. The two states of knowledge and the design of the diagrams affect the learning process that resulted from the use of interactive examples. Learning from uncontrolled random examples was not sufficient when the learning involved acquiring a new view of a concept that students had already studied formally. The examples created by rendering within controlled categories were generalized into example-spaces according to the designed categories. The random, non-structured sequence of examples was more difficult to acquire at first, but for determined students it leveraged the conceptual understanding and its implementation in problem-solving tasks.

LEARNING NEW CONCEPTS BY GENERATING EXAMPLES

The widespread use of examples in mathematics textbooks serves as a means of communication and mediation between learners and ideas. If examples are well selected, the variations between examples are the means by which students can distinguish between essential and redundant features (Herskowitz 1990, Goldenberg & Mason 2008, Watson & Shipman 2008). Computerized environments and current graphic technology have changed our way of thinking about concepts and representations in mathematics. Electronic mathematical texts introduce various uses of visual information and make possible the interaction between the reader and the electronic texts (Yerushalmy 2005). Interactive examples are an important component in texts of this type. Based on research about learning with software environments (e.g. Kieran & Yerushalmy 2004, Hershkowitz & Schwarz 1998), we conjectured that interactive diagrams embedded in mathematical activities would affect the learners awareness of what features of an object make it an example and what features can be varied to form a class of similar or related examples. In attempting to explain the cognitive processes involved when students learn new concepts or relearning aspects of known concepts with interactive examples, we watched for changes in their concept image (Tall and Vinner, 1981). In the current study, examples were not predesigned given examples nor students' self generated examples; interactive diagrams were produced upon students' control on a predesigned generator of examples. We sought to learn how the exposure to multiple random examples, and the design of user controls affect learning as reflected in

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 385-392. Thessaloniki, Greece: PME.

changes in the concept image and in the problem-solving processes. Students were given two types of examples: for the pre and post experimentation phases of the activities they had to solve problems presented with static examples on paper, designed to challenge and scaffold existing concept images; during the experimentation phase they were using interactive examples and the students decided how many examples to look at, when it was relevant to generate them, and which were the relevant examples. The present study was part of a larger research on learning with interactive diagrams in which we observe students learning algebra in the midst of activities based on experiencing examples drawn by interactive diagrams (IDs) and we report finding from work on two tasks (Figures 1 and 2).

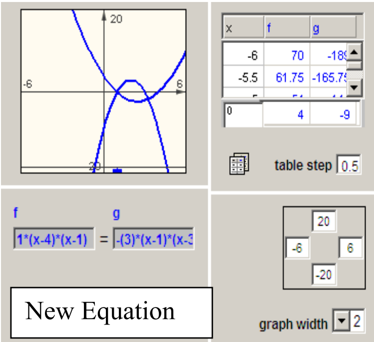


Figure 1: ID1 of Task 1: quadratic equation

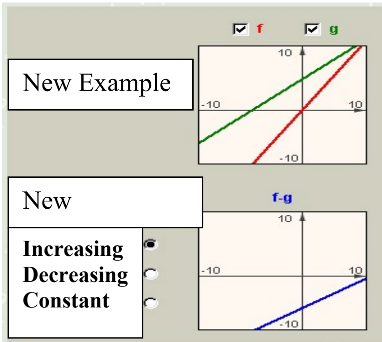


Figure 2: ID2 of Task2: difference function

Both IDs are easy to operate presenting familiar signs and representations. ID1 presents a quadratic equation as a comparison of two functions, enables users to generate a new random example of equation in multiple linked representations and to control the view of the graphs. ID2 provides a graphic representation of two random linear functions and of the difference function with the option to choose one of three types of difference graph inclinations.

Our sample consisted of 5 pairs of 15 year old students. Their teacher (the second author) encouraged mathematical inquiry. The two tasks were randomly assigned as the first or the second to each pair. We designed a three-phase interview that enabled us to study the evoked concept images arising when students learn a new mathematical concept – the difference function or take a new view of a known concept – a quadratic equation. Students were first asked to describe to their colleague the meaning of solving an equation (Task 1) and the difference between two functions (Task 2). Next, they were given paper and pencil problems to solve. This first phase helped identifying the students' concept images. In the following experimentation phase the students then received written and oral explanations about the interactive diagram and were told to ask for interactive examples for as long as they felt that they could learn from the examples. By analyzing their interactive experiences, we were able to identify learning events and explain the stimuli that prompted or discouraged learning events. Once they chose to move on to the post experimentation phase, they had to solve and explain similar but more challenging problems than those presented in the pre-experimentation phase. This stage helped us determine the effects of the experience with the interactive diagram. The entire interview took approximately half an hour during which students were observed and videotaped, and all their screens were captured.

FINDINGS

Staying with the known

The governing image that students brought to Task 1 was that solving an equation consists of manipulating the string of symbols to obtain the form of p(x)=0, using a formula to solve and viewing the graphic solution as the intersecting values of the p(x) graph with the X axis. The view of an equation as a comparison of two functions, as presented in the interactive examples, was completely new for them. The first pair regarded the graph of the two functions in the diagram as representing two unrelated equations, and conjectured that they might find common solution where both f(x)=0 and g(x)=0. Among the random examples there were a few cases in which the two graphs intersected on or close to the X axis, but they did not use these examples to elaborate their conjecture and chose to terminate the experimenting phase. The second and third pairs solved the equations algebraically and did not invest in relating the numerical results to views on the graph, and stopped generating examples after one example (Pair 2) or three examples (Pair 3). Pair 5, similar to pair 1 attempted to link the two pieces of knowledge about which both students felt certain: the symbolic procedure of solving quadratic equations and the view of a parabola as an equation in which the zeros represent the solutions. They viewed two examples but they quickly gave up and complained that the software was unclear and irrelevant because it shows two graphs to a single equation.

Acquiring a competing view to a known concept

As the rest of the pairs who were working with ID1, Pair 4 described the solution to a quadratic equation as two intersecting points of a single graph with the X axis. They struggled to find a connection between the two parabola graphs and the two algebraic solutions provided in the static example they were given at the pre-phase. Next, they experienced with the interactive examples, searching for a solution in the diagram and trying to link the X axis intersection points and the two solutions they were supposed to obtain. The turning point was due to the set of examples that depict a significant visual change in the mutual relations between the functions (Figure 3).

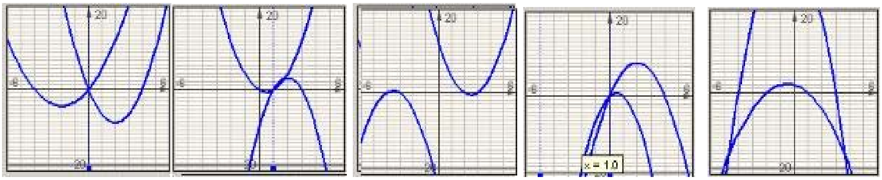


Figure 3: A sequence of five random examples

They started by identifying four solutions on each diagram and it contradicted their expectations about solutions of quadratics. Probably because they felt they were stuck they generated more examples and got a series of examples that looked significantly different: there was an example of a single intersection between the two graphs, an example of two "touching" graphs and one where the graphs were "not touching at all", which produced the realization of how the number of intersecting points of the two graphs suggest the number of solutions. Their previous discussion about the locations of the solutions on the x-axis was then replaced by one about the number of intersections. For the first time Sana views one equation as depicted by two graphs on the same screen. The notion that the two graphs might represent one equation grew out of the awareness of the existence of alternative mutual relations between the functions that resembles the familiar three symbolic situations of 2,1 or 0 solutions for any quadratic. That has led to the new image of an equation as a comparison and of the solution as the intersecting points of the graphs. During the next stage the students used their new perception to solve an equation that they could not solve procedurally: while they solved the given quadratic and linear equations using the standard algebraic solution, they solved a given cubic equation by evaluating the intersecting points between the two graphs depicted in the diagram. Clearly, comparing the successive diagrams and observing the differences had an important effect on the students' attention and awareness. This appears to confirm Davydov's claim (1972) that generalization comes from comparisons between examples, i.e., from the way in which differences enable one to see critical common features rather than seeing relevant or irrelevant properties they happen to have in common. Pair 4 moved from analyzing each diagram and observing its relevance or irrelevance to analysis of variance (the mutual positions of the graphs) and invariance (always having two graphs for one equation). Pair 4 went beyond the generalization and used it to completely reorganize their view about a known concept, about representations of equations and about the relations between graphs and solutions. Chazan, Yerushalmy & Leikin (2008) studied how challenging such reorganization is even for algebra teachers and suggest that viewing equation as a comparison of two processes requires using completely new lenses.

For Pair 5 who did not find the interactive examples of the experimentation phase relevant at all arrived at a turning point when they reached the post phase and were asked to solve on paper three equations, two of which were familiar (linear and quadratic) and one that was not (a cubic):

Solve 2x-8 =x+1 **Solve x^3-3x=$(x-1)^2$** **Solve x^2+4x=-x+1**

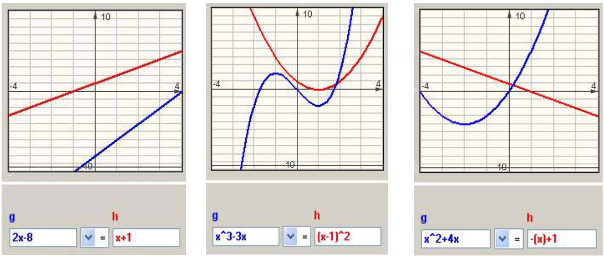


Figure 4: Task 2, post-experimentation paper tasks

They started by solving the quadratic equation the algebraic way. They regarded the fractional number that they obtained as problematic (the solutions are -0193, -5.193), so they looked for a graphic solution and assumed that the solution would be provided by the intersecting points between the two graphs and the X axis.

Iyov: Fraction, fractions - there is no solution.

Edi: But how? What do you mean there is no solution?

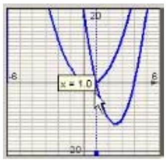
Nada: Why do you think so?

Iyov: It is obvious, you can see there is no solution… When we put both equations together we need to see a shared point for both functions on the X axis.

Nada: So what is the solution?

Iyov: there is no solution. There is not a shared intersecting point with the X axis. I mean… the straight line needs to pass at the point where the graph [intersects the x axis]…

The students reviewed the three examples and concluded that all three equations had no solution because the graphs in the given static diagrams did not intersect on the x-axis (figure 4). But they both knew that the linear equation had a solution, which contradicted their hypothesis and made them go back and recheck the interactive diagram that they had earlier considered to be irrelevant.



Edi randomly obtained the quadratic equation $2x^2$-2x=(5x-20)(x-1). He began solving the equation using the algebraic procedure but quickly realized by factoring that each side of the equation was zero at x=1. They viewed the two graphs and said: Now it makes sense… It [the vertical indication at x=1] is clear, the intersecting points of both functions are easily attainable. It is 1.

They identified the connection between the intersection of the two graphs and the intersecting points with X axis and were using the newly acquired relation to solve and check their solutions to the three given equations. The static examples in the post-phase that were designed to stimulate the students intellectually were indeed instructive and they were more so when combined with examples students have chosen themselves via the work with the interactive diagram. Apparently, the examples that pair 5 initially deemed not to be useful, relevant, or clear, unconsciously created rich visual representations that they later used when trying to solve a disturbing contradiction. They activated the numerous examples they had seen and made them accessible examples when their mind was triggered by a relevant situation.

Learning a new concept

The interviewees were not familiar with the concept of a difference function, and in most cases they failed to solve the tasks correctly during the pre experimentation phase. Their efforts to learn from examples appeared to be fruitful. We use the work of Pair 3 (Rene and Innas) to represent the work of all the students in the study. Rene argued at first that the difference between two functions is the result of "one (f) minus the second (g)," whereas Innas regarded the difference function as "the difference between the solutions of the two functions.″ Watching a given static example and attempting to solve the pre experimentation task on paper (Figure 5), they turned the two cases into two general rules: when f and g are increasing or if f is increasing and is the "larger" function, the difference increases.

Thus they already started to treat the difference as a function rather than a number, as first suggested by Innas. They assumed that increasing was analogous to "positive," and that if the difference was between the "larger" and "smaller", it must be positive and therefore increasing.

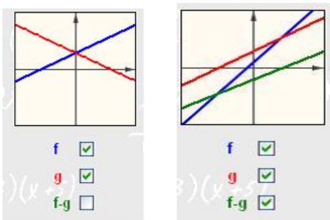


Figure 5: Static diagrams in the pre experimentation phase

Based on this conjecture the students moved to a systematic exploration with the interactive diagram (Figure 2). They started with the first category of the increasing difference graph, and the first example supported their conjecture (Figure 6, left):

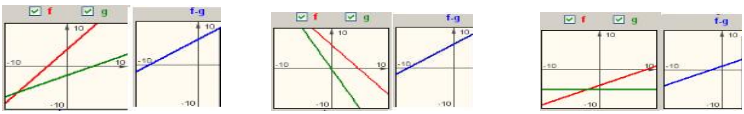


Figure 6: Confirming, contradicting and vague example generated by ID2

The second example (Figure 6, middle) refuted their conjecture, with two decreasing functions and an increasing difference. The students generated another example (Figure 6, right), which was not clear and certainly not what they had expected. They continued to generate new examples, which disproved their conjecture. They usually ignored the refuting examples and moved on fast and without explanations. Next, they moved to the second category designed in ID2 (decreasing difference function) and conjectured that they would obtain two decreasing functions if the difference results in a decreasing function. Again, the examples refuted their conjecture one after the other, and this time they expressed their frustration. "Rene: No! We wanted decreasing... You see? One is increasing and one decreasing."

They continued to generate examples, after which they made nine statements: the difference increases (1) when both functions f and g increase, (2) when f increases and g decreases, and (3) when f increases and g is constant; the difference decreases (4) when both functions decrease, (5) when f decreases and g increases, and (6) when f decreases and g is constant; the difference is constant (7) when both functions increase in parallel, (8) decrease in parallel, and (9) are constant. During the post-experimentation phase the students were again asked to complete a missing function in the f, g, and f-g triads given as paper task. At this stage they identified additional characteristics of the difference graph and were more accurate than at the beginning of the interview. In many cases they chose to go back to the interactive diagram and checked for an example that resembled the missing function in a given paper task. Thus, in Task 2 that required learning a new concept the students definitely learned from the examples; they were able to generalize based on the diagram categories and they were better solving the post-experimentation tasks. However, they remain at a descriptive level where they learned to produce a rule based on every set of similar examples, but were unable to extract a general structure regarding the difference of the slopes of any two linear functions.

IMPLICATIONS FOR THE DESIGN OF ELECTRONIC ACTIVITIES

Only two pairs of students changed their view of the solution to the equation (Task 1). These students were able to view an equation in a way that was not only new to them but also contradicted previously learned representations. All five pairs used the examples to learn the difference function (Task 2). It is important to extract from this general conclusion the two major trends in learning from generating examples using interactive generator: one is generalizing along the explicit structure of the example generator which was proved fruitful for arriving to descriptive generalizations; the second is learning by instrumenting the non-structured example generator by revealing inexplicit structures throughout organization of the examples in a way that helped to compare them and view the differences. The sequences of many examples created by students (freely or within a pre-designed category) shaped their learning. They were effective in the generalization of Task 2 and even more so in Task 1, where observing a sequence promoted inquiry into the distinctions, which then led to meaningful learning. Another question we asked ourselves was how do random un controlled examples be used to support conjectures and explain conflicts. We viewed our interviewees often finding the random examples more convincing when they confirming rather the contradicted their conjectures. Although random examples are by nature not controlled, students often used them as a personal tool to generate examples that they considered as appropriate examples that demonstrate a mental image they had or resembled a given paper diagram.

Although the focus of the study is on interactive electronic diagrams the activities include static examples and their roles had to be analysed as well. The given static examples in the pre and post phases created over-generalization and were mostly ignored when they seemed not to be relevant to the concept image already being held. However, as the interactive diagrams were simple and not intended for elaborated explorations, the design of the activities incorporating both paper tasks with static examples and interactive diagrams was of great importance in the structuring of learning.

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Thus research was supported by the Israel Science Foundation (grant No. 236/05)

THEORETICAL THINKING ON THE CONCEPT OF LIMIT: THE EFFECT OF COMPUTERS

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One of the goals of this paper was to analyze 11th grade students' reasoning and knowledge of limits through Sierpinska's model of theoretical thinking (Sierpinska's, 2002). Data were collected from 52 students through a questionnaire. These data revealed that mathematical tasks on limits might be categorized, based on Sierpinska's theory, to reflective, systemic and analytic thinking. The second goal of the study was to examine whether students working with mathematics applets based on Computer Algebra System (CAS) algebra can improve their performance in reflective, systemic and analytic tasks more than students working in a traditional environment. The analysis showed that students who worked with the mathematics applets improved their performance in analytic tasks.

INTRODUCTION

One of the main goals of this paper is to describe a conceptual framework for analyzing students' mathematical reasoning and knowledge of limits and to describe an intervention program which integrated computers in the teaching of limits. Much of the research on students' understanding of mathematics concepts showed that the concept of limits is not clearly defined in their minds (Davis & Vinner, 1986; Artique, 1997; Mamona Downs, 2001). The main reason, according to Fischbein, Jehiam, and Cohen (1995), is that school mathematics do not emphasize the idea of mathematics as a coherent, structurally organized body of knowledge. As a consequence, less emphasis is placed on theoretical thinking which is necessary for the development of the concept of limits. Thus, in the present study, the proposed conceptual framework, which is mostly based on the features described in Sierpinska's model of theoretical thinking (2002; 2005), constitutes an attempt to encompass the whole spectrum of students' understanding of limits. Furthermore, the study provides an empirical verification of the proposed model through an intervention program that used mathematics applets in order to trace differences among two groups of students in the context of theoretical thinking in limits.

THEORETICAL BACKGROUND AND PURPOSE OF THE STUDY

Learning calculus was the subject of extensive research for a long time. One of the conclusion arising out of this research is that students in general develop routine techniques rather than conceptual understanding of the theoretical concepts (Asiala, Cottrill, Dubinski, & Schwingendorf, 1997). The limit concept is rich in abstraction and calls for high level of conceptual understanding which many students find hard to

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 393-400. Thessaloniki, Greece: PME.

cope with (Parameswaran, 2007). The abstract nature of limits presents major difficulties for most students and they have little success in understanding this important mathematical concept (Asiala, Cottrill, Dubinsky & Schwingendorf, 1997). A program of research into how students learn such a topic can point to pedagogical strategies that can help improve this situation. This paper is an attempt to contribute to such a program, and thus we first refer to Sierpinska's (2005) theoretical perspective and then we explore through this theoretical perspective the development of the limit concept to two groups of students.

Categories of Theoretical thinking

Sierpinska provides a detail explanation of the categories of theoretical thinking (see Sierpinska, 2005). We modified Sierpinska's ideas in such a way as to reflect the perspective we used in testing students' theoretical thinking while they are working on tasks involving limits. According to Sierpinska (2005), reflective thinking is purposeful thinking which aims at facilitating further understanding knowledge and in this case the knowledge of limits. In this study, we examined students' understanding of the symbolic expression of limits and their understanding of the idea of infinity together with the way they link a limit to the behaviour of the function when x becomes very big.

Systemic thinking is the thinking about systems of concepts. Thus, the meaning of limits is established based on the definition of properties. When students are advised to think about the meaning of limits, they rarely think about the definitional conditions that the concept satisfies (Parameswaran, 2007). They usually think about some examples and this, of course, cannot lead them to producing generalizations (Ferrini-Mundy, & Graham, 1991). Thus, in this study we focussed on the informal definition of the limit concept so that to determine the students understanding of a limit per se and then tried to figure out how the students understand the nature of a limit.

The analytic thinking refers to the development of specialized representational systems such as the signs, the symbolic notations and specialized terminology. Limits appear to develop concurrently with methods for representing and manipulating them symbolically (Asiala et al., 1997). In the present study, we investigated whether students could recognise when a limit exists at a specific point on a given representation and whether they could provide its value.

The Effect of Computers in Developing Calculus Concepts

In addition, we examined how Computer Algebra System (CAS) and especially mathematics applets can help students to have better results in limit tasks by improving their abilities in reflective, systemic and analytic thinking.

A number of research studies supported that CAS can help students in the investigation of problems and notions through different forms of representations (Porzio, 1999; Kaput, 1996). Through the use of CAS, teachers can help students understand mathematics and learn strategies to develop abstract structures in algebra (Arnold, 2004). In addition, CAS motivates students to focus mainly on the interpretation and representation of notions rather than on the execution of different procedures (Arnold, 2004; Pierce and Stacey, 2007). The use of computers, in combination with specific software which executes the process internally, helps students to obtain the properties of the object produced before, during, or after studying the process itself, instead of having to encapsulate the process first (Tall, 1993). Based on these results, we hypothesized that specific mathematics applets in a CAS environment may contribute to the development of students' understanding of the concept of limit and advance their theoretical thinking.

Purpose of the Study

In this study, we hypothesized that the concept of limits has distinct aspects that represent the theoretical model, and we proposed to investigate the relationship among reflective, systemic and analytic theoretical thinking as it unfolds through students' responses to tasks that refer to limits. We also hypothesized that students' theoretical thinking can be enhanced through the use of software such as mathematics applets. Thus, the aims of the study were as follows:

1. To investigate whether different tasks in the context of limits can be categorized as reflective, systemic and analytic.
2. To examine whether students working with mathematics applets based on CAS algebra can improve their performance in reflective, systemic and analytic tasks than students working in a traditional environment.

METHOD

Participants

Two groups of secondary school students from three intact classes were involved in the study. One of these classes was the experimental group (group 1) while the rest two classes comprised the control group (group 2). Fifteen students from one of the intact classes formed the experimental group, while 37 students from the rest intact classes formed the control group. All students attended high school in grade 11 and all of them specialized in mathematics. The experimental group was taught the concept of limits through the use of mathematics applets and the control group was taught the same concepts and content in the traditional form i.e, following the activities dictated by school textbooks.

Procedure

A pre test was applied to all students just before the beginning of the instruction on limits. After the pre test, a three periods intervention program was organised. Each period lasted for 45 minutes. The post test, which was the same as the pre test, was administered at the end of the instruction. The instruction as well as the tests were administered during these students' mathematics courses. Students belonging in the experimental control group investigated the concept of limit with the use of mathematics applets. The following applet (http://www.calculusapplets.com/informallimits.html) is an example of the applets used during the instruction. In this example (see Figure 1), students had to choose from the applet menu specific functions and find the limit of these functions at a certain point. The applet presented at the same time the numerical value of the function at the specified point (in the cases where the limit existed). Students could check whether the limit of the function was the same with the numerical value of the function at that point.

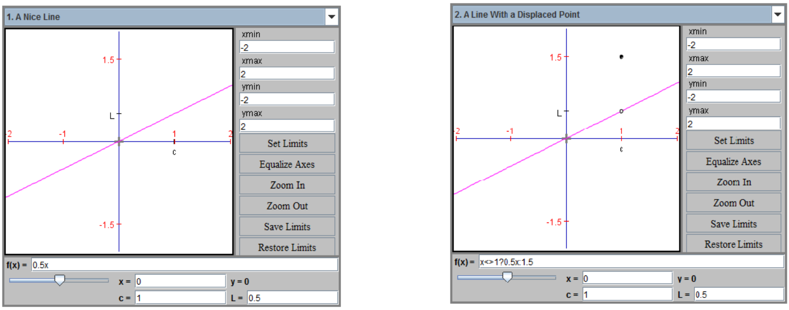


Figure 1: An applet example

Tasks in the test

For the purposes of the study, the researchers constructed a test to serve as a pre and post test. The test included 13 questions which reflected Sierpinska's model of theoretical thinking. Specifically, four tasks aimed to investigate students' abilities in reflective thinking, three in systemic thinking, and six in analytic thinking. The tasks on reflective thinking investigated students' abilities to identify the conceptual meaning of limits. The tasks that aimed to investigate students' abilities to operate in systemic thinking explored their ability to define limits. Finally, the tasks that were presented in order to investigate students' ability to operate in analytic thinking focused on symbolic and graphical representations. Examples of the reflective, systemic and analytic tasks are shown in Table 1.

Scoring and Analysis

Students' fully correct responses were marked with 1 and the incorrect responses with 0. If a student gave a partly correct response, for example if he/she gave a correct answer but wrong justification, this again was marked with 0.

The confirmatory factor analysis (CFA) was applied in order to investigate whether the model of theoretical thinking with its components fits our data. Multivariate Analysis of Variance (MANOVA) was used to compare the performance of experimental and control group students.

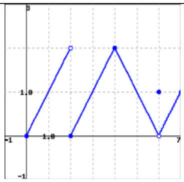
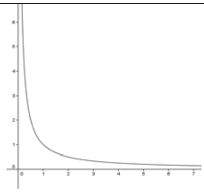
Reflective thinking	Systemic thinking	Analytic thinking
	A limit of a function is a number past which a function cannot go. True/ False	
The limit of the function at $x = 4$ is:...... Explain your answer.		What do you understand by $\lim_{x\to\infty}\frac{1}{x} = 0$?

Table 1: Examples of tasks.

RESULTS

The results are presented in relation to the aims of the study. First, we examined the hypothesis, implied by the first aim of the study, i.e., whether the reflective, symbolic and analytic components of the theoretical model constitute distinct modes of the limit concept. Second, we investigated if the performance in the components of theoretical thinking of students who worked with mathematics applets improved more than the performance of students received the traditional instruction.

