



# PROCEEDINGS

of the

35th Conference of the International  
Group for the Psychology  
of Mathematics Education

**Developing Mathematical Thinking**

**Editor: Behiye UBUZ**

Volume 4

Research Reports [Qrstuv-wxyZ]

Orta Doğu Teknik Üniversitesi [Middle East Technical University]  
Ankara-TURKEY



Ankara-Turkey  
July 10-15, 2011

---

Cite as:

Ubuz, B. (Ed.), (2011). Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education, Vol 4, Ankara, Turkey: PME.

Website: <http://www.pme35.metu.edu.tr>

The proceedings are also available on CD-ROM

Copyrights©2011 left to the authors

All rights reserved

ISSN 0771-100X

ISBN 978-975-429-262-6

Volume 4 ISBN 978-975-429-298-5

Cover design & Overall Printing Layout: Ulker BORA

Logo Concept: Behiye Ubuz

Logo: AG Tanıtım

Production: Can Tekin Matbaacılık

Composition: ARBER Professional Congress Services







# TABLE of CONTENTS

## VOLUME 4

### *Research Reports*

---

<b>Quintas, Sandra Maria; Tomás Ferreira, Rosa Antónia; Oliveira, Hélia Margarida .....</b>	<b>4-1</b>
<i>Didactical knowledge in statistics: A study with secondary teachers</i>	
<b>Rach, Stefanie; Heinze, Aiso .....</b>	<b>4-9</b>
<i>Studying mathematics at the university: The influence of learning strategies</i>	
<b>Radford, Luis.....</b>	<b>4-17</b>
<i>Embodiment, perception and symbols in the development of early algebraic thinking</i>	
<b>Rangnes, Toril Eskeland .....</b>	<b>4-25</b>
<i>Moving between norms in school mathematics practice and building company practice</i>	
<b>Reiss, Kristina M.; Barchfeld, Petra; Lindmeier, Anke; Sodian, Beate; Ufer, Stefan.....</b>	<b>4-33</b>
<i>Interpreting scientific evidence: Primary student's understanding of base rates, sampling procedures and contingency tables</i>	
<b>Ribeiro, C Miguel; Carrillo, José .....</b>	<b>4-41</b>
<i>Discussing a teacher MKT and its role on teacher practice when exploring data analysis</i>	
<b>Rivera, Ferdinand .....</b>	<b>4-49</b>
<i>Explaining differences in second grade students' patterning competence using parallel distributed processing</i>	
<b>Roh, Kyeong Hah; Lee, Yong Hah.....</b>	<b>4-57</b>
<i>Development of students' understanding of the logic in the epsilon-n definition of limit</i>	
<b>Rott, Benjamin .....</b>	<b>4-65</b>
<i>Problem solving processes of fifth graders: An analysis</i>	
<b>Rowland, Tim; Thwaites, Anne; Jared, Libby .....</b>	<b>4-73</b>
<i>Triggers of contingency in mathematics teaching</i>	
<b>Ruthven, Kenneth; Hofmann, Riikka; Mercer, Neil.....</b>	<b>4-81</b>
<i>A Dialogic approach to plenary problem synthesis</i>	

<b>Sack, Jacqueline; Vazquez, Irma .....</b>	<b>4-89</b>
<i>Development of a learning trajectory to conceptualize and represent volume using top-view coding</i>	
<b>Saldanha, Luis A. ....</b>	<b>4-97</b>
<i>Aspects of conceiving stochastic experiments</i>	
<b>Sanchez, Victoria; Garcia, Mercedes .....</b>	<b>4-105</b>
<i>Socio-mathematical and mathematical norms in pre-service primary teachers' discourse</i>	
<b>Santi, Giorgio; Andra, Chiara .....</b>	<b>4-113</b>
<i>A semiotic characterization of intuitions</i>	
<b>Schubring, Gert .....</b>	<b>4-121</b>
<i>Searching the roots of pme – the case of ,experimental pedagogy' in Germany</i>	
<b>Seah, Wee Tiong .....</b>	<b>4-129</b>
<i>Effective mathematics learning in two Australian primary classes: Exploring the underlying values</i>	
<b>Seffah, Rachid; González-Martín, Alejandro S.....</b>	<b>4-137</b>
<i>The concept of series in undergraduate textbooks: Tasks and representations</i>	
<b>Selden, John; Selden, Annie .....</b>	<b>4-145</b>
<i>The role of procedural knowledge in mathematical reasoning</i>	
<b>Sevimli, Eyup; Delice, Ali .....</b>	<b>4-153</b>
<i>Is what you prefer what you do? Representations in definite integral</i>	
<b>Shin, BoMi .....</b>	<b>4-161</b>
<i>An analysis of teachers' pedagogical content knowledge on probability</i>	
<b>Shvarts, Anna; Chumachenko, Dmitry .....</b>	<b>4-169</b>
<i>Representations in the development of mathematical concepts</i>	
<b>Silfverberg, Harry Untamo .....</b>	<b>4-177</b>
<i>Structural elements in collections of different types of functions recalled by student teachers</i>	
<b>Silvestre, Ana Isabel; Ponte, João Pedro.....</b>	<b>4-185</b>
<i>Missing value and comparison problems: What pupils know before the teaching of proportion</i>	
<b>Sinclair, Nathalie; Kaur, Harpreet .....</b>	<b>4-193</b>
<i>Young children's understanding of reflectional symmetry in a dynamic geometry environment</i>	
<b>Spinillo, Alina Galvao .....</b>	<b>4-201</b>
<i>Number sense in children: Understanding number as an operator when adding and subtracting</i>	

<b>Spyrou, Panagiotis; Zagorianakos, Andonis .....</b>	<b>4-209</b>
<i>Distinction between function and relation: A research study on pupils, students and teachers</i>	
<b>Stein, Mary Kay; Kaufman, Julia; Tekkumru-Kisa, Miray .....</b>	<b>4-217</b>
<i>Curriculum as a lever for teacher improvement</i>	
<b>Sullivan, Peter; Grootenboer, Peter; Jorgensen, Robyn .....</b>	<b>4-225</b>
<i>Considering student experience and knowledge of context in planning mathematics learning</i>	
<b>Swidan, Osama .....</b>	<b>4-233</b>
<i>How did the indefinite integral function become an accumulation function?</i>	
<b>Thomas, Michael O. J.; Yoon, Caroline .....</b>	<b>4-241</b>
<i>Resolving conflict between competing goals in mathematics teaching decisions</i>	
<b>Toluk Ucar, Zulbiye; Yavuz, Birseli .....</b>	<b>4-249</b>
<i>Elementary school students' intuitive understanding of the inequality signs</i>	
<b>Tomaz, Vanessa Sena; David, Maria Manuela .....</b>	<b>4-257</b>
<i>Classroom activity promoting students' learning about the use of drawings in geometry</i>	
<b>Triantafyllou, Chrissavgi .....</b>	<b>4-265</b>
<i>Mathematical literacy skills in a workplace context: The case of reading and interpreting data</i>	
<b>Triantafyllou, Chrissavgi; Vourvoulakis, John; Assimakis, Nicholas .....</b>	<b>4-273</b>
<i>Mathematical tools in engineering students' decision-making</i>	
<b>Turner, Fay Alison .....</b>	<b>4-281</b>
<i>Mathematical content knowledge revealed through the foundation dimension of the Kq</i>	
<b>Van Harpen, Xianwei Yuan; Presmeg, Norma .....</b>	<b>4-289</b>
<i>Insights into students' mathematical problem posing processes</i>	
<b>Verhoef, Nellie C.; Tall, David O. ....</b>	<b>4-297</b>
<i>Lesson study: The effect on teachers' professional development</i>	
<b>Viholainen, Antti .....</b>	<b>4-305</b>
<i>Critical features of formal and informal reasoning in the case of the concept of derivative</i>	
<b>Viirman, Olov .....</b>	<b>4-313</b>
<i>Discursive practices of two university teachers on the concept of 'linear transformation'</i>	
<b>Vollstedt, Maike .....</b>	<b>4-321</b>
<i>On the classification of personal meaning: Theory-governed typology vs. empiricism-based clusters</i>	

<b>Warren, Elizabeth Anne; Miller, Jodie; Cooper, Tom J.</b> .....	<b>4-329</b>
<i>Exploring young children's functional thinking</i>	
<b>Wille, Annika</b> .....	<b>4-337</b>
<i>Activation of inner mathematical discourses of students about fractions with the help of imaginary dialogues: A case study</i>	
<b>Williams, Gaye</b> .....	<b>4-345</b>
<i>Relationships between elements of cognitive, social, and optimistic mathematical problem solving activity</i>	
<b>Wilson, Sue</b> .....	<b>4-353</b>
<i>Pre-service teacher action research using bibliotherapy to address mathematics anxiety</i>	
<b>Zeybek, Zulfiye; Galindo, Enrique</b> .....	<b>4-361</b>
<i>Exploring mathematics teachers' motivation to change: A case study</i>	
<b>Author Index Vol.4</b> .....	<b>4-369</b>





# Research Reports

## Qrstuv-wxyZ







# **DIDACTICAL KNOWLEDGE IN STATISTICS: A STUDY WITH SECONDARY TEACHERS**

Sandra Quintas, Rosa Antónia Tomás Ferreira, Hélia Oliveira

PhD student, University of Lisbon, University of Porto-CMUP, University of Lisbon-CIE, Portugal

*This study is focused on analysing teachers' didactical knowledge in Statistics, in particular their knowledge and perceptions of students' difficulties in learning Statistics, at the secondary level. Data were collected through a questionnaire sent to mathematics teachers working in all schools with secondary education (10<sup>th</sup> to 12<sup>th</sup> grade levels) in the northern region of Portugal. 120 validated responses were received. Data analysis suggests that attention must be paid to teacher education and professional development programs. It seems of utmost importance that teachers develop their didactical knowledge in Statistics in order to better provide students with meaningful learning experiences in the classroom.*

## **TEACHERS AND THE TEACHING OF STATISTICS**

Statistics has been a secondary school curricular topic, in Portugal, since the sixties. In 1991, Statistics was introduced in the second and third cycles of the basic education curriculum (students aged 10-11 and 12-14 years-old, respectively); more recently, it is a curricular topic from first to ninth grade level. In all cases, Statistics is taught under the scope of the Mathematics discipline.

Quantitative information is present in many domains of our current global society. Statistical concepts are crucial to decide whether that information is reliable, yet several studies have suggested that adults do not think statistically to analyse situations and make informed decisions (Ben-Zvi & Garfield, 2004). One contributing factor to this situation may reside on the learning experiences typically offered in schools: statistical procedures have been emphasized (such as the computation of central tendency measures and the elaboration of graphs), at the expense of more cognitively demanding tasks such as small statistical projects or investigations, which involve formulating a question or questions that can be addressed with data; designing and employing a plan for collecting data; analyzing and summarizing the data, and interpreting the results from analysis (e.g., Burrill, 2008; Graça Martins & Ponte, 2010; Ponte & Fonseca, 2001; Scheaffer, 2000).

In fact, traditional teaching approaches do not facilitate an adequate development of statistical thinking (Ben-Zvi & Garfield, 2004). Several studies suggest that students may learn some statistical procedures (such as computing the standard deviation or finding interquartile ranges) but they fail in understanding different representations of the concepts involved in those computations as well as relationships of those

concepts with other statistical ideas (Garfield & Ben-Zvi, 2008). Though acknowledging the usefulness of technical procedures, Scheaffer (2000) stresses the need for a genuine “understanding of analyses and communication of [statistical] results” (p. 158; see also Ponte & Fonseca, 2001).

Research has suggested that both teachers and students fight against cognitive difficulties concerning basic statistical concepts such as sampling and central tendency measures (Groth & Bergner, 2005; 2006). In addition, many teachers have not worked Statistics in their teacher education programs, thus being little comfortable with data analysis procedures and statistical ideas (Ben-Zvi & Garfield, 2004). Teaching Statistics does require a profound knowledge of this area; however, such knowledge is not enough to ensure effective teaching (Ponte & Chapman, 2006; Shulman, 1986). This study, which is part of a larger ongoing investigation, aims at analysing secondary mathematics teachers’ didactical knowledge, namely their knowledge and perception of students’ difficulties in learning Statistics. In particular, the study focuses on the concepts of sample (including sampling procedures, variability, and representativeness), measures of central tendency and of spread, regression line, and correlation coefficients.

### **Didactical knowledge in Statistics**

There are not many theoretical frameworks related to teachers’ professional knowledge in Statistics. However, Statistics has a growing importance in today’s society, and the school must respond to the challenge of helping students become consumers and producers of statistical information (Almeida, 2000). Groth (2007) and Burgess (2006) have dedicated some work on improving or refining existing theoretical frameworks in Mathematics Education research, looking for relevant aspects pertaining to the teaching of Statistics. Based on the work of Ball and colleagues (Hill, Schilling, & Ball, 2005), those two authors have proposed models for the notion of statistical knowledge for teaching.

Batanero and Godino (2005) have emphasized the notion of didactical knowledge in Statistics, which includes, amongst other aspects: (1) teachers’ ability to reflect on the evolution of statistical concepts along the curriculum; (2) teachers’ knowledge about students’ difficulties, errors, and learning obstacles, as well as students’ strategies to solve statistical problems; and (3) teachers’ ability to analyse available resources which may contribute to improve their classroom practices. This perspective resonates with Ponte’s (1999) notion of teachers’ didactical knowledge, a type of knowledge connected to practice and “essentially oriented towards action” (p. 61). According to this author, didactical knowledge encompasses four large domains:

- (1) knowledge of the content that is to be taught, including connections amongst mathematical concepts and connections with other areas and their reasoning, argumentation, and validation forms;
- (2) knowledge of the curriculum, its goals and objectives, and its horizontal and vertical articulation/alignment;

(3) knowledge of the students, their learning processes, interests, and most frequent needs and difficulties, as well as knowledge of social and cultural factors that may influence students' performance at school; and

(4) knowledge of the instructional process, namely the planning and teaching of lessons, and the assessment of teachers' own practices. (p. 61)

In this study, we focused on Ponte's (1999) domains of content knowledge and knowledge of students, as far as Statistics is concerned.

## **METHODOLOGY**

The study followed a quantitative and descriptive approach. It was based on the elaboration of a questionnaire which was administered to mathematics teachers from 145 schools with secondary education (10<sup>th</sup> to 12<sup>th</sup> grade levels) in northern Portugal. The questionnaire, built based on recent theoretical and empirical studies, was aimed at knowing teachers' perspectives on the teaching and learning of Statistics, analysing aspects of teachers' didactical knowledge, and identifying teachers' needs in terms of professional development. Respondents were teachers who had taught 10<sup>th</sup> grade mathematics since this is the secondary grade level in which Statistics is addressed.

The questionnaire encompassed 25 questions, most of them of closed-answer, arranged in three sections. The first section addressed demographical data and teachers' perceived needs of professional development in Statistics; the second section focused on teachers' perspectives about the teaching and learning of Statistics and on their practices. In this paper, we analyse three of the six questions of the questionnaire's third section. This section was centred on teachers' didactical knowledge in Statistics, namely teachers' opinions about students' hypothetical answers to selected tasks, and on aspects related to teachers' statistical knowledge. The tasks were created by the first author or adapted from literature.

We obtained 120 validated responses, 30 of them from male teachers. We used descriptive statistics techniques to treat data from the closed-answer questions and content analysis to address data from the semiclosed-answer questions. The data from the two other sections of the questionnaire indicated that most teachers use the textbook and the graphing calculator as resources in teaching Statistics. Also, most of the respondents believe to be in need of professional development with a focus on didactical knowledge, mainly on the creation of tasks/resources and on their use with students in the classroom (Quintas, Oliveira, & Ferreira, 2009).

## **ANALYSIS OF RESULTS**

In this section we present the results obtained in three of the six questions in the third section of the questionnaire.

### **Wrapped candy**

Figure 1 depicts a question which, despite being related to probabilities, aims at analysing teachers' choices about variability and representativeness of samples.

Teachers’ responses are summarized in Table 1. A third of the respondents did not make any choice and only 21% of them responded correctly.

A container has 100 paper-wrapped candies in three different colours. 20 candy are wrapped in yellow, 50 in red, and 30 in blue. Ana took 10 candy out of the container, counted the number of candies wrapped in red and registered it on the board. She placed the candy back inside the container so that four colleagues of her could repeat the process. A group of students presented the following six hypotheses as responses to the question: *Which list is the most likely for the number of red candies?* Which of the following answer(s) do you consider adequate? Briefly provide the reason for your choice.

☐A) 8,9,7,10,9

☐B) 3,7,5,8,7

☐C) 5,5,5,5,5

☐D) 2,4,3,4,3

☐E) 3,0,9,2,8

☐F) None of the prior options

Figure 1: Question 2 of the questionnaire – Wrapped candy

	A	B	C	D	E	F	More than one option	No response
Responses (%)	2	23	21	0	0	14	18	23

Table 1: Relative frequency of teachers’ responses to Question 2

Most teachers who responded to this question considered it as a proportion problem. In fact, 21% of the teachers who chose option C argued that 50% of the candies were wrapped in red. Some of the teachers who chose option B preferred it over option C: “In each withdrawal, we should get approximately 5 red candies, but it would be a lot of coincidence to get exactly 5 in each one”. Similarly, many teachers who chose more than one list included options B and C. The majority of respondents who indicated option F considered that all provided lists had the same probability. The analysis of the data suggests that there are many teachers who tend to trust more in the notion of sample variability than in the notion of sample representativeness. Furthermore, several teachers seem to hold an inappropriate intuition about the distribution of red candies that is implicit in the situation of Question 2.

Resistance to outliers

The third question (Figure 2) in the questionnaire addressed central tendency measures: mean, mode, and median. This question has two parts: the first one aims at analysing teachers’ statistical knowledge about those measures; the second one aims at knowing whether teachers believe that students, in general, would have difficulties in responding to the same task. Teachers’ responses to both parts of this question are summarized in Figure 3.

Amongst the central tendency measures (mean, mode, and median), which one(s) is (are) more resistant to *outliers* (discrepant values)? (Choose 1 option)

☐ A: mean      ☐ B: mode and median      ☐ C: none of the prior options

Drawing on your experience, which if the options do you believe most students would tend to choose? (Choose on of the options)

☐ A: mean      ☐ B: mode and median      ☐ C: no opinion

Figure 2: Question 3 of the questionnaire – Resistance to outliers

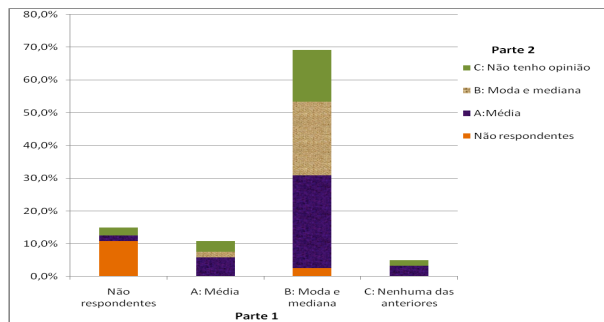


Figure 3: Relative frequency of teachers' responses to Question 3

85% of the respondents answered to the first part of this question, and 69% of them chose the right option (option B). About 39% of the teachers who responded to the second part of the question indicated that students, in facing the very same question, would tend to choose the mean. However, 29% of the respondents to this second part of the question referred that students would choose option B, and 23.3% manifested no opinion about how students would react to that question. Thus, teachers' perceptions of students' likely responses were almost evenly divided amongst all possibilities offered. Analysing teachers' responses to both parts of the question, we can see that, out of the 69% of teachers who responded correctly to the first part, 28% referred that most of their students would likely choose option A (mean) and 23% indicated that the majority of their students would also choose option B (mode and median). The data from this question suggest that some teachers may have difficulties in understanding some properties of the various measures of central tendency. In addition, many teachers consider that their students, in general, are likely to have difficulties in answering the question.

### Meaning of standard deviation

Question 4 of the questionnaire addressed the notion of standard deviation. Figure 4 depicts the question, and in Figure 5 we illustrate teachers' answers to the first part of the question (the most satisfactory answer given by a hypothetical student).

After dedicating two consecutive lessons to standard deviation, a mathematics teacher posed the following question to his class: “What might be the meaning of a high value of the standard deviation?” getting several responses. In your opinion, amongst the following options, what is the answer you consider most satisfactory (signal it in the first column of the table) and, on the contrary, what is the answer you consider least satisfactory (signal it in the second column of the table)? Justify both your options.		
+	-	Students’ responses
		A: The data are quite scattered
		B: A graph with points varying quite a lot amongst themselves
		C: The data are quite far away from the mean
		D: Small variability in data with the existence of one or more discrepant values

Figure 4: Question 4 of the questionnaire – Meaning of standard deviation

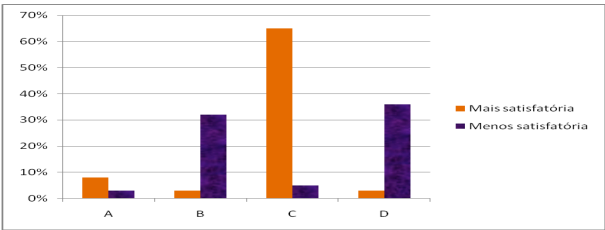


Figure 5: Relative frequency of teachers’ responses to Question 4

Though 17% of the participants did not respond to this question, most of the teachers (about 65%) chose option C for the most satisfactory response given by a hypothetical student, against 8% who chose option A. Amongst those who chose this latter option, the most common reason for the choice made is the (hypothetical) student’s relative knowledge of the meaning of standard deviation: “Option A: the student knows that the more scattered the values, the greater the distance from the mean”. The teachers who chose option C indicated similar arguments, though based on the definition of standard deviation: “Answer C is the answer that pleases me the most because the standard deviation computes a ‘kind’ of mean of the deviations from the arithmetic mean”; “In answer C, students have understood that the standard deviation is related to the deviations from the mean”.

As far as the least satisfactory response is concerned, teachers were almost evenly divided between option B (32%) and option D (36%). The most common reasons for the choice of option B are related to its alleged ambiguous phrasing: “B is little

explicit: Graph? Points varying amongst themselves?”; “Answer B would be the least correct since the deviation measures the spread of the data not of a graph”. The choice of option B for the least satisfactory response was usually anchored in its lack of consistency with the definition of standard deviation typically found in school textbooks: “Taking into account the definition given at the secondary level, C is the one that corresponds to the correct answer and D is the one that resonates the least”. Some responses to this question have revealed incoherence and misconceptions about the notion of standard deviation. For example, a teacher wrote that “Although I like answer D, I find it incomplete. It would be preferable that D also mention the number of times that those discrepant values occur”.

## CONCLUSIONS

This study allowed us to identify some aspects which teachers need to deepen their knowledge. These aspects, related to statistical ideas central to the Portuguese secondary curriculum, include essentially properties of central tendency measures and the notion of standard deviation.

Regarding teachers’ knowledge about the students, this study’s results resonate with prior research (Groth & Bergner, 2005; 2006). Teachers believe students have difficulties in dealing with basic statistical concepts but this study suggests that teachers need to improve their ability to assess students’ answers to different statistical tasks. This is an important vehicle for teachers to develop their didactical knowledge in Statistics, particularly concerning students’ learning processes and frequent learning difficulties.

Data from this study supports the need for professional development endeavours addressing topics in Statistics, as recommended elsewhere (Fernandes, 2009). The participants themselves recognized the need to develop their didactical knowledge, with a strong emphasis on the classroom and on reflective practices (Quintas, Oliveira & Ferreira, 2009). Such professional development programs may help teachers develop their statistical knowledge, directly related to the curriculum they teach, and to reflect on how students learn Statistics.

## References

- Almeida, R. (2000). *Imagens sobre o ensino e a aprendizagem da Estatística*. Tese de Mestrado, Universidade de Lisboa, Portugal.
- Batanero, C., & Godino, J. (2005). Perspectivas de la educación estadística como área de investigación. In R. Luengo (Ed.), *Líneas de investigación en Didáctica de las Matemáticas* (pp. 203-226). Badajoz, Spain: Universidad de Extremadura.
- Ben-Zvi, D., & Garfield, J. (2004). Statistical literacy, reasoning, and thinking: Goals, definitions and challenges. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 3-15). Dordrecht, The Netherlands: Kluwer Academic Publishers.

- Burguess, T. A. (2006). A framework for examining teacher knowledge as used in action while teaching statistics. In A. Rossman & B. Chance (Eds.), *Work cooperatively in statistics education: Proceedings of the 7<sup>th</sup> International Conference on Teaching Statistics*. Salvador, Brazil.
- Burrill, G. (2008). Fundamental ideas in teaching statistics and how they affect the training of teachers. *Proceedings of the ICMI Study 18 and 2008 IASE Round Table Conference*. Monterrey, Mexico.
- Fernandes, J. (2009). Ensino e aprendizagem da Estatística: realidades e desafios. *Proceedings of XIX EIEM*. Vila Real, Portugal: SPCE.
- Garfield, J., & Ben-Zvi, D. (2008). Learning to reason about variability. In J. Garfield & D. Ben-Zvi (Eds.), *Developing Students' Statistical Reasoning: Connecting Research and Teaching Practice* (pp. 201-214). Dordrecht, The Netherlands: Springer.
- Graça Martins, M. E., & Ponte, J. P. (2010). *Organização e tratamento de dados*. Lisboa, Portugal: ME-DGIDC.
- Groth, R. (2007). Toward a conceptualization of statistical knowledge for teaching. *Journal for Research in Mathematics Education*, 38(5), 427-437.
- Groth, R., & Bergner, J. (2005). Preservice elementary teachers' metaphors for the concept of statistical sample. *Statistics Education Research Journal*, 4(2), 27-42.
- Groth, R., & Bergner, J. (2006). Preservice elementary teachers' conceptual and procedural knowledge of mean, median and mode. *Mathematical Thinking and Learning*, 8, 37-63.
- Hill, H., Schilling, S., & Ball, D. (2005). Developing measures of teachers' mathematical knowledge for teaching. *Elementary School Journal*, 105, 11-30.
- Ponte, J. P. (1999). Didáticas específicas e construção do conhecimento profissional. In J. Tavares, A. Pereira, A. Pedro, & H. Sá (Eds.), *Investigar e formar em educação. Proceedings of XI EIEM* (pp. 59-72). Porto, Portugal: SPCE.
- Ponte, J. P., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 461-494). Rotterdam, The Netherlands: Sense Publishers.
- Ponte, J.P., & Fonseca, H. (2001). Orientações curriculares para o ensino da estatística: Análise comparativa de três países. *Quadrante* 10(1), 93-115.
- Quintas, S., Oliveira, H., & Ferreira, R. (2009). Estudo exploratório sobre perspectivas e práticas de professores de Matemática, do ensino secundário, no domínio do ensino da Estatística. *Proceedings of XIX EIEM*. Vila Real, Portugal: SPCE.
- Scheaffer, R. (2000). Statistics for a new century. In M. J. Burke & F. R. Curcio (Eds.), *Learning mathematics for a new century* (pp. 157-173). Reston, VA: NCTM.
- Shulman, J. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.



# STUDYING MATHEMATICS AT THE UNIVERSITY: THE INFLUENCE OF LEARNING STRATEGIES

Stefanie Rach, Aiso Heinze

Leibniz Institute for Science and Mathematics Education, Kiel, Germany

*Many students studying mathematics at university experience great difficulties in their first year. In Germany, universities have to deal with drop-out rates of 40-50% of first-year students with a major in mathematics. Possible reasons are differences between school and university concerning the character of mathematics taught and the learning culture. In a study with  $N = 118$  students with a major in mathematics, we investigate individual determinants of students' competency development in their first semester. First results indicate that the type of learning strategies mainly used by students has an impact on mathematics achievement, interest and self-concept. Except for the case of high achieving students, these types of learning strategies are not related to individual variables at the beginning of the study.*

## INTRODUCTION

The transition from school to university is experienced as an interesting and exciting period for many first-year students. This transition is challenging for students studying mathematics due to requirements including a significant time investment, self-organized learning, and academic mathematics itself. Many students fail to cope with these unexpected challenges when starting at the university. Hence, during the last decades these specific challenges were investigated from different perspectives focusing on the differences between mathematics at school and academic mathematics and the corresponding learning processes at the university (e.g., de Guzmán, Hodgson, Robert, & Villani, 1998; Hoyles, Newman, & Noss, 2001).

In Germany, universities and mathematics departments are faced with comparatively high drop-out rates (up to 50%) of first-year students in mathematics. According to surveys, students report that this is mainly caused by the enormous pressure to perform and the lack of motivation (Heublein et al., 2009). For a deeper insight into these reasons, we started a research project on the development of students' competencies and the teaching and learning processes during the first semester. In this contribution, we focus on first results concerning individual cognitive and non cognitive variables and their impact on the learning of mathematics in the first six weeks of an Analysis I course (Analysis I encompass real analysis and has a theory-based level emphasizing definitions, theorems and proofs. Calculations like in calculus courses play only a minor role).

## THEORETICAL BACKGROUND

Based on a review of research literature, we postulate that the difficulties of first-year students in mathematics are mainly rooted in two fundamental differences between mathematics learning at school and mathematics learning at university: (1) the character of mathematics that is taught and (2) the individual learning strategies necessary to use the learning opportunities effectively.

### Character of Mathematics at School and at University

Many mathematics educators agree that mathematics taught in high school is not just academic mathematics in a simplified form but has its own character (e.g., Hoyles, et al., 2001). The main reason is that learning mathematics in high school must contribute to the aim of general education. This means, in particular, that students should learn how to use mathematics for solving everyday problems. Accordingly, in school curriculum there is a specific emphasis on mathematical content which is relevant for the application of mathematics as tool (e.g., percentages, algebraic manipulations) but which is hardly interesting from a scientific mathematical perspective (e.g., Dörfler, & McLone, 1986). At university, students with a major in mathematics learn mathematics as a scientific discipline. This means that the mathematical content is organized and presented in a specific axiomatic and rigorous manner. In the first semesters, applications of mathematics for solving real-world problems hardly play any role.

Hoyles et al. (2001, p. 841) characterizes these two sides of mathematics: “It is a tool in the study of the sciences, and it is an object of study in its own right.” These two sides of mathematics are reflected in high school and at university in quite a different way which has serious consequences for the role of important characteristics of mathematics like proving, rigor or formalism. For example, most of the mathematical concepts in school are encountered in an informal way (Engelbrecht, 2010) so that students develop and work with a *concept image* (Tall & Vinner, 1981). A formal *concept definition* frequently does not play a prominent role. At university, it is just the opposite: concepts are introduced by a formal definition which is necessary to meet the standards of rigor. In the case of mathematical proof, a similar situation can be observed. Proofs are essential when dealing with mathematics as a scientific theory because they give evidence for statements and explain internal relations. Considering mathematics from an instrumental perspective (as a tool), proofs play a minor role and they are often omitted (it is enough to know that there exists a proof).

### Teaching and Learning Mathematics at School and at University

The teaching and learning of mathematics at school and at university differs in two important aspects. On the one hand, the formal organization of learning opportunities and, on the other hand, the individual learning strategies necessary for an effective use of the learning opportunities. At German universities, the Analysis courses for

first-year students consist of three different learning opportunities a week: two 90 minute lectures given by a mathematics professor, a set of 3-5 challenging exercises as obligatory homework (self-study phase in small study groups) and a 90 minutes tutorial per week where a senior mathematics student discusses the solutions of the homework with a group of 20-30 students. In particular, the self-study phase during which the students work in small groups on their homework is considered an important learning activity since students are individually involved in mathematical problem solving processes.

Regarding effective learning strategies at school and at university we can observe a necessity of additional learning strategies at the university essential for successful competence acquisition. These learning strategies correspond with the fact that mathematics at university is taught as scientific discipline. First, students need to apply elaboration strategies to understand the formally presented mathematical content in the lectures. New mathematical concepts cannot be grasped through formal concept definitions, so it is necessary that students connect the concept definition to an already existing concept image from an intuitive use of this concept in school (e.g., in the case of limits) or that they individually develop a new concept image (Engelbrecht, 2010). Second, students need to elaborate and reflect on problem-solving strategies. As Dreyfus (1991) criticizes, mathematics is taught as a completed theory and students are not involved in the trial and error process of creating new knowledge. In particular, problem-solving strategies are kept implicitly in the lectures. Accordingly, students do not get a model how to approach proof problems. Based on research on example-based learning, self-explanation has to be proven as an effective learning strategy (e.g., Chi et al., 1989). Therefore, one can assume that students performing self-explanations in the self-study phase when working through problem solutions or proofs show a better achievement than students who only comprehend solutions or proofs without self-explanations.

## **RESEARCH QUESTIONS AND DESIGN OF THE STUDY**

Based on the theoretical background, our subsequently presented study is guided by the following research questions:

- What are the individual learning prerequisites of students starting to study mathematics? Here we address the final school grade (overall and in mathematics), the prior knowledge in analysis, interest in mathematics and mathematics self-concept.
- How do the individual interest in mathematics and the mathematics self-concept develop in the first six weeks?
- What type of learning strategies do students mainly apply in self-study phase (homework, self-organized study groups) after six weeks?
- What impact do the learning strategies have on the achievement, interest in mathematics and the mathematics self-concept?

- Is the use of the learning strategies influenced by the individual learning prerequisites students bring from school?

### Sample and Methodology

Our on-going study is conducted at the Department of Mathematics of the University of Kiel (Germany) from October 2010 to February 2011 (winter term). The sample consists of 118 students majoring in mathematics which started in October 2010.

On the first day of the winter term in October 2010 we collected data for learning prerequisites consisting of the final school grade (overall and in mathematics), the prior knowledge in the field of analysis, interest in mathematics and mathematics self-concept. In December 2010, after six weeks, we collected data for achievement in analysis, interest in mathematics, mathematics self-concept and learning strategies applied during self-study phases.

We used approved questionnaires adapted from Schiefele, Moschner, and Husstegge (2002) for measuring interest in mathematics (6 items, Cronbach's  $\alpha = .83$ ) and mathematics self-concept (4 items,  $\alpha = .80$ ). On both questionnaires students had to rate statements on a four-point Likert scale (0 = strongly disagree, 1 = disagree, 2 = agree, 3 = strongly agree). The tests for prior knowledge in the field of analysis and the analysis achievement test consist of 10 and 12 items ( $\alpha = .63$  and  $\alpha = .73$ ) respectively (open items as well as multiple choice items, see appendix for two sample items). Information about the learning strategies the students mainly applied was collected by a questionnaire. Students should report which of the following three learning types fits best to their own behaviour in self-study phases:

- "I study the exercises intensively and try to solve them. I try to comprehend the solutions of other students. I give rarely self-explanations." (reproduction type)
- "I study the exercises intensively and try to solve them. I try to comprehend the solutions of other students. I explain the solution to myself and/or to other students even if I rarely find solutions by myself." (self-explanation type)
- "Often, I can solve the exercises or I find ideas for solutions. Then I explain the solution to myself and/or other students." (self-solver type)

### RESULTS

The data for the learning prerequisites indicate that the mathematics students start university with quite good prerequisites. The mean value for their final school grade in mathematics is  $M = 12.0$  ( $SD = 2.13$ ) of maximal 15 points and the means for interest in mathematics  $M = 2.17$  ( $SD = 0.41$ ) and mathematics self-concept  $M = 1.91$  ( $SD = 0.45$ ) are comparatively high. The mean value of the overall final school grade reflects only a moderate level:  $M = 2.22$  ( $SD = 0.55$ ) where 1.0 is the best and 4.0 is the worst possible grade. It seems that they have a specific strength in mathematics.

Interestingly, the results for the test on prior knowledge in analysis are comparatively weak with  $M = 4.64$  ( $SD = 2.16$ ) of maximal 10 points. Since this test asks for knowledge which is part of the school curriculum but is not really emphasized in

school, this weak results supports the assumption that mathematics in school and academic mathematics have a different character. Consistent with this finding is the fact that the average value for students' mathematics self-concept decrease significantly in the first six weeks to  $M = 1.59$  ( $SD = 0.54$ ,  $t(117) = 7.94$ ,  $p < .001$ ,  $d = 0.66$ ). It seems that in this period the students become aware that learning mathematics at university is much more challenging than they were used to from their school experience. Regarding the interest in mathematics, the mean values remain on a comparatively high level ( $M = 1.98$ ,  $SD = 0.51$ ,  $t(117) = 4.64$ ,  $p < .001$ ,  $d = 0.41$ ). Although there is also a decrease, the mean value is still on the "agree"-level ( $= 2.00$ ).

As described previously, data for the applied learning strategies in self-study phases were collected by a self-report. About 35% of the students think that their learning strategies fits best to the reproducing type, whereas 52% choose the self-explanation type and 13% the self-solver type. Based on the empirical evidence from research on example-based learning, we expect that students of the self-explanation type show a better achievement after six weeks of the winter term than students of the reproducing type (e.g. Chi et al., 1989). Obviously, students of the self-solver type are expected to be stronger than both other learning strategies types.

Results of an ANCOVA with prior knowledge in analysis as covariate indicate that the type of learning strategies have an significant impact on the achievement in analysis after six weeks (see Table 1,  $F(2,114) = 38.03$ ,  $p < .001$ ,  $R^2 = .49$ ). However, post-hoc tests show that indeed students of the self-solver type are significantly better than the other two types ( $p < .001$ ) but there are no significant differences between students of the reproducing type and of the self-explanation type ( $p = .743$ ).

Mean (max. 12 pts.)	Reproducing Type	Self-Explanation Type	Self-Solver Type
Achievement in Analysis	4.93	5.41	8.31

Table 1. Achievement in analysis after six weeks, estimated marginal means with prior knowledge in analysis as covariate.

Regarding the development of interest in mathematics and mathematics self-concept, students of the self-solver type show increasing (interest) or stable (self-concept) values (see Table 2). For students using learning strategies of the other two types, interest and self-concept decreases within the first six weeks. Here it is remarkable that students of the reproducing type show a significant lower self-concept after six weeks than students of the self-explanation type ( $M = 1.31$  ( $SD = 0.46$ ) and  $M = 1.61$  ( $SD = 0.48$ ),  $t(100) = -3.12$ ,  $p = .002$ ,  $d = -0.63$  for self-concept).

M (SD)	Reproducing Type		Self-Explanation Type		Self-Solver Type	
	First day	after 6 weeks	First day	after 6 weeks	First day	after 6 weeks
Interest in Mathematics	2.05	1.75	2.23	2.01	2.23	2.44
	(0.43)	(0.48)	(0.38)	(0.46)	(0.41)	(0.39)
	$t(40) = 4.65,$ $p < .001, d = 0.65$		$t(60) = 3.88,$ $p < .001, d = 0.52$		$t(15) = -2.83,$ $p = .013, d = -0.52$	
Mathematics Self-concept	1.77	1.31	1.93	1.61	2.23	2.23
	(0.46)	(0.46)	(0.37)	(0.48)	(0.54)	(0.41)
	$t(40) = 6.71,$ $p < .001, d = 1.00$		$t(60) = 6.04,$ $p < .001, d = 0.76$		n.s.	

Table 2. Development of interest in mathematics and mathematics self-concept in the first six weeks of the first semester divided by learning strategy types.  
Likert-Scale: 0 = strongly disagree, 1 = disagree, 2 = agree, 3 = strongly agree

Since the type of learning strategy has an impact on achievement in analysis, on mathematics self-concept and on interest in mathematics after six weeks, it is interesting if the learning strategies are influenced by learning prerequisites students bring from school or if they were developed independently from these during the first six weeks of the winter term. It turned out that there are no significant differences between students of the reproducing type and students of the self-explanation type regarding the final school grade (overall and mathematics), prior knowledge in analysis, mathematics self-concept and interest in mathematics. However, students of the self-solver type differ significantly from the other types in their prior knowledge in analysis and in the mathematics self-concept already at the beginning of their study. It seems that this small group ( $N = 16$ ) starts with better prerequisites.

DISCUSSION

Our findings indicate that first-year students with a major in mathematics at German universities can be characterized as students with a good school achievement in mathematics, high interest in mathematics and quite a good self-concept in mathematics. However, already after six weeks experience in studying mathematics, many students show a decreasing self-concept which is probably caused by the big challenges that go along with the transition phase from school to university. As we elaborated in the theoretical background, we assume that coping with these challenges requires specific learning strategies which differ from that in school. In particular, the ability to apply elaboration strategies like self-explanation (or explanation to others) is necessary to understand the presented scientific mathematics and to learn from sample solutions or other sources. As presented in the results section, the self-reported learning strategy type of the students has indeed a significant influence on the achievement in analysis after six weeks. Interestingly, we do not find the expected achievement differences between students of a reproducing learning type and students of a self-explanation type (but they differ significantly in

their mathematics self-concept). Here, we assume that the self-reported learning strategy type is open for social desirability effects so that students may report that their learning strategies are on a higher level than it is really the case.

A noticeable result is that those students mainly using reproducing or self-explanation learning strategies after six weeks do not differ in their learning prerequisites at the beginning of their study. It seems that the type of learning strategies mainly used in self-study phases are not determined by the individual learning prerequisites. So, there should be good chances for an intervention teaching specific learning strategies in the first weeks at the university which support students to cope with the challenges of academic learning in mathematics.

## References

- Chi, M. T. H., Bassok, M., Lewis, M., Reimann, P., & Glasser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. *Cognitive Science*, 13, 145–182.
- Dörfler, W., & McLone, R. (1986). Mathematics as a schoolsubject. In B. Christiansen, A.G. Howson and M. Otte (eds.), *Perspectives on Mathematics Education* (pp. 49-97). Reidel: Dordrecht.
- Dreyfus, T. (1991). Advanced Mathematical Thinking Processes. In D. Tall (Ed.), *Mathematics education library: Vol. 11. Advanced mathematical thinking* (pp. 25–41). Dordrecht: Kluwer Academic Publisher.
- Engelbrecht, J. (2010). Adding structure to the transition process to advanced mathematical activity. *International Journal of Mathematical Education in Science and Technology*, 41(2), 143–154.
- Guzmán, M. de, Hodgson, B. R., Robert, A., & Villani, V. (1998). Difficulties in the passage from secondary to tertiary education. In G. Fischer (Ed.): *Vol. Extra Volume ICM 1998. Documenta Mathematica, Proceedings of the International Congress of Mathematicians Berlin 1998, August 18 - 27* (pp. 747–762). Rosenheim: Geronimo.
- Heublein, U., Hutzsch, C., Schreiber, J., Sommer, D., & Besuch, G. (2009). *Ursachen des Studienabbruchs in Bachelor- und in herkömmlichen Studiengängen: Ergebnisse einer bundesweiten Befragung von Exmatrikulierten des Studienjahres 2007/08*. Hannover: Hochschul-Informations-System GmbH.
- Hoyle, C., Newman, K., & Noss R. (2001). Changing patterns of transition from school to university mathematics. *International Journal of Mathematical Education in Science and Technology*, 32(6), 829–845.
- Schiefele, U., Moschner, B., & Husstegge, R. (2002). *Skalenhandbuch SMILE-Projekt*. Universität Bielefeld; Abteilung für Psychologie.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151–169.

## APPENDIX

Translated sample item of the test of prior knowledge in the field of analysis:

*Does a smallest positive number exist? Which of the following statements is true?*

- 1. Yes, because you can find a number in  $\mathbb{R}$ , arbitrarily close to 0.*
- 2. No, because for every positive number there exists another number between 0 and this number.*
- 3. Yes, because the positive real numbers are bounded below.*
- 4. No, because the smallest positive number is not real, but rational.*

Translated sample item of the analysis achievement test after six weeks:

*Prove that for all  $s \in \mathbb{R}$  with  $s > 2$  the series  $\sum_{k=1}^{\infty} \frac{1}{k^s}$  is convergent.*



# EMBODIMENT, PERCEPTION AND SYMBOLS IN THE DEVELOPMENT OF EARLY ALGEBRAIC THINKING

Luis Radford

Laurentian University, Canada

*Placed in the context of early algebra research, this article deals with the question of the development of algebraic thinking in young students. In contrast to mental approaches to cognition, we argue that thinking is made up of material and ideational components—such as (inner and outer) speech, forms of sensuous imagination, gestures, tactility, and actual actions with signs and cultural artifacts. Drawing on data from a longitudinal classroom based research program where 8-year old students were followed as they moved from Grade 2 to Grade 3 to Grade 4, our developmental research question is investigated in terms of the manner in which new relationships between embodiment, perception, and symbol-use emerge and evolve as students engage in patterning activities.*

## INTRODUCTION AND THEORETICAL FRAMEWORK

The idea that young students—even with limited knowledge of arithmetic— can start learning some algebraic concepts has received increasing experimental support in the past few years. It has been found that, with suitable instructional support, young students can understand some aspects of pattern generalization—e.g., to describe the terms of a sequence according to the position they occupy therein (e.g., Becker & Rivera, 2008; Moss & Beatty, 2006; Warren & Cooper, 2008). However, despite the growing research on early algebra, many research questions remain open. For instance, little is known about how algebraic thinking develops in young students. This article seeks to contribute to this research question through the analysis of data collected in the course of a 3-year longitudinal classroom based research program.

Generally speaking, developmental questions of the kind we are dealing with here are not easy to investigate. They can only be formulated and tackled against the backdrop of explicit theoretical views about thinking and development. In our case, our research is framed by a theoretical Vygotskian perspective on teaching and learning—the *theory of knowledge objectification* (Radford, 2008a). A central feature of this theory is that, in contrast to mental cognitive approaches, thinking is not considered as something that solely happens ‘in the head.’ Thinking is rather considered as made up of material and ideational components: it is made up of (inner and outer) speech, objectified forms of sensuous imagination, gestures, tactility, and our actual actions with cultural artifacts. This does not mean that thinking is a *collection* of items. We consider thinking as a dynamic *unity of material and ideal components*—a tangible social practice materialized in the body (e.g. through kinaesthetic actions, gestures, perception, visualization), in the use of signs (e.g. mathematical symbols, graphs, written and spoken words), and artifacts of different

sorts (rulers, calculators and so on). Within this context, to ask the question of the development of algebraic thinking is to ask about the appearance of new structuring *relationships* between the material-ideational components of thinking (e.g., gesture, inner and outer speech) and the manner in which these relationships are organized and reorganized. Now, in the theoretical perspective articulated here, development is not considered to follow any pre-established or innate path. Rather, it is considered to be cultural through and through. Our research question therefore does not simply concern the appearance of new forms of psychic functioning but also the contextual conditions that make these forms possible in the first place. It is against this theoretical framework that the question of the development of young students' algebraic thinking is investigated in the following sections.

## METHODOLOGY: DATA COLLECTION AND ANALYSIS

Our data comes from a 3-year longitudinal research program conducted in an urban primary school in which a class of 25 8-year old students was followed as the students moved from Grade 2, to Grade 3, to Grade 4. The data was collected during regular mathematics lessons designed by the teacher and our research team. To collect data, we used four or five video cameras, each filming one small group of students (groups of 2 or 3). The data that is presented here comes from episodes of what happened when the students were dealing with questions about pattern generalization. We focus in particular on a student, Carlos, whose developmental path is representative of our findings. In tune with our theoretical framework, to investigate the development of early algebraic thinking we conducted a *multi-semiotic data analysis*. Once the videotapes were fully transcribed, we identified salient episodes of the activities. Focusing on the selected episodes, we carried out a low-motion and a frame-by-frame fine-grained video microanalysis to study the role of and the relationship between gestures, language, and mathematical signs.

## RESULTS AND DISCUSSIONS

### First episode: Grade 2

The first algebra activity that the students tackled in Grade 2 revolved around the sequence shown in Fig. A.



Figure 1



Figure 2



Figure 3



Figure 4

Fig. A. The first four figures of a sequence given to the students in a Grade 2 class.

In the first part of the activity, the students were asked to extend the sequence up to Figure 6. Carlos, one of the students, started counting the squares aloud, accompanying the counting process with a rhythmic upper body movement and pen-pointing gestures. He counted all the squares in an orderly way, beginning with the squares in the top row, from left to right, then those in the bottom row (see Fig. B, pic. 1-2). Then he drew Figure 5 in an orderly manner, starting from the bottom row,

left to right. Although Figure 5 contains almost the right number of squares, it certainly does not conform to the two-row arrangement of the given terms of the sequence (See Fig. B, pic. 3).

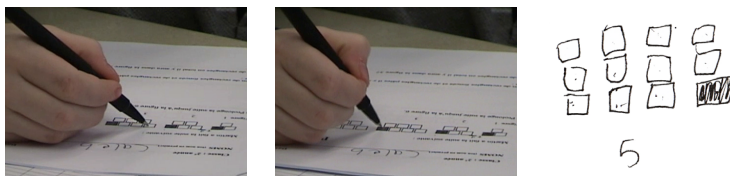


Fig. B. In pics 1 and 2, while counting aloud, Carlos sequentially points to the squares in the top row of Figure 3. Pic 3 shows Carlos's Figure 5.

To come up with an interpretation of Carlos's actions, let us note that, generally speaking, to extend a figural sequence, the students need to grasp a regularity that involves the linkage of two different structures: one *spatial* and the other *numerical*. From the spatial structure emerges a sense of the squares' *spatial position*, whereas their numerosity emerges from a numerical structure. While Carlos attends to the numerical structure in the generalizing activity, the spatial structure is not coherently emphasized. This does not mean that Carlos does not see the figures as composed of two horizontal rows. As in the case of other students, Carlos's emphasis on the numerical structure somehow leaves in the background the geometric structure. This emphasis reappeared when he finished drawing Figure 5: since the shape of the figure did not provide him with a clue about its numerosity, he might have felt the need to count the squares again. We could say that the *shape* of the terms of the sequence is used to facilitate the counting process (as he always counted the squares in a figure in an spatial orderly way), but that the geometric structure does not come to be related to the numerical one in a meaningful efficient way. Carlos's process can be contrasted to Kyle's, where shape is emphasized but numerosity is not well attended. Kyle drew Figure 5 as having two rows but drew 4 squares on the bottom and 4 squares on the top row. These examples—as well as those reported by Rivera (2010) with other Grade 2 students—suggest that the linkage of spatial and numerical structures constitutes an important aspect of the development of algebraic thinking. That such a linkage is less natural than it may appear at first sight can be made evident if we resort to studies in special education. It is well known that children with Down syndrome tend to reproduce figures such as Figure 5 in terms of their shape without much attention to numerical details; in contrast, children with Williams syndrome tend to present more analytical thinking, which focuses on the numerical in detriment to the spatial (Brigaglia, 2010). Or as Bellugi, Lai, and Wand (1997) note, William Syndrome subjects are typically impaired at reproducing global forms, while Down Syndrome subjects tend to produce global forms without local information. Coming back to our Grade 2 students, it is interesting to note that in extending the sequences, the students did not use deictic spatial terms, like “bottom” or “top.” (There was one exception: Kyle, who talked once about the “top row,” without hence

using it in a systematic manner.) In the cases in which the students did succeed in linking the spatial and numerical structures, the spatial structure appeared ostensibly only, i.e., in the embodied realm of action and perception (Radford, 2010, in press). The geometric structure reached the realm of language the next day, when the teacher discussed the sequence with the students. Indeed, during the debriefing of the first day, it was agreed with the teacher that it would be important to bring to the students' attention the linkage of the numerical and spatial structures. To do so, the teacher drew the first five terms of the sequence on the blackboard and refereed to an imaginary student who counted by rows: "This student," she said to the class, "noticed that in Figure 1 (she pointed to the name of the figure) there is one rectangle on the bottom (and she pointed to the rectangle on the bottom), one on the top (pointing to the rectangle), plus one dark rectangle (pointing to the dark rectangle)." Next, she moved to Figure 2 and repeated in a rhythmic manner the same counting process coordinating the spatial deictics "bottom" and "top," the corresponding spatial rows of the figure, and the number of rectangles therein. To make sure that everyone was following, she started again from Figure 1 and, at Figure 3, she invited the students to join her in the counting process, going together up to Figure 5 (see Fig. C).

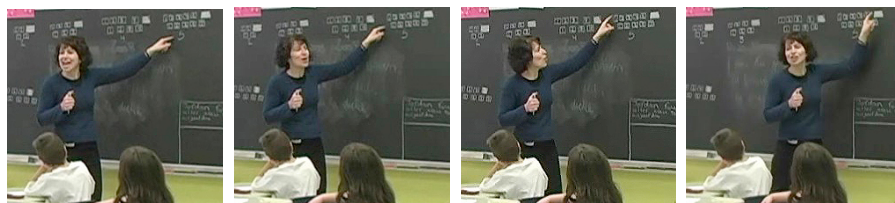


Fig. C. The teacher and the students counting rhythmically say (see Pic 1) "Figure 5", (Pic 2) "5 on the bottom", (Pic 3) "5 on top", (Pic 4) "plus 1."

Then, the teacher asked the class about the number of squares in Figure 25. Mary raised her hand and answered: "25 on the bottom, 25 on top, plus 1." The class spent some time dealing with "remote" figures, such as Figure 50, and 100. Schematically speaking, the students' answers were " $x$  on the bottom,  $x$  on the top, plus 1" where  $x$  was always a *specific* number. Since at the time the students were able to make systematic additions up to 25, the teacher made calculators available to them and asked the students to explain the steps to calculate the total of squares in specific figures. Schematically speaking, the students' answer was " $x + x + 1$ " (where  $x$  was always a *specific* number). The students came back to small-group work and continued their work. In one of the questions, they had to explain how Pierre should proceed to build a big figure of the sequence. The goal of this and other similar questions was to give an opportunity to the students to objectify a numerical-spatial regularity of the given terms of the sequence and to use it to imagine and deal with remote (or even unspecified) terms. Carlos wrote: "Pierre wants to build Figure 10,000. Pierre has to put 10,000 on the bottom[;] on the top he has to put 10,001." In our PME 34 paper we dealt with the nature of the students' emergent algebraic

thinking. What we want to discuss here is the question of development. As stated in our theoretical framework, conceptual development is marked by the appearance of new *relationships* between the material-ideational components of thinking; it brings forward new forms of psychic functioning. If during the first day Carlos and other students were emphasizing the analytic process of counting squares one by one, from the second day on, their perception of the figures and the counting processes changed. The link between the spatial and geometric structures was achieved and, as illustrated by Mary's and Carlos's answers, spatial deictics became part of their linguistic repertoire. These changes bear witness to the appearance of new relationships between gesture, speech, perception, imagination, and counting. A new unity of material and ideational components of thinking was forged. Thus, the students were able not only to imagine remote figures (e.g., Figure 100)—which would be difficult to imagine within the relationships of ideal and material components of thinking underpinning pure analytic, one-by-one counting procedures— but also to devise formulas to calculate the number of squares in figures beyond perception (e.g., “ $100+100+1$ ”).

The joint counting process in which the teacher and the students engaged during the second day is, of course, an instance of a *zone of proximal development*. The explicit use of rhythm, gestures, and linguistic deictics by the teacher, followed later by the students, opened up new possibilities for the student to use efficient and evolved cultural forms of mathematical generalization that they successfully applied to other sequences with different shapes. The joint counting process made it possible for the students to *notice* and *articulate* new forms of mathematical generalization. In particular, they became aware of the fact that the counting process can be based on a relational idea: to link the number of the figure to relevant parts of it (e.g. the squares on the bottom row). This requires an altogether new perception of the number of the figure and the figures themselves. The figure appears now not as a mere bunch of ordered squares but as something susceptible of being decomposed, the decomposed parts bearing potential clues for algebraic relationships to occur. But it is not only perception that is developmentally modified. In the same way as perception develops, so do speech (e.g., through spatial deictics) and gesture (through rhythm and precision). Indeed, perception, speech, gesture, and imagination develop in an interrelated manner. They come to form a new unity of the material-ideational components of thinking, where words, gestures, and signs more generally, are used as means of objectification, or as Vygotsky put it, “as means of voluntary directing attention, as means of abstracting and isolating features, and as a means of [...] synthesizing and symbolising” (1987, p. 164).

### Second episode: Grade 3

As usual, in Grade 3 the students were presented with generalizing tasks to be tackled in small groups. The first task featured a figural sequence,  $S_n$ , having  $n$  circles horizontally and  $n-1$  vertically, of which the first four terms were given. Contrary to what he did first in Grade 2, since the outset, Carlos perceived the sequence taking

advantage of the spatial configuration of its terms. Talking to his teammates about Figure 4 he said: “here (pointing to the vertical part) there are four. Like you take all this [i.e., the vertical part] together (he draws a line around), and you take all this [i.e., the horizontal part] together (he draws a line around; see Fig. D, pic 1). So, we should draw 5 like that (through a vertical gesture he indicates the place where the vertical part should be drawn) and (making a horizontal gesture) 5 like that” (see Fig. D, pics 2-3).

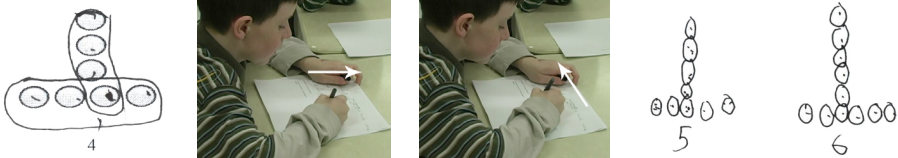


Fig. D. To the left, Figure 4 of the given sequence. Middle, Carlos's vertical and horizontal gestures while imagining and talking about the still to be drawn Figure 5. To the right, Carlos's drawings of Figures 5 and 6.

When the teacher came to see the group, she asked Carlos to sketch for her Figure 10, then Figure 50. The first answer was given using unspecified deictics and gestures. He quickly said: “10 like this (vertical gesture) and 10 like that” (horizontal gesture). The specific deictic term “vertical” was used in answering the question about Figure 50. He said: “50 on the vertical... and 49...” When the teacher left, the students kept discussing how to write the answer to the question about Figure 6. Carlos wrote: “6 vertical and 5 horizontal.” In developmental terms, we see the evolution of the unity of ideational-material components of algebraic thinking. Now, Carlos by himself and with great ease coordinates gestures, perception, and speech. The coordination of these outer components of thinking is much more refined compared to what we observed in Grade 2. This refinement is what we have called a *semiotic contraction* (Radford, 2008b) and is a symptom of learning and conceptual development.

### Third episode: Grade 4

To check developmental questions, in Grade 4 we gave to the students the sequence with which they started in Grade 2 (see Fig. A). This time, from the outset, Carlos perceived the terms as being divided into two rows. Talking to his teammates and referring to the top row of Figure 5, he said as if talking about something banal: “5 white squares, ‘cause in Figure 1, there is 1 white square (making a quick pointing gesture) ... Figure 2, 2 [squares] (making another quick pointing gesture); 3, (another quick pointing gesture) 3.” He drew the five white squares on the top row of Figure 5 and added: “after that you add a dark square.” Then, referring to the bottom row of Figure 4: “there are 4; there [Figure 5] there are 5.” When the teacher came to see their work, Carlos and his teammates explained “We looked at Figure 2, it’s the same thing [i.e., 2 white squares on top] ... Figure 6 will have 6 white squares.”





Fig. E. Left, Carlos' drawings of Figures 5 and 6. Right, Carlos's formulas.

In his answer to the question about explaining what Pierre has to do to build a big figure of the sequence, Carlos wrote: "He needs [to put as many white squares as] the number of the figure on top and on the bottom, plus a dark square on top." The algebraic formula that he provided is shown in pic 3 of Fig. E. From a developmental perspective, we see how Carlos's use of language has been refined. In Grade 2 he was resorting to particular figures (Figure 1,000) to answer the same question. Here he deals with indeterminacy in an easy way, through the expression "the number of the figure." He even goes further and produces two symbolic expressions to calculate the total of squares in the unspecified figure.

### SYNTHESIS AND CONCLUDING REMARKS

This paper seeks to contribute to the question of the development of young students' algebraic thinking. Framed by the theory of objectification, it was suggested that thinking is a *unity of material and ideal components*—inner and outer speech, forms of sensuous visualization and imagination, gestures and tactility, etc. Development is considered to consist of the refinement of previous, and the appearance of new, structuring *relationships* between the material-ideational components of thinking. Within this framework, early algebraic thinking is considered to be based on the student's possibilities to grasp patterns in culturally evolved co-variational ways and use them to deal with questions of remote and unspecified terms. Cognitively speaking, for this to occur, the students have to resort to a coordination of numeric and spatial structures. The awareness of these structures and their coordination entail a complex relationship between (inner or outer) speech, forms of visualization and imagination, gesture, and activity on signs (e.g., numbers and proto-algebraic notations). Our data offer a glimpse of the evolution of algebraic thinking. It shows how in Grade 2 "spontaneous" perception was successfully transformed through the joint work of the teacher and the students. This joint work, we suggested, might be conceptualized as occurring in a zone of proximal development out of which the students created new psychological functions. For as Schneuwly notes, "Teaching does not implant new psychological functions in the child. It makes the tools available and creates the conditions necessary for the child to build them" (1994, p. 288). A substantial refinement of the new *relationships* between the material-ideational components of algebraic thinking was accomplished in Grade 3. In Grade 4 we witnessed the marked evolution of language to deal with indeterminate terms of sequences and the appearance of a new component—symbolic activity (see Fig. E). At this point we are investigating how the new symbolic activity gives rise to abstract algebraic notations.

*Study funded by The Social Sciences and Humanities Research Council of Canada.*

## References

- Becker, J., & Rivera, F. (Eds.) (2008). Generalization in algebra: the foundation of algebraic thinking and reasoning across the grades. *ZDM*, 40(1).
- Bellugi, U., Lai, Z., & Wang, P. (1997). Language, communication, and neural system in Williams syndrome. *Mental Retardation and Developmental disabilities*, 3, 334-342.
- Brigaglia, P. (2010). *Insegnamento/apprendimento della geometria in situazioni di disabilità..* Università di Palermo, Unpublished Ph.D. Dissertation. Palermo, Italy.
- Moss, J., & Beatty, R. (2006). Knowledge building and knowledge forum: Grade 4 students collaborate to solve linear generalizing problems. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proc. 30<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 193-199). Prague: PME.
- Radford, L. (2008a). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education*. (pp. 215-234). Rotterdam: Sense Publishers.
- Radford, L. (2008b). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM*, 40(1), 83-96.
- Radford, L. (2010). Elementary forms of algebraic thinking in young students. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proc. 34<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 73-80). Belo Horizonte: PME.
- Radford, L. (in press). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early Algebraization*. Berlin: Springer-Verlag.
- Rivera, F. D. (2010). Second grade students' preinstructional competence in patterning activity. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proc. 34<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 81-88). Belo Horizonte: PME.
- Schneuwly, B. (1994). Contradiction and development: Vygotsky and paedology. *European Journal of Psychology of Education*, 9(4), 281-291.
- Vygotsky, L. S. (1987). *Collected works (Vol. 1)*. R. W. Rieber, & A. S. Carton (Eds.). New York: Plenum.
- Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns. *Educational Studies in Mathematics*, 67, 171-185.



# MOVING BETWEEN NORMS IN SCHOOL MATHEMATICS PRACTICE AND BUILDING COMPANY PRACTICE

Toril Eskeland Rangnes  
Bergen University College

*As the pupils move between school and commercial workplace, they meet different norms as to what can be acceptable ways of working with mathematics. This paper explores how different norms for working with scaling in school and in building company can give different support for learning mathematics. Conversations are analysed and discussed in relation to Gorgorió, Planas & Vilella's (2001) definition of sociomathematical norms.*

## INTRODUCTIONS AND AIMS

The object of this paper is to shed light on conversations and actions where pupils in lower secondary school (8th graders) work with mathematics in a practical setting and how this can contribute to develop different aspects in mathematics learning.

Essential in this paper is that the conversations are taken from a project which aims for lower secondary school pupils to learn mathematics through meetings with representatives from a building company, a trade which they initially do not know much about. The pupils meet two kinds of practices, the building company's, guided by production, efficiency and profitability, and the school's, governed by learning goals in mathematics as defined in the curriculum and effectuated by the teacher. Johnsen-Høines (2010) describes the pupils' movement between school and company as a learning loop. This movement is not confined to location – rather, it is about how the moving between is present in conversations both in school and in the building company. One of the participating teachers' arguments for this kind of teaching was the diversity in her class; some pupils have preferences for practical work, and most of her teaching was theoretical with textbook. When changing the way of teaching from what the pupils are used to, the pupils' understanding of what is expected of them is challenged. What activities that are possible to do in mathematics teaching are made negotiable and what procedures that are acceptable can become the object of discussion based on the pupils' and teachers' expectations and values. I see this in the light of the development of common sociomathematical norms. In this paper, sociomathematical norms are defined according to Gorgorió, Planas & Vilella (2001) as

being the whole of the implicit and explicit norms within the mathematics classroom, resulting from the juxtaposition of the social norms and the norms of the mathematical practice together with individuals' values, expectations, emotions, attitudes and beliefs. Among them are those that establish who 'has' the knowledge in class, or who regulates the valorisation of various forms of mathematics different from the 'official' one (p. 42).

Pupils and teachers form their conceptions of what is valid knowledge and what activities are acceptable and wanted through years of experience with mathematics in school. When different conceptions meet, this may lead to conflicts, but it can also open up for new insight. Norms are rarely up for explicit negotiation but are often implicitly present. This paper aims to analyse conversations in order to identify different sociomathematical norms with a view to clarifying what blocks and what opens for further learning.

## **THEORETICAL FRAMEWORK**

A sociocultural perspective on learning forms the basis of the study. Knowledge is in this perspective closely associated with practice communities and the individual's capacity/ability to participate in these (Lave & Wenger, 2003; Dysthe, 2001). Through participation in a practice, one learns from practice and about practice. Planas & Gorgorió (2004, p. 21) say that the acquisition of concepts and skills is not essential in the process of becoming a mathematical learner, but they emphasize that the active participation in the reconstruction of a specific kind of discourse is necessary. In school and company, practice communities are formed with different goals for doing mathematics. Where working life has profit and efficiency as goals, the school's goal is for the pupils to learn mathematics. Different norms are developed as to what can be done and what activities related to mathematics are valued in school and in company (Wedeg, 2006). For example, a carpenter in a building company will over time develop a norm of flexibility as to when it is best and most effective to use a visual estimate, and when it is necessary to find exact measurements in any given case. Mathematics teaching that is influenced by the "exercise paradigm" (Mellin-Olsen, 1995), may develop a practice where such practical and flexible considerations are not valued; what is valued, may be to quickly solve and get correct answers to a long row of exercises.

Yackel and Cobb (1996) originally developed the concept of sociomathematical norms. They exemplify the concept as:

.... normative understanding of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant in a classroom are sociomathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm (Yackel & Cobb, 1996, p. 461).

They relate their work closely to the inquiry tradition while at the same time stressing that sociomathematical norms are not isolated to this kind of mathematics teaching; "what counts as an acceptable mathematical explanation and justifications, are established in all classrooms regardless of instructional tradition" (Ibid, p. 462).

The teacher is underpinned as an important contributor to the formation of sociomathematical norms by Yackel and Cobb. They still articulate that normative understandings are continually regenerated and modified by the students and the teacher through their ongoing interactions (Ibid, p. 474). Planas and Gorgorió (2004,

p. 23) argue that “important discursive issues (are) lacking” in Yackel and Cobb’s analyses of norms in the mathematical classroom. In real contexts there will be several possible interpretations. A particular interpretation may be considered as valid or appropriate in a specific context although there is no universal understanding of how this norm should be interpreted (Planas & Gorgorió, 2004, p. 23). Often the teacher takes for granted that real context problems should be about, for example functions, while the pupils may see the problem from a practical realistic perspective where mathematics is not needed (Ibid). Gorgorió, Planas & Vilella’s definition (2001) encompass social norms and mathematical practice in their understanding of sociomathematical norms. McClain & Cobb (2001) characterize social norm “as general norms that are necessary for engaging in classroom discussions” and “include explaining and justifying solutions, attempting to make sense of other explanations given by others, and challenging others’ thinking” (p. 106). I include norms which the pupil carries with him from home and leisure and the norms of classroom mathematical practice that are learned through practice expressed both implicitly and explicitly as social norms. Norms are also connected to relations between participants, who have the right to define what is included and what is excluded, whom you can ask and how one regards one’s own skills in relation to others’ (McClain & Cobb, 2001; Planas & Gorgiό, 2004). Sociomathematical norms have traditionally been strongly linked to verbal mathematical argumentation. In this paper, I choose to apply sociomathematical norms not only to oral mathematical argumentation, but also to norms as to what actions are acceptable as mathematical activity in a context where pupils move between school and company. These norms are negotiated between participants on the background of the participants’ beliefs and values in and about mathematics (McClain & Cobb, 2001).

The focus of this paper is on identifying and investigating sociomathematical norms that pupils meet when they are engaged in realistic mathematics education. Conversations between pupils and between carpenter and pupils are analyzed, and the perspectives discussed above provide a theoretical background.

## METHOD

In the project “Learning Conversation in Mathematics Practice” (LCMP)<sup>1</sup> of which this study is a part, teacher educators / researchers cooperated with teachers in a municipality where an object was to carry out the aim about practical learning in mathematics in cooperation with local enterprises (Hana et al., 2010). Teacher students actively participated in developing the teaching. One of these students wanted to cooperate with an enterprise in her first year as a teacher. On the teacher’s request, I was her conversation partner in the teaching project. The teacher clearly

---

<sup>1</sup> This is part of my ongoing PhD.-study in the research project *Learning Conversation in Mathematics Practice*, leader: Marit Johnsen-Hoines). The study is financed by the Research Council of Norway and Bergen University College.

has the main responsibility for the design of the teaching plan and the implementation of the teaching. My role as a researcher was explained to the pupils; communication with me was allowed, but the teacher was the one responsible and had the authority.

Teacher and researcher met the carpenter in the building company for an initial clarifying conversation. It was made clear by the teacher that it was the practical application of mathematics in the company that was of interest.

My colleague Gert Hana and I followed two groups of pupils with video cameras and sound recorders for seven sessions over a period of five months, one of which was at the company. The two groups were organised by the teacher. The grouping was not based on ability to do mathematics; the only criteria were the participants' ability to participate in conversations and that they had consented to take part. In the conversations presented in this paper the active participants are from one of the groups.

The assignment given to them by the teacher and the building company was to construct 3D models of a *rorbu*, a combined boathouse and seaside cottage, popular as holiday resorts in this island district. Initially, the building company sent the pupils several construction drawings of *rorbu* of different sizes. These were to be taken as a basis for the pupils' own construction drawings and suggestions for possible room plans. The group would take their construction drawing to the company, and discuss their drawing with a carpenter. Back in the classroom, the pupils would realize their construction drawing in a 3D model in scale 1:25.

The excerpts presented in this paper are from a mathematics lesson two days before visiting the building company, when the pupils are working with constructing a drawing of the disposition of the floor space in a *rorbu* and one excerpt from the visit and the consultation with the carpenter. These excerpts are selected because of the contrast in how the pupils and the carpenter worked with scale. The analyses have been inspired by Bakhtin (1981, 1986) in order to study the dialogs and the pupils' learning loop (Johnsen-Høines, 2010). As the study is ongoing, the analysis is still in progress. Possible interpretations have been discussed in the research group that I am part of. Alternate interpretations have to be continually tested against each other; due to limitations of space this cannot be demonstrated in a paper of this format.

## MATHEMATICAL ACTIONS AND NORMS IN SCHOOL AND COMPANY

Building a 3D model of a *rorbu* in cooperation with a building company was seen by the teacher as a possibility for the pupils to learn more about scale.

In the following excerpts, two boys, Daniel and Jonas, and a girl, Hilde, have received three drawings from the building company, originally drawn in scale 1:100 with the actual measures of outer walls marked. The pupils discover that the drawings have been downsized in the photocopier, so that the scale is not correct. In this way, they meet work life mathematics where one has to find the figures one needs and sort

out the information oneself in order to solve problems, the way Wedege (2006) notes that work life mathematics often is like.

The drawings from the company are to be used as basis for the pupils' own work with room planning of a floor in a *rorbu*. The pupils have had relevant discussions in relation to the assignment; interpretation of the drawing from the company, what scale their own sketch should be in, how large the *rorbu* should be, area when the scale is changed, priorities and sizes of different rooms, and what kind of family the *rorbu* will be built for. They choose to double the length of the outer walls compared to the company's original drawing in scale 1:100. None of the pupils mention that the sketch will then be in scale 1:50.

The pupils take a bed length of two meters as a reference in order to find how big the bedrooms should be. Daniel takes two meters, divides it by 100 and multiplies with two to find the bed length in the correct scale. The answer is 0.04 meters, but he omits the measure unit and expresses doubt as to whether 0.04 can be correct. He involves the other pupils in his thinking by saying out loud what he is doing and indicating that he wants support. Daniel then rephrases the statement to a more general expression and puts N instead of the length of the bed. Jonas asks questions and challenges Daniel with 'why'-questions:

Daniel: N divided by a hundred times two ( $(N/100) \times 2$ ). (Points at the sketch they are making).

Jonas: N? Why N?

Hilde says "that is like X", which Jonas seems to accept until he once again questions the numbers Daniel are using. Daniel is forced to explain himself. He chooses two meters as the value of N and goes on:

Daniel: See, if you divide with a hundred it fits with this drawing (points to drawing from company in scale 1:100). And this one is twice as big as this, so then we multiply by two (points to the sketch and then to the construction drawing from the company).

Daniel concludes that it must be 0.04. Hilde reacts critically and indicates how small 0.04 will be in the drawing. It seems she interprets the number as 0.04 cm. Daniel takes the ruler, tries with 4 cm and concludes that it can't be that long, 4 cm "is extreme!" he says. Even though he has reasoned correctly up to this point, he has explained and defended his reasoning, the answer is rejected on the basis of the visual impression; 2 meters in scale 1:50 (4 cm) is more than double the size (around three times the size) of the downscaled drawing from the company, which is marked 1:100.

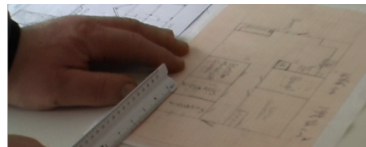
The pupils demonstrate their social norms for doing mathematics in the classroom through their discussions (McClain & Cobb, 2001). When the pupils worked on the outer walls, they knew that 6 m would be 6 cm in a drawing in scale 1:100 and the double of that would be 12 cm. Daniel applies this knowledge and uses algebra, which the class has just been given their first introduction to. When he gives reasons

for his solutions, he moves between different representation forms. He concretizes by pointing to the company's and the group's drawings, he uses the bed length as a basis for calculation and moves between the actual length and the variable  $N$  to express generally how he can find any length in this scale. The pupils are also explicit about their sociomathematical norms (Gorgorió et. al, 2001). They compare themselves with others and state that they use mathematics and discuss in order to solve the problems, as opposed to the neighbouring group, who they claim just draw without applying a single mathematical concept. At the same time, they are pressed for time and discuss whether they should just do something. They all argue in turn against this, like Jonas' argument why they "can't just do something":

Jonas: If we just draw, there will be a room that is one square meter!

A room of one square meter will be much too small when a bed is 2 meters long, is how I interpret his intention in this quote. A risk of "just drawing something" is that they could get much too small rooms without being aware of it. It seems the pupils are outlining two choices; to work mathematically and not finish on time, or to sketch something without knowing if the measures are right, but at least having a product to show the carpenter. They choose to finish a sketch without knowing the size of the rooms. When meeting the carpenter, they therefore have a sketch of a room plan without measures of the rooms, just the measures of the outer walls.

When asked by the researcher how big the smallest bedroom is, the pupils excuse themselves, saying they "have just done something and don't know". The carpenter responds by introducing and demonstrating a tool unknown to the pupils; an architect's scale.



**Fig.1. The carpenter demonstrates the use of an architect's scale.**

Carpenter: Let's see – you have drawn it in 1:50 here (talks while he takes the architect's scale, finds the scale and lengths). In this case it is just 1.50 and that's a bit small. You should have at least about 6 square, but...minimum...probably seven. But then you can have a bunk bed there. The most realistic here would be to divide this into two rooms (points to three small rooms lying side by side).

The pupils here meet a person who analyzes their drawing, who can quickly find how big the rooms are and who has information about how big they should be if there is to be room for one or two beds. He moves easily between using lengths and squares as measurements without use of measure units, just as Daniel did. He uses beds to illustrate sizes for consideration, just like the pupils used the bed. But instead of calculating the measures, he uses an architect's scale. This has at least six different

scales marked and makes it easy to visually demonstrate the relation between scales. One has to know exactly what scale one is working in for the tool to be useful. It won't do to say "the double of 1:100" – one has to know that the scale will then be 1:50.

The carpenter's use of mathematics is related to the goals of his work: efficiency and accuracy. At the same time, as a carpenter he has to relate to the customer's wishes, see possibilities and consider everything in relation to rules and regulations of authorities. His norms are shown in his communication with the pupils both through his solution focusing and his "should"-orientation and through his use of efficient tools. The pupils and the teacher adopt the architect's scale and use it as a tool in further work with scales, something that made the work considerably more effective. It seems that teacher and pupils pick up what they together see that they can make use of in learning mathematics, without necessarily adopting the carpenter's set of norms.

## CLOSING REMARKS

School and work life have different goals for using mathematics. I find evidence of differences between the pupils' norms for how to do practical work in a mathematics context and the carpenter's norms when he uses mathematics in a work life context within the same topic: scale applied to model building. The question is what the pupils' movement between a school culture and a company culture can contribute in relation to mathematics learning and new insight on the pupils' part. The pupils demonstrate some qualities and norms in the communication between themselves that are seen as important within mathematics learning; challenge/question, argue, generalize and apply different forms of representation (Yackel & Cobb, 1996, McClain & Cobb, 2001). It seems the pupils are on the borders of what they are capable of handling, and they lose track when they try to critically assess the result in relation to the practical situation. The carpenter demonstrates an action oriented and solution oriented use of mathematics where tools are used with insight about the scale one works in, and where the figures are continually linked to what is actually measured. By applying an architect's scale, the user is continually given direct feedback on measures in the current scale. At the same time, the use of this tool can hide some of the underlying mathematics. By moving between, the pupils can develop insight into the mathematics and at the same time get a visual understanding of scale. This also provides possibilities for developing a flexibility with regard to choice of tools. Meeting different norms for how to work with mathematics can enable the pupils to develop a critical competence where they can assess school mathematics in relation to work life's application, and perhaps also build a preparedness to be critical to applications of mathematics in society,

## References

- Bakhtin, M. (1986). *Speech genres and other late essays*. Austin: University of Minnesota press.
- Bakhtin, M. (Ed.). (1981). *The dialogic imagination*. Austin: University of Texas Press.
- Dysthe, O. (2001). Sosiokulturelle teoriperspektiv på kunnskap og læring. In O. Dysthe (red.), *Dialog, samspel og læring*. Oslo: Abstrakt.
- Gorgorió, N., Planas, N., & Vilella, X. (2001). Immigrant children learning mathematics in mainstream schools. In G. de Abreu, A. J. Bishop, & N. C. Presmeg (Eds.), *Transitions Between Contexts of Mathematical Practices*. Hingham: Kluwer Academic.
- Gorgorió, N., & Planas, N. (2005). Social representations as mediators of mathematical learning in multiethnic classrooms. *European Journal of Psychology of education*, XX (1), 91-104.
- Hana, G. M., Hansen, R., Johnsen-Høines, M., Lilland, I. E., & Rangnes, T. E. (2010). Learning Conversation in Mathematics Practice – school-industry partnerships as arena for teacher education. In *Educational Interfaces between Mathematics and Industry* (p. 281–290). ICMI/ICIAM study, Lisbon, Portugal.
- Johnsen-Høines, M. (2010). Interpretative Research as Collaborative Inquiry. In B. Sriraman, C. Bergsten, S. Goodchild, G. Palsdottir, B. D. Søndergaard, & L. Haapasalo (Eds.), *The First Sourcebook on Nordic Research in Mathematics Education*. Charlotte NC: Information Age Publishing.
- Lave, J., & Wenger, E. (2003). *Situeret læring – og andre tekster*. København: Reitzel.
- McClain, K., & Cobb, P. (2001). Supporting students' ability to reason about data. *Educational Studies in Mathematics*, 45, 103–129.
- Mellin-Olsen, S. (1996). Oppgavediskursen. *Tangenten* 7 (2), 9–15. Caspar.
- Planas, N., & Gorgorió, N. (2004). Are Different Students Expected to Learn Norms Differently in the Mathematics Classroom? *Mathematics Education Research Journal*, 16(1), 19–40.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education* 27(4), 458–477.
- Wedeg, T. (2006). Menneskers matematikholdige kompetencer. In O. Skovsmose & M. Blomhøj (Eds.), *Kunne det tænkes? – om matematiklæring*. Albertslund: Malling Beck A/S and authors.