The Distinct Nature of the Components of theoretical thinking

From a structural point of view, three factors, the reflective, the systemic and the analytic, should be able to model the performance of students on the tasks addressed. Figure 2 represents the model which best describes Sierpinska's theoretical model (2002). Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model. CFA showed that each of the tasks employed in the present study loaded adequately (i.e., they were statistically significant since z values were greater than 1.96) on each factor, as shown in Figure 2. CFA also showed that the observed (students' performance on tasks) and theoretical factor structures (the components of theoretical model) matched for the data set of the present study and determined the "goodness of fit" of the factor model (CFI=0.953, x^2= 74,52, df= 61, x^2/df=1.22, RMSEA=0.06), indicating that the reflective, the systemic and the analytic components can represent three distinct functions of students' thinking.

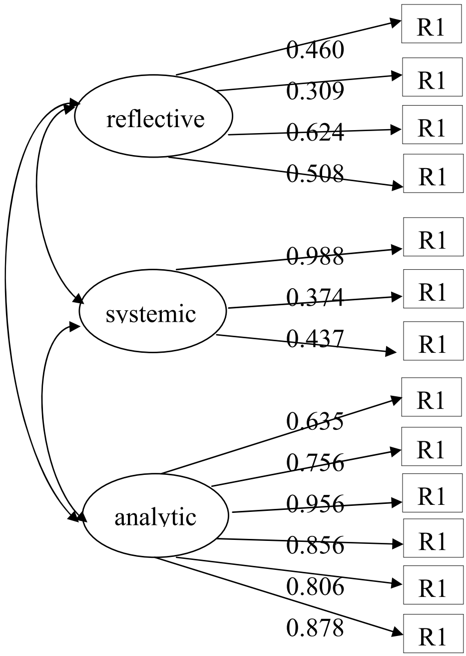


Figure 2: Model of the performance of students on the tasks addressed

The Effect of Mathematics Applets on the Components of Theoretical Thinking

In order to assess if students in the experimental group performed better than students in the control group, we conducted MANOVA with the performance of students in reflective, systemic and analytic tasks in the pre and post tests as dependent variables. The means of each group of students in the pre and post test in each component of theoretical thinking is shown in Table 2. The analysis revealed a significant effect for the experimental group in the analytical tasks $F(1,50)=4.42$, $p<.05$, and a non significant for the control group in the reflective, systemic and analytic theoretical thinking of students. This means that students who were taught with computers the limit concept gained more in the analytic tasks of the tests than students in the control group.

	Group 1		Group 2		Group 1		Group 2	
	Pre-test	SD	Pre-test	SD	Post-test	SD	Post-test	SD
reflective	0.383	0.26	0.338	0.27	0.617	0.26	0.568	0.30
systemic	0.667	0.21	0.546	0.24	0.600	0.20	0.486	0.19
analytic	0.481	0.29	0.463	0.30	0.762	0.19	0.598	0.27

Table 2: Mean values of students' responses.

Discussion

The first aim of this study was to examine whether it is possible to identify between different types of tasks that represent the reflective, systemic and analytic thinking of students in understanding the concept of limit. It can be deduced from the data presented in this study that mathematical tasks may be categorized based on Sierpinska's theory, to reflective, systemic and analytic thinking. The second aim of the study was to examine whether computers can improve students' development of theroretical thinking. The results showed that the intervention program helped students in the experimental group to gain more in analytic tasks. These results confirm previous studies (Arnold, 2004; Pierce and Stacey, 2007) which provided evidence that computers enhance students' understanding of calculus concepts through the opportunities they offer in representing concepts in different forms.

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INVESTIGATING PRE-SERVICE MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF NUMBER PATTERNS

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The purpose of this paper is to investigate the development of two pre-service mathematic teachers' pedagogical content knowledge with special attention to students' understanding of and difficulties with finding the rule of number patterns. A case study was conducted to explore this development through a school practicum course and how the observations of number pattern lessons in schools contribute to the knowledge of student difficulties. The analysis of data indicated that the observation and discussions of number pattern lessons resulted in considerable change in the way pre-service teachers addressed student difficulties in their lessons.

INTRODUCTION

Pattern is emphasised as an important topic in mathematics by many mathematicians and mathematics educators (Reys *et al*., 1984, Fox, 2005) and generalising about patterns is thought to be particularly important because the structure of mathematics can be observed by searching for patterns and relationships (Hargreaves *et. al.*). Learning about patterns gives students the opportunity to make verbal and symbolic generalisations, therefore to make the transition from arithmetic to algebra (English & Warren, 1998). Nevertheless, literature points out some student difficulties on finding rules of patterns (MacGregor & Stacey, 1995; English & Warren, 1998; Lannin, 2002). Research indicates that although students could continue to find the next term of a pattern from the previous one, they had difficulties in formulating the algebraic rule of the pattern (English & Warren, 1998; MacGregor & Stacey, 1995; Lannin, 2002). Orton & Orton (1999) mentioned that encouraging students to generate the next term in a pattern based on the previous one might prevent them from seeing the general structure of the pattern.

Numerous studies have been conducted which examined students' strategies of finding rules of number patterns (e.g. Hargreaves *et. al*; Stacey, 1989). Pre-service teachers' understanding of number patterns has received less attention from the research community. For instance, Zazkis & Liljedahl (2002) investigated the attempts of a group of pre-service elementary school teachers to generalise a repeating visual number pattern. Their results indicated that students' ability to express generality verbally was not accompanied by, and did not depend on, algebraic notation. In an attempt to contribute to this growing literature, this study investigates the development of pre-service mathematics teachers' knowledge of

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teaching number patterns. For this investigation we use pedagogical content knowledge (PCK) framework which will be explained in the next section.

THEORETICAL FRAMEWORK

In teacher education research, the notion of pedagogical content knowledge (PCK) was proposed by Shulman (1986) as a new domain of teacher knowledge and it has been a useful framework for exploring what teachers need to know or to develop for an effective teaching of a particular subject. Shulman (1986) defined PCK as a blend of content and pedagogy and further explained it as "the most useful forms of [content] representation . . . the most powerful analogies, illustrations, examples, explanation, and demonstrations—in a word, the ways of representing and formulating the subject that makes it comprehensible to others" (p.9). Many researchers defined components of PCK and agreed on the two key components of Shulman (1986): (a) knowledge of instructional strategies and (b) student conceptions with regard to the subject matter (For a detailed list of components see Park & Oliver, 2008). In this study, we will focus on one of these two components of PCK: *knowledge of students' understanding of and difficulties with specific mathematics topics*. This component is particularly important since understanding students' thinking in specific content domains provides a basis for teachers to reconceptualise their own knowledge more broadly (Carpenter, Fennema & Franke, 1996). Furthermore, there is research evidence that teachers' knowledge of students thinking has led to more effective teaching (Cobb, Wood, Yackel, 1990; Kovarik, 2008).

The purpose of this paper is to investigate the development of two pre-service elementary mathematics teachers' PCK with special attention to students' understanding of and difficulties with finding the rule of number patterns.

METHODOLOGY AND CONTEXT OF THE STUDY

This case study is part of a wider study which focuses on the development of pre-service mathematics teachers' PCK of number patterns in terms of four components of PCK suggested by Grossman (1990). The participants of this current study are two female pre-service mathematics teachers who will be entitled to a certificate for teaching mathematics in secondary schools when they graduate from a four-year teacher training program in a university in İzmir, Turkey. Data was collected during the university component of "School Practicum II" course. The instructor of this course is the first author of this paper. Instructor was tutoring eight pre-service teachers who taught number patterns to their peers as part of micro-teaching activities. After that pre-service teachers observed number pattern lessons during school placements for two weeks. During the university component, they watched these lesson videos and discussed difficulties with number patterns encountered by students. Following this, they taught number pattern lessons again. Micro-teaching lessons and pre-service teachers' discussions during the university component of school practicum course were recorded. In addition to that, semi-structured interviews during which pre-service teachers reflected on their teaching were conducted. The data were analysed to explore how they addressed students' difficulties with finding the rule of number patterns and how they developed this knowledge of student difficulties during "School Practicum II" course. Following research questions were formulated in specific terms to be explored in this study:

- What kinds of student difficulties with number patterns do two pre-service teachers report during the interviews?
- How do two pre-service teachers address students' difficulties with finding the rule of number patterns during their first micro-teaching lessons?
- How do two pre-service teachers' observations of number pattern lessons in schools contribute to their knowledge of student difficulties with finding the rule of number patterns?
- How do two pre-service teachers address student difficulties with finding the rule of number patterns during their second micro-teaching lessons?

RESULTS

In this section, results obtained from the analysis of data will be reported in two sub-sections. Each sub-section is devoted to the results from a case. For each case, results are presented in response to the research questions above.

Case of Didem

Concerning the first research question of what kinds of student difficulties with number patterns reported by the participants during the interview, Didem described student difficulties with number patterns in a general sense. She stated that students will have difficulties with seeing the relationship between numbers. In relation to the second research question, the analysis of her micro-teaching video indicated that she did not address the difficulties with finding the general term of a number pattern. For instance, she did not explain how the algebraic rule of the pattern was found. She started her lesson with an example which requires to find the 100^{th} element of the pattern: "1, 3, 5, 7,…". She modelled this number pattern but she did not use it to generalise the pattern to the n^{th} term. She focused on the differences between the consecutive elements instead of the relationship between n and the corresponding n^{th} term. She found the 100^{th} element using the verbal explanation of the rule of the pattern: "2 times the number minus 1". This is followed by another example of a pattern which has the rule of $3n$-1. She found the first six terms of this pattern. When the instructor asked her whether the difference between the terms of the pattern and the quotient of the general term is always the same, she replied as follows:

Didem: Let's see. For instance, $4n$+1. For n=1 we get 5, for n=2 we get 9, for n=3 we get 13, for n=4 we get 17. The difference is always 4 and the quotient of n in the general term is also 4. For $2n$, the difference was 2 and for $3n$-1 the difference was 3. Yes, we can generalise such a relationship.

As can be seen above, she discovered such a relationship during her micro-teaching lesson. She was then asked how to find the algebraic rule of a pattern where the differences between the consecutive terms are not the same. She then wrote the number pattern "5, 9, 17, 29,…" and investigated its general term. However, she could not find it. In summary, Didem's subject knowledge fell in short and she did not address the student difficulties with generalising a pattern to its general term.

After the discussions on their observations of number pattern lessons during the school placements, Didem stated that she realised students' difficulties concerning to finding the algebraic rule of a pattern. When evaluating her first lesson, she realised that she did not explain how the algebraic rule was found:

> Didem: During the number pattern lessons which I observed in schools, I realised that students had difficulties in understanding how the general term is found. They asked a lot of questions about that. It is an unknown and it's a new concept for them. It's not possible to teach it by giving the general term directly and asking students to find various terms, like I did in my lesson.

Didem also reflected on the way she used a model for patterns. She said the following:

> Didem: As a result of my observations, I realised that I mostly focused on the numbers, not visual models, to teach patterns. As I understood from the teacher's examples, the curriculum expects teachers to use visual models and relate these models to numbers. However, I didn't use the models that way.

As a result of her observations in school, it can be claimed that Didem became aware of students' difficulties with finding the algebraic rule of the pattern. The analysis of her second micro-teaching lesson also indicated that the way she addressed the student difficulties with number patterns has improved. First of all, she explained how the algebraic rule of the pattern can be found. She started with an example of a pattern: "3, 4, 5,…" and then "1, 3, 5, 7, …". She prepared a table by writing the numbers of the terms in the first column and the corresponding terms in the second column and focused on the relationship between these as ordered pairs rather than focusing on the relationship between the consecutive terms of the patterns. She then asked her peers to find which element of the pattern "n+2" would be 11.

Compared to her first micro-teaching she used more visual models to explain patterns. However, she did not relate the visual models to the elements of the pattern to construct the algebraic rule. She just used the model merely to visualise the numbers. Another aspect of her second micro-teaching lesson is her emphasis on the relationship between the differences between the consecutive elements of the pattern and the quotient of the n in the general term:

> Didem: Can I claim that the difference between consecutive numbers is the same as the quotient of the n in the general term? What was the amount of

> increase for 3, 4, 5,… It's 1 and the quotient of n in the general term n+2 is also 1. What was the amount of increase for 1, 3, 5, 7, …It' 2 and the quotient of a in the general term 2a-1 is also 2. For 3n+1, the quotient is 3 which is the differences between consecutive numbers of the pattern

During the interview which was conducted after her second micro-teaching session, she reflected on her teaching. She mentioned that she considered student difficulties with number patterns especially the difficulties with the unknown n:

Researcher: Did you consider student difficulties with number patterns?

Didem: During my second lesson, yes. For my first lesson, I thought students would easily understand how to find the algebraic rule because they are familiar with the unknown n from algebra. However, during my observations in school, I realised that it's not an easy job. A lot of students were not clear about it after the lesson.

Overall, it can be claimed that the observation of number pattern lessons was useful and helped Didem to improve the way she addressed student difficulties with finding the algebraic rule of a number pattern. Although she improved her teaching by focusing on the relationship between n and the corresponding terms of the patterns instead of the differences between consecutive elements, she could not use visual models of the patterns in an effective way. In addition to that, she did not use any example of a pattern which does not have a constant difference between its consecutive elements.

Case of Filiz

Concerning the first research question of what kinds of student difficulties with number patterns reported by the participants during the interview, Filiz mentioned difficulties concerning finding the algebraic rule of the pattern. The analysis of her first micro-teaching, in response to the second research question, indicated that she herself had such difficulties during her teaching. For instance, she found it difficult to discover the algebraic rule of the pattern "0, 2, 4, 6, 8…" which was originally "2, 4, 6, 8…" in her lesson plan. Furthermore, when asked by the instructor, she could not find the rule of the pattern "1, 5, 13, 29,…" which does not have a constant difference between its consecutive elements. She mentioned that it is not always possible to find the rule of such a pattern, therefore such patterns can be constructed based on the previous terms:

Filiz: This is one of those patterns which can only be constructed by knowing the former elements. What did we do for the earlier patterns? We considered value of n as an unknown. Therefore we could reach any term we wanted. However, for patterns like "1, 5, 13, 29, …", since we can't use the term numbers, we construct the elements of the patterns like a staircase. We use the former to obtain the latter.

She explained this relationship between consecutive terms verbally, not algebraically: "double the previous term and add three".

Similar to the way Didem used the models, Filiz used the model to illustrate the increase between the consecutive elements. In other words, she simply added three units to the previous figure. During the interview, she stated that "using the visual model, it is easier to see that the numbers increase in three's".

In relation to the third research question, observations in school contributed to Filiz's knowledge of student difficulties with finding the algebraic rule of the pattern. She mentioned that the teacher was dependent on the textbook and did not address potential student difficulties concerning finding the rule of the pattern.

Before her second micro-teaching lesson, Filiz evaluated her first lesson. She said that she did not address students' difficulties and her subject knowledge felt in short. She also added that she should have used the model before giving the number pattern instead of modelling the numbers afterwards. However, in her evaluation, she did not mention anything on how a model could be useful to find the rule of the pattern.

In relation to the fourth research question, we found that Filiz improved her teaching in terms of addressing the student difficulties with finding the algebraic rule of the pattern. She started the lesson with the pattern "2, 4, 6, 8, …". Although she used a visual model, she found the rule of the pattern using a table of values. When finding the rule, it was observed that she was more aware of the need to find an algebraic rule:

> Filiz: I can find the next term using the previous term. But what if I should find the 378th term? Shall we find every element one after the other up to 378? But, it would be difficult and takes time. Therefore we should find a rule.

She then explained that *n* determines the n^{th} term of the pattern. When finding the pattern using the table, she focused on the relationship between *n* and the corresponding term instead of focusing on the differences between the consecutive terms and obtaining the terms from the previous ones like she did in her first lesson. She continued with another example of a pattern "2, 5, 8, 11,..". She again modelled this pattern discovered the rule using trial and error focusing on numbers instead of the model:

> Filiz: Let's try the rules. We will try to guess. Look at the first term. There are 2 units. Can I say two times? Look at the second one. There are 5 units. 2 times 2 is 4. It doesn't give 5. So we did not find the right rule. Let's try 3 times. 1 times 3 is 3, 2 times 3 is 6, 3 times 3 is 9. All of them is 1 less than 3 times. So it's $3n$-1.

As can be seen above, she constructed the algebraic rules of the two patterns above focusing on the relationship between *n* and the corresponding term instead of constructing the elements from the previous ones. In that sense, she improved her teaching. However, she did not mention anything about the relationship between the increase in consecutive terms and the quotient of *n* in the general term.

In summary, Filiz's lesson improved in two aspects: first she focused on the relationship between *n* and the corresponding term instead of consecutive elements.

Second, she expressed the general term algebraically instead of expressing verbally. However, she still could not use the visual model to discover the rule of the pattern.

DISCUSSION AND CONCLUSION

The discussion section attends to two aspects. The first is concerned with the development of two pre-service teachers' knowledge of student difficulties with patterns. The second is related to effectiveness of the school observations and discussion on these observations in the university component of school practicum course. With regard to the development of pre-service teachers' PCK, the data indicated that both pre-service teachers improved their knowledge of student difficulties concerning to finding the rule of a pattern. Furthermore, they reflected this improvement in their micro-teaching lessons. The most obvious improvement in their teaching was related to the way they constructed the terms of the pattern. During the first lessons, both pre-service teachers constructed the terms from the previous ones, in other words, focused on the relationship between consecutive elements. On the other hand, during their second lessons they focused on the relationship between n and the corresponding n^{th} term. Therefore, they improved their teaching in terms of constructing the algebraic rule. Despite these improvements, pre-service teachers' lessons were not effective in two aspects. The first is concerned with the use of visual models. They merely used the models to visualise the increase in consecutive terms. For instance, they did not use the visual models to reveal the relationship between n and the corresponding n^{th} term. The second is concerned with privileging trial and error method. Pre-service teachers found the algebraic rules using trial and error.

With regard to the issue of effectiveness of the activities in the school practicum course, it can be concluded that observations in real classroom settings and discussions of these observations in the university component resulted in an improvement of two pre-service teachers' PCK in relation to student difficulties. Observing the students in the classroom helped pre-service teachers discover the difficulties encountered by students in a more realistic context. For this improvement, school teachers also played an important role. Observing lessons of these teachers contributed to the development of pre-service teachers' knowledge in many aspects such as content knowledge, curriculum expectations and different teaching approaches. On the other hand, the teachers who were observed did not use the visual models effectively and this affected the way pre-service teachers used the models. This reveals the importance of the choice of teachers for the school practicum course.

The study also implied that the PCK framework, with its other components, could be useful in diagnosing or monitoring the development of pre-service teachers. In this paper, we paid special attention to the "content" aspect of the framework rather than using PCK in a more general sense. Research is needed to explore the development or investigation of PCK by bringing the content dimension into play.

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GESTURES AND VIRTUAL SPACE

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We introduce the notion of a virtual space, which is a mathematical and cognitive space that is created through gestures. The notion of a virtual space is illustrated in a case study of two teachers who create a virtual space while constructing the graphical antiderivative of a function. We discuss some features of a virtual space, including the way it is physically constrained by elements of a person's gesture space, and its relationship to various types of gestures – namely deictic, iconic, and metaphoric gestures. We also discuss potential advantages and disadvantages of virtual spaces in mathematical thinking and learning.

BACKGROUND

The learning and teaching of mathematics are complex processes that make use of a variety of semiotic resources activated by both students and teachers. According to Radford (2009) thinking does not occur solely in the head but also in and through a sophisticated semiotic coordination of speech, body, gestures, symbols and tools. He thus proposes a multimodal view of cognition, for which he uses the term *sensuous cognition*. Arzarello has also postulated how an enlarged notion of semiotic system, which he calls a *semiotic bundle,* comprising semiotic resources and their mutual relationships, gives rise to multimodal thinking (Arzarello, Paola, Robutti, & Sabena, 2009). While the semiotic bundle includes the classical semiotic registers (Duval, 2006), such as symbolic algebra and graphs, recent research has also highlighted its extension to the important role gestures can play in shaping and constituting mathematical thinking (Arzarello, 2008). McNeill (1992) has classified these gestures as beat, iconic, deictic, metaphoric, and cohesive, with the key distinction between iconic and metaphoric gestures being that while both refer to a visual image, for metaphoric gestures the image pertains to an abstraction (Roth, 2001). Since mathematical content is primarily abstract one would expect metaphoric gestures to be common, but it is not yet clear to what extent this is the case.

There have been a number of attempts to describe the space of gestures, with McNeill (1992) speaking of it as a space in the frontal plane of the body in which gestures are usually performed. Arzarello (2008, p. 162) extends this and introduces the concept of an APC space, where "different perspectives can be combined in a shared environment for cognition. I call it the cognitive space of action, production and communication (APC space)", noting that "gestures constitute an important ingredient of learning, hence of the APC space" (*ibid*, p. 170). In this paper we introduce and advance the novel notion of a *virtual space*. This *virtual space* is simultaneously a mathematical and a cognitive space that is linked to, and constrained by, the gesture space, and related to the APC space. We will illustrate

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 409-416. Thessaloniki, Greece: PME.

this concept with the case study of two teachers and the deictic, iconic, and metaphoric gestures they performed as they constructed a graphical antiderivative of a function. We also consider the potential implications of this virtual space for research into mathematical thinking and learning.

VIRTUAL SPACE: A FRAMEWORK

The distinction between a virtual space and gesture space is perhaps best illustrated using the following hypothetical example. When thinking about the number "3", a student might point sequentially to three places within her gesture space, as shown in Fig. 1. The sequential nature of these three gestures creates a virtual space (see Fig. 2) that resembles a number line, with "three" occupying a position to the right of "two", which is likewise to the right of "one". The virtual space in this example is made up of mathematical objects, properties, and relationships such as discrete quantities, an ordering system, and left-to-right orientation. However, this virtual space is constrained by the student's physical gesture space, which is borne out in that this "number line" is actually a curve, limited by the reach of the student's arm. Thus, although a virtual space is primarily a mathematical and cognitive space, it is also a function of the physical gesture space in which it is created.

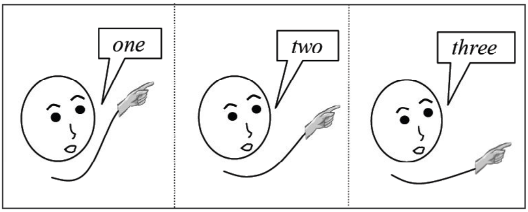

Figure 1: Gestures are used to create a virtual space

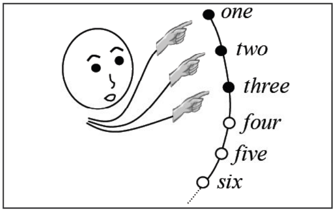

Figure 2: The virtual space

A virtual space is created together with gestures in a dialectic process. Gestures initially create a virtual space through their mathematical properties, which are conferred onto the virtual space while the gestures are performed, and are maintained cognitively after the gestures are completed. Once created, however, no physical markers of the existence of the virtual space remain, unlike the way that a diagram drawn on a piece of paper has a permanent trace. Instead, the virtual space is maintained cognitively in the minds of those who use it. Thus, a virtual space can be considered to be a cognitive space, in addition to its being a mathematical space that is constrained by a person's physical gesture space. The virtual space provides a framework for interpreting subsequent gestures that are performed within it, as gestures are interpreted as mathematical objects within the virtual space. Thus, the mathematical properties associated with the initial gestures generate the virtual space, and are later used to make sense of further gestures enacted within the virtual space.

ANALYSIS OF A VIRTUAL SPACE: THE ANTIDERIVATIVE GRAPH

Two female secondary school mathematics teachers—Ava, who had seven years of teaching experience, and Noa, who had three years of teaching experience—worked together on an antiderivative for approximately one hour, during which time they were videotaped and audiotaped. The teachers knew each other well, but neither had taught calculus at Yr 13 level (the last year of secondary school in New Zealand).

The task required them to design a method for finding the distance-height graph of a hiking track from its gradient graph, which is given in Fig. 3. This is mathematically equivalent to finding the graphical antiderivative of a function presented graphically. The task was designed using Model-Eliciting Activity design principles (Lesh, Hoover, Hole, Kelly, & Post, 2000).

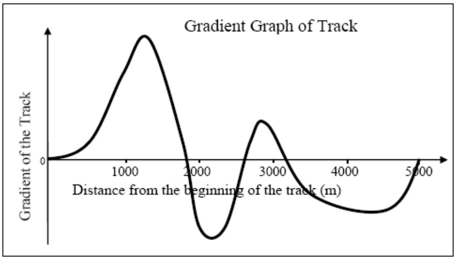

Figure 3

In this paper, we present and analyse a portion of Ava and Noa's activity in which they create and work within a virtual space in order to solve the task. It is significant to note that before creating a virtual space, Ava and Noa had already given two verbal descriptions of the features of the gradient of the hiking track, but had not yet attempted to draw a graph of the track. In their first verbal description, they focused on the positive and negative parts of the gradient graph, which they interpreted as corresponding to uphill and downhill portions of the track. Their second verbal description was more detailed, and focused on identifying changes in the steepness of the gradient. They accompanied both of these verbal descriptions with deictic (pointing) gestures, using their finger or pen to trace along the portions of the gradient graph they described.

The portion of Ava and Noa's activity we now present amounts to their third description of the track. This description is similar to the first two descriptions in that it utilises speech and deictic gestures, but it also includes the semiotic resource of a virtual space, which the teachers created through gestures. The analysis is divided into four sections in order to show through time how Ava and Noa created and used the virtual space to construct the distance-height graph of the track.

Section 1: Noa creates the virtual space

After Ava and Noa have verbally described some features of the gradient of the track, Noa points to the beginning of the gradient graph (marked "A" on Fig. 4) with her right hand (see Fig. 5). She starts to describe the track again, only this time she performs iconic gestures with her left hand (see Fig. 5).

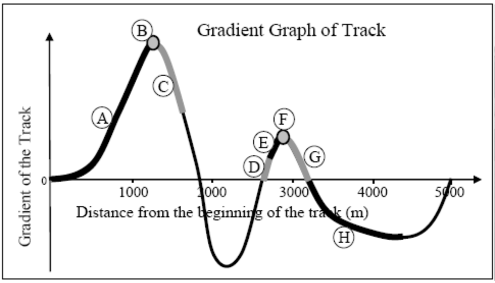

Figure 4

Figure 5

1 Noa: So this bit is just saying... Steep hill (points to A in Fig. 4, gestures steep positive gradient) and still going up (points to B in Fig. 4, gestures gradient flattening out) but not as sharply. So really, really steep section (points to A in Fig. 4, gestures steep gradient that flattens out).

2 Ava: Steep hill (points to B in Fig. 4)

3 Noa: And then the point of inflection (still pointing to B in Fig. 4, gestures less steep gradient) and then starting to get not so steep (points to C in Fig. 4, gestures flattening gradient as in Fig. 5)…

Noa's gestures of the changing slope of the track create a virtual space (indicated in white in Fig. 5) with the following mathematical properties: (a) A horizontal scale with a left-to-right direction, (b) a vertical scale denoting height, (c) a plane made up from orthogonal horizontal and vertical scales, (d) the origin of the track (e) the steepness and sense of the gradient of the track (indicated by the angle of the flat hand), (f) the height (vertical position) of the track (indicated by the vertical position of the hand), (g) the movement along the track (indicated by the motion of the hand).

Noa's virtual space appears to serve two communicative functions. First, it provides a visual way for Noa to communicate to Ava her thoughts about the shape of the hiking track. However, the virtual space is not merely a conduit for Noa's well-formed thoughts. Instead, it appears to function as a temporary blackboard that Noa can use to communicate her emerging understandings to herself. Indeed, Noa gestures the gradient of the track twice, the first time hesitatingly, the second time more confidently, and each time watching her own gestures carefully, which suggests that she is using the virtual space to understand and develop her own thoughts.

Section 2: Ava recreates Noa's virtual space within her own gesture space

Ava then copies Noa's gestures, all the time watching Noa's gestures instead of looking at her own as Noa did in the previous excerpt (see fig. 6). However, Ava's gestures always lag a few seconds behind Noa's, during which time she appears to be checking her interpretations of Noa's gestures before copying them. Thus, she is not copying merely the physical movements of the gestures, but more importantly, the mathematical properties that Noa has associated with these gestures.

At this point, the gestures that Noa and Ava are performing become metaphoric as well as iconic. That is, the gestures come to represent the abstract concept of the gradient of a tangent line, as well as representing the shape of the hiking track. Thus, Ava recreates Noa's virtual space within her own gesture space.

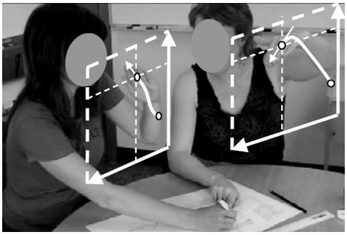

Figure 6

Section 3: Ava gestures within Noa's virtual space

After Ava successfully recreates Noa's virtual space within her own gesture space, Ava stops gesturing and watches Noa continue to gesture her interpretation of the shape of the track. During this time, Ava reaches over and gestures within Noa's virtual space on two separate occasions (see Fig. 7, (#5) and Fig. 8, (#7)).

4	Noa:	And then we start going back up (points to D in Fig. 4, gestures positive gradient) gentle (points to E in Fig. 4, gestures a positive gradient) to a harder point (points to F in Fig. 4, gestures a less steep positive gradient).
5	Ava:	But not as hard as it was over there (points in the air as in Fig. 7).
6	Noa:	But not as hard (high?) as it was over there.
7	Ava:	And then gradient is getting easier (points to G in Fig. 4, gestures a less steep positive gradient). So, flattening off (Ava points as if to extend the slope Noa is indicating, as in Fig. 8).

Figure 7

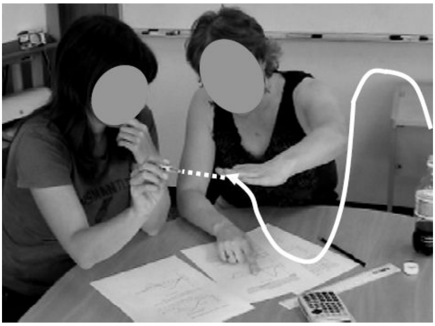

Figure 8

The first instance (Fig. 7) occurs after Noa remarks that the gradient of the track is getting "harder" (#4) at the place she is currently gesturing. At this remark, Ava gestures to another place within Noa's virtual space, where she believes Noa had previously gestured. Ava remarks that the portion of the track Noa is currently gesturing is "not as hard" (#5) as the portion of the track Noa had previously gestured, and at which Ava believes she is presently pointing. This is significant because it demonstrates that Ava is sufficiently comfortable with Noa's virtual space to be able to point to an unmarked place within it and describe its mathematical properties. Admittedly, the white superimposed diagram in Fig. 7 shows that Ava's gesture is not accurate – she has not located the exact spot where Noa had previously gestured a "steeper" part of the track. However, Noa seems to understand Ava's inaccurate gesture in her acknowledgement: "but not as hard as it was over there" (#6). Ava's ease with which she works within Noa's virtual space is all the more apparent when we consider that she might have pointed to a clearly marked part of the gradient graph (Fig. 4) as she had been doing for some time, instead of choosing to point to an unmarked place within the virtual space.

In the second instance (Fig. 8), Ava extends the virtual curve that Noa's hand is acting out in the virtual space to emphasise and agree with Noa's statement that the gradient is "flattening off" (#7). In doing so, she contributes to the construction of the graph in the virtual space that Noa initiated. At this stage, Ava's comments and actions suggest that she is no longer merely following Noa's gestures, but that she understands the virtual space sufficiently to co-create them.

Section 4: From virtual space to paper

Noa stops gesturing for a while, then re-enters the virtual space (see Fig. 9) to gesture her interpretation of the track over segment H in Fig. 4. However, since a virtual space retains no permanent traces of previously enacted gestures, she has no physical indicator of where she left off, and therefore gestures the portion corresponding to segment H in a way that is grossly out of synchronisation with the other portions of the track in her virtual space (in Fig. 9, the dotted curve shows where the portion corresponding to segment H would have been had Noa re-entered her virtual space at the same place she left it). However, this does not appear to be a problem for Ava and Noa, as it seems to be sufficient for them to enact the shape of the track one segment at a time, rather than see the whole track at once.

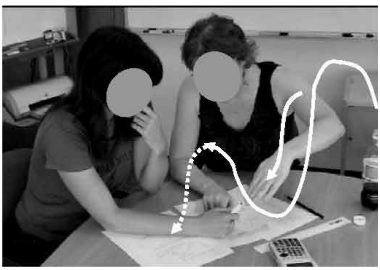

Figure 9

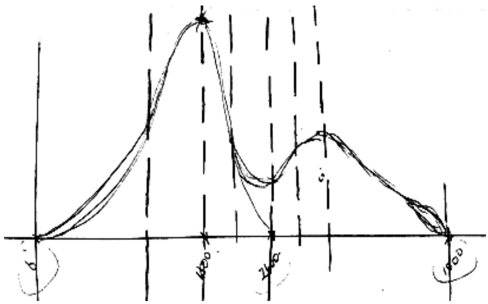

Figure 10

After gesturing the shape of the track in the virtual space they created, Ava and Noa draw a distance-time graph of the track (see Fig. 9) in what amounts to their fourth description of the track. When drawing the graph of the track on a piece of paper, they are much more concerned with the accuracy of the track, particularly with the height of the track. However, the basic shape of the drawn track is very similar to the shape of the track that was gestured in the virtual space. In this sense, the virtual space can be seen as an intermediary semiotic resource that facilitated the connection between the verbal description of the track, and the visual drawing of the track.

DISCUSSION

We consider the virtual space described above as novel in the sense that the teachers have interacted with the graphical representation of the derived function in a conceptual way (Thomas, 2008) to construct a semiotic resource that is part of a new representational system, the virtual space. They have not engaged in what Duval (2006, p. 106) calls treatments, "transformations of representations which happen within the same register", nor have they performed conversions, "transformations of representation which consist of changing a register without changing the objects", since the mathematical object is new – an antiderivative function. What they have done is to build a virtual representation (or sign) of antiderivative from a graphical representation of derivative.

One of the potential benefits of the virtual space to mathematical thinking and learning comes from its ephemeral nature. Because the virtual space records no permanent records of mistakes, it may reduce anxiety about making mistakes, and encourage greater experimentation than more permanent spaces like those created through drawing. Its low demand on accuracy may also lighten the cognitive load for those using it. When drawing the graph of the track, Ava and Noa found it necessary to consider the height of the graph in addition to its slope. They considered whether the track returned back to sea level at the local minimum (traces of this are evident in Fig. 10). They also considered the horizontal proportions of the track, which led to them drawing vertical lines to create segments along the x-axis of the newly drawn graph that corresponded to those in the gradient graph (see Fig. 10). However, Ava and Noa did not have to think about these additional issues while constructing the track in the virtual space as there was not a great demand for accuracy. This lightened their cognitive load considerably by reducing the number of mathematical issues they had to consider in the virtual space, which in turn allowed them to concentrate on the changes in the track's slope.

A potential drawback of the low demand for accuracy within a virtual space is that it may lead to significant miscommunications and misconceptions. Indeed, in the excerpt above, Noa's gestures were sometimes incorrect (see #1, #3, #4) and these mistakes were not picked up as they might have been had they been drawn on paper where they could have been studied for longer.

The example of Ava and Noa demonstrates that even though a virtual space has no permanent physical markers, it is nonetheless significantly shaped by physical factors. For Ava and Noa, the availability of the left hand made it possible to gesture the shape of a tangent line as it moved from left to right. It would have been very difficult to create the same virtual space without the left hand – it is possible that a virtual space created with the right hand in this context would have resulted in a right-to-left orientation, with the teachers starting at the end of the track and working backwards. Similarly, it would have been more difficult for Ava to copy Noa's virtual space if she had not been sitting beside her. If they had sat opposite each other, they would have had to adjust their virtual spaces to accomodate the opposing orientations.

In this example, iconic gestures were initially used to create the virtual space – they were gestures whose shape was the image of the slope of the track at each point. However, these gestures took on a metaphoric quality once the mathematical properties of the virtual space were established in that they were not only gestures representing the slope of the track, but more generally, representative of the gradient of a graph. We suggest that this is true in general – that is, that a virtual space gives an iconic gesture a metaphoric quality by conferring onto it a set of mathematical properties that enable it to be considered not simply an iconic shape, but a mathematical object in a mathematical space.

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DIFFERENT PROFILES OF ATTITUDE TOWARD MATHEMATICS: THE CASE OF LEARNED HELPLESSNESS

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In a previous study, aimed at grounding a definition of attitude toward mathematics on students' narratives about their own relationship with mathematics, we proposed a characterization of attitude based on three deeply interconnected dimensions: the individual's emotional disposition toward mathematics, his/her view of mathematics, his/her perceived competence in mathematics. From this multidimensionality follows the need of overcoming the positive/negative dichotomy for attitude, in favour of considering different profiles. In this paper we will apply this model to investigate learned helplessness as an attitude toward mathematics. The analysis highlights that learned helplessness may correspond to different profiles, that require different remedial actions.

PROFILES OF ATTITUDE TOWARD MATHEMATICS

The 'attitude' construct is widely used by researchers and practitioners in mathematics education. In the field of research on affect in mathematics education several studies have highlighted critical issues about research on attitude, and have underlined the need for a theory as well as for new observational tools (Di Martino & Zan, 2001; Zan et al., 2006). As concerns the use of the construct by practitioners, a study carried out with teachers (Polo & Zan, 2005) suggests that the diagnosis of 'negative attitude' is often a causal attribution of students' failure made by the teacher and perceived by him/her as global and uncontrollable, rather than an accurate interpretation of students' behaviour, capable of steering the teacher's future action.

To make this diagnosis a useful instrument for dealing with students' difficulties in mathematics, we deemed necessary to clarify the construct 'attitude' from a theoretical viewpoint, while keeping in touch with the practice that motivates its use. With this aim, we have carried out a research aimed at 'grounding' a characterization of attitude toward mathematics on students' narratives. We investigated how students tell their own relationship with mathematics, by proposing the essay *"Me and mathematics: my relationship with maths up to now"* to more than 1600 students ranging from 1st to 13th grade, more precisely: 874 from 51 classes of 14 primary schools (grade 1-5); 368 from 24 classes of 8 middle schools (grade 6-8); 420 from 29 classes of 10 high schools (grade 9- 13).

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 417-424. Thessaloniki, Greece: PME.

The analysis of the essays (Di Martino & Zan, submitted) highlights that, to describe their own relationship with mathematics, students refer to three deeply interconnected dimensions: emotional disposition toward mathematics, view of mathematics (i.e. their belief system about mathematics), perceived competence in mathematics.
One of the outcomes of our study is therefore a multidimensional characterization of attitude toward mathematics (fig. 1), that underlines the need to give up the positive/ negative dichotomy, in favour of considering different profiles of attitude, each obtained through a characterization of the three dimensions.

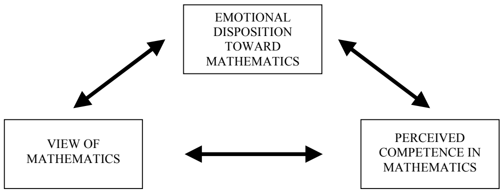

Fig. 1: Multidimensional model for attitude toward mathematics

In our model of attitude, however, emotional disposition and perceived competence are always associated with a particular view of mathematics, that we may define as either *positive* or *negative*. We consider positive, for example, the *relational* view of mathematics, as juxtaposed to an *instrumental* view (dichotomy introduced by Skemp, 1976). An interesting result from our study is that in more than 1600 essays we did not find any student with a profile of attitude characterized by a negative emotional disposition and the other two dimensions positive. In other words, a negative emotional disposition is always linked to (and perhaps a possible consequence of) either an instrumental view or a low perceived competence.

Amongst the different profiles of attitude, we will here consider and discuss those profiles characterized by a perceived lack of control over success in mathematics, i.e. the profiles associated to *learned helplessness* in mathematics.

LEARNED HELPLESSNESS

The perception of not being able to have a full control over a situation has been studied by the psychologist Martin Seligman (1975), who used the expression *learned helplessness* to indicate a perceived lack of control over an event, which results from prior exposure to similar negative events perceived as uncontrollable.

The very first studies (Seligman & Maier, 1967; Overmier & Seligman, 1967) highlight the *origins* of learned helplessness: human beings or animals that have experienced lack of control over a situation seem to *learn* not to have control, and remain passive when they face a similar situation.

Later studies propose a reformulated model of the learned helplessness theory (Abramson, Seligman & Teasdale, 1978; Peterson, Maier, & Seligman, 1993), analysing the *consequences* of that psychological situation, with a particular focus on depression, and sketching out a cognitive therapy to overcome it.

Peterson et al. (1993) observe that people characterized by learned helplessness have a *pessimistic explanatory style*, i.e. they tend to attribute failure to internal, universal and permanent causes, and success to external, specific and temporary ones. The cognitive therapy suggested by psychologists is based on the learning of a new explanatory style, a more optimistic one.

The important role of causal attribution was acknowledged in the rise of learned helplessness. According to the attribution theory developed by the psychologist Bernard Weiner (1986), individuals tend to explain their own success and failure in terms of three dimensions: locus (*internal* / *external*); stability (*stable* / *unstable*); controllability (*controllable* / *uncontrollable*).

Learned helplessness was also studied in the field of education (Dweck, 1975; Deiner & Dweck, 1978, 1980): through a comparison of helpless and mastery-oriented response patterns in schoolchildren, Dweck (1975) discovered the role of students' implicit beliefs about the nature of intelligence in their approach to intellectual tasks.

In mathematics education, researchers underline that helplessness in mathematics is associated with causal attribution of failure to lack of ability, a cause perceived as internal, stable and uncontrollable (McLeod & Ortega, 1993). Middleton and Spanias (1999) underline the importance – to minimize learned helplessness - that students learn to attribute success to effort and not to ability, and highlight the crucial role of the teacher to ensure that these beliefs are reflected in the classroom climate, for example through a teaching approach based on inquiry, open-ended problems, and group collaboration.

In the end, studies on learned helplessness in mathematics underline the importance of attributing failure to causes that are perceived as internal, stable and uncontrollable ones, with particular reference to lack of ability. As a consequence, they emphasise the role of *math self concept*, which includes beliefs of self-worth and perceived competence in mathematics (Pajares & Miller, 1994).

Learned helplessness in mathematics education has been mainly studied in the context of students' motivation, as an obstacle to invest resources and therefore to achievement. In a school-based context, a learned helplessness framework enables us to interpret particularly frequent failing behaviours in mathematics, such as avoiding answers or answering randomly. In these cases, traditional remedial actions based on correction of mistakes and repetition of topics show their limits and seem to be scarcely effective (Zan, 2002). Due to the frequency of these behaviours in mathematics, it becomes important to analyse this phenomenon, in order to provide interpretation tools, so that more goal-directed and effective interventions may be planned.

LEARNED HELPLESSNESS IN MATHEMATICS IN THE LIGHT OF THE MULTIDIMENSIONAL MODEL OF ATTITUDE

In the light of the described multidimensional model of attitude, the perception of not being able to have control over one's own success in mathematics can be detected in those profiles which are characterized by a low perceived competence.

As students' biographical narratives show, learned helplessness is often associated with causal attributions: this confirms the dialectic dialogue existing between the perception of not being able to have control over one's own success in mathematics and the process of causal attribution, already highlighted by the literature.

> '*When I am in trouble, I say: "I can't do it!" and I give up and feel negative emotions. I am a child who is not able to advise himself, and when I can't do something I feel like crying.*' [5th grade]

> '*At primary school I was not that good at mathematics, so in grade 3 I realised I was not good and then I closed my head up, saying that maths was not for me.*' [6th grade]

> '*I don't get on well with mathematics, because although I make efforts or study it, it does not get into my head and it is too difficult for me and I can't do anything about it.*' [10th grade]

The examples above confirm that learned helplessness might be related to a low perceived competence, constructed through failures attributed to internal causes. It is interesting to show how in these cases, besides an experience characterised by repeated failures, an *absolute* judgement (low perceived competence) is often brought about or anyway supported by a *relative* datum, such as a comparison with others:

> *'But when I sat next to some friends of mine who were able to finish a problem in less than a second, I always felt like crying and I just wanted the ground to swallow me up. I have always been fond of problems. They are like a challenge for me, a run against time and against fastest children.'* [6th grade]

> *'If I am alone I don't get worried and I correct my mistakes, but if I am at the blackboard or I am correcting an exercise aloud in the classroom and I make a mistake, I feel hopeless because everybody look at me and I understand that everybody could make it except me.'* [6th grade]

This highlights the role of the social context in the rise of learned helplessness (McLeod & Ortega, 1993).

We agree with Deuser and Anderson (1995), who underline that studies about learned helplessness need not to underestimate the dimension of controllability – or rather, of the *perception* of controllability – in the process of causal attribution. If one attributes success in mathematics to causes perceived as not controllable, this implies that investing resources becomes useless and the individual gives up being in charge of his/her own learning.

However, the perception of not being able to have control over success in mathematics is not always associated with attribution of failure to internal causes. For example, sometimes it is explicitly linked to the teacher (hence, an external factor, and not necessarily a stable one):

> *'When I moved to grade 10, I had a different teacher and I did not have a good relationship with him, I mean I could not understand his lessons, and therefore since then mathematics started to be a real problem to me'.* [11th grade]

Another more frequent external attribution of failure associated with learned helplessness is a view of mathematics as a discipline which is *per se* uncontrollable: in these cases the perception of lack of control over success in mathematics seems to descend from the individual's view of mathematics, rather than from a low self-concept.

The perception of mathematics as uncontrollable not only emerges from causal attributions of failure: it also stems from the individual's explicit theories of success, i.e his/her beliefs about what one must do to succeed in mathematics (Nicholls et al., 1990). In particular, students' narratives often ascribe success in mathematics to factors like an innate predisposition or an extraordinary memory, both perceived as uncontrollable:

> *'To understand maths you don't need to study, but rather to have a flair for it'* [9th grade]

> *'This subject is too difficult for me because there are too many rules to be learned by heart'* [10th grade]

Generally, students' narratives explicitly recall the view of mathematics also to motivate the quality of the relationship with the discipline itself: in actual fact, as we said earlier, the dimensions of attitude are deeply interconnected. In particular, a view of mathematics as an uncontrollable subject matter is the most frequent in cases of negative relationships with mathematics:

> *'I think that my relationship with mathematics has always been "dark and gloomy"; I have never mastered the subject and since the very beginning of primary school I felt unsure; even if I knew something I was always full of doubts. That's it: I don't know the 'why' of mathematics, why that scheme, that procedure and not another one; because, as my dad says: "In arithmetic you can't make up things". I sometimes invent things and I get wrong; I'd really like to know the reasons, the causes, because they seem to me a lot of abstract rules, stuck here and there.'* [6th grade]

> *'I started to think about Maths as something impossible and not understandable. The more I studied it , the more I hated it, and after some months I stopped studying it'* [10th grade]

The perception of mathematics as an uncontrollable discipline is often associated with a ‘negative’ view of the discipline itself, characterised as a set of rules that need to be memorised and then applied in order to get to the correct answer:

> *‘Maybe there are too many theorems and stuff for lower secondary school students, and it is impossible to learn all of them by heart and actually every time there was a written test, everybody had the formulas for the trapezium, parallelepiped etc. written on their book or hand’* [11th grade]

We remarked how a low perceived competence is often associated with the attribution of failure to external causes. Moreover, our analysis suggest that also in the case of internal locus the process of causal attributions of success / failure interacts with the individual’s view of mathematics. In other words, the process of attribution is influenced by the individual’s view of mathematics, and hence by external causes.

In the end, the dimension of (internal/external) locus in the process of causal attribution does not identify significantly different views of mathematics, which rather emerge if we look at learned helplessness in the light to our model of attitude.

The need to consider the dimension ‘view of mathematics’ in the analysis of learned helplessness reflects a more general epistemological standpoint: we actually believe that constructs that mathematics education borrows from other disciplines may become a useful instrument for both theory and practice only if they are combined with the epistemology of *our* discipline, i. e. they should be able to take into account typical problems in mathematics education.

CONCLUSIONS

The model of attitude toward mathematics we used, allowed us to identify a particularly meaningful profile of learned helplessness: that in which the perception of lack of control on success in mathematics is associated with a ‘negative’ view of the discipline. In this case, a teaching intervention aimed at helping students overcome their low competence cannot go without modifying their view of mathematics.

Thus, the identification of this profile has important implications for practice. If problem solving and an inquiry-based approach appear to be generally promising to overcome learned helplessness (Middleton & Spanias, 1999), a more precise diagnosis allows the teacher to direct his/her choices more accurately, as concerns both the problems to be used and the ways of organising problem solving activities. A student’s learned helplessness needs to be faced differently, depending on the *prevailing component*, i.e. a low perceived competence due to a low self concept, or rather a ‘negative’ view of mathematics.

In the first case, the important thing is to try and reinforce the student's low self concept by shifting the ideas of success from the production of a correct answer to the enactment of meaningful thinking processes. As concerns tasks, possible suitable problems might be those that do not immediately evoke curricular activities, that enable exploration and give value to partial answers. The activity might be organised in a way that forces pupils to be in charge of their own thinking processes, for instance by favouring work in small homogeneous groups.

In the latter case, i.e. with students who have an instrumental view of mathematics, a suitable intervention should promote a relational view of the discipline, in which the rules of mathematics are not viewed as unconnected products (to be memorised) but rather as the result of reasoning processes. An effective strategy might be that of using open-ended problems, problems with multiple solutions, problems that require a conjecture before the actual use of an algorithm, or even problems where numerical data are omitted. The activity should be organised so that the construction of a relational view of mathematics might be favoured, for instance through work in (not necessarily homogeneous) groups, followed by a final collective exchange, in which the variety of solution processes may be highlighted.

In the end, the multidimensional model of attitude toward mathematics and the consequent idea of profiles, allow for a more precise diagnosis of learned helplessness as well as for a more sophisticated interpretation of the failing behaviours that go with it. More refined observation and interpretation might lead to a more focused, and then effective, intervention.

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TEACHERS' TREATMENT OF EXAMPLES AS LEARNING OPPORTUNITIES

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In this paper we focus on teachers' learning opportunities, which were identified within a larger study[1] that examined teachers' treatment of examples in and for their classrooms. Through classroom events that were fully documented we identify such opportunities and discuss their nature and potential in terms the development of teachers' craft knowledge. We distinguish between opportunities that lend themselves to refinement of mathematical understanding and those that address critical pedagogical considerations and choices on the part of the teacher. We conclude with reflections on the limitations and challenges of capturing such moments in the practice of teachers and suggest that treatment of examples is a powerful site for such learning.

TEACHER KNOWLEDGE AND THEIR TREATMENT OF EXAMPLES

Exemplification in mathematics education is strongly connected to three aspects of teacher knowledge: knowledge of mathematics, knowledge of students, and pedagogical content knowledge (concurring with Shulman, 1986; Harel, 2008). The kind of mathematical knowledge teachers hold affects what is taught and how it is taught. In particular, the mathematical aspect of an example relates to the extent to which it satisfies certain mathematical conditions (e.g., for a concept, or a procedure) that it is meant to illustrate. Knowledge of students refers to teachers' understanding the ways in which students come to know and the effect of their existing knowledge on their construction of new knowledge. It also relates to teachers' sensitivity to students' strengths and weaknesses; with respect to examples this implies teachers' awareness of the consequences of students' over-generalizing or under-generalizing from examples, and to students' known tendency to notice irrelevant features of an example instead of attending to its critical features. Pedagogical content knowledge includes "ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p.9). Clearly, examples are inseparable from their representations, and indeed they are meant to help make mathematics comprehensible to learners.