# INTERPRETING SCIENTIFIC EVIDENCE: PRIMARY STUDENT'S UNDERSTANDING OF BASE RATES, SAMPLING PROCEDURES, AND CONTINGENCY TABLES

Kristina Reiss<sup>1</sup>, Petra Barchfeld<sup>2</sup>, Anke M. Lindmeier<sup>1</sup>, Beate Sodan<sup>2</sup>, Stefan Ufer<sup>3</sup>

<sup>1</sup>TU München, Heinz Nixdorf Chair of Mathematics Education

<sup>2</sup>LMU München, Developmental Psychology

<sup>3</sup>IPN Kiel, Mathematics Education

*Research results from mathematics education and developmental psychology suggest that children's building up of scientific understanding may be characterized as a discontinuous path. Severe deficits as well as early cognitive competencies add up to a complex pattern. In particular, probability judgments and conditional reasoning may presuppose sophisticated mathematical understanding but may as well be accessible in a specific everyday context. This duality was addressed in a study with 2<sup>nd</sup>-, 4<sup>th</sup>- and 6<sup>th</sup>-grade students. It aimed at describing the development of cognitive prerequisites for a successful handling of scientific evidence. The results presented here are based on standardized individual interviews focusing on the understanding of stochastic concepts. In a mathematics learning environment as well as in an everyday context, children were asked to solve problems using base rates, sampling procedures, and contingency tables. Their answers showed similar deficits and competencies in both environments thus giving evidence that the context did not play a significant role. More importantly, there was only a minor increase in competencies during primary education independent from the specific context. This result may indicate a preference for deductive thinking in the early grades and might initiate a discussion how stochastic thinking can be supported in the early mathematics classroom.*

## INTRODUCTION

### Theoretical background and research questions

Active participation in modern societies requires the ability to understand scientific information and to discern scientific argumentation. This ability includes domain specific knowledge as well as a basic comprehension of general research methodologies and an epistemological understanding of the interrelation between theory and evidence. This understanding should encompass the ability to evaluate specific situations from different points of view and to deal with conflicting information. It is plausible that these abilities have a stronger relation to stochastics than to other fields of mathematics in the classroom.

There is empirical evidence that young children are able to evaluate data and to draw adequate conclusions. Ruffman, Perner, Olso, and Doherty (1993) as well as Koerber, Sodan, Thoerner, and Nett (2005) have found that pre-school children could

correctly interpret simple covariance patterns and have recognized their specific influence, for example, on causal hypotheses. Research on the evaluation of contingency tables revealed an inadequate strategy use even for primary and secondary school students (e.g., Shaklee, Holt, Elek & Hall, 1988). It is unclear whether these deficits are based on an insufficient understanding of the qualitative or quantitative structure and how the evaluation of base rates might contribute to this knowledge.

The research described above was performed in the context of developmental psychology and took advantage of children's everyday life and their general knowledge. Moreover, there were some studies, which analyzed these competencies within a mathematics context. These studies could also indicate that an adequate handling of data was in principle possible for primary school children (e.g., Fischbein & Schnarch, 1997; Green, 1998; Martignon & Wassmer, 2005). A competency model by Reiss and Winkelmann (2009) based on solutions of such items by more than 1000 4<sup>th</sup>-graders supported these results.

This parallelism is not self-evident. In particular, it is widely acknowledged that a specific context might matter and show diverse results. The role of a specific context has been addressed in a number of studies, but their results are inconsistent. Wason and Johnson-Laird (1972) confronted children with the *Wason selection task*, a logic problem. They reported that the solution rates of this task increased, when children found everyday situations on their cards instead of information in form of numbers and letters. The realistic context turned out to be helpful for solving the task. Other research suggests a realistic context may impede a task. Chin and Malhotra (2002) as well as Schauble (1996) found that children relied on their declarative context knowledge even if it contradicted empirical data. Accordingly, adding a context may mean enhancing cognitive load but may as well mean to foster understanding. Moreover, a context asks for specific knowledge and presupposes verbal abilities.

The ideas presented in the preceding paragraphs guided an interdisciplinary research project between mathematics education and developmental psychology. It aimed at identifying children's notions of base rates, sampling procedures, and contingency tables and at describing their competencies with respect to a specific presentation in a mathematics context or in an everyday context. The following research questions were addressed:

- How do primary school children understand the concepts of base rate and sampling when evaluating scientific evidence?
- Are they able to apply their understanding of base rates for the evaluation of data presented in contingency tables?
- Are there differences in students' understanding when problems are presented in a formal or a content-oriented everyday context?
- Which developmental steps can be identified during primary-school years?

These research questions and their answers are particularly interesting in view of standards for school mathematics, which have been implemented in many countries (e.g., National Council of Teachers of Mathematics, 2000; Kultusministerkonferenz, 2004). The standards regard data analysis and probability as a topic for all grades and also for the primary-school mathematics classroom. It is thus relevant to learn more about children's ideas of data and their working with data.

## **DESIGN OF THE STUDY**

### **Methods**

Children's understanding of base rates and sampling procedures was evaluated in a series of tasks presented in an interview situation. First, students were confronted with data from an experiment without base rate information. They had to judge, if the data was suited to verify the hypothesis that should be tested by the experiment. Second, children were given examples of data attained from a singular case, from a small sample, and from a large sample. Depending on the specific problem, they were asked whether conclusions from these data were useful or not. Moreover, they had to draw conclusions from similar data and to judge the correctness of their statements afterwards. The interviewers asked the children to present their ideas first but interfered with specific questions when the children did not solve a problem correctly ("prompting"). All problems aiming at the understanding of base rates and the sampling procedure were introduced in an everyday context (see Shtulman & Carey, 2007).

In order to get to know their understanding of contingency tables, students were presented problems in two different contexts and were asked to evaluate them. In an everyday and content-oriented context, the problem was situated in a story about the growth of plants and the application of specific fertilizers. A blue and a yellow fertilizer had to be judged with respect to their effect on a healthy growth of trees (see Figure 1 for an example "tree growing well" vs. "growth discontinued" and Figure 2 for another example "flower growing well" vs. "growth discontinued"). In the more formal mathematics context, the children were asked to make their choice between cubes and balls, which could be red or blue (Figure 1). The different shapes make the objects distinguishable by touch. The children were told that some other child had taken an object from a bag without looking into the bag, had put it back after marking its shape and color in a table, and had repeated this experiment forty times. The children were asked to evaluate the chance of getting a specific object when drawing another object from this bag. Figures 1 and 2 show examples with different numbers of cubes and balls in different colors. The children were always supposed to take a blue object from the bag and therefore were asked a question like "Imagine I need to get a blue object. Should I better draw a cube or a ball from the bag?" Afterwards, the interviewer let them judge how certain they were about their ratings and encouraged them to verbalize their answers.

The experiments were implemented as imagined activities (“Imagine ...”), however, red and blue objects, namely cubes and balls as well as a bag were available when the children were presented the tasks and were used for an introduction into the type of problems.

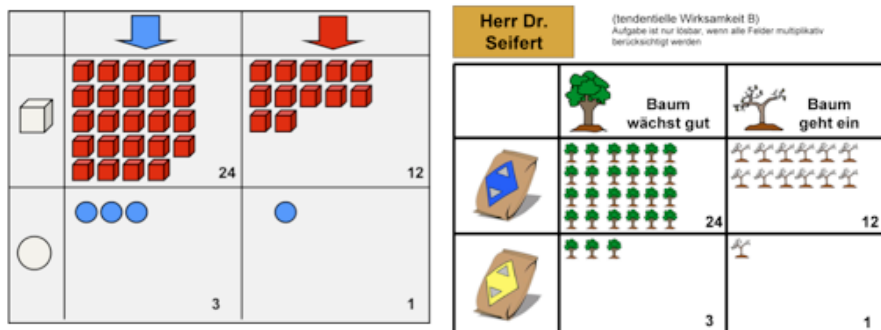


Figure 1: Sample task material in a content-oriented and a formal context

Experiments in both contexts encompassed impossible, sure, and improbable events as well as nearly equiprobable events (see Figures 1 and 2). In all experiments, children were encouraged to give reasons for their choices.

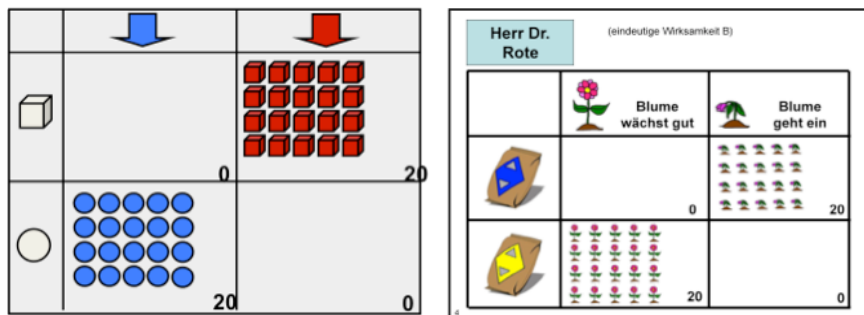


Figure 2: Sure events in a content-oriented and a formal context

## Sample

The sample comprised 158 primary school children (90 male, 68 female; 52 children from grade 2, 53 children from grade 4, 53 children from grade 6). The children were asked to solve problems as described above in individual interview situations. Each interview encompassed 19 problems and lasted for approximately 40 minutes. Children were interviewed in a school setting outside their classrooms by trained interviewers. Videotapes of the interviews allowed for differentiated coding and an in-depth analysis of the children’s arguments after transcription.

## RESULTS

### Understanding of base rates and sampling procedure

The data suggest that many 2<sup>nd</sup>-graders have a rough understanding of base rates. Nearly 30% showed a spontaneous grasp of this concept, nearly 70% showed their understanding after a specific prompting. There was a clear growth between 2<sup>nd</sup> and 6<sup>th</sup> grade, since 46% of the 4<sup>th</sup>-graders and 55% of the 6<sup>th</sup>-grade students showed spontaneous understanding and 90% in grade 4 and 97% in grade 6 showed it after prompting. The differences between grades 2 and 6 were significant, all other differences failed to be significant ( $\chi^2(2;N=105) = 7.06$ ,  $p=.03$  between grades 2 and 4,  $\chi^2(2;N=106) = 2.30$ ,  $p=.32$  between grades 4 and 6; tests were conducted using Bonferroni adjusted alpha levels of .025 per test). Moreover, primary school children understood the sampling procedure in principle. Between 10% and 15% of the students through all grades showed a good comprehension and between 50% and 80% had at least a basic knowledge about the sampling procedure. The difference between 2<sup>nd</sup>-graders and 4<sup>th</sup>-graders was significant ( $\chi^2(3;N=105) = 11.10$ ,  $p=.01$ ), however, the difference between 4<sup>th</sup>-graders and 6<sup>th</sup>-graders was not significant ( $\chi^2(3;N=106) = 1.11$ ,  $p=.78$ ; tests were conducted using Bonferroni adjusted alpha levels of .025 per test).

### Analysis of contingency tables

The analysis of contingency tables seemed to be difficult for all students. It is not surprising that these tasks were hardly solved by the 2<sup>nd</sup>-graders, though, also 6<sup>th</sup>-graders showed severe difficulties. Table 1 provides the solution rates for all items in the formal and in the everyday context. Tasks with sure ( $p = 1$ ) or impossible ( $p = 0$ ) results are marked with a grey background. The children were presented eight problems of the type described above in a mathematical context and four problems in an everyday context condition. There were varying probabilities for drawing a blue cube or a blue ball (see Table 1 for the details).

The results show that all children had difficulties to judge the contingency tables correctly. Out of the eight items presented, children in grade 2 successfully finished  $M = 3.31$  ( $SD = 1.39$ ), children in grade 4 successfully finished  $M = 3.66$  ( $SD = 1.81$ ), and children in grade 6 successfully finished  $M = 4.25$  ( $SD = 1.90$ ). Children succeeded primarily when solving items, which did not necessarily presuppose the comparison of all four fields for their solution (see, e.g., the items on Figure 2). The differences between grades 2 and 4 ( $p = .56$ ) as well as between grades 4 and 6 ( $p = .19$ ) were not significant, however the difference between grades 2 and 6 was significant ( $p = .02$ ). These results are based on a univariate one-factorial ANOVA with  $F(2,155) = 4.002$ ,  $p = .02$ ,  $\eta^2 = .05$ .

There was a similar pattern if only those four items were analyzed for which parallel items in a mathematics context and in an everyday context were available. Results for the everyday context have a mean of  $M = 1.31$  ( $SD = 0.67$ ) correctly finished items in grade 2, a mean of  $M = 1.49$  ( $SD = 0.85$ ) correctly finished items in grade 4, and a

mean of  $M = 2.00$  ( $SD = 1.00$ ) correctly finished items in grade 6. There are differences between the grades (univariate one-factorial ANOVA  $F(2,155) = 9.35$ ,  $p < .01$ ,  $\eta^2 = .11$ ), which are not significant for grades 2 and 4 ( $p = .52$ ), but are significant for grades 4 and 6 ( $p = .01$ ) and grades 2 and 6 ( $p < .01$ ).

Item	P (ball/cube)		Solution rate (SD) formal context			Solution rate (SD) everyday context		
			Grade 2	Grade 4	Grade 6	Grade 2	Grade 4	Grade 6
1*	1	0	0.62 (0.49)	0.58 (0.50)	0.58 (0.50)	1.00 (0.00)	0.96 (0.19)	0.98 (0.14)
2	0	0.5	0.79 (0.41)	0.75 (0.43)	0.83 (0.38)			
3	0.36	0.67	0.63 (0.49)	0.75 (0.43)	0.87 (0.34)			
4*	0.75	0.67	0.06 (0.24)	0.17 (0.38)	0.21 (0.41)	0.04 (0.19)	0.13 (0.34)	0.21 (0.41)
5	1	0.5	0.85 (0.36)	0.81 (0.39)	0.87 (0.34)			
6*	0.41	0.46	0.15 (0.36)	0.28 (0.45)	0.43 (0.50)	0.15 (0.36)	0.40 (0.49)	0.60 (0.49)
7*	0.68	0.67	0.13 (0.34)	0.19 (0.39)	0.26 (0.45)	0.04 (0.19)	0.09 (0.30)	0.15 (0.36)
8	0.63	0.69	0.08 (0.27)	0.11 (0.32)	0.19 (0.39)			

Table 1: Item characteristics, solution rates, and standard deviation for contingency tables in both contexts (\* parallelized items)

### Context characteristics

The data give evidence that most items were difficult for the children, likewise in both contexts. For items 1, 4, 6, and 7 parallelized versions were available for the formal and the everyday context. An important difference was identified for task 1 (see Figure 2 for this task). It offered a sure and an impossible alternative, which was obviously easier to solve in the everyday than in the formal context. Item 4, 6, and 7 did not prove to be more difficult in a specific presentation but showed only slight differences. Accordingly, the results do not indicate that the context mattered systematically.

### Development

The data from the different grades revealed a rather slow development with respect to a successful handling of contingency tables. Children understand the concept of base rate and are able to conceptualize a sampling procedure but they do not necessarily apply their knowledge while evaluating contingency tables. This statement is true for all grades, even for grade 6. There is a bottom effect for a number of items in all

grades (see items 4, 7, 8; Table 1). It becomes evident from the results, that 6<sup>th</sup>-graders perform somehow better than 4<sup>th</sup>-graders and 4<sup>th</sup>-graders perform better than 2<sup>nd</sup>-graders, but the differences are small and not all these differences are significant.

## **DISCUSSION**

The results indicate that children have a basic understanding of what base rates mean and are able to apply this concept in a simple everyday context. The increasing number of correct solutions from grade 2 to grades 4 and 6 substantiates that children learn to even better cope with such tasks. In particular, between 2<sup>nd</sup> and 6<sup>th</sup> grade, students show a significant gain in their competence. In particular, children recognized that single cases allow limited conclusions and data from large samples is in general more reliable. Moreover, it was interesting that high solution rates did even increase when children were prompted to think about their answers and to give arguments for their solutions. It is plausible that their answers can be seen as related to primary intuitions as Fischbein (1975) called it.

The study gave also evidence that analyzing contingency tables remained a difficult task for primary school children. Their strategies were mostly inadequate for the problems and revealed a lack of understanding. In particular, primarily those items were correctly solved which included a sure or an impossible event. Accordingly, our results support Wollring's (2007) hypothesis that young children tend to apply strategies, which focus on differences between data, but fail to take their ratio into account. Moreover, our research suggests that this strategy use is independent of an everyday or a formal context. In both contexts, children showed similar difficulties and solution rates. These results have some importance for learning and instruction at school. They reveal a gap between children's ways of understanding and the demands of school standards, which should be thoroughly analyzed from a mathematics education point of view.

However, it can be regarded as an important result that even second graders showed in principle an understanding of the necessity of base rate information. Thus, it appears that elementary school children can follow the general logic of contingency table analysis with some help, but that they have difficulty in integrating and analyzing the numerical information. Further research is necessary to explore whether analysis of contingency tables is possible, in principle, under reduced task demands, for instance working memory demands, and demands involved in number representation. Moreover, it might be interesting to take preliminary abilities for the analysis of contingency tables into focus. It is plausible that an appropriate understanding of base rates and sampling procedures might be a basis for explorations or the evaluation of different strategies. For mathematics education, it is important that further research better takes into account what actually happens in primary classrooms. Analyzing data and probability phenomena is in some countries a rather new topic and differences between classrooms and teachers might be meaningful. The interrelations between these components should be addressed in research studies.

## References

- Chin, C. A., & Malhotra, B. A. (2002). Epistemologically authentic inquiry in schools: A theoretical framework for evaluating in inquiry tasks. *Science Education*, 86, 175–218.
- Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht: Reidel.
- Fischbein, E. & Schnarch, D. (1997). The evolution with age of probabilistic intuitively based misconceptions. *Journal for Research in Mathematics Education*, 28, 96–105.
- Green, D. R. (1982). *Probability concepts in 11–16 year old pupils*. University of Technology, Loughborough, England Report of research, CAMET.
- Koerber, S., Sodian, B., Thoermer, C., & Nett, U. (2005). Scientific reasoning in young children. Preschoolers' ability to evaluate covariation evidence. *Swiss Journal of Psychology*, 64, 141–152.
- Kultusministerkonferenz (2004). Bildungsstandards im Fach Mathematik für die Jahrgangsstufe 4 (Primarstufe). Bonn: KMK.
- Martignon, L. & Wassner, C. (2005). Schulung frühen stochastischen Denkens von Kindern. *Zeitschrift für Erziehungswissenschaften*, 8, 202–222.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Reiss, K., & Winkelmann, H. (2009). Kompetenzstufenmodelle für das Fach Mathematik im Primarbereich. In D. Granzer, O. Köller, A. Bremerich-Vos, M. van den Heuvel-Panhuizen, K. Reiss & G. Walther (Hrsg.), *Bildungsstandards Deutsch und Mathematik. Leistungsmessung in der Grundschule* (S. 120–141). Weinheim: Beltz.
- Ruffman, T., Perner, J., Olson, D.R., & Doherty, M. (1993). Reflecting on scientific thinking: Children's understanding of the hypothesis-evidence relation. *Child Development*, 64, 1617–1636.
- Schauble, L. (1996). The development of scientific reasoning in knowledge-rich contexts. *Developmental Psychology*, 32(1), 102–119.
- Shaklee, H., Holt, P., Elek, S., and Hall, L. (1988). Covariation judgment: Improving rule use among children, adolescents, and adults. *Child Development*, 59, 755–768.
- Shtulman, A. & Carey, S. (2007). Improbable or impossible? How children reason about the possibility of extraordinary events. *Child Development*, 78, 1015 – 1032.
- Wason, P. C., & Johnson-Laird, P. N. (1972). *Psychology of reasoning: Structure and content*. Cambridge: Harvard University Press.
- Wollring, B. (2007). Den Zufall festhalten – Spielräume und Dokumente bei Zufallsexperimenten für die Grundschule. Lernumgebungen und Versuchsumgebungen zur Stochastik. *Beiträge zum Mathematikunterricht* (S. 472–475). <http://www.mathematik.uni-dortmund.de/ieem/BzMU/BzMU2007/GDMGesamt2007.pdf>



# DISCUSSING A TEACHER MKT AND ITS ROLE ON TEACHER PRACTICE WHEN EXPLORING DATA ANALYSIS

C. Miguel Ribeiro<sup>1</sup>, José Carrillo<sup>2</sup>

Research Centre for Spatial and Organizational Dynamics (CIEO), University of Algarve (Portugal)<sup>1</sup>, University of Huelva (Spain)<sup>2</sup>

*This article considers teacher knowledge in managing mathematically critical situations and the role of what can be termed a mathematical summary in the analysis of a teaching episode, viewed from the perspective of Mathematical Knowledge for Teaching (MKT) (Ball, Thames & Phelps, 2008). The analysis is based on an episode of content review and, from a perspective which aims to understand the teacher's logic (rather than merely identify gaps in their knowledge). We discuss the importance of approaching mathematically critical situations in order to contribute to eradicating mathematical innumeracy (statistics) and to promote a kind of practice which is "mathematically demanding" as well as "pedagogically exciting".*

## INTRODUCTION/MOTIVATION

Only in recent years in Portugal has greater attention been given to the contents which make up the topic of Data Analysis. This increased attention can be seen in the explicit inclusion of the topic in the new 'Programa do Ensino Básico' (Basic Teaching Syllabus) (Ponte et al., 2007). The chief goal in teaching this topic is specified as "developing students' ability to read and interpret data presented in tabular or graphical form, and enabling them to collect, organise and represent data so as to find solutions to problems in various contexts relating to their daily lives" (p. 26). The inclusion of this topic, coupled with little (if any) training on the part of the teachers in this field, has unsurprisingly led to varied degrees of success in dealing with the topic in the classroom.

The treatment it has received is directly related to the teachers' mathematical knowledge for teaching and the way they put this into operation – statically or dynamically. In this paper we conceptualise such mathematical knowledge following the systematisation of the research group led by Ball (e.g., Ball, Thames and Phelps (2008) and Hill, Rowan and Ball (2005)), in particular their conceptualisation of Mathematical Knowledge for Teaching (MKT). Within this framework, it is essential that teachers are possessed of a full, sound knowledge of the content they intend to impart if they are to ensure a corresponding comprehension on the part of their students.

This paper analyses a sample of actual classroom performance by an experienced teacher and considers the role played by MKT with respect to the opportunities made available (or not) to the students for developing their knowledge. As a result of the

analysis, we hope to gain a better understanding of how this knowledge shapes their teaching and how its deployment influences the possible student outputs, with a view to considering the implications for teacher training.

## **MKT AND MATHEMATICAL SUMMARY**

In the last few decades there have arisen various conceptualisations and ways of addressing the professional knowledge of mathematics teachers. Essentially, these originate in the three categories identified by Shulman (1986) focussing explicitly on content knowledge (subject matter knowledge, pedagogical content knowledge, and curricular knowledge). From among the various approaches to mathematics teachers' knowledge (necessary and sufficient to teach mathematics) that have emerged in recent years (some focussing more on issues relating to content, others on pedagogical questions), we opted for that of MKT and its various sub-domains put forward by Deborah Ball and associates. The selection of this conceptualisation over others derived from the nature of our aim, which was to identify, from observed practice, what knowledge the teachers were deploying at each specific moment, and consequently the system for making this identification played a key role. Also advantageous was the fact that MKT embraces a focus of knowledge in action. We especially wanted to explore the kind of mathematical knowledge that teachers require to fully tackle every aspect of each topic, and to ensure learning takes place.

The model developed by Ball and associates also provides a more specific classification, dividing Content Knowledge and Pedagogical Content Knowledge each into three sub-domains. The former is comprised by Common Content Knowledge (CCK), that is, typical 'schoolchild' maths, Specialised Content Knowledge (SCK) and Horizon Content Knowledge. The latter is formed by Knowledge of Content and Teaching (KCT), Knowledge of Content and Student, and Knowledge of Content and Curriculum (KCC).

The teacher should understand how the various mathematical areas relate to one another and how the requirements of any particular topic develop as students' progress up the school (HCK). Further, it is insufficient for the teacher to have knowledge of solely 'how to do', they equally need to know 'how to make understandable' (SCK). In other words, content knowledge needs to be complemented by an understanding of how to make said content accessible to students, and this includes knowing where and why students might encounter difficulties. In the case of Data Analysis, an example of CCK might be the knowledge concerning how to draw a pictogram incorporating a set of data, that there are impossible random generalisations or that it's only possible to infer something when data comes from a representative sample of the population. With respect to SCK, on the other hand, the teacher has also a responsibility to understand the role of each variable in the pictogram so as to be able to teach the students to successfully construct their own. Amongst other things, SCK includes – in this instance – the knowledge on the effect of changing the scale employed in the pictogram, and the

question of representativeness by which a sample approximates to the total population and how this affects the strength of inferences. They need also a knowledge related with proportionality, in order to know (be able to explain to pupils) the why of the characteristics of the sample in order to allow generalisations.

In addition to knowledge of content, teachers should also have a thorough knowledge of the curriculum and pedagogy. Knowledge of Content and Teaching (KCT) corresponds to the type of knowledge which the teacher draws on in order to organise the different ways the students explore mathematical contents, such as determining the sequencing of tasks, choosing examples, and selecting the most appropriate representations for each situation. Regarding Knowledge of Content and Students (KCS), Ball et al. (2008) relates it to the need for the teacher to anticipate what the students are likely to think, their difficulties and motivations as well as listening to and interpreting their comments. The teacher must be aware of the students' capacity to understand in such a way that it could allow him/her to go further in deepening the students' knowledge. With respect to Knowledge of Content and Curriculum (KCC), the authors agree entirely with Shulman (1986, p. 10) that teachers should have a complete picture of the diversity of programs for teaching certain subjects and topics at a particular level/year group, and a variety of educational materials they can draw on. They should also be able to recognise the varying circumstances which suggest the adoption of one approach over another. In general terms, their curricular knowledge should be what can be termed both vertical and horizontal in its scope.

This knowledge, or its lack, has a direct influence on practice, and the use of different factors which can be included in its analysis, the richer the analysis. One aspect which can be revealing is what can be termed the mathematical summary. This summary can be explored through, amongst others, the components considered by Watson (2007) for analysis of practice. These components can focus on aspects relating to how the teacher regards their role – which also exteriorises the MKT which they have or believe they have – using the enunciation of the task of teaching (Ball et al., 2008; Thames, 2009) unfolding at each moment, noting the role of the teacher and students during the course of the lesson and the type of interactions which take place (and which can be expressed through dialogue, writing, and different forms of mathematical representation). Combining all these theoretical elements allows us to explore/focus on questions of mathematical content in the classroom and ways to approach such content, leaving aside other aspects such as management and behaviour.

## **CONTEXT AND METHOD**

This paper draws on data collected within the scope of a broader research project concerned with the professional development of teachers from the point of view of various facets of their professional knowledge. Here we look at the MKT of a primary teacher, Maria, with 18 years experience. It takes an instrumental case study approach (Stake, 2005) combined with a qualitative methodology, and includes

consideration of the summary to an episode involving the review of a topic of Data Analysis in year 4.

In the situation under discussion here, the teacher explicitly aims to review what she considers an inference from the data presented in a pictogram. From the analysis of this episode, we seek to deepen our understanding of the phenomena under consideration and to arrive at some kind of theoretical construct that can amplify our knowledge of practice, the factors which influence it, and how they influence it (with a view to also considering potential perspectives for improvement).

Data collection took the form of audio and video recordings of lessons, with the focus on the teacher. The audio recordings were transcribed and complemented with video viewings, which enabled a fuller record of the teacher-student interactions to be made. Informal conversations were also conducted before and after each class, corresponding to the lesson image and an initial *in situ* self-analysis respectively. The transcriptions were then divided into episodes (Ribeiro, Monteiro & Carrillo, 2009), each associated with the teacher’s immediate goals, and the MKT, along with the mathematical summary behind each, were analysed.

**ANALYSIS AND DISCUSSION OF THE ROLE OF MKT IN THE MATHEMATICAL SUMMARY OF PRACTICE**

At an earlier point, Maria had constructed a pictogram with the aid of the students, using smiley faces to represent the preferences of the 12 class members with respect to visiting one of the Continents. The episode began with the teacher asking the students how the distribution would be affected were the number of students involved four times that of those present. After a while, one of the students went to the board to explain how, in his opinion, the distribution would turn out, indicating the number of smiley faces which would need to be added to each choice. The teacher and students then gave their confirmation that quadrupling the number of faces was correct. Other students then went to the board to show how they would solve the problem and the sequence repeated itself. This coming and going occupied the larger part of the time given over to mathematics.

Below is an extract from the transcription of this episode.

Line		Transcription
1409	T	How do we distribute the quadruple of these ones here?
1410		I'll sk someone who hasn't said anything, Ana, how would you do the
1411		distribution of the forty-eight?
...	...	...
1521	T	Seven! Let's be careful, Tiago. Make sure your partner is
1522		doing the distribution of the quadruple correctly.
1523		Have you placed any in America?
1524	S	No.
1525	T	Were there seven?
1526	S	Yes.
...	...	...

1564 T Keep calm! Is the distribution she did for Oceania correct?  
 1565 (T indicates the number originally written for Oceania on the board)  
 1566 Twelve?  
 1567 S It is!  
 1568 T Why is it?  
 1569 S Three times four is twelve.  
 1570 T Three times four is twelve.  
 1571 (T indicates the number originally written for Oceania on the board)  
 1572 Everything OK then! It was the quadruple, three times four is twelve.

Figure 1 – Extract from transcription of an episode in which the teacher's (declared) aim is to draw an inference from the data represented in a pictogram.

The joint analysis of the mathematical summary and MKT enables us to study the teacher's actions and what is emphasised during her teaching (the mathematical, or other, focus) as well as possible training needs in Data Analysis. It thus allows a detailed analysis of practice and the role of MKT in this teacher's practice.

With respect to the mathematical summary, this episode can be described as follows: teacher elicits facts (what is the quadruple of 12?); students find solution using procedure; students "find solution" without knowing procedure (solution based on opinion); teacher asks for definition (how to calculate the quadruple); teacher indicates the identification of relationships (between the actual and "required" number of smiley faces); teacher provides an explanation (asking students to locate the error, which is to be found in the count of 47 instead 48); teacher requests student verbalisation (explaining what they did in their own words); teacher requests definition (multiplication); teacher provides summary of lesson. Although this sequence forms part of an episode in which the teacher's objective is to practise reading a pictogram and to derive from this 'some sort of inference' (as stated in the interview before the lesson), the mathematical summary of the episode illustrates that the mathematical content is confined to counting and how to quadruple a given number.

Looking at Maria's practice from the perspective of MKT, on the other hand, a certain (in)numeracy is in evidence, which will certainly lead to an incomplete understanding of Data Analysis on the part of the students. Some of this lack of knowledge is evident throughout the episode, while others are associated with specific moments (here referred to by the corresponding transcription line). With respect to the task, and in terms of what can be considered "pure" mathematical knowledge, Maria shows she knows how to calculate the quadruple of twelve and to interpret data represented in a pictogram. But in this respect, she reveals a certain innumeracy when she seeks to make inferences for another population (line 1409), in that she assumes that the inference can be made using direct proportionality. A lack of SCK can also be perceived, in conjunction with the CCK, whereby she appears ignorant that a sample should embody certain characteristics for generalisations to be made from it.

Regarding Pedagogical Content Knowledge, and specifically KCT, Maria considers it important that all the students verbalise their thoughts and ideas, attaching great importance to the students' ability to voice their opinion (even if this is based purely on their preferences). In terms of the knowledge that can be considered to fall within KCS, she displays another lacuna in the instructions she offers the students, which will lead them to the idea that, if they want to make any inference about the quadruple number of students, then they should multiply each value in the pictogram by four, or randomly add three quarters of the forty-eight students, as the remainder are already to be found in the pictogram. Evidently, this lack of knowledge in terms of KCS relates intrinsically to those relating to CCK and SCK, and, depending on the analytical focus (teacher's content knowledge (explanation) or students' understanding), this lack of knowledge can be considered in any of these sub-domains. This aspect illustrates why the sub-domains cannot be seen as hermetically intact parts (the whole is greater than the sum of the parts), and highlights the complexity of the teaching process (and consequently teacher training).

Such instances of (in)numeracy are mirrored in the quality of the students' learning. Nevertheless, they are unaware of this, as the teacher believes she is offering challenging tasks that will take their knowledge to a higher level. (This knowledge is not realised as the premises on which it is based are for the most part false.)

## FINAL NOTES AND IMPLICATIONS

The lack of knowledge in terms of MKT results in a mathematically limited exploration (directly or indirectly) of the prepared tasks, similarly as referred by Charalambous (2008), in each instance. Maria focuses her classroom performance on obtaining quick answers to direct questions (Tomás Ferreira, 2005), prioritising, as the gaps in MKT show, objectives which might be "*pedagogically exciting*" but which are not always "*mathematically demanding*".

The treatment Maria gives the topic reveals in itself how she approaches Data Analysis, and, in the context of aiming for mathematically competent students (with a good degree of statistical literacy), the need for further training in this area is clear, as is the need for further studies into such mathematically critical situations. The identification of these critical situations, along with discussion of their associated mathematical summaries, aim to contribute to obtain a broader knowledge and understanding of such gaps in knowledge and the logic teacher applies. Aiming to understand especially the reasons behind such gaps, and not only to identify areas where knowledge is lacking, so that training can be improved. Fuller knowledge of these areas, and the situations in which they arise in the classroom, would (one hopes) lead to a re-structuring of training programs to provide a specific focus on them, and also to teachers becoming more active and reflective professionals, better informed of their own MKT (Ribeiro & Carrillo, 2011), and so in a position to improve their practice. This implies, thus, a teacher's deeper understanding of the

mathematical knowledge to teach it well (Ball, Lubienski & Mewborn, 2001), and for that, a more focused attention from teachers trainers' to these aspects.

In our view such training is most effective when it is based on reflection on actual practice (the teacher's own or that of others) and on such mathematically critical situations as have been identified, so that teachers genuinely feel the situations are their own and take ownership of the ensuing discussion (Tichá & Hošpesová, 2006). This awareness, drawing on difficult situations faced by others (and even oneself), through the use of video recordings (Maher, 2008; Sherin & Hans, 2004) and/or students productions (Kazemi & Franke, 2004) and subsequent discussion, may promote reflections about their critical features and lead to an improvement in teachers' MKT and a more decidedly mathematical focus in the discourse on their practice. Such awareness (and overcoming of knowledge gaps) will promote (Ribeiro, 2011) the preparation and implementation of richer mathematical tasks and reduce teachers' fear of being asked "why", increasing their confidence to respond in a way that is both mathematically correct and understandable for the pupils.

Both qualified and trainee teachers could benefit from this system of analysis, not only through the identification of the critical situations but also in terms of bridging theory and practice, and promoting a dialogue based on a common language and a shared understanding.

### Acknowledgements:

This paper has been partially supported by the Portuguese Foundation for Science and Technology (FCT).

This paper forms part of the research project "*Mathematical knowledge for teaching with respect to problem solving and reasoning*" (EDU2009-09789), General Directorate for Research and Management of National Plan I + D + i. Ministry of Science and Innovation of Spain.

### REFERENCES

- Ball, D. L., Lubienski, S. & Mewborn, D. (2001). Research on teaching mathematics: the unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of Research on Teaching* (4<sup>th</sup> ed., pp. 433-456). New York: Macmillan.
- Ball, D. L., Thames, M. H. & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Charalambous, C. Y. (2008). Mathematical knowledge for teaching and the unfolding of tasks in mathematics lessons: Integrating two lines of research. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.), *Proceedings of the 32<sup>nd</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 281-288). Morelia, Mexico: PME.
- Hill, H. C., Rowan, B. & Ball, D. L. (2005). Effects of teachers' mathematics knowledge for teaching on student achievement. *American Education Research Journal*, 42(2), 371-406.



- Kazemi, E. & Franke, M. L. (2004). Teachers learning in mathematics: using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, 7, 203-235
- Maher, C. (2008). Video recordings as pedagogical tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *International Handbook of Mathematics Teacher Education* (Vol. 2, pp. 65-83). Rotterdam: Sense Publishers.
- Ponte, J. P., Serrazina, L., Guimarães, H., Breda, A., Guimarães, F., Sousa, H., Menezes, L., Martins, M. E. & Oliveira, P. (2007). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação – DGIDC.
- Ribeiro, C. M. (2011). Future primary teachers MKT in Geometry: some examples concerning rectangles. Paper to be presented at *NORMA 11*, Reykjavík, Iceland.
- Ribeiro, C. M. & Carrillo, J. (2011). *Knowing mathematics as a teacher*. Paper to be presented at the CERME 7, Rzeszów, Poland.
- Ribeiro, C. M., Monteiro, R. & Carrillo, J. (2009). Professional knowledge in an improvisation episode: the importance of a cognitive model. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of CERME6* (pp. 2030-2039). Lyon, France: ERME.
- Sherin, M. G. & Hans, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher Education*, 20, 163-183.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15 (2), 4-14.
- Stake, R. E. (2005). Qualitative Case Studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (Third edition ed., pp. 443-466). Thousand Oaks: Sage Publications.
- Thames, M. (2009). *Coordinating Mathematical and Pedagogical Perspectives in Practice-Based and Discipline-Grounded Approaches to Studying Mathematical Knowledge for Teaching (K-8)*. Unpublished PhD, University of Michigan, Ann Arbor.
- Tichá, M. & Hošpesová, A. (2006). Qualified pedagogical reflection as a way to improve mathematics education. *Journal of Mathematics Teacher Education*, 9, 129-156.
- Tomás Ferreira, R. A. (2005). *Portuguese student teacher's evolving teaching modes: A modified teacher development experience*. Unpublished PhD, Illinois State University IL.
- Watson, A. (2007). The nature of participation afforded by tasks, questions and prompts in mathematics classrooms. *Research in Mathematics Education*, 9(1), 111-126.



# EXPLAINING DIFFERENCES IN SECOND GRADE STUDENTS' PATTERNING COMPETENCE USING PARALLEL DISTRIBUTED PROCESSING

F. D. Rivera

San Jose State University, USA

*In this research report, I explore the implications of parallel distributed processing in explaining differences in figural patterning competence among 19 Grade 2 students after a classroom teaching experiment.*

Recent research results on figural patterning activity in prekindergarten and elementary school children indicate varying levels of generalization competence from having no structural awareness to full structural awareness relative, of course, to tasks presented to them in various settings. Structural awareness is manifested in various cognitive actions that include the following five layers: I. seeing wholes; II. seeing details; III. inferring relationships; IV. extending those relationships to other domains; and V. reasoning. I address the following basic research question: *After a teaching experiment on figural pattern generalization that emphasized multiplicative thinking in second grade, how do we characterize the nature and content of students' generalizations?* By *generalization*, I assume the following description offered by Peirce (1960): "Generalization in its strict sense, means the discovery, by reflection upon a number of cases, of a general description applicable to them all. This is the kind of thought movement which I have elsewhere called formal hypothesis, or reasoning from definition to definitum. So understood, it is not an increase in breadth but an increase in depth" (p. 256). With increase in depth comes "an increase of definiteness of the conceptions [we] apply to [the] known things" (ibid).

The notion of *abduction* is now commonplace in patterning studies and is distinguished from *induction*. In processes of generalization involving figural patterns with a few known stages (e.g. Figure 1), abduction involves constructing plausible structural hypotheses on the basis of the given stages that would then be applied the unknown stages. Induction involves testing the constructed abduction on the known stages and checking to see that the abduction could be used to reasonably extend and explain the figural pattern. Such explanations, of course, are not causal (i.e. they do not explain why the stages are the way they are) but are assumed to hold for the largest stipulated frequency.

## REVIEW OF LITERATURE

I highlight two sets of results drawn from recent studies conducted with elementary students in order to convey fundamental differences with the intended analysis that I pursue in this study. The first set was drawn from findings in clinical interviews in

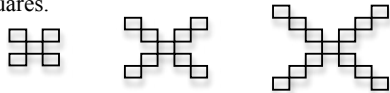
the absence of any teaching intervention on patterns, while the second set took place in the context of a teaching experiment.

*Task 1.* Below are four stages in a pattern. Each stage has two rows of squares, a top row and a bottom row.



A. Draw Stage 5. B. Draw Stage 6. C. Draw Stage 10. D. Describe Stage 25 using words and pictures. E. Describe Stage 100 using words and pictures.

*Task 3.* A square has four corners or vertices. Below are three stages in a growing pattern of squares.



A. Draw Stage 4. B. Draw Stage 5. C. Describe in words and pictures Stage 10. D. Describe in words and pictures Stage 25. E. Describe in words and pictures Stage 100.

*Task 2.* One dog has one pair of ears. Two dogs have two pairs of two ears. A. How many ears do three dogs have in terms of “pairs”? B. How many ears do four dogs have in terms of “pairs”? C. Fill in the table below.

# of Dogs	# of Ears in Words	# of Ears in x	Total
1	1 pair of ears	1 x 2	2
2	2 pairs of ears		
3	3 pairs of ears		
4			
5			
10			
25			
50			
100			

*Task 4.* Below are four stages in a pattern. Stage 1 has two rows of squares, a top row and a bottom row.



A. Draw Stage 5. B. Draw Stage 6. C. Describe Stage 25 using words and pictures. D. Describe Stage 100 using words and pictures. E. Maria describes Stage 50 as follows: “I see 50 groups of 2 squares plus 1 square.” Is she correct? How can you tell? Larry describes Stage 50 as follows: “There are 50 squares in the bottom row and 51 squares in the top row.” Is he correct? How can you tell? Which student is correct, Maria or Larry? How do you know?

**Figure 1 Four Pattern Generalization Tasks Used With Grade 2 Students**

Rivera’s (2010) analysis of 21 Grade 2 USA students prior to a teaching experiment on patterns indicates differing levels of patterning ability. Depending on the patterning tasks presented to them, they: exhibited no structural awareness; manifested partial awareness, especially approximate and global structure sense; and consistently employed narrow specializing (i.e. faulty generalizing on the basis of the last known stage in a pattern without connecting to the other given stages of the pattern). Further, while there was some success in obtaining consistent structural extensions on two figural patterns out of five figural patterning tasks of varying complexity, none of the students could verbalize an explicit rule in both cases of correct and incorrect far generalizations. Walkowiak’s (2010) cross-case analysis of three students coming from different elementary and middle school levels (i.e.

Grades 2, 5, and 8) on two patterning tasks in a nonteaching experiment context shows the salience of a combined use of spatial features in figural patterns and knowledge of number relationships in the construction and justification of an explicit rule. She also saw that, while both the second and fifth graders in her study were inclined to employ more figural than numerical reasoning compared with the 8<sup>th</sup> grader, the choice of reasoning might likely be influenced by classroom experiences.

Radford's (2010) analysis of one group of 3 Canadian Grade 2 students was drawn from a five-lesson teaching experiment on pattern generalization. The students were initially presented with a figural pattern consisting of 4 beginning stages. They were then asked to extend the pattern for the next two consecutive stages and to obtain a rule that would have enabled them to predict outcomes for the remote stages in the pattern. Radford's analysis of the students' thinking focuses on the emergence of an explicit rule as a result of their embodied actions; variables were intuited on the basis of the tools they used to convey their generalization, from gestures and initial spatially-driven verbal description to the use of a calculator and then to a more refined verbal description. Over a four-lesson teaching experiment, Vale and Pimentel's (2010) analysis of selected Grade 3 Portuguese students ( $n = 21$ ) focused on how the students obtained explicit numerical generalizations of two figural patterns by employing an inductive table and a visual grouping strategy that both conveyed what they interpreted to be the invariant structures of the patterns. Carraher, Martinez, and Schliemann's (2008) analysis of 15 Grade 3 USA students has been drawn from a 35-lesson teaching experiment that initially exposed the students to expressing everyday relationships in variable form. Hence, when the students were later presented with two patterning tasks that involve counting the maximum number of people that could sit in separate and adjacent square tables, the manner in which they constructed their explicit generalizations focused on relationships they inferred on the individual stages. Suffice it to say, the three studies depict elementary students whose pattern generalization ability could be characterized as manifesting layers II, III, and V of structural awareness (see Introduction).

In Rivera (2010), I reported findings drawn from my Grade 2 class ( $n = 21$ ) prior to the TEPGMT (see Review). The intended analysis that is pursued in this study involves assessing the same students' pattern generalization ability after a teaching experiment on pattern generalization that emphasized multiplicative thinking (TEPGMT) based on the conceptual framework below.

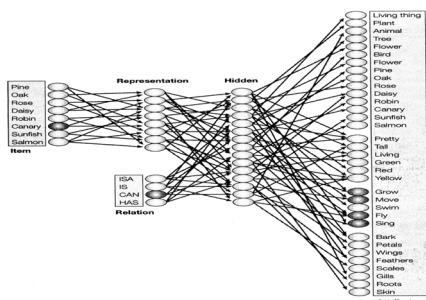
### **CONCEPTUAL FRAMEWORK: PARALLEL DISTRIBUTED PROCESSING**

One enduring research endeavor in social and developmental cognitive psychology involves how categories are formed, where categories refer to words or generic statements that convey generalities about certain classes of objects. In the preceding sections, I have noted the salience of establishing a valid structure as a goal of generalization involving figural patterns. Such a structure involves the abduction-induction of invariant, essential relationships, including the need to be algebraically

useful, that is, the interpreted structure should be expressed explicitly in closed, functional form. While existing studies done with elementary school children indicate varying levels of success (primarily determined by task) with or without the use of a teaching intervention, what is still not clarified to some degree are those underlying factors that could *sufficiently* explain differences in individual students' ability to generalize on the basis of a few known stages in a figural pattern.

In empirical studies involving category formation, such factors have been extrapolated using several different approaches. In this study, I pursue the implications of what is known as the *parallel distributed processing* (PDP) approach that basically assumes a nonlinear, dynamic, connectionist, and emergent complexity in the development of semantic processing by positing how "multiple correlations in the environment" enable individual learners to employ "perceptual and attentional mechanisms capable of extracting these regularities and establishing correspondences among correlated structures" (Sloutsky, 2003, p. 247). Such correspondences, of course, "could be a product of development and learning" (ibid, p. 249), that is, they are capable of conceptual reorganization with more experiences. For instance, the skill of perceiving relevant properties through continued experiences would result in the development of a learned perceptual system that easily sees the more relevant and essential properties. Also, the relevant (weighted) processing units are nonlinear but organized in layers and patterns of graded values that are connected in a feed-forward manner (Rogers & McClelland, 2008).

Figure 2 illustrates the model of a nonhierarchical PDP network that traces how general propositions are encoded and stored relevant to the concepts or items under discussion. The model consists of four layers of units, and activation occurs from left to right so that whenever connections occur, it means the units on the left are mapped onto the units on the right. Inputs consist of item-relation pairs relative to some context, which then spread forward through the Hidden units that activate the relevant output units that are correct (while turning off the rest that do not matter). The spreading is modulated by connection weights that evolve through more learning. For example, the pair Canary-Can activates the attributes Grow, Move, Fly, and Sing. Further analysis would then involve the role of the representation units. Each unit under the *Item/Context* layer is mapped to more than one unit in the *Representation* layer that then cause distributed patterns of activity across the units. It is sufficient for the time being to say that generalizations (or, more generally, conceptual knowledge) occur as patterns of activation across many units and are further determined both by the many different representations they generate in the process and the environment in which they occur. Central to PDP is the principle called *coherent covariation*, which refers to the clustering of properties that reliably



**Figure 2 PDP Network Model (Rogers & McClelland, 2004, p. 56)**

## METHODOLOGY

*Participants, Clinical Interviews, and Interview Protocol* Twenty 2<sup>nd</sup> grade students (6 girls, 14 boys; 20 Hispanic-Americans, 1 African-American; mean age of 7.5 years) participated in the TEPGMT, but only 19 were clinically interviewed after the TEPGMT. Both myself and a graduate student conducted the interviews. Each individual interview took about 30 minutes; six tasks were orally presented to the students one at a time. Due to space constraint, I report results drawn from four (out of six) tasks as shown in Figure 1. They were asked to think aloud and to write their answers on clean sheets of paper. Manipulatives were provided. On each task, they were asked to generate near (e.g. stages 4, 5, and 6) and distant (e.g. stages 10, 25, and 100) generalizations. When a distant generalization was offered, the interviewer probed further by asking for a general description that might reveal a plausible structure relevant to the task. Tasks 1 and 4 were also presented first and last to assess whether particular features (such as color) influence the context of the their generalizations.

*Context of the TEPGMT* Results of the clinical interviews conducted with the same group of Grade 2 students (see Review above) were used to develop the TEPGMT, which took place over 10 consecutive 55-minute sessions. Prior to the TEPGMT, the students spent three weeks formally learning the concept of multiplication of two single-digit whole numbers for the first time. They initially learned both the set model and rectangular array model of multiplication and later used them in dealing with a variety of everyday and mathematical situations that involve multiplicative reasoning. During the TEPGMT, numerical and figural patterning activities focused on ways in which relationships could be expressed multiplicatively. The TEPGMT provided the shared context of training among the students, which I assessed in the clinical interviews.

*Data Collection and Analysis* Videotaped interviews and all relevant student work were collected. Data analysis used the following grounded theory steps: a grounded analysis template was set up for each task; individual student responses were

(and naturally) fit together. One's existing knowledge base determines properties that appear to be coherently covarying and changes are likely to take place through more learning. Further, outcomes are determined by the strength (i.e. weights) of the connections that are themselves influenced by learning.

categorized accordingly; emergent themes were established and empirically verified with the construction of case studies and relevant transcripts from the interviews.

RESULTS

Table 1 provides summaries of the student responses on the four tasks presented in Figure 1. Figure 4 shows three illustrations of the student generalizations relative to Tasks 1 and 4. Three of the four students who could not generalize Task 1 were able to generalize Task 4 due to the influence of the shaded corner square, while the remaining 14 students saw no differences between Tasks 1 and 4. The table in Task 2 assisted 14 of them in stating a generalization. Thirteen students established the same generalization (description B) in Task 3 with description E producing a limited recursive relation. Overall, while 16 out of 19 developed structural generalizations in explicit terms, none of the students’ final verbal descriptions transitioned from seeing “groups” to stating them as expressions involving “times.” Further, because the class was not exposed to variable expressions, none of their explicit generalizations transitioned to symbolic direct expressions.




	<p>“There’s 5 on top and 4 on the bottom.”</p> <p>“There’s 101 on top and 100 on the bottom.”</p>
	<p>“There’s two groups of 4 squares plus the extra square.”</p> <p>“There’s two groups of 100 squares plus the extra square.”</p>
	<p>“Four groups of 2 plus the square on the corner.”</p> <p>“100 groups of 2 plus the shaded square.”</p>

Figure 4 Illustrations of Generalizations Relative to Tasks 1 and 4

DISCUSSION AND CONCLUSION

Referring to Figure 2, in this study the layer *Item/Context* refers to different figural patterns and the situations in which the training/learning occurred, while the layer *Representation* pertains to different types of generalizations produced on the basis of seeing them in either additive or multiplicative contexts. The four individual units under the layer *Relation* were kept the same and interpreted as terms that supported comparison and structural similarities within stages in a figural pattern or across

Task 1		Task 2	
Extended stages 5 an 6 based on an interpreted structure	15	Extended stages 4 and 5 based on an interpreted structure	14
Could not extend	4	Could not extend	5
Generalizations		Generalization	
A. Adding the top and bottom rows with the top row having 1 more square than the bottom row	A. 14	Number of pairs x 2	14
B. Adding two equal rows, where	B. 1		

each row has $n$ squares, plus the corner square			
<i>Task 3</i>		<i>Task 4</i>	
Extended stages 4 and 5 based on an interpreted structure	17	Extended stages 5 and 6 based on an interpreted structure	18
Could not extend	2	Could not extend	1
Generalizations A. Middle square plus the four surrounding squares plus the growing four legs each having $(n - 1)$ squares B. Middle square plus four legs each having $n$ squares C. 5 middle squares plus growing four legs each having $(n - 1)$ squares D. Middle square plus two pairs of legs with each pair totaling $2n$ squares E. Adding two diagonal squares	A. 1  B. 13  C. 1  D. 1  E. 1	Generalizations* A. Adding the top and bottom rows with the top row having 1 more square than the bottom row B. Adding two equal rows, where each row has $n$ squares, plus the corner square C. Pairs of diagonal squares plus the corner square (*One student offered two generalizations)	A. 14  B. 4  C. 1

**Table 1 Summary of Student Responses on the Tasks Shown in Figure 1 ( $n = 19$ )**

several figural patterns. Finally the layer *Attribute* consists of individual units that refer to generic descriptors for configurations or parts (e.g. “top,” “bottom,” “legs,” “groups,” etc.) and the numbers that represent cardinalities of items counted (e.g. “5,” “101,” etc.). The connection weights from one layer to the next depended on individual students’ contextual experiences with the items. For example, since the teaching experiment focused on ways in which a multiplicative approach could be used to interpret figural patterns, it was expected that faster connections occurred between item/context/relation (as an input) and representations and between representations and the relevant attributes. The PDP approach helps explain the persistence of differences in pattern generalization competence. At the very least, it is sensitive to context – “the environment provides both the input that characterizes a situation as well as the information about the outcome that then drives the process of learning” (Rogers & McClelland, 2008, p. 694). For example, when confronted with a figural pattern, students individually activated connections that reflected their experiences, with some patterns perceived as being more coherently covarying than others. In other words, familiar patterns led to faster parallel-based processing compared with unfamiliar patterns. The PDP model also could be used to explain why “failure” to “properly” generalize occurs. The relatively small number of students who did not generalize in functional form activated (less desirable) connections that had a global character, that is, they operated at structural awareness level I of seeing wholes. For example, relative to Task 1, the 4 students in Table 1 saw a pattern whose stages could be described as having “more on the top” and “less on the bottom,” which indicates that they chose the more general attributes “more”

and “less” over other specific ones. Interestingly, 3 of the same 4 students in Task 4 activated parallel connections that enabled them to produce direct expressions due to the addition of “color” in the attribute, which further captures the strength of the parallel feature of the PDP model that accounts for possible complexity of a system that evolves with more learning.

## REFERENCES

- Carraher, D., Martinez, M., & Schliemann, A. (2008). Early algebra and mathematical generalization. *ZDM*, 40(1), 3-22.
- Peirce, C. (1960). *Collected papers of Charles Sanders Peirce* (C. Hartshorne & P. Weiss, eds.). Cambridge, MA: Harvard University Press.
- Radford, L. (2010). Elementary forms of algebraic thinking in young students. In M. Pinto & T. Kawasaki (eds.), *Proceedings of PME 34* (Vol. 4, pp. 73-80). Brazil.
- Rivera, F. (2010). Second grade students’ preinstructional competence in patterning activity. In M. Pinto & T. Kawasaki (eds.), *Proceedings of the PME 34* (Vol. 4, pp. 81-88). Brazil: PME.
- Rogers, T. & McClelland, J. (2008). Precis of semantic cognition: A parallel distributed processing approach. *Behavioral and Brain Sciences*, 31, 689-749.
- Sloutsky, V. (2003). The role of similarity in the development of categorization. *Trends in Cognitive Science*, 7(6), 246-251.
- Vale, I. & Pimentel, T. (2010). From figural growing patterns to generalization: A path to algebraic thinking. In M. Pinto & T. Kawasaki (eds.), *Proceedings of PME 34* (Vol. 4, pp. 241-248). Brazil: PME.
- Walkowiak, T. (2010). An examination of algebraic reasoning: Elementary and middle school students’ analyses of pictorial growth patterns. In P. Brosnan, D. Erchick, & L. Flevaras (eds.), *Proceedings of PMENA 32* (Vol. VI, pp. 194-202). Ohio, US: Ohio State.



# DEVELOPMENT OF STUDENTS' UNDERSTANDING OF THE LOGIC IN THE EPSILON-N DEFINITION OF LIMIT

Kyeong Hah Roh

Arizona State University

Yong Hah Lee

Ewha Womans University

*The purpose of this study was to gain insights into how students might come to develop their understanding of logic in the  $\varepsilon$ - $N$  definition of the limit of a sequence. We focused on two students in an advanced calculus course to report how they comprehended the  $\varepsilon$ - $N$  definition by developing their understanding of the relationship between  $\varepsilon$  and  $N$  in defining limits while working with two other students in a group and interacted with the class.*

## INTRODUCTION

The  $\varepsilon$ - $N$  definition of the limit of a sequence is fundamental in studying advanced mathematics; however, research (Cornu, 1991; Davis & Vinner, 1986) indicates that many students experience difficulty understanding the  $\varepsilon$ - $N$  definition. Also, students' alternative understanding of the concept of limit is inconsistent across contexts (Artigue, 2000; Tall & Vinner, 1981; Williams, 1991), and is not solid enough for advanced mathematical activities (Alcock & Simpson, 2004). Students' difficulties with the  $\varepsilon$ - $N$  definition tend to be influenced by their images of limit, which are not compatible with the  $\varepsilon$ - $N$  definition (Roh, 2008). Furthermore, the difficulties are related to students' confusion over the logical structure of the  $\varepsilon$ - $N$  definition (Duran-Guerrier & Arsac, 2005; Roh, 2010a).

The aim of this study is to explore how students might come to understand the  $\varepsilon$ - $N$  definition of the limit of a sequence by developing their understanding of logic. In particular, we addressed the following research question: "How do students develop their understanding of the relationship between  $\varepsilon$  and  $N$  in the  $\varepsilon$ - $N$  definition?"

## CONCEPTUAL FRAMEWORK

To investigate our research question, we focused on the essential components of the relationship between  $\varepsilon$  and  $N$  in defining the limit of a sequence, suggested by Roh (2010a) as follows: (1)  $N$  is coordinated with  $\varepsilon$  before completing the variation of  $\varepsilon$ ; (2)  $\varepsilon$  is arbitrarily chosen; and (3) the arbitrariness of  $\varepsilon$  implies that  $\varepsilon$  decreases towards 0. We used these components as our conceptual framework to comprehend to what extent students understand the logical structure of the  $\varepsilon$ - $N$  definition.

## RESEARCH METHODOLOGY

This study was conducted in the spring of 2010, as part of a semester-long teaching experiment (Steffe & Thompson, 2000) in an introductory real analysis course at a public university in the USA. The first author of this paper served as an instructor of the course. Eleven mathematics or secondary mathematics education students

volunteered to participate in this study, working in small groups. The course consisted of two regular sessions (75 minutes in length) and a recitation (50 minutes in length, mainly for cooperative proof writing) for 15 weeks. Instruction in the course mainly followed an inquiry approach, in which students were often asked to make and justify conjectures and to evaluate arguments.

In the days of this study, the instructor implemented the  $\varepsilon$ -strip activity (Roh, 2010b). First, students were asked to determine if given sequences  $\{a_n\}_{n=1}^{\infty}$  (see Table 1) have limits.

(i) $a_n = 1/n$	(iii) $a_n = \begin{cases} 1/n & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$	(v) $a_n = \begin{cases} 1/n & \text{if } n \text{ is odd;} \\ 1 & \text{if } n \text{ is even.} \end{cases}$
(ii) $a_n = \begin{cases} 1/n & \text{if } n \leq 10 \\ 1/10 & \text{if } n > 10 \end{cases}$	(iv) $a_n = \begin{cases} 1/n & \text{if } n \text{ is odd;} \\ n & \text{if } n \text{ is even.} \end{cases}$	(vi) $a_n = (-1)^n / n$ (vii) $a_n = (-1)^n (1 + 1/n)$

Table 1: Sequences provided in the course

Second,  $\varepsilon$ -strips were introduced as strips, each with constant width and indefinite length. Half of the width of an  $\varepsilon$ -strip was called  $\varepsilon$ , and a red line was drawn in its centre to mark a possible limit value on the graph of a sequence. The students were then asked to put  $\varepsilon$ -strips of different widths on the same graph of a sequence to cover a possible limit value. They were also asked to determine how many points on the graph of the sequence were outside and then inside each  $\varepsilon$ -strip. After providing enough opportunities to work with the graphs of sequences and the  $\varepsilon$ -strips, the instructor introduced  $\varepsilon$ -strip definitions A and B to students as follows:

$\varepsilon$ -strip definition A: a certain value  $L$  is a limit of a sequence when for any  $\varepsilon$ -strip centred at  $L$ , infinitely many points on the graph of the sequence are inside the  $\varepsilon$ -strip.

$\varepsilon$ -strip definition B: a certain value  $L$  is a limit of a sequence when for any  $\varepsilon$ -strip centred at  $L$ , only finitely many points on the graph of the sequence are outside the  $\varepsilon$ -strip.

Afterwards, students applied  $\varepsilon$ -strip definitions A and B to sequences and evaluated if the use of A or B resulted in the correct answer for the limit of a sequence. Third, the students were asked to evaluate the validity of the following arguments.

Ben's argument: Choose an  $\varepsilon$ -strip with  $\varepsilon = 0.1$  and align its centre at  $y = -0.05$  on the graph of this sequence  $\{1/n\}$ . Then infinitely many points on the graph of the sequence are inside the  $\varepsilon$ -strip. Hence, accepting  $\varepsilon$ -strip definition A as a definition of limit, we should determine the value  $-0.05$  as a limit of the sequence  $\{1/n\}$ .

Emma's argument: Choose an  $\varepsilon$ -strip with  $\varepsilon = 0.1$  and align its centre at  $y = -0.05$  on the graph of this sequence  $\{1/n\}$ . Then only finitely many points on the graph of the sequence are outside the  $\varepsilon$ -strip. Hence, accepting  $\varepsilon$ -strip definition B as a definition of limit, we should determine the value  $-0.05$  as a limit of the sequence  $\{1/n\}$ .

Fourth, after evaluating Ben and Emma's arguments, students were asked to re-evaluate  $\varepsilon$ -strip definitions A and B. Finally, the  $\varepsilon$ - $N$  definition was introduced as

follows: A sequence  $\{a_n\}_{n=1}^{\infty}$  converges to  $L \in \mathbb{R}$  if for any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $|a_n - L| < \varepsilon$ . The students were then asked to discuss if the  $\varepsilon$ - $N$  definition is equivalent to  $\varepsilon$ -strip definition B.

Our analysis in this paper focused on two students, Alan and Oliver, who mainly worked with two other students, Alex and Mary; all four students had already encountered the concept of limit in calculus but did not have any experience with rigorous proofs using the  $\varepsilon$ - $N$  definition. A video camera was used to record student activities in this focus group, and students' written work was pen-casted (synchronized with their voice). We first divided the entire activity into 5 episodes as any change emerged in student understanding of definitions of limit. We then analysed if each student grasped the essential components of the relationship between  $\varepsilon$  and  $N$  in terms of our conceptual framework. In particular, Alan and Oliver's conversation with other students, or with the instructor, was examined to support their development of the understanding through interaction with the class.

## RESULTS AND ANALYSIS

Most students in the class properly determined the convergence or divergence of given sequences (i) ~ (vii) using the word "approach." However, describing that the sequence  $\{1/n\}$  approaches, but does not converge to  $-0.1$ , the students realized a need for a more precise way to account for the limit of a sequence. At that moment, the instructor introduced  $\varepsilon$ -strip definitions A and B.

### Episode 1: Initial evaluation of $\varepsilon$ -strip definitions A and B

Alan initially claimed that  $\varepsilon$ -strip definitions would not be proper for descriptions of the limit of a sequence. However, he was not able to construct a counterexample to support his claim until Alex suggested an oscillating divergent sequence for  $\varepsilon$ -strip definition A. Also, Alan did not come up with any idea for  $\varepsilon$ -strip definition B.

Alex: I don't think it's [A] actually the legitimate definition for limit because that would also be the case if it's convergent to two different values. In that case the sequence doesn't actually have a limit, but that condition [A] would imply that it does.

Alan: Umm ... I don't know. The issues that I see arising ... are primarily with part A. [...] I'm fairly sure there's a counterexample. But I couldn't think of one for A although Alex did give one probably, which is a sequence converging to two values. A problem I see with B is that ... actually I did not really think about B. So, [laugh] yeah.

Oliver agreed with Alan that neither  $\varepsilon$ -strip definition A nor B would be a proper description for limit, and suggested a tangent function as a counterexample. However, once Alex pointed out that a tangent function "does not really approach anything," Oliver realized that he was thinking of vertical asymptotes of a function instead of the limit of a function at infinity.

Oliver: So, for A and B, it works in this specific situation  $\{1/n\}$ . But as you guys were doing, I was looking for a counterexample. Umm, one of the first that I thought about is the graph for  $\tan\theta$ , where it [goes] all the way through [traces in air a function with vertical asymptotes] and so there're limits there  $[\theta = \pm\pi/2]$ . But how would we apply these specific points to that?

Alex: Well, the tangent doesn't have a limit. It doesn't really approach anything.

Oliver: Sorry, [...] I was thinking more of the  $x$  value  $[\pm\pi/2]$ . But that was the type of thing. I was looking at the counterexamples.

## Episode 2: Evaluation of Ben and Emma's arguments

Immediately after Ben and Emma's arguments were introduced to the class, Alan laughed and said, "That was really a lot easier than our counterexample." At this moment, Alan considered Ben and Emma's arguments as attempts to disprove that  $-0.05$  is the limit of the sequence  $\{1/n\}$ , and believed that testing one  $\varepsilon$ -strip was enough to account for disproving it. On the other hand, Oliver thought Ben's argument as invalid, and suggested a revision to the argument by changing to "any arbitrary"  $\varepsilon$ -strip. Accordingly, Alex recognized the arbitrariness of  $\varepsilon$  in  $\varepsilon$ -strip definition B and that choosing only one  $\varepsilon$ -strip instead of "any"  $\varepsilon$ -strip led to the erroneous conclusion in Emma's argument as well. It should be noted that Alan came to understand the significance of the arbitrariness of  $\varepsilon$  in the  $\varepsilon$ -strip definitions after interacting with Oliver and Alex.

Oliver: I guess one way to change this would be for any arbitrary  $\varepsilon$ , [...] you can find infinitely many values within it as long as  $\varepsilon$  is greater than 0.

Alan: Actually, no, there is, there is a uh [silence]

Oliver: I'm just giving an example how to change it so it fixes the definition a little bit more – [...] because they're choosing an  $\varepsilon$ -strip, so, rather than choosing an  $\varepsilon$ -strip, any value of  $\varepsilon$  greater than 0 would have to satisfy infinite values within that  $\varepsilon$ -strip.

Alex: [...] The second one [Emma] I am not sure is a legitimate disproof because it [B] says for any  $\varepsilon$ -strip, so, in this case, if you were supposing that  $-0.05$  was the limit, then it would have to work for any  $\varepsilon$ -strip. Because it works for one [ $\varepsilon$ -strip], that doesn't mean – [Oliver nods]

Alan: Well, no, no! The disproof though is a counterexample. So, you only have to choose one  $\varepsilon$ -strip for which this is the case, but it's not the limit.

Alex: I know. Well, what B is saying is that if you are trying to show that  $L$  is a limit, then it needs to work for any  $\varepsilon$  value. The counterexample is supposing that  $L$  is  $-0.05$ , which would mean that it would have to work for any  $\varepsilon$  value, not just this one. If we chose  $\varepsilon$  equals 0.01, then this limit value  $[-0.05]$  wouldn't work. [Alan: Uh-huh (affirmative)] So, even though I do think B is false, this [Emma's argument] is not a legitimate counterexample.

### Episode 3: Re-evaluation of $\varepsilon$ -strip definitions A and B

After determining Ben and Emma's arguments as invalid, Oliver and Alan evaluated that both A and B were necessary in defining the limit of a sequence. Such a change in their evaluation appeared in the conversation with Mary, who still thought  $\varepsilon$ -strip definition B was not proper: she chose the sequence  $(v)$ , whose odd terms were defined in the form of  $1/n$  and even terms as 1. She believed the sequence was a counterexample that shows why  $\varepsilon$ -strip definition B was not proper for a description of limit. In fact, she did not seem to account for the arbitrariness of the  $\varepsilon$ -strips, and only imagined an  $\varepsilon$ -strip to cover both 0 and 1. In this case, only finitely many points were outside the  $\varepsilon$ -strip, but neither 0 nor 1 was the limit of the sequence. Responding to Mary, Alan and Oliver demonstrated with a narrower  $\varepsilon$ -strip to cover only points clustered around 0. Hence, infinitely many points were outside the  $\varepsilon$ -strip, and therefore, accepting  $\varepsilon$ -strip definition B, the sequence did not converge to 0.

Mary: I thought this  $[(v)]$  was a counterexample for B.

Oliver: No, umm.

Alan: It's for A because there are infinitely many points within the  $\varepsilon$ -strip, but the sequence  $[(v)]$  doesn't converge because every even value is equal to 1 –

Mary: Oh, but wouldn't that  $[(v)]$  be a counterexample for B, too?

Oliver: B actually limits this  $[(v)]$ . If it weren't for B and it were only A, only A would allow this situation to occur where you can have this happen and this  $[0]$  would still be considered a limit – [Alan: Right.] Because if we ignored B, we can say that 0 is a limit of  $[(v)]$  because there are infinitely many values within this  $\varepsilon$ -strip. But because B is saying that there can only be finite values – [Alan: Right.] and there are infinitely many points outside of the  $\varepsilon$ -strip, so, that's why B is necessary to limit this situation.

After Oliver identified that  $\varepsilon$ -strip definition B “limited” the sequence  $(v)$  to be convergent, Alan proposed combining  $\varepsilon$ -strip definitions A and B. Oliver accepted Alan's idea since it forced them to describe both points inside and outside the  $\varepsilon$ -strip.

Alan: So we can say A and B as our definition and somehow that would work. [silence]

Oliver: Well, [it] seems you need to consider both to make it work. [...] So, what are our ideas? [laughs] I think what's our argument is that A and B are both necessary to describe this. You can't just have one or the other.

### Episode 4: $\varepsilon$ -strip definition B as a definition of the limit of a sequence

Although the students seemed satisfied with combining  $\varepsilon$ -strip definitions A and B, they continued to seek a counterexample for  $\varepsilon$ -strip definition B. Alan insisted its necessity because otherwise  $\varepsilon$ -strip definition B alone should be considered proper. Alan's comment, though unintended, made Oliver reconsider  $\varepsilon$ -strip definition B to be a proper definition of limit, and later, Alex and Alan came to agree with Oliver.

Alan: I'm just really, I really feel and I have no way, like my gut says that there is a counterexample to part B alone, but I cannot for the life of me think of one. [...]

The problem that I am having is that if we cannot find a counterexample for B on its own, then you don't need A and B. You can just use B [Alex: Mhmm]. Um...so that's just why I think we need a counterexample for B. [Alex: Yeah.]

Oliver: You [Alan] made the comment, and I think this is a good comment. If we can't disprove part B, then part B is the only one that we need. [Alan and Alex agree.] And in all honesty, the more I think about [reading B], to me, it seems like B ... is the definition we should be using in this situation. [...] I'm on [sic] the opinion that B works for this. Um, but I know you got your feeling that this [B] doesn't work [...]. Until we come up with one [counterexample], we should accept this [B] –

Alan: Okay, I mean, I'm always open to be wrong, so.

Alex: Yeah, I agree. B doesn't seem like it should be right. But since we haven't found any evidence that it's false, [Alan: Yeah.] that's what we should be going with for now.

Oliver repeated the importance of the arbitrariness of  $\varepsilon$ , coordinating the number of points outside the  $\varepsilon$ -strip. Also, he was aware of the variation of  $N$  as  $\varepsilon$  decreases.

Oliver: The important aspect obviously is the arbitrary values of  $\varepsilon$ . So, we're saying that no matter what those widths are, you're gonna run into a situation where you have infinite many inside and finitely many outside. You may have to count a little bit more with the smaller strips, but there's still gonna be a finite value.

The students admitted that because they were unable to find a counterexample for  $\varepsilon$ -strip definition B, they considered B a proper description for limit. The instructor encouraged them to think if they needed to check points both inside and outside an  $\varepsilon$ -strip. Oliver then realized that if finitely many points were outside an  $\varepsilon$ -strip, then the rest of the points would be inside the  $\varepsilon$ -strip. Consequently, he accepted  $\varepsilon$ -strip definition B, but more notably, showed in a rigorous way, why B is proper.

Roh: Is your argument B itself is enough as a description for the limit of a sequence?

Oliver: Only because we cannot find a counterexample, okay? [laughs]

Roh: I heard another group saying we need both A and B. So, we have to check infinitely many points are inside and only finitely many points are outside? [...]

Alan: No. [...]

Oliver: Well, if you have a sequence that is infinitely long, [...] there is an area where one part of it is finite that would imply that the other area would be infinite because you still have an infinite set and infinitely plus a finite value is still infinite. [...] That's why I feel that B is concise enough for the situation. You don't need to say that there's infinite set inside if the entire sequence we're using is infinite.

### Episode 5: Understanding of the $\varepsilon$ - $N$ definition via $\varepsilon$ -strip definition B

When the  $\varepsilon$ - $N$  definition was introduced, Alan and Oliver easily recognized the similarity between the  $\varepsilon$ - $N$  definition and  $\varepsilon$ -strip definition B. Alan compared  $N$  in the

$\varepsilon$ - $N$  definition with the finite number of points outside the  $\varepsilon$ -strip in  $\varepsilon$ -strip definition B. Agreeing with Alan, Oliver also demonstrated that in the case of sequence (ii), the points inside the  $\varepsilon$ -strip satisfied the inequality  $|a_n - L| < \varepsilon$  in the  $\varepsilon$ - $N$  definition.

Alan: I sorta took the approaching of unpacking the [ $\varepsilon$ - $N$ ] definition of a convergent sequence, [...] and sorta come to the conclusion similar to where B comes in. [What] I noticed is the large  $N$  basically. So by defining that large  $N$ , we're saying that there is some numbers that are smaller than  $N$ , which are outside of that  $\varepsilon$ -strip. That's how I thought of it [ $\varepsilon$ - $N$  definition].

Oliver: Yeah. [...] It [ $\varepsilon$ - $N$  definition] seems very similar to B in that 1 to the large  $N$  is finite amount [Alan: Uh-huh (affirmative)]. Because you're just starting with a natural number, 1 to large  $N$  is finite whereas anything greater than large  $N$  is infinite which fits what we were discussing about finite and infinite earlier. [...] If the large  $N$  would be 8, then you start getting the situation where this [the graph of (ii) outside the  $\varepsilon$ -strip] is finite, this [the graph of (ii) inside the points] is infinite. And any value of  $a_n$  minus that  $[1/10]$  will be less than the width of  $\varepsilon$ .

In fact, Alex encountered difficulty determining a specific  $N$  for a non-constant type of sequence (e.g.  $\{1/n\}$ ). Responding to Alex, Alan and Oliver made a note of  $N$  varying while  $N$  is coordinated with the arbitrarily chosen  $\varepsilon$ . Also, they made this coordination of  $N$  with  $\varepsilon$  before completing the variation of  $\varepsilon$ .

Alex: Well, in a situation like this [(ii)], where  $a_n$  actually does approach a specific value and it stays at after a certain point, then you can pin down a specific  $N$ , which I guess is like 8 or 9 here. But if it was like the  $1/n$  case, then what can we call  $N$ ?

Alan: You can actually, yeah,  $N$  is sort of arbitrary.

Oliver: Yeah. [...] You can say that there's still a large  $N$  where  $|a_n - L| < \varepsilon$  and that fits for each of these situations. [...] It could be anything because you could have, since arbitrary values of  $\varepsilon$ , the value of  $\varepsilon$  actually changes with the  $N$  we pick. [Alan: Yeah.] Or, you can still choose any large  $N$  and there is still infinitely many values of small  $n$  that fit within the  $\varepsilon$ . [...] The thing to keep in mind is that for all  $n > N$ , so you can choose any  $a_n$ , any value of that's greater than large  $N$ .

## DISCUSSIONS

This study adds to literature by reporting a case that students successfully developed their understanding of the  $\varepsilon$ - $N$  definition, which has been known as difficult for students to understand. Students' evaluation of  $\varepsilon$ -strip definitions A and B changed as they engaged in the  $\varepsilon$ -strip activity: neither A nor B is proper (Episode 1)  $\rightarrow$  the combination of A and B is proper (Episode 3)  $\rightarrow$  B alone is proper (Episode 4). By comparing  $\varepsilon$ -strip definition B with the  $\varepsilon$ - $N$  definition, they eventually understood the  $\varepsilon$ - $N$  definition in Episode 5. It is worth noting that Oliver and Alan came to understand the  $\varepsilon$ - $N$  definition as they understood the relationship between  $\varepsilon$  and  $N$  in  $\varepsilon$ -strip definition B. This enabled their construction of the logical equivalence between the  $\varepsilon$ - $N$  definition and  $\varepsilon$ -strip definition B. In particular, they became aware

of the coordination of  $N$  with  $\varepsilon$  before completing the variation of  $\varepsilon$ , the arbitrariness of  $\varepsilon$ , and the decrease of  $\varepsilon$  to be implied from the arbitrary nature of  $\varepsilon$ . These were the three essential components in our conceptual framework for the relationship between  $\varepsilon$  and  $N$  in defining limit. Concerning implications of the teaching and learning of the  $\varepsilon$ - $N$  definition of the limit of a sequence, it is worth noting that the  $\varepsilon$ -strip activity was effectively used in this study. In particular, students' discourse in evaluating Ben and Emma's arguments in Episode 2 played a crucial role in understanding the arbitrariness of  $\varepsilon$  in defining the limit of a sequence.

## ACKNOWLEDGEMENT

This study was supported by the National Science Foundation under a grant (#0837443).

## References

- Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, 57, 1-32.
- Artigue, M. (2000). Analysis. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 167-198). Dordrecht: Kluwer.
- Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153-166). Dordrecht: Kluwer.
- Davis, R., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5, 281-303.
- Durand-Guerrier, V., & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational Studies in Mathematics*, 60, 149-172.
- Roh, K. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, 69, 217-233.
- Roh, K. (2010a). An empirical study of students' understanding of a logical structure in the definition of limit via the  $\varepsilon$ -strip activity. *Educational Studies in Mathematics*, 73, 263-279.
- Roh, K. (2010b). How to help students conceptualize the rigorous definition of the limit of a sequence. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 20, 473-487.
- Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, 22, 219-236.



# PROBLEM SOLVING PROCESSES OF FIFTH GRADERS: AN ANALYSIS

Benjamin Rott

Leibniz University of Hanover

*This paper reports on an exploratory study which investigates (mathematical) problem solving processes of fifth graders (ages 10 to 12) from German secondary schools. An overview about the problem solving processes of 32 students working in pairs on a geometry task shows a significant correlation between the students' problem solving behavior and their success (or failure). The videotapes which supplied the raw data were coded using an adapted version of the protocol analysis framework from Schoenfeld (1985).*

## BACKGROUND

According to Halmos (1980, p. 519) "the mathematician's main reason for existence is to solve problems, [...] what mathematics *really* consists of is problems and solutions." There are different definitions for the term "problem solving". Most of them include a starting point, a goal and the way between those two, to which the problem solver – in contrast to algorithmic or routine tasks – has no immediate access (cf. Dörner 1979; Schoenfeld 1985). Mayer & Wittrock (2006, p. 287) put it this way:

"When you are faced with a problem and you are not aware of any obvious solution method, you must engage in a form of cognitive processing called *problem solving*. Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver [...]."

It is important to note that the attribute "problem" depends on the solver, not on the task. A difficult problem for one student can be a routine task for another (maybe older or more experienced) one. Thus, doing research on problem solving should focus on the problem solving *process*.

When it comes to teaching mathematics, a specific training in mathematical problem solving strategies should be an essential component of the school curriculum (cf. Schoenfeld 1992b; NCTM 2000). In German schools, problem solving "did not attract the interest it deserved as a genuine mathematical topic." (Reiss & Törner 2007, p. 431) Only since the TIMSS and PISA studies, the Standing Conference of German Educational and Cultural Ministers (KMK 2003) established new standards for education. This guideline contains content and process standards, where problem solving is part of the latter. It "encompasses working on given and individually posed problems, using heuristic strategies, principles, and tools, checking results for plausibility, and generating ideas for problem solving." (Reiss & Törner 2007, p. 439)

In mathematics education research, problem solving is an important topic since Polya's (1945) seminal work "How to solve it". Polya presents four steps, which might assist a problem solver achieving a solution. These steps are (1) understanding the problem, (2) devising a plan, (3) carrying out the plan, and (4) looking back. Polya's book was well received by both the mathematics and the mathematics education community. "However, [...] Polya's ideas had hardly any impact on the work of scientists from both communities. In particular, there is no evidence that this approach significantly influenced classroom work in Germany." (Reiss & Törner 2007, p. 4; cf. Schoenfeld 1992b, 51 ff.)

At present, there is a lack of research regarding the development of problem solving abilities, especially within younger children (cf. Heinze 2007, p. 15). Also, we have little knowledge about the genuine problem solving abilities of younger students. To gain a little more insight into students' problem solving behavior, we started the project presented below. Research questions are (among others) whether the students use Polya-like steps and whether they profit from it.

## DESIGN OF THE STUDY

Our support and research program MALU<sup>1</sup> is an enrichment project for interested fifth graders (ages 10 to 12) from secondary schools in Hanover. Since November 2008 students come to our university once a week. A group of 10 – 16 children is formed every new term.

The sessions usually follow this pattern: After some initial games and tasks, the students work in pairs on one to three mathematical problems for about 40 minutes and are videotaped in doing so. They eventually present their results to the whole group. Altogether, we had 45 students working on about 30 problem tasks in four terms. The fifth term starts in February 2011.

The students work on the tasks without interruptions or hints from the researchers, because we want to study their uninfluenced problem solving attempts. We decided not to use an interview or a think-aloud method, because we think it would interrupt the students' mental processes and be kind of artificial. To get an insight into their thoughts, we let the children work in pairs to interpret their communication. Also, working in pairs makes the students feel more comfortable being videotaped (cf. Schoenfeld 1982, p. 10) and is as common as working alone in school environments.

The tasks are selected to represent a wide range of mathematical areas and to allow the use of different heuristics. In this article I present a task that displayed a wide range of student's solutions (see below) – from no structured approach at all to mathematical reasoning. Overall we have videos of 32 children working on that task.<sup>2</sup>

---

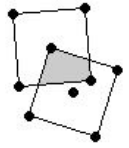
<sup>1</sup> *Mathematik AG an der Leibniz Universität* means to *Mathematics Working Group at Leibniz University*.

<sup>2</sup> There are six more children who worked on that problem, but their problem solving processes were not recorded or those recordings malfunctioned.

### Beverage Coasters

The two pictured squares depict coasters. They are placed so, that the corner of one coaster lies in the center of the other.

Examine the size of the area covered by **both** coasters.



To solve this problem (cf. Schoenfeld 1985, p. 77), one could examine special cases and – among other possibilities – argue by rotational symmetry or congruent triangles (see Figure 1).

For the interpretation, the products and the processes were coded separately: The products were classified into four categories of explanation and the processes into problem solving episodes (see below for details).

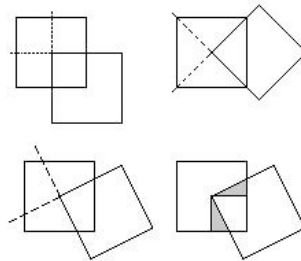


Figure 1: Possible approaches to the Coasters task

### METHODOLOGY

The students' work results – their *products* – were graded in four categories: (1) *No access*, when the problem solvers didn't get an answer or a wrong one, e.g. calculating the circumference instead of the area of the sought-after quadrilateral. (2) *Basic access*, when the students had the conjecture “one fourth” for one or both special case/s. (3) *Advanced access*, when the students had the conjecture “one fourth for every location of the two squares”, but no (correct) explanation for the general case. And (4) *full access*, when the problem solvers presented correct reasons for the general conjecture.

It is important to note, that the two members of a pair could – and sometimes did – gain a different rating, when their written results differed. These codes were carried out by a fellow Ph.D. student and me, we reached agreement in all cases.

The students behavior – the *processes* – were coded using a framework for the analysis of videotaped problem solving sessions presented by Schoenfeld (1985, chapter 9). His intention is to “identify major turning points in a solution. This is done by parsing a protocol into macroscopic chunks called *episodes* [...]” (ibid., p. 314) An episode is “a period of time during which an individual or a problem-solving

group is engaged in one large task [...] or a closely related body of tasks in the service of the same goal [...]” (ibid., p. 292)

Schoenfeld (1992a, p. 189) continues: “We found [...] that the episodes fell rather naturally into one of six categories:”

- (1) *Reading* or rereading the problem.
- (2) *Analyzing* the problem (in a coherent and structured way).
- (3) *Exploring* aspects of the problem (in a much less structured way than in Analysis).
- (4) *Planning* all or part of a solution.
- (5) *Implementing* a plan.
- (6) *Verifying* a solution.

For our study, we adapted the framework with the following modifications:

In the first place, our children – unlike the university students Schoenfeld observed – showed a great deal of non-task related behavior. So we added new categories of episodes comprising acts of *digression*, when our students talked about cartoon characters or TV series instead of working on the mathematical content, or *writing*, when they needed minutes to write an answer without achieving any new information or making any kind of progress. But this is not important for the results presented in this paper, because I'll focus on the task-related episodes, which are (2) – (6) of Schoenfeld’s list.

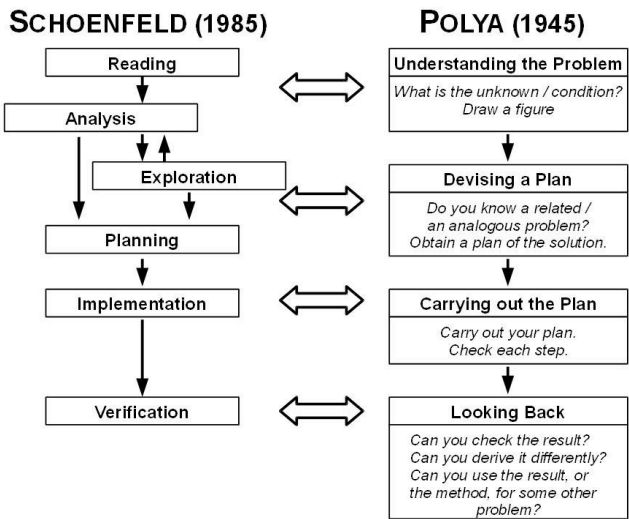


Figure 2: Analogy between Schoenfeld’s episodes and Polya’s steps

Secondly, we had some problems figuring out the differences between some episodes, especially *Analysis* and *Exploration* (as predicted in Schoenfeld 1992a, p. 194). We solved those difficulties by assuming an analogy between Schoenfeld's framework and Polya's famous list (cf. Schoenfeld 1985, chapter 4). Applying Polya's questions to the problem solving processes helped us deciding whether to code *Analysis* or *Exploration*. See Figure 2 for a summary.

The coding of the videotapes was done independently by a research assistant and me. When our codings didn't coincide (which they did most of the time), we attained agreement by recoding together (cf. Schoenfeld 1992, p. 194).

With our students, it is not uncommon to have two different episode-codings within one pair, when the children worked separately sitting next to each other. Therefore, to avoid an imbalanced weighting of the data, all 32 processes were counted independently, even if the two members of a pair worked together all the time and got the same episode-coding.

## RESULTS

The Beverage Coasters task was definitely one of the more difficult ones in the MALU project for our fifth graders. From the 32 analyzed results, 14 were coded as *no access*. Those students didn't know how to calculate an area<sup>3</sup> and tried doing so by computing the circumference (9 students) or by multiplying all four side lengths or sums of those lengths (5 students). They didn't correlate the area of the sought-after quadrilateral with those of the squares. Only one of those students showed behavior that was coded as *planning*, all the other children got stuck to their first idea and didn't evaluate or question their actions – a behavior Schoenfeld calls “wild goose chase”. None of the students attempted to *verify* his or her solution.

Six students showed *basic access* by calculating the area (nearly) correctly without correlating it with the area of the squares (3 students) or by stating the conjecture “one fourth” examining only special cases (3 students). Again, only one of those children *planned* his approach and no one *verified* it.

*Advanced access* was identified in eight processes. These children formulated the general conjecture and all but two of them drew at least one special case. Two students talked explicitly about rotating one square, one even cut out a paper square and actually did rotate it. Two of those students formulated a *plan* and *implemented* it before *verifying* their solution. The other six used an unstructured approach.

Two pairs had *full access* and found correct solutions. One of these four showed *planning*, two *verified* their work, only one of them applied an unstructured approach.

---

<sup>3</sup> Calculating areas is content of the curriculum for fifth graders in Germany.

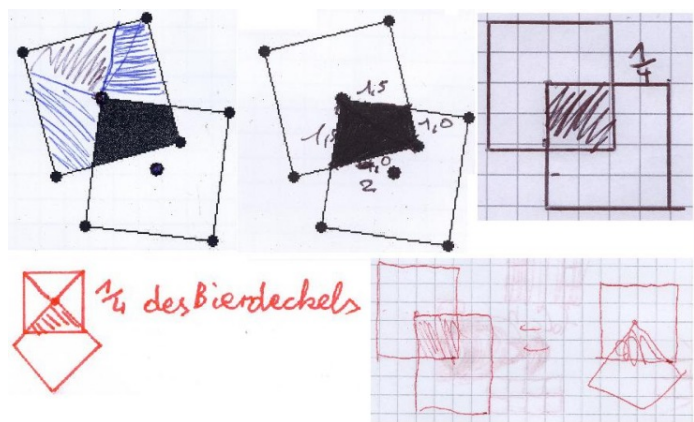


Figure 3: Some of our student’s drawings

Schoenfeld (1992a, p. 195) summarizes his results as follows:

“Approximately 60% of the protocols were of the type [...], where the students read the problem, picked a solution direction (often with little analysis or rationalization), and then pursued that approach until they ran out of time. In contrast, successful solution attempts came in a variety of shapes and sizes – but they consistently contained a significant amount of self-regulatory activity, which could clearly be seen as contributing to the problem solvers' success.”

To apply Schoenfeld's result to our data, I used a chi-square test to check the interrelation between episodes of the process and success or failure of the product. The null hypothesis is “no correlation between process type and problem solving success”. Because of the small data base, I had to subsume the *product* categories by twos. The *process* categories are “wild goose chase”, when the process consists of only *exploration* or *analysis & exploration*, on the one hand and all other occurrences of episodes, where the students showed *planning* or *verifying* activity, on the other hand. The entries in Table 1 consist of the observed numbers, the expected numbers are added in brackets. The test ( $\chi^2 = 4.401$ ) shows a significant correlation ( $p = .036 < .05$ ) between the problem solvers’ behavior and their success.

process / product categories	no & basic access	advanced & full access	sum
wild goose chase	18 (15.625)	7 (9.375)	25
miscellaneous	2 (4.375)	5 (2.625)	7
sum	20	12	32

Table 1: Contingency table – process behavior and product success

Titze, Klika & Wolpers (1997, chapter 3) report on problem solving studies: The majority of unsuccessful problem solvers fail for lack of understanding the problem properly. Furthermore, 'looking back' in terms of Polya is observed very rarely (cf. also Schoenfeld 1985, chapter 4).

The latter is definitely true for our sample: only four children showed signs of *verification* during their processes while working on the coasters task. To verify the former, I used another chi-square test (see Table 2): this time, the success of the product was correlated with the existence or absence of *analysis* episodes. The test ( $\chi^2 = 5.723$ ) affirms a significant correlation ( $p = .017 < .05$ ) between trying to understand the problem and being successful in solving it.

process / product categories	no & basic access	advanced & full access	sum
no <i>analysis</i>	12 (8.75)	2 (5.25)	<b>14</b>
<i>analysis</i>	8 (11.25)	10 (6.75)	<b>18</b>
sum	<b>20</b>	<b>12</b>	<b>32</b>

Table 2: Contingency table – existence / absence of *analysis* and product success

## DISCUSSION AND CONCLUSIONS

The chi-square tests show that there are significant interrelations between the behavior during a problem solving process and its outcome. The results by Tietze, Klika & Wolpers (1997) and Schoenfeld (1985, 1992a) are applicable to our fifth graders: lacking an adequate understanding of a task or using an unstructured approach hinders a successful solution. In the long run, this could mean that using Polya-like steps (analyzing the task, devising a plan, carrying it out and verifying the solution afterwards) is indeed helpful for children working on a problem. This could imply training programs to enhance the problem solving competencies of students.

At this time, only a small amount of our data has been analyzed. A first coding of other tasks' processes into episodes in terms of Schoenfeld seems to support the findings presented in this paper; however, at least one other coding by independent raters has to take place, before they can be discussed properly. In addition to the work presented here, I plan to take a closer look at the revised set of episode categories, especially at *digression* – maybe it is possible to identify incubation and illumination in terms of Poincaré (1914) and Hadamard (1945).

I also intend to look for heuristic elements and elements of self-regulation used by the students; both seem to be important factors when it comes to understand problem solving behavior (cf. Schoenfeld 1985; 1992b).

## References

- Dörner, Dietrich (1979). *Problemlösen als Informationsverarbeitung*, Stuttgart: Kohlhammer, 2., revised edition.
- Hadamard, Jaques (1945). *The psychology of invention in the mathematical field*. Princeton University Press. Unchanged reprint from 1954.
- Halmos, Paul R. (1980). The Heart Of Mathematics. In *The American Mathematical Monthly*, Vol. 87, No. 7 (Aug. - Sep., 1980), S. 519 – 524.
- Heinze, Aiso (2007). Problemlösen im mathematischen und außermathematischen Kontext. Modelle und unterrichtskonzepte aus kognitionstheoretischer perspektive. In *Journal für Mathematikdidaktik (JMD)*, 28(1), 3-30.
- Kultusministerkonferenz (2003). *Bildungsstandards im Fach Mathematik für den mittleren Schulabschluss*. Bonn: KMK.
- Mayer, Richard E. & Wittrock, Merlin C. (2006). Problem Solving. In Alexander, Patricia A. & Winne, Philip H. (Ed.): *Handbook of Educational Psychology*. Routledge, 287 – 304.
- National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston: NCTM.
- Polya, George (1945). *How to Solve it*. Princeton, NJ: Princeton University Press.
- Poincaré, Henri (1914). *Wissenschaft und Methode*. Berlin: Teubner.
- Schoenfeld, Alan H. (1982). On the Analysis of Two-Person Problem Solving Protocols. Paper presented at the Annual Meeting of the American Educational Research Association (66th, New York, NY, March 19 – 23, 1982).
- Schoenfeld, Alan H. (1985). *Mathematical Problem Solving*. Orlando, Florida: Academic Press, Inc.
- Schoenfeld, Alan H. (1992a). On Paradigms and Methods: What do you do when the ones you know don't do what you want them to? Issues in the Analysis of data in the form of videotapes. In *The Journal of the learning of sciences*, 2(2), 179 – 214.
- Schoenfeld, Alan H. (1992b). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In Grouws, Douglas A. (Ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan, 334 – 370. Online edition.
- Tietze, Uwe-Peter; Klika, Manfred & Wolpers, Hans (1997). *Mathematikunterricht in der Sekundarstufe II – Band I*. Braunschweig: Vieweg. 2. revised edition, 2000.
- Reiss, Kristina & Törner, Günter (2007). Problem solving in the mathematics classroom: the German perspective. In *Zentralblatt für Didaktik der Mathematik (ZDM)*, 39, 431 – 441. Springer: Berlin, Heidelberg.



# TRIGGERS OF CONTINGENCY IN MATHEMATICS TEACHING

Tim Rowland, Anne Thwaites and Libby Jared  
University of Cambridge

*Our research in the last decade has been into classroom situations that we perceive to make demands on mathematics teachers' disciplinary knowledge of content and pedagogy. Amongst the most visible of such situations are those that we describe as contingent, in which a teacher is challenged to deviate from their planned agenda for the lesson. Our research has been grounded in classroom practice, and our findings and the associated grounded theories have been open to enhancement and revision in the face of new classroom data. We propose a classification of the origins of contingent classroom episodes: namely the students; the teacher him/herself; and pedagogical tools and resources.*

## INTRODUCTION

Mathematics teaching rarely proceeds according to plan, if ever. Alan Bishop (2001, p. 244) recounts an anecdote about a class of 9- and 10-year-olds who were asked to give a fraction between  $\frac{1}{2}$  and  $\frac{3}{4}$ . One girl answered  $\frac{2}{3}$ , explaining "because 2 is between the 1 and the 3, and on the bottom the 3 lies between the 2 and the 4". The girl's answer was probably not one that the teacher had expected, even less one that s/he had hoped for. The teacher could ignore or effectively dismiss the girl's proposal (e.g. "You couldn't be sure that way"), or alternatively they could take it seriously, perhaps probing or instituting some investigation into the generality of her reasoning. It follows that the teacher who is aware of these and other options is caught up in making split-second decisions or choices as s/he conducts the lesson.

For some years now, our research into mathematics teachers' disciplinary content knowledge has focused on situations in which that knowledge 'plays out' in the classroom. Amongst the most visible of these situations (Corcoran, 2008) are those that we describe as 'contingent', in which a teacher encounters something unexpected, requiring them to 'think on their feet'. In this paper we propose a three-part classification of the sources of these contingent moments, and give examples of each of the three types.

## THEORETICAL BACKGROUND

It is reasonable to suggest that every lesson arises from a set of imagined scenarios in the mind of the teacher. Leinhardt (1993) used the term 'agenda' to capture the intended lesson structure, and the contexts and pedagogic principles that frame it. Morine-Dersheimer (1978-9) coined the term 'lesson image' to suggest a broad vision of what the teacher expects to happen. The lesson image is unlikely to be detailed, but would include the content focus and related learning objectives for the lesson, how

the content would be introduced, what tasks the students will engage in, and how they might be expected to respond to them. Schoenfeld explains:

A teacher's lesson image is, in a very expansive sense, the teacher's envisioning of the possibilities and contingencies related to a lesson. The teacher's lesson image includes knowledge of his or her students and how they may react to parts of the planned lesson; it includes a sense of what students are likely to be confused about, and how the teacher might deal with that confusion; and more (Schoenfeld, 1998, p.17).

Schoenfeld uses his introduction to an undergraduate problem solving course to exemplify lesson image, including the following illuminating comment (ibid., p. 18):

I can tell you, before the class starts, how things are likely to unfold – down to an extremely fine level of detail. It is not that I follow a rigid plan and coerce the students into it; there are many branch points and contingencies. However, I know what most of them are likely to be. And, there are few surprises.

Insofar as words can ever do so, we use the term 'contingency' with the same intended meaning (e.g. Rowland, 2010). That is, to capture the idea that the teacher cannot know in advance everything that will happen in the lesson, and at times s/he will be taken by surprise, necessitating some kind of improvisation. On the other hand, as Schoenfeld indicates, some aspects of the unknown can be predicted. In the same way that the person in tune with their environment can *predict* the weather, yet not *know* it in advance, the events of a lesson remain uncertain until they occur. Nevertheless, knowledge of individual students, common errors and misconceptions, and what students generally find difficult (e.g. Ryan & Williams, 2007) can inform prediction, and reduce surprise<sup>1</sup>. It is to be expected that the novice teacher is not very well-placed to anticipate contingent events: they lack the experience from which (potentially, at least) to learn about how students respond to certain pedagogical stimuli, or even to exemplify in practice (and thereby believe) the 'theory' that they have read, or been told, in their professional training.

Schoenfeld's theory of Teaching-in-Context models teacher behaviour in classroom instructional settings, with reference to factors such as the teacher's goals, knowledge and beliefs. Schoenfeld's analysis segments a lesson into 'action sequences' initiated by 'triggering events'. These events can be planned or unplanned: these triggers typically activate some component of the teacher's belief system (e.g. a commitment to pursuit of meaning and enquiry-driven learning) which, along with lesson goals, teacher knowledge and so on, contributes to the determination of that action sequence. Our own interest in contingency first arose from analysis of elementary mathematics teaching. Contingent action was one of four dimensions of what we now call the Knowledge Quartet, which is a framework for observing, describing and discussing the role of teachers' content (including pedagogical) knowledge, as enacted in their practice. In this paper we propose three categories of triggering events in relation to contingent moments and episodes in the mathematics classroom.

## METHOD

We draw on a programme of research at the University of Cambridge (*SKIMA*: subject knowledge in mathematics) from 2002 to the present. This programme initially investigated the mathematics content knowledge of primary trainee teachers, and the ways that this knowledge became visible, both in their planning and in their teaching in the classroom. The most recent *SKIMA* phase extended to systematic observations of secondary mathematics teaching. The resulting theory of knowledge *in* mathematics teaching, the Knowledge Quartet (KQ), categorises events in mathematics lessons with particular reference to the subject matter being taught, and to teachers' mathematics-related knowledge. Each phase of the research has entailed observing and videotaping a corpus of lessons taught during extended school-based placements towards the end of the participants' pre-service teacher education. Participants also supplied documents recording their lesson planning. Latterly, the methodology has included stimulated-recall interviews with the participants after their lessons. We took a grounded approach to the data for the purpose of generating theory (Glaser & Strauss, 1967). Each of four dimensions of the KQ is the synthesis of a set of codes which emerged from our scrutiny of mathematics classroom data. We name the four units of the KQ *foundation*, *transformation*, *connection* and *contingency* (for further details see Rowland, Huckstep & Thwaites, 2004). The fourth of these, contingency, is the subject of this paper. Our account is informed and significantly enhanced by extensive 'theoretical sampling' (Glaser & Strauss, 1967) since 2004, whereby the application of the grounded theory (the KQ) to the analysis of many lessons, in the UK and beyond (notably Cyprus, Ireland, Italy, Spain, Norway and the US) tested the theory, exposed shortcomings, and made possible further refinement and improvement.

## FINDINGS

Our scrutiny of the multiple data-sources available to us since 2002, as described above, lead us to propose that contingent moments in mathematics teaching are triggered by three types of situations, or events. These first of these is *responding to students' ideas*; the second is initiated by the teacher him/herself, as a consequence of *teacher insight*; the third is initiated by neither the student nor the teacher, but by some pedagogical tool or artefact present in the classroom, when the teacher is *responding to the (un)availability of tools and resources*. We describe each of these triggers in turn, and provide at least one example from our data.

### **Responding to students' ideas**

This refers to teachers' responses to contributions by students to the (mathematical) development of the lesson. These contributions are typically oral, but could be written. When a student articulates an idea, this points to the nature of *their* knowledge construction, which may or may not be what the teacher intended or anticipated. The great majority of triggers of contingency are of this kind: each of the examples of teacher improvisation within instruction given in Schoenfeld's

monograph-length paper (op. cit.), including the four in-depth cases, are triggered by a student's comment.

Our data include three sub-types of triggers in this category. The first is the *child's response to a question* from the teacher; the second type is a *child's spontaneous response to an activity or discussion*; the third is when *a child gives an incorrect answer* to a question, or in the course of a discussion. Moreover, our data show that the teacher's response to unexpected ideas and suggestions from students is one of three kinds: *to ignore*, *to acknowledge but put aside*, and *to acknowledge and incorporate*.<sup>2</sup> We have given examples of these triggers and responses at previous PME meetings e.g. Rowland et al., 2004; Rowland, 2010; therefore the exemplification here will be quite brief.

*Chantal.* Chantal's lesson with a Year 1 class (pupil age 5-6) was videotaped in our first phase of data collection. To begin the lesson, the class was split into two groups and proceeded to count in ones, with the two groups alternating. One child observed that one group would be counting in odds, the other in evens. Although Chantal had not included this issue in her planning, she developed it further with the class, bringing their attention to final digits. She followed this up using a 1-100 number square: Chantal gave some two-digit examples and invited the children to say whether they were odd or even. Having told the children to begin counting at 1 (odd), a child asked if they could start at zero. Chantal then suggested they start at zero, telling them that zero is even. This lesson introduction was enhanced because of the *spontaneous response* of one child to the *counting activity*. Chantal responded by *acknowledging the child's idea*, and *incorporating it* into the lesson.

### Teacher insight

This kind of trigger, like the third type to follow, is much rarer in our novice-teacher data, although seasoned teachers might recognise it in their own experience. Schön's (1983) term 'reflective practitioner' conjures up the notion of teachers as professionals who learn from their own actions and those of others. In particular, Schön's 'reflection in action' is a kind of monitoring and self-regulation of teachers' actions as they perform them. The triggers in this category are the results of reflection in action, in which the teacher becomes aware, in the course of the lesson itself, that something is amiss. The first author recalls a lesson in which his students were intended to 'see' how the number of factors of a positive integer  $n$  can be found from the powers in its prime decomposition. He introduced his exposition with the example  $n = 72$ , reasoning that this integer is relatively small, yet rich in factors. As soon as he had written  $72 = 2^3 \times 3^2$  on the board he realised that this was not such a good example, since both 2 and 3 play dual roles in the decomposition, obscuring the significance of the indices as opposed to the specific primes. 72 was hastily replaced by 6125, and the reason for doing so was explained to the students later.

*Máire.* A preservice elementary teacher, Máire, was observed teaching a lesson on whole-number division to a class of 3rd and 4th Class girls (age 9-10 years) in Ireland

(see Corcoran, 2007, for further details). Máire had written worksheets on division, set in a fantasy Harry Potter scenario. The first two problems were as follows:

1. A pack of cards costs 3 Galleons. Ron has 18 Galleons. How many packs can he buy?
2. Fred and George want to buy magic worms to put in everyone's bed. They had 44 Galleons, and each worm costs four Galleons. How many worms could they buy?

Now there are two fundamental division problem structures (e.g. Vergnaud, 1983), variously called partition (or sharing) and quotition (or measurement, or grouping). In each of the two Harry Potter problems under discussion, the problem structure is quotition. They both begin with a certain supply of Galleons, and a fixed quota (3 Galleons, 4 Galleons), whereas the 'answer unit' is packs for one problem, worms for the other. However, in exploring how to resolve the problem, Máire drew on the language and concepts of partition. Máire had provided butter beans as manipulatives, to represent the Galleons. As Megan counted out 18 butter beans for the first problem, Máire asked:

Máire: How many groups does she [Megan] need to break it into and can you tell me why? Hannah, what do you think?

Hannah: Into three groups.

Máire: Into three groups. Well done. And why? You can read the question again if you want.

Máire's query here, to the *number of groups* and not to their size, pointed inappropriately to a partition structure, and Máire proceeded to congratulate Hannah ("Well done") on her inappropriate suggestion. However, Máire was then inspired to ask Hannah to explain ("and why?"). The interaction then takes a different direction.

Hannah: Because there's three packs of cards.

Máire was pulled up short at this point. She knew that there are *not* three packs of cards. Máire had inadvertently directed the pupils to the wrong division structure, she realised that this was so, and set about re-directing the structure of the solution.

Máire: It's not that there's three packs of cards. But what is it about the cards?

Hannah: It costs three galleons.

We see an instance of reflection-in-action in this episode, in a contingent moment. Máire brought about a significant and pedagogically important shift in the discourse, and Máire's in-the-moment pedagogical insight was the trigger for this shift.

### **Responding to the (un)availability of tools and resources**

Mathematics teachers, especially at the school level, frequently use artefacts to mediate abstract, intangible concepts, to help learners engage with, and make sense of, them. Such artefacts, 'apparatus', or resources, are intended as (external) representations of abstractions, though materials such as these are not culture-free, and could be viewed as tools in a cultural-historical sense. These resources include digital as well as analogue tools. Such materials have the potential to trigger contingent moments: on the one hand such a resource might be included as a central

tool in the plan, but then unexpectedly become unavailable; on the other, a resource which did not feature in the planning might be adopted opportunistically.

*John.* John was a pre-service teacher education participant in the most recent phase of our research, which aimed to test the application and adequacy of the KQ in the context of secondary mathematics teaching. In the first of his two videotaped lessons with a high-attaining Year 9 (age 13-14 years) class, John reminded the class how equivalent expressions of quadratic functions can be achieved by completing the square (CTS), and then aimed to demonstrate applications of CTS, including sketching the graphs of quadratic functions. John had planned to use a digital resource, *Autograph*, with data-projection technology, as a pedagogical tool in the graph-sketching phase of the lesson, aligned with an inductive, enquiry-based approach. In the event he was unable to activate the software<sup>3</sup>. The electronic resource that had underpinned his plan for this portion of the lesson, as well as the epistemological framing of the related action sequence, was no longer available to him. He resorted therefore to a more deductive exposition, albeit with interactive questioning, using a whiteboard. This technology-free exposition made different demands on John's mathematics-related knowledge, as he explained in the post-lesson interview.

TR: Might you have approached it differently if Autograph had loaded up?

John: I think the way that I was going to go about it was, because when I am sketching it, I'm the person who is giving them "trust me this is what it looks like" [so] I toyed with the idea of maybe showing the graph and then seeing if we could figure out where these points came from. What I also could have done with Autograph is, I wanted them to recognise that if you factorise a quadratic, you get your two solutions, you can find the minimum point by symmetry ... I would have just used it for verification mainly, I think.

TR: ... do you have any general views about the role of the computer in that way?

John: I was going to use it mainly for verification in this but ... if the IT had been there, I would have used the computer because of this ability to zoom in and move the graph around and all this sort of stuff. So as well as for verification, more sort of exploration ... but again I lost Autograph, so that went.

In this instance, then, the unavailability of the (digital) tool compels John to draw upon alternative knowledge resources, and forces a significant epistemological accommodation on his part.

*Chloë.* We mention briefly a participant in a much earlier study. Chloë was teaching Year 1/2 (age 5-7) pupils a particular class of strategies for adding and subtracting near-multiples of 10: adding/subtracting a multiple of 10, then compensating. Her planning, following the procedure and some of the examples in official guidance for teachers (DfEE, 1998), involved *symbolic* exposition and recording e.g. " $24-9=15$  because it is the same as  $24-10+1$ ". However, just as the lesson was about to begin, she noticed a large 1-100 square lying on a table. She mounted it on a vertical board and proceeded to demonstrate the compensation strategies *spatially*, by reference to

horizontal and vertical ‘moves’ on the grid. At the same time, she still wanted the children to record their work later symbolically, in accordance with her planning and the official guidance. Even at the end of the lesson, many children remained uncertain about how to use and apply the strategies. Chloë adopted the 1-100 square voluntarily as a resource, but without recognising how it would perturb her planning. This lesson was the subject of three-way analysis at a PME33 Research Forum (Ball et al., 2009).

## CONCLUSION

Like many other forms of human activity, teaching is a kind of improvised performance within a loosely structured framework of routines (Leinhardt, 1993). Contingent moments within classroom instruction are to be expected. Drawing on empirical data, Morine-Deshimer (1978-9, p.84) comments on “the amount of discrepancy that exists between the teacher plan and the classroom reality”. Etienne Wenger actually perceives contingent, what he calls ‘emergent’, aspects of teaching as underpinning a number of pedagogical issues, and writes:

Pedagogical debates traditionally focus on such choices as authority versus freedom, instruction versus discovery, individual versus collaborative learning, or lecturing versus hands-on experience. But the real issue underlying all these debates is the interaction of the planned with the emergent. Teaching must be opportunistic because it cannot control its own effects. (Wenger, 1998, p. 267)

While this is true, in that teaching activates student responses, our research shows that teachers’ experiences of their lessons can be ‘emergent’ in other respects. In particular, we have identified the articulation of students’ ideas as the most prevalent trigger of contingency in mathematics teaching, but not the only one. Teachers’ self-regulating awareness of the content and effects of their own teaching can also trigger a re-evaluation of their intended lesson agenda; the unexpected availability, or lack of, pedagogical resources can force epistemological and pedagogical digressions with (seemingly) no time for analysis of the likely consequences. Our findings lead us to suggest that the consideration of contingency - including its possible triggers, consequences, and demands on teachers’ knowledge resources - has an important but, as yet unrecognised, place in mathematics teacher education.

## Notes

<sup>1</sup> There is some correspondence here with Piaget’s distinction between assimilation and accommodation, although the analogy probably should not be pushed too hard. Unexpected events in a lesson need not cause undue discomfort if they ‘fit’ within the teacher’s knowledge structure e.g., that such events are known to happen in classrooms, the related mathematics is familiar etc.

<sup>2</sup> The second and third of these are implicit in the *a priori* possibilities raised by Schoenfeld (1998, pp. 3-4)

<sup>3</sup> There was nothing unusual about this: many teachers in service since the mid-1980s would recognise situations when the technology let them down, notwithstanding conscientious preparation and rehearsal.



## References

- Ball, D. et al. (2009) Research Forum 1: Teacher knowledge and teaching. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.) *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 121-152. Thessaloniki, Greece: PME.
- Bishop, A.J. (2001) 'Educating student teachers about values in mathematics education'. In F-L Lin and T. J. Cooney (Eds.) *Making Sense of Mathematics Teacher Education*. (pp. 233-246) Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Corcoran, D. (2007) "You don't need a tables book when you have butter beans!" Is there a need for mathematics pedagogy here? In D Pitta-Pantazi and G. Philippou (Eds.) *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 1856-1865). Nicosia, Cyprus: Department of Education, University of Cyprus. (Compact Disk)
- Corcoran, D. (2008) *Developing mathematical knowledge for teaching: A three-tiered study of Irish pre-service primary teachers*. Unpublished PhD thesis, University of Cambridge, UK.
- DFEE (1998) *National Numeracy Strategy, Framework for Teaching Mathematics: reception to Year 6*. London, DFEE.
- Glaser, B.G. & Strauss, A.L (1967) *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine de Gruyter.
- Leinhardt, G. (1993) On teaching. In Robert Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 1-54). Hillsdale, NJ: Erlbaum.
- Morine-Dersheimer, G. (1978-9) Planning in classroom reality: An in-depth look. *Educational Research Quarterly*, 3(4), 83-99.
- Rowland, T. (2010) Back to the data: Jason, and Elliot's quarters. In M.M.F. Pinto & T.F. Kawasaki (Eds.). *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 97-104. Belo Horizonte, Brazil: PME.
- Rowland, T., Huckstep, P. and Thwaites, A. (2004) Reflecting on prospective elementary teachers' mathematics content knowledge: the case of Laura. In M. J. Høines and A. B. Fugelstad, (Eds.) *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*. Vol. 4, pp. 121-128. Bergen, Norway: Bergen University College.
- Ryan, J. & Williams, J. (2007) *Children's Mathematics 4-15: Learning from errors and misconceptions*. Maidenhead: Open University Press.
- Schoenfeld, A.H. (1998) Toward a theory of teaching-in-context. *Issues in Education* 4(1), 1-94.
- Schön, D.A. (1983) *The reflective practitioner*. New York: Basic Books.
- Vergnaud, G. (1983) Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-175). New York: Academic Press.
- Wenger, E. (1998) *Communities of Practice: Learning, Meaning and Identity*. Cambridge: Cambridge University Press.



# A DIALOGIC APPROACH TO PLENARY PROBLEM SYNTHESIS

Kenneth Ruthven

Riikka Hofmann

Neil Mercer

University of Cambridge

University of Cambridge

University of Cambridge

*In this case study, a classroom episode featuring a dialogic approach to plenary problem synthesis is analysed, drawing on ideas and methods from recent research, including reassessment of the triadic model of instructional interaction. The overarching framework of teacher-led class discussion is identified, as well as more specific discourse features associated with student contributions and their handling.*

## BACKGROUND TO THE MODELLING OF CLASSROOM DIALOGUE

Our current research involves devising resources to help mathematics and science teachers develop effective use of dialogic approaches to classroom teaching and learning. A ‘dialogic’ approach is one that takes different points of view seriously (Scott, Mortimer & Aguiar, 2006), encouraging students to talk in an exploratory way that supports development of understanding (Mercer & Sams, 2006). One of the classroom activity structures that we have been examining is the commonplace one in which a problem situation is introduced to the whole class, students tackle it together in small groups, and a plenary discussion then synthesises ideas. Orchestrating this type of discussion to advance the learning of the whole class is acknowledged to be a significant challenge: pedagogical goals include facilitating public expression and respectful examination of students’ thinking; focusing – but not funnelling – discussion to prevent it becoming overly fragmented and incoherent; and guiding students towards accepted disciplinary norms of reasoning and communication (Franke, Kazemi & Battey, 2007; Stein, Engle, Smith & Hughes, 2008). Helping teachers to meet this challenge involves finding productive and efficient ways to ‘read’ and shape classroom discourse. What, in operational terms, can serve as crucial indices of dialogic discussion and as effective mechanisms for sustaining it?

Until recently there has been a pervasive assumption that the triadic structure of conventional instructional discourse acts against dialogic talk. In defining this triadic structure, it is generally agreed that the opening *move* of a classroom *exchange* is an *Initiation*, typically (but not necessarily) by the teacher, in which typically (but not necessarily) a question is posed. The second move of an exchange is a *Response*, typically (but not necessarily) from a student nominated by the teacher. In the third move of an exchange, the initiator acknowledges the response, and possibly reacts to it. Where the pedagogical function of the exchange is to allow students to apply or rehearse their knowledge, the teacher’s positioning within the exchange is typically authoritative: thus, in the third move, s/he provides an *Evaluation* of, or *Follow-up* on, the student response to the initiation. Hence this triadic structure is often referred to as IRE or IRF. Frequently, an E- or F-move is followed by a further I-move within

the same teacher turn. In this way an initial *nuclear* exchange can give rise to further *bound* exchanges that combine to form a single coherent *sequence* of talk. Likewise, while a move within an exchange is often accomplished through a single conversational *turn* by one speaker, it may include further turns from other speakers.

Researchers have come to realise that triadic interaction pervades classroom discourse across the spectrum of pedagogical activity from inquiry to instruction. They have suggested that fostering interactive, multivocal, dialogic discourse depends on using the triadic structure in particular ways, such as shifting from providing authoritative evaluation within the E- or F-move towards promoting further reflection or argumentation. Also highlighted have been increasing the ratio of exploratory and accountable talk by students to monologic and leading talk by the teacher (Truxaw & DeFranco, 2008), and carefully managing sequence initiation, knowledge authority, and reflexive commentary (Nassaji & Wells, 2000).

### THE CASE UNDER STUDY AND METHODS OF ANALYSIS EMPLOYED

This particular episode of plenary problem synthesis was chosen for study because it appeared to be a promising example to bootstrap reflection on dialogic teaching in action. Analysing this case would help us to scaffold discussion of it with teachers during professional development activity, and provide input to construction of an analytic instrument for purposes of research and training. The episode involved a first-year secondary-school class of around 30 pupils (aged 11/12 years) of above-average attainment, taught by a teacher who had recently become involved in the *epiSTEMe* project. The class had undertaken preparatory activities aimed at establishing suitable ground rules for effective classroom discussion. This particular episode occurred while the class were undertaking a subsequent probability module.

In line with an *epiSTEMe* design principle, the problem situation combined scientific with mathematical ideas. Earlier, the class had been introduced to the genetic model of how people come to have attached or detached earlobes. Earlobe type is determined by a pairing of genes, one inherited from the mother, one from the father. There are just two versions of this gene, known as alleles, and represented as *e* and *E*. Only people who inherit an *e-e* pairing have attached earlobes; others have detached. A parent cannot pass on an allele that is not in their own pairing. If a parent has both alleles, then these are equally likely to be passed on. After whole-class activity intended to help students grasp this model, they worked together in small groups on the following problem: *A couple are expecting their first baby. Both parents have a mixed pairing of e and E alleles. How likely is their baby to have this same pairing?*

To undertake analysis, classroom talk was transcribed from a video-recording of the lesson, and examined using methods drawing on the apparatus of discourse analysis and ideas about dialogic talk outlined in the previous section. In particular, we classify moves as I, R, or F; we also use F/I to indicate a move combining both these functions within a single turn. Limitations of space mean that we restrict ourselves to

presenting only the major sequences in the course of this 15-minute plenary episode with a view to conveying broadly how it unfolded from start to finish.

### ANALYSIS OF THE MAJOR SEQUENCES IN THE PLENARY EPISODE

- A1 T: So we have, both parents have one of each. [*Writing in margins of table drawn on board*] So the father has a big ee and a little ee, and the mother has a big ee and a little ee. And the question is, how likely is their baby to have the same pairing.
- A2 Ss: [*Overlapping responses including*] Very very likely. / Fifty per cent. / A hundred per cent. / One third.
- A3 T: So, we've been offered what? [*Recording on board*] A third. What else?
- A4 Ss: [*Overlapping responses including*] A half. / A whole.
- A5 T: [*Recording on board*] Shall we write fifty per cent? [*pause*] And what was the other one?
- A6 Ss: [*Overlapping responses including*] One whole. / One hundred per cent.

In the I-move [A1] of the opening sequence, the teacher revoices the problem, distancing herself from authorship ("we have", "the question is"). This functions to request student groups to report their answers. With the teacher neither nominating a respondent nor intervening during contributions, the R-move [A2] encompasses brief student turns that report differing answers. In her F/I-move [A3], the teacher chooses one answer to start a listing, and requests that the others be repeated. From the R-move [A4] the teacher selects a further answer for listing in her F/I-move [A5], employing the equivalent expression used in the earlier R-move [A2], then requests the last answer. In the R-move [A6], students offer equivalent expressions for this.

- B1 T: Okay. Would somebody like to sell us on a third please. So one of the groups that thought a third. Who thought a third?
- B2 Ss: [*Overlapping responses*]: We did. / We did.
- B3 T: You did. And you did. So we've got two groups. Who have we not heard from really. Vin. Can you tell us why you think a third, please.
- B4 S[Vin]: Because there's really three ways of forming pairs, a small ee and a big ee, two big ees, and two small ees. [*inaudible*] So it'll be a third that they've got the same pairing, small ee big ee.
- B5 T: Okay. What's this reminding you of? Anything? Coins. Who said coins?

In the ensuing I-move [B1], the teacher frames the next sequence as examining the persuasiveness of thinking behind the first proposed answer in the list. In a subsidiary exchange, she identifies groups favouring this answer and chooses a student speaker. She then repeats her request [B3], eliciting an extended R-move [B4] from the student. Her F/I-move [B5] accepts this as a response to the original request, then refers students back to an analogous problem that the class tackled earlier.

- C1 T: Okay. Who thought fifty per cent?
- C2 Ss: [*Overlapping responses including*] I do now. / We thought it.
- C3 T: Hex.

- C4 S[Hex]: Both parents have a mixed ee, a little ee and a big ee, and so you could either have a big ee or a little ee, it depends. Two big ees or a little ee and a big ee.
- C5 S: You what? *[Other student comments overlapping inaudibly]*
- C6 T: Nan.
- C7 S[Nan]: You can get a big ee, like the, whatever sort of earlobe it is you have, and you could have like two big ees and two small ees *[inaudible]*.
- C8 T: Okay. Tia, do you want to add to that?
- C9 S[Tia]: I'm starting to think it's a third because we've got a big ee and a small ee, that's one possibility and then you've got a big ee and a big ee, that's another possibility, and then you've got a small ee and a small ee, that's another possibility, so three possibilities. You've got to have one of those.

The opening of the next sequence echoes the preceding one, although the turns forming the teacher's I-moves [C1, C3] are briefer. On this occasion, however, the student R-move [C4] prompts an F-move in which other students express incomprehension [C5], and a teacher I-move nominating a new contributor [C6]. The ensuing R-move [C7] still does not develop a clear response to the request; a further F/I-move [C8] from the teacher asks another student to expand. In the R-move [C9] this student positions her thinking as being tentative, and develops a line of reasoning that leads to the first answer on the list, rather than to the second one as requested in the opening I-move. This elicits an F/I-move [D1] in which the teacher refers back to a probability principle introduced in earlier lessons. Because this refocuses the discussion, and introduces a fresh request, we treat it as opening a new sequence.

- D1 T: Are those three equally likely?
- D2 Ss: *[Overlapping responses including]* No. / Yes.
- D3 S[Tia]: Because they've got *[inaudible]*.
- D4 S: No, no they're not.
- D5 T: Hold on a minute.
- D6 S[Tia]: Because you've got a big ee and a big ee and a small ee and a small ee. So there's not more big ees than small ees, and there's not more small ees than big ees.
- D7 S[Nan]: But you only need one big ee to get the big ee type but you need two little ees. *[Other student comments overlapping inaudibly]*.

The teacher's opening F/I move [D1] raises the question of whether the principle applies to the set of possibilities just identified. The initial turns in the ensuing R-move [D2] evidence disagreement amongst students, then an attempt at explanation [D3]. The intervention by another student [D4] starts to counter this contribution, but the teacher reinstates the earlier speaker [D5]. This student continues her contribution [D6], eliciting a further counter from another student [D7]. These counters function as student F-moves [D4, D7] to another student's earlier R-move [D3, D6] as well as further student R-moves in response to the teacher's original question. We treat the redirection through the ensuing teacher I-move [E1] as opening a new sequence.

- E1 T: *[Pointing to table drawn on board]* We have here the mother and the father so can this help us do you think?
- E2 Ss: *[Overlapping responses including]* No. / Yes. / I don't know.
- E3 T: *[Pointing to table drawn on board]* So the mother can give us a big ee and the father could give us a big ee. The mother could give us a big ee and the father could give you a little ee. The mother could give us a little ee and the father a big ee.
- E4 S: *[Overlapping]* Oh, there's four there.
- E5 T: And the mother could give us a little ee and the father could give us a little ee.
- E6 S: So it's a quarter.
- E7 S: No it's a half. *[Lesson is interrupted]*

In the I-move [E1], the teacher draws attention to a further available resource, prefigured by her when first presenting the problem [A1]. Her request is a very open one, and in the ensuing R-move [E2] students simply register differing reactions. The teacher's F-move then launches use of the resource for systematic analysis of the problem [E3, E5]. In doing so the teacher positions herself as knowledgeable, and embarks for the first time on an extended substantive turn. In parallel, however, in what might be termed *prevoicing*, students volunteer contributions that anticipate her conclusion [E4] and pursue its implications through statement [E6] and counter [E7].

- F1 T: Vin, does this make any difference to you?
- F2 S[Vin]: *[pause]* It's actually four possibilities instead of three possibilities.
- F3 T: There's a difference isn't there between the combinations we've got here, the genes they might have, and what sort of earlobes they might have.

Following an external interruption to the lesson, the teacher returns to the student who originally spoke in favour of the first answer. Her I-move [F1] requests him to describe whether his thinking has changed. His R-move shows his recognition that the four ordered pairings shown in the table constitute the appropriate set of outcomes rather than the (three) unordered combinations. The teacher's F-move [F3] highlights this distinction, and makes a further contrast with the (two) earlobe types.

Several student-originated sequences follow; in one of these a student raises the third of the listed answers, the only one that has not yet received public attention.

- G1 S[Bet]: I think they're going to have one of each, because if its mum's got one of each and its dad's got one of each you have to raise all the possibilities, right? When I first looked at it I didn't like look as deep as we have, like I said that its mum's got one of each and its dad's got one of each, so it's definitely going to have one of each. Because that's how I first saw it, but like, now we've like dug in a bit deeper, whereas, because like, say your dad has brown hair and your mum has brown hair and people would say you would have brown hair, so that's how I looked at it, his mum's got both, one of each and his dad's got one of each so you'll have one of each.
- G2 T: So you started off looking at it as though if they've both got one of each you're bound to have one of each. How do you think now?

- G3 S[Bet]: Well I kind of still think that, because if you look there's like two possibilities.
- G4 T: Are these not possible to happen then [*pointing to two pairings in table*]?
- G5 S[Bet]: They are possible but I still think that you are probably going to have one of each because if you look, there's two like, there's small ee and a big ee, and then a big ee and a small ee, so I still think that you're more likely to have one of each.
- G6 T: So can we put a number on this more likely? Bob.
- G7 S[Bob]: Can we say fifty fifty, because there's four chances there [*pointing to table*] of what's going to happen, and two of them are the same as what the mother and father have, so that's fifty fifty.
- G8 T: We can if we all agree.

After a teacher I-move that simply nominates a speaker, the student's R-move [G1] reports her current thinking and offers an account of how that thinking has evolved. The teacher's F/I-move [G2] revoices the student's shift away from a lay view of inheritance, and requests clarification of the student's current thinking. The student's R-move [G3] appears to continue to endorse some form of the lay view. The teacher's F/I-move [G4] is then a substantive one that focuses on those gene pairings that, according to the lay view, could not arise in these circumstances. In her R-move [G5] the student accepts that these pairings are "possible", and appeals to the displayed table to argue that the mixed pairing is "probable", refined to "more likely". The teacher's F/I-move [G6] requests quantification from another student. In his R-move [G7] the student provides supporting reasoning for his answer but invites approval of it ("Can we say"). The teacher's F-move [G8] appeals to class consensus as the criterion for accepting an answer, distancing herself from knowledge authority.

The teacher then polls the class, revealing continuing differences and uncertainties. Reacting to some restlessness, the teacher emphasises the goal of everyone being persuaded of an agreed answer [H1]. A little later in the ensuing sequence the teacher responds to further restlessness by elaborating norms of engagement for students taking the majority view [H8]. This ensuing sequence is again student-originated:

- H1 T: There is an answer to this and we need to work out what it is. And we all need to believe it. Because this is really quite important. [*pause*] Lea.
- H2 S[Lea]: If you do four time three then you get twelve so if you've got a large ee and a little ee you're going to get the same sort of thing, so it's sort of the same, which makes one third doesn't it?
- H3 Ss: [*Overlapping responses*] Whaaaat?
- H4 T: Do correct me if I'm wrong here Lea, what Lea's saying is that this one and this one [*pointing to two pairings in table*] are the same basically. And therefore she thinks that she's saying that there is still only three different outcomes and therefore they're a third each.
- H5 S[Lea]: Yeah I'm a bit confused.
- H6 T: Kit.
- H7 S[Kit]: There's three different outcomes but there's two ways of getting one outcome so that outcome has a higher probability than the other two.

H8 T: Okay. So Lea, does that make, does that make any difference do you think? Kit's saying that although there are only three outcomes *[pause]* All those people who are getting restless think of something that you can tell us that will convince people of what's going on, convince them of what you believe, because the majority of you are saying that there's a fifty per cent chance they will have the same grouping as their parents. We have some people who don't agree, and they have good reasons for not agreeing, but, if you're sitting there fiddling, think of something you can say to help them understand. *[pause]*. So we were just saying that, what Kit said was, although these *[pointing to two pairings in table]* are the same, you've still got one of each, there are two different ways that it can happen. Is that right Kit? There are two different ways that this *[pointing to two pairings in table]* can happen.

H9 S[Kit]: Yes.

H10 T: So these ones have two chances, whereas there's only one way of a big ee big ee and one way of a little ee little ee, there are two ways for this to happen, and so they have a fifty per cent chance. Is that what you're saying?

H11 S[Kit]: Yes.

H12 T: Rather than a third.

The teacher's I-move [H1] nominates a student. Her R-move [H2] offers an analogy that prompts an F-move [H3] from other students voicing incomprehension. The teacher's F/I-move [H4] attempts repair through speculative revoicing, referring back to the student but eliciting an uncertain R-move [H5]. An I-move [H6] nominates another student. Her R-move [H7] highlights the crucial mathematical issue. Framing discussion as being between these two students, the teacher's ensuing F/I-moves [H8, H10] revoice this contribution, and refer back to its author, eliciting R-moves [H9, H11] granting approval. The teacher's final F-move [H12] makes the contrast with the answer that students had endorsed earlier. In this sequence, then, the teacher's extended turns serve mainly to revoice and interanimate student contributions.

After more student-originated sequences (which limited space prevents us examining here), the teacher concludes the episode [J1] as the close of the lesson approaches.

J1 T: We're going to leave it unresolved for the minute, so all of you *[pause]*, all of you need to give it some thought, please, before the next lesson.

## A FUNCTIONING APPROACH TO PLENARY PROBLEM SYNTHESIS

This case study has examined a dialogic approach to plenary synthesis in action. The opening sequence [A] identifies different viewpoints within the class, and the ensuing sequences [B, C, D] set out to elicit these more fully. This establishes an overarching dialogic framework within which teacher moves serve predominantly to organise and support student articulation of mathematical thinking, and to strategically prompt students to relate that thinking to examples [B5] or principles [D1] or tools [E1] that the class has already encountered. Equally, however, the teacher has anticipated that a particular representation of the problem is likely to support productive discussion. As contributions from a succession of students fail to develop a persuasive account of the mathematically accepted viewpoint the teacher originates a sequence [E] in which



she then actively enters the discussion. Her introduction of the representation triggers student contributions that enable her to resume [in F] the more detached position that is the default for the teacher. Likewise, in the following student-originated sequences [such as G, H], although initiative remains with the teacher, her moves predominantly serve to elicit, revoice and interanimate student contributions, and to strategically bound and focus the discussion. When she does enter the discussion more directly [as in G4, G6], she quickly moves to distance herself from knowledge authority [G8].

Thus the great majority of substantive contributions to the discussion are made by students, and these are relatively lengthy: typically around 20 to 40 words. While initial ideas come from earlier groupwork, students' reflexive commentaries [e.g. C9, G1] and tentative formulations [e.g. C7, G7] show how the unfolding discussion becomes more exploratory. Direct evaluation of, or feedback on, contributions comes only from students. Contrasting multivocal responses also provide more indirect feedback. Indeed, the relatively loose regulation by this teacher of background conversation and discussion turn-taking seems to play an important part not just in sustaining engagement of students, but in eliciting multivocal responses [as in A, B, C] and spontaneous voluntary contributions [most crucially in E]. As a majority view emerges and some students become restless, the teacher emphasises dialogic norms that call for them to continue formulating potentially helpful contributions [H1, H8].

Finally, as regards our analysis, although the apparatus of sequences and moves has provided a useful heuristic framework, we have noted how – as multiple voices and viewpoints enter an exchange – limitations of a simple IRF model are exposed. While previous research has shown how such a model could be elaborated, we remain somewhat sceptical about the practical and intellectual cost/benefit ratios of doing so.

## References

- Franke, K.L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225-356). Charlotte, NC: Information Age Publishing
- Mercer, N., & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education*, 20(6), 507-528.
- Nassaji, H., & Wells, G. (2000). What's the use of 'triadic dialogue'? An investigation of teacher-student interaction. *Applied Linguistics*, 21(3), 376-406.
- Scott, P., Mortimer, E. & Aguiar, O. (2006). The tension between authoritative and dialogic discourse: a fundamental characteristic of meaning making interactions in high school science lessons. *Science Education*, 90(4), 605-631.
- Stein, M.K., Engle, R.A., Smith, M.S., & Hughes, E.K. (2008). Orchestrating productive mathematical discussions: Helping teachers learn to better incorporate student thinking. *Mathematical Thinking and Learning*, 10(4), 313-340.
- Truxaw, M. P. & DeFranco, T.C. (2008). Mapping mathematics classroom discourse and its implications for models of teaching. *Journal for Research in Mathematics Education*, 39(5), 489-525.



# DEVELOPMENT OF A LEARNING TRAJECTORY TO CONCEPTUALIZE AND REPRESENT VOLUME USING TOP-VIEW CODING

Jacqueline Sack

University of Houston-Downtown

Irma Vazquez

Houston Independent School District

*Development and adaptation of a learning trajectory to develop volume concepts for third graders is described. Prior to the intervention, most of the children in the study were unable to accurately enumerate cube structures or prism figures drawn with unit cubes if any of the cubes were hidden. Subsequent to the intervention, utilizing a dynamic computer interface, Geocadabra (Lecluse, 2005), most were successful and individually able to articulate or represent the concept.*

## INTRODUCTION

This research team intends to fill a gap in the K-12 curriculum in spatial development (National Research Council, 2006) through new and modified curricular activities guided by the Spatial Operational Capacity (SOC) framework developed by van Niekirk (1997) based on Yakimanskaya's (1991) work. Using design-research (Cobb, et al, 2003) principles this study's immediate goal is to describe children's mathematical sense making of these activities. It is conducted in a dual-language urban elementary school within one of the largest public school districts in the mid-southwestern United States. This paper focuses on the refinement of learning trajectories to support children's conceptual understanding of the volume formula for rectangular 3D arrays using top-view numeric coding.

## THEORETICAL FRAMEWORKS

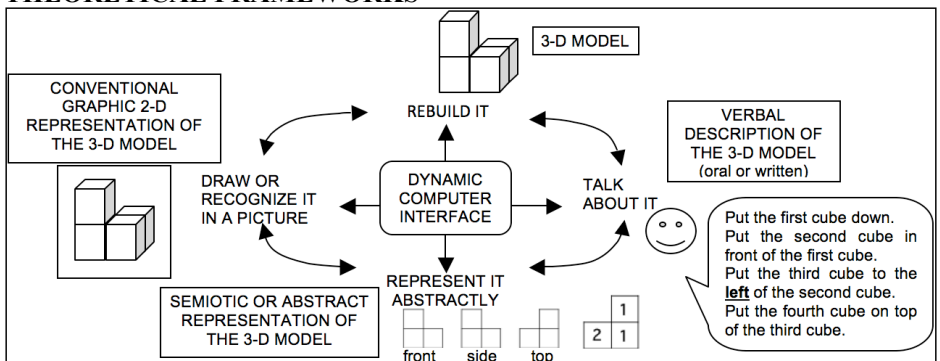


Figure 1. Multiple representations within 3-D visualization

The spatial operation capacity (SOC) framework (van Niekirk, 1997; Sack & van Niekirk, 2009) that guides this study exposes children to activities that require them to act on a variety of physical and mental objects and transformations, as prescribed by the National Council of Teachers of Mathematics (NCTM, 2000) to develop the

skills necessary for solving spatial problems. The framework (see Figure 1) uses: full-scale figures, that, in this study, are created from loose cubes or Soma figures, made from 27 unit cubes glued together in different 3- or 4-cube arrangements (see Figure 2); conventional 2D pictures that resemble the 3D figures; semiotic representations such as front, top and side views or numeric top-view codings that do not obviously resemble the 3D figures; and, verbal descriptions using appropriate mathematical language (Sack & Vazquez, 2008).

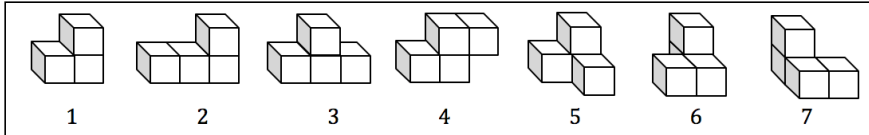


Figure 2. The Soma set can be made by gluing unit cubes together.

The project utilizes a dynamic computer interface, Geocadabra (Lecluse, 2005). Through its Construction Box module, complex, multi-cube structures can be viewed as 2-D conventional representations or as top, side and front views or numeric top-view grid codings (see Figure 3).

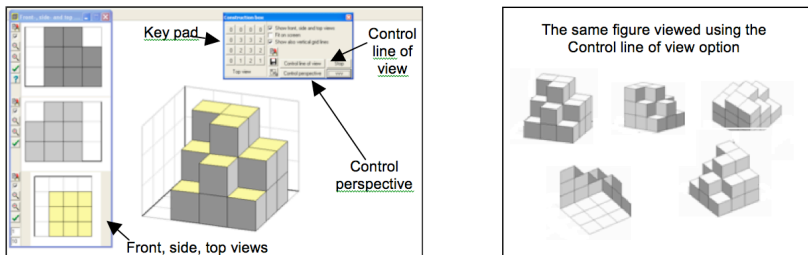


Figure 3. The Geocadabra Construction Box

## METHODOLOGY AND CONTEXT

Design research methodology (Cobb, et al, 2003) guides this study's instructional decisions based on learning trajectories developed from an instrumentalist standpoint (Baroody, et al, 2004). This conceptual and problem-solving approach aims for "mastery of basic skills, conceptual learning, and mathematical thinking" using any "relatively efficient and effective procedure as opposed to a predetermined or standard one" (p. 228). Each lesson is part of a design experiment followed by a retrospective analysis in which the research team determines the actual outcomes and then plans the next lesson. This may be an iteration of the last lesson to improve the outcomes, a rejection of the last lesson if it failed to produce adequate progress toward the desired outcomes, or a change in direction if unexpected, but interesting, outcomes arose that are worthy of more attention.

Since the project's inception in 2007-2008, a university-based researcher and two teacher-researchers form the research team that works with a group of children weekly for one hour in teacher-researcher, Vazquez' 3rd-grade classroom within the

school's existing after-school program. English and Spanish parent/guardian and student consent-to-participate forms are sent home to parents of all 3rd grade children. All respondents are accepted into the program. The research team uses socially mediated instructional approaches to support a problem-solving environment that fosters students' creativity according to readiness and interest. The following data sources are used: formal and informal interviews, video-recordings and transcriptions, field notes, student products and lesson notes.

## RESULTS AND DISCUSSION

During introductory lessons the children interact with loose cubes and the Soma figures solving problems with the 3D models and with 2D task cards that illustrate combinations of the Soma figures requiring figure identification and classification. By the middle of the second month, the children begin to use the Geocadabra Construction Box to digitally reproduce figures printed in a customized manual (e.g., see Figure 4). These activities provide the children opportunities to coordinate numeric top-view codings with 2D pictures. Children who need scaffolding may first build the 3D figures using loose wooden cubes.

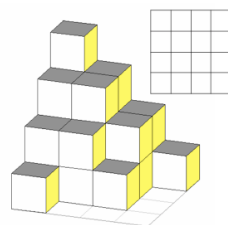


Figure 4. Task 1f

During Years 1 and 2, the research team's instructional planning was guided by the children's interest in top-view coding. Using Geocadabra, children created their own 2D task-cards of structures they had assembled with two Soma figures. Using these task cards, they drew coding puzzles for peers to decode and check. Later, they developed their own ways of coding Soma assemblies that had holes or overhangs and negotiated a non-conventional coding system for the whole class to use (Sack & Vazquez, 2009). The children became very proficient at moving among the SOC framework's 2D pictures, top-view numerical coding and 3D digital dynamic representations on Geocadabra.

During Year 2, an unexpected and interesting event occurred when five Year-1 children returned for help with the concept of rectangular volume in their regular academic classes where they were struggling with the formula. Using an initial learning trajectory based on the work of Battista (1999), the children attempted to solve volume problems by folding nets drawn on grid paper. They struggled to connect the dimensions of the flaps of each net with the height of its corresponding 3D figure. Using a contextual scenario, "Ms. Moral's Shoes," 24 shoeboxes must be shipped to a nearby city and the children, using loose wooden cubes to model the shoeboxes, were expected to find all possible combinations of rectangular arrays with 24-cubic-unit volumes. The research team was surprised to see them record their findings as numeric top-view codings rather than directly with the length-width-height formula. Connections between top-view coding and the volume formula evolved through guided discussion among the teacher and participant children. Figure 5 shows how the children recorded their work.

As a result, the team designed a new Year-3 learning trajectory to explicitly develop pre-formula volume conceptualization. The objectives were 1) to coordinate the number of cubes in a 2D conventional picture with the sum of the numbers in its numeric top-view coding; and, 2) to discover and integrate the numeric top-view coding representation with the volume formula for 3D rectangular arrays using and reinforcing emergent multiplication skills.

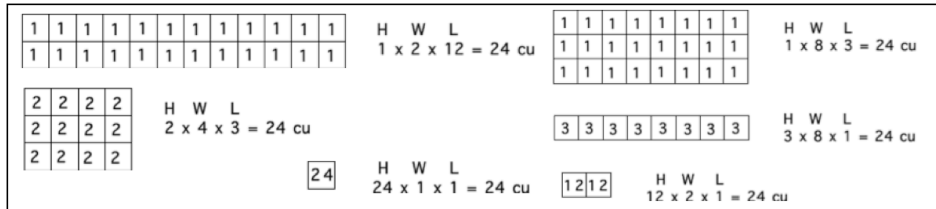
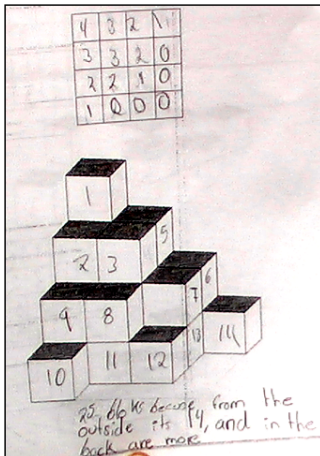


Figure 5. Recordings of 24-cube prisms

During the Year 3 children's initial work with the computer and the Geocadabra manual, they were individually challenged to determine how many unit cubes made up the figure in Task 1f (see Figure 4). After noticing wide discrepancies in children's interpretations, Evan used the Construction Box to build and project the figure to enable whole-class discussion.

Excerpts from field notes, November 2, 2009:



[The teacher] asked the class how many cubes were in the figure. Most said 24, but some said 25, 13 and 11. Those who said 24 seemed to add the numbers in the grid. . . . They pointed out that the top row (of the grid) had  $4 + 3 + 2 + 1 = 10$  cubes, and then  $8 [3 + 3 + 2]$  in front of that [row] and so on until they determined 24 cubes in all. The child who said 25 had enumerated incorrectly, but was able to explain where the 13 came from. He labeled his picture [shown to the left] as follows: 1, 2, 3, 5, 6, 7, . . . (he skipped #4 and got 14 instead of 13). Of note: He clearly counts each cube rather than visible faces. This is an exceptional step that may be supported by the fact that they have used the actual 3D cubes for the past 6 lessons in conjunction with 2D pictures. Several children said that 13 came from looking at only the visible cubes in

the picture and that there are several more hidden behind and/or below what is visible.

Evan showed how he obtained 11 by counting only the grid squares that had non-zero numbers in them. He counted squares rather than the number of cubes in the squares.

Alan said that 11 came from counting only the cubes with black faces in the picture.

The research team was surprised that none of the children had declared 30 or 20 cubes, based on counting all visible cube faces or all visible lateral faces, since this

was found to be a common counting misconception reported in Battista's study on 3D-array enumeration (1999). Those who counted only visible cubes or non-zero grid positions were convinced that there were more cubes when Evan rotated the figure to show the initially-hidden cubes on the left side and back of the figure, demonstrating the impact of the software's dynamic Control-line-of-view function.

Next, the children worked at their own computer stations to create figures that had exactly 24 cubes. This required coordination of the numbers in the numeric top-view Construction Box grid with the 2D figure each had created on the screen. Some used the Control-line-of-view option to rotate the screen figure to verify the 24-cube sum. These figures were formatted into task cards used to reinforce coordination of the numeric top-view grid with the 2D picture. Each child drew a grid on blank paper and, using a task card with someone else's 2D figure, determined its numeric coding (see Figure 6). Then, using only the written coding, each child built the figure on the computer to verify the numeric solution. To address the second objective (integrating top-view coding with the volume formula), the class was asked to build a rectangular array using 24 loose cubes and then find as many different 24-cube arrays as possible (see Figure 6). The children continued to find other combinations of 24-cube 3D arrays including their permutations.

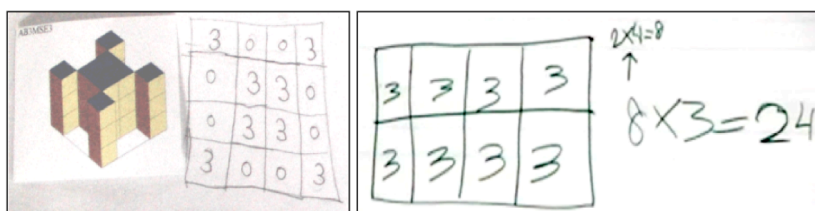


Figure 6. Sample codings of a task card and of a 24-cube array

For Year 4, having started with a pre-interview with a new group of 17 children, the research team is also administering periodic paper and pencil assessments as the trajectory evolves in order to track the progress of every single child in the current cohort. This will provide additional close-grained data to confirm prior results.

For the pre-interview, each child was shown only one item at a time (Figure 7, in order from 1 through 6, and A through E. Items 1-4 intend to determine whether orientation or partially hidden cubes affects a child's enumeration of cubes. Only one child enumerated items 1-4 incorrectly. Items 5-6 contain hidden cubes and only 1 child was able to enumerate these correctly. Sixteen children counted only the visible cubes in item 5. For item 6, thirteen children counted only the visible cubes; four recognized that there were hidden cubes but enumerated them incorrectly.

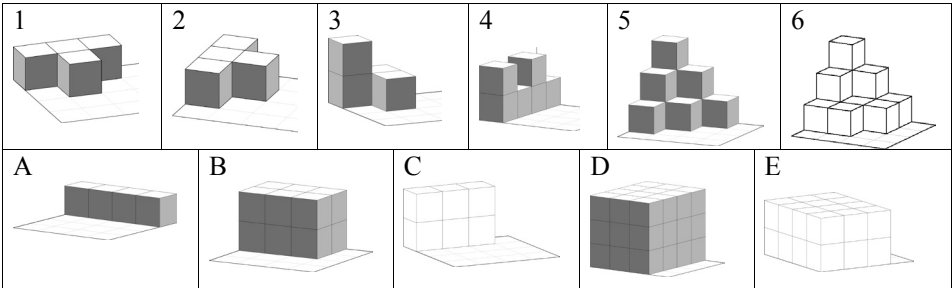


Figure 7. Pre-interview items, September 13, 20, 2010

Items A-E intend to determine if children can visualize repeated rows or faces since these prism structures are not presented in prior mathematics classes but is a new objective for third grade as defined by the state education agency. Items A and C were enumerated correctly by all children. Nine children enumerated item B correctly as two layers, either top-bottom or front-back. Only three children enumerated items D and E correctly. Errors included counting all visible faces; one layer of cubes combined with face counting; two vertical sides with top faces (repeated counts); and, guessing how many cubes were invisible.

The Construction Box activities were used at the start of Year 4 with strong emphasis on determining the number of cubes in each figure in the Geocadabra manual. Since figures like Task 1f were more complicated than those used for the pre-interview, new figures were designed for an interim assessment (see Figure 8 and Table 1).

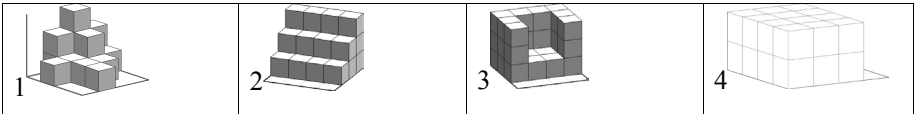


Figure 8. Interim assessment items, November 29, 2010

Assessment Item	Correct coding and enumeration	Incorrect enumeration with correct coding	Incorrect enumeration and coding
1	8	1	2
2	11	1	0
3	5	5	1
4	6	2	3

Table 1. Response frequencies for interim assessment, November 29, 2010

After only four one-hour sessions using the Construction Box interface, eight out of eleven children enumerated assessment item 1 correctly while on similar pre-assessment items 5 and 6 only one out of seventeen was successful.

For assessment item 2 some children enumerated based on their top-view coding grid; others recognized the layer structure in the 2D picture having four sets of 3 at

the back, four sets of 2 in the middle and four 1s in the front. The side view of the structure may have enabled some to recognize the figure's vertical layering. Figure 9 shows how one child used different numeric representations for her solution, reinforcing developing multiplication skills.

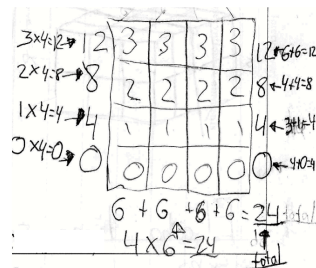


Figure 9. A solution for Item 2

Assessment items 3 and 4: Ten and eight children coded these figures correctly respectively, indicating a high degree of coding capacity. The mathematics component of their school day focuses heavily on multiplication proficiency and these children vary considerably in their numeracy skills, which could account for the variation in numerical accuracy on these two assessment items.

The research team continues to challenge the Year 4 group with additional coding and de-coding activities and will include additional interim assessments to track progress in rectangular array enumeration. These activities within the learning trajectory have a strong problem-solving component and have enabled the children to integrate their developing visual capacities with other mathematics strands.

## CONCLUSIONS

Activities for third-grade children that move to-and-fro among the three visual SOC representations facilitate their conceptualization of the volume formula for rectangular prisms. The initial coding activities that involve non-specific cube structures (e.g., Figure 4) were refined and now include rectangular arrays. To enumerate and justify children may add the numbers in the top view coding or cumulate the number of cubes in each layer or stack of cubes in the 2D pictures. In this way, the coding of rectangular array structures fosters children's understanding of the volume formula in concert with their emerging multiplication skills.

Outhred, et al, (2003), referring to the complexity of representing volume measurement compared to rectangular area measurement, state that "the process is more complex because students have to coordinate three dimensions and diagrams cannot show the layer structure clearly" (p, 84). It is significant that young children in this study were able to use top-view coding as a viable representation and enumeration of 3D arrays. Using mental imagery, they explain that each number in the rectangular grid represents the height of a stack of cubes on a space, a measurement rather than a numeracy construct. Battista (1999) reports a common miscounting error when children enumerate cubes in arrays shown in conventional 2D pictures. They may count the same cube three times if it is located on a vertex or twice if located on an edge based on the number of visible faces. When presented figures, such as that in Task 1f (Figure 4), none of the children in this study counted all visible faces when asked for the total number of cubes as some had in the Year 4



pre-interview. Moving to-and-fro among the SOC visual representations supported by the Geocadabra computer interface may avert this misconception.

## References

- Baroody, A. J., Cibulskis, J., Lai, M., & Li, X. (2004) Comments on the use of learning trajectories in curriculum development and research. *Mathematical Thinking and Learning*, 6(2), 227-260.
- Battista, M. (1999). Fifth graders enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, 30 (4), 417-448.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9-13.
- Lecluse, A. (2005). *Geocadabra* [Computer software].  
<http://home.casema.nl/alecluse/setupeng.exe>
- National Research Council (NRC). (2006). *Learning to think spatially: GIS as a support system in the K-12 curriculum*. Washington, D.C.: Author.
- National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Outhred, L., Mitchelmore, M., McPhail, D., & Gould, P. (2003). Count Me into Measurement: A program for the early elementary school. In Clements, D. H., & Bright, G. (Eds.) *Learning and Teaching Measurement: Sixty-fifth yearbook*, NCTM, Reston, VA, 81-99.
- Sack, J., & van Niekerk, R. (2009). Developing the spatial operational capacity of young children using wooden cubes and dynamic simulation software. In Craine, T., & Rubenstein, R. (Eds.) *Understanding Geometry for a Changing World: Seventy-first yearbook*. NCTM, Reston, VA, 141-154.
- Sack, J., & Vazquez, I. (2008). Three-dimensional visualization: Children's non-conventional verbal representations. In Figueras, O., Cortina, J. L., Alatorre, S., Rojano, T., & Sepulveda, A. (Eds.) *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX*. Mexico: Cinvestav-UMSNH. 4, 217-224.
- Sack, J., & Vazquez, I. (September, 2009). Elementary children's 3-D visualization development: Representing top views. *Proceedings of the Thirty-First Annual Meeting of PME-NA*, Atlanta, GA.
- van Niekerk, (Retha) H. M. (1997). *A subject didactical analysis of the development of the spatial knowledge of young children through a problem-centered approach to mathematics teaching and learning*. Ph.D. diss., Potchefstroom University for CHE, South Africa.
- Yakimanskaya, I. S. (1991). *The development of spatial thinking in school children*. (P. S. Wilson and E. J. Davis, Trans.), Vol. 5 of Soviet Studies in Mathematics Education, Reston, VA: NCTM.



# ASPECTS OF CONCEIVING STOCHASTIC EXPERIMENTS<sup>1</sup>

Luis Saldanha

Arizona State University

*This article reports on a teaching experiment that engaged a group of high school students in designing sampling simulations within a computer micro world. The simulation-design activities provided a vehicle for exploring students' efforts to conceptualize situations as idealized stochastic experiments. The study highlights challenges that students experienced and that shed light on aspects of conceiving a trial of an experiment.*

## **Background**

Mills' (2002) review of the literature on the use of computer simulations in statistics education reveals that a panoply of instructional approaches and software resources have been developed and used over the last quarter century for teaching a variety of statistical topics. Mills also noted the paucity of empirical research into the impact of simulation-based instruction on students' understanding of statistical concepts. Nevertheless, several notable empirical studies of students' statistical reasoning or achievement in relation to their participation in instruction involving the use of computer simulations have been conducted (delMas, Chance, & Garfield, 1999; Kuhn, Hoppe, Lingnau, & Wichmann, 2006; Pratt, 2000; Sedlemeier, 1999; Stohl & Tarr, 2002). A subset of these studies (Kuhn et al., 2006; Pratt, 2000; Stohl & Tarr, 2002) engaged students in *designing* components of simulations, in the context of modelling given stochastic situations within specially designed micro worlds. Two salient commonalities among this focused subset of studies frame the study reported here. The first commonality was the prominence of the idea of repeated sampling and long-run relative frequency as a basis for supporting students' understanding of ideas of likelihood. The second commonality was the use of simulation environments involving virtual versions of canonical random devices such as spinners and dice. As such, they provided students with pre-determined models that provided a certain transparency to the modelled phenomenon.

## **purpose and method**

In the study reported here repeated sampling and long-term relative frequency of outcomes were also seen as foundational to a coherent understanding of likelihood. However, this study differs from the aforementioned ones in the nature of the tasks and the micro world employed in it, which in turn reflects the different purpose of this research. Instead of having students work with predetermined models and virtual

---

<sup>1</sup> The research reported here was supported by the National Science Foundation (Grant No. REC-9811879). Any conclusions or opinions stated here are those of the author and do not necessarily reflect official positions of NSF.

versions of canonical random devices in the modelling process, this study's tasks and micro world aimed to provide them with a metaphor for thinking about simulating stochastic experiments in terms of a population, a sample, and a repeatable method of selecting the former from the latter. This set-up—in which the relationship between the modelled situation and the modelling tool was not so transparent—provided opportunity to investigate aspects of conceiving contextual situations *as having a stochastic structure*. This study thus differs significantly from those cited above in that its primary goal was to gain insight into conceptual problems of learning to model such situations explicitly in terms of a stochastic experiment—an important area of reasoning about which relatively little is known in the stochastics education research literature. This goal was advanced by conducting a classroom teaching experiment that explored the thinking of a group of high school students as they engaged with tasks designed to foster their creation and use of computer simulations for making informal inferences and testing hypotheses in situations involving the construal of a stochastic experiment.

Eight students participated in the classroom teaching experiment conducted in their intact introductory statistics course. Instructional activities entailed designing sampling simulations within the Prob Sim micro world (Konold & Miller, 1996). Prob Sim's user interface employs the metaphor of a “mixer” for a population and a “bin” for a sample; it enables the user to easily specify a population's composition and size, the size and selection method of a sample (with or without replacement), and the number of trials of the simulated sampling experiment to be conducted (see Figure 1, right panel)<sup>2</sup>. The simulation-design activities presented students with a contextual situation in non-statistical terms and asked them to investigate whether a specified event of interest might be unusual (see Figure 1, left panel). Their task was to design a simulation, guided by the constraints of the Prob Sim interface, to run the simulation and then, on the basis of its results, draw a conclusion about the event's unusualness. The left panel of Figure 1 displays a representative example of a simulation design activity that will be discussed in the next section of the paper. These activities built on the foundational idea of repeated sampling (as a strategy for exploring a hypothesis about particular kinds of sampling outcomes) that had been developed with students in a preceding phase of the experiment. Activities unfolded over four 45-55-minute class periods held on consecutive school days. They typically involved whole-class discussions directed at having students think and describe how they would use Prob Sim to investigate and resolve the question raised in a given situation. A retrospective analysis of the classroom discursive data was conducted (all sessions were videotaped), documenting evidence of students' understandings and challenges they experienced in their efforts to conceive of contextual situations in terms of a stochastic experiment. Following Cobb and Whitenack's (1996) method of videotape analysis, analytical procedures involved

---

<sup>2</sup> The software also displays the outcome of each simulated sampling experiment as a raw data list, and it provides summary analyses of the aggregated outcomes and displays their distribution as a relative frequency histogram.

generating and applying descriptive and explanatory codes in an effort to capture students' salient understandings and images.

**INVESTIGATING “UNUSUALNESS”**

Ephram works at a theatre, taking tickets for one movie per night at a theatre that holds 250 people. The town has 30,000 people. He estimates that he knows 300 of them by name.

Ephram noticed that he often saw at least two people he knew. Is it in fact unusual that at least two people Ephram knows attend the movie he shows, or could people be coming because he is there? (The theatre holds 250 people)

Assumptions for your investigation:

Method of investigation: “Gut level” answer:

Result:

Conclusion:

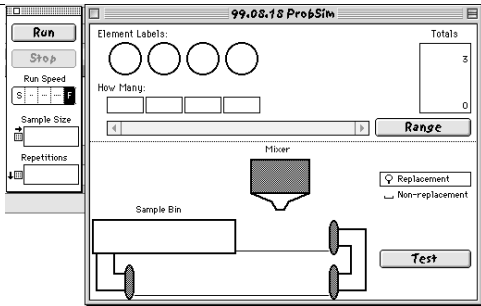


Figure 1. The “Movie Theatre” task (Konold, 2002), involving the design of a sampling simulation (left panel) in the Prob Sim micro world (right panel).

## RESULTS

The classroom discussions around the movie theatre situation revealed difficulties that students experienced in re-conceiving a given situation as an idealized stochastic experiment. This was a significant challenge for most students; as illustrated in the following data excerpts, their progress was tentative and tightly embedded within their classroom interactions with the instructor.

### Constructing Assumptions

The following sequence of data excerpts is drawn from the classroom discussions centered on explicating the assumptions for simulating the movie theatre situation (Figure 1). Excerpt 1 illustrates David's sense of overwhelm by the possibilities for assumptions.

#### Excerpt 1 (Lesson 6)

428. David: I didn't get this question 'cause it, there are so many different things that could happen. Like, what if only half the town goes to see movies? Or uhh what if it's the same 2 people every night, that he sees? It says he knows 300, but couldn't, like, the same 2 people go see the movie every night?
429. Nicole: Yeah
430. Instr: Sure, that's right. So that's where you lay—
431. Peter (to David): Good thinking!

432. Instr (continues): you settle all of this in your assumptions. Like, one of the assumptions that you have to make in order to look at this in the abstract, without actually knowing him and the town, is that it's a random process by which the verand—theatre gets filled every week. (3-second pause) Now, it may not in fact be! But in, that's an assumption that you could make that will let you proceed.
433. David: Oh, ok.
434. Sarah: You also have to assume that he sees everyone that goes to the movie.
435. Instr: Very good! Because if he only sees a small fraction of the people going in, people could be there and he might not see them. (3-second pause) All right. So we're not saying he does, but we're saying in order to proceed we'll make this assumption. Ok, all right. Does that make sense, David?
436. David: Yeah.

Excerpt 1 indicates that David was disabled from deciding what to assume because he felt lost in a sea of possible choices. David seems to have thought of an assumption as a hard fact about the situation, rather than as a working supposition upon which to proceed further. Thus, David's difficulty appeared to be in looking beyond the information given in the situation and reconfiguring it in terms of aspects that are not explicitly given per se but which are nonetheless necessary to presume. The need to reason hypothetically about a situation as a starting point for designing an investigation of an issue, together with the absence of clear constraints on what could be hypothesized, made the tasks seem too open-ended and ambiguous to some students. The classroom discussions were intended to help students learn to deal with such ambiguity by providing them with opportunities to unpack their implicit assumptions (i.e., line 433) and create new assumptions.

The class's difficulties in deciding what to assume about the given situations were ongoing in these discussions. Decisions were rarely made in a clear-cut manner, instead they often emerged out of relatively arduous negotiations embedded within messy interactions. The subsequent excerpts illustrate this. Excerpt 2 begins with David struggling to make sense of the underlined part of the central question that was posed as part of the movie theatre situation: "*Is it in fact unusual that at least two people Ephram knows attend the movie he shows, or could people be coming because he is there?*".