Another significant aspect of teachers' knowledge has to do with knowing-to act in the moment (Mason & Spence, 1999). This aspect of knowing relates to teachers'

[1] This study was funded by the Israel Science Foundation (grant 834/04 O. Zaslavsky PI)

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ways of responding to actual (often unexpected) classroom events that require an immediate action on the part of the teacher; it heavily relies on teachers' increasing awareness and ongoing reflection. With respect to examples, the act of choosing and generating examples for teaching may occur spontaneously, that is, with no pre-planning.

There is a growing interest in the mathematical knowledge needed for teaching (e.g., Ball et al, 2005; Silverman & Thompson, 2008). In addition, increasing attention is given to the nature and roles of examples in learning and teaching mathematics (Zazkis & Chernoff, 2008; Zodik & Zaslavsky, 2008; Zaslavsky & Zodik, 2007; Bills et al, 2006). This paper addresses these two significant themes. We focus on teacher knowledge and use of examples for teaching mathematics.

WHAT ARE LEARNING OPPORTUNITIES?

Our experience indicates that most mathematics teacher education programs do not explicitly and systematically prepare prospective teachers to select and generate instructional examples in an educated way. Thus, we consider the knowledge teachers construct with respect to their choice and treatment of examples in the mathematics classroom as craft knowledge; since it is constructed mostly through their practice, as an outcome of their reflection in and on action and their increasing awareness to alternative ways that better serve their teaching goals (e.g., Kennedy, 2002). In our study, we aimed at capturing moments of teachers' gaining awareness regarding their choices and use of examples, which may be considered opportunities for teachers to learn from their practice.

In our earlier work (Zodik & Zaslavsky, 2008) we observed that some of the examples teachers use are planned in advance (i.e., pre-planned) while others are chosen or constructed on their feet either in response to classroom interactions or as part of the way they planned to conduct the lesson. Two main types of occasions in which teachers generate examples spontaneously 'on-the-fly' were identified: (i) as a response to students' actions, such as raising a falsifiable claim; (ii) in reaction to their recognition in the course of the lesson of certain limitations of a pre-planned example. These occasions lend themselves to fostering teacher awareness to mathematical and pedagogical subtleties, which constitute learning opportunities for them (Mason, 1998).

THE STUDY

Goals

The goal of our study was to identify and characterize moments in teachers’ practice that constitute learning opportunities for them in the context of treatment of examples in their mathematics classroom.

Data Sources

Data sources included both randomly and carefully selected mathematics lessons of 5 experienced secondary teachers (with at least 10 years of teaching experience). The 'carefully selected' classroom observations were lessons that teachers invited us to observe based on their views of what a 'best case' of use of examples in their classroom may be. That is, lessons which the teacher anticipated in advance that would illustrate a particularly good way of example use in her or his classroom. Grade levels varied from 7^{th} to 9^{th} grades (ages 13-15). In most of the observations, pre and post lesson interviews were conducted with the teacher. In addition, we collected relevant documents and managed a research journal.

Analysis

The research is an interpretive study of teaching that follows a qualitative research paradigm, based on extensive classroom observations, aiming at making sense of teachers' practice.

For the part of the study on which this paper relies, we looked for classroom events that indicated a state of surprise or uneasiness on the part of the teacher. For these events, we closely examined the teacher's spontaneous reactions, with respect to treatment of examples, and analysed the change that occurred compared to the original plan, with a focus on what the teacher became aware of in this process. One of the strongest evidence of teachers' learning was a shift from a pre-planned example to a spontaneous one that addressed an unexpected classroom situation, which pointed to a problematic aspect of the pre-planned example. Other evidence was expressed by teachers' utterances in the classroom. We looked for supporting evidence in the post lesson interview.

It should be noted that there were cases that teachers shared with us some insights they gained in the course of their teaching that for them served as learning opportunities, however, had no observable indicators for such learning. Clearly, capturing moments in which teachers crystallize their knowledge for teaching mathematics is not a trivial task. Our analysis applies only to observable cases, which indicated actual teacher-learning.

FINDING

We identified two main types of learning opportunities, both related to teachers' use of spontaneous examples; these opportunities varied, depending on the kind of example that was treated in the lesson: A mathematically correct but pedagogically inappropriate example (see Zaslavasky & Zodik, 2007); a mathematically incorrect example that seems pedagogically appropriate (see below, the classroom event).

One kind of learning that was triggered by treatment of examples and occurred in the course teaching was manifested by a shift in teachers' awareness to the limitations of an example and their realization that an example that seemed appropriate may, in fact, be inappropriate, mathematically or pedagogically, for a certain purpose in a particular context. The main triggers for such learning were the teacher’s own reflection and/or interactions with his or her students. These learning opportunities indicate some growth in teachers' mathematical knowledge, pedagogical content knowledge, and/or sensitivity to students. In some cases that were documented, these opportunities were not used to the fullest, to the extent that they could be considered missed-opportunities.

Capturing teacher learning through generating a spontaneous example

We turn to two classroom events that convey the nature of teachers' learning from treatment of examples. The first (and more detailed) case was triggered by a student's invalid inference that challenged the teacher's mathematical knowledge; the second case was triggered by the constraints of a technology enhanced learning environment that resulted in the teacher's rethinking issues of representation.

Case 1: Learning is triggered by students' reactions. The following classroom event (taken form Zodik & Zaslavsky, 2007) conveys the subtlety of relying on textbook examples and constructing a (counter-) example in response to an unforeseen need that arises. It also illustrates a meaningful learning opportunity for the teacher.

In a geometry lesson dealing with the SAS congruency theorem, students were asked to determine for a number of pairs of triangles whether they are congruent according to SAS. This in itself presents a logical difficulty in inferences for cases that do not satisfy SAS. Unless there is additional explicit or implicit information, the conclusion can only be that it is a non-decisive situation.

The teacher [T1] brought examples from the textbook to illustrate several pairs of triangles that differ with respect to their givens. Figure 1 is one of the textbook examples that the teacher chose to discuss in the classroom; it appeared on a worksheet which she had prepared in advance. Thus, we consider it a pre-planned example.

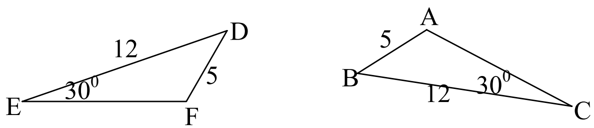

Figure 1: A textbook example of a pair of triangles with 3 congruent elements

This example (Figure 1) should be considered in the context of dealing with issues of logical inferences. On the one hand, the two triangles appear to be congruent, judging iconically by the drawing; however, they do not satisfy the SAS conditions. That is, one cannot apply the SAS congruence theorem to infer that they are congruent. This was actually the teacher's purpose in dealing with them in the first place. Moreover, two triangles with the givens in Figure1 may be congruent but are not necessarily congruent; thus, to better represent the case of two non-congruent triangles satisfying the givens in Figure 1, a different drawing would be more appropriate for some students (see Figure 2).

The teacher focused on the inference, ignoring the extra "noise" that the drawing brings into the discussion. Her aim was for the students to realize that they cannot infer that the pair of triangles consists of congruent triangles by SAS. However, one of her students was attending only to what appeared in the drawing (Figure 1) and insisted that these particular triangles were indeed congruent. This presented the teacher with the need to come up with another case of two non-congruent triangles with the same givens.

Teacher: I can give you a counter-example and show that they are not congruent. There are actually an infinite number of counter-examples.

The teacher realized that the initial example drew the student's attention to the appearance rather than to the logical inference she was after, namely, that since the conditions of SAS are not met they could not infer that the triangles in Figure 1 are congruent. The student agreed that the inference can not be made from SAS, but insisted that the triangles were congruent, relying on the drawing that actually represented the case in which the two triangles were congruent; thus, the teacher decided to make the following construction (Figure 2). At first, she was sure that there were an infinite number of different counter-examples, and that she had a large degree of freedom in generating a pair of triangles serving her new purpose. However, as she began sketching, expecting to have many possible pairs of triangles that could serve as counter-examples, she realized that there was "just one more":

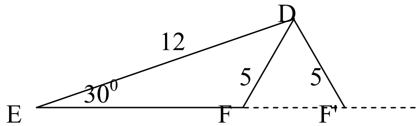

Figure 2: The teacher's spontaneous counter-example

Teacher: I have an infinite number of triangles. Wait, no, I don’t – I only have one pair of triangles that are not congruent.

Student: Why?

Teacher: I'll try to go your way. You understand that we don't have SAS conditions, right? So we can construct two triangles, this [points to ΔEDF] can be one and that [points to $\Delta EDF'$] can be the other.

Student: It's not the same as in the first pair of triangles!

In this event, the teacher reached a point where as she tried to construct a "general case" she became aware of the fact that there were only two distinct cases of pairs of triangles with the specific givens. This was an insightful moment for her (as she expressed also in the post-lesson interview). Yet, there was still a mis-match between what she was trying to illustrate and where the student was focusing his attention and what he attributed to the visual representation in Figure 1. It seems as if the student regarded the teacher's second example as a completely different case, and was not able to use it to move on. He insisted on treating the first example somewhat iconically by attending to the visual entailments that indicated that the pair of triangles were congruent, regardless of whether they satisfy the SAS conditions or whether there is another pair of non-congruent triangles with the same givens.

In terms of the teacher's acting in the moment (Mason & Spence, 1999): The second example was clearly a spontaneous one. She generated it on her feet as the need for it came up. Being experienced, some rather sophisticated aspects were already automatic for her (construction in Figure 2). However, as she began constructing this familiar case, that was part of her accessible example space, it appears that she began monitoring her construction. This led her to an analytical mode in which she inferred, much to her surprise, that there were not as many examples as she had thought. Support to this explanation, can be seen in the post-lesson interview where she had difficulty articulating how she came up with the second (immediate) example. As the teacher expressed in her own words, after realizing the way she operated:

> Teacher: Now I say to myself that there could be many other examples that I seem to be confident about and pull out immediately but if you would confront me about them I would get confused and would need to think more about them.

In the interview she began to ponder on the moment of conflict and realized the gap between her strong intuition about this example and her rather weak logical support for it. In terms of the mathematics, the teacher became aware of the fact that in the case she used there were only two possible connections between the triangles satisfying the givens. They can either be congruent (as represented in Figure 1) or non-congruent in the way that is represented in Figure 2. She was still unable to generalize for different cases of pairs of triangles of this sort, but she clearly stated that "I need to better understand what went on mathematically". From the student perspective, after the lesson she tried to make sense of the difficulty he encountered, and at a certain point in the interview she said "… and I didn't understand what he didn't understand. How could I?".

Note that in addition to the above incorrect accretion, both pre-planned and spontaneous (counter-)example are mathematically incorrect, because in fact such triangles do not exist. There is no triangle with the given measurements (Figures 1 & 2). Given the two measurements (12 and 30°) the length of the side opposite the 30° angle cannot be less than 6. Interestingly, this problem was not noticed by the textbook author or by the teacher, and the issue of existence was not addressed at all throughout the lesson. Thus, this was not a problem for the students either. Turning this into a learning event for the teacher would require some external intervention (in the spirit in which we designed and successfully implemented teacher education activities based on this case).

Case 2: Learning is triggered by the constraints of the learning environment. In an investigative oriented lesson with a technology enhanced graphical tool, the teacher [T1] offered various examples for the students to explore, with a specific goal in mind for each example. Her main goal was to connect between the symbolic representation of a system of two (linear) equations and its solution, to their graphical representations. Thus, she wanted the students to examine the case of a system of two linear equations that have an infinite number of solutions (i.e., ordered pairs that satisfy both equations) and their graphical representations as two coinciding straight lines. Based on her previous experience in a regular classroom with no technological tools, she spontaneously suggested the following two examples, and asked the students to find commonalities between these two pairs of linear functions (as indicated later, she actually thought of them as systems of equations):

$$(i)\begin{cases}2x+4\\4x+8\end{cases}\quad,\quad(ii)\begin{cases}-2x+4\\-4x+8\end{cases}$$

As she was giving the specific instructions she said:

Teacher: just a second…, what's going on here? Why? This is not good… Well, try it anyway.

In the post-lesson interview the teacher expressed her awareness to the different representations used in the traditional classroom setting compared to the graphical tool. Thus, in the following lesson she gave the students the following examples for investigation in a non-technological setting (on grid paper):

$$(i)\begin{cases}y=2x+4\\2y=4x+8\end{cases}\quad,\quad(ii)\begin{cases}y=-2x+4\\2y=-4x+8\end{cases}$$

This case highlights the complexity of acting 'on-the-fly'; the teacher could not come up with more appropriate examples in the moment, although she immediately sensed that there was a problem. Only after reflecting and contemplating on the conceptual constraints of the graphical tool, she realized what she considered the limitation of the graphical tool for her pedagogical goal. Note that she did not choose to confront the students with the implications of representing functions and equations with these different tools.

CONCLUDING REMARKS

The two classroom situations illustrate the affordances of some spontaneous examples as learning opportunities for the teacher, both mathematically and pedagogically, enhanced by monitoring and reflection in-action and on-action. The analysis of the nature and conditions of these learning opportunities sheds light on ways in which teachers' craft knowledge develops.

Every learning situation involves some change; in the context of learning through treatment of examples, "the change" is mainly associated with increased awareness (Mason, 1998) to mathematical and pedagogical implicit demands of an appropriate example for a particular purpose. The change may (but need not) be manifested in an actual change in their choice of an example. In Zazkis & Chernoff (2008) terms, when you study experienced teachers learning "… the information does not have to be 'new', but 'newly realized' or 'newly attended to' …" (p.196).

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Poster Presentations

TEACHING “COMPLEX NUMBER” IN FRANCE AND TURKEY

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This research relates to the comparative analysis of the teaching of the complex numbers in France and Turkey. For this study we refer to the framework of the Didactic Transposition, developed by Chevallard (1992), we also refer to his anthropological approach of the knowledge, which enables us to take into account institutional dimensions. With regard to the more precise analysis of the objects of teaching, we refer to work of Robert (1998) on the tools for analysis of the mathematical contents.

Our research will aim thus to bring brief replies to the following questions:

1. What are the institutional relations with the object “complex number” such as they appear in the textbooks in France and Turkey?
2. Taking into account these institutional relations, what are the choices made by the teachers, in the two countries?
3. What are the errors frequently made by the students? And which subjacent obstacles appear by these errors?
4. Which representations are present in the teaching of the complex numbers? And what are the possible changes between those (indicated or not)?

With this intention, first of all we compared the curriculums and the instructions. To try to see the real life of the complex numbers in the classes, we analyzed the questionnaires proposed to the students and to the teachers to determine their institutional relations with the teaching of the complex numbers. Our choices of analysis allowed construction of diagnostic tools made up of the seven tasks which take into account tool and object dimensions of the complex numbers and recover situations of use of the complex numbers bringing into play various registers of representation which makes it possible to study the institutional relations of the students.

The poster will display in both pictorial and written formats the methodology, analysis, results, and discussion of this research and its findings.

References [A list will be made available at the session]

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COMPUTATION AND NUMBER SENSE DEVELOPMENT: A TEACHING EXPERIMENT

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Mathematics education in Portugal has suffered significant changes in the past decade. In 2007 the new portuguese mathematical curriculum was published, bringing a new perspective on the purpose and goals of mathematical education. However, and considering the differences between the former and the present curriculum there has been an effort to build materials that can help teachers bring these new directives in to the classroom. This poster aims to present, in a graphical display, how a task built for this purpose was applied in a 5th grade class. Furthermore, and regarding the development of number sense, it will set out to deepen the knowledge on the choice of different computation processes and strategies and how that relates to the nature of the task.

FOCUS OF THE POSTER

This investigation aims to deepen the knowledge on the number sense development of fifth grade students, giving special attention to its problem solving ability and to the use of different computation processes in the classroom. Number sense may be considered as the attribution of a real meaning to mathematical symbols (Fosnot et al, 2001). To McIntosh et al. (1992) number sense involves a general comprehension of numbers and operations, as well as the ability to use this knowledge in a flexible way in solving problems.

This research stands on the assumption that all human interaction is mediated by interpretation (Bogdan & Biklen, 1991). Data collection will occur in two main spheres: the classroom, observing and video recording lessons to further understand students' practices; and through clinical interviews to gain access to students' practices regarding different computation processes.

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DO NORWEGIAN STUDENTS GET THE OPPORTUNITY TO LEARN ALGEBRA?

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RESEARCH QUESTIONS AND AIMS

1. What characterizes items in TIMSS 2007 (pop. 2) where Norway has the lowest score as compared with the international average?
2. Why are Norwegian students' achievements low on these particular items?

Our aim is to discuss these questions in the light of curriculum indicators collected in TIMSS and to relate our discussion to the theoretical concept "opportunity to learn".

METHOD

By using categories applied in TIMSS, we have analysed 10 items in TIMSS 2007 (pop. 2) where the score for Norway is lowest as compared with the international average. Additionally, we have examined Norwegian teachers' response to the question of how much instructional time they use on different domains, and compared the percentage with the international mean. Finally, we have looked at the curriculum analysis conducted by the Norwegian expert group in relation to the TIMSS items, as an indication of the priorities in the intended curriculum in Norway.

THEORY

Students' opportunities to learn have been widely discussed in research in mathematics education (Hiebert & Grouws 2007). Hiebert & Grouws state that *opportunity to learn*, even if it seems to be a quite general notion, can be a powerful concept if traced carefully through its implications. This same concept has also played an important role in international comparative studies like TIMSS.

FINDINGS

Seven of the ten items where the Norwegian score is lowest, as compared with the international average, is in Algebra. Evaluations of the Norwegian expert group in TIMSS, as well as our analysis of the teachers' answers on curriculum related questions, indicate that in Norway relatively little time is devoted to Algebra. Our conclusion is that the achievement of the Norwegian students on the selected items to a large extent can be explained by priorities stated in the curriculum and that students learn what they are given the opportunity to learn.

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DIFFERENT APPROACHES TO LEARN MULTIPLICATION

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The approach to learn multiplication in different cultures by comparing Thai context and Japanese context regarding to multiplication and initial attempt of Multiplication Teaching by Thai teachers' teaching methods different from the former time, would be presented in this poster.

For the study of approach to learn multiplication both in Thai and Japanese contexts, the researcher participated in participative observation of schools under research project including 3 phases of Lesson Study: 1) cooperatively plan, 2) cooperatively do, and 3) cooperatively see. According to the interview of Thai teachers teaching in grade 1-6, comparison of Thai and Japanese textbooks, and the analysis of classroom videos titled "Multiplication" of Thailand and Japan, the findings were as follows: 1). Different approaches to learn multiplication regarding to meaning, algorithm and sequence of lesson of Thailand and Japan was very different, 2). Lesson study affecting the approach to learn multiplication of grade 2-5 of Thai schools in the research project. Traditionally, the students had to learn multiplication by addition, memorization of multiplication table and steps of multiplication focusing on correct and quick calculation. Now, they learned the meaning of multiplication by considering the number of members in the set and number of set, think about multiplication table as well as various problem situations of multiplication, think about how to calculate, ranking in order of lesson would help the students to find simplicity in developing mathematical ideas (based on two worlds of (conceptual) embodiment and (proceptual) symbolism) (Tall, 2007) 3). The components of Lesson Study in Thai context affecting the changes of approach to learn multiplication by which the teachers used Japanese textbook comparing to Thai textbook in designing research lessons and materials, participating in participative observation, and providing feedback in classroom from many groups such as the teacher team, research team, and experts both from inside and outside country.

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MATHEMATICAL RIDDLES AMONG THE MUSHARS: LINKED TO A HISTORICAL TRADITION?

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Riddles are part of the folk mathematics of many cultures. It has been suggested that riddles are important for literacy and numeracy education (Rampal et al., 1998). In India, mathematical riddles are a strong part of the folk culture and may have links to the historical tradition of algebra. In this report, we present examples of riddles known to members of the Mushar (rodent hunters), which is a non-literate, oral community living largely in the northern Indian states of Bihar and Uttar Pradesh. Members of this community were nomads till recently and now are socially marginalised, living in poverty without any regular job or access to formal education. Several of the riddles are based on indeterminate equations with positive integer solutions. Constraints are added which make them determinate equations. The solutions of indeterminate equations have played an important part in the historical development of algebra in India (Katz, 1998). This study of Mushars' knowledge of riddles was part of an ethnomathematical study of the working knowledge of arithmetic that was conducted in three Musharis (hamlets where Mushars live). Data was collected through interviews and discussions with 25 adults (20 male and 5 female). Example of one such riddle is:

A goat produces a quarter kilo (litre) milk everyday, a cow half a kilo, and a buffalo 4 kilos. 20 kilos of milk is required from 20 such animals. How many of each of these animals is needed?

Some Mushars (those who could solve the riddles) were able to not only pose and solve riddles, but also to solve other riddles similar in structure but with the numbers changed. In one instance, one of them seemingly constructed a new riddle for the researcher. The presentation will focus on the knowledge of basic arithmetic among the Mushars, which is similar to that found in oral cultures (Nunes et al., 1985). This fact and the remarks made during the interview indicate that the riddles are learnt by the Mushars from other village communities. In fact, such riddles are a common part of the folk culture in this region, suggesting a long tradition and connections with the historical tradition of algebra in India.

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NETWORKING FRAMEWORKS FOR ANALYSING TEACHERS' CLASSROOM PRACTICES: A FOCUS ON TECHNOLOGY

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This poster presents research aimed at networking two frameworks for analysing teachers' classroom practices: instrumental orchestration (Trouche, 2004) and Ruthven's (2007) five-point framework. English and Sriraman (2005) suggest it is time to "take stock of the multiple and widely diverging mathematical theories". In response, Bikner-Ahsbahs and Prediger (2006) develop the notion of *networking* theories as a means of improving coherence while respecting the richness of diverse theoretical frameworks. Networking theories in research on the use of technology in the teaching and learning of mathematics is especially pertinent. Firstly, research in this area reflects general trends in mathematics education. Secondly, tensions between theory and practice are particularly prominent. Research has highlighted the potential of digital technologies to enhance mathematics education. In contrast, the reality of classroom use remains limited. In the UK, the use of technology in school mathematics has remained weak and often fails to realise its potential.

The instrumental approach developed from attempts to theorise tool use. In contrast, Ruthven's framework is developed from attempts to theorise teachers' classroom practices. The two frameworks thus represent quite different, yet potentially complementary, perspectives on the complexity of technology integration into mathematics teachers' classroom practices. Using these frameworks, this poster analyses data gathered from interviews and classroom observations of four mathematics teachers regarding the integration of digital technologies into their classroom practices.

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PROBLEM-BASED LEARNING AND TEACHER TRAINING IN MATHEMATICS: THE ROLE OF THE *PROBLEM*

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Problem-based learning (PBL) is a constructivist learner-centred instructional approach based on the analysis, resolution and discussion of a given problem (for a neat definition of PBL we refer to Savery (2006)). PBL can be applied to any subject, indeed it is especially effective for the teaching of mathematics (Cazzola, 2008). Typically a PBL session follows these steps:

- pupils are given a problem;
- they discuss the problem and/or work on the problem in small groups, collecting information useful to solve the problem;
- all the pupils gather together to compare findings and/or discuss conclusions; new problems could arise from this discussion, in this case
- pupils go back to work on the new problems, and the cycle starts again.

In spite of researches documenting the effectiveness of PBL (e.g. see Savery (2006), Hmelo-Silver (2004)), we have to regret that such a methodology is not common in real teaching practice at school, as teachers usually rely on self perpetuating "traditional" methods (Handal, 2003). According to our experience with pre-service and in-service primary school teacher training courses, the most effective way to have teachers willing to use PBL is to have them experience themselves the joy of mathematical discovery: by undergoing PBL, teachers acquire beliefs in favour of a PBL methodology as they see that in this way they learn some *good maths* (Cazzola, 2008).

One of the critical factors for the success of the PBL approach is the selection of suitable problems. Typically problems must be engaging, not of immediate solution and usually should rely on disciplinary competences superior to the one that the teachers are supposed to teach. The purpose of this poster is to show examples of problems we use for our PBL teacher training session.

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THE PLACE OF THE QUANTIFICATION IN ARISTOTLE'S LOGIC

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The first exposition systematic of the logic was given to the IVth century before J.-C by the Greek philosopher Aristotle. He formulated the main rules in his big treaty, the Organon which is the first didactic work known for logic.

The interest of the logic for the sciences led Aristotle to the study of the categorical said propositions. Each of its propositions (for example the proposition: *Every man is white*) results from the combination of terms according to the structure: S-P (subject-predicate). This structure S-P of the categorical propositions presents four realizations according to the quality and the quantity. In the Aristotelian logic, we speak about the quantity of the subject as it is universal or particular, and of the quality of the copula as it is affirmative or negative. By combining quantity and quality, we obtain four fundamental types of propositions: universal affirmative, universal negative, particular affirmative and particular negative.

All the propositions are subdivided into four sorts who differ by the quantity:

- The universal proposition:
  - "*Every man is white*" is an affirmative universal proposition;
  - "*No man is white*" is a negative universal proposition.
- The particular proposition:
  - "*Some man is white*" is an affirmative existential proposition;
  - "*Some man is not white*" is a negative existential proposition.
- The singular proposition:
  - "Socrates is white" is an affirmative singular proposition;
  - "Socrates is not white" a negative singular proposition is.
- The indefinite proposition:
  - "The man is white" is an affirmative indefinite proposition;
  - "The man is not white" is a negative indefinite proposition is.