Excerpt 2 (Lesson 6):

482. David: Why did you throw in that last part that says "or could people be coming because he is there?". Why did you put that part? That was, that wiggled me out (motions with hands above head), I didn't know what to do. It's, like, what is that?
483. Instr: Oh! Well—
484. David: It says (reads) "or could people be coming just because he is there?"
485. Nicole: Yeah! That's my point!
486. Instr: Or for some other reason or another.
487. David: Yeah. I was, like, what is that?

488. Instr: Well, if he always saw—  
 489. Peter: We have to assume that they're not?  
 490. Instr (continues): if he always saw 30 people that he knows—a tenth of the people that he knows in this town are there every night, then something's going on, right? (2 second pause). That, I mean—(coughing)  
 [...]
   
 491. David: Yeah, maybe he's sneaking them in for free  
 492. Instr: Perhaps. Something's going on (turns on laptop and window re-appears on screen). Would you expect him to see very many people that he knows? If he knows, if there are 30,000 people and it's a random draw to fill the theatre, would you expect him to see very many people that he knows?  
 493. Nicole: Well, how many movies are there a night?  
 494. David: He only knows, like, 1% of the town, so it's kind of weird that he'd see people, 2 people every single night.  
 495. Luke: Yeah.  
 496. Instr: Yeah, he knows 1% of the town.  
 497. Instr: Well, we're going to simulate it, we don't know the answer to the question yet!  
 498. Kit (to David): I think there's one movie per night.  
 499. Instr (continues): It might be rare

The question that students found so problematic in Excerpt 2 was intended to occasion reflection on the reasonableness of the randomness assumption, so that if the event in question turned out to be unusual, issues could then be raised about whether something other than a random attendance process was at play in the scenario. David and Nicole (lines 482-487) did not, however, interpreting things this way. To them this appeared as an isolated question that made little sense and which they could not relate to the greater task. In retrospect, their difficulty is somewhat unsurprising; because the class had not yet investigated whether the event in question might be unusual, these students could not interpret this statement as broaching a possibly relevant issue. The tension that these students experienced drove the instructor to start bringing issues of underlying assumptions out into the discussion.

### **Conceiving a Population and a Sample**

In the ensuing interaction in Excerpt 2 (line 494), David expressed his “gut level feeling” that the event of seeing two or more people at the movie theatre each night is inconsistent with the given assumption that Ephram knows only 1% of the 30,000 people in the town. Thus, David's intuition seemed to touch on the idea of drawing a non-representative sample from an underlying population. Moreover, this also suggests that David was mindful of the 300 acquaintances as a proportion of the entire population, but did not seem to reason similarly about a sample to the think of the number of acquaintances as a proportion of 250 people selected. Nicole (line 493) also began entertaining possibilities for underlying assumptions, such as “how many movies are we to assume were shown each night?”, suggesting that she thought this

had important implications for how many acquaintances Ephram could reasonably expect to see at the movies each night. Thus, Nicole had not yet structured the situation in terms of an unambiguous sampling experiment in which the act of looking into a filled 250-person theatre once per night is *as if* one were recording the composition of a 250-person sample collected from the town's population (assuming random attendance).

In Excerpt 3 the discussion turned to making explicit connections between the situation and an idealized sampling experiment, constrained by the software interface. Excerpt 3 (Lesson 6):

504. Instr: Now, I'm going to, I'm gonna do this in a way that uhh the guy who wrote this program suggested (sets up Prob Sim to simulate movie theatre scenario). Put a little tiny dot to represent a person in the town who he doesn't know, and a big dot to represent a person in the town that he does know. How many of these dots are there gonna be in this mixer? (points to small dot in left-most element label on the screen)
505. Luke: 30,000
506. Instr: No
507. (Others students chime in unanimously and simultaneously): "27,000".
508. Instr: Yeah, 27,000. No, twenty nine thousand seven hundred.
509. Peter: 29,700
510. (Instructor enters this value into "how many" slot under first element label)
511. Peter (to others): Mathematical geniuses!
512. (Peter and Nicole laugh in background)
513. Instr: So how many people does—uhh that's because he knows 300 of those 30,000. Right?
514. Luke: Nods. He knows (inaudible) thousand.  
[...]
515. Instr: Ok, we're gonna take 250 of those people. Right? (assigns this value as sample size in the software field) Are we taking them with or without replacement?
516. Lesley: Without
517. Peter: With, with!
518. Luke: No, without replacement.
519. Lesley: With!
520. Nicole: No, you can't, it
521. Luke: You can't—
522. Nicole: If it's people it has to be—without
523. Peter: 'Cause they can come back the next night.
524. Instr: No, no, we're talking about one night.
525. Kit (to Peter): Yeah, but not on the same night.
526. Peter: Oh!
527. Luke: Repetitions is (inaudible).
528. Instr: One night. So, is it with replacement or without replacement? (points cursor at "replacement" option in Prob Sim window on screen)

529. Nicole: With  
 530. Lesley: With  
 531. Luke: Without replacement.  
 532. Kit: Without  
 533. Instr: If a—can a person be in a theatre twice?  
 534. Kit: No.  
 535. Luke: No.  
 536. Lesley: No.  
 537. Instr: Ok, so it's without replacement  
 538. Peter: They snuck back in and watched it again.  
 539. Nicole: Wait  
 540. Kit (to Peter): not at the same time you can't  
 541. David (to Instructor): You got it on without replacement  
 542. Nicole: it's one movie one night  
 543. Instr: Yeah.  
 544. Lesley: I don't understand.  
 545. Instr: Or, it's just one night. We don't know how many movies, but—  
 546. Nicole: Well then it's a difference! That's what I asked you  
 547. Instr: Ok, then let's say one movie one night. That's a good assumption. (2-second pause)

Excerpt 3 began with Luke proposing that the Mixer—the software's metaphor for a population—should contain 30,000 of the small dots representing an *non-acquaintance* in the population (Line 505), thus suggesting that he had not yet conceived of the population as comprised of two distinct classes: acquaintances and non-acquaintance. The instructor and other students immediately chimed in with a different answer (lines 506-509). After resolving an estimation error, the instructor explained that the population is divided into 300 acquaintances out of 30,000 total people, to justify choosing 29,700 as the number of small dots in the mixer.

The discussion then turned to assigning the sampling parameters. After proposing 250 as sample size, the instructor raised the issue of whether to sample with or without replacement (Line 515). Here, students had different opinions, but direct evidence of how they were thinking about the issue is limited. It seems, however, that a source of their differences might be attributed to students holding different assumptions about the number of movies shown each night. In line 523 Peter appeared to assume that people could return to the movie theatre on different nights, suggesting that he had not structured the situation in terms of what might occur on an individual night as a distinct unit. Rather, he seemed to be considering events that could occur across several nights. Peter's utterance in line 538 indicates that even when restricting himself to considering an individual night, he was thinking of contingencies that suggest he was unclear about *what* constituted a sample in the scenario. "Could one sample be like one full audience, or should it be thought of as all audiences in one night?" These seemed to be the kinds of underlying questions that Peter was contemplating without resolution. Peter's problem can be interpreted as a difficulty in making an idealized assumption about the situation, making

judicious choices about how to appropriately simplify the situation so as to make it amenable to model. Evidently, the constraints of Prob Sim's interface did not sufficiently help him structure the situation as an unambiguous sampling experiment in which a sample consisted of a 250-person audience that attended a single movie in one night. Excerpt 3 illustrates how the students were generally indecisive about whether to sample with or without replacement. Those who changed their minds (Lesley and Nicole), flitting from one sampling option to the other, were evidently unsettled on what to take as a sample in the scenario; their assumptions were still formative and highly unstable.

### **conclusion**

To summarize, the data excerpts discussed here illustrate how construing a contextual situation as a stochastic experiment—essentially conceiving of a *trial* of an experiment—can be a highly non-trivial activity for students. Notable student difficulties included conceiving a population and a sample drawn from it in a manner that could enable students to reconceive the situation as a repeatable sampling experiment. At the core of these difficulties were ambiguities regarding what to take as the “repeatable entity”, seemingly driven in part by students’ uncertainties concerning what idealized assumptions to make about the underlying situation and how to map such assumptions to methods of sampling (with or without replacement). These findings provide good reason not to treat a trial of an experiment as a given in statistics instruction, but to treat it, instead, as a concept that learners must construct and whose construction can entail considerable conceptual machinery and instructional challenges.

### **References**

- Cobb, P., & Whitenack, J. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Education Studies in Mathematics*, 30(3), 213-228.
- delMas, R. C., Garfield, J., and Chance, B. L. (1999). A Model of Classroom Research in Action: Developing Simulation Activities to Improve Students' Statistical Reasoning. *Journal of Statistics Education* [Online], 7(3).
- Konold, C. (2002). Prob Sim User Guide. Amherst, MA: Author.
- Konold, C., & Miller, C. (1996). *Prob Sim*. Computer Program. Amherst, MA.
- Kuhn, M., Hoppe, U., Lingnau, A., & Wichmann, A. (2006). Computational modelling and simulation fostering new approaches in learning probability. *Innovations in Education and Teaching International*, 43(2), 183–194.
- Mills, J. (2002). Using Computer Simulation Methods to Teach Statistics: A Review of the Literature. *Journal of Statistics Education*, 10 (1). Retrieved from <http://www.amstat.org/publications/jse/v10n1/mills.html>
- Pratt, D. (2000). Making Sense of the Total of Two Dice. *Journal for Research in Mathematics Education*, 31(5), 602-625 .
- Sedlmeier, P. (1999). Improving statistical reasoning: Theoretical models and practical implications. Mahwah, NJ: Lawrence Erlbaum.
- Stohl, H., & Tarr, J. E. (2002). Developing notions of inference using probability simulation tools. *Journal of Mathematical Behavior*, 21, 319-337.



# SOCIO-MATHEMATICAL AND MATHEMATICAL NORMS IN PRE-SERVICE PRIMARY TEACHERS' DISCOURSE

V. Sánchez & M. García

Departamento de Didáctica de las Matemáticas, Universidad de Sevilla, Spain

*This study is part of a wider research which seeks to investigate the existence (or not) of relationships between socio-mathematical and mathematical norms at different academic levels. Here we consider the norms that arise in the interaction between primary student teachers when they solve a mathematical task related to the mathematical definition. We hypothesize that in the colloquial mathematical discourse between these students co-exist two types of discourse: socio-mathematical and mathematical, each one with its specific norms. This co-existence originates commognitive conflicts. We have been able to identify different commognitive conflicts that arose from the confrontation of four socio-mathematical norms and three mathematical norms linked to define, which origin distinct subsequent effects.*

## INTRODUCTION

In mathematics education, the last few years have seen a growing number of studies drawing on sociocultural perspectives that have provided new ways of considering mathematical thinking and learning. In particular, within the framework of commognition, Sfard (2006, 2008) considers thinking as an individualized form of communication, being interpersonal communication the collective activity that morphs into thinking through the process of individualization. From Sfard's perspective, learning mathematics may be defined as 'individualizing mathematical discourse'. Consequently, mathematical discourses (interpersonal or individualized) begin to be the 'principal object of inquiry' (Sfard, 2008, p. 276).

Focusing on mathematical discourse, we would like to inquire about what happens in different groups of students who work together on a proposed mathematical task. Within the different mathematical discourses identified by Sfard (2008), this discourse can be considered as a colloquial mathematical discourse. Following this author, we assume that the way to distinguish discourses is to specify their respective objects, and that human communication is a rule-regulated activity. In her opinion, 'it is important to distinguish between *metadiscursive* and *object-level rules*' (Sfard, 2008, p. 201). The metadiscursive rules are considered higher-level rules that speak about the actions of the discursants, not about the behaviour of mathematical objects.

Some researchers have considered the social-psychological aspect in their studies of the different rules that regulate the interaction (Yackel and Cobb, 1996; Sfard, 2000; Yackel 2001). From a sociological perspective, these rules or prescriptions have been specified as demands or norms (Biddle & Thomas, 1996). Among the different norms that can be identified in the discourse, authors such as Tatsis & Koleza, (2008) have

focused on the social and socio-mathematical norms, considering them in the same sense as Yackel & Cobb (1996, p. 461) as ‘normative aspects of mathematics discussions specific to students’ mathematical activity’. They investigated how they were established during the interaction of pairs of pre-service teachers in the context of solving mathematical tasks. As in the case of these authors, our research deals with the interaction of pre-service teachers inter-cooperating, with no scaffolding taking place. However, in our work the way of considering socio-mathematical norms is based on the perspective of Sfard, who in a general sense considers norms to be ‘metadiscursive rules that are widely endorsed and enacted within the discourse community’ (Sfard, 2008, p. 300). Furthermore, without minimising the importance of social norms, we focus on socio-mathematical and mathematical norms, incorporating the mathematical topic addressed as a new variable.

## **THEORETICAL FRAMEWORK**

In this study, we deal with norms, understood to be taken as in Sfard’s above-mentioned sense. We consider the socio-mathematics norms and mathematical norms that arise in the interaction between groups of primary student teachers when they solve a mathematical task related to a specific mathematical topic, in our case the mathematical definition. We hypothesize that in the colloquial mathematical discourse that arises in a group of students when solving a mathematical task, two types of discourse co-exist: one is linked to socio-mathematical aspects coming from the way of considering mathematics as a subject matter in the school context. The other comes more specifically from the mathematical field. The co-existence of these discourses, each one with its specific norms, originates commognitive conflicts, defined as ‘the phenomenon that occurs when seemingly conflicting narratives are originated from different discourses – from discourses that differ in their use of words, in the rules of substantiation and so forth’ (Sfard, 2008, p. 257).

Moreover, the topic that we have considered is the mathematical definition, which we consider the end product of the process of defining. Authors such as Borasi (1991) point out some commonly accepted requirements for mathematical definitions: precision in terminology, isolation of the concept, essentiality, non-contradiction and non-circularity. She indicates that these requirements seem reasonable if a definition is required to discriminate between instances and non instances of a concept with certainty, consistency, and efficiency, or “capture” or synthesize the mathematical essence of the concept (Borasi, 1991, p.18). Van Dormolen & Zaslavsky, (2003, p. 93) express some criteria that we demand of a definition, considered logical necessities (criterion of hierarchy, criterion of existence, criterion of equivalence, criterion of axiomatization) or as part of a general culture (among others, criterion of minimality). Finally, Harel et al. (2006, p.151) indicate some features of mathematical definitions that mathematicians value and that have been described in mathematics education research literature: existent, non-contradictory, unambiguous, logically equivalent to other definitions and invariant under changes of representation.

These requirements, criteria and features allow us to identify in students' discourse the mathematical norms related with defining. In addition to these specific norms, we are aware that other mathematical norms of a more general character exist, but they are not considered in this part of our study.

We would emphasize that, at least in Spain, students enter university with many experiences that are bound to shape their learning of defining. Nevertheless, it is not explicitly mentioned in the school curriculum, but students approach it in an indirect way, through other mathematics curricular topics.

The research questions behind this study are:

- Is it possible to identify commognitive conflicts between socio-mathematical and mathematical norms in the context of solving a mathematical task related to a specific topic?

- Are there any possible effects of these conflicts (if they exist) in the comprehension of that topic?

## **METHOD**

### **Participants**

The data reported in this paper come from a wider study which seeks to investigate the existence (or not) of relationships between socio-mathematical norms and mathematical norms at different academic levels. In the part of the research reported here, participants were 14 pre-service primary teacher students aged 19 to 21 enrolled in a problem-solving course at a large university in Spain. They participated voluntarily in the study. The problem-solving course was a non-compulsory subject matter in the mathematics teacher education programme at our university. In general, the students had chosen it because they were interested in improving their mathematical background. On the course, the 14 students worked in four small groups (namely G1, G2, G3 and G4 on our research) in two-hour sessions per week. The study was developed with these groups at one of these sessions. G1 consisted of a one female and two male, G2 four and one respectively, G3 one and one, and finally G4 was formed by three females and one male.

### **The research instrument**

Taking into account that in the commognitive framework of Sfard the mathematical discourse of students is the unit of analysis, researchers have used different learning environments to access students' thinking (Wille & Boquet, 2009). The data for our study consists mainly of the transcriptions of the verbal dialogues the students maintained when they solved a mathematical task proposed in the classroom, and the written responses they agreed to consider as the final answer to the questions posed in it.

The task consists of a picture of a square (figure 1), a rectangle (figure 2) and a rhombus (figure 3), and a series of nine questions aimed at obtaining information on

the students' ideas related with the properties and characteristics of defining. Examples of these questions are as follows: What properties or characteristics can you observe in each of these figures? Is there any property or characteristic common to the three figures? Can you define each one of the figures? Can you give another definition of each of them? Finally, we asked for comments or suggestions the students wished to make. Each group of students was provided with a set of sheets of paper that included, on the first page, general information on our research and the importance of their assistance and, on the following pages, the task. The only instructions given to the students were that they should verbalise their ideas when they solved the questions and write their final answers on the provided sheets of paper.

### **Data analysis**

Once the dialogues were transcribed into written text, in the first level we selected the parts of the mathematical discourse in which conflicts (understood in a general sense as disagreement between people with opposing opinions or principles) could be identified. In the second level, following Sfard (2006, 2008), we analysed the parts of the discourse related to these general conflicts (GC) on the basis of the four properties identified by said author that 'can be considered as critical in deciding whether the given instance of discourse can count as mathematical' (Sfard, 2008, p.133): Mathematical words, Visual mediators, Endorsed narratives and Routines. In our case, the final endorsed narratives were considered from the group written responses. In a third level, we identified mathematical norms related to defining and socio-mathematical norms. In particular, socio-mathematical norms were inferred by identifying regularities in the above-mentioned properties linked to aspects coming from the way of considering mathematics as a subject matter in the school context. Mathematical norms were extracted from the characteristics of the process of defining that have been described for the above-mentioned authors (Borasi, 1991, Van Dormolen & Zaslavsky, 2003, Harel et al., 2006). Then, the mutual relationship between both norms was considered, looking for commognitive conflicts. Finally, in a fourth level, we looked for the consequences of these conflicts. We here below present a brief example from a group of students, in order to show our analytical procedure.

For instance, one of the general conflicts (first level of analysis) was identified in Group 4 (G4, lines 527-693), when the students tried to answer Question 6 (Q6: Defining each one of the figures) and they then realised what Question 7 asked (Q7: Give another definition of each of the figures). Focused on mathematical discourse (second level of analysis), initially all the students used words such as square, rectangle, rhombus with a discriminatory meaning, establishing for themselves that the name serves for naming and not for defining. Nevertheless, words such as regular are used in some sentences with a colloquial meaning (habitual, usual or repeated) and not a mathematical meaning. The following excerpt was representative of this second level of analysis:

- 538 A2: But watch out, because they ask for another definition here  
 539 A3: [reading] Could you give another definition? Gee!  
 540 A1: Ah, well! We'll leave it in a superficial way here and that's the end of it.  
 542 A2: No, here [referring to the answers to Q7] we'll answer Regular Polygon, Regular Polygon, Regular Polygon... [It is written in as the agreed upon answer for Q7]  
 545 A1: And here [referring to the answers to Q6] we'll say a 4-sided regular polygon... [It is written in three times as the agreed upon answer for Q6]  
 548 A2: I'm going to ask [the teacher] if it refers to the square. We might be bursting our brains out for nothing  
 550 A1: OK, go ahead!  
 551 Teacher: In different ways...

*[following the teacher's intervention, the students reconsider their answers, and try to add additional information to that already written in for question 7]*

- 552 A1: OK! ... You say you define the square. Regular polygon with four equal sides and that means the 90 degree angles are formed.

The above excerpt shows how the students assumed initially as narratives the same answer for the two questions. In the successive lines of the transcription, the use of the four properties (Mathematical words, Visual mediators, Endorsed narratives and Routines) allowed us to identify two discourses. On the one hand, three of the students continued with a discourse with analogous characteristics of the previous discourse. On the other hand, the discourse maintained by student A3, in which some characteristics closer to a mathematical field start to arise. Among them:

- The use of words such as regular is questioned:

- 559 A1: [writing] Regular polygon with four equal sides ...  
 560 A3: What? Make your contribution [*speaking to A2*]  
 561 A2: What is a regular polygon? The one whose sides are equal.  
 562 A4: Ah, I see!  
 563 A3: But this is then ...  
 564 A2: Are they all regular polygons?  
 565 A4: Uh! This one. Isn't it?  
 566 A3: **No. That which has its sides equal no. It has nothing to do with it, and I don't know exactly what else is.** I have thought about it before

- The inclusion of other mathematical words in order to establish differences:

- 640 A3: **The angles have to be put here**  
 641 A4: Well alright  
 642 A1: But don't the angles have to be included in some definition?

643            **A3:     If they aren't, there are no differences**

- Narratives such as that shown in the lines 552-553 are not assumed by A3 as a definition because they include an implication:

554            **A3:     But that is not a definition**

555            A1:     How isn't it? Regular polygon with four equal sides

556            **A3:     but the thing about the angles and that means...**

In level 3, from the regularities in the above-mentioned properties, we were able to identify a commognitive conflict between the socio-mathematical norm that we can synthesize in the sentence "you have to answer all the questions, it does not matter if the answers are the same" and a norm that points out the necessity of discriminating, in which some characteristics of the specific mathematical discourse of defining arise that allow us to consider it as a mathematical norm. The process of solving this conflict led to students assuming a final narrative for the first part of question 7 (defining a square) (fourth level of analysis),

"Square: 4 equal sides forming right angles" (written response Q7, G4)  
establishing differences with rectangle and rhombus and deleting of the initial written response "Regular polygon" (initial written response Q7, G4) for the three figures.

In what follows, we present some general findings from this analysis.

## FINDINGS OF THE STUDY

Based on our analysis, we have been able to identify, in our study, different commognitive conflicts that arose from the confrontation of four socio-mathematical norms and three mathematical norms linked to define, originating distinct subsequent effects that we present in the following:

- Commognitive conflict between the Mathematical Norm (MN) related with defining expressed in the criterion of minimality and the Socio-Mathematical Norm (SMN) that leads to the idea "the more you write as response to a mathematical task, the better you do". This conflict has been identified in (Group1/General Conflict2), (G4/GC2,) (G2/GC1,) (G3/GC1). It leads students to incorporate descriptive features/aspects, included in the task presentation, in some of their responses, which are not necessary but might be pertinent (for example, central angles, and vertices).

- Commognitive conflict identified in (G1, GC1) (G2, GC2) (G4, GC1) between the MN related with defining expressed in the criterion of minimality and the SMN "everything you see in a figure that goes with the presentation of a task has to necessarily indicate something". It leads students to incorporate descriptive features/aspects, coming from the task presentation, in some of their responses that are neither necessary nor relevant (for example, length of the side).

- Commognitive conflict identified in (G4, GC6) between the MN related with defining that points out that a definition is required to discriminate between instances and non instances of a concept and the SMN that indicates that "you have to answer

all the questions; it does not matter if the answers are the same". This conflict leads to non-use of discriminatory aspects of the mathematical objects. The main aim is 'refilling' the line corresponding to the response of the question.

- Commognitive conflict identified in (G1, GC5) between the MN expressed in the criterion of hierarchy that considers that in the meaning of a mathematical object the meanings of the objects that define it are included, and the SMN indicating that "text/word in a mathematical task has only a linguistic meaning". It leads to accepting that the word that expresses a property as 'one property', without assuming that the aforementioned property involves other properties that give meaning to it. For example, in this group 'regular' is accepted as a property because it is a word. The characteristics that involve 'regular' are expressed by several words so, for these students, there are several properties. When these students speak about 'regular' they are not assuming that they are speaking about all these characteristics as properties.

## **DISCUSSION AND FURTHER CONSIDERATIONS**

An approach into the problem of making operative theoretical ideas coming from socio-cultural perspectives into the practice of researchers has been presented in this work. In our case, our practice as researchers is situated in the mathematics teacher education field, and the theoretical ideas arising from Sfard's perspective. We have therefore tried to see if old questions can be posed and answered from this theory, and whether the responses bring something new to the results of the previous research.

Our results extend the work of authors such as Tatsis & Koleza (2008), who have dealt with social norms and socio-mathematical norms from a general perspective, incorporating the identification of norms related to specific mathematical contents. We have been able to identify some socio-mathematical and mathematical norms in pre-service teachers' mathematical discourse and some conflicts related to them, bringing information to researchers and teachers about the influence of those conflicts in students' mathematical learning of defining.

With respect to the conflicts, we wish to point out that not all the conflicts become commognitive conflicts. Clearly, there must be two confronted discourses, and they need expert guidance, from whoever or whatever that plays that role, to advance.

In our work we have identified SMNs that cause commognitive conflicts with some MNs. Our results lead us to pose different questions related to the different aspects that intervene in the generation and development of these conflicts. We asked ourselves if working with more groups would allow us to identify new conflicts, and the greater or lesser dependence of these conflicts with respect to one or other norm. We need to extend this work to other students to deepen our research into this result. Other questions are related with the task. We wonder whether different tasks might give rise to different SMN and MN norms that lead to other conflicts. Finally, there are some questions related to the teacher, and the different characteristics that socio-cultural perspectives give to his/her role with respect to other theoretical perspectives.

We can say that the role of teachers as experts who transmit knowledge to students, or as facilitators of knowledge that guide and stimulate students and solve cognitive conflicts, has been changed for a teacher who, among other things, is able to identify and solve commognitive conflicts. In this way, the role of teachers as experts includes now new characteristics.

To sum up, if we take into account that students, tasks and teachers are key elements in the educational field, and we consider that they affect commognitive conflicts and are also affected by them, said conflicts become relevant elements in the teaching/learning process of mathematical concepts.

### **Additional information**

The research reported here was supported by the Ministerio de Ciencia e Innovación (Spain) through Grant corresponding to the project No. PSI2008-02289.

### **References**

- Biddle, B.J., & Thomas, E.J. (1966). Prescriptions. In B.J. Biddle & E.J. Thomas (Eds.), *Role theory: Concepts and research*, (pp. 103–104). NY: Wiley and Sons, Inc.
- Borasi, R. (1991). *Learning mathematics through inquiry*. Portsmouth, NH: Heinemann Educational Books, Ins.
- Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education* (pp. 147-172). Rotterdam: Sense Publishers.
- Sfard (2000). On reform movement and the limits of mathematical discourse. *Mathematical thinking and learning*, 2 (3), 157-189.
- Sfard, A. (2006). Participacionist discourse on mathematics learning. In J. Maasz&W. Schlöglmann (Eds.), *New Mathematics Education Research and Practice*. Rotterdam:Sense Publishers.
- Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses and mathematizing*. Cambridge University Press.
- Tatsis, K., & Koleza, E. (2008). Social and socio-mathematical norms in collaborative problem-solving. *European Journal of Teacher Education*, 31 (1), 89-100.
- Wille, A. M., & Boquet, M. (2009). Imaginary dialogues written by low achieving students about origami: a case study. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.), *Proceedings of the 33rd annual conference of the International Group for the PME*, (Vol. 5, 337-344). Thessaloniki, Greece.
- Van Dormolen, J., & Zaslavsky, O. (2003). The many facets of a definition: The case of periodicity. *Journal of Mathematical Behavior*, 22, 91-106.
- Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In Marjia van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the PME*, (Vol.1,9–24). Utrecht, The Netherlands: Freudenthal Institute.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education* 27, No. 4: 390–408.
- Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, 36 (4), 317-347.



# A SEMIOTIC CHARACTERIZATION OF INTUITIONS

Chiara Andrà<sup>1</sup> and Giorgio Santi<sup>2</sup>

<sup>1</sup>University of Torino, Italy; <sup>2</sup>NRD of Bologna, Italy

*The aim of this paper is to provide a tentative theoretical framework for characterizing intuitions in a semiotic perspective. Starting from Fischbein's definition of intuitions as a self-evident and immediate form of cognition, we study them addressing Duval's structural and functional approach, and Radford's cultural-semiotic perspective. We single out some semiotic behaviours that account for the immediacy and self-evidence of intuitive thinking. We discuss examples taken from an experimentation conducted with primary school students working on fractions.*

In the common sense, the term 'intuitive' is used for a cognitive behaviour of some individuals who are able to grasp a concept or solve a problem as if they established a "direct connection" to the mathematical object or to the solution of the problem, without the need of some form of mediation. This acceptance of the term, however, is troublesome when dealing with mathematical objects, due to their special epistemological and ontological statute: in fact, they do not allow us an ostensive and direct relationship to them. Mathematical cognition is therefore intrinsically mediated by signs. Hence we believe that semiotics allows us to investigate the distinction between a mediated and an immediate cognition in mathematics, thereby providing a further characterization of intuitive thinking.

The aim of this paper is to give a first semiotic characterization of intuitive thinking. Therefore, we trace back (non) intuitive thinking to corresponding (non) direct uses of signs. We address two semiotic perspectives: Duval's structural and functional approach (Duval, 1995; 2008) and Radford's cultural-semiotic one (Radford, 2006; 2008). In Duval's perspective, direct, self-evident, and global thinking is interpreted in terms of suitable uses of systems of signs and their transformations within and between such systems. In Radford's perspective, it is interpreted in terms of the kind of socially shared activities and levels of generality accomplished by semiotic resources.

## THEORETICAL FRAMEWORK

We address intuitions in as defined by Fischbein (1998): "in mathematics there are claims that seem to be directly accepted as self-evident, while for other claims a proof is necessary to accept them as true". Fischbein defines *intuitive cognition* as a direct, self-evident, certain, and global form of thinking, in contrast with *logical cognition* that accepts statements only after an argumentative or proving process. A well known example in literature is given by the misconception that multiplication leads to an increase of involved quantities. Consider the following problem:

(a) Which is the number that multiplied by 5 gives 15?

According to Fischbein (1998), the solution is self-evident and immediate: multiplying five by three, the involved quantity increases to fifteen. On the other hand, the following problem:

(b) Which is the number that multiplied by 15 gives 5?

is non-intuitive under Fischbein's (1998) perspective. Even if the two problems have the same mathematical structure, (b) requires some form of logical reflection: the quantity of 15 must be multiplied by  $1/3$ , and it decreases instead of increasing. Some students divide  $15:5=3$ , instead of multiplying  $15 \times 1/3 = 5$ . The inverse formula is  $5:15=1/3$ , and this is not immediate in Fischbein's perspective.

Fischbein's distinction is extremely insightful, but *why does the first formulation of the problem allow for an immediate solution, whereas the second one requires a logical effort?* Fischbein (1998) answers this question from a cognitive point of view. We consider it from a semiotic perspective.

Duval's (1995, 2008) identifies mathematical thinking and learning with the *coordination of semiotic systems* according to the following operations: *treatment* (transforming a representation into another one within the same semiotic system), and *conversion* (transforming a representation into another one, in another semiotic system). In order to trigger semiotic transformations, it is necessary to identify the most appropriate representation according to the *distinctive features* of a mathematical concept. Addressing Duval's theoretical tools, we can recognize Fischbein's intuitive thinking when: (1) the accordance between the distinctive features and the representation of the mathematical object is direct, global and immediate; and (2) correspondences between different representations obtained through treatment and conversion transformations are immediate and self-evident; such transformations allow for a global control of the involved representations.

Let us come back to the example. For solving the exercise, both cases require a conversion from natural language to the arithmetic register. In (a), however, students directly transform the problem in the arithmetic register:  $5 \times \_ = 15$ . This is then possibly transformed into  $5 + 5 + \dots = 15$ , which leads to a direct (and correct) solution. It is evident that there is an immediate and global correspondence both in conversion between natural language and the arithmetic register, and in the treatment within the arithmetic register. Moreover, the semiotic transformation that leads to represent the problem as a repeated sum makes the problem even easier because supported by an additive structure: Vergnaud (1983), in fact, has shown that additive structures are cognitively easier to handle. On the other hand, (b) is not intuitive. From a semiotic viewpoint, there is neither a direct nor a global transformation as in (a) that leads to an effective representation for the solution of the problem. Using the above procedure, students may be led to do  $15 + 15 + \dots$ , but they immediately recognize that they will never reach 5 as a result: there is an inconsistency between the distinctive features of multiplication (mathematically speaking, this operation can both 'increase' and 'decrease'), and the meaning students commonly confer on its

semiotic representation as only increasing. The solution of this problem, in fact, implies two strong cognitive ruptures: (1) conversion transforms the multiplication expressed in natural language into a division in the arithmetic register; and (2) it is required to divide a smaller number by a greater one.  $5:15$  gives a ratio,  $1/3$ , and it is well known the obstacles students have to overcome when dealing with fractions (Hart, 1985; Fandiño Pinilla, 2005).

To sum up, Duval's framework tells us that there are both semiotic representations and transformations that support and foster mathematical thinking, opposed to others that impede and inhibit cognitive processes. A further question is: *how and why is it possible to identify semiotic representations that support reasoning and others that hinder it?* The answer requires a shift from a perspective that considers signs with a *representational* role to another perspective that considers what signs *allow us to do*. We therefore address Radford's (2003; 2008) cultural-semiotic approach that considers cognition as a *reflexive mediated activity*. When referring to learning processes, reflexive activity is termed as *objectification*. Cognition is considered as a shared practice (activity) that involves the individual as a whole (both mind and body), accomplished in the socio-cultural contexts he belongs to. Mathematical activity is carried out through a set of semiotic resources (mediation), termed as *semiotic means of objectification*, that direct individual's intentional acts (reflexivity). They include natural language (oral and written), gestures, objects, artefacts, bodily actions, and mathematical symbolic language. According to the semiotic means that mediate activity, mathematical objects are stratified in *layers of generality*. Radford (2003) identifies three increasing levels of generality: a *factual generalization*, when the objectification of the general scheme takes the form of a *perceptual/sensorimotor semiosis*; a *contextual generalization* when the general scheme is objectified by more abstract semiotic means that, however, bear the spatial and temporal origin of the situation they come from; a *symbolic generalization* when the general scheme is objectified by symbolic language that does not allow any relation with the spatial-temporal dimension. The learner lives a desubjectification of meaning, namely a rupture with his spatial-temporal and sensorimotor experience.

Referring to Radford's perspective, self-evidence and immediacy that characterize intuitive thinking can be traced back to: (1) the spatial-temporal, sensorimotor and perceptive activity that semiotic means of objectification accomplish, support, foster; (2) the level of generality at which the mathematical concept is objectified that can desubjectify meaning with a loss of immediacy and self-evidence; and (3) the acquaintance with the reflexive activities that objectify the mathematical object.

Problem (a) is highly intuitive because in the context of natural numbers multiplication can be interpreted as a repeated sum. The symbolic representations  $5 \times \_ = 15$  and  $5 + 5 + \dots = 15$ , along with other semiotic means of objectification (gestures, objects, bodily actions etc.), mediate multiplication in terms of a meaningful sensorimotor activity. Problem (b) implies a cognitive rupture that requires a desubjectification of meaning. There is no direct spatial-temporal

experience and sensorimotor activity that allows us to find a number that multiplied by 15 gives 5. It is necessary to objectify the notion of multiplication at a higher level of generality introducing rational numbers. The problem can be solved also using a division, with the dividend smaller than the divisor. It requires a rupture with the spatial and sensuous experience of “distributing” a bigger quantity amongst smaller ones and a leap to a higher level of generality.

### **Coordinating Duval’s and Radford’s perspectives**

It is possible to characterize intuitions as a twofold semiotic behaviour, described by a structural and functional approach, and a socio-cultural one. On the one hand, intuitive thinking rests on semiotic representations that are *immediately* transformed *within* (treatments) and *between* (conversions) semiotic systems. Such transformations, in fact, are constitutive of both mathematical thinking and its development as a field of knowledge. On the other hand, intuitive thinking rests on semiotic means that objectify mathematical concepts *through meaningful mediated reflexive activities*. We recognize intuitive thinking when semiotic means of objectification support space-time experience, perceptions, feelings, and sensorimotor activity, of the individual within his consciousness’ intentional acts.

To support the consistency of the semiotic characterization of intuitions within our theoretical framework, it is necessary to answer the following research questions. Which relationships can be established, and to what extent, between intuitive thinking (in different mathematical contexts), and:

- the features of semiotic transformations involving semiotic systems? - on the one side
- the individual’s space-time experience, feelings, and perceptions, within reflexive activities mediated by semiotic means of objectification? - on the other side

## **EXPERIMENTAL EVIDENCE**

### **Methodology**

Data come from a test administered at the end of a one-year-lasting experimentation, during 2009/10. Its purpose was to investigate students’ intuitions and misconceptions concerning fractions. It was carried out by the research group in Mathematics Education in Torino, Italy (coordinated by prof. Ferdinando Arzarello). The activities involved 9 classes, grade K-4 and K-5 (age: 9 to 11), for a total of 248 children. Students were exposed to a variety of didactical methodologies for the learning of fractions (grade K-4) and their strengthening (grade K-5): from traditional lessons to cooperative learning, from exercises to the use of stories and games, etc. Students were trained in basic concepts and algorithms regarding fractions. We focus only on students’ behaviour regarding their intuitive/non-intuitive reasoning, disregarding the effects of different didactical methodologies.

For the purpose of our research, we focus on two questions of the final test:

(i) Which is the number that multiplied by 5 gives 13? Explain your reasoning.

(ii) Which is the number that multiplied by 16 gives 7? Explain your reasoning.

A previous analysis (through pre-testing and video-recording classroom activities) verified student's prerequisites on fractions.

### Analysis of protocols and results

A first raw classification based on students' answers is: (a) students say that the solution does not exist; (b) students solve correctly (i), but either omit or solve incorrectly (ii); (c) students solve correctly both problems, with various strategies. Let us look firstly at Andrea's protocol, an example of type (a):

(i) None. (ii) None.

No number multiplied by 5 gives 13, because 13 is not present in any multiplication table. The one multiplied by 16 cannot be done, because 7 is smaller than 16, hence any number multiplied by 16 gives always a number greater than 7.

Andrea carries out a semiotic transformation into the arithmetical register, referring to multiplication tables, but he does not think of working with rational numbers. In (i) he maybe thinks of the five-times table and he does not find 13. In (ii), as in (i), he rests on the domain of natural numbers. Transformations into and within the register of rational numbers lose the immediate and global character Andrea experiences with natural numbers. Furthermore, he is bound to a reflexive activity that objectifies multiplication as an operation between integer quantities (with the misconception '*multiplication increases*'). The shift to a symbolic generalization for multiplication (using rational numbers) is not supported by a meaningful reflexive activity. As a result, this is a non-intuitive task for Andrea, since he is not able to resort to appropriate semiotic resources both in terms of transformations between registers, and in terms of reflexive activities. In fact, Andrea is conversant with the semiotics appropriate for natural numbers: transformations between registers are immediate for him, and the reflexive activity is meaningful in terms of the spatial-temporal experience conveyed by multiplication tables. The problem  $5 \times \_ = 15$  would have been extremely intuitive for him. The shift to the context of rational numbers requires a semiotics that desubjectifies meaning and hinders immediate and global cognition. Therefore, the task becomes non-intuitive.

Let us now look at two examples of type (b). Alessandro's protocol:

(i) 2,6 (ii) 2,02

(i) I did  $2,6 \times 5 = 13$ , so I obtained the number that multiplied by 5 gives 13. (ii) I did  $16 : 7 = 2,02$ , so I obtained the number that multiplied by 16 gives 7.

and Donato's protocol:

(i) 2,6 (ii) None.

(i) I did  $13 : 5 = \_$ , but the result does not give an integer number. Hence, I multiplied  $2,6 \times 10$  and it gave  $26 : 2 = 13$ .  $2,6 \times 5 = 13$ . (ii) By guessing with integer numbers and

decimal numbers the result was not feasible, and no method of computation could provide that result.

As regards (i), Alessandro does not explain how he arrived to the solution, but we infer from the answer to the second question that he did 13:5. Donato, instead, says that he tried to divide 13 by 5. Since he felt uncomfortable with non integer solutions, he maybe had thought that 26 is  $2,6 \times 10$ , and 26 is  $2 \times 13$ , therefore  $2,6 \times (10/2) = 13$ . From a structural and functional approach, conversions and treatments support reasoning to solve the problem correctly. Going deeper into details, however, even if Donato handles the register of decimal numbers as a semiotic means, it does not mediate a meaningful reflexive activity: he goes through a convoluted process in order to bind his reasoning on the multiplication by 10 that brings him into the context of integers that he feels as a meaningful activity. Our framework allows us to highlight how, although they find the correct solution, semiotics supports students' intuitive thinking in terms of formal operations, but in the case of Donato, in terms of his personal experienced meaning, he still resorts to the domain of natural numbers. Donato, in fact, comes back to a practice that requires more operations than the ones required resorting directly to decimals, but with which he is more conversant, therefore 'more intuitive'.

Just for this reason, Donato failed (ii). In this case, in fact, he can use neither the multiplication by integers as in (i), nor the multiplication by suitable decimals. Alessandro, instead, used the same algorithm of (i), but he made a mistake inverting the dividend and the divisor. In fact, he divided the greater number by the smaller one. As regards Alessandro, he performs the correct semiotic transformation, but there is not an intuitive control from a cultural-semiotic perspective, since the meaning of division is still bound to the perceptive/manipulative experience of dividing a bigger quantity into smaller ones. The use of the arithmetic register requires a cognitive rupture to access the symbolic generalization in the domain of rationals. Donato, instead, does not even attempt to carry out a conversion from natural language to the arithmetic register, writing a division. He remains bound to a contextual generalization that does not allow him to objectify a scheme of multiplication appropriate for the new situation; he guesses the answer.

As regards (i), the operation 13:5 seems to be intuitive for both students, although the functional control of representations is not always supported by a meaningful mediated reflexive activity in the domain of rational numbers. In (ii), 7:16 is not-intuitive, since the lack of semiotic control, from a cultural-semiotic perspective, in the domain of rationals hinders objectification at a higher layer of generality.

Let us finally look at two examples of type (c). Giulia's protocol:

- (i) 2,6      (ii) ~~2,3~~ 0,441

My reasoning was trying 2, *and a little bit*. Immediately I guessed  $2,7 \times 5 = 13,5$ , so it was too much, therefore I did  $2,6 \times 5 = 13$  and it was correct. Instead, in the one of 16, I had little more difficulties, in fact I obtained 7,05 while it had to be 7.

and Sara's protocol:

- (i) 2,6      (ii) 0,4375

I reasoned using the formulas for discovering the missing number in the operation:  $\_ \times 5 = 13$ , hence I carried out the inverse operation:  $13 : 5 = \_$ , because the result computed through the inverse operation of the operation you have to do gives as a result the number you don't know.

In both questions Giulia proceeded by guessing the answer, remaining bound to the multiplicative model. She "guessed" the answers, but she had a good control on the involved quantities. In fact, she did not arrive to the correct answer of (ii), but she intuitively ascribed a meaning to the result she has obtained. In the case of Sara, clearly she used operations and their inverse ones correctly. From a structural and functional perspective, Giulia used the arithmetic register in the domain of rationals (with appropriate treatments and conversions). It is interesting to observe how she was trying to objectify the multiplication with decimals: she started with a space-time situated meaning of multiplication as a repeated sum, a factual generalization, ('2, and a little bit'), but she did not fall into the misconception that 'multiplication increases'. Through the arithmetic register, as a semiotic means of objectification, she was accessing a higher layer of generality (contextual/symbolic) that allowed her to use multiplication with decimals correctly to answer even the second question. As regards Sara, instead, a strong control of conversions and treatments involving the arithmetic register in the domain of rationals allowed her to solve both problems immediately. The inverse operation is intuitive for her to perform. Sara provides a good example of symbolic generalization. In fact, she objectifies a general scheme to control direct and inverse operations using symbolic semiotic means of objectification. It is interesting how the desubjectification of meaning does not hinder her intuitive thinking. Instead, the leap to symbolic generalization provides an immediate and global solution through a meaningful reflexive activity.

The protocol of Giulia shows that the activity is intuitive for her, even if the answer is not completely correct. In fact, her reasoning is strongly supported by semiotics according to both the semiotic perspectives we considered.

## DISCUSSION

The aim of this paper was to attempt a semiotic characterization of intuitions, defined by Fischbein (1998) as a self-evident, immediate and global form of cognition. By integrating Duval's structural and functional approach and Radford's cultural-semiotic perspective, we have outlined the following features that characterize intuitions semiotically: a cognitive behaviour is intuitive when easily supported by immediate transformations of signs within and between semiotic systems, and/or when semiotic means of objectification mediate meaningful reflexive activities – in terms of spatial-temporal, sensorimotor, and perceptive experience – at suitable levels of generality. Experimental evidence, taken from activities involving decimals in primary schools, confirmed that students highlight an effective intuitive thinking



when they master semiotics in both perspectives of our framework. We can answer our research questions regarding the relationships between intuitions and semiotics. The results pointed out that there is a strong intuitive thinking when students handle semiotic transformations within semiotic systems supported by meaningful mediated reflexive activities. (Sara). There are also cases in which students successfully carry out semiotic transformations without a support in terms of their sensorimotor and perceptive experience (Alessandro). In other cases, there is a good support in terms of meaningful reflexive activities, but they do not allow accessing suitable layers of generality to successfully complete the task (Giulia and Donato). Finally, there is the case of students whose reasoning is hindered by inadequate (or not yet sufficiently evolved) semiotics in terms of both perspectives (Andrea). The case of Sara who experiences an intuitive thinking in a symbolic generalization opens the way for further investigations to outline what features of reflexive activity foster immediacy and self-evidence. We should also outline, from a structural and functional viewpoint, features of the semiotic systems – structure, discursive and meta-discursive functions, congruence etcetera – that allow to identify immediate and global transformations of signs that sustain intuitive thinking.

## References

- Duval, R. (1995). *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*. [Semiotics and human thinking. Semiotic registers and intellectual learning]. Berne: Peter Lang.
- Duval, R. (2008). Eight problems for a semiotic approach in mathematics education. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 39-61). Rotterdam: Sense Publishers.
- Fandiño Pinilla, M. I. (2005). *Le frazioni*. [Fractions]. Bologna: Pitagora.
- Fishbein, E., (1998). Conoscenza intuitiva e conoscenza logica nella attività matematica. [Intuitive knowledge and cognitive knowledge in mathematical activity]. *La Matematica e la sua Didattica*, 4, 365-410
- Hart, K. (1985). Le frazioni sono difficili. [Fractions are difficult]. In: L. Chini Artusi (ed.), *Numeri e operazioni nella scuola di base* [Numbers and operations in compulsory school]. Bologna: Zanichelli-UMI
- Radford, L. (2003). Gestures, speech, and the sprouting of signs. *Mathematical Thinking and Learning*, 5(1), 37-70.
- Radford, L. (2008). Towards a cultural theory of learning. In: L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215-234). Rotterdam: Sense Publishers
- Vergnaud, G. (1983). Multiplicative structures. In: R. Lesh & M. Landau (eds.), *The acquisition of mathematical concepts and processes* (pp. 127-174). New York: Academic Press



# SEARCHING THE ROOTS OF PME – THE CASE OF ,EXPERIMENTAL PEDAGOGY' IN GERMANY

Gert Schubring

Universität Bielefeld/Germany; UFRJ/Brazil

*PME was founded in 1976, at ICME 3, organized by ICMI. While PME is thus beyond coming of age and is reflecting its further orientation – due to the present “social turn”, the origins of investigating psychological aspects of mathematics learning have not yet been systematically studied. We are undertaking here a first such approach, concentrating on Germany where the first pertinent monographs were published in 1913 and 1916. Different endeavours, focussing in particular on the notion of error, merged into the characteristic approach of ‘experimental pedagogy’. Given the key function of ICMI for founding PME, an additional aspect is whether the forerunner of ICMI: IMUK, founded in 1908, had an impact upon promoting research into the psychology of mathematics education.*

## INTRODUCTION

Given that a genuinely empirical turn of mathematics education occurred internationally only from the 1970s on, together with its emergence as a scientific discipline (Biehler et al., 1994), and given that this empirical turn was intimately tied to the contemporary cognitive turn in psychology and pedagogy, which was so decisive for the founding of PME, should it really be possible to maintain that psychological research into mathematics instruction had been pursued before, and that means before the establishment of PME? Clearly, mathematics had always served for philosophers as a preferred subject for their investigations on thinking and reasoning. Since psychology as a discipline essentially emerged by differentiation from philosophy, during the 19<sup>th</sup> century, and since it focused more on developmental aspects of reasoning, one could expect that psychology would not only investigate learning processes, but also consider in particular the learning of mathematics. The hypothesis was, therefore, that some early moments might be found in an empirically minded psychology. The focus was on the period until World War I. Being a historical study, methods from general history are used for this paper to search documents and biographical information. For the interpretation of the data, prosopography is used to unravel characteristic biographical profiles, and for the analysis of the documents approaches from social history and from history of mathematics education are adapted.

## THE JOURNAL “DIE KINDERFEHLER” – *THE ERRORS OF CHILDREN*

Research on errors of students constituted a key element in the empirical turn of mathematics education (see Lorenz & Radatz, 1979), thus my first search was for

finding early publications on errors. To my surprise, I even encountered a journal with the issue of error in its title, and moreover beginning to be published in 1896, thus in the supposed time period. Its full title was “Die Kinderfehler. Zeitschrift für Pädagogische Pathologie und Therapie”, thus focussing on pedagogical pathology and therapy. It had four founders, a director of a primary school, a director of a lunatic asylum, a theology professor, and as chief editor Johannes Trüper, director of an asylum for “Heilerziehung”, for education of disabled children. Trüper (1855-1921), of poor descent, was trained after primary school as a teacher for these schools for lower classes and served several years as a teacher. From 1887 on, he studied pedagogy and psychiatry at Jena University. In 1890, he founded in Jena an innovative asylum for children impaired in their development, practicing a more child-oriented approach.

In their opening statement, the editors emphasized to study faulty predispositions and in general the children’s “soul”, to promote *Heilerziehung* within family, school, church, and social life, and to appeal to all for contributions who deal with mental errors of children. From the first volume on, there were papers on the abilities of children to count and on their number concepts. A paper in the second volume (1897) lamented that teaching reckoning constitutes the tender spot of the schools for the mentally disabled. The paper evidences the conviction, which is shared by all the editors and authors of the journal at least until WW I, that there is an unbridgeable dichotomy between normal and abnormal, or disabled, children: the author declared reckoning to be a matter of intellect, and intellect being absent in mentally deficient children. The terminology used in general underlined this segregating emphasis; for instance, schools for the disabled were called “Idiotenschulen” – schools for idiots.

On the other hand, in volume 2, too, there was an appeal by Trüper for empirical research: stating that the first teaching of reckoning for normal children as well as the entire such instruction for “mentally debilitated” will remain an open question, he claimed as necessary basis for progress that extensive empirical research should be done about the topic: “How emerge and develop the spontaneous conceptions of numbers in sane children and in those with pathological predispositions?” (II, 1897, p. 151). In fact, two years later, a collaborator of Trüper, the teacher Landmann, reported about his research on the development of number conceptions of “schwachbegabte” (poorly gifted) children. The report, which used ‘abnormal’ as synonymous with ‘schwachbegabt’, was based on the then dominating Assoziations-Psychologie and used operations with concrete material to evaluate the knowledge about numbers. It is telling that results for ‘abnormal’ children are related to what is known about “uncivilized” peoples (IV, 1899, pp. 195-197).

In fact, the first years of the journal show the editors and the authors as practitioners, not questioning established theories. After the fourth year, it elevated somewhat its status and functioning. An association, “for the welfare of young psychopaths”, was founded assuming the editorship; and the title changed to a more general one: “Zeitschrift für Kinderforschung” – *Journal for research on children*; it was

continued this way until its 50<sup>th</sup> volume, ceasing in 1944, during WW II. Among the by now more extensive papers on learning difficulties regarding basic arithmetic, there was in 1906 the first paper by a true professional who even became later on the key authority regarding dyscalculia: Paul Ranschburg.

## RANSCHBURG ESTABLISHING *RECHENSCHWÄCHE* – DYSCALCULIA

Paul Ranschburg (1870-1945) was born in Győr in Hungary, but this being a part of the Austro-Hungarian Empire, he was working within German culture and science. He studied medicine at Budapest University, graduating as neurologist, and obtained there his PhD degree. He founded a psychological laboratory within the institutes for therapeutic pedagogy in Budapest. After achieving a *Habilitation* and acting as *Privatdozent*, he became in 1918 a professor at Budapest University. His first paper in the journal for *Kinderforschung* dealt with comparative studies on normal and low achieving students, to measure the achievement potential of both groups (Ranschburg, 1906). Presenting a battery of simple addition tasks, he measured the needed time for the answers and deduced “Gesetzmäßigkeiten” – lawful regularities – regarding the procedures in the two groups. “Low achievers” were for him students at *Hilfsschulen*, i.e. at schools for mentally disabled children. Ranschburg’s exclusively descriptive task, “to fix and make comparable in a more sure manner the mental achievements”, points into the direction of intelligence tests. Further studies were published in the journal *Experimentelle Pädagogik*, edited by Ernst Meumann.

His main work became, however, his 1916 monograph, which coined for the first time the two terms of *Legasthenie* (dyslexia) and of *Rechenschwäche* (dyscalculia) and which should become later on in the 20<sup>th</sup> century key issues for research into the reading process and into the learning of basic arithmetic. Ranschburg emphasized the key importance to undertake psychological experiments for getting insight about the nature of dyscalculia. On the other hand, it is utmost characteristic not only for his work, but also for the entire community and the practitioners, that the basic assumption about cognitive abilities was that of stable predispositions assumed as largely innate qualities. The German terms *Begabung* and *Anlage* express these unquestioned convictions. Consequently, Ranschburg understood not only the reading aptitude as a “function of a special predisposition” (Ranschburg, 1916, p. 18), but he declared even reckoning as not being accessible for all and being the expression of a proper *Anlage* (1916, p. 22). Normal were therefore youngsters having acquired until the sixth year the capacity of understanding the four operations for numbers until twenty, doing such operations mentally. By contrast, the minority of those not able to perform this “suffers the lack of this *Anlage* to calculate; this lack I am summarizing in all its degrees as the form of deficiency as *Rechenschwäche* or *Arithmoasthenie*” (1916, p. IV). They constitute the population of the *Hilfsschulen*.

In the studies presented in his book, Ranschburg not remained restricted to addition and subtraction, but included multiplication and subtraction, too. The tasks remained yet very simple; although the number domain went up to twenty, it is remarkable that

he did not address the problems of the passage below ten, and thus of the decimal system, which today are known to constitute the key for overcoming the mere counting of isolated numbers and achieve structural insight.

In his experiments, Ranschburg measured the number of solutions obtained by each student and the intervals of time for giving the answers. Moreover, he studied the objective and the subjective certainty for a correct answer. There is the interesting hint that he declared that “the quality of the errors” has to be considered for an exact evaluation of the achievement (1916, p. 40). Unfortunately, there is no realization of this promising task. In fact, his entire approach impedes him to enter into a qualitative analysis of errors committed. On the one hand, due to his assumption of predispositions, he attributes to *rechenschwachen* students a “nervous system”, which exerts an “almost invincible resistance” against mastering these tasks (1916, p. 48). And on the other hand, he conceives of errors as a result of missing attention when he speaks of students who correct their errors “only after request”. There is no conceptualization at all of errors as particular and subjective cognitive strategies.

What is most striking, however, is how Ranschburg dealt with the treatment of the dyscalculia, today constituting even an own industry. There is but a short section of two pages at the end of his monograph. For largely normal students he recommends but year long coaching, outside the school lessons, and for pathological dyscalculia there is no other help than those segregating *Hilfsschulen* (1916, pp. 67-68).

## ERNST MEUMANN – AN APOTHEOSIS OF EXPERIMENTAL PEDAGOGY

Ernst Meumann (1862-1915) presents the climax of experimental approaches in pedagogy. He studied at first philosophy and the arts sciences, then Protestant theology and eventually psychology, obtaining his PhD for research on *Assoziations-Psychologie*. He worked as an assistant with Wilhelm Wundt at his Institute for experimental psychology; Wundt had been deeply influenced by Hermann von Helmholtz (1821-1894), the universalist scholar with innovative researches in physiology and physics. After his *Habilitation* in psychology, he obtained professorships at various universities, beginning in Zürich and ending in Hamburg, and always promoting experimental research.

His seminal publication is the three volumes of his Lectures on experimental pedagogy (1911-1914). The third volume considers in particular the relation of pedagogy to general *Didaktik* – dealing with conceptions for teaching the school disciplines in general – and the special *Didaktik* of the particular teaching subjects.<sup>1</sup> Meumann’s basic concept for investigating learning in schools is “mental work”, as opposed to physical one. Not only pedagogy needs experimental investigations for being able to improve the mental work of students, also *Didaktik* needs experimental research into students’ within the various teaching subjects. Meumann emphasized that, although the *Didaktik* of a teaching subject has to observe the logic of the respective discipline, it does not remain restricted to such a logic, but has to consider

also socio-cultural elements, which are, for instance, materialized in the school structure of the respective society (Meumann, 1914, p. 352).

Given the importance of teaching methodology for the mental work of students, experimental *Didaktik* has to investigate empirically the worth and the effects of various methods on the work of the students. Meumann described the various forms of related didactical experiments, ranging from classroom experiments, thus with a great number, over experiments with individual children, to experiments evaluating the success of particular teaching methods and analyses of special predispositions of a student for a given teaching subject (1914, p. 356).

Always emphasizing the key importance of systematic observation and empirical research, Meumann underlined in particular their need for the teaching of arithmetic. He deplored that, despite extensive literature on its methodology, the essential still remains to be done, “since this can be only resolved by the in depth psychological analysis of calculating via the experiment; this was not accessible to the elder pedagogy” (1914, p. 624-625). In particular, he deplored the missing analysis of the calculating activities of children; likewise, studies about the development of number concepts before entering school are largely missing, too. What he claims as necessary foundation for the *Didaktik* of arithmetic is insight into the “genetic constitution of the number representation and the number concept in the child” (1914, p. 637). In his 19<sup>th</sup> lecture, Meumann gave an excellent report on the state of the art, including a systematic evaluation of German and foreign publications. Remarkable is another hint to socio-cultural factors when he underlines that one cannot make absolute assertions about the development of the number concept, since it depends on the context either promoting a child or letting a child it struggling alone (1914, p. 647).

Especially relevant for this research is his short section on calculating errors of students. Citing experiments of the psychiatric Emil Kraepelin (1856-1926) with adults on arithmetic, Meumann relates Kraepelin’s categorization of their errors into “Denkfehler”, reasoning errors, and writing errors. While ‘writing errors’ clearly mean just missing attention, the notion of *Denkfehler* would have been promising for a cognitive approach. He comments, however, that one would better call the first category “Gedächtnis- oder Assoziationsfehler” (1914, p. 688), thus attributing errors but to bad memory or to a wrong association of facts. He added that such errors rely on a failure of the mechanism of association whose causes could not be explained. Meumann missed here the opportunity for a cognitive approach. Also in the later section on the teaching of mathematics, Meumann stated the almost complete missing of experimental studies. As approaches, he proposed rather vaguely and superficially to compare the effects of some basic methodologies (1914, p. 807). Studies of students’ errors or teacher-student-interactions were still far from the horizon.

## DAVID KATZ'S FIRST "PME" MONOGRAPH – ENHANCED BY IMUK

It presents a major result that a first monograph on psychology of mathematics education was published within the context of IMUK, the *Internationale Mathematische Unterrichtskommission*, the precursor of ICMI, the *International Commission on Mathematical Instruction*. In fact, it was due to the indefatigable energy of Felix Klein (1849-1925) that such a presentation of the state of the art was commissioned. Felix Klein had not only succeeded, as the first president of IMUK, in complementing the national reports on mathematics instruction in the member countries by international trend reports on key aspects of mathematics teaching (see Schubring, 2003), he had moreover, being also president of the German subcommittee, likewise organized that the national reports on mathematics instruction in Germany were complemented by studies on issues constitutive for mathematics education. Among them were studies about the textbooks in the various school types, on the use of history in mathematics teaching, on philosophy in mathematics instruction. And it shows the perspicuity of Klein that he not only envisaged psychological research as deeply pertinent, but also thought that such research had achieved sufficient substance for deserving a proper presentation.

Klein had commissioned David Katz (1884-1953) to present this dimension. Having at first studied mathematics and the sciences at Göttingen University, Katz was already known to Klein. Katz changed to psychology, obtained his PhD and *Habilitation* in Göttingen in psychology. Being professor at Rostock University, he had to leave Germany in 1933, due to racial persecution by the Nazi regime. He continued his researches in the exile, in England and in Sweden. His study for IMUK was published in 1913 as *Psychology and Mathematics Instruction*. He reported there also about his own research. Meumann, too, evaluated Katz's book in his lectures.

Katz's approach clearly was more comprehensive than earlier studies. While those with a pedagogical orientation concerned almost exclusively primary grades, Katz aimed at providing insights into the process of learning mathematics based on experimental psychology for all levels of schooling. Moreover, he had another understanding of the results of psychological research on the students hitherto labelled as 'abnormal'. He distinguished between those impaired in some of their senses – eyes, ears, voice – and the mentally disabled. And even the latter ones were for him not totally segregated, since Katz proposed, for instance, that the teaching methods developed for them constitute resources for teachers of not impaired or disabled children (Katz, 1913, p. 10). For Katz, there was a neat convergence and mutual support of experimental psychology and of experimental pedagogy.

A major focus were the development of the number concept and here in particular the traditional conflict between teaching numbers via counting and via visual representations of numbers (*Zahlenbilder*). Katz included, however, studies about the development of spatial notions, which did not figure in the earlier reported investigations, since geometry used for a long time not to be a subject in primary

schools. An own chapter was devoted to the use of differential psychology, which for Katz meant to apply the then well accepted typology of representations: ascribing to each person a particular type how he/she represents itself the proper perceptions of the outer world. He quoted more visual types, more aural, more motor. He admitted that almost no research on such a pedagogical-didactic use was available, but it was the approach he envisaged to adapt teaching to the individual's predispositions. A research on, say, students' errors was out of reach for him, too (1913, pp. 41-46).

Evidently, Katz also discussed the always controversial issue of mathematical "Begabung" – giftedness. Although somewhat influenced, too, by the physiological fundamentalism of Möbius who had tried to localize such giftedness in certain parts of the brain and had discriminated women as not being capable of understanding mathematics (Möbius, 1907), it was productive that he distinguished between gifted for mathematics as a scientist and aptitude for mathematics as a school subject. And he denounced those who maintained that even for school mathematics a special intelligence is needed (Katz, 1913, p. 59). On the other hand, he introduced 'interest' as the explaining category for the differences in students' achievements. Understanding, however, interest as an "irrational factor" and admitting the heredity of personal interests, Katz missed to unravel social influences on children adopting certain cultural views of sciences.

## OPEN QUESTIONS ABOUT THE FURTHER DEVELOPMENT

The research is so far concentrated on the period until WW I, since this was the most productive period in the international work of IMUK/ICMI. The further developments need to be investigated. Preliminary results suggest that, astonishingly, there was for a long time no progress and no updating or continuing of the researches from before the World War. Neither experimental pedagogy nor psychological research into the learning of mathematics seems to have been continued or pursued by new approaches. Such hints are drawn from a major work on the teaching of arithmetic, destined as methodology for teacher formation in arithmetic instruction, by Ewald Fettweis (1881-1967). This handbook, first published in 1929, but re-edited until the 1960s, referred also in its second and third editions of 1949 and 1950 to the same publications of Ranschburg and Katz exposed here. There is a section on *Rechenschwäche*, following Ranschburg, without more substance. Relative extensive and new is a chapter on students' errors. Although claiming to be based on psychological considerations, there are no experimental evidences but rather abstract reflections, attributing errors to: emotional irritations, uncontrollable causes, physical instances, missing attention, etc. (Fettweis, 1949, pp. 220-235).

An important indicator on the role of ICMI and the importance attributed by it to psychology is again a German contribution, this time to the series of national reports on the state of mathematics instruction commissioned by the re-established IMUK-ICMI from 1954: The volume, edited by Drenckhahn (1958), contains a contribution by Kurt Strunz, not only with the same title as Katz's book, but also following the



same structure. As sign for the dawn of a new time, the volume contains, however, as additional contribution – on developmental psychology regarding mathematics instruction – a paper by Bärbel Inhelder.

## OUTLOOK

The research so far concerned but the case of Germany. The developments in other countries need to be studied, too. According to the extensive references made by Meumann and Katz, there were experimental studies in particular in France and in the USA. On the other hand, among the more than 300 publications of the first IMUK, only the German subcommittee published on psychology.

## REFERENCES

- Biehler, R. et al. (eds.) (1994). *Didactics of mathematics as a scientific discipline*. Dordrecht: Kluwer.
- Drenckhahn, F. (Ed.) (1958). *Der mathematische Unterricht für die sechs- bis fünfzehnjährige Jugend in der Bundesrepublik Deutschland*. Vandenhoeck & Ruprecht.
- Fettweis, E. (1949). *Methodik für den Rechenunterricht*. 2nd edition. Paderborn: Schöningh.
- Katz, D. (1913). *Psychologie und Mathematischer Unterricht*. Abhandlungen über den mathematischen Unterricht in Deutschland. Band III, Heft 2. Leipzig: Teubner.
- Lorenz, J.-H. & H. Radatz (1979) *Lehrer-Schüler-Interaktionen im Mathematikunterricht; Schülerfehler im mathematischen Lernprozeß*. Forschungsbeiträge zum mathematischen Lehr-Lernprozeß 2. Bielefeld: Univ. Bielefeld, Institut für Didaktik der Mathematik.
- Meumann, E. (1914). *Vorlesungen zur Einführung in die experimentelle Pädagogik und ihre psychologischen Grundlagen*. Vol. 3, 2nd edition. Leipzig: Engelmann.
- Möbius, P. J. (1907). *Über die Anlage zur Mathematik*. Leipzig: Barth.
- Ranschburg, P. (1906). Vergleichende Untersuchungen an normalen und schwachbefähigten Schulkindern. *Zeitschrift für Kinderforschung*, 11, 5-18.
- Ranschburg, P. (1916). *Die Leseschwäche (Legasthenie) und Rechenschwäche (Arithmasthenie) der Schulkinder im Lichte des Experimentes*. Berlin: Springer.
- Schubring, G. (2003). *L'Enseignement Mathématique* and the First International Commission (IMUK): The Emergence of International Communication and Cooperation. In Daniel Coray, F. Furinghetti, H. Gispert, B.H. Hodgson, G. Schubring (Eds.), *One Hundred Years of L'Enseignement Mathématique. Moments of Mathematics Education in the Twentieth Century* (pp. 47-65). Geneva: L'Enseignement Mathématique.

## Sources

*Die Kinderfehler. Zeitschrift für Pädagogische Pathologie und Therapie*. Vol.s 1 to 4: 1896-1899. Continued as: *Zeitschrift für Kinderforschung*. Organ der Gesellschaft für Heilpädagogik und des Deutschen Vereins zur Fürsorge für Jugendliche Psychopathen Vol.s 5. 1900- 50.1944.

---

<sup>i</sup> In English, there is no good equivalent for the German term *Didaktik* – the term ‘didactics’ even having a negative connotation, while French and Italian languages have such a term.



# EFFECTIVE MATHEMATICS LEARNING IN TWO AUSTRALIAN PRIMARY CLASSES: EXPLORING THE UNDERLYING VALUES

Wee Tiong SEAH

Monash University, Australia

*Mathematics educational research on high performing countries highlights the role of cultural values. This paper reports on the qualitative phase of a project, which aims to identify some of the values underlying effective mathematics learning in the primary classroom, and to explore how these were valued by the teacher and students. Amongst the 13 values negotiated in the six hour-long lessons, 'example', 'sharing', 'resources', and 'multimodal representations' were embraced by students across all ability groups. In the negotiation of conflicting values between teachers and their students, it was found that the teacher's authority and his/her awareness of what s/he values play important roles.*

## **WHY THE FOCUS ON VALUES**

Particular mathematics pedagogical practices of high performing countries in international comparative assessments such as TIMSS and PISA have been known to be adopted by some education systems. However, a successful practice in one culture does not necessarily transfer to another culture automatically.

Problems with transplanting pedagogical practices or resources across cultures may be socio-cultural in nature. For example, commenting on the adoption of Singapore mathematics textbooks in several American states, Alan Ginsburg identified societal differences between the two countries (e.g. student exposure to mathematics tuitions after school, and differences in teacher professional development) (Gowen, 2001). These cultural differences are unlike more trivial ones such as the metric versus imperial systems of measure (which can be easily addressed), in that they reflect differences in the way societies value (mathematics) education. Similarly, Li (2007) attributed differences in algebra content in textbooks from mainland China, Hong Kong, Singapore and USA to differences in cultural values.

The role of values in shaping mathematics curricula – intended, implemented and attained – across cultures is highlighted in a report commissioned by the Nuffield Foundation (Askew, Hodgen, Hossain, & Bretscher, 2010). It started out thus:

one of the most striking things the review has shown is that high attainment may be much more closely linked to cultural values than to specific mathematics teaching practices ....[S]tudy after study shows that countries ranked highly on international studies – Finland, Flemish Belgium, Singapore, Korea – do not have particularly innovative teaching approaches .... Culture, beliefs and dispositions have all come through strongly as powerful influences in learning mathematics and we explore these in some detail in this report. (p. 12)

The level of culture at which such values manifest themselves exists beyond the nation. In fact, it appears to be more useful to acknowledge the diversity of cultures

within any nation. In Australia, for example, Suliman and McInerney (2006) found that despite research showing that students of non-English speaking background achieved better than other students, the same could not be said of Lebanese-background students' performance. The Lebanese-background students' valuing of *power* also negatively impacted on their attainment in school.

This paper reports on a study which aims to identify some of the cultural values that underlie effective mathematics lessons in Australian primary schools, and to explore how these were co-valued by the teachers and students involved. It is part of a wider, multinational research project, the Third Wave project, investigating how culturally-specific values underlying effective mathematics lessons might be harnessed to optimise mathematics education in schools. In the next section, the notion of values in mathematics education will be discussed. This will be followed by the research design and research context, before the results and findings are presented.

### **VALUES RELEVANT TO MATHEMATICS EDUCATION**

Not all values relevant to mathematics education are contextualised within ethnic groups or countries. Alan Bishop had in the late 1980s proposed the construct of values based on the mathematics discipline, in the form of three complementary pairs of mathematical values, that is, *rationalism* and *objectivism*, *control* and *progress*, and *openness* and *mystery* (Bishop, 1988). The mathematical value of *objectivism* was later replaced by *empiricism* (Bishop, Clarke, Corrigan & Gunstone, 2005).

In the mid-1990s, Bishop (1996) conceptualised values relevant to mathematics education as those which are not only embedded in the discipline, but also in the learning context and in the society within which it is situated. He proposed the categories of mathematical values, mathematics educational values (e.g. *neatness*), and general educational values (e.g. *honesty*). Seah's (2005) research highlighted the influence of the education authorities and school organisation on the learning experience in the classroom, proposing an additional category of institutional values (e.g. *professional development*).

Seah's (2005) research with mathematics teachers revealed how the values subscribed to by principals, parents and students can be in conflict with those held by the teacher, as well as how these convictions were negotiated or co-valued as part of establishing the didactic contract in the class. An understanding of the values negotiation process is thus crucial to any attempt at harnessing values underlying effective pedagogy. As such, this process has also been explored in this study.

### **RESEARCH DESIGN**

This study constitutes the qualitative phase of the sequential mixed methods design for the wider project. Working with a small group of participants in this phase has allowed for the depth of understanding required to identify underlying values and to explain how these had been negotiated in the classroom learning context. The findings arising from this study are expected to inform the construction (content and wording) of relevant data-collection instruments for the quantitative phase.

The methods adopted in this study were lesson observations, interviews and artefact analyses (of photographs and journal entries). For each class, three lessons lasting

about an hour each were visited over a month. During the lessons, each student participant was provided with a digital camera with which to record the moments when s/he felt that mathematics was learnt particularly well. The use of digital cameras as recording device served two purposes: one, photographs taken through the camera lens reveal the pedagogical context from the students' perspectives and angles. More importantly, through the production of 'photovoice' (Wang & Burris, 1997), the students became the ones nominating what constituted data. This focus on students' opinions and views reflects Loughran and Northfield's (1996) view that "quality teaching requires learner consent" (p. 124), "compared to teaching in which it is assumed that learning can be mandated" (Loughran, 2010, p. 49).

After the lessons, the students were asked to review the photographs taken. These then served as conversation stimuli during the post-lesson focus-group sessions, in which semi-structured interview questions probed for what the 'moments of effective learning' looked like, how contradicting values were negotiated, and what eventually were co-valued in the class.

On the other hand, the teacher participants maintained a journal for 4 weeks before the lesson visits, in which they shared their experiences of effective lessons taught, reflecting on what they thought were being valued by them and their students in such situations. They were also individually interviewed after each student focus-group interview, in which similar questions were posed. These semi-structured interviews also allowed the teachers to comment on what their students said in the focus-group interviews. Cross-checking of data was also achieved through preliminary analyses conducted between visits, for follow-up (if needed) in the next visits.

Interview audio-records were transcribed into verbatim format. All data were analysed through the three-stage open, axial and selective coding which typifies the grounded theory research approach proposed by Strauss and Corbin (1990).

## **RESEARCH CONTEXT**

Two teachers, Kellie and Yasmine, from a government primary school in suburban Melbourne took part in this study. Many of the parents associated with this school are Generation Y-ers representing diverse ethnicities. That classes in the school were not streamed, that both Kellie and Yasmine were teaching in Grade 5, and that they planned their lessons together, all contributed to the similarity across the two classes.

Kellie was an experienced classroom teacher and a mathematics leader within the school. She possessed 8 years of teaching experience across different grade levels in 2 different schools. On the other hand, Yasmine was a 'first year out' teacher; having completed a pre-service primary education degree a year before. There were much communication and co-planning between these two teachers.

Each teacher was invited to nominate 6 students, 2 each of whom had been perceived by the respective teachers as being of high, average and below average abilities. The different ethnicities of the students served this study well, in that it allowed for a more comprehensive list of values operating in the Australian classroom to be made.

To identify the values associated with effective mathematics lessons, participants were asked periodically to specify what were regarded as being important to them in

relation to the ‘moments of effective learning’, and why. Often, these reasons would reveal underlying, cultural values. For example, schools in Japan and England may value *lesson study*, but it is only through ‘peeling back the layers’ would this apparent similarity give way to the culturally-different values underlying the adoption of this pedagogical approach (see Askew, Hodgen, Hossain, & Bretscher, 2010).

**WHAT THE STUDENTS AND THEIR TEACHERS CO-VALUED**

The values that were nominated by the students as being associated with moments of effective learning are listed in Table 1 according to teacher, gender, and ability levels. In the Table, students are represented by codes made up of two alphabets and a numeral. The first alphabet (K or Y) represents a student’s teacher. The second alphabet (H, M or L, which represents high, medium and low respectively) associates the student with the ability level perceived by his/her teacher. The numeral differentiates between the two students in each ability group in each class.

Values	Kellie’s students		Yasmine’s students	
	Male	Female	Male	Female
Examples	KH1	KM1, KL2	YH1, YM2	YH2, YM1, YL1, YL2
Sharing	KH1, KL1	KH2, KM2, KL2		
Resources			YM2	YH2, YM1, YL1
Multimodal representations	KL1	KH2, KM1		
Explanation	KL1	KM1, KM2, KL2		YL1
Fun	KH1		YM2	YH2
Doing mathematics			YH1, YM2	
Efficiency				YH2, YM1
Competition			YH1	YH2
Questions		KM1		YM2
Certainty				YL2
Assistance				YH2
Individual effort			YM2	

Table 1. Values associated with effective mathematics lessons.

A total of 13 different values appear to be co-valued by the 2 teacher participants and their 12 students in 6 lessons, these being *examples*, *sharing*, *resources*, *multimodal representations*, *explanation*, *fun*, *doing mathematics*, *efficiency*, *competition*, *questions*, *certainty*, *assistance*, and *individual effort*. They have been listed in Table 1 in groups according to their being embraced by students across all three ability levels, any two ability levels, or unique to any one ability level. In particular, the values of *examples*, *sharing*, *resources*, and *multimodal representations* were embraced by students from all three ability levels, with implications for better catering to the learning styles and needs of mixed-ability classes.

It can be argued that the valuing of *examples*, *resources*, and *multimodal representations* during effective moments of mathematics learning emphasises the importance of concrete and semi-concrete support materials. Askew, Hodgen, Hossain and Bretscher's (2010) review had revealed that "one major finding ... is the evidence of more use of formal and abstract strategies by Chinese pupils than by American counterparts" (p. 32); to what extent then does the Australian mathematics education culture reflect this difference between the East and the West? What might the implication be, given that the valuing of *challenge* was not evident in this study (which echoes Ainley, Kos and Nicholas' (2008) finding that Year 8 students in Australia did not feel challenged in mathematics lessons)?

*Sharing* was also valued by students across all perceived ability groups. This value was expressed through either peers sharing (e.g. student KM2), or students' own sharing with their respective peers (e.g. student KH2). It was also reflected in the effective learning that arose from groups of friends working together on mathematical tasks (e.g. student KH1). This association of *sharing* with effective mathematics learning is perhaps not surprising in the Australian (educational) culture, in which children starting formal schooling are already expected to show-and-tell regularly in front of their classmates.

In this qualitative phase, data had been gathered from only 2 classes; no claim can be made yet that they represent all the values associated with effective mathematics pedagogy in the Australian classroom. Rather, the quantitative phase of this project will be designed to derive lists of values representative of different cultural groups.

## FEATURES OF THE VALUES IDENTIFIED

A feature amongst the 13 values identified in this study has been that the same one value might be expressed as different pedagogical practices. For example, *sharing* was valued both in terms of students discussing problem-solving strategies in groups, as well as in terms of individual students explaining their reasoning by the whiteboard. This focus on values rather than on specific pedagogical activities may well enable us to better cater to mixed-ability groups of students.

There were 2 male and 4 female student participants from each class. In this context, there is no evidence from Table 1 that gender affects the values associated with

effective mathematics learning. This aspect of values nomination will be probed further in the quantitative phase of the project.

As is evident in Table 1, values associated with effective mathematics learning may be common across all student ability groups (*examples, sharing, resources, and multimodal representations*), between the high and average ability groups (*fun, doing mathematics, and efficiency*), and between the average and below average ability groups (*explanation*). These may also be unique to high ability students only (*competition, and assistance*), to average ability students only (*questions, and individual effort*), or to below average students only (*certainty*). The high and below average ability groups did not share any value in common.

While equal representation of students in the three ability groups has allowed us to identify what were valued across these groups, the prevalence of individual values would not be immediately obvious. For example, the current data do not demonstrate how *fun* and *explanation* were highly valued in mathematics lessons (see Seah & Ho, 2009). They did show, nevertheless, that both these values were embraced by average ability students; their dominance (see Seah & Ho, 2009) can be deduced by considering the relative proportion of these students in the population generally.

## HOW CONTRADICTING VALUES WERE NEGOTIATED

The teacher and his/her students bring to the lesson different values, some supporting one another, but others in conflict. As discussed earlier, values enacted through pedagogical activities in the mathematics classroom would have been negotiated between the teacher and his/her students. It is an aim of this study to explore the nature of such values negotiation processes, as situated meanings and subjective intentions are brought to bear.

The following accounts by Kellie demonstrate how she typically negotiated about differences in values between herself and her students. Here, she was talking about her valuing of (mathematical) *language* in mathematics learning:

Sometimes I will actually be very frank with them and say, “you need to know this language, because you are not going to understand it when you get to another teacher.” And they go, they sit up and go, “okay, now I really need to know. She’s [Kellie] been fairly harsh.” And once I do that, they go, “okay,” and they take notice, and they think that’s important, whereas if I don’t put an importance to it, they’d just go, “okay, that’s not that important.” (Kellie, KI3 0138-0256)

whereas there were also times when

I kinda listen to them [ie her students] a little, and goes, “well, this is what they think is important or they do, so I try to manipulate it to the way they like it. (Kellie, KI3 0000-0048)

Quite clearly, Kellie was aware of her authority as teacher in the classroom, and certain of the pedagogical values she embraced. Here, she valued student command of mathematical language in that it would position the students well to understand the

pedagogical discourse of other mathematics teachers. While her students appeared to respect her valuing of *language*, Kellie would also give in to certain aspects of this valuing without sacrificing the ‘big picture’.

Yet, such contestation of values as they relate to mathematics pedagogy was not always a straightforward process; it could be a work-in-progress, as demonstrated by Yasmine’s experience. The problems she faced in dealing with the contradictory values of *explanation* and *listening* were attributed by her to a relative lack of professional experience, wherein “I think a lot of it comes up to, probably being first year [in the job]” (Yasmine, YI3, 1131-1135).

## DISCUSSION

This study constitutes the qualitative phase of a wider project. The nature of this phase has allowed for the identification of 13 different culturally-based values related to effective mathematics learning in two Australian primary school classes. While there was no evidence of gender effect, it will be probed further in the quantitative phase of the study. Student ability appeared to be a factor governing what were co-valued. Nevertheless, four values (i.e. *example*, *sharing*, *resources*, *multimodal representations*) were found to be embraced by students across all ability groups.

This study also seeks to understand how teachers and students responded to situations when their values were not in agreement with one another. The data collected suggested that with professional experience and with the authority teachers command in class, teachers’ explicit knowledge of what they value in (mathematics) education is a key factor in guiding the way in which contradictory values are negotiated.