Among these four propositions Aristotle was interested only in two types: the universal and the particularly individuals.

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A STUDY OF STUDENTS' CONCEPTIONS UNDERSTAND IN PROPERTIES OF DISTRIBUTION AND ASSOCIATION

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ABSTRACT

This study reports on an investigated data of 80 fourth-grade student's conceptions in understanding the principles of distribution and association when they are introduced three mathematical task involving different representations and contexts according to Baek's (1998, 2008) classification scheme for children's multiplication strategies. The analysis has revealed the following outcomes: focusing on the correct rates of judging true/false sentence, performance of using properties to solving different representation problems, thinking routines of properties, and influence factors.

From the analysis of the result, aims of the research were to find how students identify the property components of resolute multiplication questions under the context of combine words and graphic presentation. were there the abilities to partition and group number for students? Could students judge, understand, apply the properties of multiplication and explain why the partitioned and grouped strategies could work? Could they write down the expression of mathematics to solve mathematical problem.

In conclusion, results of the study of student's problem-solving strategies and thoughts in this paper suggest that students can develop powerful algebraic reasoning. The representation presented can help students make them understand mathematical properties more explicitly. Explicit discussions of properties can, in turn, help students to generalize and justify them, and develop algebraic reasoning.

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A STRATEGY TO ENHANCE STUDENTS' UNDERSTANDING OF FACTORS AND MULTIPLES: CASE STUDY OF A 4TH GRADER

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The concepts of factor and multiple are central in the learning of multiplicative relation in natural numbers. Curriculum often introduces these two concepts separately and later asks students to make connection of these two concepts by observing the relations of division and multiplication in equal-quantity-groups context. Instruction following the curriculum design may also cause students misconceptions of these two concepts. For example, students may view factors as an equivalent concept of division. In order to enhance students' understanding of factors and multiples, this study employed the "rectangular numbers" activity (Skemp, 1991) in teaching these two concepts.

Originally, the activity of "rectangular numbers" was designed to teach students the concept of prime numbers. We found out that this activity can also be used to help students understand the relations between factors and multiples by focus on multiplicative decompositions of "rectangular numbers". We introduce the definitions of factors and multiples as "A is a multiple of B if and only if B is a factor of A" while students operate the rectangular numbers in the activity. We argue that the introduced definitions can allow students to understand the relation between factors and multiples by "the law of equivalence concept" (Vygosky, 1996). After students recognize the definitions, we encourage them to use the definitions to describe diverse rectangular numbers they found. The results of a case study of a fourth-grade student showed that

- The student was capable of figuring out all the multiplicative decompositions of the given numbers by means of manipulating pieces, using multiplying table, observing the symmetric arrangement of the pieces, and applying the commutative of multiplication.
- The student was also capable of describing the multiplicative relation among numbers correctly by using the words "multiple" and "factor".
- In addition, the student can flexibly use multiplication, division and symmetry to figure out all factors of the given numbers (in context or not).

The results confirms that the proposed instructional strategy can provide students rich experiences in understanding the multiplicative relation and the concepts of factors and multiples which may not be achieved by following traditional curriculum design.

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CRITERIA FOR EVALUATING AN ARGUMENT

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To evaluate the efficiency of an argument is an important competence of mathematics literacy. Healy and Holyes (2000) noted that students held two views of proof: those would receive the best mark and those they would adopt for themselves. Reid and Roberts (2004) found that students would accept an argument on the basis of familiarity and clarity. These seem that communication, background knowledge, and the format of arguments may influence students' evaluation. In light of these ideas, we explored what criteria people use to accept or reject an argument.

There were 35 college, 30 8^{th} grade, and 30 6^{th} grade students in this study. They were interviewed individually with six arguments which were composed of format (formal or narrative), reasoning (deductive or inductive), and correctness (correct or incorrect) that was nested under the reasoning factor. These arguments all claim "When you add any 3 consecutive numbers, your answer is always a multiple of 3" is true. Participants were instructed to rate whether each argument was efficient on seven-point scale and then to report the criteria they used. This paper focused on the criteria they reported.

The results show that, students used several criteria simultaneously when rating an argument. We counted the criterion students ever used through six arguments and Table 1 demonstrated the percentage of 3 groups. All groups considered whether the claim is correct, but more college and 8^{th} students considered whether the argument is valid than 6^{th} students. Over 40% of college and 8^{th} students would care about the first part of an argument, such as "*Let 2x be even numbers…*". These college students knew the inductive argument with formal form is "*not for all cases*", but they would give bonus points for "*at least the first part is right*". These 8^{th} students though "*it looks better if you saying that in the beginning*". 6^{th} students seldom take the premise into consideration. Most 6^{th} students ever rated the argument low because "*I can't understand*". There were fewer college and 8^{th} students saying that but they deemed an argument was better if "it's easy to understand".

	college	8^{th}	6^{th}
Correctness	91%	80%	77%
Validity	86%	63%	10%
Suppose	49%	40%	7%
Not necessary	11%	50%	43%
Knowable	20%	30%	70%
Appearance	37%	13%	50%

Table 1: Frequent criteria students used

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CONNECTING VISUAL AND PROPORTIONAL SOLUTIONS TO A COMPARISON TASK REGARDING STEEPNESS

Diana Cheng and Polina Sabinin

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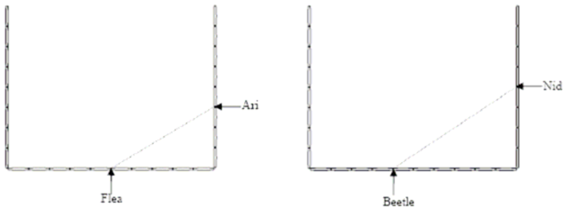
Proportional reasoning is a major focus of middle school mathematics instruction, and is linked to students' further studies in algebra. Difficulty with proportional reasoning is well documented in the literature (Lesh, Post, & Behr, 1988; National Council of Teachers of Mathematics, 2000; Noelting, 1980). This qualitative study investigates grade 7 students' solution methods on a problem related to steepness. One possible way to lead students to think about steepness using proportions is by providing them with tasks in which it is difficult to determine steepness solely by looking at angles. We show that in small-group problem solving settings, students are able to challenge each other to reasoning about the comparison problem multiplicatively.

Two spiders, Ari and Nid, live in rooms with tiles on the two walls and on the floor. Ari wants to catch a flea from 3 tiles high, while Nid wants to catch a beetle from 4 tiles high. Whose web will be steeper?



Circle the best answer.

A. Ari's web to the flea is steeper.
B. Nid's web to the beetle is steeper.
C. Both webs have the same steepness.
D. You cannot tell which web is steeper.

Figure 1: Steepness comparison task

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AN INVESTIGATION OF THE INFLUENCE OF IMPLEMENTING INQUIRY-BASED MATHEMATICS TEACHING ON MATHEMATICS ANXIETY AND PROBLEM SOLVING PROCESS

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Inquiry-based mathematics teaching has proved useful in furthering students' mathematical learning (e.g. Siegel, Borasi & Fonzi, 1998), but its effect on mathematics anxiety and problem solving has not been investigated. This paper reports the influence of implementing inquiry-based teaching on 7th graders' mathematics anxiety and problem solving process. The participants were sixty-six 7th grade students who were selected from two classes (34 and 32 students respectively). The mixed method design was used in this study. Both qualitative and quantitative data were collected. The qualitative data consisted of video-taped records of classroom teaching, student interviews, students' worksheets and teacher's reflective journals. As to the quantitative data, we collected the pre-, mid- and post-test results of "Mathematics Anxiety Questionnaire" (MAQ).

The results indicated that inquiry-based mathematics teaching had positive effects on students' mathematics anxiety and problem solving. A repeated measures analysis of variances of MAQ revealed a statistically significant reduction in mathematics anxiety in the pre-, mid- and post-test. On the other hand, an analysis of qualitative data showed that students with different levels of mathematics achievement (high, medium, low) improved their problem solving abilities differently. The high mathematics achievers were able to think creatively, develop multiple strategies and chose them in tackling a particular problem while doing problem solving. The medium and low mathematics achievers were able to solve problems more accurately and develop their own strategies.

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SPOKEN MATHEMATICS AND LEARNING OUTCOMES IN INTERNATIONAL MATHEMATICS CLASSROOMS

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Western educational literature has identified effective mathematics instruction with the promotion of student mathematical talk in classrooms. Our research examines spoken mathematics in both Asian and Western classrooms. This poster reports our analysis of spoken mathematics in private student-student classroom conversations in 85 lessons in seventeen classrooms located in Australia (Melbourne), China (Hong Kong and Shanghai), Japan (Tokyo), Korea (Seoul), and the USA (San Diego) and in student post-lesson video-stimulated interviews. The Spoken Mathematics Project is one component of the Learner's Perspective Study (Clarke, 2006). For the analysis reported here, the essential details relate to the standardization of transcription and translation procedures. Three video records were generated for each lesson (teacher camera, student camera, and whole class camera), and it was possible to transcribe three different types of oral interactions: (i) whole class interactions; (ii) teacher-student interactions; and (iii) student-student interactions, and, where necessary, translate these into English. A stratified analysis was conducted in five stages focusing on the significance of the situated use of spoken mathematical language in these classrooms. Stages one and two focused on the frequency of public utterance and the spoken use of key mathematical terms (Clarke & Xu, 2008). We consider teacher-student interactions to be public from the point of view of the student. Stages three and four repeated these analyses for student-student interactions and documented students' oral articulation of the relatively sophisticated mathematical terms that formed the conceptual content of the lesson. This distinguished one classroom from another according to how such student mathematical orality was afforded or constrained in both public and private classroom contexts. The fifth stage addressed those learning outcomes evident in student use of mathematical terms in post-lesson video-stimulated interviews. This poster presents the results of the last three analytical stages in the form of three graphs. From the analyses undertaken in this project, we suggest that the instructional practices of the teachers in the various classrooms assigned spoken mathematics a very different function in the learning process, leading to demonstrably different learning outcomes.

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THE GRAPHIC CALCULATOR IN THE LEARNING OF FUNCTIONS IN THE SECONDARY EDUCATION

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The concept of function is one of the most important in mathematics. However, students frequently have difficulties in problem solving that evolves conversion with different representations of functions. In Portugal the graphic calculator is used by students in classes since 1997, following the suggestion of the syllabus but very little is known about how they integrate it in their learning processes. The poster illustrates a PhD research project which main goal is to analyze the contribution of the graphic calculator in the learning of functions, in secondary education. The theoretical framework of this study focuses on three fields: instrumental approach, theory of representation and the learning of functions. Research has shown us that the calculator is not an efficient mathematical tool for itself, and that only through a complex process - instrumental genesis - the students can transform the artifact into an instrument that will help them solve problems by combining the various sources of information available (Guin & Trouche,1999; Rivera, 2007; Kidron, 2008).

This study intends to answer to the following questions: (i) Which are the processes that students use in problem solving with functions and how they evolve with the use of the graphical calculator?; (ii) Which are the difficulties that students face in problem solving with functions and how the graphical calculator helps them to exceed these difficulties?; (iii) How can we characterize the process of instrumental genesis for the graphic calculator in the classroom? For this, I will develop a qualitative research using the case study method, through a longitudinal study of two years following a class and, particularly 4 students within the class, in grades 10 and 11. Data collection includes participant observation, documents produced by the students, and interviews with them. A graphical display will describe the research project in terms of the main goals, theoretical background and methodology.

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MATHEMATICAL THINKING APPROACHED FROM AN 'AESTHETIC OF RECEPTION' PERSPECTIVE

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Considerable research has been conducted into the notion of mathematical thinking. Research literature reports on how mathematical thinking develops and mathematical processes involved, e.g. generalizing, reasoning, conjecturing, abstraction (Schoenfeld, 1992; Polya, 1945; Mason et al., 1982). However, this does not seem to address the pragmatic question raised by mathematics teacher Pitt (2002) of how mathematical thinking can be recognized in the classroom nor how teachers can ensure students are engaged in rich mathematical thinking. This paper reports on a study addressing this. It built on Jauss' (1982) framework of 'Aesthetic of Reception' from literary analysis. Jauss argues the focus should be on the result a text has on the reader, not on procedural processes involved. A text awakes existing memories from earlier texts, and triggers emotions, expectations and assumptions called the existing 'horizon of expectations'. The new text varies, corrects, alters or reproduces this horizon. Jauss argues the artistic character of a text is determined by the distance of change in the expansion in the reader's 'horizon of expectations'. The study explored the parallel between horizon of literary and mathematical expectations through the analysis of case studies. It addressed three research questions: does this framework allow to recognize and to describe students' mathematical thinking in the classroom; does it allow to account for different intensities of richness of students' mathematical thinking; what are the limitations of its validity? Data consisted of a Year 7 lesson observation and case studies of different responses of two students to the same mathematics task. Evidence was taken from classroom field notes, audio data and interviews with students and teacher. The study found that the framework provided descriptors for students' mathematical thinking and for the difference in their richness. However, questions remain about the process of defining 'mathematical expectations'. The poster will include an outline of Jauss framework, research questions, a diagram of the horizon of literary and mathematical expectations in context to the case studies, and the findings of the study.

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THE CAPACITY OF SOLVING PROBLEMS USING ADDITION AND SUBTRACTION AND THE DEVELOPMENT OF NUMBER SENSE IN GRADE 2

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The development of number sense is an essential aspect of learning mathematics in the first school years, enabling students to solve problems involving addition and subtraction with positive whole numbers (McIntosh, Reys & Reys, 1992). This study is based on an experiment at grade 2, carried out throughout the school year 2007/08. Its aims is to understand how students develop some of the aspects of number sense in contextualized problem solving using addition and subtraction of positive whole numbers, contemplating the various meanings of these operations.

The theoretical framework includes three main domains. The first is the notion of number sense (McIntosh, Reys & Reys, 1992); the second is the theoretical orientations of addition and subtraction (Carpenter & Moser, 1983; Fosnot & Dolk, 2001; Fuson, 1992), and the third domain is about a design experiment that involves an emergent approach in which psychological constructivist analysis of individual activity is coordinated with interactionist analysis of classroom interactions and discourse (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003).

The methodology of this study is qualitative and interpretative. The object of study is a group of five children in elementary school, at grade 2. Data collection includes (i) participant observation; (ii) interviews; and (iii) written documents, namely, reports of classroom episodes, tasks and student's involvement in classroom activities.

The poster presents the aim, a synthesis of theoretical ideas underlying this subject, the context and methodology of this research project and some student work illustrative of the main results.

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FROM PIE TO NUMBER: LEARNING-BY-DOING ACTIVITIES TO IMPROVE CHILDREN'S FRACTION UNDERSTANDING

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We designed an educational intervention mostly based on the numerical concept of fractions. 4^th^- and 5^th^-graders received wooden circles representing fractions from halves to twelves, and played card games involving fractions during ten weeks. First results showed improvement in conceptual and procedural understanding of fractions.

This paper will be presented in pictorial form (poster).

INTRODUCTION

Fractions are well known to constitute a stumbling block for primary school children. Various hypotheses have been proposed in order to explain those difficulties. One of them is that fractions can denote different concepts, such as proportion, part of a whole, measure and number. The numerical representation of fractions seems to be one of the most difficult concepts to catch for primary pupils. Besides, the use of fractions demands to articulate conceptual knowledge with complex manipulation procedures. We introduced games about the numerical concept of fractions in order to improve the conceptual understanding of 4^{th} and 5^{th} graders. There were no formal lessons as we insisted on the learning-by-doing principle.

METHOD

Participants were 137 4^{th} graders and 142 5^{th} graders of eight classes from four schools of the French Community of Belgium. Half of the classes received the experimental instructions, while the other half pursued their usual lessons. Pupils were familiarised with wooden circles divides from half to twelves during the first experimental lesson. Afterwards, during ten weeks they played five games such as memory, bagger-my-neighbour, and blackjack with cards representing fractions. The wooden circles helped them representing and manipulating fractions while playing games. There were four levels of difficulty for each game.

RESULTS AND DISCUSSION

When comparing pre- and post-tests, the first results show an improvement in conceptual and procedural understanding of fractions. This underlines the importance of manipulating concrete objects while learning fractions. This also enables us to insist on establishing various conceptual representations of fractions in the curricula, and not only the part of a whole representation.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 452. Thessaloniki, Greece: PME.

THE MENTAL REPRESENTATION OF FRACTIONS: ADULTS' SAME-DIFFERENT JUDGMENTS

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Although fractions have interested educational psychologists for a long time, little is known about the processing and representation of fractions in normal, numerate adults. We investigated the issue by measuring performance and response time in a same-different decision task. The results show that access to the magnitude of fractions remains slow and error-prone.

This paper will be presented in pictorial form (poster).

INTRODUCTION

A few recent studies have investigated how adults process fractional notations in comparison tasks. Bonato *et al*. (2007) concluded that adults process the numerator and denominator separately and do not access the magnitude of the fraction. Conversely, Meert *et al.* (2008) produced data suggesting that the magnitude of the fractions is used, at least in limited conditions. The aim of the present research was to examine the time course of processing in a same-different task.

METHOD & RESULTS

We used a same-different judgment task with two conditions. In the nominal condition, fractions were categorized as same if their numerators and denominators were identical (e.g., 1/2 1/2). In the semantic condition, fractions were classified as same if their values was equivalent (e.g, 1/2 2/4). Eighty 1st and 2d year university students were tested. Stimuli included a large range of fractions with denominators up to 20.

Overall, no effect of equivalence was found on “different” responses in the nominal condition, suggesting that magnitude is not accessed automatically. RTs were slower in the semantic condition, and performance was weak for equivalent fractions, suggesting that participants often fail to identify numerical equivalence.

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AN INTEGRATIVE THEORETICAL SYSTEM FOR MATHEMATICS EDUCATION: THE ONTO-SEMIOTIC APPROACH TO MATHEMATICAL KNOWLEDG AND INSTRUCTION

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The Onto-Semiotic Approach (OSA) is a theoretical framework that adopts semiotic and anthropological assumptions about mathematics, and socio-constructivist and interactionist principles for the study of teaching and learning processes. The OSA was started by the "Theory of Mathematics Education" Research Group at the University of Granada (Spain) in the beginning of the nineties, and is now developed and applied by others Spanish and Latin-American research groups. The set of theoretical notions that at present compose OSA are classified into five groups, each of them allowing specific level of analysis for the teaching and learning processes of particular mathematical themes:

(1) *System of* (operative and discursive) *practices*. Here we assume a pragmatist – anthropological conception of mathematics, both from the institutional (socio-cultural) and personal (psychological) viewpoints. (2) *Configuration of mathematical objects and processes* that emerge and intervene in mathematical practices. (3) *Didactical configuration*, conceived as the articulated system of teachers' and students' activity when interacting within a configuration of mathematical objects and processes linked to a problem–situation, is the main tool to analyze mathematical instruction. (4) *Normative dimension* is the system of rules, habits, norms that restrict and support mathematical and didactical practices, and which generalizes the notions of didactical contract and socio-mathematical norms. (5) *Didactical suitability* is defined as general criteria of adaptation and appropriateness of the educational authorities' students' and teachers' actions, as well as of the knowledge and resources used in a specific study process.

The Onto-semiotic Approach allows a coherent articulation of diverse theoretical models usually applied in Mathematics Education research (didactical phenomenology, ethno-mathematics, anthropological theory, didactical situation, conceptual fields, semiotic representation registers, socio-epistemology, etc.)

Acknowledgement

This research work has been carried out as part of the project, SEJ2007-60110/EDUC, MEC-FEDER.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 454. Thessaloniki, Greece: PME.

EXAMINING STUDENT PERFORMANCE ON NON-ROUTINE WORD PROBLEMS

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Teachers of Mathematics in Slovakia often experience that many students may know how to carry out basic mathematical procedures when problems are presented in symbolic form but are not able to apply these procedures in solving problems presented in words. The results of the 2003 Program for International Student Assessment (PISA) revealed that the students of Slovakia are particularly weak in mathematical problem solving performance and scored below the average score of other industrialized countries.

According to Mayer and Hegarty (1996) the difficulty for students lies first of all in understanding the situation being described in the problem and building of a coherent mental representation of it. The authors distinguished between two general approaches in solving word problems, which they labeled the direct translation strategy and the problem model strategy. In the direct translation strategy, the problem solver selects numbers and key words from the problem and prepares to perform arithmetic operations on it. The resulting solution plan is likely to be incorrect for problems in which the key words prime incorrect operations. In contrast in the problem model strategy the problem solver seeks construct a mental model of the situation described in there. The resulting solution plan is likely to be correct even for problems in which the key words prime incorrect operations.

The purpose of this poster is to present the results of an empirical study which was carried out at primary and secondary grammar schools in Slovakia in a total of 40 classes. The study investigated Primary Grade 5, Primary Grade 8, and Secondary Grade 2 students' written performance in solving non-routine word problems. Our goal was to examine the hypothesis that when confronted with a word problem, students who have worse school results in mathematics are more likely than students with greater mathematical knowledge to use the Direct Translation Strategy.

The poster will present the quantitative analysis of two problems used in the study (water lilies and meeting-up). Furthermore, the poster will also provide examples of wrong answers most frequently given by students and some explanations to our findings.

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FACILITATING DEEP SUBJECT MATTER KNOWLEDGE IN PROSPECTIVE SECONDARY SCHOOL TEACHERS

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In this short oral presentation, I will describe initial results on a study involving a mathematics course specifically designed to facilitate prospective secondary school teachers' acquisition of deep subject-matter knowledge which can later profitably be connected to PCK and curriculum knowledge in their pedagogical studies. In the course, the student teachers are charged with the planning and implementation, in groups, of innovative tutorial sessions for first year university mathematics courses. This allows the students to experience both the profundity of knowledge required to teach, and how deep knowledge may come about in the process of teaching. Furthermore, the active, discursive nature of both the tutorials and their planning should contribute to widening the students' views on the role of reasoning and conviction in mathematics.

The course is running for the second year, but the research into possible effects on students' subject matter knowledge is still at its inception. In the presentation, I will present results from student interviews based on a qualitative coding scheme which draws out common experiences as well as critical incidents. The data will be discussed in relation to several frameworks, including Schulman's classical work, the work of Ball on primary teacher's knowledge for teaching (e.g. Loewenberg Ball, Thames and Phelps, 2008) and the knowledge quartet (e.g. Rowland, Huckstep and Thwaites, 2005).

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SCHEME – ORIENTED EDUCATION ON ELEMENTARY LEVEL

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Scheme-oriented education (especially on elementary level) comes out of the idea that most knowledge and abilities which help us in our everyday life are gained not through aimed learning but through various repeating life experience acquired as consequence of our life needs. The tool of the scheme – oriented education is a set of environments. The poster presents the environment called BUS.

Many children learn the essentials of calculating, properties of geometrical shapes and objects and basic logical figures already on pre-school level through various repeating life experience gained as the consequence of their life needs. On entering school, this spontaneous way of acquiring knowledge changes to the way of instruction and drill. Scheme-oriented education proposes that the spontaneous way of acquiring knowledge should continue and gradually be enriched by the traditional school practices.

The tool of scheme-oriented education is a set of environments. What is understood by an environment is a segment of life (later mathematical) pupil's experience transformed into a set of objects and relations, which make it possible to pose interesting problems. Every environment contributes to the growth in several areas of the pupil's mathematical development. BUS is one of these environments coming out of the pupil's experience with travelling by public transport.

The poster will present the construction of the environment BUS in 6 stages: 1. introduction, 2. simple problems solved using dramatization, 3. creation of a sign language, 4. work with the sign language and the table, 5. completing the table, 6. division of passengers to female and male.

The didactic commentary points at the needs and abilities developed in the individual stages. The poster will also contain picture documentation of implementation of the game BUS and present pupils' solutions appended by didactic commentary.

Acknowledgement. The paper was supported by the research project – "Teaching profession in the changing education requirements" No. MSM 0021620862.

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STUDENTS' MATHEMATICAL REASONING WHEN EXPLORING INVESTIGATION ACTIVITIES

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Research has indicated that a continuous experience in carrying out mathematical investigations in the classroom may support the development of students' mathematical reasoning and their understanding of mathematical processes and may become an important element of the learning process (Ponte, 2007). Nevertheless, research results are still insufficient to guide an efficient implementation of this kind of activities, mainly at university level.

The main aim of the study is to understand the mathematical reasoning processes used by university students when exploring investigation activities and to know how it may influence students' learning of concepts and procedures and the development of problem solving skills. The study also aims to connect research with teaching practice. This is done through a teaching experiment using investigation activities. The participants are 2nd year students of the Naval Academy who attended the Numerical Analysis course. The investigation activities are the way I choose to gain information on students' performance on reasoning strategies and problem solving that may represent a shift from elementary to advanced mathematics. Hence, four investigation activities are designed and proposed to the students as a means to activate their reasoning processes, to promote their contact with mathematical processes typical of investigation activities and to provide a basis for learning the concepts, processes and strategies of numerical analysis.

The study uses a qualitative research approach based on case studies. Data collection methods include participant observation, audio tape recording of the interviews to the cases, the written reports produced by students and questionnaires. Preliminary results show that reasoning based on established experiences dominates while plausible reasoning is rare and of limited range (Lithner, 2008). Most of the students prefer an informal approach and have tendency to avoid visual methods and arguments.

The purpose of this poster is to document the teaching experiment described above. The poster begins with a presentation of the study including the aims, context and methodology. The focus is then on the teaching experiment that supports the study and some examples of the students' exploration of the proposed investigation tasks.