Given the small participant size in this study, these findings are by no means generalisable. Rather, the findings will inform the construction of relevant data collection instruments in the next, quantitative phase of the project, such as the questionnaire survey, the interview protocol, and the lesson observation protocol.

Nevertheless, these findings highlight the roles which (cultural) values play in facilitating effective mathematics pedagogy. An understanding of what these are and how they are co-valued in the didactic contract as they relate to mathematics teaching and learning should not be a complement to other approaches to mathematics education research and professional practice, but an integral part of these.

## References

- Ainley, J., Kos, J., & Nicholas, M. (2008). *Participation in science, mathematics and technology in Australian education*. Victoria, Australia: ACER.
- Askew, M., Hodgen, J., Hossain, S., & Bretscher, N. (2010). *Values and variables: Mathematics education in high-performing countries*. London: Nuffield Foundation.
- Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

- Bishop, A. J. (1996, June 3-7). *How should mathematics teaching in modern societies relate to cultural values --- some preliminary questions*. Paper presented at the Seventh Southeast Asian Conference on Mathematics Education, Hanoi, Vietnam.
- Bishop, A., Clarke, B., Corrigan, D., & Gunstone, D. (2005). Teachers' preferences and practices regarding values in teaching mathematics and science. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Research, theory and practice* (Vol. 1, pp. 153-160). Sydney, Australia: Mathematics Education Research Group of Australasia.
- Gowen, A. (2001, October 18). East meets west in math classes: 4 schools import curriculum from Singapore. *Washington Post*, p. T14,
- Li, Y. (2007). Curriculum and culture: An exploratory examination of mathematics curriculum materials in their system and cultural contexts. *The Mathematics Educator*, 10(1), 21-38.
- Loughran, J.J. (2010). What expert teachers do: Teachers' professional knowledge of classroom practice. Sydney & London: Allen & Unwin, Routledge.
- Loughran, J.J. & Northfield, J.R. (1996). Opening the classroom door: Teacher, researcher, learner. London: Falmer Press.
- Seah, W. T. (2005). *The negotiation of perceived value differences by immigrant teachers of mathematics in Australia*. Unpublished PhD dissertation, Monash University, Australia.
- Seah, W. T., & Ho, S. Y. (2009). Values operating in effective mathematics lessons in Australia and Singapore: Reflections of pre-service teachers. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 57-64). Thessaloniki, Greece: International Group for the Psychology of Mathematics Education.
- Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.
- Suliman, R., & McInerney, D. M. (2006). Motivational goals and school achievement: Lebanese-background students in south-western Sydney. *Australian Journal of Education*, 50(3), 242-264.
- Wang, C., & Burris, M. A. (1997). Photovoice: Concept, methodology, and use for participatory needs assessment. *Health Education and Behavior*, 24(3), 369-387.



# THE CONCEPT OF SERIES IN UNDERGRADUATE TEXTBOOKS: TASKS AND REPRESENTATIONS

Rachid Seffah and Alejandro S. González-Martín

Université de Montréal (Canada)

*The concept of infinite sum holds a fundamental place in mathematics. This privileged place is due, among others, to its large number of applications and to its importance in defining other concepts. However, there are very few studies about the teaching and learning of this concept. In order to portray how this concept is taught, in this paper we analyse the way in which infinite sums are introduced in postsecondary textbooks in Québec. Our results show that textbooks generally introduce this concept by privileging both the use of algorithmic tasks, which do not directly address the concept of series itself, and the use of the algebraic register.*

## INTRODUCTION

The study of infinite sums (or series) is an important topic in the mathematical curriculum of postsecondary studies (Bagni, 2000). This concept, useful for introducing other key concepts, also has many applications. For instance, infinite sums are used as a basis to introduce the concepts of Riemann sum and of definite integral, both essential in calculating the area under a curve. One of the easiest applications of series in mathematics is the writing of numbers with infinite decimals (e.g.  $0.999... = 0.9 + 0.09 + 0.009 + ...$ ). Furthermore, the concept of infinite sum has many applications outside of the field of mathematics as it is useful for modelling different phenomena in various scientific domains (physics, economy, biology, etc.). Essentially, this concept is fundamental in the mathematical education of a large range of scientists and professionals. Given the importance of series in introducing and understanding many other mathematical concepts and considering their many applications in a variety of scientific areas, we believe that they deserve to be the focus of research work in mathematics education. Consequently, it is quite surprising that there are very few studies about its teaching and learning (Nardi, Biza & González-Martín, 2009). The concept of series usually appears indirectly in research results about the concepts of convergence, limit, and numerical sequence, but there are not many works focusing on the concept of series itself. We summarise in the following some of the main results on the teaching and learning of series.

Initial research has shown that the problems generally used in traditional teaching do not guide the students in constructing an adequate notion of convergence for numerical series (Robert, 1982), and also that traditional teaching generally does not make use of visual representations, which can lead to erroneous constructions of this concept (Boschet, 1983). More recent research suggests that the use of visual reasoning and the graphic register could help students to better understand series

(Codes & Sierra, 2007). These findings coincide with others in more general research stating that the use of visual aspects in teaching mathematical notions could allow for better constructions and a greater understanding of the targeted concepts for learners (Tall, 1991). Moreover, according to Alcock & Simpson (2004), traditional teaching usually presents infinite sums as reduced exclusively to their algorithmic aspects and restricted to the sole use of the algebraic register. This approach might later produce some difficulties in understanding the concept of integral (Bezuidenhout & Oliver, 2000; González-Martín, 2006).

Our literature review has led us to the following conjectures according to which: 1) even 30 years after Robert's and Boschet's works, postsecondary teaching practices still favour the application of criteria to study the convergence or the divergence of series, and the procedures required to calculate the sum of certain convergent series, however little importance is given to the applications of the concept or to the construction of meaning; 2) postsecondary teaching practices privilege the study of series while restricting their representation to the algebraic register, with hardly any connections to the graphic register. Consequently, the study of these two conjectures should provide data about the institutional approaches in the teaching of series since, as mentioned above, no extensive research about these practices has been done up to date. The confirmation of our conjectures would result in negative effects in the understanding of series by undergraduate students, as indicated by previous research.

In order to verify our conjectures, we decided to analyse textbooks used in postsecondary education to introduce numerical series, as well as some teaching practices. Moreover, this study is being conducted both in Canada and in UK, which will allow us to compare practices in two different countries. A first glance of the results in the UK is shown in Nardi, Biza & González-Martín (2009), and a comparison of some of the results in both countries can be found in González-Martín, Nardi & Biza (in press). Some initial trends emerging after the analysis of seven textbooks appeared previously in González-Martín, Seffah & Nardi (2009). The present paper will focus on the analysis of textbooks developed in Canada, to give the first extensive portrait of the ways in which series are introduced in postsecondary education. We analysed seventeen textbooks used over a period of eighteen years in postsecondary education in Québec. We combined an institutional and a semiotic perspective to our analysis of the practices privileged by the textbooks, as well as the registers used. This perspective will also allow us to make some inferences about the learning that can be achieved by students.

## **THEORETICAL FRAMEWORK**

We want to analyse the learning and understanding of series through three dimensions: epistemological, cognitive, and didactic (Brousseau, 1983). The research described in this paper shares our results about the third dimension listed, as associated with the ways in which the concept of series is presented to the students in both pedagogical and curricular terms. We analyse the ways in which textbooks

introduce the concept of series in order to see whether these methods foster a sufficient understanding of the targeted concept.

In developing our analysis of textbooks we chose to consider Chevallard's (1999) anthropological theory, which recognises that mathematical objects are not absolute objects, but entities which arise from the practices of given institutions. These practices are described in terms of tasks, in terms of techniques used to solve these tasks, and in terms of a discourse which both explains and justifies the techniques. In this sense, to understand the meaning in an institution of "knowing/understanding an object", is to identify the practices which bring this object into play. This theory helps to better understand and to interpret the choices made in organising the teaching of a given concept, and their consequences on the significance of the concepts taught as well as on the learning achieved by the students.

The use of this theory enabled us to distance ourselves from a strictly cognitive approach, and to consider the place of institutional issues which we recognise as essential, especially in the organisation and presentation of content in textbooks.

In order to analyse the cognitive activities asked of the students in the textbooks, and taking into consideration the results of our literature review which indicates the advantages of visualisation, we decided to use Duval's theory of the registers of semiotic representation (Duval, 1995). According to this author, teaching restricted to a single register does not facilitate an adequate understanding of a mathematical concept. In mathematics, it is essential to distinguish between an object and its representations and to achieve understanding, the use of different semiotic representations of mathematical objects is absolutely necessary. This is due to the fact that each register gives only partial information about the object it refers to. However, it is not enough to simply be able to use these semiotic representations; Duval states that the coordination of at least two different representations is necessary to understand a mathematical concept (Duval, 1995).

Consequently, in the analysis of our 17 textbooks we paid particular attention to the kind of tasks privileged by them in introducing the concept of series, in addition to the use and the coordination of both the algebraic and the graphic registers.

## METHODOLOGY

In Canada, each province has its own Ministry of Education, which organises education on the provincial level. In Québec, between secondary education (where students are 12-16 years of age) and undergraduate education (where students are 19 years old or more), there are two years of postsecondary education called *collégial*. *Collégial* studies prepare students to pursue university studies, and the concept of series is first taught in the first year of *collégial* for the students pursuing a scientific-technical career. In *collégial* studies, textbooks are a main source of teaching practices.

For our analysis of textbooks, we chose textbooks appearing in the official programs

of postsecondary establishments of Montreal (the city with the largest student population in the province of Québec) over the last eighteen years. For textbooks that had several editions, we considered the different editions only if the chapter for series had undergone substantial changes. If not, we chose to only consider the earliest edition. Table 1 below shows the year of edition of our seventeen textbooks (the whole references appear in González-Martín, Nardi & Biza, in press):

A	B	C	D	E	F	G	H	I
1992	1993	1993	1994	1995	1995	1996	1997	1997
J	K	L	M	N	O	P	Q	
2000	2001	2002	2002	2002	2004	2006	2008	

Table 1: The seventeen textbooks of our sample.

During our analyses, we were mainly interested in answering the following questions:

- Do the textbooks use visual representations and what semiotic registers are favoured?
- Is there coordination between the algebraic and graphic registers in the textbooks?
- What types of tasks (exercises) are privileged?
- Do these tasks guide the students towards constructing meaning of the mathematical concept?

To tackle these questions, we constructed an analysis grid considering the following elements: the number, type and role of the visual representations (e.g. graphs, drawings, etc.) and the ratio of visual representations per page; the number and type of applications of the concept of series (e.g. real life applications, applications in other disciplines, etc.); the number, type and category of tasks (e.g. exercises aiming to study convergence, to calculate the sum, etc.). Therefore, our analysis grid included both qualitative and quantitative data. To analyse the role of visual representations in the textbooks, we adapted the classification offered by Elia & Phillipou (2004), they studied the role of images in problems. As we were interested in the use of visualisation in building the concept of series, we created new categories to analyse the role of images in the theoretical explanations of the textbooks:

- Non-conceptual (NC): does not relate to a mathematical concept (e.g. the portrait of a mathematician).
- Conceptualised (C): does relate to a mathematical concept and it is used to explain a theoretical notion (e.g. a graph in the proof of a given theorem).
- Bland- Conceptualised (BC): does relate to a mathematical concept, but it is not essential for understanding the theoretical explanation (e.g. sketch in a margin showing the graph of the sinus function, to remind the student of its shape).

To analyse the types of exercises, we will use in this paper two subcategories in the exercises without context:

- Algorithmic exercises (AE): exercises concerning the study of the convergence or the divergence of a series, and/or the calculation of its sum.
- Other case (OC): other types of exercises.

The following section shows the results of our analyses pertaining to our research questions.

## DATA DISCUSSION

Regarding the use of the visual representations in the theoretical sections to introduce and explain series, the number of them used by textbooks has increased, especially in the most recent textbooks (from 2002 on), but the ratio number of visual images per page is still very low, with an average of 0.17 in our sample (see table 2):

	A	B	C	D	E	F	G	H	I
Number of images/graphs	2/0	2/1	1/2	1/2	2/1	2/2	1/2	8/3	2/2
(ratio)	0.09	0.15	0.08	0.05	0.08	0.10	0.05	0.22	0.10
Number of Conceptualised images/graphs	0/0	0/1	1/2	0/2	0/1	0/2	1/2	0/2	1/2
	J	K	L	M	N	O	P	Q	
Number of images/graphs	2/1	1/4	4/7	3/2	10/5	10/2	11/19	11/2	
(ratio)	0.08	0.37	0.21	0.18	0.31	0.22	0.46	0.26	
Number of Conceptualised images/graphs	0/1	0/4	3/2	0/2	3/5	1/2	8/5	3/2	

Table 2: Number of images and graphs in the theoretical sections

The use of the graphic register in the theoretical content of the textbooks is mainly reserved to support the understanding of the Integral Test (stating under which conditions the behaviour of a series is the same as the behaviour of an integral), like in figure 1. The role of these graphs is conceptualised. As table 2 shows, almost all the textbooks only use 2 C-graphs, which are a variation of the graph shown in figure 1. Textbooks N and P use five variations of the same graph.

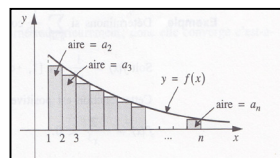


Figure 1

In none of the seventeen textbooks, the student is explicitly asked to produce or to interpret a graphical (or any other kind of visual) representation of the concept of series. For this reason, we argue that the coordination of the graphic and the algebraic registers is absent in all of the seventeen textbooks that we have analysed.

Moreover, in all the exercises of the 17 textbooks, the Integral Test is reduced to the application of an algorithm without referring to the graphic register. This situation does not favour the construction of visual representations for the concept of series, so the coordination of registers is also absent. According to Duval (1995), this lack of coordination might have a negative effect on the students’ construction of the notion of series. This fact seems to support González-Martín’s (2006) conjecture that undergraduate students have no visual images associated with the concept of series.

Concerning the kind of activities which are privileged in the textbooks, table 3 shows that the number of exercises with a context is very small in our sample:

	A	B	C	D	E	F	G	H	I
Exercises	49	154	272	132	92	228	320	75	166
With a context	0	0	6	3	1	7	3	0	2
Algorithmic exercises	33	142	219	124	76	194	240	71	127
Other activities	16	12	47	5	15	27	77	4	37
	J	K	L	M	N	O	P	Q	
Exercises	104	45	293	0	217	167	173	51	
With a context	0	11	11	0	2	2	5	0	
Algorithmic exercises	99	29	253	0	182	132	127	44	
Other activities	5	5	29	0	33	33	41	7	

Table 3: Exercises in the seventeen textbooks.

Presenting applications do not seem to be a priority in the textbooks as this aspect is generally neglected. Instead, the favoured applications are: economics, medicine, modelling of the bouncing of a ball, and writing numbers with infinite decimals. Moreover, six of the textbooks do not show any application for series. This means that series are usually presented as a concept which does not seem to fill any purpose, so practices seem to be centred on the learning of convergence criteria. According to Chevallard (1999), the learning achieved in these cases will be completely deprived of any practical purpose, and the techniques will be associated with the application of criteria, but not with the concept of series itself.

Furthermore, if we compare the number of algorithmic exercises to the total number of tasks given in each textbook, we can see that the emphasis is placed on these algorithmic tasks, which as mentioned previously, involve techniques which require the students to apply criteria rather than to handle the process of convergence and the

notion of the infinite which is inherent to series. The percentage of tasks concerning the use of criteria or of the sum of a series is greater than 80% in eleven of the textbooks. It seems that this fact supports our conjecture about the approach generally privileged by postsecondary practices to introduce the concept of series.

## FINAL REMARKS

Our analyses seem to reinforce the conjecture that the approaches generally favoured by postsecondary textbooks for introducing the concept of series are traditional, with few visual representations and the predominance of the algebraic register. The coordination of the graphic and the algebraic registers is absent in all the textbooks of our sample, even if this coordination is fundamental to achieve understanding of the concept (Duval, 1995). In addition to that, the student is never explicitly asked to interpret or to produce a visual representation of the concept of series. Consequently, students might experience great difficulty in constructing or interpreting visual images linked to this concept. Moreover, little importance is given to the applications of the concept of series and the tasks featured in the textbooks imply the use of techniques which are not always directly linked to the concept of series itself, but to the application of algorithms required to study convergence or to calculate a sum.

The fact that there is little research about the teaching and learning of the concept of infinite sum might be one of the reasons why practices for the teaching of this concept have remained relatively unchanged for the past 30 years. We believe that this situation justifies more research about the teaching and learning of series, which could provide a basis for changing these practices.

## Acknowledgements

This project is funded by the program Nouveau chercheur du Fonds Québécois de Recherche sur la Société et la Culture (FQRSC) – ref. 128796.

## References

- Alcock, L. & Simpson, A. (2004). Convergence of sequences and series: interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, 57, 77-100.
- Bagni, G. T. (2000). Difficulties with series in history and in the classroom. In J. Fauvel & J. Van Maanen (Eds.), *History in Mathematics Education. The ICMI Study* (pp.82-86). Dordrecht: Kluwer Academic Publisher.
- Bezuidenhout, J. & Olivier, A. (2000). Students' conceptions of the integral. In T. Nakahara & M. Koyama (Eds.), *Proc. 24<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 73-80). Hiroshima, Japan: PME.
- Boschet, F. (1983). Les suites numériques comme objet d'enseignement. *Recherches en didactique des mathématiques*, 4(2), 141-163.
- Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en didactique des mathématiques*, 4(2), 165-198.

- Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en didactique des mathématiques*, 19(2), 221-266.
- Codes, M. & Sierra, M. (2007). Actividad rectángulos: Un ejemplo de aplicación de metodologías activas en el aula universitaria de matemáticas. In *Actas de las IV Jornadas internacionales de Innovación Universitaria*. Ed. Universidad Europea de Madrid.
- Duval, R. (1995). *Sémiosis et Pensée Humaine. Registres sémiotiques et apprentissages intellectuels*. Neuchatel : Peter Lang.
- Elia, I. & Philippou, G. (2004). The functions of pictures in problem solving. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 351-358). Bergen, Norway: PME.
- González-Martín, A. S. (2006). *La generalización de la integral definida desde las perspectivas numérica, gráfica y simbólica utilizando entornos informáticos. Problemas de enseñanza y de aprendizaje*. PhD Thesis. Univrsiy of La Laguna (Spain).
- González-Martín, A. S., Nardi, E. & Biza, I. (in press). Conceptually-driven and visually-rich tasks in texts and teaching practice: the case of *infinite series*. *International Journal of Mathematics Education in Science and Technology*.
- González-Martín, A. S., Seffah, R. & Nardi, E. (2009). The concept of series in textbooks: a meaningful introduction? In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.), *Proc. 33<sup>rd</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 105-112). Thessaloniki, Greece: PME.
- Nardi, E., Biza, I. & González-Martín, A. S. (2009). Introducing the concept of infinite sum: the role of visualisation and exemplification. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.), *Proc. 33<sup>rd</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 193-200). Thessaloniki, Greece: PME.
- Robert, A. (1982). L'acquisition du concept de convergence des suites numériques dans l'enseignement supérieur. *Recherches en didactique des mathématiques*, 3(3), 307-341.
- Tall, D. (1991). Intuition and Rigour: The role of Visualization in Calculus. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in Teaching and Learning of Mathematics* (pp. 105-119). MAA Notes n° 19. MAA: Washington DC.



# THE ROLE OF PROCEDURAL KNOWLEDGE IN MATHEMATICAL REASONING

John Selden and Annie Selden

New Mexico State University, USA

*This theoretical paper treats mathematical reasoning as a sequence of actions, each of which is enacted in response to a situation in a partly finished reasoning process. If similar situation-action pairs occur often, they may be (tacitly) reified into an automated situation-action pair, without the need for explicit recollection or a warrant. Such situation-action pairs are a form of “active” procedural knowledge, the smallest of which we call behavioral schemas. Six properties of behavioral schemas are described and several examples from a proof construction course are provided. Finally, it is argued that one can see the role of such schemas by solving a linear equation or constructing a simple proof, and noting that many of one’s own actions are taken in an automated way, without the need to provide warrants.*

It has been noted that procedural knowledge has been somewhat neglected in recent mathematics education research (Star, 2005). Here, in this theoretical paper, we introduce a new way of looking at procedural knowledge that suggests it can play a major role in understanding and teaching mathematical reasoning generally, from solving linear equations to constructing proofs.

We treat procedural knowledge as knowing how to do something, in contrast to conceptual knowledge, that is, knowing that or why something is true. Procedural knowledge can occur in two forms, either as a description of how something is done or as the ability to actually do the something when the need arises. We will focus only on the second, more active form, that includes what Mason and Spence (1999) have called “knowing to act in the moment”. We suggest that this kind of procedural knowledge is retained in procedural memory, much of which is known not to degrade with age in healthy subjects (Backman, Small, & Wahlin, 2001).

## THE RESEARCH SETTING

The setting in which we have developed our ideas about reasoning is a course on proof construction designed to improve the proving skills of beginning graduate and advanced undergraduate mathematics students. The students are given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students construct their proofs at home and present them in class. The proofs are then critiqued, sometimes extensively, and suggestions for improvements in the notation used and style of writing may also be given. There are no formal lectures, and all comments and conversations are based solely on students’ work. The specific topics covered are of less importance than giving students opportunities to experience as many different types of proofs as possible and having

them develop beneficial ways of reasoning. This setting allows us to see into the development of students' reasoning. Although the topics in this setting are different from those concerned with, say, solving equations, we see no grounds to suspect that students' development of methods of reasoning should be different.

## SITUATIONS AND ACTIONS

Mathematical reasoning can be done by an individual or a group, but for simplicity, we will discuss only individual reasoning. We see this as an activity, that is, as a sequence of actions, that are either physical (such as writing or drawing) or mental (such as attempting to recall a definition or theorem). Each action is paired with, and is a response to, a situation in a partly completed reasoning process. By a situation we mean a reasoner's inner, or interpreted, situation as opposed to an outer situation that may be visible to an observer. This distinction was illustrated by Norton and D'Ambrosio (2008), who described two middle school students, Will and Hillary, trying to work a task involving a fraction such as  $\frac{2}{3}$  (the outer situation). Hillary had (in her knowledge base) a *partitive fractional scheme*, as well as a *part-whole fractional scheme*, while Will had only the latter scheme. Hillary was able to solve the task while Will was not. Will was only able to solve the task after he had developed a partitive fractional scheme, and had presumably experienced a richer inner situation.

## BEHAVIORAL SCHEMAS

If a person engages in several reasoning sessions, such as proving several theorems or solving several equations, then he or she is likely to experience a number of similar situations yielding similar actions, such as collecting the  $x$ 's to solve a linear equation. The first such situation-action pair is likely to have a conscious warrant based on, say, heuristics, logic, strategy, or known mathematics. However with time and (sometimes considerable) repetition, the need for a conscious warrant may disappear. The situation may then become linked, in an automated way, to a tendency to carry out the corresponding action; and the individual will not be conscious of anything happening between the situation and the action. We see such automated situation-action pairs as persistent mental structures and have called the smallest of them *behavioral schemas* (Selden, McKee, & Selden, 2010). By a small situation-action pair, we mean one that is not equivalent to any sequence of smaller such pairs. While the word "schema" has been used in several ways in the literature, we only mean such a persistent mental structure.

Seeing procedural knowledge in this way suggests it may occur widely throughout mathematics. For example, in situations involving a function  $f$  from  $X$  to  $Y$  and a subset  $A$  of  $Y$ , most university mathematics students will know that  $f^{-1}(A)$  is defined (conceptually) to be  $\{x \mid f(x) \in A\}$ . However, in the context of constructing a proof, for many students this will correspond to the procedural knowledge that, given

$b \in f^{-1}(A)$ , it is legitimate to claim  $f(b) \in A$ , and vice versa. However, we have had some students who could state the concept definition, but in practice did not exhibit

the corresponding procedural knowledge by carrying out the indicated action in the appropriate situation.

Although in a behavioral schema we are referring to a person's inner situation, we have found in teaching that we can often gauge approximately what the inner situation is from the outer, observable, situation and the ensuing action. We describe such events from our teaching towards the end of the paper.

The idea of behavioral schemas is at variance with the widely held belief that the conscious intention to act is the cause of an action. If this were so, then, in a behavioral schema, the conscious intention to act would occur between recognizing a situation and the corresponding action. However, Libet, Gleason, Wright, and Pearl (1983) have shown that the belief that conscious intentions drive actions is at least partly an illusion. They compared the onset of a brain wave called the readiness-potential (the beginning of an act) and subjects' reports of when they intended to act, and found that subjects' intentions could not have caused their actions. This finding was later replicated by Lau, Rogers, Haggard, and Passingham (2004) using fMRI technology instead of subjects' self reports (that might not have been accurate).

## **THE ROLE OF CONCEPTS IN THE GENESIS OF BEHAVIORAL SCHEMAS**

Although behavioral schemas are a kind of procedural knowledge and are retained in procedural memory, their genesis requires developing a way of recognizing particular kinds of situations, and in response, enacting particular kinds of actions. It is possible that neither the kind of situation nor the kind of action for a potential behavioral schema exists as a concept in the surrounding culture. In that case, constructing a behavioral schema requires noticing similarities among situations and among the corresponding actions, and eventually reifying these into what amounts to conceptions (usually without any need for formal designations).

To illustrate the need for the construction of such conceptions, we describe a student, Marsha, from our Fall 2008 proof construction course. Marsha, who planned to become a primary school teacher, appeared to be bright but was well behind the other students in mathematical preparation, and hence, required extensive tutoring by us. In one tutoring session, Marsha was attempting to construct a proof and the first author was suggesting things to do only as necessary. Marsha had a partly constructed proof that had already called on everything one might get from the logical structure of the theorem and definitions involved. Since she had reached an impasse, the first author suggested that Marsha might "explore" the mathematics related to the theorem in the course notes. There is no standard terminology for this idea, but what was being suggested was that she write very brief accounts of everything related to the theorem on the scratch work blackboard in an effort to generate a new idea. It turned out that Marsha had no idea of this concept and responded only after considerable prompting and explanation. Thus, for Marsha to develop a behavioral schema with the idea of explore as an action, she first needed to construct a conception of exploration.

## PROPERTIES OF BEHAVIORAL SCHEMAS

(1) Within very broad contextual considerations, behavioral schemas are immediately available. They do *not* normally have to be recalled, that is, searched for and brought to mind. This distinguishes them from most conceptual knowledge and episodic or declarative memory, which generally *do* have to be recalled or brought to mind.

(2) Behavioral schemas operate outside of consciousness. A person is not aware of doing anything immediately prior to the resulting action – he/she just does it. This eases the burden on working memory. Furthermore, the enactment of a behavioral schema that leads to an error is not under conscious control, and one should not expect that merely understanding the origin of the error would prevent its future occurrences. Compound behavioral schemas, that is, several behavioral schemas “chained together,” are also largely not under conscious control.

(3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs.

(4) A behavioral schema that would produce a particular action cannot pass that information, outside of consciousness, to be acted on by another behavioral schema. The first action must actually take place and become conscious in order to become information acted on by the second behavioral schema. That is, one cannot “chain together” behavioral schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action. For example, if the solution to a linear equation would normally require several steps, one cannot give the final answer without being conscious of some of the intermediate steps.

(5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person’s actions, but such influence is normally not precise enough for doing mathematics. Also, non-conscious stimuli that lead to action usually originate outside of the mind, not within it (as often happens in proof construction). This suggests consciousness is required for much of mathematical reasoning.

(6) Behavioral schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing detrimental behavioral schemas, many of which have been tacitly acquired, requires similar, perhaps longer, practice (Selden, McKee, & Selden, 2010).

Some experienced teachers may have noticed that giving a counterexample to a student who consistently makes an errorful calculation, like  $(3a+b)/3c = (a+b)/c$  or  $\sqrt{a^2+b^2} = a+b$ , is often not very effective. This can be so even when the student seems to understand the counterexample. Our theoretical perspective suggests an explanation. If an incorrect algebraic simplification is caused by the enactment of a behavioral schema, then the resulting action, the incorrect simplification would

follow directly from the situation, that is, would not be under conscious control. To change the student's behavior, one might try to change the detrimental behavioral schema not only by providing a counterexample, but also a number of relevant problems and some monitoring.

## RELATED FINDINGS FROM PSYCHOLOGY

The above consideration of behavioral schemas appears to be consistent with some current work in psychology. At present, in cognitive psychology there is considerable interest in *dual-process theory*, that is, in distinguishing System 1 (S1) cognition from System 2 (S2) cognition (Stanovich & West, 2000). S1 is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory. In contrast, S2 is slow, conscious, effortful, evolutionarily recent, and puts considerable call on working memory. Leron and Hazzan (2009) have described and considered these two systems as rough equivalents of intuitive and analytical thinking. S2 thinking often monitors S1 thinking and takes over when necessary to avoid error. Certain errors can be explained by S2 not doing this. We think that when a person constructs a behavioral schema, he or she moves part of his or her cognition from S2 to S1, thus unburdening working memory and making S2 monitoring less important. In a sense, one might see behavioral schemas as residing between S1 and S2.

There are also social psychologists who point out the automated nature of much of everyday life (e.g., Bargh, 1997). The automated actions that they describe seem very like the actions arising in what we are calling behavioral schemas, except that these psychologists seem not to have reified the linking of situations to actions into objects (i.e., behavioral schemas) as we have.

## EXAMPLES OF BEHAVIORAL SCHEMAS DEVELOPED BY STUDENTS

### Mary's reaction to considering fixed, but arbitrary elements

There are theorems, particularly in real analysis, that involve universal quantifiers. For example, proving a function  $f$  is *continuous* at  $a$  involves proving that for all  $\varepsilon > 0$  there is a  $\delta > 0$  so that for all  $x$ , if  $|x-a| < \delta$  then  $|f(x)-f(a)| < \varepsilon$ . For such proofs, one needs to consider a fixed, but arbitrary  $\varepsilon$ . Students are often reluctant to do this. We conjecture this is because they do not feel it right or appropriate to do so.

Mary, an advanced mathematics graduate student, was interviewed about events that took place two years earlier when she was taking both a pilot version of our proof construction course and Dr. K's graduate real analysis course. In the homework for Dr. K's course, Mary needed to prove many statements that included phrases like 'For all real numbers  $\varepsilon > 0$ ,' where  $\varepsilon$  represented a variable (the situation). In her proofs, Mary needed to write something like 'Let  $\varepsilon > 0$ ,' where  $\varepsilon$  represented an arbitrary, but fixed number (the action). Dr. K, whom we also interviewed, often discussed Mary's proofs with her, and in particular, thought she carried out this action based on his authority.

When Mary was interviewed about this situation-action pair she said the following:

Mary: At that point [early in Dr. K's real analysis course] my biggest idea was, well he said to "do it", so I'm going to do it because I want to get full credit. And so I didn't have a sense of why it worked.

Interviewer: Did you have any feeling ... if it was positive or negative, or extra ...

Mary: Well, I guess I had a feeling of discomfort ...

Interviewer: Did this particular feature [having to fix  $\varepsilon$ ] keep coming up in proofs?

Mary: ... it comes up a lot and what happened, and I don't remember [exactly] when, is that instead of being rote and kind of uncomfortable, it started to just make sense ... By the end of the semester this was very comfortable for me.

After completing each such proof, Mary told us that she attempted to convince herself that considering a fixed, but arbitrary element resulted in a correct proof. However, only after repeatedly executing this situation-action pair, and convincing herself that her individual proofs were correct, did she develop a feeling of appropriateness.

### Willy's focusing too soon on the hypotheses

We have observed that after writing little more than the hypotheses, some students turn immediately to focusing on using the hypotheses, rather than unpacking the conclusion to see what is to be proved, after which they often cannot not complete a proof. For example, late in our Spring 2008 proof construction course, Willy was asked to prove the theorem: *Let  $X$  and  $Y$  be topological spaces and  $f: X \rightarrow Y$  be a homeomorphism of  $X$  onto  $Y$ . If  $X$  is a Hausdorff space, then so is  $Y$ .* Because only ten minutes of class time remained and Willy had indicated that he had not yet proved the theorem, we asked him to "do the set-up", that is, present the formal-rhetorical part of the proof (Selden, McKee, & Selden, 2010).

On the left side of the blackboard, Willy wrote:

*Proof.* Let  $X$  and  $Y$  be topological spaces.

Let  $f: X \rightarrow Y$  be a homeomorphism of  $X$  onto  $Y$ .

Suppose  $X$  is a Hausdorff space.

...

Then  $Y$  is a Hausdorff space.

Then, on the right side of the board which was for scratch work, he listed one after the other: "homeomorphism, one-to-one, onto, continuous ( $f$  is open mapping)". He then looked perplexedly back at the left side of the board. Even after two hints to look at the final line of his proof, Willy said, "And, I was just trying to just think, homeomorphism means one-to-one, onto, ..." After some discussion about the meaning of homeomorphism, the second author said, "There is no harm in analysing

what stuff you might want to use, but there is more to do before you can use any of that stuff”, meaning that the conclusion should be examined and unpacked first.

We inferred that Willy was enacting a behavioral schema in which the situation was having written little more than the hypotheses, and the action was focusing on the meaning and potential uses of those hypotheses before examining the conclusion. We conjectured that Willy and other students, who are reluctant to look at, and unpack, the conclusion feel uncomfortable about this, or perhaps feel it more appropriate to begin with the hypotheses and work forward.

### **Sofia’s reaction to not having an idea**

Sofia was a diligent first-year graduate student; however, as our Spring 2008 proof construction course progressed, an unfortunate pattern in her proving attempts emerged. When she did not have an idea for how to proceed, she often produced what one might call an “unreflective guess” only loosely related to the context at hand, after which she could not make further progress. Although we could sometimes speculate on the origins of Sofia’s guesses, we could not see how they could reasonably have been helpful in making a proof, nor did she seem to reflect on, or evaluate, them herself. We inferred that Sofia was enacting a behavioral schema: she was recognizing a situation, that is, that she had written as much of a proof as she could, and had a feeling of not knowing what to do next. This situation was linked in an automated way to the action of just guessing any approach that usually was only loosely related to the problem at hand without much reflection on its usefulness.

Using our idea of behavioral schemas, we devised an intervention that was used in tutoring sessions with Sofia. We attempted to deflect implementation of her “unreflective guess” schema, by suggesting that she write the first and last lines of a proof, unpack the conclusion, and then do something else, such as draw a diagram, review her class notes, or reflect on everything done so far. These suggestions and guidance helped Sofia construct a beneficial behavioral schema. As the course ended, this intervention of directing Sofia to do something else was beginning to show promise. For example, on the in-class final examination Sofia proved that if  $f$ ,  $g$ , and  $h$  are functions from a set to itself,  $f$  is one-to-one, and  $f \circ g = f \circ h$ , then  $g = h$ . Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-oriented parts (Selden, McKee, & Selden, 2010) of at least a few proofs by the end of the course, and suggests her “unreflective guess” behavioral schema was weakened.

### **CONCLUSION**

This paper treats mathematical reasoning (of any kind) as a sequence of (physical or mental) actions arising from situations in a partly completed reasoning process. It suggests that in some cases the need for a warrant linking commonly occurring situations to actions may eventually disappear, leaving what we call behavioral



schemas, a form of procedural knowledge. The use of such behavioral schemas converts some S2 reasoning to S1 reasoning and leaves much of working memory available for working creatively with conceptual knowledge. While the examples provided give one some idea of the role of procedural knowledge in reasoning, that role can be seen directly by solving a linear equation and noting that most of one's actions were immediate responses to situations. They probably did not involve having to recall something and were not actually accompanied by the reasons that one could have provided. We suggest that much reasoning is done in this way, but usually will also have a conceptual part.

Attention to behavioral schemas suggests a number of research questions: Is the development of a behavioral schema often like Mary's relative to authority, feelings, or length of time needed? How can detrimental behavioral schemas be altered? Do some students have behavioral schemas that "control" when they construct diagrams or search for relevant conceptual knowledge?

## References

- Backman, L., Small, B. J., & Wahlin, A. (2001). Aging and memory: Cognitive and biological perspectives. In J. E. Birren & K. W. Schaie (Eds.), *Handbook of the psychology of aging* (5th ed., pp. 349-377). San Diego, CA: Academic Press.
- Bargh, J. A. (1997). The automaticity of everyday life. In R. S. Wyer, Jr. (Ed.), *The automaticity of everyday life* (pp. 1-61). Mahwah, NJ: Lawrence Erlbaum Associates.
- Lau, H. C., Rogers, R. D., Haggard, P., & Passingham, R. E. (2004). Attention to intention. *Science*, 303, 1208-1210.
- Leron, U., & Hazzan, O. (2009). Intuitive versus analytical thinking: Four perspectives. *Educational Studies in Mathematics*, 71, 263-278.
- Libet, B., Gleason, C. A., Wright, E. W., & Pearl, D. K. (1983). Time of conscious intention to act in relation to onset of cerebral activity (readiness-potential): The unconscious initiation of a freely voluntary act. *Brain*, 106, 623-642.
- Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to-act in the moment. *Educational Studies in Mathematics*, 28(1-3), 135-161.
- Norton, A., & D'Ambrosio, B. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education*, 39(3), 220-246.
- Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioral schemas and the proving process. *International Journal of Mathematical Education in Science and Technology*, 41(2), 199-215.
- Stanovich, K. E., & West, R. F. (2000). Individual differences in reasoning: Implications for the rationality debate? *Behavioral and Brain Sciences*, 23, 645-726.
- Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, 36(5), 404-411.



# IS WHAT YOU PREFER WHAT YOU DO? REPRESENTATIONS IN DEFINITE INTEGRAL

Eyup SEVIMLI and Ali DELICE  
Marmara University

*In the process of problem solving, being able to select and use the appropriate representation affects performance. Hence, the representations preferred and used in the process of solving definite integral problems and the consistency between these two were investigated. This research is a case study within the qualitative interpretive paradigm. The participants were 45 second year teacher trainees studying mathematics education at a state university. The Representation Preferences and Transition Test was administered to the trainees and it was analyzed descriptively. The findings indicated that teacher trainees lacked sufficient levels of representation awareness in the process of solving definite integral problems. The most preferred and used representation type in the problem solving process was algebraic representation. However, there was inconsistency between the representation types that were preferred and used. We discuss the findings in light of the literature and propose several suggestions to increase success at problem solving.*

## INTRODUCTION

The curriculum emphasises the importance of the necessity to equip students with mathematical problem solving knowledge and skills (Duval, 2002). One of the most necessary cognitive processes in the process of problem solving is the ability to choose and use an appropriate representation (Schoenfeld, 1992). For a mathematical concept or problem, representations could be used as effective solution tools when within or between transformations are possible (Monaghan, Sun & Tall, 1994). An awareness of the language in which the problem is stated has a crucial role in comprehending and solving the problem. Hence, another important thing here is the individual's awareness of his/her knowledge of representations. The learner who can understand the language in which the problem is stated should also be selective in preferring the appropriate representation. The learner should be able to manage the process via the most appropriate representation for the problem characteristics. Calculus problem solving processes also have a content for which various representations could be used. Hence, multiple representations centred teaching approach, which began with the Calculus reform in 1994, points to the necessity for the conceptual understanding of calculus problem solving processes (Goldin & Kaput, 1996). Many researchers reported challenges encountered by learners in understanding the concept of definite integral as part of the analysis course (Orton, 1983). The reason to this is believed to be a lack of knowledge and awareness of the representations observed during the process of solving definite integral problems (Sevimli & Delice, 2010). The present study aimed to identify mathematics teacher trainees' preferred representations and the representations they used

and to investigate the influence of representation awareness on the problem solving process for definite integrals that can have several representations.

## **THEORETICAL FRAMEWORK**

The knowledge and use of multiple representations have been dealt with many researchers in various contexts. In its most general sense, representations that are known as the language of mathematics can be defined as “the process of modelling abstract concepts or structures in the real world which are open to discussion” (Duval, 2002; Kaput, 1998). In this study, multiple representations approach is grounded in external multiple representations and the concept of multiple representations means “equations, data tables, graphics, geometric figures, diagrams and written texts that can be directly obtained by the research”. Numeric, graphic and algebraic representations that are frequently used in traditional mathematics and accepted as “Formal Mathematical Systems” by Goldin and Kaput (1996) constitute the multiple representations in this study. Making use of multiple representations could be seen as an advantage because it enables learners to approach solutions from different perspectives and because it facilitates comprehension of the concept (Keller & Hirsch, 1998). Transformation between representations of a mathematical concept is a presupposition for successful problem solving (Duval, 2002). Moreover, the success or failure of the problem solving process depends on the representation chosen for the solution (Keller & Hirsch, 1998), because representations are not always beneficial in the problem solving process; a representation that does not fit with the model in the problem could sometimes obstruct the solution process (Duval, 2002). Therefore, the representations that are preferred and used in the problem solving process, and the relationship between the preferred and used representations are important. Research set around the multiple representations approach focus on either one or more of the variables of learner, teacher and learning context. Findings of the research studies on representation preferences point to the fact that in traditional classrooms algebraic representations are predominantly preferred (Keller & Hirsch, 1998). Studies also indicate that teachers’ and learners’ knowledge, attitude and beliefs, as well as the curriculum and the order of teaching environments influence learner preferences (Patterson & Norwood, 2004). Software or materials used in the classroom have been found to increase the frequency of the preference and use of numeric and graphic representations (Kendal & Stacey, 2003). Thus, while learner representation preferences for the concepts of functions, limits and derivatives are widely documented in literature, research on learners’ representation preferences and their success in representation transformations in definite integral problems is limited. The structure of the concept of integral is as conducive to the use of multiple representations as the concepts of function, limit and derivative. In definite integral problems, “problems of the area below the curve” can be solved by using graphic; “problems of the sum of variance ratios” can be solved by using numeric; and “problems of basic integral calculations” can be solved by using algebraic thinking algorithms and representations (Orton, 1983; Sevimli & Delice, 2010). This study seeks to answer the question “are representations preferred and used in the process of solving definite

integral problems consistent?” and aims to bring a new perspective to the literature in terms of representation awareness.

## **METHOD**

### **Research Design and Participants**

The study adheres to a non-positivist paradigm with an interpretative approach. A multi-methods approach was adopted by using more than one research technique. The study mainly made use of qualitative techniques and quantitative techniques were used as to obtain supplementary data. Case study (Yin, 1994) was the main strategy of the study, because the relationships between representations that are preferred and used in the process of problem solving were deeply examined. The participants were 45 second year mathematics teacher trainees studying at a state university in the spring semester of the 2008-2009 academic term. All teacher trainee participants had taken courses Analysis I and II in the previous year.

### **Data Collection Tools**

Data collection tools of the study were interviews, document analysis and a performance test. In order to carry out content analysis of the courses Analysis I and II, interviews were administered with the tutor and 6 teacher trainees selected using purposeful sampling among the participants. Both tutor's and students' course notes were also examined. Data obtained from interviews and document analysis were used for discussion as supplementary data and the detailed analysis are not presented here. The main data for the study was obtained by the Representation Preferences and Transition Test (RPTT) developed by the researchers. The RPTT had two aims, which were to identify representations which were preferred and representations which were used. By representation preference, the participants were expected to identify the representation types which they believed would facilitate the process of solving a given definite integral problem. In order to identify representations used in problem solving, participants' processes of solving a given integral were examined in terms of the representations they used. RPTT was designed to identify participants' tendencies in multiple-representation types; hence it was important that each problem in the test had a different representation characteristic. The test consists of nine items each of which represent a different objective of the course. There are input and output representations in each of these questions. Input representations defining the givens of the problem and output representations which the solution of the problem includes. The test has two sub-categories: 'Transition within representations' sub-category contains question types in which input and expected output representations are the same. 'Transition between representations' sub-category contains question types in which input and expected output representations are different. Textbooks, exam questions, school grades and relevant literature were taken into account to determine 71 questions which then were reduced to nine according to the predetermined objectives. The test was found to have face and content validity after the analysis made by five experts in mathematics (education). The testing time was determined to be 45 minutes. The inter-rater reliability

was taken as a measure for the reliability of the testing instrument for which randomly selected 12 answer sheets from RPTT were evaluated by three experts who have PhDs in mathematics education. High correlations between the answers of assessors are interpreted as sufficient reliability.

RPTT was administered twice, each a week apart, in order to identify the preferred representations and the representations that were used respectively. For representation preference, the participants were asked to choose the appropriate option among three given representation types (numeric, graphic, algebraic) for each problem. When choosing one of the representations, participants were not allowed to make any calculation and the test time was kept short. For the second administration of the test the participants were asked to solve the problems.

**An example from RPTT:** Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	0	1	2	3	4
Leakage (gal/h)	20	35	58	84	110

Give an upper estimate of the total quantity of oil that has escaped after 4 hours.

What representation do you prefer in problem solving?

Numeric	Graphic	Algebraic

Data analysis

Each participant’s representation preferences for each question were analysed separately within categories of numeric, graphic, algebraic or mixed. A mixed representation is said to exist when, more than one representation are used in relation to the same question. In order to identify the representations used in problem solving, same processes were analysed for same categories. Then, consistency between the preferred and used representations to solve each problem was analysed for each participant. When the preferences were reflected to problem solving process correctly, then they were coded as Preference is Represent Done (PRD); and when preferences and the representations used in problem solving were different, they were coded as Preference is not Represent Done (PnRD). If none of the representations were preferred or none of the representations were used, then they were coded as Preference Not Represent (PNR).

For example, if a participant who had identified his/her preferred representation as numeric and in problem solving used numeric or a mixed representation including numeric, then it was coded as PRD; if graphic was used then it was coded as PnRD; if the problem was not solved, then it was coded as PNR.

FINDINGS

The study aimed to identify teacher trainees’ representation preferences. Their preferences were investigated for problems of transition within the representation and transformation between representations; and their general preference tendencies were

evaluated based on the problems in these two sections. Trainees were observed to predominantly prefer algebraic representations (56%) for problems of transition within the representation (Table 1). Numeric, graphic and mixed representations were preferred at similar levels with low percentages. In problems of transformations between representations, graphic representation was preferred most (33%). For this type of problems, preference percentages of the other representations were not significantly different. General preference tendencies determined for all the responses in the test indicated that most preferred representation type was algebraic representation. Other representation types preferred for definite integral problems were respectively graphic (27%), mixed (21%) and numeric representations. The least preferred representation type for both within representation and between representations was numeric representation. As a result, for general preference tendencies numeric representations were the least preferred (15%).

Preferences	Algebra	Graphic	Numeric	Mixed
<i>Within</i>	56	16	16	12
<i>Between</i>	27	33	14	26
Total	37	27	15	21

Table 1: Representations preferences for definite integral problems.

In order to identify representation types teacher trainees' used in solving definite integral problems, RPTT was administered for a second time. Representations used in solving problems of transition within representation and transformation between representations were investigated and general representation usages were identified for these two sections. For transition within representation problems, participants were observed to mainly use algebraic representations (51%) (Table 2). Numeric, graphic and mixed representations were used 20% or less. In problems of transformation between representations, similar to transition within representation problems and with a higher percentage, algebraic representations were used. Table 2 presents that other representation types were used with low percentages also for transformation between representations problems. All answers given in the test indicated that the most used representation type was algebraic representations. Other representation types used in definite integral problems were graphic (18%), mixed (16%) and numeric representations.

Use	Algebra	Graphic	Numeric	Mixed
Within	51	15	14	20
Between	56	20	11	13
Total	54	18	12	16

Table 2: Representations used in solving definite integral problems.

When preferred and used representations were compared, the least observed representation type for both situations was numeric representation. Another parallel (similar) finding in preferred and used representations was that algebraic representations were preferred and used with high percentages. One of the differences between preferred and used representations was that while preferred representation types had similar frequencies, for used representation types there was a tendency towards algebraic representation. Trainees who had preferred graphic representation did not use their preferences in the problem solving process. Moreover, the percentages of numeric and mixed representations were less when they were used than when they were preferred. Even trainees who had not considered algebraic representation during the preference process used it in problem solving.

Differences between preferred and used representations indicated differences between trainees’ thoughts and applications in the problem solving process. Hence, the consistency between preferred and used representations of each trainee during the problem solving process was explored. In problems of transition within representation, 65% of the trainees projected their preferences on the problem solving process. In problems of transformation between representations, 51% of the solutions were consistent in terms of preferred and used representations. Moreover, 37% of the trainees tried to solve the problems with representations different than their preferred representations. In terms of all the answers in RPTT, answers where preferred representations were also used in the solution constituted 58% of all answers and answers where preferred and used representations were not related constituted 31% of all the answers (Table 3).

Consistent	PRD	PnRD	PNR
Within	65	24	9
Between	51	37	12
Total	58	31	11

Table 3: Consistency between preferred and used representations.

Analysis of course notes and interviews revealed that the teaching mostly encouraged algebraic representations; that the topics were covered respectively via presentation and “Definition-Theory-Proof-Practice-Test”; and that real life problems were not included.

**DISCUSSION**

Participants’ answers to Representation Preference and Transition Test indicated that algebraic representations were preferred more than other representation types. Teacher trainees believed that algebraic representations would be more beneficial in the process of problem solving. One of the reasons of the high preference of algebraic representations is the heavy reliance on this representation throughout the course. Upon analysis of teacher trainees’ course notes and the content of teaching, it was observed that algebraic representation was frequently used while some representation types (numeric) were never used. Kendal and Stacey (2003) obtained similar results in their study on derivative problems; and concluded that the teaching process as well as

representation types in course books affects student preferences. On the other hand, least preferred representations were numeric representations. Trainees' believed that numeric representations were not necessary for the process of problem solving. Teacher trainees stated that they did not often encounter definite integral problems expected to be solved using numeric approaches during the teaching process. We should investigate why trainees did not prefer and use these representations and why they were not successful at problems for which these representations should be used. Hence, tutor's attitude and behaviour towards multiple-representations might influence teacher trainees' preferences (Patterson & Norwood, 2004).

When representations used in solving definite integral problems were explored, in comparison to preferred representations, algebraic representations were mainly used again and with a higher percentage. However, the percentage of the algebraic representations used was higher than that were preferred. This might be a result of the differences in epistemological beliefs for the concept of representation (Duval, 2002). Trainees' knowledge of representation affects their preferences and skills of use as much as tutor's classroom practice and attitudes (Patterson and Norwood, 2004). Trainees had inadequacies and differences in terms of their knowledge of benefiting from different representations. Kendal and Stacey (2003) also found that algebraic representations were used most in derivatives while graphic representations were used the least. Trainees tended to use algebraic representations in all kinds of problems. This indicated that teacher trainees were used to and were willing to use algebraic representation even though expected gains from the problem were different. Given that teaching of analysis generally relied on a single representation type (algebraic representation), trainees' inconsistencies between preferred and used representations made sense (Keller & Hirsch, 1998). As the definite integral was conceived using algebraic approaches, learners experienced challenges in problems that could be solved using different representation types. Results presented in Table 3 led to two interpretations; either the trainees lacked knowledge of multiple representations or failed to manage cognitive processes involved in problem solving.

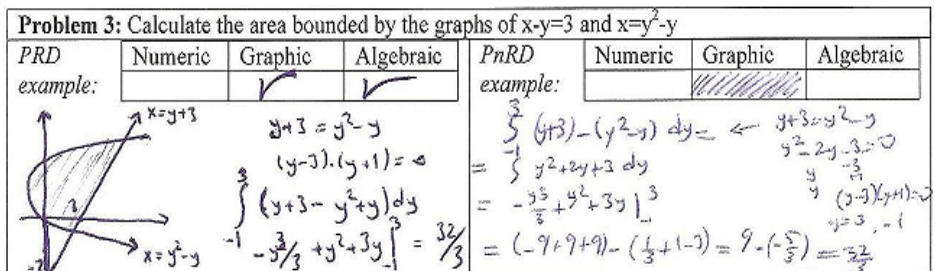


Figure 1: Example for consistency between preferred and used representations

Preferred representation is the representation which is believed to contribute to problem solving. The fact trainees could not project preferences on their solutions might suggest that their meta-cognitive skills were insufficient (Figure 1). Another indication might be



that their preferences could change during the solution process. While this could sometimes be regarded as richness, inconsistencies observed in preferred and used representations in a problem designed for the use of a specific representation might explain for complications experienced during mental processes.

## CONCLUSION

While teacher trainees believed that the use of algebraic representations were necessary to solve definite integral problems, they were persistent in using the algebraic representation even in problems which should be solved using other representations. A widespread distribution was observed in representation preferences alone, whereas in the process of problem solving there was an accumulation towards a single representation type (algebraic). This resulted in inconsistencies between preferred and used representations. The number of teacher trainees who preferred and used more than one representation type was considerably limited. This might have been due to teaching which heavily relied on algebraic representations, evaluation methods that were based on assessing procedural knowledge and trainees' lack of meta-cognitive awareness. Future research might replicate the present study by designing a learning context with an emphasis on different representations of the definite integral concept. Moreover, the relationship between trainees' meta-cognitive skills and their awareness of representations could be explored.

## References

- Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education*, 1(2), 1-16.
- Goldin, G. A., & Kaput, J. J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Kaput, J. J. (1998). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. *Journal of Mathematical Behavior*, 17 (2), 265-281.
- Keller, B. A., & Hirsch, C. R. (1998). Student preferences for representations of functions. *International Journal in Mathematics Education Science Technology*, 29 (1), 1-17.
- Kendal, M., & Stacey, K. (2003). Tracing learning of three representations with the differentiation competency framework. *Mathematics Education Research Journal*, 15(1), 22-41
- Monaghan, J., Sun, S. & Tall, D. O. (1994). Construction of the limit concept with a computer algebra system, In *Proceedings of the 18th international conference of the International Group for the Psychology of Mathematics Education* (pp. 279-286). University of Lisbon, Portugal.
- Orton, A. (1983). Student's understanding of integration. *Educational Studies in Mathematics*, 14 (1), 1-18.
- Patterson, N., & Norwood, K. (2004). A case study of teacher beliefs on students' beliefs about multiple representations. *International Journal of Science & Mathematics Education*, 2(1), 5-23.
- Schoenfeld, A. (Ed.). (1992). *Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics*. New York: Macmillan.
- Sevimli, E., & Delice, A. (2010). Spatial visualization and representations in problem solving process: definite integral case. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proc. 34<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 103). Belo Horizonte, Brazil: PME.
- Yin, R. (1994). *Case study research: Design and methods*. Thousand Oaks, CA: Sage.



# AN ANALYSIS OF TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE ON PROBABILITY

BoMi Shin

Chonnam National University

*The purpose of this study is to analyze teachers' pedagogical content knowledge on probability. This study has categorized the knowledge into two parts: (a) subject matter knowledge, and (b) knowledge of students' understanding. To analyze teachers' subject matter knowledge on probability, I gave the teachers a paper and pencil test. To examine teachers' knowledge of students' understanding on probability, I asked the teachers to suggest possible reasons for students' responses through group discussion. I compared the results of the paper and pencil test and group discussion with those of previous studies and summarized teachers' pedagogical content knowledge on probability into 4 characteristics.*

## INTRODUCTION AND REVIEW LITERATURE

Many studies have found that teachers' pedagogical content knowledge (PCK) affects classroom practice and is modified and influenced by practice (Lamprianou, & Lamprianou, 2009). Chick, Baker, Pham, & Cheng (2006) proposed a framework of PCK as three categories. These three categories include the ability to deconstruct knowledge to its key components, awareness of mathematical structure, content knowledge including connections and ideas, and knowledge of students' thinking such as knowledge of misconceptions and individual learning styles. According to Liu, & Thompson (2004), teachers' understanding of significant mathematical ideas has profound influence on their capacity to teach mathematics effectively. Misailidou (2008) suggested that teachers need to be aware of their students' misconception in order to design teaching strategies that will help their students reorganize their thinking.

In particular, as there are a number of different results from deterministic intuition based on logical thinking in probability, it is important to put an emphasis on relations between students' prior intuition and structures of probability (Borovcnik, & Peard, 1996). That is, teachers should have knowledge of students' misconceptions and of key contents of probability so as to design effective probability teaching strategies as teaching probability needs to begin from students' intuition on probability (Batanero, & Sanchez, 2005). However, Stohl(2005) pointed out that there are few studies on teachers' subject matter knowledge of probability and on teachers' knowledge of students' understanding of probability. This study aims to analyze teachers' pedagogical content knowledge of probability in two aspects of teachers' subject matter knowledge and their knowledge of students' understanding

of probability. It is expected that the results of this study can provide suggestions on design of curriculum on probability for teachers.

## **METHODOLOGY**

The subjects of this study were 44 mathematics teachers who have been working in middle and high schools in large cities of South Korea for three to five years. To identify teachers' subject matter knowledge on probability, I presented them with a 20 minute paper and pencil test. To examine teachers' knowledge of students' understanding on probability, I presented the results of the paper and pencil test of students who participated in Shin (2007) and asked the teachers to suggest possible reasons for students' incorrect responses through group discussion for about 40 minutes.

I used the questionnaire which was designed to identify characteristics of students' misconceptions on probability in Shin (2007) for the paper and pencil test. In addition, I presented the teachers with the responses of students from the same questionnaire used to examine teachers' subject matter knowledge of probability in order to analyze teachers' knowledge of students' understanding on probability. Therefore, I was able to indirectly identify the relation between teachers' knowledge of probability and teachers' knowledge of students' understanding. The results of the test were analyzed based on the percentage of correct answers and response of choices, teachers' questions during the test, and the results of the previous studies including those of Fischbein, & Schnarch (1997).

Horn (2009) explained that conversations between teachers had contributed to revealing PCK. Therefore, this study divided the subjects into 11 small groups with 4 teachers in each group in order for teachers to show their knowledge of students' understanding of probability in detail. I then asked the subjects to suggest possible reasons for students' incorrect answers through group discussion and to record the results of the discussion. This study analyzed teachers' knowledge of students' understanding of probability based on the process and the results of the group discussion.

## **RESULTS**

### **Teachers' subject matter knowledge on probability**

#### **A. The center of frequency data is considered by priority.**

Mathematical probability of pulling out one red candy is  $\frac{1}{2}$  in Question 1. However, it does not actually mean that 5 red candies appear exactly when 10 candies are pulled out with replacement. In Question 1, the subjects will not choose ③ as a correct answer if they have appropriate understanding about probability (Zawojewski, & Shaughnessy, 2000: 259). The subjects should keep in mind the fact that the central value of frequency data is 5 but they should carefully consider variation of the data at the same time.

However, 50% of the teachers chose ③ as a correct answer and some of them asked whether Question 1 should be approached mathematically or statistically. In the next discourse, teacher B regarded that precisely 5 red candies are always chosen as a correct answer mathematically even when actually 10 candies are pulled out with replacement (\*).

Teacher B: I wonder whether this question should be mathematically or statistically answered.

Researcher: Let us imagine that you would actually pull out 10 candies.

Teacher B: *(very embarrassed)* I do not know the intention of the question. (He chose ③)

Researcher: Would you say that ③ *always appears in a real situation*?

Teacher B: We couldn't be certain of other numbers' appearing, could we?

Researcher: Choose the one you think is the most appropriate when pulling out 10 candies in an actual situation.

Teacher B: *The answer will vary according to whether I think of it mathematically or statistically (\*). Don't you think the wording of the question is wrong?*

In considering the percentage of the response to ③ and teacher B's case, teachers tend to think the center of frequency data is more important. Traditionally, school mathematics has been taught in a deterministic context (Jones 2005), the subjects have been taught through such a curriculum for a long time, and they have been teaching mathematics through the same curriculum. The situation in which they regarded the center of frequency data as more important than variation supports the results of Batanero, Henry, & Parzysz (2005), who pointed out that a mathematics curriculum of a deterministic context can be a limit to sufficient understanding of variation as an important concept of probability.

## **B. For questions regarding the reversal of the order of time, conditional probability is considered.**

Fischbein, & Schnarch (1997) identified whether the subjects have the time-axis fallacy that case *B* precedes case *A* in a conditional probability of  $P(A|B)$  using the Questions (1), (2) of 3. As a result, most of the students gave correct answers to question (1) and incorrect answers to question (2). Fischbein, & Schnarch explained it as the typical time-axis fallacy. However, most of the teachers in this study did not solve (1) of question 3 correctly.

43% of the teachers regarded a conditional 'when' as a conjunction 'and' then answered the probability of (1) is  $\frac{1}{6}$  ( $=\frac{2}{4} \times \frac{1}{3}$ ). In Question 2, in considering that 60 % of the teachers chose ③ as a correct answer, most of them seemed to confuse the conditional 'when' with the conjunction 'and'. However, 59% of them used conditional probability to obtain a correct answer when solving (2) of Question 3. That is, teachers tended to choose an incorrect answer in (1) and the correct answer in (2) unlike the results of Fischbein, & Schnarch (1997).

Although both (1) and (2) contain the conditional ‘when’ which shows that conditional probability can be applied in solving these questions, there was a difference in the percentage of teachers’ using the conditional probability to solve each question. 38% of the teachers used the conditional probability to solve (1) whereas 59% of them used the conditional probability to solve (2). Meanwhile, 43% of the teachers confused the conditional with the conjunction in (1) while only 9.09% of the teachers confused them in (2).

According to this, the teachers tended to use the conditional probability more in questions regarding the reversal of the order of time rather than in questions with the conditional ‘when’. It is different from the results of Lee (2005) in which the conditional probability is generally used to solve the problems with conditionals. The teachers had less time-axis fallacy related to the conditional probability in comparison with the results of previous studies but had relatively more misconception of confusing the conditional with the conjunction.

Code	Fischbein, & Schnarch (1997)				This study	Type of misconception
	7th	9th	11th	Preliminary teachers		
(1)correct (2)correct	50	35	30	39	32	
(1)correct (2)incorrect	30	35	70	44	7	time-axis fallacy
(1)incorrect (2)correct	5	5	0	17	27	
(1)incorrect (2)incorrect	15	25	0	0	34	

Table 1: Comparison of the Responses to Question 3 (unit %)

Teachers’ knowledge of students’ understanding on probability

A. Teachers have difficulty in understanding misconceptions related to variation.

In respect to Question 1, Zawojewski, & Shaughnessy (2000) stated that the misconception of students who chose ③ as an answer, was ‘excessive concentration for the center’. In addition, they also stated that the misconception of students who chose ⑤ as an answer, was ‘the outcome approach’. In Question 1, students who chose ③ as a correct answer have a tendency to focus on the center of frequency data and students who chose ⑤ as a correct answer have a tendency to focus on variation of the data.

Most of the teachers (10 groups of 11) recognized the misconception of students who considered only the center of data as properly diagnosing the reason for students to choose ③ by saying ‘because students think only mathematically’ or ‘because the probability that one red candy is pulled out is  $\frac{1}{2}$ ’. However, only four groups

explained the reason why students chose ⑤ as ‘because the average of 1, 3, 5, 7 and 9 is 5’ or ‘because students predicted every number would be different for a random sampling’. The other five groups did not explain the reason at all and the other two groups explained the reason as ‘because students chose it with no definite idea’.

Teachers recognized the misconception that occurs when only the center is focused on and variation is ignored as in ③ rather than the misconception that occurs when variation is excessively inferred. This situation is closely related to the cognitive inclination of the teachers who considered the center more importantly than variation of frequency data in the test about teachers’ subject matter knowledge on probability.

### **B. Teachers don’t recognize students’ time-axis fallacy.**

For Question 3 consisting of sub-questions (1) and (2), the teachers were expected to focus on the difference of the percentage for correct answers of sub-questions and then recognize students’ time-axis fallacy. According to Shin (2007), 39% of the students presented the correct answer to (1) while only 22% presented the correct answer to (2). This difference of percentage for the correct answer to (1) and (2) which have the same conditional ‘when’ proves that there are different structures of probability in (1) and (2). The teachers need to focus on what are the different structures on probability.

However, the teachers did not perceive that the differences were based on the time-axis fallacy. The teachers properly explained that the reason why students chose  $\frac{1}{6}$  as a correct answer in (1) was ‘because students confused the conditional with the conjunction’. On the other hand, they were not able to explain the valid reason for students’ incorrect answers in (2). Teachers recognized students’ misconception for confusing the conditional with the conjunction but did not recognize the time-axis fallacy. Such a characteristic can be a limit to teachers’ designing teaching strategies for conditional probability.

## **DISCUSSION AND RESULTS**

In this study, most of the teachers considered only the center of frequency data as important and did not recognize variation of the data. They hardly understood students’ misconceptions that occur due to excessive inference regarding variation. In order for teachers to have abundant pedagogical content knowledge about variation of frequency data, the following points need to be considered when designing the curriculum for teachers. Teachers should specifically experience variation by dealing with probability through frequency data statistically as well as theoretically (Shaughnessy, Canada, & Ciancetta, 2003). In addition, they should understand the Law of Large Numbers which explains characteristics related to variation of frequency data and recognize the meaning of the law which contains key ideas and thoughts of probability (Pratt, 2005).

According to Na, Han, Lee, & Song (2007; Lee, 2005), conditional probability is generally related to misconceptions of confusing conditionals with conjunctions and

time-axis fallacy. In this study, teachers had more misconception of confusing the conditional ‘when’ with the conjunction ‘and’ than time-axis fallacy. This result is contrary to the results of Fischbein, & Schnarch (1997). I found that the characteristic of teachers’ understanding on conditional probability influenced teachers’ recognition of students’ misconception related to conditional probability. Teachers were able to explain the reason why students had the misconception of confusing the conditional with the conjunction but did not recognize at all that students might have the time-axis fallacy. In the design of the curriculum on probability for teachers, conditional probability needs to be specifically dealt with in terms of proper teaching strategies as well as subject matter knowledge.

## References

- Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G. A. Jones (Eds.), *Exploring probability in school: Challenges for teaching and learning* (pp. 15-37). USA: Springer.
- Batanero, C., & Sanchez, E. (2005). What is the nature of high school students' conceptions and misconceptions about probability. In G. A. Jones (Eds.), *Exploring probability in school : Challenges for teaching and learning* (pp. 241-266). USA: Springer.
- Borovcnik, M., & Peard, R. (1996). Probability. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook in mathematics education* (Part 1, pp. 239-288). Dordrecht: Kluwer.
- Chick, H., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers’ pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proc. 30<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague, Czech: PME.
- Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education*, 28, 96-105.
- Horn, I. (2009). The development of pedagogical content knowledge in collaborative high school teacher communities. In Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.), *Proc. 31<sup>st</sup> annual meeting of the North American Chapter of the Int. Group for the Psychology of Mathematics Education* (Vol. 5, pp. 1451-1458). Atlanta, GA: Georgia State University: PME.
- Jones, G. A. (2005). Introduction. In G. A. Jones (Eds.), *Exploring probability in school: Challenges for teaching and learning* (pp. 1-12). USA: Springer.
- Lamprianou, A. T., & Lamprianou, I. (2009). Teacher effects on the probabilistic thinking of pupils in England. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proc. 33<sup>rd</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 9-16). Thessaloniki, Greece: PME.
- Lee, J. (2005). *A study on the understanding of conditional probability concept*. Unpublished master’s thesis, Seoul National University, Seoul: Korea.
- Liu, Y., & Thompson, P. (2004). Teachers’ personal and pedagogical understanding of probability and statistical inference. In D. McDougal (Eds.), *Proc. 26<sup>th</sup> Annual Meeting*

- of the Int. Group for the Psychology of Mathematics Education (pp.1-7). Toronto: PME-NA.
- Misailidou, C. (2008). Assessing and developing pedagogical content knowledge: A new approach. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proc. 32<sup>nd</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 391-398). Morelia, México: PME.
- Na, G., Han, D., Lee, K., & Song, S. (2007). Mathematically gifted students' problem solving approaches on conditional probability. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.), *Proc. 31<sup>st</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 1-8). Seoul, Korea: PME.
- Pratt, D. (2005). How do teachers foster students' understanding of probability? In G. A. Jones (Eds.), *Exploring probability in school: Challenges for teaching and learning* (pp. 171-189). USA: Springer.
- Shaughnessy, J. M., Canada, D., & Ciancetta, M. (2003). Middle school students' thinking about variability in repeated trials: A cross-task comparison. In A. D. Pateman, N. A., Dougherty, B. J., & Zilliox, J. (Eds.), *Proc. 27<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 159-165). Honolulu, Hawaii: PME.
- Shin, B. (2007). *Using simulation for a didactic transposition of probability*. Unpublished doctoral dissertation, Korea National University of Education, ChungBuk: Korea.
- Stohl, H. (2005). Probability in teacher education and development. In G. A. Jones (Eds.), *Exploring probability in school: Challenges for teaching and learning* (pp. 345-366). USA: Springer.
- Zawojewski, J. S., & Shaughnessy, J. M. (2000). Data and chance. In E. A. Silver, & P. A. Kenney (Eds.), *Results from the 7<sup>th</sup> Mathematics Assessment of the National Assessment of Educational Progress* (pp. 235-268). Reston, VA: NCTM.

Questions	Choice	Misconception	Number of the students	Number of the teachers
1. In a box there are 50 red candies, 30 blue candies and 20 yellow candies. After checking the colors of 10 candies from the box, mix them with the others, and pick 10 candies again. Repeat these trials 5times. How many red candies could you get in each trial? (Zawojewski, &	① 8, 9, 7, 10, 9		13	4
	② 3, 7, 5, 8, 5 (correct answer)		47	16
	③ 5, 5, 5, 5, 5	focusing on the only center	44	22
	④ 2, 4, 3, 4, 3		17	
	⑤ 1, 3, 5, 7, 9	outcome approach	8	2

Shaughnessy, 2000, p.259)				
2. When you throw 2 coins with same probability of heads and tails, p is the probability that both coins have H and q is the probability that one has H when the other has H. Then? (Shaughnessy, Canada, & Ciancetta, 2003, p.225)	① $p > q$ .		13	1
	② $p < q$ (correct answer)		66	17
	③ $p = q$	confusing conditionals with conjunctions	53	26
3. An urn contains two green balls and two blue balls. We pick up two balls at random, one after the other without replacement. (1) What is the probability that the second ball is blue, given that the first ball is also blue? (Fischbein, & Schnarch, 1997, p.99)	$\frac{1}{6}$		34	19
	$\frac{1}{3}$ (correct answer)		50	17
	$\frac{1}{2}$		4	
	$\frac{1}{4}$		14	1
	$\frac{2}{3}$		3	2
	$\frac{1}{2}$		12	3
	no answer			
3. (2) What is the probability that the first ball is blue, given that the second ball is also blue? (Fischbein, & Schnarch, 1997, p.99)	$\frac{1}{6}$		18	4
	$\frac{1}{3}$ (correct answer)		29	26
	$\frac{1}{2}$		10	
	$\frac{1}{4}$		18	1
	$\frac{1}{2}$		34	8
	$\frac{5}{12}$		1	
	no answer		19	5

Appendix: Comparison of answers of the students (Shin, 2007) and the teachers



# REPRESENTATIONS IN THE DEVELOPMENT OF MATHEMATICAL CONCEPTS

Shvarts Anna and Chumachenko Dmitry

M.V. Lomonosov Moscow State University

*The present study provides analysis of mathematical concepts as components of personal knowledge. To describe the process of development of school mathematical concepts, we conducted an empirical comparison of representational system of concepts by students with different level of math background. The study revealed that intermediate level students often represent concepts by images of plots transformation or by memorizing the process of calculation, while the most frequent forms of representation for advanced students are verbal, static visual and algebraic representations. These results indicate that development of math concept arises from adequate mathematical activity and then ability to consciously express the meaning of concept emerges.*

## INTRODUCTION

This paper addresses the nature of mathematical concepts which are traditionally considered to be the most abstract and rigor among all possible concepts. We will discuss mathematical concepts as neither logic nor mathematical entities but as psychological constructs functioning within one's head, constituents of individual knowledge and instruments of individual reasoning.

L.S. Vygotsky distinguished between scientific and everyday concepts (Vygotsky, 2008). According to his theory, these two types of concepts develop in antithetic ways. Everyday concepts develop in practice due to concrete individual experience. Children use words referring to such concepts in discourse long before they can verbally express their meaning. Scientific concepts are acquired in school education and develop verbally, coming the way from verbal definitions to certain specific content. The term is presented to a student together with the meaning of the concept, and concept is represented within the system of other concepts from the very beginning of the study. According to Vygotsky, this is the main specific character of scientific concepts.

This distinction made by Vygotsky is in agreement with the distinction of modal and amodal representations in cognitive science. According to classical approach in cognitive psychology, categorical knowledge is stored in amodal form, as a system of interconnected nodes. The content of nodes and type of links vary between models (i.e. propositional structure, semantic network) (Velichkovsky, 2006). Initial models formed under direct influence of computer metaphor didn't assume any modal, embodiment components within semantic memory.

Alternative modal approach becomes more popular nowadays (Lakoff, Johnson, 1980, Barsalou, 2003). This approach assumes long-term storage of sensory and perceptual information. Barsalou's theory of perceptive symbol systems suggest that concept is an aggregate of all experience related to a given object (Barsalou, 2003). For mathematical concepts a possible analogy of such perceptual experience is not clear, because mathematical objects are abstract and cannot be directly perceived by the human senses.

Cognitive-linguistic approach (Lakoff, Johnson 1980) supposes that embodied experience standing behind every word can be transferred from one area of discourse to another by language means. R.Nunez and G.Lakoff (see for example, Lakoff, Nunez, 2000) suggest that mathematical concepts refer to embodiment experience by metaphors and metonymies. But this idea doesn't make clear, how this experience is represented for consciousness and used in problem solving, because linguistic approach doesn't analyse the role of visualization in representation of concept.

Special research on this topic demonstrates that schematic but not rich images help in problem solving (Hegarty, Kozhevnikov, 1999, Calvin, et. al., 1995). N.Presmeg (1992) assumes that transition from rich secondary images to abstract concepts is continuous and forms a spectrum of schemes characterized by progressively increasing abstraction and loss of details.

Some researchers consider visual representations as one of the multiple coexisting representations of the concept (Gagatsis, Shiakalli, 2004). It is supposed that a concept can be acquired, only if all representations were acquired and coordinated among themselves. However, it is difficult for a student to transform material from one representation into another, and visual representations bring some problems themselves. The perception of a plot or a diagram strongly depends on the previous experience and knowledge of the perceiver (Arcavi, 2003). Moreover, besides institutional visual representations, that are pictured in books or offered by teacher, students have spontaneous and uncontrolled representations which may be not only wrong (Aspinwall, et. al., 1997) but may also contain an essence of the solution of problems of a certain type (Hitt, 2008).

Let us consider the question about gradual abstraction of image schemata and the role of spontaneous and functional visual representation in problem solving from theoretical position of the theory of activity. P.Ya.Galperin (1955) regards an 'image' (by which he means any representation) as a product of interiorized action. According to him, "Every image conceals action inside. It is generalized, short, automated *mental* action of identification of the content of that image." (Galperin 1955, p.425), So, the theory of activity treats a concept as a composition of various mental actions. And representations are "death masks" of these actions. To teach any representation we should organize the activity of students in a way to stimulate required actions. According to V.V.Davydov (2000), practical acquisition of material precedes any pictorial or verbal expression of a concept. This is in agreement with

suggestion made by Dubinsky (2000) He assumes that math concepts are underlined by actions that are encapsulated into process and then process encapsulated into object. Theory of activity allows to explain schematic characteristic of image as absence of details irrelevant for corresponding action.

Following Vygotsky's thread of argument, we should treat mathematical concepts as scientific ones: their meaning arises from school education, but not from child's everyday practice. However, this point has to be clarified empirically. So, the first question of our study is whether mathematical concepts follow the way from verbal definitions to specific content, or whether they arise from practical actions. Also we would analyze contents and specifics of the images, standing up for mathematical concepts at different stages of their acquisition.

## METHOD

The first stage of our research was the qualitative analysis. We conducted the semi-structured interview about the ways of understanding of new math concepts with 7 students. Different ways of understanding of mathematical concepts have been revealed. See Table 1 for concept representations we used in questionnaire.

At the second stage of the research we investigated the representations of the mathematical concepts by students of different mathematics level. Low and middle groups consisted of students of Psychological department of the Moscow State University. Students were divided into groups by their results of the entrance math's test. Subjects in high-level group were students of Moscow mathematical departments. Numbers of participants in groups were 19, 21 and 20.

At first students read a list of various ways of representing mathematical concepts and reflected descriptions of different representations of concept *the rational number*, offered by the investigator. In our research we used 12 basic mathematical concepts from a school course of algebra: *negative number, root, product, log, subtraction, function, parabola, absolute magnitude, fraction, division, cosine, natural number*. For each concept subjects had to mark in a questionnaire the representation, corresponding to their understanding of the given concept then to range the chosen units.

After filling in the questionnaire, subjects had to describe all visual representations that arose in their mind them.

## RESULTS AND DISCUSSION

### Differences of representational system of students with different math level

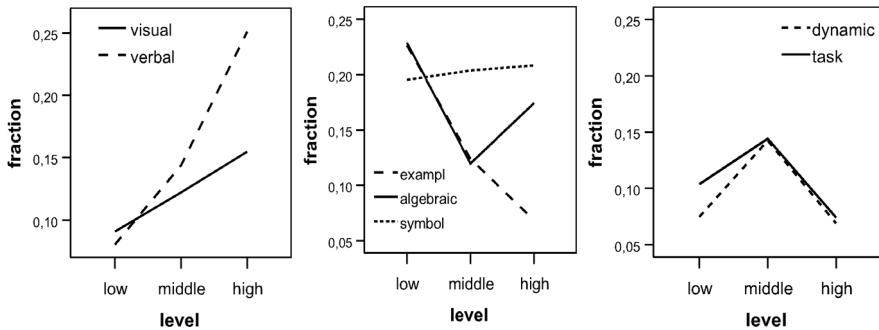
There are 13 scales reflecting various ways to represent mathematical concepts in the questionnaire (each scale was based on interviews with students).

The analysis of points and profiles of the representations of concepts allowed us to form some indexes uniting similar representations (Table 1).

Visual representation	This concept elicits an image, I represent it to myself visually.
Symbolic representation	This concept is represented to me as a mathematical symbol.
Dynamic representation	To me this concept is connected with a dynamic image: the picture on which something moves, which is used somehow.
	This concept is connected with action, but not on a picture.
Verbal representation	This concept is represented to me as an element in system of definitions, in relation with other concepts.
	This concept — the object possessing a set of properties.
	I understand this concept by means of another mathematical concept, I refer the given object to kindly familiar class of objects.
Algebraic representation	This concept — some mathematical object. It is possible to operate with it according to known rules
	I understand this concept through algebraic formula which is clearing up meaning of given concept.
Representation through problem solving	I understand this concept as technique, a method of the decision of mathematical problems.
	This concept — result of calculations, algorithm applications.
Representation through an example	I understand this concept through a concrete example.
	This concept causes associations with its application in life, in practice, during the decision of applied problems.

Table 1: Types of representation and indexes

Changes in a role of each representation in groups of different math's level are presented on the Plot 1.



Plot 1: Fraction of representation's types in varying math level.

The difference between groups was tested by Mann-Whitney test in SPSS 14.0 for each representation. Let us consider the dynamics in use of representations.

Symbolic representations in all groups take about 20%. That is students simply associate the given concept with the corresponding mathematical symbol.

In low-level group mathematical concepts are associated with concrete mathematical examples and examples from life (index of representation through an example). Also algebraic representations were often mentioned by students of low-level group. Already in middle-level group these representations are less often ( $p < 0.005$ ). We observed almost no example representations in high-level group. So, this representation's type can be considered as primary and isn't presented in system of representations of good mastering concept.

The quantity of visual representations accrues from low to high group (distinctions between high and low groups are on a significance value  $p = 0.01$ ). But if we consider also dynamical visual representations (scale in questionnaire about connection with a dynamic image, see Table 1) it appears that the quantity of visual representations in middle and high groups are almost equal (19,2% and 19,3%) whereas in low group it is 4,3%.

Furthermore, the quantity of verbal representations concepts increases with proficiency. The increment occurs from low group to middle ( $p < 0.01$ ), and from middle group to high ( $p < 0.05$ ). In high group verbal ways of representations become dominating.

The proportions of representations through a problem and dynamic representations were of the most interest to us. Both these indexes show that concept represented not in the stiffened kind, but through its inclusion in activity. There is a high percentage of these representations at middle group comparing with two other groups ( $p < 0.05$ ).

It is of critical importance that, according to our data, mature mathematical concepts are not represented only through relations to other concepts or as a set of attributes (as it follows from amodal theories of semantic memory). In their representations of

concepts advanced students use verbal, visual and algebraic representations. Our data is completely coordinated with a view of mathematical concept as a system of various representations, developed, for example, in Gagatsis, Shiakalli (2004).

Our data provides evidence for a primary role of practice in development of scientific concepts. Those students, who have not acquired mathematical concepts, nevertheless are able to use them in problem solving and know how to work with their visual representations. The ability to pass the meaning of concept verbally and to include it in a system of other concepts appears at the advanced level of material mastering.

Visualizations of advanced students are static. We do not doubt that advanced students are able to restore (de-encapsulate, as E.Dubinsky call it) to life the process, associated with this visualization. But static conventional visualization can be treated as an object for next subsequent reasoning (Dubinsky, 2000).

In general, the process of mastering of knowledge can be reconstructed as follows. At first students get acquainted with the symbols and some examples of the studied concepts, they remember some pictures offered by the teacher or their own associations. However at this stage the concepts are not used in an adequate mathematical activity (or activity isn't storage in long-term memory in the form of declarative knowledge). Further the adequate mathematical activity is started: problems are solved, calculations are made, schemes and graphs are used... Students keep dynamic images of this activity and storage it. At the final stage, the activity component is interiorized and may be restored as necessary, concepts are represented for consciousness as conventional visual and algebraic signs. Besides, well acquired concept possesses verbal representation which can be carried to reflective level. This is not a definition learned by heart, but possibility to design definition and to include it into the system of other concepts.

### Specificity of visual representations on different math level

Now let us consider specificity of visual representations of mathematical concepts at students of different level of math. Among all visualizations (60 in low, 156 in middle and 144 in high group) we marked **unique**, **incorrect** and "**metaphorical**" images. We call unique those representations, which only one person has pointed out. Incorrect images represent concept in a wrong way or are not related to concepts at all. Metaphorical visualizations associate concept with some other experience, mostly everyday one.

- There are 29,5% of **unique** visualizations in low group, 17,3% in middle group and 7,6% in high group (differences are significant at level  $p=0,001$ ,  $\chi^2=15,095$ .)
- There are 13,1% of **metaphorical** visualizations in low group, 14,7% in middle group and 0% in high group ( $p<0,001$ ,  $\chi^2=22,803$ ).
- There are 14,8% of **incorrect** visualizations in low group, 3,2% in middle group and 0,7% in high group. ( $p<0,001$ ,  $\chi^2=21,991$ ).

Already at the middle math's level there is almost no incorrect visualization, and at the high level it completely disappears. The number of unique and metaphorical visualizations decreases from middle to advanced group: students with high mathematical level rarely mention them.

Unique visualizations in middle group reflect individual associations and schemes of work with the given concept. Much of them were personal rich images, clear only to the respondent: the concept a root associated with a carrot root, division was represented as cutting of a cherry by a knife, etc. Unique representations of advanced students are qualitatively different. They are institutional representations of the advanced side of school concept. For example, the concept of absolute magnitude was represented as a picture of amplitude of complex number (this terms are identical in Russian). There were no rich images in high group. This result is in agreement with Hegarty, Kozhevnikov (1999), Calvin, et. al. (1995) data that rich images do not facilitate mathematical problem solving.

Meaning of institutional representation isn't learned and isn't transferred to students directly, and is developed during work with the given concept. Our data confirms the role of functional representation in development of mathematical concepts, and not only institutional representation to which the most part of the literature on visualization is devoted (Hitt, 2008).

The rate of visual representations at high mathematic level coincides with that at the middle group. But these visualizations are standardized, cleared from individual way of concept development. Thus, visualizations of students from advanced group are forms of the conventional mathematical knowledge.

## **CONCLUSIONS**

Our research has revealed that mathematical concepts are multiply represented. We have demonstrated that advanced students use verbal, visual and algebraic representations. According to our results, mathematical concepts in intermediate and advanced math students include a substantial share of visual representation. So, visual representations are the essential component of a concept whereas formal definitions are not necessarily included.

Our results emphasize the role of actions with mathematical concepts in the acquisition process. We have demonstrated that intermediate students use dynamic representations and represent concepts through a problem. The use of such representations in the advanced group is much more atypical. As for visual representations, intermediate students use a lot of spontaneous representations, whereas advanced students apply only institutional ones. This confirms that institutional visual representations are not delivered to students directly through the teaching process, but are rather developed through the work with each given concept.

Students who have completely acquired mathematical concepts represent them in a system with other concepts and by verbal descriptions of properties. However, these

verbal representations are used neither in the low-level nor in the high-level groups of students. Thus, we have no evidence that scientific concepts are being acquired in a special way as described by Vygotsky.

## References

- Arcavi, A. (2003). The role of visual representation in the learning of mathematics. *Educational Studies in Mathematics*. 2003. (Vol. 52. No3. 215–241).
- Aspinwall, L., Shaw, K. & Presmeg, N. (1997). Uncontrollable mental Imagery: graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33, 301-317.
- Barsalou, L. W. (2003) Abstraction in perceptual symbol systems, *Phil. Trans. R. Soc. Lond. B* 358(1435), 1177-1187.
- Campbell, K., Collis, K., Watson, J. (1995) Visual processing during mathematical problem solving. *Educational Studies in Mathematics* 28, 177-194.
- Davidov, V. V. (2000). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula*. Moscow: Pedagogical society of Russia.
- Dubinsky, E. (2000). Mathematical literacy and abstraction in the 21st century. In Maggie Niess, ed., *Vision for Science and Mathematics Education in the 21st Century. School Science and Mathematics*.
- Gagatsis, A., Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology* (Vol. 24, No. 5, 645-657).
- Galperin, P. (1955). On the formation of mental images and concepts. *Proc. Conf. On Psychology*. APN RSFSR.
- Hegarty, M., Kozhevnikov, M. (1999). Types of visual-spatial representation and mathematical problem solving. *Journal of Educational Psychology*, 91(4), 684–689.
- Hitt, F., Gonzalez, A. & Morasse, C. (2008) Visualization and students' functional representations in the construction of mathematical concepts. An example: The concept of co-variation as a prelude to the concept of function. *11th International Congress on Mathematics Education (ICME-11)*.
- Lakoff, G., Johnson M. (1980). *Metaphors we live by*. Chicago: University of Chicago Press.
- Lakoff, G., Núñez, R. (2000). *Where mathematics comes from: how the embodied mind brings mathematics into being*. New York: Basic Books.
- Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23, 595-610.
- Velichkovsky, B.M. (2006). Cognitive science: Foundations of epistemic psychology. (Vol. 1) Moscow: Smysl.
- Vygotsky, L. S. (2008). Thought and language. Moscow: Labyrinth.



# STRUCTURAL ELEMENTS IN COLLECTIONS OF DIFFERENT TYPES OF FUNCTIONS RECALLED BY STUDENT TEACHERS

Harry Silfverberg

University of Tampere, Finland

*During the upper secondary school studies the student gets acquainted with a variety of functions e.g. polynomial functions, rational functions, power functions, exponential functions, logarithm and trigonometric functions, absolute value functions, derivative and integral functions, etc. In an undergraduate course of mathematics, the participants were asked to recall all the types of functions they were familiar with and to describe how they taught the types of functions were interlinked to each others. Two techniques of collecting data were used - a listing and a concept map technique. The purpose of the study was to investigate how this extensive range of function concepts which students should have been learnt in upper secondary school mathematics is structured in the minds of the student teachers.*

## INTRODUCTION

In school mathematics, the concept of function plays a central role during the whole lower and upper secondary school education. In the educational research of the learning of the concept of function three separate, yet interlinked, main points of view can be discriminated: an evolutionary point of view (Jones 2006; Kleiner 1993; Luzin 1998a, 1998b), a point of view of concept formation (Sfard 1992; Slavit 1997; Tall 1992), and a point of view of the role of the representations of functions (Carlson 1998; DeMarois & Tall 1996; DeMarois & Tall 1999; Kenehan 2007, Verstappen 1982).

Despite the fact that the field of problems related to the understanding of the concept of function has been fairly extensively researched so far it has paid too little attention to the question how the repertoire of different types of functions is organised into the learner's memory. As far as I know, this question which is the focus of this paper has so far been studied little (however, c.f. Grevholm 2008; Williams 1998).

The repertoire of different types of functions which students meet during the studies of school mathematics is wide. A student who has completed the upper secondary school studies has become familiar with a variety of functions like polynomial functions, rational functions, power functions, exponential functions, logarithmic and trigonometric functions etc. Moreover the student can also reasonably be expected to have got acquainted with absolute value function, piece-wise defined function, derivative and integral function, composite function, outer and inner function, monotonic functions, continuous functions, and many more.

The purpose of this study was to investigate how this extensive range of function concepts which students should have been learnt in upper secondary school mathematics is structured in the mind of the mathematics student who is pursuing his/her university studies in mathematics and planning to become a mathematics teacher. An ability to organise knowledge into systematic structures belongs to the pedagogical content knowledge required of a teacher. Without this ability it is difficult to master extensive knowledge contents and to perform the planning of instruction in such time constraints which we nowadays have in upper secondary school mathematics teaching.

## **METHODOLOGY**

The data for this study were collected in Finland from 31 students who were making the so called pedagogical studies (60 ECTS) in the department of education in the autumn term 2010. One of the participants had not done any university level mathematics courses but the others had studied on average 58 ECTS mathematics ranging from 15 to 110 ECTS. From one to four years before this testing all of these students had successfully completed their studies of mathematics at upper secondary school. Most of them (29 from 31) had studied the so called long course of mathematics at Finnish upper secondary school which is considerably more extensive and demanding than the optional shorter course of mathematics.

Two methods were used in collecting the data: (1) the listing technique, which involved the handing out of a questionnaire in order to find out how many different types of function the informant was totally able to recall one after another, and (2) the concept map technique, where the informant was asked to construct a concept map starting from the central concept of “Function” and to link to it all the different types of functions which he/she considered relevant. Both tests were pilot-tested with another group of students before the use. The use of two tests added significantly both the reliability and the validity of the study. The first test instructed the informant to produce the widest possible range of different functions of various types and names, while the second test implied special attention being paid to the relationships between the concepts presented. Two testes also gave us an opportunity to compare the repertoire of the recalled functions given in both testes.

*Listing method:* The test followed the idea of the “Drawing test” by which Burger and Shaughnessy (1986) in their studies of the van Hiele levels investigated pupils’ ability to produce consecutively as many different triangles as possible and the interpretations which pupils’ drawings seemed to give to the possible variation of the shape of the triangle and its conceptualization. The following extract presents the first item in the questionnaire which required the informants to recall various functions.

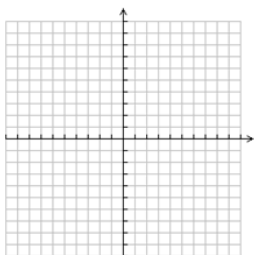
ANSWERER: \_\_\_\_\_ DEPT. \_\_\_\_\_

IN UPPER SECONDARY SCHOOL I STUDIED

a shorter course of mathematics / an advanced course of mathematics

The functions can be classified into different types on the basis of their properties. In the following tasks, your task is to recall all the different types of functions you know by name.

1. Draw/sketch on the coordinate system an example of the graph of such a function which you know to belong to some functions, which you can name.



Of what type is the function illustrated by the picture?

Supplement:

It is \_\_\_\_\_ function.

Give two examples of the equations of the functions of this type:

1. \_\_\_\_\_

2. \_\_\_\_\_

**Figure 1: The item 1 of the questionnaire.**

In a corresponding way in items 2-20, the informants were asked to produce examples of functions which were always of a type different from all those presented by the informant earlier. The questionnaire was designed to enable a maximum of 20 types of function to be featured, but the informants were still asked to produce as many examples as they possibly could of the functions of different types. In the first four items, an example of a graph of function, the name of a type of function, and two examples of equations of functions were required, while in the items 5-20 only the name of a type of function was asked.

*Concept map method:* In the second part of the collecting of data the informants were asked to draw a concept map of all the different types of function which they were familiar with. They were requested to use the concept “Function” as the central concept in the map. The answers which they had given in the first part of the test were not collected from them before they could proceed to the second part. The concept map technique is used in a teacher training a lot and the method was familiar to the interviewees.

From the results of test 1 e.g. from the lists of different types of functions recalled by teacher students we studied the following issues: (a) the different types of functions each student could remember, (b) the order of the types of the functions in the list (supposed to indicate the differences in the easiness to recall the functions), and (c)

the way the functions seemed to link to each other. It was interpreted that the respondent had a link between two or more functions if those were listed consecutively and the functions belonged to the same mathematical context. From the concept maps produced in test 2 we studied the above mentioned features (a) and (c) by checking from each map the variety of the different types of functions it had and the so-called structural elements included in each map. By structural elements we mean general tools like hierarchical structures, pairs of function/operation and its inverse function/operation, themes which grouped functions together, etc. by which students tried to organise the range of concepts which they recalled and presented.

## RESULTS

The inference from the answers given by the teacher students in the "listing" test is that generally speaking students - with few exceptions - were not able to produce so well organised and systematic presentation of functions of different types which we expected. Moreover, some informants confused the names of graphs and functions, giving functions such as parabola function, circle function, ascending and descending functions, function of the straight line, etc. Fairly obviously, for many students the graph of a function was the primary underlying representation of the function concept, not the function rule whether given in the form  $y = f(x)$  or  $f(x) = \dots$  or as a mapping  $x \rightarrow f(x)$ .

The mean number of different functions in the lists in test 1 was 9,0 (min 4, max 18) and in the pictures in test 2 11,7 (min 4, max 22). In both of the tests only few informants were able to recall for instance such functions as monotonic function, composite function, power function, absolute value function. Table 1 contains the types of function which appear in the listings of at least five students.

	Technique					
	Listing			Mind mapping		
Recalled functions	$f(n=31)$	$f[\%]$	$\bar{n}$	$f(n=31)$	$f[\%]$	
Trigonometric functions	23	74,2	3,4	28	90,3	
Logarithmic function	22	71,0	5,7	23	74,2	
Exponential function	20	64,5	4,8	19	61,3	
Sine-function	20	64,5	3,3	19	61,3	
Cosine-function	18	58,1	4,8	19	61,3	
Polynomial	18	58,1	2,9	25	80,6	

function (in general)					
First-degree polyn. function	11	35,5	3,2	14	45,2
Linear function	11	35,5	4,3	7	22,6
Root function	11	35,5	6,1	3	9,7
Second-degree polyn. function	10	32,3	4,0	12	38,7
Constant function	9	29,0	4,0	11	35,5
Derivative function	8	25,8	6,1	10	32,3
Integral function	8	25,8	6,0	10	32,3
Piecewise defined function	6	19,4	6,8	6	19,4
Continuous function	5	16,1	10,8	6	19,4
Non-continuous function	5	16,1	11,0	9	29,0

**Table 1. The most commonly mentioned type of functions and the average order in the list of each function (type).**

Especially in the test 1, answers mostly seemed to consist of separate facts which had occurred to the informants. The students seemed not to be handling general tools for organising the range of concepts, e.g. following hierarchical structures, analysing functions as pairs of “function – inverse function”, grouping by themes, etc. Instead, in figures produced by the concept map technique it is practically impossible to avoid at least some degree of conceptual organisation (cf. Grevholm 2008; Williams 1998). From the mathematical point of view the techniques applied here by the informants were often quite deficient and the resulting concept maps scanty and insufficient, in many cases also partly erroneous. True, there were also some richly structured concept maps.

In these maps, the object of the research was the occurrence of structural elements – such as (1) trigonometric functions – sine-cosine-tangent, (2) composite function – inner function-outer function, (3) logarithmic function - exponential function, (4) derivative function-integral function, etc. - based on hierarchy or other connection between concepts. Table 2 shows the most common organising elements found in the concept maps.

STRUCTURAL ELEMENTS		TECHNIQUE	
		Listing	Concept map
PRINCIPLE	FUNCTIONS		
CONCEPT AND SUBCONCEPT(S)	Polynomial functions of the first degree, of the second degree etc.	16/31	8/31
	Polynomial f. -linear f.	2/31	2/31
	Polynomial f. -constant f.	2/31	1/31
	Trigonometric functions and at least sin and cos	23/31	20/31
INVERSE FUNCTIONS OR OPERATIONS	Logarithmic f.-exponential f.	6/31	8/31
	Exponential f. - root f.	2/31	2/31
	Derivative f. – integral f.	9/31	8/31
DICOTOMIES GENERATED BY A PROPERTY	Continuous f.-discontinuous f.	3/31	5/31
	Linear f.-nonlinear f.	1/31	5/31
	Even f. – odd f.	2/31	1/31
	Functions of one variable – functions of multiple variables	3/31	3/31
LINKED BY THE CONTEXT	Density f. – cumulative distribution f.	2/31	3/37
	Translation-reflection-rotation	0/31	2/31
	Ceiling f. – floor f.	1/31	1/31
	Piecewise defined-discont. f.	0/31	5/31

**Table 2. Structural elements in the concept maps drawn up by the students.**

The structural elements listed above appeared in the total of the concept maps but only few maps contained more than two or three examples of them. Quite clearly, the selected research method also had its impact on the results. Obviously, the concept map technique in particular seems strongly to invite some informants to include in the scheme everything they can possibly remember without much regard to whether

the final inner structure of the scheme is systematic enough or not. The functions given in the answer sheets had perhaps been written down in the order in which they had happened to occur to the informants. Our purpose is that at the following stage of the study we will ask students in an interview situation to give an account why they arranged recalled functions in that order which in the listings and mind maps is seen. This way we can make sure that the interviewees have consciously used the structural elements which we found in the data.

## DISCUSSION

What, then, is the view provided by the study of what stays in the students' memory of the mathematics instruction concerning functions at upper secondary school? On the basis of the data examined here it can be stated that for too many students only single concepts seem to stay, with only a loose organising structure. These university student-informants seemed to lack the efficient tools for conceptual organisation, such as following hierarchical structures from upper concepts to sub-concepts, thinking about functions as pairs of function/operation and its inverse function/operation, interlinking functions within one and the same context and seeing how properties of functions lead to important classes of functions. However, it was delighting to notice that there were also many (about every fourth student) students who mastered the studied knowledge area and could perceive its conceptual structure rather well. Nevertheless the instruction of mathematics both at the secondary school level and at the university level faces challenges if we want that all our mathematics teacher students will get better expertise in such large knowledge areas which were considered in this study. A try to lift the structural elements more explicit in the teaching of mathematics can be seen on the one hand just an aid in the focusing of students' attention to the links in the conceptual knowledge (Hiebert & Lefevre 1986) but on the other hand also as a kind of Ausubelian attempt to offer students more efficient ways to handle extensive the knowledge blocs of mathematics in general.

## References

- Burger, W.F. & Shaughnessy, J.M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for the Research in Mathematics Education* 17 (1), 31-48.
- Carlson, M.P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.) *CBMS Issues in Mathematics Education 7. Research in Collegiate Mathematics Education III: Conference Board of Mathematical Sciences* (pp. 114-162).
- DeMarois, P. & Tall, D. (1996). Facets and layers of the function concept. In L. Puig & A. Gutierrez (Eds.) *Proc. of the 20<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Valencia, Spain:PME.

- DeMarois, P. & Tall, D. (1999). Function: Organizing principle or cognitive root? In O. Zaslavsky (Ed.), *Proc. of the 23<sup>rd</sup> Conf. of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 257-264). Haifa Israel:PME.
- Grevholm, B. (2008). Concept maps as a research tool in mathematics education. Paper presented in a conference: Concept Mapping: Connecting Educators. *Proc. of the Third Int. Conference on Concept Mapping, Tallinn, Estonia & Helsinki, Finland 2008*. <<http://cmc.ihmc.us/cmc2008papers/cmc2008-p301.pdf>> (Last read on Jan. 11 2011)
- Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics. An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp 1-27). Hillsdale, NJ: Lawrence Erlbaum.
- Jones, M. (2006). Demystifying functions: The historical and pedagogical difficulties of the concept of the function. *Undergraduate Math Journal*, 7(2), 1-20.
- Kenehan, G. (2007). Examining function understanding from contexts using concept maps. In T. Lamberg & L. R. Wiest (Eds.) *Proc. of the 29<sup>th</sup> annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 93-99) Stateline, Lake Tahoe: University of Nevada, Reno.
- Kleiner, I. (1993). Functions: Historical and pedagogical aspects. *Science & Education*, 2(2), 183-209.
- Luzin, N. (1998a). The evolution of ... function: Part 1. *The American Mathematical Monthly*, 105(1), 59-67.
- Luzin, N. (1998b). The evolution of ... function: Part 2. *The American Mathematical Monthly*, 105(3), 263-270.
- Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In G. Harel & E. Dubinsky (Eds.) *The Concept of Function: Aspects of Epistemology and Pedagogy* MAA Notes (Vol. 25, pp. 59–84). Washington, DC: MAA.
- Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics*, 33(3), 259-281.
- Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495–511). New York: Macmillan.
- Verstappen, P. (1982). Some reflections on the introduction of relations and functions. In G. van Barneveld & H. Krabbendam (Eds.) *Proc. of Conf. on Functions* (pp.166-184). Enschede, The Netherlands: National Institute for Curriculum Development.
- Williams, C.G. (1998). Using concept maps to assess conceptual knowledge of function. *Journal for Research in Mathematics Education*, 29(4), 414-421.



# MISSING VALUE AND COMPARISON PROBLEMS: WHAT PUPILS KNOW BEFORE THE TEACHING OF PROPORTION<sup>1</sup>

Ana Isabel Silvestre  
Gaspar Correia School, Portugal

João Pedro da Ponte  
Institute of Education  
University of Lisbon; Portugal

*This paper analyses grade 6 pupils' mathematical processes and difficulties in solving proportion problems before the formal teaching of this topic. Using a qualitative methodology, we examine pupils' thinking processes at four levels of performance in missing value and comparison problems proposed in a written test and in an oral interview video and audio recorded. The results show that pupils tend to use scalar composition and decomposition strategies in missing value problems and functional strategies in comparison problems. Pupils' difficulties are related to a lack of recognition of the multiplicative nature of proportion relationships.*

## INTRODUCTION

Pupils' ability in proportional reasoning is essential for their mathematical development. This reasoning is fundamental to solve daily life problems and also for learning advanced mathematical topics as well as other fields of study, including natural and social sciences (Post, Behr & Lesh, 1988). Pupils' difficulties in this aspect of mathematical reasoning are well known (Bowers, Nickerson & Kenenhan, 2002; Van Dooren, De Bock, Hessels, Janssens & Verschaffel, 2005). Furthermore, as Lesh, Post and Behr (1988) note, there are many people that solve direct proportion problems without using proportional reasoning.

The Portuguese mathematics syllabus indicates that pupils at grades 5-6 must understand the notion of proportion and develop proportional reasoning. This document points that pupils at grades 1-4 already work on mathematical tasks involving proportional relationships. In their planning, teachers must take into account such pupils' prior informal knowledge. Thus, it is important to know pupils' ability to solve proportion problems before the teaching of this topic. In this paper, we discuss pupils' mathematical processes, including representations and strategies, in solving missing value and comparison problems, as well as their difficulties.

## Direct proportion

Proportional relationships may be investigated under different perspectives. In a psychological perspective, Vergnaud (1983) stresses the isomorphism of measures. In this model variables remain independent and the transformations within or between variables keep proportional relationships between numerical values (Figure 1).

---

<sup>1</sup> Paper produced within Project IMLNA – Improving Mathematics Learning in Numbers and Algebra, supported by FCT – Fundação para a Ciência e Tecnologia, Portugal (contract PTDC/CED/65448/2006).

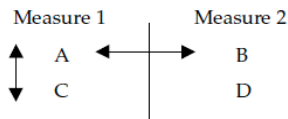


Figure 1: Isomorphism of measures model (Vergnaud, 1983)

In a mathematical perspective, a proportional relationship between two variables is represented as an equality of two ratios  $\frac{a}{b} = \frac{c}{d}$  ( $a$  and  $c$  are values of a variable,  $b$  and  $d$  values of another variable) or as a linear function  $y = mx$  with  $m \neq 0$ . Finally, a curriculum perspective stresses the use of representations, leading pupils to learn first to solve problems using equalities between ratios and then using linear functions. Generally, work with these two representations remains unconnected. Stanley, McGowan and Hull (2003) argue that the usual teaching approach for the development of proportional thinking in which pupils “solve proportions” is outdated and should be replaced by another in which pupils engage in activities that help them discover that proportion is the variation of two quantities related to each other.

### Types of problems and pupil’s strategies

Missing value problems present three numerical values and ask for the fourth value, whereas comparison problems present two or more pairs of numerical values and request their comparison. Several studies identified pupils’ strategies in solving these problems. For example, Post, Behr and Lesh (1988) and Cramer, Post and Currier (1993) identified the following: (i) the ratio unit, the most intuitive strategy that pupils use since the early years of schooling (computation of ratio units on division problems and computation of multiple ratios in the multiplication unit), (ii) factor or factor of change scale (Hart, 1984), known as “often like” strategy that is related to the numerical aspects of the problems but is used by many children, (iii) comparison of ratio problems associated with comparison, which allows comparing ratio units through two divisions, and (iv) cross product algorithm, also known as “rule of three”, which, while effective, is a mechanical process devoid of meaning in the context of the problems. In addition, Post, Behr and Lesh (1988) identified the strategy of graphic interpretation. Another rather informal strategy, that appears both in additive and multiplicative reasoning, is composition/decomposition (Christou & Philipou, 2002; Hart, 1984). Lamon (1993) classifies reasoning strategies as “within” and “between” variables, distinguishing between scalar reasoning (concerning the transformations within the same variable) and functional reasoning (establishing relationships between variables). In her view, the distinction between these two types of relationships is important because they involve different cognitive processes.

### METHODOLOGY

This study follows a qualitative approach (Denzin & Lincoln, 1998). The participants are four grade 6 pupils, both 11 years old, belonging to two different classes. Before the proportion chapter, all pupils in these two classes took a diagnostic test on the

topic. In each class, it was selected a pupil with satisfactory performance and another with difficulties in solving problems. Semi-structured interviews were conducted with these four pupils, video and audio recorded. Based on the strategies for solving proportional problems identified in the literature, we created a system of categories of analysis (Table 1). This repertoire of strategies was complemented by a pictorial strategy that was detected in this study.

Level	Strategy	Description
1	Multiplicative	<ul style="list-style-type: none"> <li>Establishes a multiplicative relationship between variables. Understands the meaning of ratio.</li> <li>Establishes a multiplicative co-variation relationship among variables.</li> </ul>
2	Additive and multiplicative	<ul style="list-style-type: none"> <li>Computes the unit ratio and uses it in additive processes.</li> <li>Composes and decomposes numbers involving addition, multiplication and division.</li> </ul>
3	Additive	<ul style="list-style-type: none"> <li>Composes numbers using addition.</li> </ul>
4	Pictorial	<ul style="list-style-type: none"> <li>Represents pictorially objects or sets of objects and counts them.</li> </ul>

Table 1: Categories of analysis for pupils' strategies

## RESULTS

### Solving missing value problems

Margarida bought three books from the collection "Once upon a time" for 12 euro. If Margarida has 48 euro, how many books she can buy?

This problem, posed in an interview, has a simple context and its data involves multi-plies of 3. These are some of the pupils' answers:

S1 said: "1 book costs 4 €. If 10 books is 40 € more 2 (8 €) is 12 books.  
Answer: She can buy 12 books."

$$\begin{array}{r|l} 9 & 36 \\ 18 & 72 \\ 36 & 144 \end{array}$$

1 I: How did you think?

2 S2: I multiplied by 2. It is the double. (...) I think that this way will not do. (...) 48 euro, is here. (Points between 36 and 72.)

1 S3: 4 euro is one book, right? (...) I do 48 divided by 4 [euro] of one book. (computes mentally.) It gives... 40 [euro] is 10


	[books]. 44 [euro] is 11 [books]. 48 [euro] is 12 [books]. She can buy 12 books.
1	S3: I'll do 3 on 3 [books] until it gives? 

Table 2 – Pupil’s answers to problem 1 (interview)

S1, S3 and S4 give correct answers. S1 calculates the unit ratio she but does not use this functional relationship to determine the missing value, choosing to compose numerical values, as ten times, the double and their sum. S2 uses a composition strategy which involves the successive computation of the doubles of the numerical pairs. Using this strategy, she cannot find the missing value. S3 develops a multiplicative functional strategy, using the price of each book, computing mentally the missing value (12 books), starting with a reference value (40 euro correspond to 10 books). Finally, S4 uses a strategy which involves the pictorial representation of the books and the successive addition of the price of 3 books until 48 euro, providing the missing value (12 books).

A car takes 30 minutes to go 50 miles. At the same speed, how long it takes to go 125 miles?

This question of the diagnostic test has a simple context and data that involve multiples of 5. Here are some of the pupils’ answers.

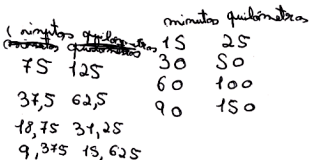

S1  50 Km - 30 m. 25 Km - 15 m.  50 quilómetros - 30 minutos 100 quilómetros - 60 minutos 125 quilómetros - 75 minutos  R: Siem, levará 75 minutos a percorrer	S2  
S3  30 m = 50 Km      15 = 25 Km 60 m = 100 Km <del>125 m</del> 75 m = 125 Km	S4  

Table 3 – Pupil’s answers to problem 2 (diagnostic test)

S1 determines the missing value using numerical composition/decomposition strategy. However, her written record does not allow us to understand if the numerical composition/decomposition involves additive or multiplicative reasoning. To obtain

the values 100 km and 60 minutes, respectively, the pupil may have done: (i) the additions  $50+50$  and  $30+30$ ; or (ii) the multiplications  $2 \times 50$  and  $2 \times 30$ . Then, she may have decomposed the initial values, building a new representation in columns (upper right corner). The numerical values that are written by S1 on the third row of her first table seem to come from the addition of the numerical values of the second lines of the two representations,  $125\text{km}=25\text{km}+100\text{km}$  and  $75\text{min}=15\text{min}+60\text{min}$ . S2 and S3 answered in a similar way. S4 answered incorrectly computing the difference between the distances as the value of the travelling distance (missing value).

### Solving comparison problems

A comparison problem proposed on the diagnostic test has a simple context and the values of one variable are multiples of the values of the other variable. The pupils' answer is shown on Table 4.

Luis and Rosa will do chocolate shake for brothers and cousins. The tables represented their recipes.							
<table border="1"> <tr> <th colspan="2">Luis' Recipe</th></tr> <tr> <td>Milk (cups)</td><td>12</td></tr> <tr> <td>Chocolate (spoon)</td><td>3</td></tr> </table>		Luis' Recipe		Milk (cups)	12	Chocolate (spoon)	3
Luis' Recipe							
Milk (cups)	12						
Chocolate (spoon)	3						
<table border="1"> <tr> <th colspan="2">Rosa' Recipe</th></tr> <tr> <td>Milk (cups)</td><td>20</td></tr> <tr> <td>Chocolate (spoon)</td><td>5</td></tr> </table>		Rosa' Recipe		Milk (cups)	20	Chocolate (spoon)	5
Rosa' Recipe							
Milk (cups)	20						
Chocolate (spoon)	5						
In which one the chocolate flavor is stronger?							

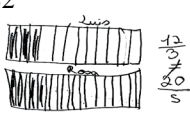
<p>S1</p> <p>Receita de Luis - 3:12 = 0,25 = quantidade de chocolate em 1 copo.  Receita de Rosa - 5:20 = 0,25 = quantidade de chocolate em 1 copo.  R: O leite sabe a mesma coisa nas duas receitas.</p> <p>Luis recipe – <math>3:12=0,25</math> amount of chocolate in a cup...</p> <p>The milk tastes the same</p>	<p>S2</p>  <p>Rosa's recipe has a stronger chocolate flavor</p>
<p>S3</p> <p><math>3 \times 4 = 12</math>      ? = 4  <math>3 \times 4 = 12</math>  <math>3 \times 4 = 12</math></p> <p>R: Nenhum, tinham todas a mesmo sabor.</p> <p>None, they had all the same flavour</p>	<p>S4</p> <p>É na receita de Luis (porque leva mais leite) que o leite sabe mais a chocolate</p> <p>It is in Luis recipe (because it takes less milk, # of cups) that the milk tastes more to chocolate</p> <div style="text-align: right;"> <math display="block">\begin{array}{r} 20 \\ - 5 \\ \hline 15 \end{array} \quad \begin{array}{r} 12 \\ - 3 \\ \hline 9 \end{array}</math> </div>

Table 4 – Pupil's answers to the problem 3 (diagnostic test)

S1 and S3 answer correctly. S1 represents the data as a division – using the symbol “:” – and computes the missing value. This is a functional strategy because it involves the relationship between the two variables. It shows the pupil understands the meaning of the invariant (0,25) which she does not yet see as the proportion constant but that she uses to make a qualitative judgement about the flavour of the mixtures. S3 begins by representing part of the data by an equation ( $3x=12$ ). Then he becomes involved on the exploration of the relationship between the quantity of chocolate and the quantity of milk, and he seems to have obtained mentally the invariant factor that he uses to judge the flavour of the mixture. This is a functional strategy. S2 and S4 answer incorrectly. S2 makes an incorrect pictorial representation of the data, because it is not a whole:part relationship but a part:part relationship. Then, he writes another representation (a fraction) that does not correspond to the pictorial representation. It is based on this representation that she says that Rosa’s mixture has a stronger taste of chocolate. S4 establishes a relation based on subtracting the quantities of milk and chocolate.

The next problem shows a mixture of liquids, on a simple context, involving small numbers. The pupils’ answers are shown on Table 5.

At art’s class, Inês and Maria joined black ink (black square) with white paint (white square) to prepare gray ink. Which girl prepared the darker gray ink?

Inês

■

□

□

Maria

■

□

□

□

□

1	S1:	It’s like there’s only one in black... (Make the next record. Corresponds to two attempts.)
		<div><div><div><div>■</div><div>□</div></div><div><div>■</div><div>□</div></div></div><div><div><div>Inês</div><div><div>■</div><div>□</div><div>□</div></div></div><div><div><div>Maria</div><div><div>■</div><div>□</div><div>□</div><div>□</div><div>□</div></div></div></div></div></div>
1	S2:	It’s the same colour.
2	I:	The grey is similar?
3	S2:	It is... Or not? This [Maria mix] has more one [cup] white. It has the same black, it is equal.
1	S3:	It is this one [Inês mix] because Maria’s ink has more white, is clearer.
1	S4:	I don’t know... Well... (Points to Maria’s mix) This has more ink? And... It has the same quantity of black ink... But which is more gray... I have no idea!

S3 answers correctly because he establishes a comparison between the mixtures attending to the quantities of white and black ink, knowing that the darker mixture is the one with less white ink. S1 and S2 answer wrongly saying that the colour of the mixtures is equal, arguing that they have the same amount of black ink, which means they are using only a part of the information they were given. However, S1 seems to be focused on the amount of black ink, because with her representations (Inês's mixture) she tries to find the exact quantity of white ink that Maria used. S4 says that he does not know which of the mixtures is the darkest and shows that he is thinking about the total amount quantities of ink instead of the tone.

## CONCLUSION

The results show that the pupils tend to solve correctly missing value problems using composition/decomposition strategies involving additive and multiplicative relationships at the same time. They tend to calculate doubles of the initial and intermediate values, getting more accurate values, as the solutions of problem 2 show. Then, they decompose the initial values on halves and find the missing value using addition. We may think that this strategy involves specific relationships using halves, as in problem 2 (two and a half times). However, pupils also use this strategy in relationships involving integer numbers. This is what happens in S1's solution to problem 1, where she uses reference values that she knows well, obtained by multiplication, and gets the missing value through addition. Pupils' use of composition/decomposition strategies face a difficulty when it is not possible to determine the missing value by numerical composition through doubling. In fact, in problem 1, S2 could not use other knowledge to find the missing value. Another difficulty involves the lack of understanding that the difference between the given values of a variable does not correspond to the numerical value of the other variable.

On comparison problems, pupils show different performances, with more difficulties on the problem that involves the tone of the color. It is likely that they never had to think deeply about this situation, and they focus their attention on data in a partial way, despite the fact that the operations described on the problem are simple to carry out on an art class. On the problem about chocolate flavor, pupils tend to use a functional strategy. The representation of the relation between the values of the variables involves the ratio, indicated by ":", but also the use of numerical columns where pupils explore the multiplicative relationship. One pupil reveals that he is able to explain the meaning of the constant of proportionality in this problem. The difficulties identified relate to interpreting the part:part relationship as a part:whole relationship as well as using different representations of the relationship part:whole. Another difficulty concerns the lack of understanding of the multiplicative relationship, establishing incorrectly additive relationships between the variables.

We suggest that the pupils, before the formal teaching of direct proportion, tend to solve missing value problems using strategies that involve simultaneously addition and multiplication, and it is possible to identify rudimentary strategies that involve

pictorial elements and unitary count. On comparison problems, pupils are able to use multiplicative functional strategies. However, correct answers depend on pupils' comprehension of the representations, of the nature of the relationship between the variables and of the context described in the problem.

This knowledge about grade 6 pupil's mathematical processes and difficulties in solving proportion problems, before its formal teaching, is important to teachers as it allows them to help pupils to develop proportional reasoning from their prior knowledge. On the one hand, teachers may promote the sophistication of pupil's strategies. For instance, composition/decomposition strategies may evolve and become scalar multiplicative strategies, reinforcing the comprehension of the multiplicative nature of proportions. On the other hand, knowing the difficulties and helping to identify the concepts that are not well understood and may lead to poor pupil performance, provide teachers with hints about the work they need to propose to their pupils.

## References

- Bowers, J., Nickerson, S., & Kenehan, G. (2002). Using technology to teach concepts of speed. In B. Litwiller & G. Bright (Eds.) *Making sense of fractions, ratios and proportions* (pp. 176-187). Reston, VA: NCTM.
- Cramer, K., Post, T., & Currier, S. (1993). *Learning and teaching ratio and proportion: Research implications*. [http://education.umn.edu/rationalnumberproject/93\\_4.html](http://education.umn.edu/rationalnumberproject/93_4.html)
- Christou, C., & Philippou, G. (2002). Mapping and development of intuitive proportional thinking. *Journal of Mathematical Behavior*, 20, 321-336.
- Denzin, N. K., & Lincoln, Y. S. (1994). Introduction: Entering the field of qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 1-17). London: Sage.
- Hart, K., (1984) *Ratio: Children's strategies and errors*. London: NFER-Nelson.
- Lamon, S. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education*, 24, 41-61.
- Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations for the middle grades* (pp. 93-118) Hillsdale, NJ: Lawrence Erlbaum.
- Post, T., Behr, M., & Lesh, R. (1988). Proportionality and the development of prealgebra understandings In *Algebraic concepts in the curriculum K-12*. Reston, VA: NCTM.
- Stanley, D., McGowan, D., & Hull, S. H. (2003). Pitfalls of over-reliance on cross multiplication as a method to find missing values. *Texas Mathematics Teacher*, 11, 9-11.
- Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2005). Not everything is proportional: Effects of age and problem type on propensities for overgeneralization. *Cognition and Instruction*, 23(1), 57-86.
- Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.) *Acquisition of mathematics concepts and processes* (pp. 127-174). New York, NY: Academic Press.



# YOUNG CHILDREN'S UNDERSTANDING OF REFLECTIONAL SYMMETRY IN A DYNAMIC GEOMETRY ENVIRONMENT

Nathalie Sinclair and Harpreet Kaur

Simon Fraser University

*Abstract: This paper examines the effect of the use of dynamic geometry environments on children's thinking about symmetry. Using a pair of black box tasks, kindergarten children were able to develop an understanding of symmetry that showed awareness of the properties of reflectional symmetry through the behaviour of dynamic images.*

## INTRODUCTION

Symmetry is a central idea in mathematics (for example, see Weyl, 1952; Schattschneider, 2006). One impact of Euclid's Elements on the school geometry is that symmetry usually plays a peripheral role in the curriculum (Sinclair, 2008), this despite the effectiveness of transformational approaches (see Usiskin, 1969) and the modern Kleinian conceptualisation. In addition to the mathematical motivations for increased emphasis on symmetry, the psychological research also suggests that young children show a strong capacity for attending to and identifying symmetry—a capacity that should be developed through their school geometry experiences.

In this paper we report on an exploratory study conducted with a class of kindergarten children (ages 4-5) working with reflecting symmetry using dynamic geometry software. In order to analyse the classroom interactions with these children, we develop a discursive framework related to reflectional symmetry thinking that is guided by Sfard's commognition framework. We show how the children's work with black-box reflectional symmetry sketches enabled them to develop their thinking about symmetry, and discuss the specific mediating role of DG on this thinking.

### *Children's understanding of symmetry*

Children have intuitive notions of symmetry from the earliest years (Vurpillot, 1976). Pre-school children spontaneously constructing symmetrical figures in informal play (Seo and Ginsburg, 2004). However, many concepts of symmetry are not firmly established before 12 years of age (Genkins, 1975). Vertical bilateral symmetry remains easier for students to handle than horizontal symmetry (Genkins, 1975), and oblique lines of symmetry are the most difficult (Hart, 1981; Hoyles & Healy, 1997).

Clements & Sarama (2004) encourage work with symmetry in the pre-K through grade 2 years and offer a developmental trajectory for transformations and symmetry in which children begin at the pre-K level to create shapes that have line symmetry, then work in kindergarten and grade 1 to identify symmetry in 2-D objects. In grade 2, children identify the mirror lines of shapes with line symmetry. In their approach,

children make “flips” with pattern blocks, visualise these flips, and predict the outcome of flip motions. The DG approach we describe takes a different approach in that the reflections have already been performed and the goal is to investigate their behaviour. This will entail different developmental possibilities.

## **THEORETICAL PERSPECTIVE**

In previous research, we have found Sfard’s (2008) ‘commognition’ approach suitable for analysing the geometric learning of students interacting with DGEs (see Sinclair, Moss & Jones, 2010; Sinclair & Yurita, 2008). For Sfard, learning corresponds to a change in discourse: learning geometry thus corresponds to changing the way one communicates about geometric objects and relationships.

In the context of identifying shapes, Sfard has proposed the following three levels of discourse characterised by different types of routines and word uses, which Sinclair & Moss (to appear) use in their study of children’s interactions with DG triangles:

- 1<sup>st</sup> level: the word ‘triangle’ is used as a proper noun. Shapes will be called triangles if they are similar to previously-seen (and therefore, prototypical) triangles. The routine of identification involves visual object recognition.
- 2<sup>nd</sup> level: the word triangle is used as a family name, that is, the name of a category of elementary objects; identification is made according to visual family recognition as well as through an informal properties check. Children can justify their identification (because it has three sides) but their justifications may also include extraneous properties.
- 3<sup>rd</sup> level: the word “triangle” is used as the name of a category of objects, and identification is made through visual family resemblance first, and then verification/refinement of properties. Students at this level use definitions as necessary and sufficient conditions.

The identification of objects with reflectional symmetry could proceed in a similar fashion through the three levels of discourse. In the 1<sup>st</sup> level, children might identify an object as ‘symmetric’ if it looks like proto-typical symmetric objects such as hearts and ladybugs—shapes with vertical line symmetry. At the 2<sup>nd</sup> level, children will be able to say that a shape is symmetric if it satisfies certain conditions, so that identification will be made according to visual family recognition. Thus, they may be able to identify a novel shape as being symmetric and justify this decision by stating informal properties. Or, they may be able to ascertain that an object is not symmetric if it doesn’t have the same elements on both sides of a line of symmetry. They might also be able to identify a shape as symmetric when its line of symmetry is oblique.

At the 3<sup>rd</sup> level, children can say that a shape is symmetric first through visual family recognition and then through verification of necessary and sufficient conditions. These conditions include the more formal property that the line of symmetry is the perpendicular bisector of the segment between any pre-image point and its corresponding image point. We were not expecting children to attain a 3<sup>rd</sup> level of discourse, as we did not introduce the formal notion of perpendicularity. However,

we were interested in investigating how they might move to a 2<sup>nd</sup> level of discourse, and to examine the informal language they would use to talk about symmetry.

## exploring reflectional symmetry

### *Participants and tasks*

We worked with kindergarten children from a University Lab pre-K-6 school in an urban middle SES district. There were 22 children per class from diverse ethnic backgrounds and with a wide range of academic abilities, with 25% being special needs learners. We worked with the children for three days on a variety of geometric concepts. Each lesson lasted approximately 30 minutes and was conducted in a small group (half class at a time) with the children seated on a carpet in front of a screen. Two researchers, and the classroom teacher, were present for each lesson. The first author conducted the lessons. Lessons were videotaped and transcribed. The lesson presented in this paper focused on one kindergarten lesson on reflectional symmetry. The children had one previous lessons involving *Sketchpad*, where they had worked with triangles, but they had never received formal instruction related to symmetry.

### *dynamic reflectional symmetry sketches*

In this study we used two different sketches to explore reflectional symmetry with the children. We began with the “discrete symmetry machine,” shown in Figure 1a, adapted from an original design by Michael Battista. In this sketch, dragging a square on the right side of the mirror makes the corresponding square on the left side move in a symmetric fashion. The squares have different colours (purple, red and yellow). Dragging the bottom right (yellow) square down makes the corresponding square go down as well, by the same amount—see Figure 1b. The file contains challenge tabs, each of which display a configuration made of squares, some of which are symmetric, and some not (see Figures 2a, 2b, 2c and 2d).

The purpose of using this sketch was to enable children to focus on the relationship between corresponding squares of the symmetric figure, and to build an understanding of symmetry through its behavioural properties. In contrast to manipulatives such as paper-folding and Miras, we note that the action of reflection in this sketch remains in the plane

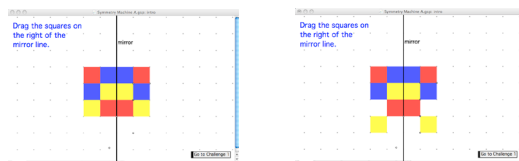
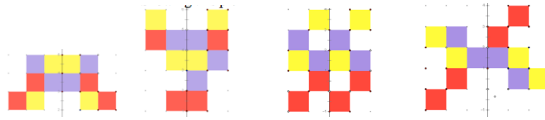


Figure 1a and 1b: The discrete symmetry machine.



Figures 2a, b, c, d: Challenge configurations

We also used the “continuous symmetry machine” (see Figure 3a and b). Two points appear on the screen. As one point is dragged, the other moves in a symmetric fashion—both leave traces as they move. The lines of symmetry is not visible.



Figures 3a and 3b: The continuous symmetry machine

The purpose of this sketch was to move to a more abstract presentation of symmetry, where the focus of the children would be more on the continuous behaviour of the point and its image, and where more familiar shapes could be created.

### **classroom discussion**

We begin with the introduction of the discrete symmetry machine, then look at how the children solved the symmetry challenges, and finally examine their interactions with the continuous symmetry machine.

### **Introducing the discrete symmetry machine**

The teacher began by asking the children whether they had every heard the word symmetry before, which they had not. She then introduced the discrete symmetry machine, and explained how it works.

12 Teacher Symmetry machine lets me only move these squares over here. For example, you want me to move the red square. Red square goes up.

13 Angela: It goes the same (inaudible)

Five children were invited to pick a square to move. The teacher asked them to explain exactly where they wanted to move their squares (up, down or right, left). Following this, the teacher invited the students to describe the machine.

47 Teacher: Does somebody want to tell me what the symmetry machine does?

48 Allan: It does what you do on the one side, to the other side.

49 Teacher: It does on one side what you do on the other side. Somebody else?

50 Rabia: It does what you want it to do.

51 Teacher: Yes, that's true.

52 Alastair: It's like the mirror image on the other side.

Eric later expands on Alastair's idea:

56. Eric: If you, if you, because from now you saw it's like a mirror, so that's why you can't move that side, It has to be always symmetrical.

Angela's first statement that the red square "goes the same" provides an initial description of the behaviour of the symmetry machine. As more squares are moved, Allan offers a description linking the movement of the two squares: "It does what you do on one side, to the other." These very general and qualitative descriptions begin to point to properties associated with the symmetry machine. Rabia's statement is interesting in that she seems to be pointing to the idea that the squares on the left respond as one would want, remaining symmetrical. Alastair provides a link to the idea of a mirror, which Eric expands upon and uses as a justification for why one cannot move squares on the left of the line. Although the children have not yet identified objects as being symmetrical, they are developing a way of talking about how the symmetry machine behaves.

### **Solving symmetry machine challenges**

The teacher showed the children the first symmetry challenge (Figure 2a) and asked the students whether they could make it using the symmetry machine. Several students said "no," and then one said "actually, yes," and another concurred. This shows that many of them are using 1<sup>st</sup> level discourse around symmetry. The teacher then invited one student to the computer and asked others to direct the student explaining exactly which square to move and how. So, for example, Paul asked Ryan to "Move the yellows at the top, up one." After the children complete this challenge, the teacher offered a second one (Figure 2b). One child said that it was impossible to make it but most others disagreed.

111 Teacher: Why do you think?

112 Ryan: One of the purple (inaudible) needs another purple.

113 Teacher: Okay just a sec. Allan, can you tell me what you think?

114 Allan: It is not symmetrical. It needs the other purple right here

Despite Allan and Ryan's explanations, Alvin still wanted to try. From the configuration in Figure 2a, he told Eric to move the red square (3<sup>rd</sup> row) down one.

142 Angela: Put the purple down, the other purple one will follow it. [referring to the purple squares in the 2<sup>nd</sup> row]

143 Teacher: Okay, let's watch to see what happens, the purple one down.

144 Student: Another down. [referring to the purple squares, so they are not in 4<sup>th</sup> row]

145 Teacher: So, is that what you thought would happen?

146 Angela: Yes

147 Teacher: Is that a problem?

148 Sss: Yes

149 Teacher: Why is it a problem?

150 Alvin: You cannot move one, you can only move two.

151 Teacher: Yeah, so

152 Alastair: No, no, but they are not symmetrical.

153 Teacher: This one is not symmetrical? Can you explain why it is not symmetrical?

154 Alastair: That purple has to be there.

The children agreed that it was impossible to make this picture, pointing to the purple squares as being problematic. The teacher offered the next challenge, which was a symmetric one. The children all readily agree they could make it, and easily did so. Then teacher introduced the fourth challenge (Figure 2c).

205 Teacher: I want to show you one more

206 Sss: Oooh. We can't make it. We can't make it. Oh, we can, we can, we can.

207 Teacher: We can? Okay, whose turn is it?

208 Eric: Yea, we can't make it.

The teacher decided to go ahead and let the children try it, despite the fact that most through it could not be made. They quickly decided, after trying to move a yellow square (from the initially configuration of Figure 1) on the top row, that it would not be possible. They do not say it explicitly, but in stating that the picture could not be made after trying to move the yellow square, we infer they were attending to the fact that one yellow square could not be further from the mirror than the other.

232 Teacher: Okay, what do you think about this picture?

233 Alastair: It is not symmetrical.

234 Teacher: Why is it not symmetrical?

235 Eric: You mean, you cannot fold it [...].

236 Ryan: Because these two red ones and these two purple and yellow and red don't move all the way down.

At this point, the children are able to identify shapes as being symmetrical or not, even if they are novel ones. While Eric uses the idea of folding, Ryan seems to attend more to the corresponding squares needing to be at the same horizontal level, which is a different property than the equidistant one elicited by the previous challenge.

### **Working with the continuous symmetry machine**

The teacher introduced the continuous symmetry machine (with line of symmetry). The children were excited to see the designs left by the traces. Ryan immediately said "that's symmetrical."

255 Teacher: How do you explain it's symmetrical?

256 Alvin: It has two ...

257 Ryan: Yeah, when we click, they are moving in the same direction. That's symmetric.

Ryan uses the property of “moving in the same direction” to justify the symmetry. This way of describing it stems from the behaviour of the point on the screen; contrast this with a more static justification for symmetry such as ‘they are the same on both sides.’

Children took turns coming to the computer to drag the point and create designs. They make a ball, a bridge, and a butterfly. When Angela made a butterfly, and started filling it in, the teacher asked why it was symmetric.

288. Ryan: If you cut it in half, it is the exact same thing, except different colours.

Here Ryan’s previous dynamic description of symmetry changes in that he task about cutting the butterfly in half rather than moving points. This shift may be due to the fact that the previous designs were less recognisable as objects (and were not necessarily joined together). ‘Symmetry’ has become a work that describes an object rather than a process.

After a few other children had their turn, the teacher switched to a different tab, with an oblique (but invisible) line of symmetry. She slowly dragged the point so that it formed a heart. She then made a butterfly. She asked if the butterfly was symmetric. Most children answered “yes.”

321 Teacher: It doesn’t matter if it is tilted like that?

322 Sss: No.

323 Eric: Because now you have a diagonal line.

Recall that there was no visible mirror line, so Eric is identifying the line of reflection based just on the shape that was produced. The children’s general consensus that the obliquely-drawn shape was indeed symmetrical suggests that their thinking about symmetry was not restricted to vertical line symmetry.

## ***Discussion***

The discrete symmetry machine focused the children’s attention on the behaviour of the corresponding squares, so that the idea of symmetry involved having the same thing on both sides, which included being the same distance away from the line and being on the same ‘level.’ These are informal ways of describing the perpendicular bisector relationship of the line of symmetry with the segment between the two corresponding points. While they were initially not able identify symmetric configurations, they were eventually able to use these informal properties to correctly identify asymmetrical shapes. The mirror metaphor (which was certainly provoked in part by the presence of the word “mirror” on the sketch, related well the way in which one side responded to what the other side was doing. The idea of folding comes later, as Eric begins to see the set of squares as one object having two parts.

The transition to the continuous symmetry machine drew out from Ryan more attention to the fact that the points are moving in the same direction, where the word “direction” is more precise than the previous descriptions offered by the children.

Similar to the shift made by Eric, Ryan goes from seeing the two points behaving in relation to each other to seeing the whole design as one object that could be cut in half. Angela's filled-in butterfly may have helped to shift his description.

The results of this exploratory study provide some refinement of Clements and Sarama's (2004) developmental trajectory, and point to the way in which the tools used to explore a concept such as symmetry affect the ways students conceptualise it.

## REFERENCES

- Clements, D. & Sarama, J. (2004). *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education*. Mahwah, NJ: Erlbaum.
- Genkins, E. F. (1975). The concept of bilateral symmetry in young children. In M. F. Roszkopf (Ed.), *Children's mathematical concepts: Six Piagetian studies in mathematics education* (pp. 5-43). New York: Teachers College Press.
- Hart, K. (1981). *Children's understanding of mathematics: 11-16*. London: John Murray.
- Hoyles, C. & Healy, L. (1997). Unfolding Meanings for Reflective Symmetry, *International Journal of Computers in Mathematical Learning*, 2, 27-59.
- Schattschneider, D. (2006). Beauty and truth in mathematics, in Sinclair, N., Pimm, D. & Higginson, W. (eds), *Mathematics and the aesthetic: new approaches to an ancient affinity* (pp. 41-57). New York, NY, Springer.
- Seo, K.H. & Ginsburg, H. P. (2004). What is developmentally appropriate in early childhood mathematics education? Lessons from new research. In D. H. Clements, J. Sarama, & DiBiase, A.-M. (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 91-104). Mahwah, NJ: Erlbaum.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, England: Cambridge University Press.
- Sinclair, N., Moss, J. and Jones, K. (2010). Developing geometric discourse using DGS in K-3. *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, Belo Horizonte, Brazil: PME.
- Sinclair, N. (2008). *The history of the geometry curriculum in the United States*. IAP - Information Age Publishing Inc.
- Usiskin, Z. P. (1969). *Effects of teaching Euclidian geometry via transformations on student achievement and attitudes in tenth grade geometry*. Univ. of Michigan, Ann Arbor, Mich.
- Weyl, H. (1952). *Symmetry*, Princeton. Princeton, NJ, Princeton University Press.



# NUMBER SENSE IN CHILDREN: UNDERSTANDING NUMBER AS AN OPERATOR WHEN ADDING AND SUBTRACTING

Alina Galvão Spinillo

Federal University of Pernambuco, Brazil

*This study investigated children's number sense by exploring their understanding of number as operators when solving addition and subtraction tasks. One hundred and twenty six 6-8 years old children solved a task in which they had to judge whether two successive, inverse operations (transformations) carried out on an initial quantity (static measure) would provoke an increase or decrease in this quantity, or leave it unchanged. Children's performance and justifications showed that initially they intuitively grasp the effect of adding and subtracting one quantity from another. However, only at the age of 8 do they become capable of making compensations and understanding that one operation can cancel the other one out. The 6- and 7-year-olds had difficulties with the inverse relationship between addition and subtraction.*

## INTRODUCTION

Number sense is difficult to define since it is a multifaceted entity that involves many dimensions. It refers to a general, intuitive understanding of numbers, the relationships among them, and operations, including the ability to make judgments and inferences about quantities; the ability to develop useful, flexible, and efficient strategies; and knowledge about the effect of using a number as an operator on other numbers (e.g., Greeno, 1991; Reys at al., 1999; Ribeiro & Spinillo, 2006; Sowder & Schappelle, 1989; Spinillo, 2006; Yang, 2003). Manifestations of number sense include using numbers flexibly when mentally computing, estimating, judging number magnitude, and judging reasonableness of results, moving between number representations, and relating numbers and operations (Markovits & Sowder, 1994; Sowder, 1995).

Numbers may be considered as static measures or as transformations (Nunes & Bryant, 1996; Vergnaud, 1982). For instance, in the word problem There were 30 children at Mary's birthday party. Later, when they were cutting the cake, 9 children arrived. After the cake was cut, 30 children left. Did the number of children in the party go up, go down or stay the same?, the first 30 refers to a static measure while 9 and the second 30 refer to transformations. According to our analysis, numbers as transformations are operators, since they have an effect on numbers as static measures. In order to solve problems like the one above, children may use the notion of compensation in relation to the transformations applied to a given initial quantity. In other words, they need to understand that two successive and inverse operations (transformations), when applied to an initial quantity (static measure) may provoke

an increase or decrease in this quantity, or may not change it at all. Based on the methodological paradigm usually adopted in research on number sense, problems similar to this one were presented in this investigation to children who did not need to carry out precise numerical calculations, but rather make judgments about numerical situations.

This research is part of a wider investigation on number sense in which different aspects of the mathematical knowledge of elementary school children are explored (numbers, operations, and measures). This present study investigates children's number sense by exploring their understanding of numbers as operators when solving addition and subtraction tasks that, in the literature, are referred to as change situations, in which one quantity is transformed by adding to it or subtracting from it (Nunes & Bryant, 1996; Vergnaud, 1982).

## **METHOD**

### **Participants**

One hundred and twenty-six low-income children attending the first, second, and third grades at elementary schools in the city of Recife (Brazil) were equally divided into three age groups: 6, 7 and 8 years old.

### **Procedure and experiment design**

The children were individually interviewed in two sessions, which were audio recorded. In the first session (pre-test) the task examined their ability to identify addition and subtraction operations. In the second session the task explored children's understanding of numbers as operators. Justifications were asked for each response given in the trials in both tasks. The trials were in writing, presented and read aloud by the interviewer.

### **The pre-test**

The objective of the pre-test was to select children capable of correctly identifying operations of addition and subtraction, who had the intuitive notion that increasing an initial quantity was an addition operation, and that decreasing said quantity was a subtraction operation. A lack of understanding about these ideas means not understanding the principles governing these arithmetical operations.

The general instruction was, "There's a machine that does secret mathematical operations: a number goes in the machine and another number comes out. You don't need to make any calculation at all, you only have to discover what operation the machine did, and to explain how you found that out."

Only children who met the following two criteria were allowed to participate: they had gotten all 12 trials correct and provided at least 75% appropriate justifications which expressed understanding that, when the final quantity was bigger than the initial one, the operation involved was that of addition, and when the initial quantity was smaller than the final one, the operation was that of subtraction. Example:

Interviewer: The number 152 went in and the number 120 came out. What operation did the machine do: addition or subtraction? How did you find that out?

Child: Subtraction. Because there was a really big number in there and then it started going backwards until it reached 120. If a little number went in and a big number came out, it'd be addition. Here a big one went in and a little one came out, so that's subtraction.

Interviewer: But 120 is not a small number.

Child: But it is smaller than 150 anyway.

### **The task**

This task examined children's intuitive knowledge about the effect of a number (as an operator) on another number, understanding that two successive and inverse operations, when applied to an initial quantity, can cause that quantity to increase or decrease, or cause no change to that quantity. The general instruction was, "I will show you some problems. You don't need to make any calculations at all; you only have to discover whether, in these problems, the quantity went up, went down, or stayed the same; and to explain how you found that out."

There were three types of trials, with four trials in each. In Type 1, the result involved an increase in the initial quantity; in Type 2, the result involved a decrease in the initial quantity; and in Type 3 the result indicated no change in the initial quantity (see examples below).

### **DATA ANALYSIS**

Three types of justification were identified:

Justification 1: No justification was given or the child simply repeated part of the question. Example:

Type 2 trial (decrease): There were 30 children at Mary's birthday party. Later, when they were cutting the cake, 9 children arrived. After the cake was cut, 30 children left. Did the number of children in the party go up, go down or stay the same?

Interviewer: Why do you think it went up?

Child: Because there were 30, there were 9.

Justification 2: The child considered only the last operation, so that, when this was addition, s/he considered that the number increased; when this was subtraction, the child considered that the number decreased. Example:

Type 3 trial (no change): There were 48 people in a movie theatre. Then 5 people left. Later, 5 people arrived. Did the number of people in the theatre go up, go down, or stay the same?

Interviewer: Why do you think it went up?

Child: Because 5 people arrived, so it went up.

Interviewer: But 5 people left.

Child: But that was before.

Interviewer: Before?

Child: Before the people had arrived. I guess they didn't like the movie. So in the end 5 arrived, more people arrived.

In some trials a correct response was given; however, it was for the wrong reason, as can be seen in the following example:

Type 2 trial (decrease): There were 24 chocolates in a box. Mary put 5 chocolates into the box. Then she ate 24 chocolates. Did the number of chocolates in the box go up, go down, or stay the same?

Interviewer: Why do you think it went down?

Child: Because in the end she ate 24.

Interviewer: But she had put in 5 more chocolates.

Child: But she ate them after that. When you eat, that takes away. Sometimes you eat everything up.

Justification 3: both operations were considered. The child understood that (i) if the amount added was greater than the amount taken away, then the number went up; (ii) if the amount taken away was greater than the amount added, then the number went down; and (iii) if the amount added was the same as the amount taken away, the number remained the same, taking into account the compensation based on the inverse effect of one operation on the other. Almost all (95%) Type 3 justifications were accompanied by correct responses. Examples:

Type 1 trial (increase): A bus left the bus station with 16 people inside. At the first stop, 4 people got off. At another stop 16 people got on. Did the number of people on the bus go up, go down, or stay the same?

Child: I think it went up, because more people got on the bus than got off. A few people got off and a lot of people got on.

Type 3 trial (no change): There were 36 apples in a basket. John ate 2 apples. Later, his mother put 2 apples into the basket. Did the number of apples in the basket go up, go down, or stay the same.

Interviewer: Why do you think it remained the same?

Child: He ate 2 apples, didn't he? Then his mom put in 2 to make up for what he took out of the basket. So it stayed the same as before.

## RESULTS

According to Mann-Whitney Test ( $p < .03$ ), the 8-year-olds performed better (76,1%) than the 6- (61,3%) and 7-year-olds (63,2%), whose performance did not significantly differ from each other's.

Trials	6 years	7 years	8 years
Type 1 (increase)	63.9	66.5	77.9
Type 2 (decrease)	65.3	64.7	74.3
Type 3 (no change)	54.8	57.4	76.1

Table 1: Percentage of correct responses (out of 168).

The Wilcoxon Test revealed that the performance of the 6- and 7-year-old children was significantly worse on Type 3 trials than on the other types ( $p < .05$ ). For these children, it was difficult to understand that one operation canceled out the other, and they applied the rationale of Type 2 justifications (taking into account only the last operation) when solving Type 3 trials. Eight-year-old children, on the other hand, had the same high level performance on all three types of trials.

The Mann-Whitney Test did not identify significant differences between the groups of 6- and 7-year-olds in relation to each type of trial. However, the percentage of correct responses on all three types of trials was significantly higher in the group of 8-year-olds ( $p < .003$ ), especially in relation to the Type 3 trials (Table 1). It seems the most significant development takes place at age 8, especially in relation to understanding that one operation can cancel out another, a type of reasoning which is necessary to solve Type 3 trials.

As shown in Table 2, confirmed by the Mann-Whitney Test, the type of error present in Type 2 justifications tends to diminish in the group of 8-year-olds ( $p < .002$ ), a type of error which is frequently observed in the groups of 6- and 7-year-olds. The opposite occurs in relation to Type 3 justifications, which are given more often by the 8-year-olds than by the 6- and 7-year-olds ( $p < .002$ ).

Types of justification	6 years (n=504)	7 years (n=504)	8 years (n=504)
J1 (no justification, repetition)	47,6	44,1	38,7
J2 (last operation)	32,1	30,7	15,5
J3 (both operations)	20,3	25,2	45,8

Table 2: Percentage of justifications per age group.

According to the Wilcoxon Test ( $p < .02$ ), the justifications of 6- and 7-year-olds are concentrated in Type 1 (47,6% e 44,1%), while those of 8-year-olds are concentrated in Type 3 (45,8%).

Types of justification	Trial Type 1 (Increase) (n=168)	Trial Type 2 (Decrease) (n=168)	Trial Type 3 (No change) (n=168)
J1 (no justification, repetition)	35.7	32.7	30.3
J2 (last operation)	31	35.7	34.5
J3 (both operations)	33.3	31.6	35.2

Table 3: Percentage of justification per type of trial.

As displayed in Table 3, the use of the three types of justification did not vary in function of the type of trial, as revealed by the Wilcoxon Test ( $p > .05$ ).

## DISCUSSION AND CONCLUSIONS

The findings in this study indicate that children seem to understand the general idea according to which, when the initial quantity increases, the operation involved is that of addition, and when the initial quantity decreases, the operation involved is that of subtraction, as shown in the pre-test. These intuitive notions about addition and subtraction, although necessary, are clearly not sufficient to guarantee that the child will be able to understand and account for the effect of inverse and successive operations on numbers. In a developmental perspective, it seems that initially the child grasps the effects of adding and subtracting one quantity from another, but only later becomes capable of making compensations and understanding that one operation can cancel out another. This notion of compensation appears to be understood around the age of 8, when children are able to understand that (i) if what is added is more than what is taken out, then the quantity increases; (ii) if what is taken out is more than what is added, then the quantity decreases; and (iii) if the amount added is the same as the amount taken out, then nothing changes. The 8-year-old children, besides performing better than the 6 - and 7-year-olds, also gave explanations that expressed a rule about the effect of the operations on numbers.

The main difficulty of children aged 6 and 7 is to understand the inverse relationship between addition and subtraction. This difficulty is illustrated by the Type 2 justifications, which expresses a type of reasoning that does not involve compensation between the operations, but takes into account only the last operation done. Aware of this type of mistake, teachers could explore didactic situations which would make the inadequacy of this form of reasoning evident to students.

Elementary school teachers should develop in their pupils an intuitive sense about the inverse relations between addition and subtraction, and the idea that there is compensation between them. Teachers should also take into consideration that, at age 8, children have an initial understanding of these relations, and that this understanding may be related to future, more complex acquisitions, such as notions about negative numbers.

Research supported by a grant from the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) of the Brazilian Ministry of Science and Technology.

## References

- Greeno, J. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, 22 (3), 170-218.
- Markovits, Z. & Sowder, J. (1994). Developing number sense: an intervention study in grade 7. *Journal for Research in Mathematics Education*, 25 (1), 4-29.
- Nunes, T. & Bryant, P. (1996) *Children doing mathematics*. Oxford: Blackwell.
- Reys, R., Reys, B., McIntosh, A., Emanuelsson, G. & Der, C.Y. (1999). Assessing number sense of students in Australia, Sweden, Taiwan, and the United States. *School Science and Mathematics*, 99 (2), 61-70.
- Ribeiro, L. M. & Spinillo, A. G. (2006). Preschool children's number sense. *Proc. of the 30th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, p. 417). Prague: PME.
- Sowder, J. T. & Shappelle, B. P. (1989). *Establishing foundations for research on number sense and related topics: report of a conference*. San Diego: Diego State University Center for Research in Mathematics and Science Education.
- Sowder, J. T. & Kelin, J. (1993). Research on number sense and related topics. In D. Owens (Ed.), *Research ideas for the classroom: middle grades mathematics* (pp. 41-57). New York, N.Y.: Macmillan.
- Sowder, J. (1995). A compreensão de número na escola de primeiro grau. In L.Meira & A.G. Spinillo (Eds.), *Anais da I Semana de Estudos em Psicologia da Educação Matemática* (pp. 19-27). Recife, Brasil: Universidade Federal de Pernambuco.
- Spinillo, A. G. (2006). O sentido de número e sua importância na educação matemática. In M.R.F. de Brito (Ed.), *Solução de problemas e a matemática escolar* (pp.83-111), Campinas: Editora Alínea.

- Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T.P. Carpenter; J.M. Moser & T.A. Romberg (eds.); *Addition and subtraction: a cognitive perspective* (pp. 60-67). Hillsdale, NJ.: Lawrence Erlbaum Associates.
- Yang, D-C. (2003). Teaching and learning number sense: an intervention study of fifth grade students in Taiwan. *International Journal of Science and Mathematics Education*, 1 (1), 115-134.



# DISTINCTION BETWEEN FUNCTION AND RELATION: A RESEARCH STUDY ON PUPILS, STUDENTS AND TEACHERS.

Panagiotis Spyrou\* and Andonis Zagorianakos\*\*

University of Athens, Greece\*; Manchester Metropolitan University, England\*\*

*The main focus of the research is on pupils', students' and teachers' awareness of the distinction between the concepts of function and arbitrary relation. This issue is linked to the discrimination between dependent and independent variables (Sierpinska, 1992). The research is based on data collected from 12th grade pupils, a sample of students in the Department of Mathematics at the University of Athens and a sample of teachers of mathematics in secondary schools. An epistemological approach reveals that the origin of the difficulties is located in understanding the "single-valued" condition of the definition. Based on the research results, we argue for an action conception of function through problems in which the dependent variable is a magnitude subjected to measure.*

## INTRODUCTION AND THEORETICAL FRAMEWORK

The concept of function is essential in mathematics. Early theoretical frameworks for analyzing the students' understanding of function focused on various modes of representation and the translation between them. An alternative route of investigation of the process conception of function is concerned with analyzing the understanding of the function *properties* by the students, since they are more able to comprehend properties such as 1-1, onto, invertibility, once a process conception of function is achieved (Bagni, 2005; Ronda, 2009; Slavitt, 1997; Vinner, 1983).

The difficulties in understanding the concept of function, which are widespread and connected with the development of the concept through its history, are known as *epistemological obstacles* (Sierpinska, 1992). The literature on such obstacles is particularly rich (e.g. Freudenthal, 1983; Sfard, 1992; Dubinsky & Harel, 1992; Sierpinska, 1992; Even & Tirosh, 1995; Even, 1999). Sierpinska (1992) argues that the comprehension of the concept of function presupposes the overcoming of some main epistemological obstacles through corresponding mental *actions*. Our study is concerned with the awareness of students and teachers of a particular act of understanding: "Discrimination between the concepts of function and relation" (U(f)-12, *ibid.*, p. 49). Moreover, an epistemological obstacle connected with the previous discrimination has to do with the observation that the asymmetry of the dependent and the independent variable in the context of a function is linked with an unconscious scheme of thought: "Regarding the order of variables as irrelevant" (EO(f)-5, *ibid.*, p. 38). The first approaches of the concept of function by Bernoulli, Euler and Cauchy were manifested as a covariance of two magnitudes, with a vague

distinction of the order of variables (ibid). It was Dirichlet in 1837 who formulated for the first time a general definition of function:

If a variable  $y$  is so related to a variable  $x$  that whenever a numerical value is assigned to  $x$  there is a rule according to which a unique value of  $y$  is determined, then  $y$  is said to be a function of the independent variable  $x$ . (Boyer, 1968, p. 600)

In order to understand the epistemological shift that was provided by the Dirichlet definition, it is useful to think of the progress of science during the evolution of the concept of function in that era; the main claim of modernity in order to establish the objectivity of knowledge for nature during the Renaissance was that “every thing that can be measured exists” (Cassirer, 1923, p. 357) and that the “conception of being” in nature is “interacted with ...the conception of measure”. Throughout the progress of science every new magnitude could enter the list of natural entities if and only if a new measurement relationship was established with the previously known entities.

[T]he reality stands over against the reality of a mediated perception as something through and through mediated; as a system... of abstract intellectual symbols which serve to express certain relations of magnitude and measure, certain functional coordination and dependencies of phenomena. (p. 357)

The Dirichlet definition states the necessity of the dependent coordinate being uniquely determined. The novelty of the definition is the “single-valued” condition (Burgess, 1966, p. 6). Therefore, the Dirichlet definition expressed precisely the insight of a *mediated* measure in the concept of function (the function can be more generally considered as a mediated *estimation*, when the range is a two value set). To estimate the dependent variable  $y$  and to achieve it although there is no immediate access to  $y$ , is to estimate it through  $x$ . Therefore the independent variable is the mediating variable which gives access to the dependent variable, resulting in the *priority* of the latter. This priority is clearly manifested in the case of ‘many to one’, that is when there are many points where the function has the same (single) value (Kelley, 1955, p.11). This feature of the “single-valued” condition that the Dirichlet definition allows, also determines the *asymmetry* of the dependent and the independent variable and it is the core of the mathematical concept of function, as expressed by Fraenkel (1966/1919), best known for his work on his contributions to axiomatic theory (Zermelo–Fraenkel), in his thermograph example:

The function  $T = f(t)$  marked by the thermograph is single-valued, for to every moment  $t$  corresponds a uniquely defined temperature. If, however, we ask at what time a certain temperature has been reached (say, within the temperature-range in question) then the answer is given by a function –the inverse of the function  $T = f(t)$ – which in general is not single-valued because the same temperature may be reached at different moments. The concept of a single-valued but not uniquely invertible function is much used in mathematical analysis. (p. 23)

The teaching of the “single-valued” condition seems to be critical for the understanding of the concept. A common practice to tackle it in our educational system is the invocation of Venn diagrams and the vertical line (VL) test (that is, to

check if there is more than one intersection with the graph when vertical lines are drawn), which are used as mnemonic rules by some teachers (Lerman, 2005). Some researchers also refer to the VL test (Bagni, 2005; Slavits, 1997). We suspected that the usage of the VL test, even when applied correctly, conceals the broader apprehension of the “single-valued” condition, when the function appears as “many to one”. Closely related to the understanding of the “single-valued” condition is the fact that the dependent and the independent variables are not symmetric in the definition of function, in other words that it is always the first which is uniquely determined by the second and *not always* vice-versa (Sierpinska, 1992). We wanted to detect if the lack of understanding of the general claim of the “single-valued” condition is an issue for pupils who attend the last year of their secondary education, if the obstacle persists during the studies of prospective teachers of mathematics and finally, to check the teachers’ approach to the “single-valued” condition and their interpretation of its necessity. Thus, we formed the following research questions (accompanied by their relation to the questions of the questionnaire that was given to the research participants—see Appendix):

[1] Can the pupils, students, or even teachers recognize the “single-valued” property as the main difference between the concepts of function and of an arbitrary relation? (1st, 2nd question of the questionnaire)

[2] Can the pupils, students, or even teachers distinguish the order of the variables  $x$  and  $y$ , and the asymmetry that they have? (3rd, 4th question of the questionnaire)

## METHODOLOGY

The methodology was initially customized in order to test the appearance of the obstacle in nineteen 12<sup>th</sup> grade pupils, a few months before their graduation exams in a typical secondary school of Athens, in December 2009. This particular class was chosen because mathematics is a main course in their curriculum. During the same period we carried out the same research with the two teachers of mathematics at the same school. A second research study followed, concerning 16 students of the Department of Mathematics of the University of Athens and 4 more teachers at high schools (April 2010). We decided to design a survey in two phases.

**Phase 1** included the completion of a 4 questions questionnaire by the participants in the research (pupils, students and teachers). The format of the questions in the questionnaire included two types of questions (see Appendix):

[1] Crosscheck items corresponding to questions (key: Q=questionnaire):

**A)** (Q1) Participants were given 3 correspondences on graphs and another 3 with table values. They were asked to find out which represented functions. We omitted Venn diagrams in order to focus in the meaning of given values and to think the “single-valued” (Fraenkel, 1966/1919) condition.

**B) (Q3)** Participants were given 4 functions on a graph. They were asked which of the 4 would still be functions if the lines on the graph (not the coordinates system) were rotated by  $90^\circ$ .

[2] Open-type questions with short answers such as the following:

**A) (Q1)** Participants were asked to make the necessary changes to the graphs and table values to change the arbitrary relations into functions.

**B) (Q2)** Participants were asked to give two examples of (arbitrary) relations which were not functions, one algebraic and one represented graphically.

**C) (Q3)** Participants were asked to justify whether the 4 functions on the graph remained functions when turned  $90^\circ$ .

**D) (Q4)** Participants were asked in which case(s) the  $90^\circ$  rotation of a function's graphical representation represents a function and to give a general rule.

The questions in the questionnaire were designed to correspond to the research questions posed in the theoretical framework, as follows:

**A)** The 1<sup>st</sup> question asks the participants to distinguish which correspondences are functions and which are arbitrary relations. The 2<sup>nd</sup> question asks them to give two counterexamples of a relation, one graphical and one with an analytical description, which do not fit the function definition. The answers to both of these questions correspond to the 1<sup>st</sup> research question.

**B)** The  $90^\circ$  turn on the graph, which is the concern of the 3<sup>rd</sup> and the 4<sup>th</sup> questions occurs as a consequence of the permutation between the dependent and independent variables. Therefore, the participants are tested in their ability to realize the asymmetry of the  $x$  and  $y$  variables, and the decisive operation of the “many to one” condition, i.e., the 2<sup>nd</sup> research question.

**Phase 2** of the research took place after about one week from the collection of the questionnaires, concerning the pupils and the students, and included semi-structured interviews with 10 pupils and 10 students who volunteered. Since we sought to explore the less than transparent causes of the potential deficiencies that the pupils and/or the students might present, we took into account the case that the correct answers given in the questionnaires might not necessarily mean that the concept of function is comprehended. We were also interested in tracing possible links with the educational experience of the participants, an area where the interviews were thought to be more effective. Participants (pupils and students) were informed of the objectives of the research and gave their permission for their interviews to be recorded. During the process the interviewer emphasized that the purpose of the interview was not to examine the participants but was to explore what they thought when they answered the questionnaire, regardless of whether the answers were right or wrong. Each interview was based on a clarification of the answers given on the questionnaire and the problems that the pupils had encountered in completing it. This assessment showed a consistency in the interview results. The interviews of the 6

teachers took place either just before or during their completion of the questionnaire, and they were not as structured in order to understand their general beliefs that determined their teaching.

## DISCUSSION

The results from the questionnaires and the interviews confirmed the problematic areas anticipated at the outset of the research. It is evident from the results of the questionnaires and the interviews that pupils and students experienced difficulties in answering all four questions. Furthermore, most of the correct answers were based on stereotypical examples like the circle example (2<sup>nd</sup> question) or the VL test (1<sup>st</sup>, 3<sup>rd</sup> questions) (Bagni 2005, Slavit 1997), which (as the interviews revealed) showed that in most cases pupils and students did not comprehend their connection with the claim of the definition to which they correspond (the “single-valued” condition). The research showed that the primary importance granted to the visual representation can lead pupils to forget the significance of certain elements of the definition (Bagni 2005). As an effect the claims of the definition appeared as disconnected components, thus needing mnemonic rules in order to be retained. The necessity of such mnemonic rules seemed to emerge from a teaching of function that is inconsistent with their experiences in the real world (Ronda, 2009), teaching that presents it as a symbolic convention. A typical example is the following one, where the pupil insists on using the VL test instead of checking the values on the table:

Interviewer: So what did you do in order to answer the 1<sup>st</sup> question?

Pupil 4: I drew the points of the table on a Cartesian plane and then I used a ruler to “scan” the plane in order to check if there are cases where one  $x$  corresponds to more than one  $y$ .

During the interviews the pupils as well as the students gave exclusively 1–1 examples of functions and they could not give examples of not invertible functions (‘many-to-one’); we were surprised especially by the students, since they had been using the Dirichlet function for quite a few university courses. Furthermore, all but one student identified the functions that are not invertible with the monotonous ones. From the interviews with the students it was also apparent that their first encounter with the concept of function during their secondary education played a decisive part in their concept image and their misconceptions (e.g. student 6 and student 11: “These answers we gave would have been correct if we had been still attending the secondary school”).

As the interviews of the pupils and the students implied, the subsequent interviews with teachers of mathematics seemed necessary in order to trace potential causes of their shortage in understanding of the “single-valued” condition of the definition. The interviews with the teachers of mathematics, who also responded to the questionnaire just before the interviews took place, showed a potential source of these deficiencies. The interviews showed that all teachers except teacher 6 focused on 1–1 functions, while introducing the definition of function to the pupils:

Teacher 2: I tell the students “We start observing functions from our everyday life, from our environment: the car has got one plate, your identity card has got one number, only one, it doesn’t have another one, a commodity on the shelves of a supermarket has got a single price, not two. It also has a single bar code. Every single tin of anchovies has got a single bar code. And you may correspond its’ unique price to it.”

Consequently, all teachers except teacher 6 could not justify the necessity of the ‘many-to-one’ condition of the definition (although this condition appears in teacher 2’s supermarket commodity example). Three of the teachers were not concerned with this query at all, favoring mnemonic rules such as the VL test in terms of being effective towards the final exams; one of the teachers (teacher 2) was focused on showing that the ‘one-to-many’ relations are not functions, without realizing that he was almost exclusively dealing with 1–1 functions. We also met the authoritative approach (Harel & Rabin, 2010), when another teacher responded as follows:

Teacher 5: The definition is the definition! We can’t possibly know why someone thought to set a definition like that!

There was only one teacher who showed that he understood the necessity of the ‘single-valued’ condition of the definition:

Teacher 6: The answer is that ... under every natural phenomenon there is a measure ... In order to quantify, to study the natural phenomena we need to measure ... in other words, in order to construct a credible tool, which will not correspond 7 in one instance and 5 in another instance, under the same circumstances! At least this is what I say to the pupils.