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THE FACTORS RELATED TO MATHEMATIC ACHIEVMENT OF THE BEGINGING LEARNER IN ELEMENTARY SCHOOLS

Li-Yu Hung (National Taiwan Normal University)

Pi-Hsia Hung (National Tainan University)

Li- Hwa Chin (Kaohusing Municipal Neiwei Elementary School)

Shu-Li Chen (National Taitung University)

This study aims to investigate the factors related to mathematics achievement of different graders in the elementary school and to find out the predictive factors to learn math in the beginning of the elementary school. Three research questions are proposed: (1) what are the personal factors related to pupils' math? (2)what are the environmental factors related to pupils' math achievement? (3) Are the related factors to pupil's math achievement varied by grades?

There are 563 pupils participating in this study, who are selected from 8 elementary schools and three different area of Taiwan (i.e. north, south and east). Two different sizes of schools are selected in each area and the number of school selected to represent the proportion of the schools in the area. The number of the 1st-graded students is 258, and that of the 3rd-graded is 291. All the participating students were administrated in The Test of Math Achievement and The Primary Reading Test for the first graded, The Reading Comprehension Test for Elementary School in the end of two semesters (i.e. January and June). The Chinese version of Reven Progressive Test and The Survey of Learning Experience, included leisure activity related to reading or math, homework, after school program, interest, and confidence, were administrated in the second semester (April). The teachers of all the participating classes filled out the Teacher Survey which are selected from the significant items in TIMMS(2003) or PIRLS(2001, 2006). The administrative representative of each school (i.e. director of academic affair or director of guidance and counselling) filled out the School Survey which are selected from the significant items in TIMMS(2003) or PIRLS(2001, 2006).

The major findings are conclude: (1)IQ and reading are significantly related to math achievement, and reading comprehension is more powerfully predictive to math than IQ in the 3rd grade. Gender failed to be significantly related to the math. Self-confidence and interesting math are significantly related to the math achievement. (2)Most factors of school environment are similar among schools, so only few significant ones were found(e.g. area and school size). However, many factors of home environment are related to math achievement(e.g. parental education, job, the number of books at home, accessibility to computer at home. (3) The related factors or the relationship to math achievement are found differently between G1 and G3. The related factors frequently reported in literature, such as reading, items of home economical situation, and parent's involvement are found in the G3 but not in the G1.

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 459. Thessaloniki, Greece: PME.

VALUE DIFFERENCES AND THEIR IMPACT ON TEACHING

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In order for Thai teachers to change and shift from their usual ways of teaching and learning to conduct "Lesson Study" and "Open approach" is challenging because both are new innovations for improvement of education which their ways and their focus very different from Thai school context. To bring new innovations into school teachers may have trouble and have impact on their teaching. This study is a part of three-year professional development project (2006 - 2009) implementing lesson study and open approach conducted by the Center for Research in Mathematics Education. Unlike Japanese lesson study, this project modified Japanese lesson study by incorporating open approach and emphasizing "collaboration" in every phase of lesson study cycle. Thus, teacher, researchers and outside experts participated in *collaboratively* designing research lesson, *collaboratively* observing their friend teaching the research lesson, and *collaboratively* doing post-discussion or reflection on teaching practice (Inprasitha and Loipha, 2007). Every week, 78 teachers from 4 schools participated in each phase of lesson study cycle through 3 academic years.

This study was aimed to investigate while a teacher has to struggle or confront value differences, how the teacher manages these different values and the impact on their teaching and learning in classroom? The participants in this study were 7 mathematics teachers from a school in a three-year professional development project. This study was a qualitative research, employing ethnographic study. The researcher, as important instrumental, was in the school for 3 years, making participative observation and field note for recording to teaching and learning in every phase of lesson study cycle and data analysis was done by means of analytic description.

Preliminary findings of the study found that teachers have to manage value differences such as some teachers rejected the new innovation and reverted back to the way that they feel more confident, even if it can not improve the student's learning as Thai education reform needs. On the others hand, some teachers have been trying to change the ways of teaching and culture working with colleagues to improve their teaching and student's learning even though they think it is hard to do and have to try more.

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This work was granted by the Commission on Higher Education granting. Ms.Thanya Kadroon was supported for CHE-PhD-THA from the Commission on Higher Education.

RITUAL, ARBITRARY AND IMPRACTICAL: DO STUDENTS' PROOF SCHEMES MIRROR CLASSROOM EXPERIENCES?

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Mathematical competitions are a rather unconventional context in which mathematics education researchers have the opportunity to explore student development with regard to key aspects of mathematical thinking such as proving and argumentation. In Greece the Hellenic Mathematical Society (HMS) competition is an annual, national event taking place in three, increasingly difficult steps (Thales, Euclid and Archimedes). Participants are pupils between the ages of 13 and 18 and participation to the next step presupposes success in the previous. Although pupils are encouraged by their schools to take part, participation is voluntary. The problems that the participants are asked to solve are often outside the material covered by the Greek National Curriculum and different from the tasks encountered in the typical Greek secondary classroom. However the Thales step, as the easiest of the three, provides participants with the opportunity to engage with challenging tasks that enrich what the students have already encountered in their ordinary mathematics classrooms. It is thus the one step of the HMS competition that also provides an opportunity for collecting data on how a particular group of secondary students (keen on mathematics and with a positive image of their ability in the subject) performs in a series of challenging, intriguing, yet not totally alien to the typical Greek secondary classroom, tasks. The small-scale study we draw on here aims to examine performance in such tasks of 74 students (44 in Year 10 and 30 in Year 11) with particular regard to the proof schemes (Harel & Sowder, 1998) evident in their responses to 8 tasks (4 for each Year in Algebra, Geometry and Elementary Number Theory). Our initial scrutiny of the student scripts suggests findings that resonate with those already reported in the literature (e.g. Mariotti, 2006) such as the dominance of empirical proof schemes (e.g. seeing evidence as proof) but also suggests some less widely-reported tendencies such as: a 'ritualistic' and not always coherent, consistent or meaningful approach to algebraic manipulation of the task's givens; the frequent appearance of unfounded assertions; and, missing opportunities to benefit from the insight that may be gained from approaching a problem initially in an empirical, intuitive, example-based manner. Our analyses currently explore potential origins of these tendencies, such mathematical practices students experience in the classroom.

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NEGATIVE NUMBERS AND METAPHORICAL REASONING

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Negative numbers are taught as a topic in grade 8 in Sweden. At higher levels this knowledge is to be used in algebra and calculus. This study explored how 99 pre-service preschool and elementary school teachers solved tasks involving operations with negative numbers. Based on the works of Lakoff and Núñez (1997) the use of metaphors in mathematical reasoning was assessed. Concerning their level of secondary school mathematics the study population is representative of a population of Swedish students leaving secondary school. A positive correlation between level of secondary school mathematics and correct solutions was also hypothesized.

In a written test two of the items were tasks that had previously been identified as the two most difficult types of tasks involving negative integers (Küchemann, 1981). Item 1: [(-3) – (-8) =] and item 2: [(-2) · (-3) =] . Students' answers were as follows: for item 1: [5] 70% ; [-11] 25% ; [-5] 4% ; [5 and -11] 1%

for item 2: [6] 55% ; [-6] 44% ; [-12] 1%

The strategies used to solve these items where categorised as being either metaphorical or formal (i.e. by means of sign rules, induction, laws of arithmetic or symbol manipulation). For item 1 students who solved the task directly by means of metaphorical reasoning (n=14) *all* gave an incorrect answer [-11], whereas all students who *first* transformed the expression formally from (-3)-(-8) to (-3)+8 and *then* used metaphorical reasoning (n=12) arrived at the correct answer [6]. Among students using *only* formal reasoning 85.5% were successful. For item 2 only 2 students used metaphorical reasoning and they both answered incorrectly [-6]. No correlation was found in this population between levels of mathematics (number of secondary math courses passed) and achievement on these negative number items.

In conclusion: The results indicate that, when it comes to operating with negative numbers, metaphorical reasoning is not sufficient in itself but needs to be supplemented with formal reasoning, and inappropriate means of reasoning, be they metaphorical or formal, do not disappear automatically but need to be explicitly addressed in elementary as well as secondary school courses.

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SPATIAL SENSE: A MISSING LINK IN PRESERVICE TEACHERS' KNOWLEDGE FOR GEOMETRY TEACHING

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This poster session contributes to research on the tasks of mathematics teacher preparation and their impact. Specifically, the study examines preservice secondary mathematics teachers' responses to geoboard tasks in the secondary mathematics methods course. These university-level mathematics students report surprising gains in spatial understanding of essential geometric concepts (area, area formulas, and the Pythagorean theorem) typically taught in the middle grades.

BACKGROUND, THEORETICAL FRAMEWORK & METHODOLOGY

At the PME 2004 Research Symposium on the Nature of Mathematical Knowledge for Secondary Teaching (Stacey et al, 2001), international researchers agreed that secondary mathematics teacher preparation must address the procedural, rule-bound nature of prospective secondary teachers' understanding of school mathematics. This qualitative study of 20 teacher-learning journals in the author's secondary mathematics methods course analyzes the impact of geoboard manipulative-tasks on prospective teacher understanding of school geometry using a modification of Perkins and Simmons' (1988) levels of disciplinary understanding framework.

RESULTS, DISCUSSION AND FUTURE RESEARCH

The poster describes three geoboard tasks: area on the geoboard, inducing area formulas on the geoboard, and area versions of the Pythagorean Theorem and its generalized form. For each task, prospective teachers report gains in spatial understanding of geometric ideas previously understood only algebraically. One brief sample reflection previews the poster findings: "On the geoboard, finding the area by counting units was especially helpful. Before this I saw area of a rectangle as $\text{area}_{\text{rectangle}} = b \bullet h$, and that is all it meant to me." The study concludes with a discussion of spatial ability, research on its importance for student achievement, and recommendations for future research on Euclidean and non-Euclidean spatial-thinking tasks in the preparation of secondary mathematics geometry teachers.

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STUDENTS’ UNDERSTANDING OF THE NEGATIVE NUMBERS ON THE NUMBER LINE MODEL

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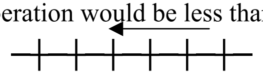
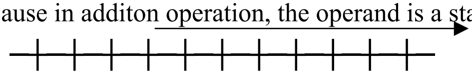
It is difficult for many students to understand the operation with negative numbers. In the textbook, there is used to the number line model for students to understand the operation with negative numbers. The previous researches (cf. Bruno and Martinon, 1996; Lytle, 1994) have been not enough to show how students understand them.

The purpose of this paper is to investigate how students understand the relation of negative numbers with the number line model. And 129 students in seventh Grade were given some questionnaire tests to represent the number line model to satisfy the operation with negative numbers. As some of results, there became clear as follows;

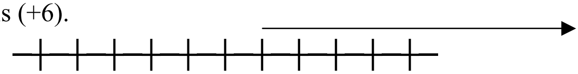
(1) They represented the number line to keep on the misconception. For example, in the problem ((-1)-(-2)), they answered (-3) as the correct answer. Because they thought that result of operation would be less than (-1) if they subtract (-2) from (-1).



(2) They apply the mistake rule to relate the operation with the models. For example, when some were asked to represented the operation((-2) ×(-3)), they wrote (-2) as a start point. They thought that the operand must be a start point in multiplication operation, because in additon operation, the operand is a start point.



(3) They represeted the operation as another the operation to keep same answer. For example, when some were asked to represented the operation((-2)×(-3)) by using the number line, they interpreted as (+2)×(+3). Because the answer of (-2)×(-3) and (+2)×(+3) is (+6).



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THE NUMBER ANALYSIS IN THE KINDERGARTEN

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Most didactical approaches of numerical notions in the kindergarten ignore one of the fundamental traits of number: the analysis of number. The analysis and synthesis of number find theoretical support in the work of Piaget and Resnick (Piaget & Szeminska, 1941, Resnick, 1983).

The experimental planning includes two groups of 6-year-old children, who attend the second class of kindergarten: the experimental group (or research group) and the control group. The research group attended a new approach of the number concept, which stresses on the analysis of number, while the control group attended the usual didactic method of the approach of number used in Greek schools, which includes activities of logic type such as classifications, corresponding, serial-ordering as a core axis.

The findings which came out of this research verify the assumption that the toddlers who were tried in the educational approach of the number which is called “approach of number analysis” (research group) have demonstrated a more mature ability on the concept number of and also in problem solving (addition, subtraction, partitioning), than those who were tried in the usual teaching of the logical-mathematical approach of the number (control group). Moreover, the former group has been able to apply the knowledge of analysis, especially with the analysis-synthesis use of finger patterns in many aspects of the number concept. All in all, the above results can withstand time as it was proved by the experiment using the same children as subject when they reached the 1st class of elementary school. The foundation of this approach and the content of beam of instructive activities are of utmost importance (Kosyvas, 2001).

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THE ZEROTH PROBLEM AND STATISTICAL REASONING

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Research indicates that statistical reasoning evolves from everyday intuitions, observations and reasoning processes (Schwartz, Goldman, Vye, & Barron, 1998). In this paper, I focus on contextual structuring as an important but little described part of statistical modelling in learning and teaching situations. Mallows (1998, pp. 1-3) expounds the relevance of context to the statistical process and indicates the importance of everyday reasoning, to solve what he calls the *Zeroth problem* – getting to understand a situation in terms of its contributing factors, before data is gathered. In addition Beyeth-Marom, Fidler and Cumming (2008) presents a construct 'Statistical Cognition' which lends weight to researching the relationship between descriptive, everyday reasoning and normative reasoning as demanded by the field of Statistics.

In the presentation, I will argue that the development of statistical reasoning depends on increasing ability to structure everyday contexts in terms of variables to be measured, and the ability to conjecture relationships between concomitant variables – even before data is gathered or interpreted. I will discuss two examples from my PhD study where students attempt to structure situations in order to make them "treatable" with statistics, namely: (1) What is a reasonable price to pay for a pre-owned car? (2) Is South African minimum wages scandalous? I will show that spending time on solving the *Zeroth problem* as the initial stage of statistical modelling can facilitate the development of sound statistical reasoning from sound everyday reasoning. I will interpret my findings in terms of the construct of Statistical Cognition.

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IMPROVING CHILDREN´S DIVISION WORD PROBLEMS SOLVING SKILLS

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Children’s difficulties with division are well known in the literature (Correa; Nunes & Bryant, 1998, Li & Silver, 2000). One may ask whether these difficulties would be overcome if children had concentrated experience with problem-solving situations involving discussions about the role played by the remainder, about the importance of keeping the equality of the parts and about the inverse co-variation between the size of the parts and the number of parts. This idea was tested in an intervention study carried out with 100 third grade low-income Brazilian children (8 to 11 years old) who experienced difficulties with division. They were equally assigned to an experimental and a control group. All participants were given a pre- and a post-test consisting of 12 division word problems. Children in the experimental group were given a specific intervention involving a large variety of problem solving situations. They were asked to discuss their thought processes while the examiner explicitly talked with them about the remainder, the equality of the parts and about the inverse relation between the size of the parts and the number of parts. No significant differences were found among the groups in the pre-test. After the intervention, children in the experimental group gave more correct responses than in the pre-test and were also able to offer justifications expressing an understanding of the operational invariants in the concept of division. The control group showed no improvement when compared in the two testing occasions. The conclusion was that discussion explicitly focusing on the role played by the remainder and on the inverse relations can help children overcome their difficulties with the concept of division.

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THE USE OF NON-VISUAL REPRESENTATIONS IN PRIMARY SCHOOL MATHEMATICS

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The majority of representations in mathematics teaching are of visual modality, yet teachers are usually advised to make use of all sensory modalities (Hasemann, 2003, pp. 63f). To learn more about adequate use of non-visual material it could prove useful to inspect a situation where visual materials are not an option: teaching blind children. The development of materials for these children has to be based on knowledge about their perceptional and cognitive preconditions. The results of a thorough theoretical review on the foundations of arithmetical learning of blind children are to be presented here. This includes an integrative perspective on current research from neuropsychology and cognitive psychology on haptic perception (e.g. Heller & Ballesteros, 2006), auditive perception (e.g. Stevens & Weaver, 2009), mathematical cognition (e.g. Dehaene, Spelke, et al., 1999; Szücs & Csépe, 2005) and mental representation (e.g. Cornoldi & Vecchi, 2000; Kosslyn, 2005). It is shown that haptic and auditive representations are important for the formation of mental representations in *all* children, blind and sighted. Examples of auditive and haptic materials for first graders will be provided.

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THE GENDER DIFFERENCES IN CRAM EDUCATION EFFECTS ON MATHEMATICS ACHIEVEMENT AND LITERACY

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Pi-Hsia Hung

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The performances of Asian students in international assessments such as Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA) have received much attention. Educational researchers have tried to explain the outcomes from different facets. This study attempts to focus on a certain educational context that is specific in Asian culture, the cram school offers additional after-school instruction to enhance students' academic performance.

The purpose of this paper is to compare the gender differences of variances accounted by time in cramming variable on senior high school student mathematics literacy and mathematics achievement. The sample included is drawn from PISA 2006. There were 8815 students in the PISA 2006 mathematics assessment. Among these participants, 4078 students also accepted the Taiwan Assessment of Student Achievement on Mathematics (TASA). The purpose of TASA is to investigate the student mathematics achievement. The PISA assessment focuses on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms.

Only about 40% students don't go to cram schools. For females, the variances accounted by time in cramming are 10.4% of TASA and 7.2% of PISA, but the variances accounted are 7.2% of TASA and 4.7% of PISA. In other words, TASA is more cram education sensitive and the females could be benefit more than males from the cram education. Tsai and Kuo (2008) found that most of cram school students conceptualized learning or learning science as memorizing school knowledge, preparing for tests, or practicing tutorial problems and processing calculations. This study suggested that research could further analyze gender differences in learning features (such as study and motivational approaches) in terms of their conceptions of learning or learning mathematics among the cram school students.

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PROFESSIONAL LEARNING PROCESS IN AN EDUCATION PROGRAM: REFLECTIONS ABOUT RESEARCHES AROUND THE USE OF CALCULATORS IN MATHEMATICS CLASSROOM

Nielce Meneguelo Lobo da Costa and Maria Elisabette Brisola Brito Prado

UNIBAN BRASIL UNIBAN BRASIL

This study aims to discuss upon the professional learning process and the potentialities of using technological resources, particularly calculators, in teacher education process in a way that it could interfere in Mathematics teacher praxis. The research was developed which six students into an education program to Public High School teachers. It was structured half in presidential classrooms and half online, around six mathematical themes, e.g. Teaching and Learning Geometry; New Technologies and applicability in Mathematics Classes, etc. Beyond, a classroom research had been conducted by each student and resulted in a final report.

The theme "*New Technologies…*", aimed to: *"enhance teacher comprehension, from the theory and practical perspective on issues related to the complex relation of teaching with technological tolls.* Jahn e Healy (2006, p. 5)

Based on qualitative methodologies and using register data in the virtual environment, an interpretative analysis was performed essentially using documents related to the student's researches on the use of calculators.

In common, the students' researches showed that their pupils realized the fact that calculator will not solve the problems for them, but with the tool it was possible to explore problems, got conclusions and they would not be able to do without it. Other results will be exemplified in the poster presentation.

Regarding potentialities to interfere in the praxis, was evident the importance for a teacher to experience didactic situations using calculators, first as an apprentice and after building his own sequence to apply in the classroom. (Ponte, 1989)

Finally, we can conclude that calculators may be a didactic resource to develop mental calculation and a tool to support Mathematics teaching, establishing a different and more significant learning possibility.

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VIDEO ANALYSIS: A TOOL FOR UNDERSTANDING STUDENTS' DECODING OF MATHEMATICS TASKS RICH IN GRAPHICS

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CENTRAL BOX WITH THE KEY THEMES

Video has numerous strengths as a research tool (Ratcliff, 2003) and can provide researchers with a holistic view of participants' performance and behaviour. In particular, video can assist in capturing subtleties; facilitate the re-coding of data from different perspectives; and support in-depth, fine-grained data analysis. This study seeks to gain a greater understanding about how students process and make sense of mathematics tasks which are rich in graphics (e.g., maps). Studiocode[1], a qualitative research tool, will be employed for video editing students' engagement with these tasks. This tool will allow for the annotation of video and recording the frequency of specific events which occur while students are engaged in the task.

SURROUNDING TEXT BOXES

Setting: This study draws on data from a 3-year longitudinal project period which examined 40 primary-aged students' interpretation of graphics in mathematics[2]. The participants were interviewed at ages 10, 11 and 12. Video transcripts of students solving and explaining their sense making on mathematics tasks will be analysed.

Methodology: Using qualitative data mining techniques, video transcripts will be analysed within a multiliteracies framework which allows for the analysis of the dynamic relationships between linguistic, visual, audio, gestural and spatial elements in video (The New London Group, 2000). Screenshots from Studiocode will be displayed to illustrate how the video data is to be analysed.

Mathematics tasks: Examples of the tasks presented to students will be displayed on the poster to indicate the level of graphical processing involved for the students.

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[1] See http://www.studiocodegroup.com/Default.htm for more information

[2] See http://www.csu.edu.au/research/glim/index.htm for more information.

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EXPLANATIONS AND NEGOTIATION OF MEANINGS IN THE CONCEPTIONS AND PRACTICES OF A MATHEMATICS TEACHER CANDIDATE

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This investigation presents a case study of a mathematics teacher candidate, in the final stage of her teacher education program, addressing her mathematics communication conceptions and practices. The main object of study is the communication that she promotes in her *mathematics classroom,* with focus on explanations and negotiation of meanings.

The interest for communication in the mathematics classroom has grown recently. Two important elements of such communication are explanation and negotiation of meanings (Bishop & Goffree, 1986). Both emerge as important forms of communication in the interactions between the teacher and the pupils and between the pupils themselves (Yackel & Cobb, 1998). This communication carried out in the mathematics classroom, integrates the teacher's didactical knowledge of mathematics (Ponte, 1999) and relates with other aspects of this knowledge.

This poster presents the aims of this research, the general theoretical framework for the study of communication in the mathematics classroom and some of the data collected for one of case studies (Yin, 2003). The aim is to interpret the conceptions and practices about explanation and negotiation of meanings of a mathematics teacher candidate. The conceptions and practices will be shown in the poster through extracts from interviews and classroom episodes, respectively, through text and diagrams.

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NUMBER SENSE DEVELOPMENT: A TASK CHAIN FOCUSED ON MULTIPLICATION

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The Project "Developing number sense: curricular demands and perspectives"[2] studies the development of number sense in children from 5 to 11 years old. The project team included classroom teachers and researchers that developed and experimented tasks and task chains that intended to foster number sense.

In this Poster Presentation I will present a task chain focused on multiplication and it implementation in a 2nd grade class (7-8 years old). I will center the presentation and discussion on the strategies used by children.

NUMBER SENSE DEVELOPMENT

Number sense has been considered one of the most important components of elementary mathematics curriculum. Understanding numbers and relations among numbers; understanding meanings of operations and how they relate to one another; and to compute fluently and to make reasonable estimates are curricular standards recommended both by international literature (NCTM, 2000) and Portuguese curricular documents. Having these ideas as a starting point, a team of classroom teachers and researchers developed (from 2005 to 2008) the Project "Number sense development: curricular demands and perspectives.

The task chain – *Forming groups*, *The loft wall* and *Chewing gums*

This task chain was developed as a hypothetical learning trajectory in the sense used by Simon (1995).This chain had 3 tasks and was designed with the objective of developing the understanding on multiplication and the use of different calculus strategies related with multiplication. A case study analyses the way this task chain was explored in a 2nd grade classroom. This Poster will present the 3 tasks and will illustrate some of the strategies and procedures that children used to solve them. The information will be presented in a graphical format that includes figures with the children's strategies.

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ALGEBRA ONE: THE CAREER GATEKEEPER

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BACKGROUND AND METHODS

The education of 21st century women requires a dramatic shift, prospective workers need to be well educated in mathematics and science as the United States closes the door on the industrial age. Fifty years ago, 20% of high school graduates took Algebra 1. Today, many states require that all students take Algebra 1 and are moving toward requiring Geometry and Algebra 2 for high school graduation. Our research question is: 1) To what extent is algebra 1 achievement in middle grades a predictor of young women's PSAT-Math and SAT-Math achievement in high school? The Scholastic Aptitude Test [SAT] is a college entrance examination for many US colleges and universities. The Preliminary Scholastic Aptitude Test [PSAT] is used as an achievement benchmark, a practice examination, and/or for merit awards. Both tests use a scale of 200-800 points. We report on data from a nine-year longitudinal study of 180 young women who were high achievers in middle grades mathematics. They attended a summer mathematics camp (Girls on Track) in seventh and eighth grade upon the recommendations of their teachers and their selection to take Algebra 1 in 7th or 8th grade. Data were collected from school records on a number of measures including the state end-of-course [EOC] algebra 1 test taken in grades 7 or 8, and the PSAT-Math (10th grade) and the SAT Math (11th-12th grade) tests. Descriptive statistics, including frequency distributions, were obtained and a linear regression calculated.

RESULTS AND DISCUSSION

Graphs and tables will be presented in the poster describing the results of the analysis of the data. The means of Algebra 1, PSAT-M, and SAT-M scores were 73.12 (sd=8.221), 574.52 (sd=101.14), and 616.52 (sd = 96.66) respectively. Our analysis of subjects' algebra 1 achievement found a correlation of .710 with PSAT Math scores and a correlation of .716 with the SAT Math scores. It appears that these findings suggest that algebra 1 is a gatekeeper for successful mathematical achievement of high achieving young women in high school and perhaps even college. Teaching of early algebra concepts in grades K-6 in preparation for middle grades algebra 1 may turn a gatekeeper into a gateway for the career futures of high achieving females (Blanton, et. al, 2007).

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This research was sponsored, in part, by the National Science Foundation (Grant #0624584). The views expressed here do not necessarily reflect the views of the Foundation.