To our surprise, even teacher 6, when he was referring to the function definition, was saying “for every  $x$  there is **one**  $y$ ”, neglecting to add “and **only** one”. When the verbal representation of the definition omits the unambiguous statement of the critical ‘single-value’ attribute, it may lead the pupils to identify the concept of 1–1 correspondence with the concept of function, and inactivate the “many to one” case; a practice that is added to the uncomplicated enumeration, which is already consolidated in the pupils’ minds.

Thinking on the remedy of the obstacle, we suggest that the teaching of function should be linked with action tasks (Bagni, 2005; Slavit, 1997). Even pure mathematicians consider this need: “One way to think of a function is an action, a process that takes the domain to the range” (Thurston in Bagni, 2005, p. 2-3). Therefore a possible answer for the introduction of the concept is the association with certain activities, where the independent variable is pursued and used as a means for the estimation of the target, namely the dependent variable; there are substantial arguments supporting the idea that the experience of measurement is interwoven with the reality of the pupils as well as the applications that they are taught in school. An action conception (of function) is concerned with the computation of a single quantity for a single numeric value via a given algorithm or rule of association (Slavit, 1997). Our suggestion is to further approach this action conception of function through problems, where the dependent variable is a subject of measurement, while the independent variable is directly accessible to measurement. Say, for instance that we

need to measure the height of a hill and we know the distance from it and the tangent of the angle from the observation point.

The concept of function as action of measurement would unify the components of the definition and establish them as necessities.

## References

- Bagni, G. T. (2005). Functions: Processes, properties, objects. *Scientia Paedagogica Experimentalis*, XLII(2), 205-230.
- Boyer, C. (1968). *A history of mathematics*. New York: Wiley & Sons.
- Burgess, D. C. J. (1966). *Analytical topology*. London: D. van Nostrand Company.
- Cassirer, E. (1923). *Substance and function and Einstein's theory of relativity*. Chicago-London: The Open Court Publishing Company.
- Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Washington, D.C.: Mathematical Association of America.
- Even, R. (1999). Integrating academic and practical knowledge in a teacher leaders' development program. *Educational Studies in Mathematics*, 38, 235-252.
- Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations and subject matter. *Educational Studies in Mathematics*, 29, 1-20.
- Fraenkel, A. (1966/1919). *Abstract set theory*. Amsterdam: North Holland.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel Publishing Company.
- Harel, G., & Rabin, J. M. (2010). Teaching practices associated with authoritative proof scheme, *Journal for Research in Mathematics Education*, 41 (1), p. 14-19.
- Kelley, J. L. (1955). *General topology*. New York: Springer.
- Lerman, S. (2005). *Studying student teachers' voices and their beliefs and attitudes*. Paper presented at the 15<sup>th</sup> conference of the International Commission on Mathematics Instruction, Place, Brazil.
- Ronda, E. R. (2009). Growth points in students' developing understanding of function in equation form. *Mathematics Education Research Journal*, 21(1), 31-53.
- Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25-28). Washington, D.C.: Mathematical Association of America (MAA).
- Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification—The case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 59-84). Washington, D.C.: MAA.
- Slavit, D. (1997). An alternative route to the reification of function, *Educational Studies in Mathematics*, 33, 259-281.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *Int. J. Math. Educ. Sci. Technol.*,14, 293-305.

## Appendix: The Questionnaire

❶ Which of the following relations are function relations? Make the necessary corrections to the rest of them, in order to transform them into functions.

x	y
-1	0
0	1
1	2
2	3
3	4

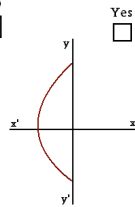
Yes ☐ No ☐

x	y
-3	2
-2	2
-1	2
0	2
1	2

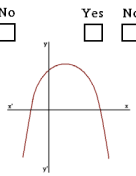
Yes ☐ No ☐

x	y
5	3
-3	2
5	1
0	1
-3	6

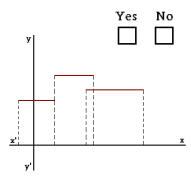
Yes ☐ No ☐



Yes ☐ No ☐



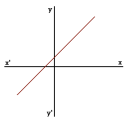
Yes ☐ No ☐



Yes ☐ No ☐

❷ Give two examples of arbitrary relations that are not functions (one described graphically and one analytically, with an algebraic formula).

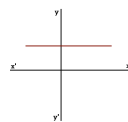
❸ What happens to the graphical representations of the following functions if the line on the graph is rotated by  $90^\circ$ ? Are they still functions? Give a short justification in your answer.



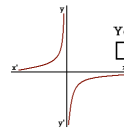
Yes ☐ No ☐



Yes ☐ No ☐



Yes ☐ No ☐



Yes ☐ No ☐

❹ In which situation(s) does a  $90^\circ$  turn of a function's graphical representation represents a function? Which general rule would you use?



# CURRICULUM AS A LEVER FOR TEACHER IMPROVEMENT

Mary Kay Stein<sup>1</sup>, Julia Kaufman<sup>2</sup> and Miray Tekkumru-Kisa<sup>3</sup>

<sup>1</sup> University of Pittsburgh, <sup>2</sup> Carnegie Mellon University, <sup>3</sup> University of Pittsburgh

*This study explores the relationship among three dimensions of curricular fidelity in the context of systemic change: teachers' use of curriculum materials, the congruence of their instructional practice to the suggested pedagogical methods in the curriculum (e.g., use of group work and manipulatives), and the quality of their instruction in terms its level of cognitive demand, attention to student thinking, and location of intellectual authority. Data associated with 13 teachers' implementation of curricula over a 3-year period of district-wide reform efforts, suggest that high levels of use and congruence are easier to attain than high-quality implementations. However, findings also suggest that mechanical use of the curricula may be a step along the pathway to quality instruction.*

## INTRODUCTION

Over the past two decades, reform in mathematics education has advocated conceptually driven, student-centered forms of instruction in which lessons are organized around cognitively challenging tasks that require students to think, reason, and problem solve (NCTM, 2000). *Curriculum* is often viewed as the key policy lever for improving instruction along these dimensions. For most countries, reliance on curriculum as a carrier of reform practices is instantiated in the existence of a national curriculum that embodies the reform practices; in the United States, despite the absence of a national curriculum, districts are increasingly mandating the use of a single curriculum that aligns with this vision for reform. However, curriculum alone has been shown to have limited influence on teachers' instructional practices (Ball & Cohen, 1996; Fullan, 1991). This is because curricula are not self-enacting and most teachers do not implement the curriculum exactly as prescribed. Typically, this problem is encapsulated as an issue of "curriculum fidelity."

In the reform implementation literature, curriculum fidelity has been defined along a variety of dimensions, including the extent to which teachers actually use the intended materials, the degree to which teachers follow the pedagogical suggestions offered by curriculum materials, and the extent to which teachers align their practice with the deep, theory-based structure of how students learn that is embedded in the materials (Stein Grover, & Henningsen, 1996; Tarr, Chavez, Reys, & Reys, 2006). To date, however, the literature has been silent on how these dimensions relate to one another and how they might develop—individually and together—over time.

The purpose of this study was to explore the relationship among teachers' *use* of curriculum materials, the *congruence* of their instructional practice to the suggested pedagogical methods in the curriculum (e.g., use of group work and manipulatives),

and the *quality* of their instruction in terms of the extent to which teachers execute lessons that maintained high levels of cognitive demand and supported students' development of thinking, reasoning, and problem solving capacities. By examining how these three dimensions relate to one another *at any given moment in time*, we aim to support the development of a more robust, multi-dimensional conceptualization of curriculum fidelity. By exploring how they develop *over time*, both individually and in relationship to one another, we hope to contribute to building a theory of developmental trajectories of teacher learning in the context of district efforts to improve teaching using curriculum as a lever.

We examine these issues by tracking a set of teachers over three years who participated in district-wide reform efforts that relied heavily on curriculum materials. Our study was guided by the following propositions:

1. *Most teachers will use the curriculum materials as their main source of lesson activities.* Both districts were embarking on highly publicized reform efforts that involved considerable effort to bring principals and teachers on board.
2. *Most teachers' practices will be congruent with the methods described in the materials.* In both districts, coaches were employed to help teachers learn new pedagogical strategies found in the materials. In addition, principals were empowered to act as instructional leaders, spending more time in classrooms to ensure that the new materials were being followed.
3. *Fewer teachers' practice will be of high quality.* Teaching that aims to help students develop deep understanding of big mathematical ideas by scaffolding students' emerging understandings as they work through challenging tasks is difficult for teachers to master (Ball, 1993; 2001; Leinhardt & Steele, 1998; Sherin, 2002).
4. *Teachers who use the curriculum and use it in a congruent way will not necessarily have high quality instructional practice.* Past research has shown that teachers can set up instructional tasks exactly as specified in the materials, yet fail to support high-level thinking and reasoning as students actually work on the task (Stein, Grover, & Henningsen, 1996). This is significant because it is not whether students are sitting in groups or using manipulatives that matters, rather it is what students are actually thinking about that determines their opportunities to learn.
5. *Teachers' practice will gradually become higher quality over the initial three years of curriculum-based reform, with those teachers who regularly use the curriculum in a congruent manner achieving quality sooner and more often than those who do not.* Here, we are suggesting that the process of trying to implement reform curriculum is, in and of itself, educative and that teachers may need to go through a period of "unenlightened" use in order to learn how to implement the more difficult aspects of reform lessons such as listening to and scaffolding student thinking.

## METHODS

Data for the present study come from a large multi-year study of the initial years of district-wide implementation of reform elementary mathematics curricula in two urban districts in the United States. In the Fall of 2006, Greene School District mandated implementation of *Investigations* and Region-Z mandated implementation

of *Everyday Mathematics* (both districts' names are pseudonyms). Six focal teachers in each of 4 case-study schools in each district were selected for observation. Schools were selected to represent the range of schools in each district with respect to teacher capacity and extent of teacher professional communities; teachers were selected to represent the range of talent and grade levels in the building. For this study we used 11 Greene teachers and 3 Region-Z teachers for whom we had 3 consecutive years of data over five school semesters (from spring of year 1 through spring of year 3).

Each teacher was observed for 3 consecutive lessons in each seasonal cycle (i.e., Fall or Spring) by trained observers who took fieldnotes and then completed detailed write ups that included both a narrative summary of the lesson and answers to a pre-defined set of questions. In addition to teacher observations, we have transcripts of teacher pre- and post-lesson interviews, observations of professional development at different levels of the system, and transcripts of interviews with principals, mathematics coaches and district leaders.

Our analysis focused on characterizing each of the 14 teachers according to use, congruence, and quality for each of five seasonal cycles: Year 1-Spring, Year 2-Fall, Year 2-Spring, Year 3-Fall, and Year 3-Spring. Each lesson write-up was coded by a trained mathematics educator according to *use* (according to the portion of the lesson that used the curriculum as the source of activities); *congruence* (a congruent/not-congruent judgment based on the mathematics educators' assessment of the lesson's alignment with a list of the curriculum's pedagogical features specifically constructed for each curriculum); and *quality* (based on judgments of the levels of cognitive demand in the materials and at the set-up and enactment phases of the lesson coupled with mathematics educators' judgment of where intellectual authority resided in the lesson and the extent to which the lesson built on student thinking; this coding system builds on earlier work by Stein et al, 1996). Inter-rater reliability was 81%, 67%, and 75% on each of the use, congruence, and quality scores respectively.

Next, the scores for each of the teacher's lessons were averaged across the three-lesson set that comprised each cycle to represent a year/semester score on each dimension. Finally, teachers were determined to be high or low users, congruent or non-congruent implementers, and high- or low-quality implementers based on cut scores that were conceptually determined. High users were defined as teachers with average semester use ratings that indicated that they used the curriculum – on average – more than 75% of the time. Highly congruent teachers were defined as teachers whose instructional practices were aligned with the pedagogical features of the particular curriculum they were using for at least 2 out of 3 lessons in the semester. High-quality teachers were defined as teachers who—on average— maintained a high level of cognitive demand of a task across lesson phases. In addition, to be considered high quality, teachers needed high ratings for either their work to uncover student thinking *or* how well they vested intellectual authority in mathematical reasoning versus the teacher/text.

FINDINGS

Use (Proposition #1).

Table 1 identifies the number of teachers whose instructional practice was judged to be high use, highly congruent, and high quality. The first line of each row identifies the number of teachers *across both districts* (n=14) who were high users, highly congruent, and high quality. The second and third lines of each row (the numbers in parentheses) specify the number of teachers at these levels for Region Z (n=3) and Greene (n= 11) respectively. Teachers’ use of the mandated curricula was relatively high throughout the first three years of implementation across both districts. As shown in the Year 1-Spring column, *all* teachers (n=14) were observed using the curricula for more than 75%, on average, during the first data collection cycle.

Table 1. Teachers with High Use, High Congruence and High Quality Per Semester

	Yr. 1 Spr.	Yr. 2 Fall	Yr. 2 Spr.	Yr. 3 Fall	Yr. 3 Spr.
High Use	14	12	11	11	10
Region Z (n=3)	(3)	(2)	(2)	(3)	(2)
Greene (n=11)	(11)	(10)	(9)	(8)	(8)
High Congruence	10	12	12	11	11
Region Z	(1)	(1)	(1)	(1)	(1)
Greene	(9)	(11)	(11)	(10)	(10)
High Quality	4	6	6	5	7
Region Z	(0)	(0)	(0)	(0)	(0)
Greene	(4)	(6)	(6)	(5)	(7)

Looking across the first row of Table 1, we see that high levels of use trailed off a little over time. The numbers in parentheses show that this happened especially in Greene where a change in superintendents between years 2 and 3 led to much less emphasis on mandated use of the curriculum.

Congruence (Proposition #2).

As predicted, congruence with the pedagogical features the curriculum was also fairly high throughout the 3-year period (see row 2 of Table 1). It is interesting to note that, despite the decline in use for some Greene teachers, these same teachers continued to adhere to the pedagogical features recommended by *Investigations* (hence the high congruence ratings), suggesting, perhaps, that they had internalized these features as part of their instructional practice

### Quality (Proposition #3).

High-quality was observed much less frequently with fewer than half of the lesson-sets being judged as high quality until the final data collection period in which exactly half were of high quality. Also, none of the teachers' practice in Region Z was ever judged to be high quality whereas by the end of the third year, 7 out of 11 teachers in Greene displayed high-quality instructional practice.

### Relationship Among Three Dimensions (Proposition #4).

The disparity between high levels of use and congruence—on the one hand—and more moderate levels of quality—on the other—suggests that some teachers were failing to achieve quality despite using the curriculum with regularity and aligning their practice to the pedagogical methods prescribed in the materials. Table 2 identifies *only* teachers who were judged to be high-use and high-congruence, but who differed in terms of quality.

Table 2. Teachers High in Use and Congruence but Different on Quality

	Yr. 1 Spr.	Yr. 2 Fall	Yr. 2 Spr.	Yr. 3 Fall	Yr. 3 Spr.
Low Quality	6	6	6	5	2
High Quality	4	5	5	3	6

As shown in Table 2, at any given time point (except Spring of the third year), the number of teachers who were faithfully using and following the curriculum yet whose lessons were judged to be of low quality was *higher* than the number who were faithfully using and following the curriculum and whose lessons were judged to be of high quality. Clearly, as predicted in proposition #4, high use and high congruence are *not* synonymous with high quality.

How might one characterize the high-use, high-congruence teachers who, nonetheless, have low-quality lessons (the first row of Table 2)? We propose the term **Mechanical** to signify that they have mastered the procedures of the curriculum, but have not internalized the intent in terms of the kind of student learning that the curriculum is trying to foster or the kind of teaching needed to support that learning.

How might one characterize the high-use, high-congruence teachers whose lessons *are* considered to be high level (the second row of Table 2)? Calling these teachers **Canonical** would call attention to all 3 dimensions of fidelity: (a) their commitment to using the curriculum; (b) their commitment to using it in a manner congruent with its pedagogical features; and (c) their ability to faithfully execute high-quality lessons that are aligned to the underlying intentions of the materials.

What about the high-quality teachers whose practice either displays low use or low congruence? In years 2 and 3 there were 5 times in which a teachers' practice was judged to be high quality but low in use of the curriculum materials. We propose the

term, **Maverick**, to refer to individuals whose lessons of high-quality yet do not rely—in any substantial way—on the curriculum materials as the source of activity in their classroom.

Finally, we propose the term **Flounderer** for teachers whose practice was judged to be low-quality and who either displayed low use or low congruence to the curriculum. The term is meant to signify that the teacher’s practice veers from what the district is encouraging and supporting, while, at the same time being low-quality. This suggests that the teacher is not on any kind of a trajectory toward improvement.

**Change over time (Proposition #5).**

How stable might these profiles of implementation be over time? And is there any discernible pathway through which teachers progress? Table 3 identifies the profile that each of the 14 teachers exhibited across the 5 data collection points. The first three teachers are from Region Z; the remainder are from Greene. The numbers following each teacher’s initials are their school code.

Table 3. Change Over Time in Teachers’ Profiles

	Yr. 1 Spr.	Yr. 2 Fall	Yr. 2 Spr.	Yr. 3 Fall	Yr. 3 Spr.
BT-1-Z	Flounderer	Flounderer	Flounderer	Flounderer	Flounderer
OG-2-Z	Mechanical	Flounderer	Mechanical	Mechanical	Mechanical
EB-3-Z	Flounderer	Flounderer	Flounderer	Flounderer	Flounderer
KE-5-G	Mechanical	Canonical	Mechanical	Canonical	Canonical
SN-5-G	Mechanical	Mechanical	Canonical	Mechanical	Flounderer
DT-6-G	Mechanical	Maverick	Mechanical	Mechanical	Flounderer
LH-6-G	Canonical	Canonical	Mechanical	Maverick	Canonical
NQ-6-G	Canonical	Canonical	Mechanical	Canonical	Maverick
XN-6-G	Mechanical	Canonical	Mechanical	Canonical	Canonical
FT-7-G	Canonical	Mechanical	Canonical	Flounderer	Canonical
LH-7-G	Canonical	Mechanical	Canonical	Mechanical	Flounderer
TS-7-G	Mechanical	Mechanical	Canonical	Flounderer	Canonical
WH-7-G	Flounderer	Mechanical	Canonical	Mechanical	Mechanical
DN-8-G	Flounderer	Canonical	Maverick	Maverick	Canonical

The most stable pattern evidenced by the table is Flounderer with both BT and EB falling into that category continuously throughout the 3 years. Second, given the high number of Mechanical implementations, use and congruence are clearly not synonymous with quality, but *they may be part of a developmental pathway toward*

*quality*. Consider the teachers showing sustained high quality implementations in year 3. Both KE and XN, for example, ended up as sustained Canonicals but both also had two semesters of mechanical implementations, including the first data collection point. Similarly, NQ and LH-6—both sustained high-quality teachers in year 3—had been Mechanical implementers at one point. (Although both were Canonical at our first data collection point, Year 1-Spring was actually the end of the first year of implementation; a fall data collection point may very well have shown both teachers beginning as Mechanical implementers.) Finally, Maverick implementations appear to be more productive later in the implementation cycle than earlier. DT's high-quality practice (Year 2-Fall) was not sustained, whereas the Maverick/Canonical pairings that occurred later in the implementation cycle (LH-6, NQ and DN) appear to be associated with sustained high-quality practice.

## CONCLUSIONS

Good curricular materials can be a strong ally in teacher learning of reform practices. However, teachers, coaches, leaders and researchers need to be able to differentiate ways in which teachers engage with those materials. This study shows that, while regular use and consistent congruence with pedagogical features of materials may be relative easy to attain, adherence to epistemological features of the materials (i.e., their intentions regarding how students learn and how teachers can support that learning) is much more difficult to enact. Moreover, we suspect that use and congruence are easier for leaders and coaches to observe and comment on, thereby possibly leading to an over-emphasis on these aspects during professional development and teacher evaluation.

This study also suggests, however, that use and congruence may be “on the road” to high quality. If this finding holds up, further longitudinal work would be warranted with respect to how Mechanicals can be supported to become Canonicals implementers. For example, arranging for Mechanical implementers to observe and work with Canonical implementers (who themselves may have recently emerged from a mechanical implementation) could help the Mechanical implementers envision “next steps” that, while still relying on curricular materials, represent greater attention to student thinking and keeping the cognitive demand of the task at a high level. Sending Mechanical implementers to observe Mavericks, on the other hand, may be a “bridge too far.” More research on the role of Mavericks within systemic reform efforts may also be warranted. These individuals may have “outgrown” the curricular materials; they also may represent a rich source of expertise and energy for system-level discussions of how to improve the curricular program overall.

Finally, we need to delve into the school and district contexts that shape teacher learning in order to understand how and why teachers travel these different trajectories. In particular, the sharp differences between Region Z and Greene suggest that there may be systematic differences in teacher support or in the curriculum materials themselves. At the school level, school #6 appears worthy of more study as

12 out of 20 of the data collection cycles were coded as high-quality implementations. Using data from the larger study and previous project publications, we will produce contextualized portraits of different teacher trajectories in our conference presentation.

## REFERENCES

- Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, 94(4), 373–397.
- Ball, D. L. (2001). Teaching, with respect to mathematics and students. In T. Wood, B. S. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 11–22). Mahwah, NJ: Erlbaum.
- Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is--or might be--the role of curriculum materials in teacher learning and instructional reform. *Educational Researcher*, 25 (9), 6–8.
- Fullan, M. (1991). *The new meaning of educational change* (2nd ed.). New York: Teachers College Press.
- Leinhardt, G., & Steele, M. D. (2005). Seeing the complexity of standing to the side: Instructional dialogues. *Cognition and Instruction*, 23(1), 87–163.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction*, 20(2), 119–150.
- Stein, M.K., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classroom. *American Educational Research Journal*, 33(2), 455–488.
- Tarr, J. E., Chavez, O., Reys, R. E., & Reys, B. J. (2006). From the written to enacted curricula: The intermediary role of middle school mathematics teachers in shaping students' opportunities to learn. *School Science and Mathematics*, 106(4), 191–201.



# CONSIDERING STUDENT EXPERIENCE AND KNOWLEDGE OF CONTEXT IN PLANNING MATHEMATICS LEARNING

Peter Sullivan  
Monash University

Peter Grootenboer  
Griffith University

Robyn Jorgensen  
Griffith University

*Abstract: In this study, 42 Australian Indigenous children were posed a series of related questions on addition and subtraction. The questions were posed both in the context of money and without any context. The data revealed an alignment between the students' capacity to work with money and to work with numbers. The money context did not increase the cognitive load or task complexity suggesting that relevant contexts can be used by teachers to support mathematics learning.*

## THE CONTEXT OF THE RESEARCH

The following is a discussion of some data collected at two Indigenous Australian Community Schools in a remote region of Western Australia, as part of the *Maths in the Kimberley* research project<sup>1</sup>. The project investigated ways of supporting the teaching of mathematics in small community-run schools, motivated by general concerns about the achievement of students in such schools.

All recent reports that synthesise results from large scale assessments of learning in Australian schools include analysis and discussion of the extent to which Indigenous Australian students are outperformed by their non Indigenous peers. Thompson, de Bortoli, Nicholas, Hillman, and Buckley (2010), for example, reported that Indigenous students were almost two full years of schooling behind non-Indigenous students in mathematical literacy. The disparity in achievement is a direct result of mismatches between the goals and processes of schools, and the experience and aspirations of Indigenous students (see Jorgensen & Sullivan, 2010).

While recognising that there are competing view on this, we agree with Hughes (2010) that the pathway to modernisation for Indigenous communities and the creation of opportunities for Indigenous students is through the learning of the conventional mathematics content, rather than through modified curricula, and we explored issues associated with the teaching of that mathematics. The project design recognised the complexity of the educational challenges in such small communities, acknowledged those who have addressed these issues previously, and emphasised collaboration with the respective communities at each stage.

## Building on the experience of Indigenous Australian students

The project recognised that one of the key challenges for teachers in remote schools is connecting the mathematical concepts they are teaching to the experience of the

---

<sup>1</sup> The Maths in the Kimberley project was funded by the Australian Research Council, led by Robyn Jorgensen, and conducted on the invitation of the Association of Independent Schools of Western Australia.

students. Of course, connection to experience is a consistent theme in advice to all teachers. Hattie and Timperley (2007), for example, reviewed a large range of studies on the characteristics of effective classrooms. They found that feedback was one of the main influences on student achievement, and the key elements were “where am I going?”, “how am I going?”, and “where am I going to next?”. To provide this sort of feedback, it is clear that teachers need to have an understanding of what mathematics the students know and can do. Similarly, Tzur (2008) argued that instruction should begin with what the students already know and are confident with, and then move to content that is unfamiliar, rather than what he claims is the common approach of starting with unfamiliar content. The clear implication is that it is best if instructional decisions are informed by, and grounded in, what the students already know and experiences with which they are familiar.

This connection with experience also forms a consistent theme in recommendations for teaching Indigenous students. Stanton (1994), for example, argued that the curriculum and pedagogies for Indigenous students could incorporate “both ways” or “common ground” approaches, and that this applies to school policy, management, structure, curriculum and pedagogy. In the case of mathematics, he suggested that the curriculum should: be negotiated; build on aspects of traditional culture; incorporate technology; recognize the interfaces with language; and utilize contexts. Similarly, Frigo, Corrigan, Adams, Hughes, Stephens, and Woods (2003), in reporting a study of schools with high proportions of Indigenous students, listed among key elements of effective numeracy teaching as presenting skills in real life contexts, and building on what the students know. The theme of understanding the students’ mathematical knowledge, capabilities and identities, especially in contexts where the teachers and students are from different cultures, is a prominent and recurring one.

It seems that both traditionally, as reflected in the language, and currently, in terms of the limited use of quantities in their everyday lives, number activities tend to be remote from Indigenous students’ experience. To overcome this, we sought to explore ways of building connections between money, with which the students have some familiarity and more abstract number ideas. This generally involves making connections to realistic contexts with which the students are familiar.

This creates its own challenges. Bransford, Brown, and Cocking (1999) argued that real-life contexts can be confusing and increase cognitive load. Cooper and Dunne (1998) found that contextualising mathematics tasks created particular difficulties for low socio-economic status (SES) students. Likewise, Lubienski (2000) found that pupils who preferred the contextualised trial materials and found them easier all had high SES backgrounds, while most pupils who preferred closed, context free tasks were low SES. In fact, many of the low SES pupils claimed to be worse off with contextualised problems, and none found the contextualised materials easier.

The challenge for teachers is to find ways to build on students’ experience without increasing the complexity of the formulation of tasks, and the cognitive load that

multi-step problems create. As highlighted previously, this means using contexts that are familiar and inviting to the Indigenous learners. This study was an attempt to examine what this might mean in project classrooms, and to explore the potential of using money to establish a the basis for important generalisable number ideas.

### **The context of the data collection**

The focus of the data collection reported below was prompted by classroom observations of lessons in which the teacher was seeking to develop generalisable number concepts such as partitioning and additive thinking. The teacher had earlier noticed that the students were adept with using money at the school fete, more so than seemed evident in their classroom mathematics. In the course of developing some learning experiences associated with partitioning, some classroom activities were presented that required the students to perceptually recognise amounts of money (see Sullivan, Youdale, & Jorgensen (2009) for a detailed report of the lesson observations). *Inter alia*, the observer noted students who were able to recognise immediately and accurately amounts made up of coins in various denominations, apparently using what Sousa (2008) described as perceptual recognition. This created the impetus for the data collection reported here.

The focus of the instruction was on ways of partitioning amounts such as \$1 (e.g.,  $70c + 30c$ ,  $60c + 40c$ ), and other amounts and how this could be used for operations such as combining particular money amounts (e.g.,  $80c + 50c$  is the same as  $\$1 + 30c$ , etc).

In order to explore this further, the research questions were:

- Are the students fluent with recognising money amounts, and if so, what amounts can they perceptually recognise?
- What is the relationship between fluency with recognising money amounts and items involving equivalent addition tasks posed using only numbers?
- What is the relationship between the coins and addition items, and equivalent subtraction items both involving coins and numbers?

The framework used for the creation of the particular items was that used by the *Early Numeracy Research Project* (ENRP) (Clarke et al., 2001) and consists of sets of growth points that describe sequential development of concepts, with four number domains. We drew on the addition and subtraction section of the framework, the relevant growth points of which were:

2. Counts on from one number to find the total of two collections.
3. Given a subtraction situation, chooses appropriately from strategies including count back, count down to and count up from
4. Given an addition or subtraction problem, strategies such as doubles, commutativity, adding 10, tens facts, and other known facts are evident
5. Given an addition or subtraction problem, strategies such as near doubles, adding 9, build to next ten, fact families and intuitive strategies are evident

This hierarchical framework has been shown to be robust and to represent growth of learning in students. Students in our project schools had been earlier interviewed based on this framework, and most students in the middle primary years were able to answers questions such as: ‘I have 8 biscuits, and I eat 3. How many do I have left?’ and; ‘What is  $10 - 7$ ?’ Nearly all of the students interviewed were able to: count by 10s past 100; count by 5s to 90; calculate  $9 + 4$ , where the 9 objects were covered, requiring counting; state the answer to  $2 + 19$ . At the same time, few students were about the answer the question “I have 12 strawberries, and eat 9. How many do I have left?”

We used these results as the basis of a task based interview, and sought to extend the complexity of the tasks from there. We considered an interview as the most appropriate way to gain insights into what the students could do. This was partly so a student could answer without concern for what other class members might say, and partly so that the researcher could monitor the progress of the student and adapt the protocol accordingly. The interviews were conducted in a quiet place separate from the rest of the class (e.g., in a separate room) so the student could focus on the tasks and questions at hand. The students were familiar with the researchers because we had spent some time in the classes before the interviews were undertaken, and during the interviews the researchers endeavoured to make the students feel comfortable, valued and relaxed so they could answer freely and honestly.

An interview protocol was developed that posed 10 sets of items, with four items in each set (see Figure 1). For example, the first set of items was as follows:

*Recognition of the value of a collection of coins:* Two 20c pieces were covered then shown for 2 seconds (“and 1 and 2”) and then covered. The students were asked “how much money is under my hand?”

*Number addition:* Students were shown a card on which was written “ $20 + 20$ ” and the question “what is 20 plus 20?” was posed.

*Calculation of difference using money:* The researcher put out two 20c pieces on the table without the students seeing them, and uncovered one 20c piece. The question was posed “There is 40c on the table altogether. How much is still covered by my hand?”

*Calculation of difference using number:* Students were shown a card on which was written “ $40 - ? = 20$ ” and the question “what number do I take from 40 to get 20?” was posed. If this was not clear the first time, the question was asked another way.

Depending on the level of response, these items were addressing the growth point levels 2, 3 and 4 described above. In posing the items, the first item in each set was presented first. In other words, all the coins recognition items were posed in sequence, and then the researcher went back to pose the number addition items in sequence, and so on.

The particular number and money amounts from the respective sets of items are shown in Figure 1:

Item	Perceptual recognition of collection of coins	Number Addition	Difference using money	Number Difference
1	20c, 20c	20+20	40c - 20c	20 + ? = 40
2	20c, 20c, 10c	20+20+10	50c - 10c	60 + ? = 50
3	50c, 10c	50+10	60c - 10c	70 + ? = 60
4	50c, 20c, 10c	50+20+10	80c - 20c	60 + ? = 80
5	50c, 20c, 20c, 10c	50+20+20+10	\$1 - 20c	80 + ? = 100
6	50c, 20c, 10c, 10c, 5c, 5c	50+20+10+10+5+5	\$1 - 10c	90 + ? = 100
7	50c, 20c, 20c, 20c	50+20+20+20	\$1.10 - 20c	90 + ? = 110
8	\$2, \$2, \$2	2+2+2	\$6 - \$2	4 + ? = 6
9	\$2, \$2, \$2, \$1	2+2+2+1	\$7 - \$1	6 + ? = 7
10	\$2, \$2, \$2, \$2, \$1, \$1	2+2+2+2+1+1	\$10 - \$2	8 + ? = 10

Figure 1: The ten sets of four related items.

The items with cents became progressively more difficult, as did the items using dollar amounts. Once a student experienced difficulty with a sequence of items, we jumped to the start of the next sequence (i.e., item 8). For example, when a student was having difficulty with an item on the value of coins involving cents we would skip forward to the items involving \$1 and \$2 coins.

## Results

The following is a selection from the responses to the interview questions. The number of students who correctly recognised the value of the coins, when the coins were shown for 2 seconds then covered, is shown in Table 1 below. Note that once a student experienced difficulty with a sequence of items, we jumped to the start of the next sequence.

Amount in coins	Number of students correct
20c, 20c	31
20c, 20c, 10c	28
50c, 10c,	31
50c, 20c, 10c	25
50c, 20c, 20c, 10c	25
50c, 20c, 10c, 10c, 5c, 5c	11
50c, 20c, 20c, 20c	5
\$2, \$2, \$2	31
\$2, \$2, \$2, \$1	32
\$2, \$2, \$2, \$2, \$1, \$1	27

Table 1: Number of students who correctly recognised amounts of coins (n = 42)

Around three quarters of the students could do the easier items, and most of them could also do the items of medium difficulty as well. Noting that the coins were only shown briefly, this suggested that many of the students were able to recognise the money readily. In other words, many were not forming a mental image and then

adding the value of the coins, but seemed to be using some other process. There were students, including some of the more successful students, who did form such mental images nevertheless.

Overall, the range of results was broad with some young students getting none of the items correct, and a few students (from grades 4 to 6) getting the majority of them correct. There was about one quarter of the students who could not say the money amounts even with the easier items. The diversity in the responses is discussed below.

To explore the nature of the responses and to allow comparison across the sets of items, the responses were compared by grade level for the equivalent items for item 4 in Figure 1 involving 50c, 20c, 10c (see Table 2 below). This was the fourth item in each set and is illustrative of the other results since the profile of responses is similar.

Grade	n	Coin recognition	Number addition	Money subtraction	Number subtraction
3	18	7	7	6	6
4	9	6	7	6	2
5	13	10	10	9	5
6	2	2	2	1	1
Total	42	25	26	22	14

Table 2: Correct responses to the items involving 50c, 20c, 10c

It appears that the overall facilities of the coin recognition, number addition, and money subtraction items were similar. Given that they were only shown the coin recognition task briefly it confirms that many students have an interesting fluency with recognition of the money amounts. The majority of the grade 4, 5 and 6 students were able to do the coin recognition for this combination, but only a minority of the grade 3s. This suggests that the questions were pitched at the appropriate level to probe these students’ capacity at the items. It can be inferred that students who completed all of these tasks would be at growth point 4 or 5 in the framework for addition and subtraction presented earlier.

Most of the students who were not able to recognise the amount of these coins were in grade 3. This suggests that the collective experience of the students influences their capacity. It seems that, whatever experiences are needed for recognising coin amounts, most students had had those experiences by the time they were in grade 4.

Our hypothesis, based on classroom observations, was that the students would be better with recognising coin amounts than with straight calculations, and that we would use this facility in designing instructional programs. This was not the case for equivalent pairs of items involving addition.

To illustrate the extent to which the responses to one are dependent on the response to the related item, the numbers in each of the four possible combinations of responses to the equivalent addition items are presented in Table 3.

	Coin recognition correct	Coin recognition incorrect
Number addition correct	24	1
Number addition incorrect	2	5

Table 3: Comparison of student responses of the pairs of addition items connected to 50c, 20c, 10c

This table shows that 29 out of the 32 students that attempted both were either correct or incorrect on both, and only 3 (less than 10%) were correct on only one. For the equivalent items in this case, it seems that the capacity to add 50, 20 and 10, is connected to the ability to perceptually recognise this amount in coins. In fact, the cross tabulation for seven sets of items (items 1, 2, 3, 4, 5, 8 and 9) is similar.

Across the ten items, the facility on the items involving the *difference* between coin amounts was also very close to the coin recognition items, and therefore, to the number addition items.

### Conclusion

In terms of the research questions, it seems that many of these Indigenous Australian primary students were able to state the amount of collections of coins quickly and readily. This was compatible with our classroom observations. It also seems that nearly all students who could do this were able to respond to number addition questions using comparable numbers. While it makes sense that this should happen, it seems that the students were more fluent with the number calculation tasks than seemed evident from observations of their classroom responses. It is possible that the interview environment is more conducive to concentration.

An inference overall is that the facility with the straight number tasks is connected directly to the ability to perceptually recognise the money amounts, although it is not clear which skill informs the other.

This fluent recognition of collections of coins seems a strength of these students, and teachers could explore the extent to which such strengths could be used to introduce students to addition tasks, for promoting confidence in solving contextual problems, for challenging those students who are ready, and for allowing students to recognise the advantages of developing number fluency. In other words, the context with which these students are familiar can be used to enrich their learning of mathematics.

Part of the challenge of teaching mathematics to students whose experience with modern mathematical ideas is limited due to culture and remoteness is that it is difficult to identify contexts that are both familiar to the students and which also exemplify key mathematical ideas. Connected to this is that some contexts increase the cognitive load on the students making learning more difficult. This research suggests that thoughtfully selected contexts can be used effectively to engage students in learning mathematics.

## References

- Bransford, J. B., Brown, A. L., & Cocking, R. R. (Eds.) (1999). *How people learn: Brain, mind, experience, and school*. London: Committee on Developments in the Science of Learning, National Research Council.
- Clarke, D., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P., Roche, A., Sullivan, P., Clarke, B., & Rowley, G. (2002). Early Numeracy Research Project (ENRP): Final report. Retrieved September 11, 2010 from: <http://www.sofweb.vic.edu.au/eyes/num/enrp.htm>.
- Cooper, B., & M. Dunne (1998). Anyone for tennis? Social class differences in children's responses to national curriculum mathematics testing. *The Sociological Review*, (Jan), 115–148.
- Frigo, T., Corrigan, M., Adams, I., Hughes, C., Stephens, M., & Woods, D. (2003). *Supporting English literacy and numeracy learning for indigenous students in the early years*. ACER Monograph 57: Camberwell.
- Hattie, J., & Timperley, H. (2007) The power of feedback. *Review of Educational Research*, 77 (1), 81-112.
- Hughes, P. (2010). Keynote address at Dare to Lead conference, Yulara NT. April.
- Jorgensen, R., & Sullivan, P. (2010). Scholastic heritage and success in school mathematics: Implications for remote Aboriginal learners. In I. Synder & J. Nieuwenhuysen (Eds). *Closing the Gap: Improving outcomes in Southern World Societies*. Clayton, Melbourne; Monash University Press.
- Lubienski, S. T. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class lens. *Journal for Research in Mathematics Education*, 31, 454–482.
- Sousa, D. (2008). *How the brain learns mathematics*. California: Corwin Press.
- Stanton, R. (1994). Mathematics “both ways”: A mathematics curriculum for aboriginal teacher education students. *For the Learning of Mathematics*, 14(3), 15-23.
- Sullivan, P., Youdale, R., & Jorgensen, R. L. (2009). The link between planning and teaching mathematics: An exploration in an indigenous community school In C. Hurst, M. Kemp, B. Kissane, L. Sparrow, & T. Spencer (Eds.), *Proceedings of the 22nd Biennial Conference of the Australian Association of Mathematics Teachers Inc.* (pp. 247-255). Adelaide: The Australian Association of Mathematics Teachers.
- Thompson, S., de Bortoli, L., Nicholas, M., Hillman, K., & Buckley, S. (2010). Challenges for Australian education: Results from PISA 2009. Melbourne: Australian Council of Educational Research.
- Tzur, R. (2008). Profound awareness of the learning paradox. In B. Jaworski & T. Wood (Eds). *The Mathematics Teacher Educator as a developing professional* (pp. 137-156).



# HOW DID THE INDEFINITE INTEGRAL FUNCTION BECOME AN ACCUMULATION FUNCTION?

Osama Swidan  
University of Haifa

*This case study was designed to analyze the processes of evolution, the personal meanings that arise through the interaction with artifact to get the mathematical meaning of indefinite integral. The study focuses on a 25 year old graduate student of science. This study is guided by the socio-cultural theory which considers artifacts of any kind to be fundamental in processes of cognition. Through a task-based interview he was asked to explain possible links between both graphs while using computer applications. Data analysis reveals three elaborations of the meaning of the indefinite integral graph: (1) a zero function correlates to zero indefinite integral function; (2) a constant function correlates to linear indefinite integral function; (3) interpretation of the vertical transformation of the indefinite integral function through the accumulation concept.*

## INTRODUCTION

The integral concept includes two ideas: the Riemann sums as defined by the limit of sums of products which are called the definite integral, and the integral defined as *anti-derivative* which is called the *indefinite integral*. The fundamental theory of calculus connects between both ideas and as a result connects between the derivative and the integral concept. Thus, in order to perform meaningful learning of calculus and especially the integral concept, awareness of these connections must be developed (Thompson, 1994).

In a previous study published and presented in the PME 33 conference, (Swidan & Yerushalmy, 2009) introduced a learning experiment regarding acquisition of the integral concept as related to an accumulation function. That learning experiment included two differently designed artifacts. Both artifacts include mathematical facts accepted by the mathematical community. The designed calculus integral sketcher (CIS) (Shternberg, Yerushalmy & Zilber, 2004) relies on the iconic view which emphasizes the link between the function graph ( $f(x)$ ) and the anti-derivative graph ( $AD(x)$ ). It allows the construction of graphs of functions and manipulation directly on the graph by dragging. The second designed artifact relies on the calculus UnLimited software integral tool CUL (Schwartz & Yerushalmy, 1996). The CUL designed artifact is considered to be a dynamic and multi – representational artifact which emphasizes graphic and symbolic views, concentrating on the representation of rectangles. Swidan & Yerushalmy (2009) identified four foci made by two secondary students, toward clarifying the connection between the function graph and the accumulation graph, when learned by the artifacts mentioned above. They revealed that the personal meaning which evolves through the use of the CIS artifact

was limited and restricted to the attention stage of exposing elements in both graphs. On the other side, their personal meaning developed into a mathematical meaning when they used the CUL artifact and its component. The restriction of the CIS to the primary stages of learning by secondary level students motivates the emergence of this study. Through it I intend to better understand the role of the iconic tools (CIS). Therefore, I decided to explore how one who already knows the symbolic familiar calculus, deals and analyzes a specific calculus topic according to an iconic view. My assumption was that a graduate student who was familiar with the mathematical content can highlight our insight to the processes of the evolution of personal meaning to acquiring mathematical meaning that relates to the links between both graphs.

### **Artifacts, signs and meanings**

According to the socio cultural theory of learning that guided the current case study; artifacts of any kind are central and play a fundamental role in cognition (Falcade Laborde & Mariotti, 2007). Vygotsky (1978) points out that the use of an artifact in learning processes reaches achievements that might otherwise remain unreached. Generally speaking, artifacts and their components are considered to be outward oriented tools, while mental activity supports and evolves by means of signs which are the product of the internalization process, called psychological tools, which are inward oriented. The outward oriented tools may transform into inward tools through social interaction. Bartolini Bussi & Mariotti (2008) claim that, within the social use of an artifact to accomplish a task, shared signs, which relate to the artifact, are produced and may be related to the content intended to be mediated. The relation between artifact and knowledge is expressed by signs, culturally determined. The relation between the artifact and accomplishing a task is expressed by signs such as gestures, speech and drawing. Signs in general and mathematical signs in particular play two roles. Radford, Bardini, Sabena, Diallo, & Simbagoye (2005) define these roles as "social objects in that they are bearers of culturally objective facts in the world that transcend the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions" (ibid, 117). Mariotti & Bussi (2008) distinguish between personal meaning and mathematical meaning. Thus, through the interaction with the artifact, the subject develops a personal meaning to the culturally determined signs. These meanings may evolve into meanings that an expert would determine as mathematical. The CIS is considered to be an artifact which allows perceptual interaction. The linked graph in the artifact (Figure 1) is a culturally determined sign and refers to the link between function and indefinite integral concept, which is defined as anti-derivative. Due to the design of the artifact, I was able to determine the link between the graphs, as a function and anti-derivative because: 1) dragging the  $AD(x)$  vertically doesn't change the location of  $f(x)$ . 2) The derivative of  $AD(x)$  gets the function  $f(x)$ . The icons and the direct dragging are outward tools. Personal meanings will arise from the interaction with the artifact by the outward tools and contribute to the evolution of mathematical

meanings which I will then analyze. I will do this by considering all signs that appear in the social interaction between the graduate student, the culturally designed artifact and the goal of accomplishing a task.

## METHOD

The artifact used in the task-based interview (Goldin, 2000) was 'The Calculus Integral Sketcher (CIS)', (Shternberg, Yerushalmy, & Zilber, 2004). Sketching a function in the CIS is carried out by choosing an icon from the icons applet, placing it on the lower Cartesian system, and manipulating it by dragging the lines or its end points. The graphs in the intervals are drawn continuously and sequentially from left to right. The sketch in Figure 1 shows a line graph drawn by using a single linear icon, and its anti-derivative graph ( $AD(x)$ ) below is drawn by the CIS. The tool allows dragging of the axes and the function graph freely as the anti-derivative graph changes accordingly. The CIS only allow dragging ( $AD(x)$ ) vertically and in this case, the function graph ( $f(x)$ ) is still stable in the upper Cartesian system. Each transformation of the function's graph ( $f(x)$ ) creates a "new" function, and the anti-derivative function ( $AD(x)$ ) is redrawn accordingly to obtain the value of zero as its leftmost value. Both graphs are linked dynamically according to culturally accepted meaning.

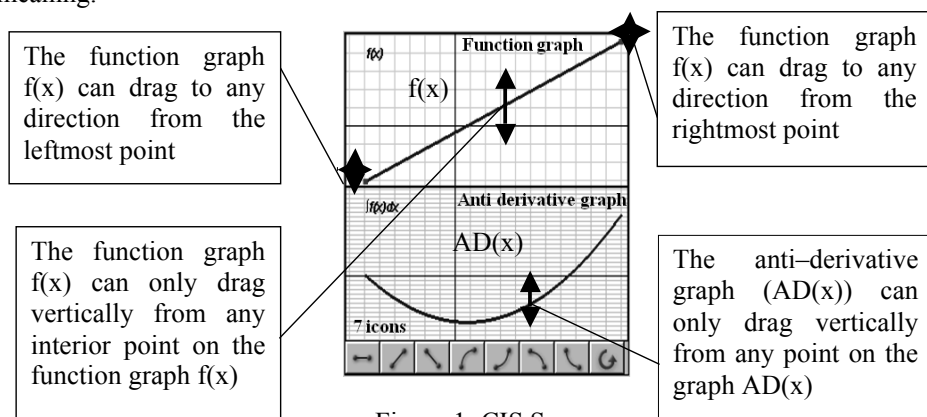


Figure 1: CIS Screen

This case study analyzes the development and evolution of personal meaning to mathematical meaning for a M.Sc. graduate student of science while accomplishing a task with a CIS artifact. The student, Amro, is 25 years old, he employed with calculus topics such as derivative and integration. Amro received a short introduction on the use of the Calculus Sketcher Integral tool (Figure 1) (Shternberg, Yerushalmy & Zilber, 2004). The interviewer (Osama Swidan, who is also the author) asked Amro to accomplish a specific task. The task presented to him was: "You need to create graphs with the 7 icons in the CIS artifact in the upper Cartesian system. The graphs that appear in the upper and the lower Cartesian system are linked. You are asked to explore and explain possible connections between the two graphs. you are

free to change the graphs at any time and also free to drag the graphs if need.” the task-based interview with Amro was video-recorded and captured his computer screens. As the interviewer, Swidan asked clarification questions about matters that arose during the interaction with the artifact. In this paper, I demonstrate the semiotic activity used by Amro in the course of the interaction with an artifact which links the function graph ( $f(x)$ ) and anti-derivative graph ( $AD(x)$ ) dynamically. I adopt the personal and mathematical meaning mentioned in Bartolini Bussi & Mariotti (2008) to analyze the data and to follow the evolution of the personal meaning to be conjecture of the mathematical meaning, in order to answer the question: whether and how the personal meanings of the graduate student that emerge from accomplishing a task with CIS artifact take on mathematical significance for him?

### DRAGGING THE FUNCTION GRAPH

Amro began the interview by interacting with the artifact. He produced a single graph of the function  $f(x) = \text{constant}$  in the upper Cartesian system and as a result the artifact produced an anti-derivative graph in the form  $AD(x) = ax$  in the lower Cartesian system (Figure 2). At one point, Amro horizontally dragged  $f(x)$  from the most right point (from A to A' in figure 2) and the anti-derivative  $AD(x)$  changed accordingly from B to B'; another time he dragged  $f(x)$  vertically. Immediately, after accomplishing the dragging of the  $f(x)$  he got the correlation between the two graphs that

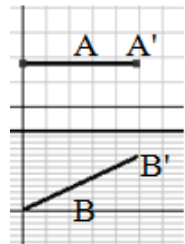


Figure 2

appears on the screen. Amro found that the correlation is a, function and its accumulated function. He dragged the graph in the upper Cartesian system vertically (up and down). Amro utilizes the direct dragging which the artifact allows and thereby drags the upper graph slowly. While he is dragging the upper graph toward x-axis he notices that the graph in the lower Cartesian system becomes equal to zero (overlapping x-axis) simultaneously when the upper graph is becoming zero.

Amro continues to drag the constant function in the upper Cartesian system whose form is  $y = a$  ( $a$  is constant) and dragged it vertically (Figure 3)

- 1 Amro: First of all, I start my exploration by choosing a special case of function  $y = \text{constant}$  [at the same time he was dragging the graph in the upper system vertically]. I am trying to find the correlation between both graphs [he indicated to upper graph and then to the lower graph]. I notice that when I am progressing in the upper function....

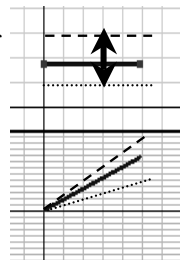


Figure 3

- 2 Osama: What do you mean by progressing?
- 3 Amro: Toward the positive side of x-axis [he indicates with the mouse along the x-axes] I see that the function is

increasing [he make gesture like going around the graph in the lower Cartesian with the computer mouse]. If the constant is decreasing [he drags the graph in the upper Cartesian system toward the x-axes] I see that the graph's slope is also decreasing, but the correlation between both graphs stays linear.

4 Osama: What is your conclusion therefore?

5 Amro: There is something accumulating [he goes around the graph in the upper Cartesian system and drags it down]. This graph is an accumulation function. [He indicates to the graph in the lower Cartesian system].

Amro searches for a correlation between both functions. He drags the constant function graph ( $f(x)$ ) vertically up and down [1]. After this, he concentrates on the lower Cartesian system and indicates on the x-axis of  $AD(x)$  toward right from the origin and concludes that the  $AD(x)$  is increasing. Through the indication action on the x-axis and his utterance "is increasing" [3], Amro follows the behavior of the  $AD(x)$  graphs. After performing the vertical dragging and the indication on the x-axis he explores stable correlations between both graphs; a constant function correlates to the linear  $AD(x)$  function and, changing the value of the constant function correspondingly changes the slope of the  $AD(x)$  linear function [3]. It seems that both the graphs can be considered as the roots of his determination. Amro dragged the graph function vertically and produced a family of graphs. He noticed that these graphs all have similar characteristics. That is, a constant function of the upper graph and a linear function of the lower graph. But he also sees the differences in the family of graphs, that is, the constant's value decreasing causes the slopes' reducing in the lower graph [3].

### **DRAGGING THE ANTI-DERIVATIVE $AD(x)$ GRAPH VERTICALLY**

Over the course of the experiment with the artifact, Amro produced an additional signal graph. This time he produced a linear graph in the upper Cartesian system, forming  $y=ax$  (Figure 4). Initially he dragged the function graph freely, and then after he dragged the anti-derivative  $AD(x)$  graph vertically (this is the only option for dragging this graph in this artifact) and he was surprised that the function graph ( $f(x)$ ) doesn't change its location.

6 Osama: What are you thinking about? [Amro is dragging the anti-derivative graph (Figure 4)]

- 7 Amro: Why the upper graph doesn't move when I drag it up and down [he is still dragging the anti-derivative graph]. [Silence] if the initial point in the upper graph locates the origin, this point [indicates to the leftmost point in the anti-derivative graph] isn't the origin. It locates a point different than zero on the y-axis. It is unacceptable for me.



Figure 4

- 8 Osama: Why, it isn't unacceptable?
- 9 Amro: Because my conclusion was that the lower graph represents something accumulating. When the leftmost point in the upper graph is zero, it means I accumulate nothing. But here the lower graph has value other than zero in the leftmost. It isn't logical.

While Amro dragged the  $AD(x)$  graph, he noticed that the function graph ( $f(x)$ ) still fixed. He expected that the leftmost point in the function graph must move and change its location. This expectation was guided by the personal meaning he developed for the  $AD(x)$  graph as an accumulation function [9]. The contrast between the changing he sees on the screen and the conclusion he got, allows him to endow the vertical transformation of the  $AD(x)$  graph with personal meaning which doesn't conjectures the mathematics meaning, as i will show in the next transcript.

- 10 Amro: [Drags the  $AD(x)$  graph] if I consider any point's value relative to the leftmost point I get the same mathematical information.
- 11 Osama: Can you give more details?
- 12 Amro: The important thing here isn't the y-value of the point itself, but the relative value of any point on the graph with the leftmost point [he is tracing on the  $AD(x)$  graph]. That is, if I choose an upper point and a lower point on this graph [indicates two different points on the  $AD(x)$  graph] and I consider the difference between both, it doesn't matter where the lower graph is located. It can be located anywhere you want [drags the  $AD(x)$  vertically (Figure 4)] because you add a parameter to all the points.

While he drags the  $AD(x)$  graph he notices that the shape of the  $AD(x)$  graph doesn't change, but, of course, its location changes vertically and the function graph is still fixed and doesn't change or move [12]. To interpret the vertical transformation of the  $AD(x)$  graph, Amro had not yet looked at the y-values on the anti-derivative graph [19], but observes the difference of any point on the anti-derivative graph to the left most point. The gesture he makes, together with the words he says, "any point on the

graph with the leftmost point," tell us that he subtracts the y-coordination of any points on the  $AD(x)$  graph with a fixed point (the leftmost point). According to the strategy of relative value that he used, his conclusion was that the location of the  $AD(x)$  graph is irrelevant because when choosing any two points on an  $AD(x)$  graph, the difference between its y-value stays the same when it is dragged vertically.

## DISCUSSION

This case study revealed the personal meaning that Amro got for the linked graph in the artifact was a correlation between a function and its accumulation function. This study suggests three elaboration stages toward the meaning of the  $AD(x)$  graph to get the meaning of accumulation function. 1) Corresponding zero function to zero  $AD(x)$  function. It seems that this corresponding was the sprout of the evolution for the personal meaning to eventually get to the accumulation meaning. My interpretation of the slowly dragging of the function graph toward the x-axis, and the overlapping the anti-derivative function, is that it shows an effort of Amro to notice the approach of the  $AD(x)$  graph to x-axis while the function overlaps it. 2) Corresponding the constant function to linear anti-derivative function. This corresponding may be considered as a central part of the process of evolution of the personal meaning to get the accumulation meaning. 3) Interpretation and justification of the vertical transformation of the  $AD(x)$  graph.

The first two stages can be considered as the development of personal meaning that evolves through the use of the artifact to get the mathematical meaning as an accumulation function. The third stage may be considered as personal meaning that doesn't fit the mathematical meaning. Mathematically speaking, there is a unique accumulation function for a fixed function  $f(x)$  at a fixed lower limit, but dragging the  $AD(x)$  vertically by the artifact produced a family of graphs different by constant. This situation is suitable for somebody to endow the graph  $AD(x)$  with the anti-derivative meaning rather than the accumulation meaning. It seems that the explanation of the vertical transformation of the  $AD(x)$  according to the strategy of the difference between y-values prevent the anti-derivative meaning from arising up.

The linked graph of function and anti-derivative graph in the Cartesian systems as a sign bears a culturally accepted meaning and the dragging option as a tool in the designed artifact play an essential role in the evolution process toward the accumulation meaning. Initially, representing linked graphs allows Amro to focus on the changing that occurs in the  $AD(x)$  graph while dragging the function graph. The dragging tool produces a family of similar function graphs which allows the graduate student to notice the differences between similar correlations. That is, to see the differences between the family of constant functions and its influences on the  $AD(x)$  graph. On the other hand, it allows him to notice the similarity between different correlations. In spite of varying the constant functions, the linearity of the  $AD(x)$  functions is still stable. It seems that the artifact supports and gives rise to generalization of the link between both graphs. Our conclusion about the artifact as

inspiring generalizations comes from Kant, who was cited in Radford (2010), that: "generalization rests on synthesizing resemblances between different things and also differences between resembling things".

## References

- Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp.746–783). Mahwah, NJ: Lawrence Erlbaum Associates.
- Falcade, R., Laborde, C., & Mariotti, M. (2007). Approaching function: Cabri tools as instruments of semiotics mediation. *Educational Studies in Mathematics*, 66, 317-333.
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517-545). Mahwah, NJ, USA, Lawrence Erlbaum.
- Radford, L. (2010). The eye as a theoretician: seeing structures in generalization activities. *For the Learning of Mathematics*, 30, 2.
- Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical Thinking. In Helen L. Chick, & Jill L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (pp. 113-120). Australia: University of Melbourne
- Schwartz, J.L., & Yerushalmy, M. (1996). Calculus Unlimited (software). *Center of Educational Technology, Israel*, <http://www.cet.ac.il/math-international/software7.htm>
- Shternberg, B., Yerushalmy, M., & Zilber, A. (2004). The Calculus Integral Sketcher: *Center of Educational Technology, Israel*. <http://calculus.cet.ac.il/Lib/index.aspx>
- Swidan, O., & Yerushalmy, M. (2009). Making sense of accumulation function in a technological environment. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 201-208. Thessaloniki, Greece: PME.
- Thompson, P.W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26, 229-274.
- Vygotsky, L. S. (1978). *Mind and Society: The Development of Higher Psychological Processes*. Cambridge: Harvard University Press.



# RESOLVING CONFLICT BETWEEN COMPETING GOALS IN MATHEMATICS TEACHING DECISIONS

Mike Thomas and Caroline Yoon

The University of Auckland

*This paper describes part of an international project considering graphical construction of antiderivative functions in the secondary mathematics classroom. We use Schoenfeld's Resources, Orientations and Goals (ROG) framework to analyse the decisions made in part of a lesson of a teacher, Adam. In this he discusses with his students constructs arising from the relationship between a function and its graphical antiderivative. We present details of Adam's ROG and see how this is related to resolution of the conflict between his competing goals and the decisions he makes. The results suggest that a beneficial professional development strategy might be to assist teachers to become more aware of their ROG and its influence on in-the-moment classroom decisions.*

## BACKGROUND

Traditional calculus teaching has largely focused on giving students a range of technical rules and procedures that they can use to solve problems. However, although students may become proficient at executing procedures efficiently through traditional instruction, they often have a poor conceptual understanding of why these procedures work, or what they mean. More recent initiatives have widely espoused the value of a graphical approach to the learning and teaching of calculus, stressing conceptual understanding through multiple representations.

Teaching calculus through multiple representations can be successful if it enables students to notice and appreciate rich connections between representations (Thompson, 1993). Berry and Nyman (2003, p. 496) concluded, "If students can develop the skill of drawing a function graph from its slope graph then their level of conceptual understanding of the derivative...will be greatly improved." They suggest that encouraging students to engage in an enactive 'walking' of the derivative graph may promote this ability. A further study by Haciomeroglu, Aspinwall and Presmeg (2009) considered students' cognitive processes as they sketched antiderivative graphs from derivative graphs, such as one similar to  $y=a/x$ . They found that flexibility and reversibility of thinking are crucial to understanding the relationship.

In this paper, we consider a calculus lesson that was set purely within a graphical representation, and designed to encourage student exploration of conceptual ideas underlying antiderivative graphs. By setting the problems within the graphical domain, students had little recourse to the algebraic techniques with which they were familiar, and were constrained to consider relationships between gradients of functions and their antiderivative graphs. However, this kind of calculus instruction can be challenging for teachers, who are often more familiar with teaching executable

algebraic procedures rather than reasoning conceptually in the graphical domain. We focus on the decisions one teacher, Adam, made while teaching a lesson on the graphical construction of an antiderivative function, given the graph of the original function. In particular, we analyse the tension Adam experienced between wanting to prepare students with the necessary tools and understanding to solve the exploratory problems, and allowing students to explore the problems themselves.

## THEORETICAL FRAMEWORK

Schoenfeld (2008, 2010) has produced a theoretical framework for goal-oriented decision making applied to teaching-in-context. The framework is based on the *Resources, Orientations, and Goals* (ROG) of teachers and the manner in which these are linked to the decisions they make in the classroom. The term *orientations* is an inclusive one, incorporating an individual's dispositions, beliefs, values, tastes and preferences. Such orientations not only shape the way we see the world but also, in any given situation, the goals that we establish in order to deal with those situations. Furthermore, attainment of goals is reached by the marshalling of resources, which consist primarily of one's knowledge, but includes anything that can be made available in the service of attaining the goal.

In pedagogical situations, a teacher enters a classroom with a basic ROG, and orients his/her self to the situation. Following this, goals are established and relevant knowledge, which may include, for example, facts, procedural and conceptual knowledge, and problem-solving strategies, is activated. Decisions consistent with the goals are then made, consciously or unconsciously, with regard to the directions to pursue and the resources to use (Schoenfeld, 2010). Of course this process is dynamic not static, so a teacher's knowledge, goals, and orientations are updated as the interactions in the classroom proceed. Priorities may change, goals may be met, and new ones, including possible subgoals, established.

The explanatory power of this theory has been shown by researchers (e.g., Aguirre & Speer, 2000; Törner, Rolke, Rösken, & Sririman, 2010). However, their focus has been primarily on the role of beliefs, and further research is needed to investigate the relationship between all teacher orientations and goals, and pedagogical practices. We employed Schoenfeld's ROG framework, outlined above, to analyse a spontaneous decision within a teacher's lesson, where he judged it necessary to use teacher-led instruction to prepare the students to attempt the subsequent activities.

## METHODOLOGY

This study was part of a larger international comparison that investigated the implementation of a sequence of four tasks on graphical antiderivatives in seven classrooms in Italy, Israel and New Zealand. In this paper, we focus on a lesson taught by Adam in New Zealand. Adam was in his second year of secondary school teaching, and taught in English proficiently, even though it was not his first language. The class was a high ability year 12 (age 16-17) mathematics class in a low socioeconomic, multicultural school in Auckland, New Zealand, with predominantly

Maori and Pacific Island students. The topic of graphical antiderivatives was not part of the New Zealand national curriculum. Consequently, Adam was only able to devote four lessons to the four tasks. We focus on Adam's second lesson, which was based on activities that asked students to draw the antiderivatives of functions presented graphically (see Figure 1), and intentionally drawn so as to be dissimilar to well known graphs, such as straight lines and parabolas.

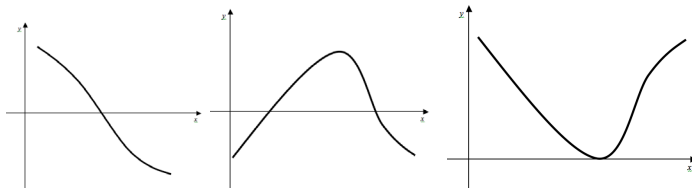


Figure 1: Examples of functions presented graphically in Task 2.

### Data collection and analysis

The lessons were audiotaped and videotaped, and student work was collected. One videocamera focused on the teacher and the whole class. There were two focus pairs of students, both of which had one videocamera focusing on their faces and another on their written work. After each lesson, the teacher participated in debriefing interviews, which were audiotaped, in which he described his experience of the lesson, explained certain teaching decisions, and planned for subsequent lessons.

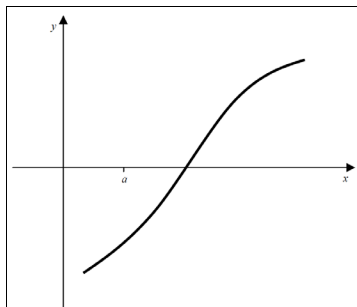
The four debriefing interviews were transcribed and coded according to Schoenfeld's ROG model, and used to create an overall description of Adam's ROG. The video files were transcribed and annotated with descriptions of the teacher's and students' behaviour during the lesson, and pictures showing gestures, boardwork, and student work. The transcript from lesson 2 was coded to identify key decision points.

Each decision point in lesson 2 was coded to infer the R, O, and G that appeared to influence the teacher's decision. The outcomes of the decision (in terms of mathematical thinking, teaching and learning) were also analysed. In addition, codes from the overall espoused ROG (from the interview) were used as supporting evidence to corroborate the ROG that we inferred from the lesson.

### RESULTS: A TEACHING DECISION VIGNETTE

In this section, we describe a vignette of a spontaneous teaching decision in Adam's second lesson. The excerpt begins 12 and a half minutes into the second lesson, where Adam introduces the first question, which is shown in Figure 2a below.

1. Consider the graph of the function  $y = f(x)$  shown below.



Consider the point  $a$  marked on the  $x$ -axis above.

Notice that the value  $f(a)$  is negative.

What does this mean for its antiderivative at the point  $x = a$ ?

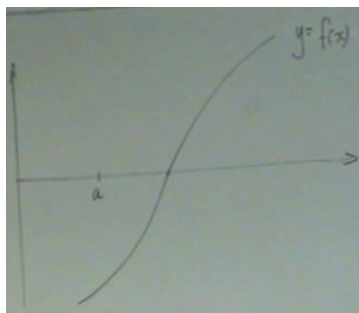


Figure 2: (a) The first question in the worksheet – students have a copy. (b) Adam draws the same graph on the board.

Adam draws a graph of the function on the board (see Figure 2b) and two minutes later, he asks the class, “How can you describe  $f(a)$ ?” He waits for 13 seconds, and appears uncomfortable when no one answers. Then Adam draws a graph of  $y = x^3$  on the board, and marks two points on the  $x$ -axis at 1 and  $-1$ . He asks the class what  $f(1)$  and  $f(-1)$  would be, and writes down the answers on the board (see Figure 3a). Next, he engages in a verbal interchange with the class:

Adam [Pointing at the board], Describe to me  $f(1)$ .

Student 1

Adam Now describe to me  $f(-1)$ .

Student  $-1$

Adam Now give me a way to differentiate  $f(1)$  and  $f(-1)$ .  $f(1)$  is?

Student Positive.

Adam Positive.  $f(-1)$  is?

Student Negative.

Adam Negative. [He marks on the right side of the graph, “pos” and on the left side “neg”, then erases the graph]. So we now have a tool to use to describe values of functions, right?

This interchange suggests that Adam’s goal in asking, “How can you describe  $f(a)$ ?” was to establish that  $f(a)$  was negative. Adam interpreted the students’ silence as evidence that they did not know how to determine whether a function was positive or negative, and made an in-the-moment decision to use the example of  $y = x^3$  to show how to do this, having initially considered using  $y = x^2$ .

After this interchange, Adam erases the graph of  $y = x^3$ , which suggests that he believes he has successfully taught the students how to “describe”  $f(a)$ . He returns to the original graph (Figure 2b), and says, “can you describe to me  $f(a)$ ?” The majority

of the class seem to be confused still. Adam then redraws the graph of  $y = x^3$  (see Figure 3b), and asks what  $f(1)$  and  $f(-1)$  are, to which students respond “positive” and “negative”. Finally, Adam returns to the original graph (Figure 2b), and asks, “where is  $f(a)$ ?” to which the class answers “negative”, which appears to satisfy Adam.

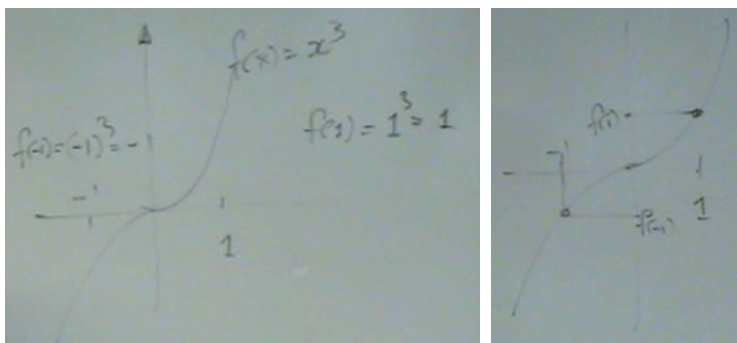


Figure 3: (a) Adam draws a graph of  $y = x^3$  on the board and evaluates it at  $x = 1$  and  $x = -1$ . (b) Adam redraws the graph of  $y = x^3$  on the board after erasing it.

This lengthy excerpt took around 5 minutes of class time. The length of this teaching excerpt was partly due to the ambiguous phrasing of Adam’s original question—“How can you describe  $f(a)$ ?”, followed by similarly ambiguous rephrasings such as, “Where is  $f(a)$ ?” If he had said instead, “Is  $f(a)$  positive or negative?”, the students may have answered readily that it is negative, which would have eliminated the need for this lengthy diversion. Adam may have used these phrasings because English was not his native language. Nevertheless, the amount of time that Adam spent on this demonstrated his desire to ensure that students were prepared for the upcoming task.

## ANALYSIS

Adam’s spontaneous decision in the excerpt above appeared to be influenced by two competing goals with associated resources and orientations.

### Goal A: To prepare students for success on future tasks

One of Adam’s primary goals was to prepare his students for the work they would encounter in the subsequent tasks. He appeared to focus on preparing students to understand the concepts that would be needed in the subsequent activities, so they could solve the activities easily: “The basic concept of derivative and antiderivative will be firmed up and then that will be a very easy task for them later on.” Table 1 summarises Adam’s five orientations and three resources that were associated with this goal of preparing students for success on future tasks.

O(A1)	Belief that students should understand the concepts that are used in a task before they attempt that task.
O(A2)	Belief that the problems the students were given were difficult, and that the first one in particular was too hard for the students.
O(A3)	Belief that it takes time to develop understandings of new ideas, and consequently

	empathises with students who don't understand the concept of derivatives (in a graphical sense) completely.
O(A4)	Belief that mathematics is hierarchical, and that mathematical knowledge develops by building on basics.
O(A5)	Belief that mathematics lessons should not be (conceptually) "heavy" for students.
R(A1)	Knowledge of which students have struggled with related mathematical concepts.
R(A2)	Pedagogical knowledge that students need to reconstruct generalisations from specific examples.
R(A3)	Mathematical knowledge of $y = x^2$ and $y = x^3$ , and which one will give both positive and negative values.

Table 1: Orientations and Resources Associated with Goal A.

### Goal B: To engage in student-centred learning as much as possible

Adam's second goal appeared to be to use student-centred learning as much as possible in his lessons. This was evident in his attempts to use student-led discussion on numerous occasions, and to have students experience the activities for themselves, rather than have the teacher "tell" them what to do. For Adam, this goal was associated with the four orientations and five resources identified in Table 2.

O(B1)	Belief that student participation is important
O(B2)	Preference that students experience the activity themselves rather than have the teacher tell them how to do the activity
O(B3)	Belief that it is better for an explanation to come from another student than from the teacher.
O(B4)	Values and is interested in students' thinking
R(B1)	Knowledge of his own tendency of wanting the students to experience the learning for themselves
R(B2)	Pedagogical knowledge of the strategy of getting students to share their solutions and explanations on the board
R(B3)	Knowledge that students will use non-technical language in their explanations, which will make it easier for other students to understand their reasoning.

Table 2: Orientations and Resources associated with Goal B.

### Competing goals: Conflict and resolution

In the excerpt of the spontaneous, in-the-moment decision described above, Adam is seen using primarily teacher-led discussion to prepare students for success in the upcoming tasks. On the surface, Adam appears to have sacrificed goal B in order to achieve Goal A. This appears to be the result of two constraints that brought the two goals into conflict.

First, Adam had only allocated a limited amount of time to the tasks, since the topic of graphical antiderivatives was outside the national curriculum, which he described as being more important, more "real". However, Adam was very reluctant to adapt to the time constraints by cutting down the amount of material. This led to intense time pressure, where he tried to cover a lot of material in a short amount of time, and made him question whether it was feasible to use student-centred learning. In the excerpt,

he used teacher-led demonstration, which he justified by explaining that it was more efficient for covering a large amount of material in a short amount of time.

Second, Adam appeared to have doubts about whether the students would be able to learn through student-centred learning, explaining that it was different to the culture of teaching and learning that the students were used to experiencing. Although he expressed a belief that student discussion is better for developing understanding than listening to a teacher, he thought the students would pay more attention to a teacher speaking than another student.

However, the competition between the two goals did not result in the complete rejection of goal B. Instead, Adam appeared to fashion a new goal (C), which was a hybrid of goals A and B (see Figure 4). This new goal appeared to consist of preparing students for success in upcoming tasks in a limited amount of time, by incorporating student participation in teacher-led explanations. Indeed, Adam's teacher-led discussion in the excerpt given above relied heavily on student participation.

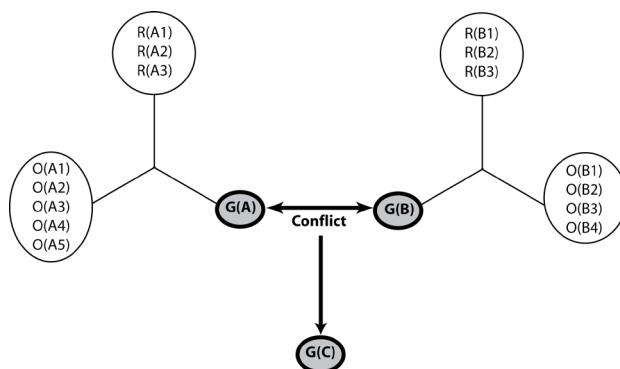


Figure 4: Adam's goals A and B come into conflict under the constraints of time, and concern about student ability. Their resolution gives rise to a new goal, C.

## DISCUSSION

In this paper we have provided some evidence of the role of a teacher ROG in specific instances of pedagogical practice, demonstrating the applicability of the framework to fine-grained analyses of teaching. The contribution of this paper is to give evidence of how the theory gives valuable insight into conflict resolution in decision-making. When faced with an in-the-moment decision about whether to give an explanation of when a function  $f$  is positive or negative Adam was faced with competing goals. The theoretical framework has enabled us to see how the resolution involved a blending, or amalgamating, of these goals to produce a new goal. The main driver in determining the new goal was Adam's orientations, which were prioritised within the constraints of the situation, and which, in a sense, produced new orientations that were relevant within those constraints (e.g. a belief that teacher-led

discussion can engender understanding within time constraints). Thus a refined ROG emerged that had a strong influence over the decision made (in this case to use the  $f(a)$  example). It is clear that Adam is a considerate teacher, who is concerned about his students' understanding, their success in the lesson and their overall yearly performance on national assessments, and his decisions are based on these concerns. However, we see that even a teacher with these orientations, and their associated goals, can subjugate his strong belief in the effectiveness of student-centred learning and revert to teacher-led demonstrations when constrained under time pressure. There may be a message here for professional development. Since the conflict resolution we have described may be subconscious, a process where teachers are explicitly involved in analysing the relationship between their ROG and their practice could raise awareness of how their ROG influences their teaching decisions, leading to a desire to change practice.

## ACKNOWLEDGEMENTS

We wish to acknowledge Tessa Miskell for her transcription and video annotation.

## REFERENCES

- Aguirre, J., & Speer, N. M. (2000). Examining the relationship between beliefs and goals in teacher practice. *Journal of Mathematical Behavior*, 18(3), 327–356.
- Berry, J. S., & Nyman, M. A. (2003). Promoting students' understanding of the calculus. *Journal of Mathematical Behavior*, 22, 481–497.
- Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N. (2009). The role of reversibility in the learning of the calculus derivative and antiderivative graphs. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31<sup>st</sup> Annual Meeting of the North American Chapter of The International Group for the Psychology of Mathematics Education*, Atlanta, Georgia: Georgia State University.
- Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior*, 20, 1–33.
- Schoenfeld, A. H. (2008). On modeling teachers' in-the-moment decision-making. In A. H. Schoenfeld (Ed.), *A study of teaching: Multiple lenses, multiple views (Journal for Research in Mathematics Education Monograph No. 14)*, pp. 45–96. Reston, VA: National Council of Teachers of Mathematics.
- Schoenfeld, A. H. (2010). *How we think. A theory of goal-oriented decision making and its educational applications*. Routledge: New York.
- Thompson, P. W. (1993). Students, functions, and the undergraduate curriculum. Presentation to the Annual Joint Meeting of AMS/MAA. January, *Contemporary Mathematics*, (publ. American Mathematical Society).
- Törner, G., Rolke, K., Rösken, B., & Sriraman, B. (2010). Understanding a teacher's actions in the classroom by applying Schoenfeld's theory teaching-in-context: Reflecting on goals and beliefs. B. Sriraman, & L. English (Eds.), *Theories of Mathematics Education, Advances in Mathematics Education* (pp. 401–420). Berlin: Springer-Verlag.



# ELEMENTARY SCHOOL STUDENTS' INTUITIVE UNDERSTANDING OF THE INEQUALITY SIGNS

Zulbiye Toluk Ucar and Birseli Yavuz

Abant Izzet Baysal University

*In this study, we investigated how students from 4<sup>th</sup> through 8<sup>th</sup> grade interpret the equal and inequality signs and how this understanding develops over grades. Data were collected through interviews and the equal sign and inequality signs tests. After administration of the tests to total of 257 students, 11 students were interviewed. Results showed that the students' understandings of the equal and inequality signs develop over the grades. Students' way of dealing with the equal sign and inequality signs showed similar patterns. However, students had difficulty in reading and recognizing the inequality signs and in attaching appropriate meanings to them. It was also found that students had an intuitive understanding of the inequality signs and this understanding was influenced by their understanding of the equal sign.*

## INTRODUCTION

What motivated this study is an eighth grade mathematics teacher's struggle in teaching "less than or equal to ( $\leq$ )" and "greater than or equal to ( $\geq$ )" signs. She explained that her eight grade students had difficulty in understanding how a number could be both equal to and smaller than another number. She reported that her students interpreted these symbols as either equal to sign or greater (less) than sign and she was unable to change this conception.

The equal sign in mathematics is one of the most important symbols and is used to show relationships between quantities or values. There are other signs used to show relationships. These are "not equal to ( $\neq$ )" sign, "greater than" sign ( $>$ ) or "less than" sign ( $<$ ), the "greater than or equal to" sign ( $\geq$ ) or "less than or equal to" sign ( $\leq$ ). Students' difficulties with these signs have been well documented. Studies investigating elementary and middle school students' understanding of the equal sign showed that many students lack a relational understanding of the equal sign and instead interpret it as a unidirectional, representing the result of an arithmetic operation or separating the answer from the operation (Behr, Erlwanger, & Nichols, 1980; Kieran, 1981; Falkner, Levi, & Carpenter, 1999; Rittle-Johnson & Alibali, 1999; Yaman, Toluk, & Olkun, 2003; Essien & Setati, 2006).

Studies have shown that students from middle school to college have difficulties with inequalities (Prestage & Perks, 2003; Blanco & Garrote, 2007; Şandır, Ubuz & Argün, 2007). Students usually interpret inequalities as equations; have a limited understanding of greater than and less than symbols; and have difficulties interpreting solutions. For instance, Blanco and Garrote (2007) found that most 17 and 18 years

old students were unable to understand the semantic difference between the concepts of equation and inequality, and grasp the meaning of inequality signs. As a consequence, they observed that students have difficulty in reading inequalities from left to right or from right to left. Blanco and Garrote also found that students restricted the concept of interval to the domain of natural numbers and integers. Prestage and Perks (2005) observed the same tendency to treat inequalities as equations among the students who were attending a postgraduate certificate of education program.

## **BACKGROUND: SIGNS AND SYMBOLS**

One of the problems in algebra is the fact that students frequently misuse and misinterpret algebraic symbols and algebraic syntax (MacGregor & Stacey, 1997). Mathematical symbols have abstract meaning (Pirie, 1998). The acceptance or use of a symbol does not always happen with its underlying meaning (Sáenz-Ludlow & Walgamuth, 1998). The acceptance or use of the equal and inequality signs by students does not mean that students interpret these signs in a mathematically appropriate way. The historical account of the development of the equal sign best exemplifies the symbol-meaning tension that exists in mathematics (Essien and Setati, 2006). Mathematicians did not immediately adopt Recorde's symbol for the equality which was first introduced in 1557. Sáenz-Ludlow and Walgamuth (1998) use the historical development of the equal sign to address the importance of the interpreting activity of the individuals. They emphasize that mathematical meanings are not directly conveyed by the symbols. Falkner et al. (1999) suggest that misconceptions at the early stage of student's perception of the equal sign should be tackled before they become more deeply established. This study broadens this suggestion and aims to identify elementary students' conceptions of the equal and inequality signs starting from 4<sup>th</sup> grade to 8<sup>th</sup> grade.

Introduction of inequality signs in elementary mathematics curriculum in Turkey begins at the third grade within the context of comparing and ordering natural numbers. At third grade, students learn to use "greater than ( $>$ )" and "less than ( $<$ )" symbols in order to show the relationship between two or more natural numbers. There is no specific emphasis on teaching of "not equal to ( $\neq$ )" sign in the program, the use of this sign depends on the teacher. Finally, students are introduced to "less than or equal to ( $\leq$ )" and "greater than or equal to ( $\geq$ )" signs as well as solving and graphing simple inequalities at eight grade algebra. With this background, we formulated the following research question: How do elementary school students conceptualize equal and inequality signs? Based on this research question, we aimed to answer the following related questions: How do the students' understandings of these signs develop over the grades? Is there a significant relationship between students' understandings of equal sign and inequality signs?