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THEORETICAL CONTROL- CONNECTIONS BETWEEN VERBAL AND SYMBOLIC LANGUAGE

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Equivalence is a core notion in algebra, and important for students in order to develop meaning for the algebraic language. Fusing symbols and notations with situations is considered another important source of meaning making (Kieran, 2007). Seeing algebra as a language, the importance of both syntax and semantic, is stressed throughout the literature. Theories of duality between two types of understanding have developed, emphasising the importance of them being complementary. That symbolic representations are more accessible to students than story problems, is a common belief. Koedinger and Nathan (2004) contradict this belief, and claim that this is a consequence of students' difficulties with comprehending the formal symbolic representation of quantitative relations, rather than a consequence of situated world knowledge facilitating problem solving performance.

By a fine-grained analysis of cognitive processes behind Norwegian 8^{th} and 10^{th} grade students' responses on transformational and generational activities, the study aims at identifying factors involved in a successful transition from situational/verbal representations to a symbolic mode of representations, emphasising equivalence relations. The study was using a mixed methods design, containing written solutions from 800 students, supplemented by one-to-one interviews. The questionnaire (given in two parallel versions) included matching story problems/ word equations/ equations, judging equivalence of pairs of equations, and recognising equivalent text problems and symbolic expressions. The coding guide consists of 2-or 3-digit codes emphasising dimension of correctness, methods/strategies used and errors made.

The poster will exemplify solution methods, effectiveness of the methods, error types, reasoning groups for judging equivalence for selected problems within the two contexts, and correlations between them. Interview results will be used to further understand the main findings from the quantitative part of the study.

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IMPORTANT FACTORS IN DESIGNING PROBABILISTIC TASKS FOR PRESCHOOLERS.

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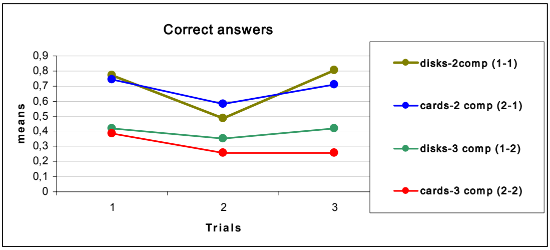
The development of probabilistic thinking has gained much interest during the last decades. Recent research has underlined that children even at the very early age of 5 understand and manipulate basic notions of probabilities (Schlottmann, 2001; Way, 2003; Nikiforidou&Pange, 2007).

While investigating children's probabilistic thinking, an important aspect that should be taken into account refers to the nature and design of the probability tasks (Pratt, 2000). In the current study we investigate whether differences in the manipulative and/or random generator (disks vs cards) as well as variations in the distribution of the sample space (2 vs 3 components) alter children's estimations of the likelihood of events.

In a total of 372 trials, the use of disks vs cards did not affect preschoolers' (N=31) predictions about which outcome was each time the most likely to come out. Thus, the number of components used (2 vs 3) had a significant impact on children's responses.



Further research has to take into account such methodological issues.

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BILLY'S COMPETENCE FOR UNDERSTANDING AND SOLVING MULTISTEP ARITHMETIC WORD PROBLEMS

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Unlike many struggling readers, Billy is proficient in numeracy. His scores on national tests are in the 83rd percentile and equally high in all content areas. Billy is one of the grade 8 students who took part in a research project where students' competence for working on multistep arithmetic word problems is investigated partly through a protocol analysis of the students' mediated work on word problems and partly through a correlation analysis of national tests in reading and numeracy.

Analysis demonstrates a strong relationship between reading comprehension and solving word problems. The correlation between reading comprehension and solving multistep arithmetic word problems is 0.645 ($p<0.001$) on national tests. Billy is one of the struggling readers, with reading scores in the 35th percentile. Although he is reasonable secure retrieving information from texts, he struggles to interpret and reflect on text content. Billy demonstrates mastery of multistep word problems on the numeracy test as well as in the verbal protocol session. Faced with the word problems in the protocol session, however, he demonstrates that although successful, he does not feel confident about whether his understanding of the problem texts is correct.

To solve word problems, students need to understand the text and to master the related mathematical content (Reed, 1999). To understand a word problem, a student needs to form a situation model containing both the social situation and the mathematical relationships embedded in the text (Thevenot, Devidal, Barrouillet & Fayol, 2007). Correlation analysis of national tests reveals that retrieving information correlates higher both to numeracy (0.703, $p<0.001$) and solving multistep arithmetic word problems (0.634, $p<0.001$) than interpreting and reflecting on text content. One possible explanation might be that although only some problems require students to interpret and reflect on text content, all problems ask students to locate information. Billy is more proficient in retrieving information, and this might explain how he can compensate for his poor text comprehension. The poster will display his competence for understanding and working on multistep arithmetic word problems within the framework of the study.

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MATHEMATICS (TEACHER) EDUCATION FOR TEACHER MOBILITY

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[2)] Paris 12 - IUFM Créteil, France

The poster focuses on teacher mobility from the perspective of mathematics teacher education. It presents results obtained in MA²ThE TE AMO, the European project researching knowledge and practices essential for successful mathematics teaching abroad. The situation is framed by the complexity of learning and teaching in multicultural and multilingual settings – the mother tongue of the teacher is not the same as the mother tongue of the students of the target country. In the preparatory stage it was necessary to devise tools that would give the teachers the feeling of confidence for successful teaching through a foreign language. In order to achieve the project aims, new, experimental university courses were developed to provide the participants with insight into new trends in mathematics education. More stress was put on methodology, awareness of different classroom practices and school culture. Project activities were piloted in five European countries: Austria, Czech Republic, Denmark, France, and Italy. For this presentation, interdisciplinary approach was adopted, combining ideas and findings related to the domain of mathematics teacher education with theories of second language acquisition (Hofmannová, Novotná, Moschkovich, 2004). The poster brings examples of Czech and French activities and useful practices from pre- and in-service teacher training, toolbox content, as well as the most interesting findings stated in the post- teacher mobility analyses.

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The contribution presents interim results gained from the Socrates Comenius 2.1 project: MA²ThE TE AMO – MAking MAThEmatics TEAchers Mobile 129543-CP-1-2006-1-IT-COMENIUS-C21. The members of the team are: F. Favilli, R. Peroni (University of Pisa, Italy), A. Ulovec, Ch. Brunner, A. Brychta, (University of Vienna, Austria), J. Novotná, M. Hofmannová (Charles Unviersity in Prague, Czech Republic), A. Jäpelt, B. Lotzfeldt (University College Lillebaelt, Skaarup College of Education, Denmark), Y. Alvez, J.-F. Chesné, M.-H. Le Yaouanq, B. Martucci (Paris 12 - IUFM Créteil, France).

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HOW TO PROMOTE MATHEMATICAL COMMUNICATION IN ELEMENTARY CLASSROOM THROUGH TEXTBOOKS

JeongSuk Pang*, JiYoung Choi*, & Wan Kang**

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"Communication" in mathematics and mathematical education is inseparable from mathematical thinking. It helps the teacher share mathematical ideas with her students, which can foster their mathematical understanding (Sfard, 2002). NCTM (2000) suggests that all students from prekindergarten through grade 12 should acquire the ability of mathematical communication, and proposes the appropriate expectations for students. However, instructional materials for attaining such ideal expectations have been insufficiently developed so that the teacher may experience difficulty in enhancing students' mathematical communication at the classroom level.

Korea's national curriculum pronounced in 2007 stresses the importance of students' mathematical communication coupled with mathematical thinking. In aligning with the curriculum, elementary mathematics textbooks for grades 1 and 2 have been developed and are applied to every elementary school this year. In order to promote mathematical communication, the new textbooks have the following four specific activities in common across all units: (a) exploration activity – investigating comprehensively the main contents of each unit, (b) problem solving – drawing out a solution for a task in an episode of mathematical history or interesting reading materials beyond typical problems in the textbook, (c) story corner – resolving daily life troubles that a variety of characters in a story run into, and (d) play station – joining games or puzzles embodying mathematical contents of each unit in cooperation with peers.

This poster illustrates in detail the four activities described above with an example of a unit called as "making a mathematical expression" of 2nd grade textbook. The poster also displays how each activity is organized and connected in the unit, and how it is expected to be used in classroom settings, as described in teachers' manual. As such, this poster provides education material developers and teachers with a specific and practical idea which helps students build mathematical communicative competence.

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MEASUREMENT STRATEGIES AND CONCEPT DEVELOPMENT OF LENGTH FOR YOUNG CHILDREN

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An experience of measuring through comparison should be prerequisite for understanding length measurement, and the conceptual understanding of the units of measurement is essential (Barrett, Jones, Thornton, & Dickson, 2003). However, even students who are proficient at calculating length lack both an appropriate sense of measure and logical reasoning of the process of measuring. A questionnaire was developed to look closely at students' reasoning ability as to measuring length. It was composed of the 4 types of questions by Battista (2006): unit-length comparison, units and unit counting, unit-length expectation, and length comparison. We tested a total of 375 students (185 second graders and 190 third graders) of 6 elementary schools.

The results of this study revealed that 'units and unit counting' is the type of questions with the highest percentage of correct answers by students. This result seems to stem from the fact that students have solved a lot of similar type of questions in mathematics textbooks. The type of questions with the lowest percentage of correct answers was 'length comparison', which was because students conjectured that 'a line is long as long as it is winding' and 'a length of an object is short if its area is small'. With regard to the type of question 'unit-length expectation', students hurried to add or multiply the numbers given in the problem in place of reasoning the relationship between lengths. In the type of question 'unit-length comparison', many students did not count the interval between marks but used 'hash mark counting'.

A noticeable result was that the reasoning level that students prefer was strongly tied to the types of questions. For instance, in the type of question 'unit-length comparison', students compared two lengths by stretching a winding line or using one-to-one correspondence between them. In the 'units and unit counting', however, students counted a unit length iteratively and exactly. This poster illustrates students' typical responses according to the four types of problems along with the explanation of their solution process. The poster also has visual display on the relationship between the reasoning level students used and the types of problems. As such, this poster emphasizes that the teacher should attend to students' difficulties measuring various objects and to consider various reasoning levels related to the types of questions.

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THINKING ABOUT DIMENSION

Nicole Panorkou

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Following on from a phenomenographic study of how dimension is experienced, the study reported here explored how individuals think about dimension. The poster will include three main parts: (a) the theoretical background together with the phenomenographic study held and the characterisation of dimensional thinking, (b) the methodology of the succeeding study leading to 'Flatland the movie' as the window on students' thinking and (c) a report on how dimensional thinking changed during the activity with Flatland.

A MAP OF DIMENSIONAL THINKING

A phenomenographic study using semi-structured interviews identified the experiences of dimension of four students and six teachers and a map of dimensional thinking was created with the characteristics of *location, measuring, abstracting dimension, representing dimension, visualising dimension and relationships across dimensions*. However, the qualitative data showed that both students and teachers' thinking about dimension is dominated by inconsistencies and naïve thoughts. The succeeding study aimed first to identify these naïve ideas held but then to perturb such thinking in order to explore limitations and potentials.

FLATLAND: A WINDOW ON STUDENTS' THINKING

The methodology is based on Noss and Hoyles (1996) notion of 'windows on thinking-in-change'. 'Flatland the movie' (2007) was the chosen methodological tool in this study. The film is based on 'Flatland', a book written by A. Abbott in 1984. The main character of the film is a square living in Flatland, a two-dimensional world. The sample consisted of two pairs of 10-year old students from London.

DIMENSIONAL THINKING-IN-CHANGE

In order to interpret accounts of the students' experiences and thinking-in-change the analysis of the data included examination of the individuals' intuitions, visualisation ability and prototypes. The findings, which built up to the definition of dimensional thinking, showed how dimensional thinking can be changed during an activity.

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DEVELOPING MATHEMATICAL ACTIVITIES FOR THE EARLY YEARS: TEACHER DEVELOPMENT THROUGH CLASSROOM RESEARCH.

Chrystalla Papademetri-Kachrimani, Maria Siakalli

European University Cyprus

This poster presents the case study of a preschool teacher involved in a 5 month in-service training programme entitled ‘Developing mathematical activities for the early years’. The aims of the programme were to train the participants to deal with their practice as teacher-researchers and to support the participants in designing scientifically justified mathematical activities.

The training programme approached the issue of professional development within an action research framework (Merthler, 2009; Mills, 2007; Stringer, 2007) based on the conviction that teacher development requires (a) collecting and analysing information regarding ones practice (b) identifying and articulating problems and (c) designing, implementing and reflecting on applications. The case study shares the small steps of a teacher’s attempt towards improving her practice in relation to designing mathematical activities. It describes how the teacher articulated a specific pedagogical problem based on information she gathered about her practice, how she designed and implemented a structured activity plan to address the problem and how this experience helped her to build on change. The story is described from the point of view of the teacher-researcher as well as the researcher-fascilitator which was the person in-charge of the programme and facilitated the teacher-researcher’s development.

The procedure involved a number of data collection methods (reflective journals, audiovisual material from real classroom environment, videotaped sessions, children’s representations etc). Overall, the analysis of this data allowed (a) reflecting on the involvement of the teacher in the program and thus on the program itself and (b) developing scientifically justified mathematical activities through a collaborative process between a teacher-researcher and a researcher-facilitator.

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THE DISCREPANCY BETWEEN THE TEACHER'S PERSPECTIVE AND STUDENTS' PERSPECTIVES ON MATHEMATICS LESSONS

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Hongik University

The didactic relationship is a ternary relation between the teacher, the students, and the pieces of knowledge at stake (Sensevy, et al, 2005). Instructional triangle in the lesson focuses on the interaction among teachers and students around content. However, in most of lesson analysis studies, the teacher's perspective tends to be emphasized, and that of students are not fully reflected. This study attempted to listen up the voice of both the students and the teacher, document their perspectives, compare students' perspectives with the teacher's perspective, and find the discrepancies between the two perspectives.

Data collection was done in an 8th grade mathematics lessons at the junior high school located in Seoul, Korea. For the study, ten consecutive lessons on the topic "systems of linear equations in two unknowns" were videotaped, and the teacher and the students were interviewed following the LPS data collection method (Clarke, et al, 2006).

The discrepancies between the teacher's and the students' behavior in the lesson were characterized into two aspects. First, the teacher focused on the fact that the different methods of solving the systems of linear equations result in the same solution while students wanted to acquire a practical recipe to choose a convenient method to solve the systems of linear equations. For the teacher who had complete mastery of the systems of linear equations, the more important fact worth being emphasized in the lesson was the uniqueness of the solution regardless of the methods adopted.

Second, the teacher tended to ignore the mistakes committed during the lesson, yet the students were annoyed by those mistakes and want the teacher to explicitly correct the error. The teacher who fully understood the content, the trivial computational mistakes were considered to be minor and thus he just continued the explanation without explicitly correcting the errors. But for students who were struggling to understand the newly introduced concepts and procedures, teacher's small mistakes were not negligible.

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ICT-BASED DYNAMIC ASSESSMENT IN SPECIAL EDUCATION

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The poster addresses a research project aimed at revealing the mathematical proficiency of students with learning difficulties (LD) by means of ICT-based dynamic assessment. The focus is on a topic that is generally recognized as rather difficult for LD students: subtraction up to 100 with ‘borrowing’ (e.g., 62 – 58 = ...).

The students involved in the project were 8-12 years old and attended a school for special education. Their mathematical level was about end Grade 2. In general, teachers use standardized written tests to assess students’ mathematical understanding and computational skills. These tests do not allow the use of any auxiliary resources and can therefore not provide information on whether the students could benefit from using these resources. To gather this information, we developed and used an ICT-based assessment instrument containing different optional auxiliary tools that students can use for solving subtraction problems with borrowing.

In a series of sub-studies we aim to find out which tool is most suited to reveal the mathematical proficiency of LD students. These tools are based on the two main didactical models that can support calculation up to 100: a group model (e.g., the 100-board) and a line model (e.g., the empty number line) (Van den Heuvel-Panhuizen, 2001). In each study data are collected with the ICT-based instrument and the standardized test.

Earlier, we conducted a study with the 100-board as an auxiliary tool (Peltenburg, Van den Heuvel-Panhuizen & Doig, 2009). Now, we present a study in which an empty numberline is used as an auxiliary tool, which operates by touch screen technology. The empty number line is a horizontal line, on which the students could put markers and add number symbols, and on which they could carry out operations by drawing jumps backward or forward. In total, 43 LD students participated in the study. The poster shows the aim and set-up of the study, a screenshot of the tool, and in a tabular form the frequencies of correct and incorrect scores of the students in both test formats: ICT test format with optional auxiliary tool and standardized test format. In addition, screen video clips of students’ operations are shown on a laptop.

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‘WHAT’S IN BETWEEN?’ AN ICT-BASED INVESTIGATION OF NUMBERLINE STRATEGIES IN SPECIAL EDUCATION

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#IQB, Humboldt University Berlin, Germany

The presented study addresses an ICT-based investigation of numberline strategies of students with learning difficulties (LD). The focus of this study is on a topic that is generally recognized as rather difficult for weak students: subtraction up to 100 with ‘borrowing’ (e.g., 62 – 58 = ...). In total, 43 students from two schools for special education participated in the study. They were 8-12 years old and their mathematical level was one to four years behind the level of their peer group in regular schools.

To get a clear picture of the students’ proficiency in solving so-called borrowing problems, it is crucial to gain insight in their solution strategies. Although a paper-and-pencil format can help to reveal this information, it provides us merely with a static image of students’ work. In contrast, ICT can, as Clements (1998) argues, offer ‘windows to the mind’ of students, since it can register detailed information on the course of students’ solution processes. For example, capturing software enables us to record audio and video data of students’ working on the computer.

To examine students’ strategies in detail, we developed and used an ICT-based assessment instrument with an optional auxiliary tool that students could use for solving borrowing problems. The tool provides students with an empty numberline, which operates by touch screen technology. The numberline consists of a horizontal line, on which the students could put markers and add number symbols, and on which they could carry out operations by drawing jumps back- and forward. While working on the computer, the students’ steps through the program were recorded on video.

Analysis of the screen videos shows that the students were quite capable of using the empty numberline. Besides, in connection to Selter (1998), it appeared that the empty number line did not force the students to care too much for conventions (p. 6). For example, using the tool did not necessarily mean that the students performed their strategy exclusively on the numberline. Some students used the tool to the degree they preferred, which could be of sufficient assistance to find the correct answer.

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MULTIPLICATION AND DIVISION OF RATIONAL NUMBERS

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According to Lamon (2007), rational numbers are one of the topics of the school curricula that are, in terms of development, "the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites." (p.629)

This poster presents preliminary results of the learning pathway of one student who is part of a larger study that aims to understand how three 6th grade students develop the sense of multiplication and division of rational numbers. An instructional unit was implemented, based on the principles of Realistic Mathematics Education (Freudenthal, 1973), the multiplicative structures (Vergnaud, 1988) and the sense of the operations (Huinker, 2002). The focus here is to identify and to analyze the strategies adopted by the student as well as his difficulties when he solved problems of multiplication and division.

The methodology adopted in the study followed the interpretive paradigm of design with multiple case study (Ponte, 2006). Data was gathered by means of audio and video records during classroom observations, analysis of students productions and in-depth interviews.

This poster also presents the objective of the study, its theoretical basis and the methodology adopted as well as some relevant aspects (strategies and difficulties) of the learning pathway of the case study under analysis.

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CHARACTERISTICS OF GRADE 8 MATHEMATICS TEACHERS IN HIGH VS LOW PERFORMANCE COUNTRIES IN TIMSS 2007

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Ronnachai Panapoi — University of Georgia

The purpose of international studies of student achievements, such as the Trends in International Mathematics and Science Study 2007 (TIMSS 2007) and Progress in International Reading Literacy Study 2006 (PIRLS 2006), should not only be for compiling a league table of countries in terms of their students' achievements but also reveal factors in explanation of student achievement (Leung, 2005.) Among which the most significant factors may be those concerning teachers' characteristics.

In the present study, the data of grade 8 mathematics teachers from 24 countries participating in TIMSS 2007 are re-analysed. The countries are selected based on their average achievement in mathematics (i.e., extremely high or extremely low) and classified into high and low performance groups for comparison purpose. Teachers' characteristics focused in this study include general background, teaching preparation, professional development, and strategies used in math class. In addition to descriptive statistics, the data of 5,512 teachers from the selected countries are analysed by means of profile analysis to investigate the similarities and differences between characteristic profiles of the two groups.

The major features on our finished poster presentation will include the characteristic profiles of grade 8 mathematics teachers from high and low performance countries, results from statistical testing of similarities and differences between the profiles, and discussion based on the findings.

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USING JAPANESE MATHEMATICAL TEXTBOOKS IN THAI LESSON STUDY

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This article is a part of the research project and development of students' mathematical thinking through Lesson Study and Open Approach. This part focuses on using Japanese mathematical textbooks in lesson study of Thai context. The objective was to present the Japanese mathematical textbook used under Thai lesson study. Basic approach of system included the lesson development and improvement in real classroom context. In Thai classroom context, the guidelines of Lesson study was applied in Thai classroom context including: a) *collaboratively* study of teaching materials (plan) b) *collaboratively* Lesson study (do) c) *collaboratively* Lesson discussion meeting (see) (Inprasitha, M. & et.all, 2007)

In this study, 1st grade was studied in the unit "How Many?" by using Japanese mathematical textbooks by GAKKOHTOSHO CO., LTD. Japanese mathematical textbooks consisted of prominent pattern were the lesson structure begins with one problem situation, students take time to represent, explain, solve, and discuss the problem and their approachs to sloving the problem, problem in Japanese textbooks are often 'worked out' and relevant representations support the entire problem-solving process. (Murata, A., 2007)

The result of the study indicated that Japanese mathematical textbook would be most used as a tool in phase *collaboratively* study of teaching materials (plan) in order to plan instructional management in classroom in Thai lesson study. The teachers used those textbooks as a major tool so that the teachers and researchers could use them in discussing together for designing open- ended situations/activities of mathematics. Besides, they also used for designing materials and supply that could be able to allow the students construct mathematical concept hidden in the activities, pattern of the introduction activity, and teachers' questions to be used in classroom in order to achieve of activities as well as conjecture the answers expected to occur in classroom for adjusting to be appropriate to Thai classroom.

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PEDAGOGICAL CONTENT KNOWLEDGE OF INTERNSHIP MATHEMATICS STUDENT TEACHERS: A CASE OF KHON KAEN UNIVERSITY, THAILAND

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Pedagogical Content Knowledge (PCK) is one of the most important knowledge for teachers. According to Shulman (1987), the special amalgam of content and pedagogy is uniquely the province of teacher, their own special form of professional understanding. Response to this idea, Park (2005) categorized the courses in mathematics teacher education curriculum of countries in Eastern Asia such as Korea, China, Taiwan, Hong Kong, and Singapore and found that most curriculums composed of Content Knowledge (CK), Pedagogical Knowledge (PK), and PCK. However, in Thailand most of the mathematics teacher education curriculum still focuses only on CK and PK separately. Even some of them seem to have PCK but they are not followed international mathematics education trend which focus on both the mathematical content and processes (NCTM, 2000). Based on this idea, Khon Kaen University has developed new teacher education program (Inprasitha, 2008) offering 8 courses (24 credits) of PCK related courses as the required courses which focused on mathematical processes.

In 2008, the firstling of those students had their internship in schools. This study aims to investigate output of this curriculum, that is, to investigate how internship students manage their classrooms when they teach in real situation from the perspective of Mathematics PCK. The study selected 3 internship students who enrolled teaching profession experience in school under the project of students' mathematical thinking development through Lesson Study and Open Approach. The data was collected by the researcher in the 2008 academic year.

The findings from the researcher's observation, the classroom management of mathematics teacher education internship student based on MPCK on three phase of Lesson Study in order to develop their students' Mathematical Thinking.

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TRACKING CHANGES IN THE TEACHING PRACTICE: INTRODUCING A PROFESSIONAL DEVELOPMENT MODEL

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The teaching process can be seen from a wide range of perspectives. Here we propose a cognitive approach, the teacher's cognitions – beliefs, goals, knowledge –, type of communication with students and actions being its dimensions. This approach is aimed not only at improving the understanding of these dimensions, but dealing with the relations among them and the way these relations influence the teaching practice (Ribeiro et al., 2009).

This research is based on a case study (with an interpretative methodology) and focuses on the professional development of two primary school teachers under a cognitive scope. Audio and video recordings were collected in three different phases, along a school year. After the first phase (state of the art) a collaborative work has been started (between the teachers and the first author), focusing on the teacher's practice and the critical features they could identify in their (both teachers) practice, complemented by what the researcher identified. All the classes were transcribed, divided in episodes (Schoenfeld, 1998) and the components of the model were assigned in each episode.

Concerning Maria (one of the teachers), from the first stage of the analysis some relations emerged between the model dimensions (e.g. when presenting certain content in an unidirectional way, the teacher calls for her self all the responsibility, which is related with her belief concerning the learning process – pupils learn in a passive way, just by earring the teacher's explanations).

Our next step is to track changes in the teacher practice in the three phases. For that we are elaborating a Professional Development Model (PDM), based on Carrillo et al, (2008). This poster will present the emergent relations between the cognitive dimensions in two situations and explain how to use the PDM in these situations to track changes in the teacher's practice.

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AN ASSESSMENT OF CREATIVE ABILITY IN MATHEMATICS

– GRADE 6 STUDENTS IN THAILAND –

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The purpose of this study is to assess students' creative ability in mathematics for grade 6 students in Thailand. Creative ability in mathematics will be assessed through the answers of creativity test emphasizing in Divergence, Fluency, Flexibility and Originality.

Usually, creativity in school education is understood as follows: "the creative ability is the ability to making things which have a new worth to the students and which are evaluated by the members of group, and is personal character" (SAITO, N. 1990a)

The creativity test was conducted across grade 6 primary school in Thailand. The total number of students is 220. The time set to 15 minutes for each problem.

Problem 1 There are two sets of two triangles (total 4 sheets). The size of each set of the triangle is same. Draw as many quadrangles as possible using some triangles.

Problem 2 Draw as many figures as possible which area is 4. Write down the length in the necessary part.

An assessment of creative ability in each problem, total score is set to 100 points.
1. Divergence – all answers which students can write. (16 points)
2. Fluency – correct answers. (14 points)
3. Flexibility – the numbers of pattern of answer. (40 points)
4. Originality – the numbers of new or original pattern of answer. (30 points)

Problem	Divergence	Fluency	Flexibility	Originality	Total
1	10.37(5.16)	2.02 (1.82)	6.44 (4.73)	0.68 (2.28)	19.51 (9.03)
2	9.14 (5.55)	1.85 (2.72)	4.73 (5.60)	0.64 (2.45)	16.34 (10.34)

Table 1: An average points of answer in creativity test and standard deviation

From the table we found that grade 6 students in Thailand have quite good creative ability in divergence, but they have to improve in fluency, flexibility and originality.

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TEACHING FOR THE DEVELOPMENT OF PROPORTIONAL REASONING

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Proportional reasoning is fundamental to solve many daily life and learning advanced mathematical topics. And, on the other hand, indicates that students' ability to reason proportionally is limited. Nonetheless, this paper describes a project to ascertain if students who are encouraged to construct their own knowledge on proportion through collaborative problem solving activities, perform better on problems involving this notion.

THE FOCUS OF THE STUDY

The poster describes an ongoing PhD research project and concerns the way grade 6 students learn about direct proportion, when they are involved in a teaching experiment based on investigative activities and problem solving - distinction of situations that involve proportion from those who not (Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998); understanding of the multiplicative nature of proportions (Post, Behr & Lesh, 1988). The central aim of the study is to understand the students' strategies and ways of data representation in common proportion problems as they are encouraged to construct their own conceptual and procedural knowledge.

INDICATION AND JUSTIFICATION FOR THE CONTENT

This study follows a qualitative methodology, an approach that facilitates a symbolic and physical proximity between the researcher and the participants in research focused on the construction of meaning (Lessard-Hébert, Goyette & Boutin, 1990). The participants in this research are students from three grade 6 classes, whose teachers develop a teaching experiment.

A graphical display will explain the empirical activities involved in this study and relate them to the theoretical background, the research design, and the methodology.

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ANALYSING ALGEBRAIC CONTENTS IN SWEDISH MATHEMATICS TEXTBOOKS FROM A HISTORICAL POINT OF VIEW

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The purpose of this study is to analyse the algebra contents presented in eight Swedish mathematics textbooks at upper-secondary level. The textural analysis focuses on three different methods presented for solving quadratic equations; factorization, completing squares, quadratic formula. In mathematics education, solving quadratic equations has been emphasized as learning algebraic manipulation procedures. On the one hand, these methods are taught as mathematics tools and instruments since students need to practice their skills for their further studies, on the other hand, being able to use these mathematics instruments does not necessarily mean that students have understood why they should use them and how they relate to each other (Skemp, 1976). Can we find ways to enhance students' conceptual understanding of algebra concepts in spite of the strong focus on procedural practicing (Hiebert & Handa, 2004)? The study concerns a review of the history of algebra in analysing the content structure. There are certain intertwined relations between algebra contents presented in textbooks and the long historical development of algebra. In which order are these algebra contents presented when comparing with its historical development? How do these books deal with the dilemma of instrumental and conceptual intentions? How do the books theoretically integrate with the fields of geometry, numbers and algebra structure? How much abstract respectively real-world algebra are presented in the books? The historical and theoretical investigations lay the ground for the analysis which consists of two parts; mathematics theoretical content and related tasks. The criteria for the analysis are based on analysing framework by Van Dormolen (1986) and Brändström (2005). Eight Swedish textbooks are being investigated.

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FACTORIZATION: A PEDAGOGICAL OR MATHEMATICAL ISSUE?

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Solving quadratic equations or expressions is one of the important algebraic components in the Mathematics B course according to the Swedish mathematics syllabus. In eight analyzed Swedish mathematics textbooks, four different solving approaches are presented: the square-root method, factorization for simple quadratics, completing the square and using the quadratic formula. This study aims at investigating which role factorization plays among these different solving methods in the analysis of the mathematics theoretical aspects involved the eight Swedish mathematics textbooks used at upper-secondary schools. Internationally, mathematics educators have been discussing the uses of different methods for solving quadratic equations for decades. Among their discussions, utilizing factorization is a debatable topic (Bossé & Nandakumar, 2005). How these four different solving methods— especially factorization— are presented in these investigated textbooks is vital for mathematics teaching in Sweden since mathematics textbooks are one of the major sources for both teachers and students.

The mathematics textbooks' analysis bases its analyzing criteria on two mathematics educators' work (Van Dormolen, 1986; Brändsström, 2005) in when relating to analyzing theoretical texts and mathematics exercises. The focus of this analyzing study is on the mathematics content and pedagogical intention. The result of this analysis shows that the mathematics contents in these books are consistent and four solving approaches are presented. The factorization presented in all the books is applied only for solving simple quadratic equations. Factorization for solving standard quadratic equations is absent. The further analysis of related mathematics exercises is still in process. This study will lay the ground for the further research on pedagogical content knowledge through interviewing with mathematics teachers.

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HOW TO PREPARE STUDENTS FOR A SUCCESSFUL FIRST YEAR AT UNIVERSITY: AN EXPERIENCE IN VISUALIZATION

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Visualization has been an area of interest for a number of researchers concerned with mathematics education, specifying that visual thinking can be an alternative and powerful resource for students doing mathematics (Presmeg, 2006).

Two research questions directed our study: To what extent can visual considerations in calculus be taught to undergraduate students? To what extent can visual considerations become a natural part of the undergraduate's mathematical thinking? In order to answer these questions, we describe the effects of an innovative experience (Laboratory discipline) developed with mathematics undergraduate Spanish students at the Complutense University. The aim of this experience is to help students during their first year in the mathematical learning on the transition's difficulties (secondary-tertiary transition phenomena). Laboratory discipline fosters the mathematical experience in advanced thinking and the development of visual thinking (the aspects of mathematical thought which are based on, or can be expressed in terms of visual images). The theoretical approach of this innovative experience is supported by the ideas of Guzmán about visualization (Guzmán, 2002). We follow the concept of visualization given by (Zimmermann & Cunningham, 1991 and Guzmán, 2002) as the process of producing or using geometrical or graphical representations of mathematical concepts, principles, or problems, whether hand drawn or computer-generated. The investigation has been qualitative in character. In this report we present partial results (related to test data). Four months after the training period in visualization methods, (which lasted for five weeks), a test was given composed of different problems. Each problem rests firmly on a concept which has a visual interpretation and focus on specific characteristics of the learning difficulties. Different categories were taken into account to analyse the data: preference for visual methods, skills, capabilities, creative possibilities for the visualization, applicability of images in solving problems and calculus. We analysed learning difficulties in visualization from both sociological and cognitive points of view.

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UNLIKE TERMS: INTERNATIONAL DEVELOPMENT AS A CONTEXT FOR ALGEBRA

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The United Nations' Millennium Development Goals (MDGs) seek to make lasting improvements in the health and economic well-being of the world's poor during the years 1990 to 2015. The MDGs include quantitative targets—e.g. halving extreme poverty and reducing maternal mortality by three-quarters—that provide an interdisciplinary context for teaching topics in introductory and intermediate algebra. The intersections of public health, economics, anthropology and math create a complex terrain for students to investigate global issues of poverty, epidemics, and the relationships between health and economics using linear and quadratic functions, rational exponentiation, and logarithmic modelling.

Teaching algebra through the MDGs also serves as a case study on interdisciplinarity in mathematics curriculum at the first-year university level. Arguably, U.S. students need to develop a greater competency and sense of agency in issues of international inequality. This poster will suggest that developing an interdisciplinary sense of agency that at once uses mathematics and is critical of it (Skovsmose & Valero, 2002) requires students to pass through a reflective process that cannot be fully assessed within the discipline of mathematics. A critical interdisciplinary math lesson must originate and return to the "personal meaningfulness principle" of model development (English, 2009; Lesh et al, 2003).

The poster format will allow the presentation of graphics published by the United Nations Development Programme (UNDP) that provide the data for the applications (e.g. UNDP, 2005, images used with permission). Samples of student work and responses to the applications will also be presented.

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ENHANCEMENT THEORY CONSTRUCTION: AN EXPERIMENT OF "SPIRAL VARIATION CURRICULUM DESIGN"ABOUT DIVISION OF FRACTION

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This poster will display results from a Ph.D. thesis (Sun, 2007), including a new curriculum framework and its rationales, the corresponding materials (the textbook, guide for teachers and worksheets for students) based on the framework and its experiment results. In order to provide the threshold, within which Chinese mathematics education advances theoretically, this study made a new attempt in the theory and its practice in curriculum design in mathematics. The framework, called "spiral variation curriculum design", conceptualized and rationalized Chinese mathematics practice experience of the variation problems with concepts \ methods connection to elicit the whole mathematical structure and its methodological system, known widely in China as one problem multiple solutions (一題多解，varying solutions), multiple problems one solution (多題一解，varying presentations) and one problem multiple changes (一題多變，varying conditions and conclusions). This practice of the variation problems is growingly regarded as a salient curriculum feature, which could be traced to any one teaching material (formal ones, such as textbooks or informal ones such as teaching plans) at school and any single learning material (such as students` exercise or worksheets) done after school in China. In contrast, this kind of Chinese strategy appears rarely in the West. The rationales of practice could be tracked to the source of Chinese deep-rooted cultural philosophy, I jing, in which the thoughts on abstracting invariant concepts from the varied situation and applying the invariant concepts to the varied situation were highlighted. The new curriculum enhances the examples and exercises with focus on invariant concepts/solutions connection by the variation problems mentioned above, in order to spirally unfold mathematics structures /solutions space, and is uplifted into a curriculum design framework, which will be shown in figures. Division of fraction was taken as the topic for experimentation. The corresponding textbook, guide for teachers and worksheets for students based on the framework above was developed. The quasi-experiment was conducted in 27 classes of six primary schools in Hong Kong as their curriculum materials are heavily influenced by UK. Significant positive results on concept understanding were obtained by the spiral variation curriculum. Some examples & achievements will be shown in figures and charts.

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METACOGNITIVE STRATEGY IN PROBLEM SOLVING BY 1ST GRADE STUDENT IN CLASSROOM THROUGH LESSON STUDY AND OPEN APPROACH

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The poster is to present metacognitive strategy in problem solving by 1st grade students. This study is a part of three-year professional development project implementing lesson study and open approach conducted by the Center for Research in Mathematics Education, Khon Kaen University, this project implemented regularly since April 2006 by applying innovation which could be used as major means for developing mathematical thinking in integrating lesson study and open approach. Unlike Japanese lesson study, this project modified Japanese lesson study by incorporating open approach and emphasizing “collaboration” in every phase of lesson study cycle. Thus, teachers, researchers and outside experts participated in *collaboratively* designing research lesson, *collaboratively* observing their friend teaching the research lesson, and *collaboratively* doing post-discussion or reflection on teaching practice (Inprasitha & Loipha, 2007). Every week, 78 teachers from 4 schools participated in each phase of lesson study cycle through 3 academic years. The major issue was to create Open-Ended Problem Situation that the students could participate in and express their competency in mathematical thinking with full potentiality. This included the example of students’ collaborative work where the researcher collaborated in observing the classroom for one year, on methodology using ethnographic study.

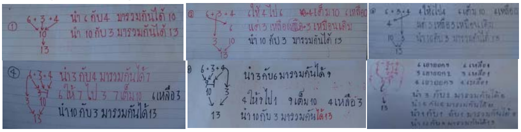

Fig. 1: Shown problem solving “6 + 3 + 4” from open-ended problem situation of 1st grade (one group of 4 students)

Finding, from fig. 1 shown students’ success of problem solving “6 + 3 + 4”, that each ways their goal was how to get 10, they use it as a tool and carry out the plan. In their six different ways they use “Tree Diagram” in order to describe of their own ways. They are attempt to work enthusiastically and they are making careful move on complex tasks. They knew and were aware how to do on task. These actions occurred from practice in their classroom.

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MATHEMATICAL ACTIVITY AS AN ACTIVITY IN THE PROCESS OF ABSTRACTION

Nisara Suthisung

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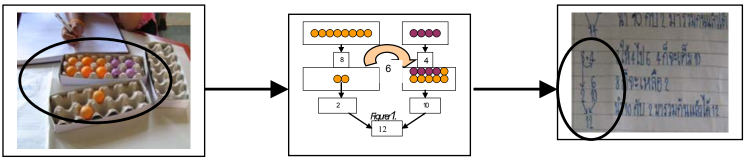
This presentation was to show the empirical evidence happened in Opened-ended problems situation, while the student doing mathematical activity such as reading, speaking, writing, picking up and manipulating of 1st grade, Kukam-Pittayasan School, KhonKaen Province, participating in the project under Center for Research in Mathematics Education, Faculty of Education, Khon Kaen University, for nearly 3 years, situated in the context of "Lesson Study" including 3 stages, and used Open Approach. It was an innovation of learning management. In this study, it focused on open-ended problems causing the problem move in mathematical activity, are considered to be an activity in the process of abstraction from the concrete experience of real life to the world of mathematics. In addition, it encompasses an activity of manipulation and manifestation in the world of mathematics Nohda (1998). On the other hand, Gray & Tall (2007) defined 'abstraction' is dually a process of 'drawing from' a situation and also the concept (the abstraction) output by that process. For order to collected data, to used ethnographic study, as well as participative observation, field note, and participation for all 3 stages of Lesson Study based on data interpretation.

It was found that in the 'How many lemons? problem' mathematical activity can be considered as the concrete manipulation of small number of ping-pong and egg trays from which the inherent procedures of counting and processes of add to up and take away to the world of mathematics, to used flexibility symbolic and into concept of addition and subtraction.



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EMOTIONAL EXPERIENCES: CHALLENGING ASPECT

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In traditional mathematics classroom, some students who labelled as "Slower Learners" are **abandoned** by their teachers. The reason is that most of the teachers seem that mathematical knowledge has a central role in problem solving while emotional experience is not considered as important. Emotional experience has influence students' mathematical problem solving (McDonald, 1989). Causality that teachers neglect to study emotional experience is because of the study of emotions and affective structures in individuals is extremely difficult to "scale up" (Goldin, 2008), and the teachers confronted difficultly task that how to know emotions works.

Interesting research about descriptive mechanism of emotional experience that generated in mathematical problem solving conducted by Inprasitha (2001). In his study, emotional experience and mathematical problem solving are intimately linked. Mathematical problem solving as an interruption situation that students bring mathematical knowledge (or concepts, procedure, etc) to make sense the interruption and lead to the generation of emotional experience.

In this study, Inprasitha's (2001, P.78) model is used for analysis of the mechanism of emotional experience of two first grade students when they solved two open–ended problem-solving situations together with friends. The results indicated that ***active emotional experiences*** of first grade students were different from passive emotional experiences that occurs in traditional classroom.

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SCHEMA CHANGE IN SMALL-GROUP MATHEMATICAL COMMUNICATION THROUGH OPEN APPROACH

Sampan Thinwiangthong — Khon Kaen University
Suladda Loipha — Khon Kaen University

In Traditional Classroom, Thai students individually do their seat works. The traditional teaching was sequences as: reviewing the previous lesson, explaining new contents by teacher, exampling by teacher, and exercising by students individually. Students have not opportunity to discuss and share mathematical ideas naturally.

This study is a part of the 3-years Research Project on "Model for Fostering Students' Mathematical Thinking by Implementing Lesson Study and Open Approach" by Center for Research in Mathematics Education (CRME) of Khon Kaen University which introduces Open Approach as a teaching approach in urban school of Khon Kaen. The sequences of teaching approach as: posing the open-ended problem, small-group working, whole class discussion, and summarizing the lesson. The small-group working is the session which students communicated mathematical ideas naturally. Open Approach provides students with opportunity to engage in small-group mathematical communication (SMC) in which defined as communication in small-group working including characteristics of mathematical communication according to Emori (2005). SMC is treated as social interaction, according to Steffe & Thomson (2000) noted that learning occurs in particular interactions in which students modify their current schema. According to Rumelhart (1984 cited in Byrnes, 2001), schema was modified by one of three processes: accretion, tuning or restructuring.

This study was conducted during the 2007-2008 academic years by the purpose of analyzing schema change in SMC. To accomplish the purpose, targeted group were selected from the 7th grade students by using ethnographic method. Collaborative lesson planning, classroom observation, clinical interview and reflection on the lessons were done. Protocols of SMC were analyzed.

Conclusions point out that students have richly opportunity in learning mathematics by changing their current schema in SMC through Open Approach.

Acknowledgement This study was funded by The Commission on Higher Education, Thailand, within the University Development Committee Program (UDC).

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MULTILINGUAL MATHEMATICS TEACHERS: USING POETRY TO ACCESS HIDDEN IDENTITIES

Lyn Webb and Lesley Foster

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South Africa is grappling with low mathematics results in primary, secondary and tertiary institutions. One of the reasons put forward to explain this poor performance in mathematics is that English is the chosen language of learning and teaching in most schools, because it is seen as a language of power and access to upward mobility (Setati, 2005; Webb, 2002). However for the majority of learners and teachers it is either a foreign language or a second language, thus the relations of power which existed in the apartheid era are perpetuated through the silencing and marginalization of the voices of all those who do not attain a high level of proficiency in English (Webb, 2002). It, therefore, becomes very important to create spaces in which teachers, who are not English first language speakers, feel free to share their 'language stories' and reflect on the multilingual realities of their mathematics classrooms - and the teaching and learning implications for mathematics.

To open up such a space teachers studying for an in-service, upgrading qualification in mathematics and science were given the opportunity to read, discuss and write reflectively on their personal language experiences and in this way give form to their feelings and frustrations (Benton & Fox, 1985). Their words bear sensitive witness to what teachers and their learners are facing in schools worldwide today. The excitement and pride, as well as pain and difficulties, captured in the poems challenge all educators to be aware of language issues in the course of their own teaching. Teachers who believe that mathematics and poetry are two extremes of a continuum are enabled to make the link between logical thinking and feeling (Benton & Fox, 1985). This experience opens a door for the teachers to express emotions in a mathematics context which they previously perceived to be 'absolutist'.

Teachers need to develop confidence in their own experiences and perceptions in order to be willing to express them in open discussions and exploratory talk within a constructivist learning and teaching environment (Benton and Fox, 1985). Reading poems written by other teachers and writing their own poems enables students to feel that their own thoughts and feelings are valued. Poetry serves as a mirror creating self knowledge as well as a window through which the teachers can view and participate in the hidden identities of others. Armed with this knowledge teachers can engage with the learning and teaching of mathematics from a new perspective, moving from alienation to engagement.

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ENHANCING STUDENTS' GEOMETRIC CONJECTURES BY SYSTEMATIC SEARCHING

Chao-Jung Wu Ju-Chen Chen

National Taiwan Normal University

Conjecturing is the first step to "discover" mathematics, and it creates connections between mathematic concepts and properties. But to make high quality conjectures is difficult. Lin and Wu (2007) found children's geometric conjecturing is related to the activation of relational-schema. In light of these ideas, this study exams the effect to enhance conjecture quantity and quality by encouraging students to search geometric invariance systematically.

54 six-grade participants were sampled from 4 elementary schools during the period in May 2008. All participants were randomly and equally divided into systematic searching (SS) group and control group. A pre-test was executed before the treatment to realize the geometric conjecturing ability of participants. The pre-test and treatment (see Fig. 1) are given geometric conditions with a figure as example. The students are asked to think about what other geometric invariance would surely exist. In SS group, the treatment is to encourage systematically searching in a framework about quantitative (e.g., angle, length, area, etc.) or qualitative (e.g., parallel, symmetry, congruence, etc.) attributes. As to control group, the participant is requested to observe the item in detail and report the invariant properties. The pre-test performance of 2 groups were analysed by Mann-Whitney Test and the result implied their conjecturing abilities were homogeneous. Furthermore, the treatment enhanced the conjecture quantity of SS group. The total proposition number and the correct proposition number of the SS group (total = 280/ correct = 264) are as twice as the ones of the control group (total = 140/ correct = 131), and their difference is significantly, $t\,(52) = 6.24$, $p < .001$, $t\,(52) = 6.21$ $p < .001$.We adopted some conjectures possessed mathematics potential as qualitative indexes. SS group always proposed more high quality conjectures than control group. For example, more participants of SS group (n = 17) identified $\overline{EF} = 1/2\overline{AD}$ or $\overline{ED} = 1/2\overline{AB}$ than control group (n = 2). We will discuss that the mechanism of systematic searching in conjecturing in this article.

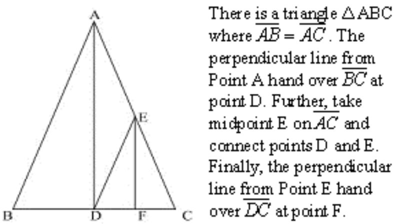

Fig.1: Given Conditions and Figure of Treatment.

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BEYOND TRADITIONAL CURRICULUM – EFFECTIVENESS OF NEW MATHEMATICS CURRICULUM ACHIEVEMENT IN CHINA

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Fu Ma — Nanjing Normal University, China
Shuhua An — California State University, Long Beach, U.S.A.

THE PURPOSES AND RESEARCH QUESTIONS

The goal of this study aimed at comparing middle school student achievement in mathematical knowledge in reformed and traditional curricula in China. This study also compared student attitudes toward mathematics between the reformed and traditional groups. The research questions asked in this study were: 1) Are traditional students' scores significantly different from reformed students on given test items? 2) What are the differences in attitudes between the reformed and traditional students?

METHODOLOGY

Subjects. The participants were 3356 middle school students at 9th grade levels in 16 schools from four provinces in China.

Data Collation and Instruments. Data were collected via an assessment on student knowledge and a questionnaire on attitudes toward mathematics.

Data Analysis. The study used a combination of quantitative and qualitative methods to analyse data. Two-Way Mixed ANOVA (GLM model) was used to compare the differences in means scores between/within groups on five content areas. Correlation Test was used to identify the relationship between attitudes and mathematics achievement.

RESULTS

Results indicated 1) there was no significant difference between reformed and traditional students' performance in the areas of Number, Algebra, and Measurement. However, students with the new curriculum achieved higher scores in statistics and probability and geometry than traditional students; 3) the reformed students have more positive attitudes toward mathematics compared to the traditional group.

SIGNICANACE OF THE RESEARCH

Over the past decades, a number of curricula have been developed to enact the vision of "reform mathematics." The question is whether these curricula have resulted in an improvement in student learning and achievement. The results of this study provided considerable evidence that the new Chinese mathematics curriculum has resulted in remarkable effects on student learning. Although it borrows constructive ideas from various reformed curricula in different cultural systems around world, it has its own strengths and characteristics that could provide insights to theory and practice.

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PRESCHOOL TEACHERS' ROLE IN PROMOTING MATH SKILLS ACCORDING TO PRESCHOOL TEACHERS' AND TEACHER EDUCATORS' PERCEPTIONS

Tugba Yalcin & Mehmet Fatih Ocal

Kafkas University Middle East Technical University

The purpose of this study is examining the perceptions of preschool teachers and teacher educators about the role of preschool teachers in promoting math skills.

There is an increased recognition of the importance of mathematics (Kilpatrick, Swafford & Findell, 2001). Many science and technology related jobs need more sophisticated mathematical skills in this global world. And according to National Council of Teachers of Mathematics (2000) *children's mathematical development is established in the earliest years*. Many experts also believe that young children possess a substantial amount of informal knowledge about mathematics (Ginsberg, 1996 cited in Smith, 1998). In this situation, teachers help her/his students reinvent mathematics for themselves (Smith, 1998). And as Vygotsky said *a reflective teacher helps the child discover and communicate ideas that wouldn't have occurred spontaneously without the adult's help* (Vygotsky, 1978).

In the poster, the data gathered through semi-structured interviews with randomly selected sample from preschool teachers (in Kars Province) and teacher educators (Kafkas University Faculty of Education) will be presented. The poster will also include some explanations to our findings.

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INVESTIGATION ON MATHEMATICS LEARNING OF THE "NEW CHILDREN OF TAIWAN"

Ru-Fen Yao

National Chia-Yi University

Current globalization and immigration have gradually changed Taiwan, where a special phenomenon appears due to the growing intermarriage. The main purpose of this research was to investigate the mathematics performance of the "new children of Taiwan", the children of "immigrant women". Taking willingness of primary schools and enrolments of the second-grade children of "immigrant women" into consideration, the researcher conducted the investigation on 14 primary schools, 28 second-grade classes, and 778 students, in which 92 children of "immigrant women" were included. The results showed that a) there were obvious individual differences among these children on math performance; b) These children had the best performances in Statistics than the other math topics, while that in Algebra was vaguer; c) The nationalities of immigrant women weren't the crucial factor to influence their children's math performance; d) in comparison with local students, these children had lower achievement in Measurement and Geometry, but had no obvious difference on Number, Statistics, and Algebra. The number of the "immigrant women" will continually go up in a short time in Taiwan. If we could take them as a part of us in this society and help those coming from foreign countries, "the mothers of Taiwan," with the adjustment, these are important issues to discuss and to learn in Taiwanese society. The researcher expects for more understanding and communication not only for research, but also for a socially affiliated society, in order to create an environment with care, acceptance, friendliness and respect for the "immigrant women" and their children, the "new children of Taiwan".

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