## RESEARCH METHOD AND DATA ANALYSIS

Quantitative and qualitative techniques were used to collect the data. Participants of the study were 257 students (57 from 4<sup>th</sup> grade, 60 from 5<sup>th</sup> grade, 50 from 6<sup>th</sup> grade, 51 from 7<sup>th</sup> grade and 39 from 8<sup>th</sup> grade) from fourth through eighth grades in two different schools. Two tests were used to collect the quantitative data. The Equal Sign Test, which was taken from Yaman, Toluk, and Olkun's (2005) study, consisted of three parts. In the first part of the test, the equal ( $=$ ) sign and inequality signs ( $>$ ,  $<$ ,  $\neq$ ,  $\leq$ , and  $\geq$ ) were given and students were asked to write what these signs mean. Second part consisted of 10 equalities and students were asked to write the missing number(s) in the equality and the last part included 10 equalities and it was asked to determine whether the given equality is true or false. The Inequality Signs Test had the same structure as the Equal Sign Test. It consisted of two parts, each part having 10 questions. Scoring of the two tests was done exactly in the same manner. Each correct response was scored as 2 point; partially correct answers were scored as 1, otherwise it was scored as 0 point. The total score that can be obtained from each test was 40. After administering two tests to the students, we conducted individual interviews with selected 11 students to give them opportunities to explain their reasoning in the test items. During the interviews, students were shown their responses on the test and then were asked to explain their reasoning while they gave those answers. All of the interviews were audio recorded for the analysis. Literature review on the equal signs and the inequality signs provided a conceptual base for the data analysis. Content analysis was conducted to extract meaning in the student's explanations.

## RESULTS

Figure 1 show how mean scores on the tests vary with respect to the grade level. As can be seen from the figure, mean scores on both tests showed an increase from Grade 4 to Grade 8.

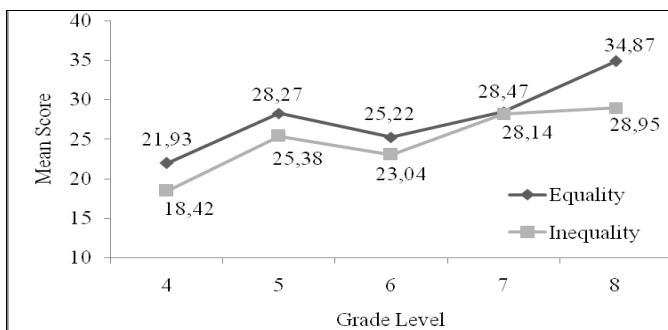


Figure 1: Changes in the mean scores of the both tests across the grades.

Figure 1 suggested a linear relationship between Equal Sign Test scores and Inequality Signs Test scores. Results of the correlation analysis showed that

performance on the Equal Sign Test and Inequality Signs Test was moderately correlated ( $r=0.54$ ,  $p<0.01$ ).

Analysis of the data obtained from interviews and the first part of the Equal Sign Test revealed that students mostly interpret equal and inequality signs as operational symbols. When asked to write what the equal sign means, almost all of the participants either just wrote the name of the symbol or its operational meaning. While almost all 6<sup>th</sup> through 8<sup>th</sup> grade students just provided the name of the symbol as a response, 4<sup>th</sup> and 5<sup>th</sup> grade students tended to write that it is used to show an answer or when doing operations. Only 8 percent of the student wrote that the equal sign means same as. During the interviews only two participants (interestingly two 4<sup>th</sup> and 5<sup>th</sup> grade students) displayed a relational understanding of the equal sign. Remaining 9 students interpreted the equal sign representing the answer or result of an arithmetic operation. An 8<sup>th</sup> grade student explained her response for the item  $3=?$  as follows.

Murat: It says 3 equals to question mark. I had written 3.

Interviewer: You also had written 6?

Murat: Their sum. I add 3 and 3, it is 6.

Another 6<sup>th</sup> grade student's unidirectional and operational understanding of the equal sign is presented below.

Interviewer: How did yo find 2 for this question? (showing  $?=1+3$ )

Selma: I added 2 and 1 to get 3.

Interviewer: You added 1 and 2 to get 3, don't you?

Selma: Yes. But, I don't know what to with this equal sign.

...

Interviewer: You wrote False for this question? ( $6=8-2$ )

Selma: 6 equals to 8. Up to here it is False. 8 minus 2, 6.

Interviewer: True or False?

Selma: Well.... 8 minus 2, 6. True.

Interviewer: Why?

Selma: Because 8 minus 2 is 6. ... But I put them in precedence order .

Interviewer: What do you mean by precedence order?

Selma For example, if 8 minus 2 was written first, it would be more correct.

Students' interpretation of the not equal to sign was similar to their interpretation of the equal sign. More than 75% of the students from each grade (except 4<sup>th</sup> grade) correctly recognized the sign and mostly interpreted it as an operational symbol. During the interviews, out of 11 students, 8 students correctly read the sign and

explained that it is used when they found the answer of an operation incorrectly. A 6<sup>th</sup> grade student justification of his answer is given below.

Eren:            this ( $\neq$ ) is not equal to sign.

Interviewer: How do you use it?

Eren:            For example, we did this addition incorrectly. We found a wrong answer. This answer is not equal to the correct answer. Our teacher always writes not equal to sign when we make a mistake in our calculation.

Only one fifth grade student was able to interpret this sign as a relational symbol. When explaining the not equal to sign, the same student drew on his understanding of the equal sign, which was also relational. This student wrote that she had no idea for the item “ $? \neq 6$ ” on the test, but during the interview she was able to answer the question.

Gamze:        Question mark not equal to 6. I had written I had no idea (surprised).

Interviewer: Tell me what you are thinking.

Gamze:        Six is equal to six. Then, numbers less than five are not equal to 6.

Interviewer: Then which numbers can come there?

Gamze:        For example, 5. Five is not equal to six.

Interviewer: Does only 5 come there?

Gamze:        No. Numbers larger than 6 and less than 6 can be written as an answer.

Interviewer: What about 6?

Gamze:        No, it can't. Because 6 is equal to 6.

Even though two students were unable to read the name of the not equal to sign; they were able to correctly answer the related items on the test.

The number of students correctly naming the less than and greater than signs increased by the grade levels. While 75% of the fourth grade students correctly wrote the name of the signs, 97% of the eight grade students were to do so. Despite this high proportion of naming of the symbols, students' marks on these signs in the test paper indicated that they had difficulty determining the direction of the inequality. Almost all of the participants who correctly named the symbols used this strategy. In addition, even if they correctly read these signs, they misinterpreted these names. During the interviews, two 4<sup>th</sup> grade and two 8<sup>th</sup> grade students displayed this misunderstanding. For example, a fourth grader explained his reasoning as follows:

Interviewer: (pointing  $5 > ?$ ) what does this mean?

Ali:            five larger a number.

Interviewer: Which number comes there?

Ali:            six

Interviewer: What else? Tell me what you are thinking.

Ali: seven. I am writing larger numbers instead of question mark.

Interviewer: So what do you think about this ( $5<?$ )?

Ali: I will write numbers less than five. .. such as 3, 2, 1.

Interviewer: What about this? ( $?>2$ )

Ali: 1 greater... it should be two because the number will be less than two.

In English,  $5>?$  is read as “5 is greater than a number.” However, in Turkish, it is read as “5 büyüktür bir sayı” (5 is greater a number), which has no sense mathematically. Instead, it should be read as “5 büyüktür bir sayıdan”. Because of this language obstacle, some students think that the word “büyüktür (larger)” refers to the number on the right hand side of the greater than sign. Moreover, inequalities involving operations ( $6<10-5$  and  $3\times3>9$ ) were found to be more difficult than inequalities ( $5>7$  and  $5<3$ ) requiring comparing and ordering numbers. In addition, the inequality  $3\times3>9$  was more difficult than  $6<10-5$ . All of the 11 students correctly explained the inequality  $3\times3>9$ , whereas only 2 fifth grade students correctly answered the inequality  $6<10-5$ , and three students corrected their mistakes during the interviews. Remaining 6 students compared 6 and 10 and explained that the inequality was true. Following excerpts from the interview display a fifth grade student’s reasoning.

Interviewer: You wrote false for this inequality on your paper. How did you decide?

Gamze: (is  $6<10-5$  true or false?) 6 is less than 5. False.

Interviewer: How did you find 5?

Gamze: I subtracted 5 from 10, it is 5. 6 is not less than 5. False.

An 8<sup>th</sup> grade student incorrectly answered the item related to  $6<10-5$  on the test. Yet, during the interview, she was not sure about her response on the test. This excerpt indicates that this student’s interpretation of the equal sign as an operational sign manifests itself in her interpretation of the less than sign.

Kumru: 6 is less than 10. True. Minus 5. I made a mistake.

Interviewer: What do you mean?

Kumru: That is true, but there is 10 minus 5 here.

Interviewer: What are you going to do?

Kumru: I think I made a mistake. There is minus 5 here. 10 take away 5, 5. 5 is less than 6. Then this is true

Interviewer: Then which one is right?

Kumru: I think my first answer was correct.

Analysis of the data related to the greater than or equal to and less than equal to sign indicated that despite their recognition of the sign, students interpret these signs in different ways. Emerging interpretations were given in Table 1.

Interpretation	N	$? \geq 6$
Not greater (less) than	3	$6 \geq 6$ because 6 is not less than 6.
Equal to greater (less)	1	$9 \geq 6$ , 6 is equal to larger, 9.
Greater (less) than	6	Any number larger than 6, except 6.
Greater (less) than or equal to	1	Any number larger than 6 including 6.

Table 1: Students' interpretations of the  $\leq$  and  $\geq$  signs

During the interviews, only 3 students (a fifth grade and two eighth grade students) correctly named greater (less) than or equal to signs, but only the fifth grade students correctly explained her reasoning for  $5 \geq ?$ .

Gaye: 5 is greater than or equal to. We can write 5 and the numbers less than 5.

Interviewer: Here? ( $6 \leq ?$ )

Gaye: 6 and numbers greater than 6.

Even though two eighth grade students correctly read the signs, they misused the signs. One eighth grade student explains the meaning of " $\leq$ " as follows.

Interviewer: What do you mean? Can you give an example?

Ekrem: It means less than or equal to. For example, 3 and 3 are less than or equal to. How would I say? Now, less than or equal to, exactly opposite of this. I can't explain.

Interviewer: If we write  $5 \leq 5$ , would it be correct?

Ekrem: They are equal... Well, to me it is wrong.

## DISCUSSION AND CONCLUSION

This study provided some evidence that elementary school students' understandings of the inequality signs develop in relation to the development of their understandings of the equal sign. Students tended to solve inequalities in the same way they solved corresponding equalities. While they were solving inequalities, they generally worked from left to right and usually ignored the operations on the right hand side. First they found the result of the operation on the left hand side of the inequality and then compared the result with the number just on the right hand side of the sign. This relationship was also evident in that those students who interpret the equal sign as a tool representing a relation were also able to interpret the inequality signs as relational symbols. It was also found that the students had problems in reading inequalities from left to right or from right to left. Even if they correctly read, Turkish language structure constituted an obstacle to identifying the direction of the inequalities. In addition, despite the fact that a symbol (e.g.  $\neq$ ) is not formally taught, teachers' occasional uses of the symbol cause students to draw the meanings of this symbol from those situations in which it is used. In this study, students frequently

said that their teachers usually use the not equal to sign to denote an incorrect answer of an operation and these students usually interpreted this symbol as an operational symbol.

Meanings of mathematical signs and symbols are not transparent. Students' extensive uses of these signs starting from early grades do not guarantee the development of appropriate meanings of these signs by the students. Although a small number of students who had a relational understanding of the equal sign intuitively applied this understanding to other relational symbols, the dominant conception of the operational understanding of the equal sign among the students prevents the development of the relational interpretation of the other signs. Starting from early grades, teachers need to provide experiences for students for extending their meaning of the equal and inequality signs from an operational sign to a relational sign. By uncovering students' existing understandings of these signs, inappropriate understandings should be challenged by the students and the teacher.

## References

- Behr, M., Erlwanger, S. & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, 92, 13- 15.
- Essien, A. and Setati, S. (2006). Revisiting the equal sign: Some Grade 8 and 9 learners' interpretations. *African Journal of Research in SMT Education*, 10(1), 47-58.
- Falkner, K.P., Levi, L. & Carpenter, T.P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, 6(4), 232-6
- Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12, 317-326.
- MacGregor, M. & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics*, 33(1), 1-19.
- Pirie, S. (1998). Crossing the gulf between thought and symbol: Language as (slippery) stepping-Stones. In H. Steinbring, B. Bussi, & A. Sierpienska, (eds.) *Language and communication in the mathematics classroom* (pp.7-29). NCTM, Reston: Virginia.
- Prestage, S. and Perks, P. (2005). Inequalities and Paper Hats. *Mathematics Teaching*, 193, 31-33.
- Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, 91, 175-189.
- Sáenz-Ludlow, A. & Walgamuth, C. (1998). Third graders' interpretation of equality and the equal symbol. *Educational Studies in Mathematics*, 35, 153-187.
- Şandır, H., Ubuz, B., & Argün, Z. (2007). 9th grade students' difficulties in arithmetic operations, ordering numbers, solving equations and inequalities. *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi*, 32, 274-281.
- Yaman, H., Toluk, Z., & Olkun, S. (2003). How the elementary school students would perceive equal sign? *Hacettepe Üniversitesi Eğitim Fakültesi Dergisi*, 24, 142-151.



# CLASSROOM ACTIVITY PROMOTING STUDENTS' LEARNING ABOUT THE USE OF DRAWINGS IN GEOMETRY

Vanessa Sena Tomaz

Maria Manuela David

Universidade Federal de Minas Gerais, Brazil

*Our aim is to contribute with research on how drawings structure geometry activity in the classroom. We present one illustrative episode of public elementary schools in Brazil. The framework of Activity Theory helped in the characterization of the episode as a system of interconnected activities and we explore the idea of miniature cycles of learning actions to focus on the mathematical learning that is taking place. We focus our analysis in one miniature cycle, apparently a detour from the main activity, to discuss the power of the visual representations for structuring and modifying the mathematical activity in the classroom and we conclude that this detour, instead of deviating the students' attention, opened the possibilities to deepen their understanding about the measure of angles, the main subject of the lesson.*

## INTRODUCTION

In our classroom observations with the purpose of investigating what mathematics is being taught and how it is being taught in public elementary schools in Brazil our attention was caught by some situations in which two teachers, Telma and Roberto<sup>1</sup>, were led to discuss and to explicitly teach the rules that should govern the use and interpretation of drawings in school mathematics, without having planned it in advance. These situations have redirected our work to focus on two issues worth of more research: how visual representations (drawings, figures) can structure the geometry activity in the classroom and how teaching practices can facilitate students' visualization of mathematical objects. The two teachers had different classroom practices and add different contributions for understanding how classroom activity can promote students' learning about the use of drawings in geometry.

In this paper we present one episode, from Telma's lessons, in which the classroom activity was influenced by an intervention of a student, as the teacher made a quite radical detour from the situation that generated the student's difficulty, using a more familiar situation for the class. The aim of this paper is to show how this detour, instead of deviating the class from the main classroom activity, has, in fact, created new opportunities for learning about the rules that govern the use of drawings in school mathematics.

## THEORETICAL FRAMEWORK

The theoretical grounding of our analysis of Telma's episode is based in the framework of cultural-historical Activity Theory (Leont'ev, 1978; Engeström, 1993) and in the theory of expansive learning of Engeström (1987).

According to Leont'ev (1978), the concept of activity represents a specific form of social existence that includes crucial changes to social reality. Leont'ev explains that activity emerges from a necessity, which drives motives towards a related object. To satisfy motives, actions are needed. These, in turn, are accomplished in accordance with the conditions that determine the operations related to each action.

Engeström (1987) added *community*, *divisor of labor* and *rules* as new elements to Activity Theory and points out that the analysis of the activity system and its constituent components and actions should be done historically and that one activity system is always connected to other activity systems through some of its components. If one of these components changes, other changes must take place to adjust the whole system. In our case, of classroom activity, we consider that the unity of analysis is the activity system of the students and the teacher in the classroom.

When discussing learning according to this theory, Engeström (1987) developed the expansive learning theory, according to which, one should always put the primacy on the second element of each of the following pairs, corresponding to the four dimensions of learning: learners as individuals and as *communities*; transmission/preservation and *transformation/creation of culture*; vertical improvement according to scales of competence and *horizontal movement and hybridization of cultural contexts*; and, acquisition and *formation of theoretical knowledge and concepts*. For Engeström (1993), an expansive transformation is accomplished when the object and motive of the activity are reconceptualized to embrace a radically wider horizon of possibilities than in the previous mode of the activity.

Engeström & Sannino (2010) claim that the theory of expansive learning sees contradictions as historically evolving tensions. Therefore, contradictions are a key concept in the expansive learning theory, as they involve tensions and produce culturally new patterns of activity.

Two other main concepts in expansive learning theory are internalization and creative externalization. According to Engeström (1993), whereas internalization is related to cultural reproduction, creative externalization relates to the explanation of how the creation of new artefacts makes the transformation of the activity possible.

Engeström & Sannino (2010) examine some studies based on the theory of expansive learning and we thought one of the approaches analyzed by them, named *cycles of learning actions*, to be the most appropriate one to our situation. Since, ultimately, we aim at discussing what sort of mathematical learning is taking place in the classroom, and as it is not possible to have a direct access to the *students' learning*, one can instead observe the *students' actions* as a way to facilitate the perception of their learning. By applying the idea of learning cycles to the classrooms and describing the students' actions, it is possible to emphasize the emergence of the process of creative externalization and of formation of new concepts, and also to describe the trajectory of an activity system out of its equilibrium conditions.

For Sannino & Engeström (2010, p.11), a new expansive cycle is initiated when a relatively stable pattern of activity begins to be questioned and, correspondingly, a cycle ends when a new pattern of activity has become consolidated and relatively stable. They add that large scale cycles involve numerous smaller cycles of learning actions, and that *miniature cycles of innovative learning* should be regarded as *potentially expansive*. This makes the idea of *miniature cycles* very helpful to deal with classroom activity without disregarding the principle of the historicity of the activity, by adapting it to the short periods of time we are dealing with. In our case, this idea is going to prove to be helpful for identifying changes or expansions of the activity's object in some miniature cycles of the lessons.

In this paper we are going to focus our analysis in one of those miniature cycles, apparently a *detour* from the main activity, to discuss the power of the visual representations for structuring and modifying the mathematical activity in the classroom. Thus, we are going to use the Activity Theory perspective to reinterpret one of Telma's lessons that we have already analyzed elsewhere, under a different theoretical perspective (Tomaz, 2006).

### **ACTIVITY SYSTEM: MEASURE OF THE ANGLE FORMED BY THE BISECTORS OF TWO ADJACENT ANGLES**

Telma has a large experience (over 23 years) as a mathematics teacher in the elementary and secondary school level. According to her students and members of the community she is a good mathematics teacher. She usually creates a good relationship with the students and manages to make them participate of the mathematical activities in the classroom. Here we report on some of her lessons in a 7<sup>th</sup> grade classroom (35 students, 13-14 years old), from a public school in Brazil.

We are going to discuss one activity that begins when the teacher is trying to clarify the students' doubts related with an exercise she had proposed to the class, about the measure of the angle formed by the bisectors of two adjacent angles. We could conclude that this activity is part of an activity system formed by a constellation of four activities, identified as miniature cycles of innovative learning (Engestrom & Sannino, 2010). These elapsed along short periods of time during the progress of the classroom activity and could be perceived when it was examined historically: 1) Using the protector to measure the angles formed by a bisector; 2) Measure of the angle formed by the bisectors of two adjacent angles; 3) Identification of the characteristics of a mathematical object from its visual representation (example: square/rectangle); and, 4) Returning to Activity 2 with new rules for the calculation of the measure of the angle of the two bisectors.

For the purpose of this paper we have elected to focus mainly in Activity 3 - Identification of the characteristics of a mathematical object from its visual representation (example: square/rectangle). Previously to Activity 3, the students

were learning about angles formed by two half lines and about the internal half lines of an angle. They were drawing and learning the definition of the bisector of an angle and were measuring the angles formed by it. At some point, the students were told to fold a piece of paper where an angle had been drawn to construct the two adjacent angles formed by its bisector and, on the following, they were oriented to use a protractor to measure those angles (Activity 1).

Some days later, the teacher proposed an exercise (Fig.1) where the students had to find the measure of an angle formed by the bisectors of two given adjacent angles (Activity 2). They were supposed to do this task at home.

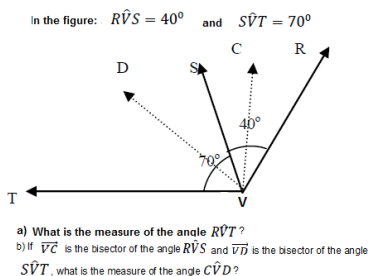


Figure 1

In the next lesson, Tereza, a student, said to the teacher that she was unable to do the task because she did not have a protractor at home. Apparently, she did not perceive that with the information on the drawing it was possible to calculate this measure by dividing 70 and 40 by two, and then adding the two halves. The subsequent action of the teacher in response to Tereza's argument, of making public the student's doubt, led to a discussion in class that can be characterized as a moment when the students and the teacher engaged in an Activity related with angles and bisector lines. The role of the visual representation was of major importance in this moment because there was an essential information ( $\widehat{RVS}$  and  $\widehat{SVT}$  are adjacent angles) which could only be found on the drawing (Fig.1).

### Activity 3(Detour): Identification of the characteristics of a mathematical object from its visual representation (example: square/rectangle)

When the teacher Telma shared the difficulty expressed by this student with the others, she has provoked a tension in the class as a whole. As several students were getting confused about what they were supposed to do in this case, the teacher decided to make a detour, creating a new situation, by using a different drawing.

01 Telma: now...look... some people told me that they did not make the task because she did not have a protractor at home... but... in fact... when we have

...[the teacher draws a rectangle on the blackboard [Fig.2] and questions]...what is this figure?

02 Joaquim: is:...a rectangle.

03 Telma: Is this a rectangle?

04 Sônia: yes... it has four angles

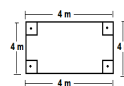
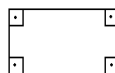


Figure 2

Figure 3

05 Telma: Yes...Is it a rectangle? ...I do not know what this is ...because for me...I could say that this is a square.

06 Student: Isn't this a square?

07 Sônia: two squares ...if we split it in the middle... exactly.

08 Telma: because I did not indicate these measures here [showing the sides of the rectangle- Fig 2]...if I had done this and had written that from here to here and here the measure is 4 meters by 4 meters [Fig 3]... measured here... the drawing that I was looking at before isn't there anymore... but if I indicated this here...look [writing the measure of the sides like in Fig. 3]

09 Sônia: ah:: It is a square!!

10 Telma: Yes a square...why is it a square? .

11 Joaquim: what is this!!!!:::

12 Telma - why? Why is it a square?

13 Tereza – because all sides have the same measure.

14 Telma - why?...why is it a square?...because all sides have the same measure and each internal angle is 90 degrees.

15 Neusa – so... why [did you make it look like a rectangle?..]

16 Telma – because I want to show to you that when you make this drawing (Fig.1)...even if the measure is not exactly 40 degrees....

17 Joaquim - the measure is 16 [the student speaks out loudly, apparently having measured the angle using a protractor]

18 Telma - ...but if I point to 40 degrees you should read like 40 degree...do you understand? [The students nodded the head affirmatively]

We can associate to this activity the following components: *Object*: rules for the identification of the characteristics of a mathematical object from its visual representation; *Subject*: math teacher and students; *Community*: math teacher, students, mathematics educations, mathematicians, textbook authors, curriculum developers; *Division of labor*: the teacher is the authority; *Rules*: use the drawing to extract the information necessary to differentiate a square from a rectangle; *Tools*: pictures of squares/rectangles.

Taking a closer look at this activity, we can notice that the stable pattern of the activity is lost and begins to be questioned when the teacher shared this difficulty with the whole class. The sequence of actions, located between lines 1-14 reflects a momentarily fluctuation of the object because the teacher starts teaching the rules that should govern the use of drawings in mathematics using a particular figure (square/rectangle). This opens up the possibility of considering another *miniature cycle of Innovative Learning* (Engestrom & Sannino, 2010) with this sequence of actions: the teacher draws a picture and asks her students to identify the mathematical object that it represents; the students analyze the picture and visualize a rectangle, because it has four right angles explicitly marked there; this response redirects the actions of the teacher to focus on the definition of a square rather than in the visual representation itself, triggering a mini cycle related with the square definition.

Just after the first responses of the students (identifying the figure with a rectangle, not square) the teacher makes a new challenge and causes *tension*, by showing that she disagreed with their answer (line 3). After Sônia's response (line 4), the teacher adds more information about possible measures of the sides of the figure (4m on all sides) and reiterates her intention to make the students understand that it is not sufficient to identify four right angles in a figure to conclude that it is a square. At this moment the students have two ways to interpret the drawing: considering that it has four equal angles (right), as they said in the beginning or that it has the same measure in the four sides. Finally, Sonia states the desired conclusion that the figure drawn by the teacher was a square. Although she did not say it explicitly, she seems to be aware that this is so because it has four (right) angles and four sides with the same measure, even though the sides do not look the same. Again, the teacher shares the conclusion of the student with the class (community) and, to make sure that all were following her, asks: "*Why? Why is it a square?*". Another student, Tereza, answers in the desired direction: "*because all the sides have the same measure.*" And the teacher resumes, completing "*because all sides have the same measure and each internal angle is 90 degrees*", thus closing the mini cycle of the definition of a square and offering to the students the opportunity to better understand the definition of a square and how they can identify it from its visual representation.

As it was the purpose of the teacher to teach her students how to interpret the drawing to recognize the mathematical object that it represents in school mathematics, she goes back and forth on the drawings representing the square/rectangle and the adjacent angles, gradually redirecting the actions of the community to finding the measure of the angle formed by the bisectors of two given adjacent angles. After the line 16, the teacher retakes the sum of the measures of the known angles (40+70). Then, she introduces some scribbling to differentiate firstly the measure of the sides of square/rectangle (Fig.4), secondly the measure of the angles (Fig.5). This shows that there were overlapping between activities 3 and 4. Therefore, both drawings can be considered, simultaneously, as tools in the two composed activities.

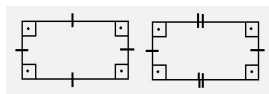
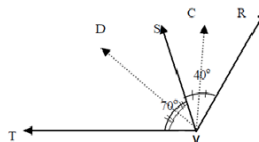


Figure 4

In the figure:  $R\hat{V}S = 40^\circ$  and  $S\hat{V}T = 70^\circ$



- a) What is the measure of the angle  $R\hat{V}T$ ?
- b) If  $\overline{VC}$  is the bisector of the angle  $R\hat{V}S$  and  $\overline{VD}$  is the bisector of the angle  $S\hat{V}T$ , what is the measure of the angle  $C\hat{V}D$ ?

Figure 5

Another important aspect in this sequence of actions is related with the manifestations of the teacher (lines 1, 3 and 5) which seem to reflect her intent to make clear for the students the historical contradiction between the mathematical abstract ideas and their empirical representations in school mathematics. But this does not seem to have any impact on the students at this moment. Only in line 17, when Joaquim (student) took the protractor and actually measured the angle and found 16 degrees, there is an explicit confrontation between the empirical measure and its representation.

The teacher directs the class to focus on the main activity object and to solve the exercise using the bisector definition to make the calculation of the angle formed by the two bisectors. At this moment, among all the possibilities previously considered for measuring angles – to use a protractor, to fold the sheet of paper or to use the symbols in the visual representation – only the last one was considered by the class.

We say that there was a fluctuation of the object since, as the activity develops, what we could identify as the rules in Activity 2 became the object of the new Activity 3. The rules of use of drawings in mathematics were transformed in the object of the teaching activity.

At the end, the teacher resumed the explanation about the calculation of this angle measure, step by step, but she did not mention anything about the rules for the use of visual representations in mathematics. The students were paying attention and showing to follow the teacher thinking.

## CONCLUSION

As the students were showing to follow the teacher thinking, this can be interpreted in two ways: as a manifestation of an internalization of the definitions of square and rectangle based on the use of the rules that govern the visual representations in mathematics; as a manifestation of their understanding of how to use these rules for the calculation of the angle of the bisectors. Thus, in this activity system it was introduced a new pattern for the interpretation and use of visual representations of mathematical objects, creating the possibility for a *potentially expansive learning*.

The actions of the teacher are worth emphasizing since, when presenting the drawing of the square/rectangle, she has introduced other ways of representing and defining those mathematical objects and she has created new opportunities for learning about the rules of use of visual representations, which were not anticipated by the exercise at hand. As we have seen, the detour provoked by the teacher, instead of deviating the students' attention, opened the possibilities to deepen their understanding about the measure of angles, the main subject of the lesson. This can be related with the teacher's experience and with the fact that the students were already used to this type of situation, which was a mark of this teacher's practice.

## Acknowledgment

The authors want to declare, first of all, their gratitude to the teacher and students involved in this study. We also wish to thank the participants of the Grupo de Pesquisa e Estudos Histórico-Culturais em Educação Matemática e em Ciências, for valuable comments, and the financial support received from the Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq and from Fundação de Amparo à Pesquisa de Minas Gerais – FAPEMIG.

## References

- Engeström, Y. (1987). *Learning by expanding: an activity-theoretical approach to developmental research*. Helsinki, Finland: Orienta-Konsultit.
- Engeström, Y. (1993). Developmental studies of work as a testbench of activity theory: the case of primary care medical practice. In: Chaiklin, S.; Lave, J. (Eds.). *Understanding practice: perspectives on activity and context*. Cambridge: Cambridge University Press, 64-103.
- Engeström, Y., & Sannino, A. (2010). Studies of expansive learning: Foundations, findings and future challenges. *Educational Research Review*, 5, 1-24.
- Leont'ev, A.N. (1978). *Activity, consciousness, personality*. Englewood Cliffs, New Jersey: Prentice-Hall.
- Tomaz, V. S. (2006) Práticas Escolares: uma releitura da transferência de aprendizagem. III Seminário Internacional de Pesquisa em Educação Matemática. Águas de Lindóia. In: *Anais do III Seminário Internacional de Pesquisa em Educação Matemática*. Curitiba: Setor de Educação. UFPR. 3.



# MATHEMATICAL LITERACY SKILLS IN A WORKPLACE CONTEXT: THE CASE OF READING AND INTERPRETING DATA

Chrissavgi Triantafillou

Technological Educational Institute of Lamia, Greece

*In this paper I trace the action of reading and interpreting data in three groups of technicians in the same workplace setting. The theoretical and analytical framework is guided by the first generation activity theory model and the three-tiered explanation of activity. The research methodology follows the ethnographic tradition. The results ascertain technicians' need for continuous reflection and interrelation with the referential situation in order to achieve their goal and proceed with their working activity. This supports the need to consider the case of reading and interpreting data as a crucial skill that has to be given special attention in the school curriculum*

## INTRODUCTION

Mathematics in the workplace is an area that has received considerable attention over the last decades. This is due to the fact that work settings provide us with several opportunities to investigate issues concerning the generation and the use of mathematical meanings and the ways that the tools and the goals of the work culture influence and shape them (e.g. Pozzi, Noss & Hoyles, 1998; Roth, 2005). Studies in advance technological settings agree that mathematical knowledge becomes visible in breakdown situations is embedded in the working activity and is hidden and crystallized in tools and practices (Wake & Williams, 2007). An important general mathematical practice in the workplace context is reading and interpreting data represented on tables, charts and graphs (Roth, 2005). In parallel, many studies identify the crucial role that these types of representations play in the construction and expression of mathematical meanings (e.g. Triantafillou & Potari, 2010). The practice of reading and interpreting data is classified as a component of mathematical literacy in the OECD's International Programme for Student Assessment (PISA). Roth (2002) argues that greater attention must be given to the practices of reading and understanding communicative tools such as texts, graphs, drawings etc. and he proposes an integrated notion of literacy that applies across the entire school curriculum. In the workplace settings Wedege (2002) refers to semi-skilled workers' reading and understanding drawings as a practice-related competence connected with critical judgments and communication skills.

The work presented in this article is part of a research project that aims to investigate discrepancies and invariants between the mathematical knowledge taught in schools and the knowledge used in a workplace setting. Under this general goal, we

conducted a 1-year ethnographic study of the mathematical practices of three different groups of technicians in a telecommunication setting. We also explored to what extent a range of technological tools, ranging from primitive (Group A) to technologically advanced (Groups B and C), influence the particularities of the emerging mathematical practices. Participants in Group A and B had vocational knowledge while technicians in Group C had academic knowledge. In our previous studies (Triantafyllou & Potari, 2006; 2010) we analysed the role of mathematical and non-mathematical tools in technicians' mathematical practices. In this paper, by considering reading and interpreting data as a central action in technicians' activity I trace this action in all three Groups of technicians. I search for similarities and differences in relation to the mathematical tools employed by them while performing this process. By exploring this issue we anticipate to expand our understanding of the role of this mathematical literacy skill in a rapidly changing technological environment and reason why we have to support this skill in classroom activities.

## **THEORETICAL FRAMEWORK**

In the present study we make the assumption that human behaviour and thinking occur within meaningful contexts where goal-directed activities are taking place, people are acting with tools (psychological and material) that carry social-historical meanings. The two main constructs we use in our work in order to capture the mathematical practices of the three groups of participants are the first-generation activity theory model and the Vygotskian view of an activity system, as reported in Daniels (2001, p. 86) and Leont'ev's (1978) work on the three-tiered explanation of activity. The first generation activity theory model, which is represented by a triangle, demonstrates relations between mediational means, the subjects and the object of the activity. The three-tiered explanation of activity focuses on its different components: activity, actions and operations. Human activity is always energized by a motive, actions translate the motive to reality and operations accomplish the actions instrumentally (Leont'ev, 1978, pp.62-65). Under this framework, mathematical practices in the workplace can be identified in the operational level when technicians draw on a range of tools to perform a certain action.

In our study the mediational means include all forms of tools used by the different groups of technicians while the subject is each group of professionals and the object is the specific motive-oriented activity. The concept of the tool has been expanded beyond the conventional view as a physical artefact to include a range of semiotic objects. For example, Vygotsky (1981) discusses psychological tools in which language, mnemonic techniques, algebraic symbols, diagrams, maps, drawings, or all sorts of conventional signs are included. The mathematical practices in the workplace can be identified in the operational level when technicians draw on a range of tools, including thinking or communicative tools (Mellin-Olsen, 1987) to perform an action. These tools frame the construction of mathematical meanings as several studies, particularly in the workplace, have demonstrated (eg. Pozzi, Noss & Hoyles, 1998).

## METHODOLOGY

The workplace was the National Telecommunication Organization in a small Greek town and a satellite earth station near Athens. This context was unfamiliar and massively complex, both mathematically and technologically. The ethnographic research method that was adopted lasted one year and the visits took place, on average, once a week giving a total of 70 hours of observations and interviewing.

*The participants:* In Group A, the participants were four technicians with vocational knowledge. The main workplace activity was to locate the fault in underground wiring in the local telecommunication network. The work of this group received marginal influence from the emerging technological innovations in the organization compared to the other two groups of our overall study. In Group B, the participants were four technicians with mostly vocational qualifications. Their main workplace activity was to install, programme and operate on terminal equipment. The work of this group was influenced by the emerging technological innovations and their tools were mostly computer mediated. In group C, the participants were three technicians with academic backgrounds who worked in the satellite earth station. Their activity was always at the edge of any technological innovation and the tools used were either measuring or computer mediated.

*Research process and data analysis:* Each group in order to translate their activity into reality had to perform certain actions. One of these actions was reading and interpreting information on a number of visual representations. By tracing technicians' operations related to this action we could identify a number of mathematical tools they employed in order to achieve their goal. In the overall study the methods of collecting data were participant observation, interviewing, and collection of artefacts, technical and academic textbooks. The observation involved shadowing and interviewing subjects informally at work. The interviews ranged from informal conversations in the beginning of the study to semi- structured interviews at the end. Field notes were kept and discussions between the researcher and the subjects were often audio-recorded. The researcher systematically kept a research diary where she reported the main issues that emerged from the observations and her field notes, the transcriptions of the audio-taped interviews, the transcriptions or field notes of the informal discussions, and her reflections. The collected artefacts were worksheets, technical maps, manuals, cables and photos. The analysis of the data was both ongoing and retrospective. For each action, the data were analysed using a grounded theory approach (Strauss & Corbin, 1998) in a qualitative content analysis.

## RESULTS

In this paper we analyse one central action in the technicians' every day activity of each group. In group A the action was to trace the route of the underground wiring-network, in group B to install terminal units in the subscribers' place and in Group C to measure the quality of the signal. To this end, they had to read and interpret data in a number of visual representations. These representations were technical maps

(Group A), schematic representations of terminal devices (Group B) and graphical displays on computer screens (Group C).

*Group A:* Technical maps were used by the technicians in Group A to trace the route of the underground wiring network and identify faults. To this end, they needed to read and interpret the data from the map correctly and move from the 2-dimensional representation of the map to the 3-dimensional reality and vice versa. Each technical map (Fig. 1) features a specific closet, as well as, the underground wiring network it distributes around the area in a certain scale. It also includes mathematical images (Evans, Tsatsaroni & Czarnecka, 2009) that indicate information about the network. For example, the symbol  $150'' = 100''/27.00$  (in the circle) indicates that the cable has 150 wire-pairs, from which only 100 are active and that the wire length between two knots is 27 metres, where the knot is a waterproof splice of a bunch of wires. The mathematical concepts and processes that were also involved in the above activity were spatial orientation, the use of

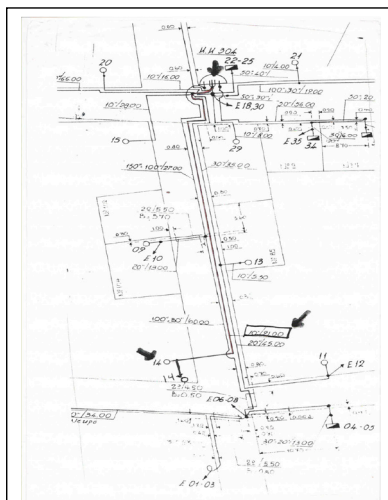


Fig. 1: The technical map

scales and units of measurement. Various imponderable factors influenced the progress of their working activity. For example, the representations in the technical map did not often depict what they would actually find in reality since the technical maps were often not up-to-date. In this case they made a series of logical hypotheses and used their practical wisdom to make decisions. Although technicians in this Group face many breakdown situations in their every day activity they cope with confidence “this is the challenge in our work”.

*Group B:* Unscaled schematic representations of terminal devices were used by the technicians in Group B in order to install these units in the subscribers’ place. These representations were diagrammatic and usually found in the manuals of these devices (Fig. 2). We consider this type of representation as layered inscription (Han, Roth, & Pozzer-Ardenghi, 2005) since different

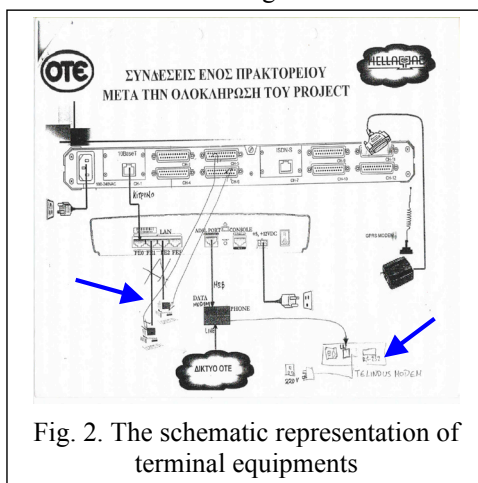


Fig. 2. The schematic representation of terminal equipments

entities are portrayed in a systemic way. The entities are naturalistic drawings (the terminal units), sketches (the pcs) and symbolic data (the telecommunication network in the clouded symbol in the bottom of the representation). This kind of representations denotes both the entities and their relationships with lines that indicate the way the devices are connected. The technicians needed to read and interpret the data from the schematic representation correctly and move from the 2-dimensional representation to the 3-dimensional reality and vice versa. But the information provided in this representation is insufficient to explain metrical and topological relations between the different entities. For example, nothing indicates how far away or close enough are these items and where they are in the subscribers' place. The absence of these data (scales, topological relations of the represented entities) causes many problems to the technicians since they could not continue with the installation of these units in the subscribers' place. In order to overcome the difficulties they faced due to the incomplete information represented on the schematic representations, they used their own drawings in order to correct them. We see, for example, that one technician made more drawings of other entities (arrow on the right) and corrected some links (arrow on the left) on the representation in order to proceed with his task. Technicians in this Group felt anxious about their every day job. The need for continuous installation of new equipment, required reading and interpreting data on schematic representations which was a necessary action for the technicians of this Group and brought about emotions of anxiety, anger and depression. "Every now and then they sent me new equipment to install. Every time I have to start from the beginning ... again and again ... For how long ... from the time we started this work we learn something new" (*Technician B3, Diary 11, 20/04/05*).

Group C: Graphs were used by the technicians in Group C when they were testing the quality characteristics of the signal before transmitting it to the satellite. The graph (Fig. 3) records authentic data and was chosen by the technician from the record files of the earth station as a typical graph that represents the most preferred method of measuring the carrier-to-noise ratio. The technician explains: "*We are measuring here how clear our carrier is in relation to the noise level*". The carrier is represented by the central curve and the noise by the fluctuations in the base of the graph. The technician refers to these fluctuations as "*noise*" or "*side harmonics*". The screen of the measuring instruments has 10 divisions on the vertical and 10 divisions on the horizontal axis. The vertical axis represents "*the power of the signal in dB*" while the horizontal axis represents the frequency in KHz. The technician must decode all the coded data given on this representation. These coded data are referring to the number represented in the vertical (5decibel or dB per division) and horizontal division or span (20 KHz) and the coordinates of the maximum point of the curve (arrow on the left in fig. 3), that is surprisingly expressed in GHz and dBm (Center 4.11400180 GHz, -67.45 dBm). The instrument refreshes its display according to the Resolution Bandwidth (RBw = 1.0 KHz) and the Sweep Time (SWP =200ms) control settings.

The technician measures the distance between the max point of the central curve and the max point of the fluctuations (the length of the arrow on the left in Fig. 3) and compares this number with international telecommunication standards. In this representation concepts from science (frequencies, dB), telecommunications (carrier, noise, resolution Bandwith) and mathematics (the scales and the graph) coexist. In technicians' measurement ( $5\text{dB/division} \times 7\text{divisions}=35\text{dB}$ ) the mathematical concept of the logarithmic function is hidden ( $\text{Carrier-to-Noise ratio in dB} = 10\log_{10}\text{Carrier/Noise}$ ). In order to find the required ratio he takes the difference between these quantities. Many other researchers in this area refer to crystallized mathematical processes (Williams & Wake 2007) in advance technological settings.

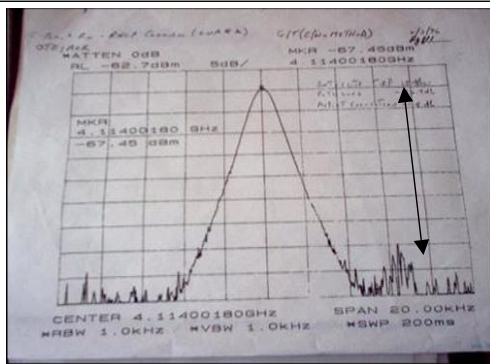


Fig.3. The graph

Let's bypass this crystallization of mathematical knowledge and focus on technicians' practices related to this action. Since this graph is a snap shot on their screen there are many control settings that allow the technicians to change either the vertical or the horizontal scales of the graph or the Resolution Bandwidth and the Sweep time. I present below the way the technician explains the control settings related to the Resolution Bandwidth and the

#### Sweep Time:

TEX. G1: The resolution Bandwidth allows us to control the fluctuation on the main curve. If there is an intense fluctuation on the graph of the carrier I should open this by increasing the RBw to see what is going on there [...] Sweep time refers to how quickly the instrument is sweeping the screen or what is the rate that the instrument is sampling the signal. In our case the rate is sampling in every 200milliseconds.

Researcher: If you change this rate how does it affect your result?

TEX. G1: Sometimes there is a case that the changes are so quick to watch so if I put the sweep time in 1second for example, this is a very slow rate if I have changes in every 100 milliseconds I could not realize them. But I always have to remember that faster sweeps of the screen have less resolution so the one control setting influences the other. (Technician C1, Diary 16, 23/07/05).

In the above extract it appears that the technician usually has to adjust many settings in order to have a clear picture of the signal. So his action relating to the reading and interpreting data on his screen involves his continuous interaction and reflection on his display. The graph represents a phenomenon that he has to explore and it is connected with decisions he has to make. The result of all these actions is crucial for his overall activity. In our interviews they always mention the significance of reading

and interpreting this type of representations: *“The graphs are my eyes. Only with them I can check what is going on in my wires”*.

Although the technicians in this Group are dealing with the most rapidly changing technological environment that involves the integration of different academic subjects and a constant adaptation to new requirements they were feeling secure about their work and they acknowledge the existence of mathematics. Maybe their strong mathematical background helps them to experience these positive emotions.

## DISCUSSION

In this paper I trace the action of reading and interpreting data in three groups of technicians in the same workplace setting, the setting of a telecommunication Organization. The data are represented on technical maps in Group A, on schematic representations of terminals that they had to install in Group B and on graphical displays on their computer screens in Group C. The action of reading and interpreting information on a variety of representations was central for each group of technicians because without managing that they could not proceed with their working activity.

The technicians of all groups had to continuously reflect (check, correct, modulate) on their data while reading and interpreting them. The reason for this continuous reflection and interrelation was either because the data did not feature the specific reality (Group A and B) or due to the absence of scales and topological relations (Group B) or because the phenomenon under investigation was not clearly represented (Group C). Due to this interrelation and reflection technicians developed a number of problem-solving strategies such as the visual inspection (Groups A, B, C) or the logical exclusion strategy (Groups A and B). Cases where representations in a workplace context cause breakdown situations are also reported in other studies (Pozzi et al., 1998). In this study we observe practitioners intervening with the data so as to correct and modulate them in order to suit the referential situation. This continuous interference with the data brings emotions of anxiety in only one group of technicians (Group B). In contrast, Group A and C face it as a challenge in their every day work activity. The explanation for the variety of emotions displayed may relate to the technicians' academic background and the need for constant adaptation to their working environment. The Technicians in Group A had only vocational knowledge but their work activity was not influenced by the technological changes. Group C's academic knowledge may have helped them to cope with the massively complex, rapidly changing technological environment. Group B's limited academic knowledge does not help them to deal with all the changes they met in their daily working activity. All the above reasons ascertain the need to consider the case of reading and interpreting data as more than a basic mathematical skill. Reading and interpreting data in a complicated technological era where complex images dominate everyone's life is a crucial skill that has to be given special attention in the school curriculum.



## References

- Daniels, H. (2001). *Vygotsky and Pedagogy*. London: Routledge Falmer.
- Evans, J. Tsatsaroni, A. & Czarnecka B. (2009). Equations in a consumer culture: mathematical images in advertising. In Tzekaki, M. Kaldromidou, M. & Sakonidis H. (Eds.) *Proceedings of the 33th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 17-24. Thessaloniki - Greece.
- Han, J. Y., Roth, W.-M., & Pozzer-Ardenghi, L. (2005). Layered Inscriptions in Science Textbooks. *Paper presented at the Korean Association for Research in Science Education*, Seoul, Korea.
- Jurdak, M. (2006). Contrasting Perspectives and Performance of High School Students on Problem Solving in Real World, Situated, and School Contexts. *Educational Studies in Mathematics*, 63(3), 283-301.
- Leont'ev, A. N. (1978). *Activity, Consciousness, and Personality*. Englewood Cliffs: Prentice Hall.
- Mellin-Olsen, S. (1987). *The Politics of Mathematics Education*. Dordrecht: Reidel.
- Pozzi, S., Noss, R., & Hoyles, C. (1998). Tools in practice, Mathematics in use. *Educational Studies in Mathematics*, 36(2), 105-122.
- Roth, W. (2002). Reading graphs: contributions to an integrative concept of literacy. *Journal of Curriculum Studies*, 34 (1), 1-24.
- Roth, W. M. (2005). Mathematical Inscriptions and the reflexive Elaboration of Understanding: An Ethnography of Graphing and Numeracy in a Fish Hatchery. *Mathematical Thinking and Learning*, 7(2), 75-110.
- Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory*. Thousand Oaks: Sage.
- Triantafyllou, C. & Potari, D. (2006). Mathematical Activity in a Technological Workplace: Results from an ethnographic study. In Novotna, H. Moraova, H. Kratka, M. & Stehlikova, N. (Eds.) *Proceedings of the 30<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 297 – 304. Prague: Charles University.
- Triantafyllou, C. & Potari, D. (2010). Mathematical practices in a technological workplace: the role of tools. *Educational Studies in Mathematics*, 74(3), 275-294.
- Vygotsky, L. (1981). The development of higher forms of attention. In J. Wertsch & M. E. Sharpe (Eds.), *The concept of activity in soviet psychology* (pp. 189–240). New York: Armonk.
- Wedge, T. (2002). Numeracy as a Basic Qualification in Semi-Skilled Jobs. *For the Learning of Mathematics*, 22(3), 23-28.
- Williams, J. S. & Wake, G. D. (2007). Black Boxes in Workplace Mathematics. *Educational Studies in Mathematics*, 64(3), 317–343.



# MATHEMATICAL TOOLS IN ENGINEERING STUDENTS' DECISION-MAKING

Chrissavgi Triantafillou, John Vourvoulakis & Nicholas Assimakis

Technological Educational Institute of Lamia, Greece

*In this paper we investigate students' argumentation in an authentic activity. The two main theoretical constructs used in our work in order to capture mathematical elements in students' argumentation process is Leont'ev's three-tiered explanation of activity and Toulmin's framework as a structural model to analyse students' arguments. The main dataset for which detailed analysis was conducted were transcripts of students' meetings and students' e-mail communication. The results of this study indicate that students in ill-structured activities use mathematical tools (processes and concepts) and produce coherent arguments.*

## INTRODUCTION

Solving problems in the real world is frequently related with decision-making (Jurdak, 2007) calls for quantitatively sophisticated reasoning (Shoenfeld, 2002) and relates to critical citizenship (Skovsmose, 1994). An important skill in solving authentic, ill-structured problems is the production of coherent arguments to justify the best possible solution (Cho & Jonassen, 2002). Research emerging from a variety of disciplines supports the notion that examining students' argumentation serves as an effective means of accessing their informal reasoning (Patronis, Potari, & Spiliotopoulou, 1999; Misailidou & Williams, 2004).

In this paper we examine issues that arise from learning and knowledge in technology education. The context of our study is undergraduate engineering education in the department of Electronics of a Greek Technological Educational Institute. The current study is part of a research project that aims to investigate the development of scientific and mathematical knowledge of a team of engineering students while they are participating in an extra-curricular activity that is related to their studies. This activity involves the construction of a prototype/model quad copter (helicopter with four rotors). In this activity the students were allowed to decide on the central object of their activity (the construction of the quad copter) and choose all the actions they have to develop in order to accomplish their goal. Two members of the teaching staff, one Computer and Informatics Engineer and one Electrical and Computer Engineer (co-authors in this paper) supervise and support the students' actions. In this paper our aim is to investigate students' decision-making relating to the appropriate materials they have to choose for their model. To this end we examine the coherence of students' argumentation and we trace the mathematical tools they employed on their decision-making. By exploring these issues we anticipate to expand our understanding of students' behaviour in a complex, ill-structured learning

environment. This paper contributes to a controversial issue in mathematics education dealing with the role of authentic problems in classrooms activities (Boaler, 1993).

## THEORETICAL FRAMEWORK

The two main theoretical constructs used in our work in order to capture mathematical elements in students' argumentation process is Leont'ev's three-tiered explanation of activity and Toulmin's framework as a structural model to analyze students' argumentation. According to Leont'ev, activity is always energized by a motive; actions translate the motive to reality and operations accomplish the actions instrumentally (Leont'ev, 1978, pp.62-65). Leont'ev attributes to the concept of tool a central meaning and considers that a tool acts at the operational level of the activity. The concept of the tool has been expanded beyond the conventional view as a physical artifact to include a range of semiotic objects. For example, Vygotsky (1981) talks about psychological tools in which language, mnemonic techniques, algebraic symbols, diagrams, maps, drawings, or all sorts of conventional signs are included. Jurdak (2006, p. 288) used the term "tools of mathematics" in his work to refer to mathematical concepts, procedures, and strategies that high school students used to delay with problem-solving activities. The *construction of single arguments* is based on Toulmin's framework with the following elements: the *claim*, the *ground*, the *warrant*, the *backing*, the *qualifier* and the *rebuttal*. The first four elements determine the soundness of the arguments while the last two their strength. Claims are statements that advance the position learners take. Grounds involve evidences, observations, statistical data etc. and warrants are the logical connections between grounds and claims that indicate how a claim is supported by the grounds. Backing supports the validity of the warrants. Qualifiers refer to the degree of strength and certainty in one's argument while rebuttals challenge any element of others' arguments (Toulmin, Rieke & Janik, 1984).

In our study arguments can be considered as operations undertaken in students' actions of choosing the best materials for their aircraft (the object of their activity). These actions are necessary in order to translate their motive-oriented activity into reality. These operations are mediated through a number of tools mainly from the fields of science, technology and mathematics.

## METHODOLOGY

The participants are nine male engineering students, five from the Department of Electronics and four from the Department of Informatics.

*The research setting:* Students' activity refers to the study and construction of a quad rotor helicopter. The first action in this activity involves the choice and provision of materials such as the frame of the construction, the rotors, the motors, electronic devices as gyroscopes and accelerometers and the suitable computer software (programming language and tools) for motor control and stability of the aircraft. In this action the teaching staff role was to guide students for searching the appropriate

materials and pieces of equipment. Students worked alone or in pairs and met with the teaching staff once a week in the Institution area to share their findings, ask questions and make their weekly report on the progress of their actions. These meetings lasted approximately from one to two hours. In parallel, there was electronic communication between the students and their teachers. In this communication students shared their findings and teachers responded to their queries.

*The research process:* The research follows the ethnographic research tradition (Eisenhart, 1988). The researcher participated in all students' meetings and as well as in their electronic communication. Field notes were kept and the discussions between the students and their teachers were audio-recorded. The main dataset for which detailed analysis was conducted were transcripts of students' meetings and students' email communication. The selected data were analysed by the researcher, and subsequently the results of this analysis were discussed and negotiated between the researcher and the teaching staff. In parallel, there were separate discussions between the researcher and the teaching staff about themes arising from students' actions. In this paper the data derive from 4 meetings with the students lasting approximately 6 hours and 60 electronic messages (e-mails) among the students and the teaching staff. Analysis of students' argumentations was based on Toulmin's method. The data were classified as claims (C), grounds (G), warrants (W) and backing (B), qualifiers (Q) and rebuttals (R); students' arguments were schematically represented using Toulmin's categories and in each category we trace the mathematical tools employed.

*The role of the researcher:* The researcher is a math teacher in the same Technological Institute. The researcher's role was identified as that of the participant observer. Often the researcher asked the students to elaborate on their positions and clarify their reasoning.

## RESULTS

Almost in all students' argumentation process, the *grounds* were data concerning internet resources (videos with quad copters, specific links referring to the desired materials, and relative forums where air modellers present their ideas). We identified mathematical tools in all elements of students' argumentation. These mathematical tools were geometric shapes when they illustrated the configuration frame of the aircraft and geometric transformations when they described the independent displacements and rotations that specify the orientation of the aircraft in space (translation and rotation in the x-axis, y-axis and z- axis). The students also realized correlations among quantities (e.g. the more range of values that an instrument indicates, the less sensitivity it could have) and reason by using their common sense (i.e. if you want to weigh something in tonnes- a large range of values- you don't care about the sensitivity of your instrument in grams). Moreover, they identified analogical relations among quantities (e.g. they choose the rotor with the smallest length since it guarantees the biggest torque) and justify these relations by using logical and scientific arguments (i.e. the smaller the rotor the easier for the motor to

roll it) avoided using mathematical relations to support their claims. Finally, since students’ activity was based on the use of digital technology devices (e.g. choosing the best computer programming behaviour i.e. 8-bit or 16-bit processor), concepts related to the base-2 number system (a system of numbers in two numerical digits 0 and 1) were often present in students’ discussions.

Several times their decision-making was straight forward without having to consider any other alternatives (as in case 1) while in some other cases they had to reconsider their first choices because new data are coming into the discussion (as in case 2). In case 2 one student rejects (*rebuts* in Toulmin scheme) his colleagues’ arguments. In other cases the students come up with heuristic solutions in order to overcome adversities that they faced (as in case 3).

**Case 1: Constructing the frame configuration.**

In the following extract two students, Fillip and James, (all students’ name are pseudonyms), present their decision about the configuration of the frame.

(*Fillip had watched videos with quad copters online*). Fillip: After having watched many videos, I have understood that probably when the frames are bigger than 80cm then the stability of the quad copter is made harder to achieve. So, I believe we should take approximately 50-60cm or 25-30cm radius [Fillip presents his 3-d model of the frame configuration as in Fig 1].

Teacher: Let’s choose a 30cm radius, a 60cm diameter. It is a bit arbitrary but we have to start from some point

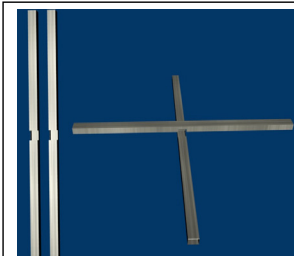


Fig. 1: The frame configuration Fillip created.

Fillip: Despite the fact that the construction of the frame is one-way, perpendicular bars, these bars can be in 2 comprehensible formulations. In plus [+] and in X which means that it will have one

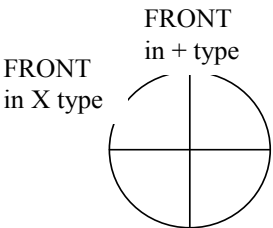


Fig. 2: Fillip’s visualization of the frame configuration.

motor in front (driver) or two. We suggest them to be in the shape of a plus. We shall put the motors and the rotors on each end and the electronic equipment in the centre.

Researcher: Why is the plus formulation better than the X?

James: Because it is used more often and control is made easier whereas in the X formulation appears to be a different approach in the way the motors work. (The above extract comes from the 2<sup>nd</sup> meeting with the students).

In Fillips' argumentation we recognize the existence of geometric shapes, such as rectangular prisms perpendicular on each other that consist the frame configuration of their aircraft (*warrant1*); the estimation of the maximum length dimension of the rectangular prisms (in this case the student uses a *qualifying* phrase); his visualization of the frame construction as a circle (*claim 1*) and his identification that two perpendicular lines could have two different spatial orientations (*claim 2*). In Fig. 2 we present Fillip's visualization of the frame configuration in the two possible spatial orientations. In each argumentation element the mathematical tools students' employed are integrated and supported with concepts from technology and the referential situation (i.e. "stability" or "control of the aircraft"). Fillip's contextual anchoring helps him to develop a dynamic image of his geometrical construction and visualize it as a circle so it could have any spatial orientation possible.

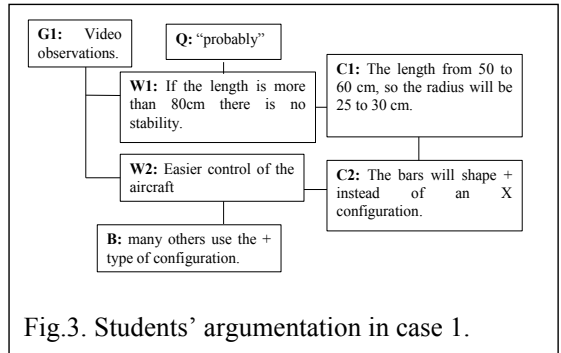


Fig.3. Students' argumentation in case 1.

### Case 2: Selecting the best motor type.

In order to select the best motor type the teachers made clear to the students that they

Fig.4. The *grounds* in students' argumentation (case 2).

Motor type 1		The motor		Motor type 2	
Kv (rpm/v)	1800	Kv (rpm/v)	700	Kv (rpm/v)	1750
Weight (g)	26.8	Weight (g)	78	Weight (g)	50
Max Current (A)	9	Max Current (A)	13	Max Current (A)	13
Resistance (mh)	270	Resistance (mh)	0	Resistance (mh)	0
Max Voltage (V)	11	Max Voltage (V)	11	Max Voltage (V)	11
Power(W)	0	Power(W)	0	Power(W)	0
Shaft A (mm)	-	Shaft A (mm)	4	Shaft A (mm)	-
Length B (mm)	17	Length B (mm)	38	Length B (mm)	32
Diameter C (mm)	28	Diameter C (mm)	40	Diameter C (mm)	31
Can Length D (mm)	5	Can Length D (mm)	10	Can Length D (mm)	9
Total Length E (mm)	31	Total Length E (mm)	78	Total Length E (mm)	65
Price:	13.79\$	Price:	10.95	Price:	5.49\$
	Motor type 1		Motor type 3		Motor type 2

suggestions. We present here Alex's and Marcos's suggestions and the new considerations that arise when a student rejects their arguments by providing new *grounds* that they have to consider in their decision-making.

Alex's argumentation: Guys, I believe that those are the best, in terms of price and quality, for the specific purpose [he presents the links that refer to motor type 1 and 2 in

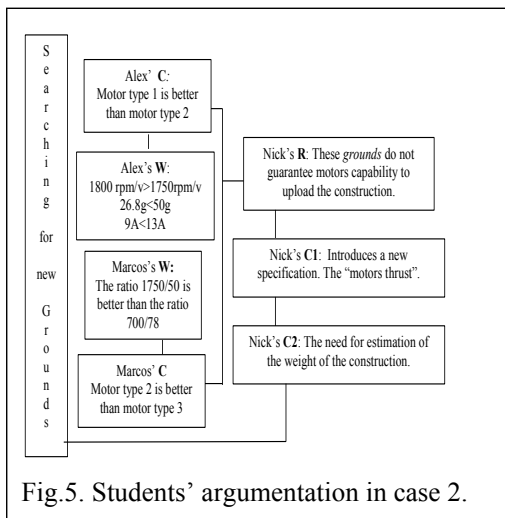
Fig.4]. The second [he refers to motor type 2] may be better in terms of price but the first combines everything, high rate of rounds, adequately low Ampere consumption and it is light enough [he refers to motor type 1].

Marco's argumentation: I give my best choices [he presents the links that refer to motor type 2 and 3 in Fig. 4]. My first choice is the one with 1750 rpm/v [rounds per minute per Volt][motor type 2] because it has a better rate rpm over weight since it has more rounds per minute and less weight from the other.

Nick: When we are looking for motors we must check its thrust. How much weight it is able to uplift. I saw one with 11000rpm/v but if you give it a weight it does nothing. However in this case we have to estimate the weight of the whole construction. (The above data come from student's electronic communication).

The main mathematical tools in Alex's and Marcos' argumentation are reading, interpreting and comparing numerical data. Alex compares numerical values that

seem to be important in this case while Marcos prefers to compare ratios of numerical values (the rounds per minute-to-weight ratio). Nick introduces a new specification "motors thrust" (*claim 1*). Nick *rebutts* students' claims by giving an example to illustrate that the large number of rounds per minute does not guarantee their goal (make the aircraft capable to fly). A new circle of argumentation begins. Since "motors thrust" depends on the weight of the aircraft, they have to estimate the weight of the aircraft before constructing it (*Nick's claim 2*). Nick' rebuttal forces students to



enter into a new circle of argumentation during which a new mathematical tool (the weight estimation) enters their argumentation process. Often one student used arguments in order to defend or reject his colleagues' choices. For example, there was a disagreement with their teachers about the maximum possible consumption of the motor. Their teacher argues that 13 or even 9 Amperes (as shown in Fig. 4) was extremely large consumption and he suggested motors with consumption up to 500mA. The students insisted on their selections and supported their choices by referring to other air modellers choices: "All the others use this type of motors why don't we?".

### Case 3: Converting the output of an electronic device (the gyroscope).

The gyroscope measures the angular velocity of the aircraft around the three axes (how many degrees per second the aircraft rotates) and it gives 3 outputs -pitch, yaw

and roll- for the rotation in the x, z, y axes respectively (Fig. 6). The teacher in the following extract challenges students' activity by addressing the following problem: How could we change the degree per second ( $^{\circ}/s$ ) output to a simple numerical output in degrees? A student recommends a method to handle the above change.

Teacher: We should take the angle from the gyroscope in a certain time interval otherwise we will not be able to combine it with the data from the other devices.

Athan: We could take the absolute angle using the rate value. For example, if it shows that it has changed 2 degrees per second [the instrument indicates  $2^{\circ}/s$ ], for 5 seconds it changed by 10 degrees ( $2 \times 5 = 10$ ) during the time we measured. (The above extract comes from the 4<sup>th</sup> meeting with the students).

The ground in Athan's argument is one specification of the device (Fig. 6). His claim is that the problem that teacher addresses is manageable, his warrant is an example that illustrates the way they could handle this situation and his backing is his

<p>Gyroscope Module Pitch, yaw, and roll gyro output 0.83 and 3.33 mV/<math>^{\circ}/s</math> sensitivity, respectively, <math>\pm 300^{\circ}/s</math> range</p>
---

Fig. 6. The grounds in Case 3.

understanding of what rate of change actually means. In this case the mathematical skill of reading and interpreting data is not enough but the restrictions of the activity require intervening with the data in order to suit the referential situation.

## DISCUSSION

In this paper we investigate students' argumentation in an authentic technological activity. We consider argumentations as operations undertaken by students in their decision-making actions about the best materials for their quad copter model. Leont'ev's (1978) three-tiered explanation of activity allows us to identify mathematics at the operational level in students' argumentation process. We identified mathematical tools in all elements of students' arguments on Toulmin's model (Toulmin et al., 1978). These tools were mathematical processes (estimation; reading, interpreting and comparing numerical data; using ratios); and mathematical concepts (base-2 number system; geometrical transformations; rate of change). All the above mathematical tools were embedded in the situation and were interrelated with tools from students' scientific and technological knowledge. This students' anchoring to the referential situation appears to be not the weakness but the strength in students' argumentation process since it seems to help them to develop dynamic images of geometrical shapes, heuristic methods to overcome adversities, and help them monitor arguments uttered by others. But we also realized that students avoid using formal mathematical tools (e.g. algebraic relations) to warrant their claims and it seems to agree with other research findings that mathematics plays a minor role in students' real life decision-making (e.g. Jurdak, 2006, pp. 296-298). But in our study, when students judged that this type of knowledge was necessary they used their formal mathematical knowledge in a creative way.



Students produced coherent arguments in all their decision-making processes. Moreover, they felt secure to reject (rebut) their colleagues' and their teachers' arguments. These results are consistent with many relative research findings in ill-structured activities (Cho & Jannassen, 2002). Successful argumentation in this case depends on the degree of how well one's argument makes others' opinions change, so rebuttals are critical elements in any decision-making.

The findings in this study support the case of using authentic, ill-structured problems in mathematical classes. In this way we could help students to develop well-formulated arguments and defend their own solutions and encourage them to justify their claims by using formal mathematical tools. This will help students to participate thoughtfully in societies that depend on mathematics, science and technology.

## References

- Boaler, J. (1993). The Role of Contexts in the Mathematics Classrooms: Do they make Mathematics more "Real"? *For the learning of mathematics*, 13(2). Pp. 12-17.
- Cho, K.& Jonassen, D. (2002). The Effects of Argumentation Scaffolds on Argumentation and Problem Solving. *Educational Technology Research and Development*, 50(3), pp. 5-22.
- Eisenhart, M. (1988). The Ethnographic Research Tradition and Mathematics Education Research. *Journal for Research in Mathematics Education*, 19(2), 99-114.
- Jurdak, M. (2006). Contrasting Perspectives and Performance of High School Students on Problem Solving in Real World, Situated, and School Contexts. *Educational Studies in Mathematics*, 63(3), 283-301.
- Leont'ev, A. N. (1978). Activity, Consciousness, and Personality. Englewood Cliffs: Prentice Hall.
- Misailidou, C. & Williams, J. (2004). Helping children to model proportionally in group argumentation: Overcoming the constant sum error. *Proceedings of the 28<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 321-328.
- Patronis, A., Potari, D. & Spiliotopoulou, V. (1999) Students' argumentation on a socio-scientific issue: Implications for teaching. *International Journal of Science Education*, 21(7), 745-754.
- Schoenfeld, A. (2002). Making mathematics work for children: Issues of standards, testing, and equity. *Educational Researcher*, 31(1), pp. 13-25.
- Skovsmose, O. (1994). Towards a philosophy of critical mathematical Education. Dordrecht Kluwer Academic Publishers.
- Toulmin, S, Rieke, R. & Janik, A. (1984). An introduction to reasoning. Second Edition. N.Y. Macmillan Publishing Co., Inc.
- Vygotsky, L. (1981). The development of higher forms of attention. In J. Wertsch & M. E. Sharpe (Eds.), *The concept of activity in soviet psychology* (pp. 189-240). New York: Armonk.



# MATHEMATICAL CONTENT KNOWLEDGE REVEALED THROUGH THE FOUNDATION DIMENSION OF THE KQ

Fay Turner

University of Cambridge

*In this paper I present findings from a completed four-year developmental research project in which the Knowledge Quartet (KQ) framework was used as a tool to identify and to support development of the mathematical content knowledge of a group of early career elementary teachers. I focus on the propositional knowledge (Shulman, 1986), of one participant as revealed through the Foundation dimension of the KQ during observations and discussions of her teaching. These findings suggest that supported reflection on teaching which is focused on mathematical content may enhance the development of mathematical content knowledge for teaching.*

## INTRODUCTION

What constitutes mathematical content knowledge for teaching, and how an understanding of this might be used to support the development of teachers, has been a consistent thread in meetings of the International Group for the Psychology of Mathematics Education (PME) over the last ten years (Rowland, Martyn, Barber and Heal, 2001, Turner, 2008) and was the subject of a Research Forum coordinated by Deborah Ball and colleagues at PME in Thessaloniki in 2009. Shulman’s (1987) conceptualisations of subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge (CK), and more recently the refinements of these categories by Ball, Thames and Phelps (2008), have underpinned much of the debate. In Ball *et al*’s refinements *common content knowledge* (CCK), which is knowledge of mathematics that non-teachers might be expected to have, and *specialised content knowledge* (SCK) that is necessary only for teaching are seen as two aspects of SMK. *Knowledge of content and teaching* (KCT) and *knowledge of content and learners* (KCL) are seen as two aspects of PCK. These categories of knowledge are frequently referred to in this paper and Table 1 below gives an overview of these for easy reference.

Shulman (1987)	Subject Matter Knowledge SMK		Pedagogical Content Knowledge PCK	
	Common Content Knowledge CCK	Specialised Content Knowledge SCK	Knowledge of Content and Learners KCL	Knowledge of Content and Teaching KCT

Table 1: Categories of mathematical content knowledge

One approach to identifying and developing mathematical content knowledge which was discussed at the forum in Thessaloniki was the Knowledge Quartet (KQ) framework (Rowland and Turner, 2009). The KQ framework was developed empirically from observations of mathematics teaching and the categorisation of situations in which mathematical content knowledge was revealed (Rowland, Huckstep and Thwaites, 2005). In order to make the framework more useful to mentors and tutors when observing mathematics teaching, the original 18 ‘situations’ or codes which emerged (Glaser and Strauss, 1967) were classified into four ‘superordinate’ categories or dimensions based on associations between them. These categories make up the four dimensions of the Knowledge Quartet: *foundation*, *transformation*, *connection* and *contingency*.

The codes and dimensions of the KQ refer to situations in which content knowledge is revealed rather than to categories of knowledge per se. However, links can be made to the categories set out in Table 1 above. The *foundation* dimension encompasses situations in which propositional SMK and PCK become apparent. The prefix ‘propositional’ suggests that this knowledge is in the form of *knowing-about* (Mason and Spence, 1999) and is knowledge which may be gained ‘in the institution’, from personal research or experience. The three remaining dimensions of the KQ categorise types of situations in which teachers’ propositional PCK is activated to make it accessible to learners in the process of teaching. Knowledge relating to these three dimensions may be conceptualized in terms of Mason and Spence’s (1999) notion of *knowing-to*. The *Transformation* dimension encompasses situations in which the *active* forms of PCK are revealed through demonstrations, representations and examples used by teachers. The *Connection* dimension encompasses situations in which a teacher’s knowledge of connections in mathematics (SMK) is made visible in their teaching and also encompasses situations in which knowledge of how to sequence mathematics teaching and make connections for learners (PCK) becomes visible. Finally, the *Contingency* dimension relates to situations in which teachers respond to the unplanned-for and the unexpected in their teaching, when they are seen to draw on combinations of aspects of both SMK and PCK.

In this paper I focus on the identification and development of one beginning teacher’s mathematical content knowledge as revealed through situations under the *foundation* dimension of the KQ, i.e. examples of situations in which her propositional SMK or PCK were revealed.

## METHODOLOGY

My research questions were concerned with whether reflection on practice with a focus on mathematical content would help early career elementary teachers to develop their mathematical content knowledge and whether the Knowledge Quartet framework would be an effective tool in helping them to focus their reflection in this way. The study took place over four years, beginning with 12 participants in their

postgraduate teacher preparation year. As expected, this cohort reduced to nine in the second year, to six in the third year and finally to four in the fourth and final year of the study. This attrition was predicted, and a consequence of the participants' relocation and changes in commitment to the project. The study was based on a model of teacher professional development through reflection (Schön, 1983) in which the teachers used the KQ framework as a tool to support reflection on, and discussion about, their mathematics teaching. Videotapes of the participants' lessons were used to aid recall and to allow in-depth reflection on their teaching.

The participants were introduced to and familiarised with the KQ framework in their teacher preparation year. One lesson taught by each of the 12 participants was videotaped and analysed during their final teaching placement. These videotapes were used in one-to-one stimulated recall interviews with the participants using the KQ to focus on the mathematical content of each lesson. During their first year of teaching, the participants were videotaped teaching mathematics on two occasions and given focused feedback structured by the KQ framework during post-lesson reflective interviews. They later watched the videotapes and wrote reflective accounts of these lessons. This was intended to support and develop their own use of the framework so that in the following year participants were able to use the framework more independently. During the third year of the project, three lessons were observed and videotaped. There were no post-lesson reflective interviews and the participants completed independent reflective accounts of these lessons after watching them on DVD. In this phase of the study, the participants also wrote periodic reflective accounts of their mathematics teaching more generally. Participants attended a number of group interviews or meetings over the four years of the study. One of these was held at the end of the first year, two during the second year, three during the third year and one in the final year of the study. Individual interviews and observations in the last year of the study gave final indications of the development in participants' mathematical content knowledge as it was evidenced in their teaching.

Lesson observations were analysed using the dimensions and constituent codes of the KQ. Transcripts of interviews and the participants' written reflective accounts were analysed using the computer-aided qualitative analysis software NVivo. This gave rise to a hierarchy of emergent codes and themes (Glaser and Strauss, 1967) which informed the final analysis of the data. This analysis constituted the writing of four case studies charting developments in the teachers' mathematical content knowledge and changes in their conceptions of mathematics teaching. This paper reports findings about just one aspect of content knowledge from one of these case studies, Kate. Similar developments in knowledge revealed through the *foundation* dimension of the KQ were also found in the other case studies (Turner, 2010).

At the beginning of the study, Kate was a student on the one year postgraduate course for preparation in elementary school teaching at the University of Cambridge. She had gained the top grade in the national mathematics examinations (GCSE) at age 16, and the audit given at the beginning of the course suggested she was confident in her

own mathematical content knowledge. Kate's final teaching placement was in a Year 1 class (5-6 years). She took up her first post in a Year 1/2 class (5-7 years) and taught a Year 2 class (6-7 years) in years 3 and 4 of the study. Despite confidence in her own mathematical understanding, Kate initially expressed a lack of confidence in her ability to teach mathematics.

## FINDINGS

Analysis in relation to the *foundation* dimension of the KQ suggested a number of issues in Kate's mathematical content knowledge as revealed through her early teaching. On a number of occasions during the lesson observed in her teacher preparation year, Kate used a number line to help children complete addition calculations such as ' $8 + 8$ ' and ' $3 + 4$ ' by beginning at one of the numbers and then counting on the second number. This pre-supposed that children had reached the 'count on' stage in addition. However, my observation of the children's independent use of the number lines suggested that some were still at the 'count all' stage (Carpenter and Moser, 1984). At this point Kate's teaching did not reflect knowledge of stages in addition strategies (KCL). In the post-lesson reflective interview I asked Kate if she remembered the stages children go through in learning addition:

At first not knowing that you can just start at numbers, that you have to count the one, two, three ... so you have to count three to get up to three before you can carry on. (Kate, post-lesson reflective interview)

Kate's focused reflection on her teaching helped her to remember this aspect of KCL that had been introduced during her university mathematics methods course but which she hadn't applied to her teaching. The following year, Kate demonstrated that she was applying her knowledge of stages in addition strategies when, in the first of her lessons observed that year, she modelled a number of methods for solving addition problems, involving both 'count all' and 'count on' strategies.

An issue relating to Kate's KCT was also revealed in the lesson that I observed during her teacher preparation year. The lesson was about doubles and near doubles and Kate consistently recorded doubling as additions, e.g.  $3 + 3$ , focusing on additive rather than multiplicative reasoning. When demonstrating the doubling of two-digit numbers, Kate recorded the doubled number of tens and of units separately, e.g. double 13 was recorded as 2 tens and 6 units. This focused on *column value* rather than *quantity value* (Thompson, 2003). When discussing this, Kate recognised that her recording of this procedure might have been problematic:

Maybe it wasn't the most helpful thing to do at all because the easy way to do it is to think about it as a column method and if you are not doing it that way then it is probably quite hard. (Kate, post-lesson reflective interview)

Kate realised that her recordings reflected a column method for addition and did not support the mental methods recommended during her teacher preparation course and in curriculum guidance (DfEE, 1999). Kate's supported reflection on her teaching

enabled her to draw on KCT that had not been available to her in the planning and teaching of this lesson.

A lesson observed in the middle of Kate's first year of teaching revealed a significant issue in her SCK. This related to her knowledge of the two structures of subtraction described by Haylock (2006) as *partition* and *comparison*. In English primary schools, teachers are expected to explain both of these structures which are described in the curriculum guidance (DfEE, 1999) as 'take away' and 'difference' respectively. Kate's stated learning objective for the children was 'to find the difference between two numbers'. After some discussion, demonstrations and use of representations which all clearly related to the *comparison* structure of subtraction, Kate asked the children "What is the difference between nine and five?" and instructed them to show their jottings on small wipe-able boards. Some children's jottings represented the partition structure of subtraction e.g. one child explained that he had drawn nine lines and then erased five of them. Kate accepted all responses as correct and did not distinguish between representations of the *partition* structure and those which modelled the *comparison* structure. In the post-lesson interview I asked Kate if she thought that the child's recording of a subtraction, in which he drew a number of dots and then crossed some out, represented finding the difference. Kate said that she thought it did, revealing some confusion in her SCK, specifically her understanding of the two structures of subtraction.

I discussed this confusion with Kate during the post-lesson reflective interview and she appears to have continued to reflect on this particularly in relation to how knowledge of these different structures might inform her teaching. In a later reflective account of her mathematics teaching she wrote:

I had an interesting discussion with Fay about my subtraction subject knowledge – she pointed out that 'take away' is a different kind of subtraction than 'find the difference', not just different vocabulary for the same thing. This has made me think a lot about the examples I choose – so that when I ask 'take away' questions, I am not using numbers which are more appropriate to 'find the difference' methods. (Kate, reflective account)

Reflection on, and discussions of, her teaching structured by the KQ seemed to have helped Kate to recognise the implications of the two structures of subtraction for her teaching and to develop her SCK. Kate's confidence in her mathematical content knowledge for teaching developed over the study. However this confidence appeared to depend on her familiarity with teaching a particular mathematical topic. In the third year of the study, Kate felt that her understanding of number was the most secure area of her mathematical content knowledge because she had more experience of teaching this:

I like the number bits. I feel less confident in things like shape and data handling and length. I don't feel as confident in these because I have not taught them as often. I quite like the fact that some of the mathematical operations and things and the number bits is relatively complicated, but I understand all the complications. (Kate, group interview)

It was interesting that Kate recognised that number was ‘relatively complicated’, even at the level of teaching 6-7 year old children and it is likely that this recognition was aided by her reflections on the content of her mathematics teaching. It was not only the *experience* of teaching different topics that developed Kate’s confidence in teaching them but also her *reflection* on that experience.

The final observation of Kate teaching mathematics was made near the end of the study and this suggested that she had become confident at drawing on her SCK when teaching. The lesson had been planned by another teacher as a power point presentation and was about addition of two-digit numbers. Kate had only briefly looked at this and had not had time to make any amendments before using the presentation in her lesson. When a slide was displayed showing  $23 + 12$  as  $(20 + 10) + (3 + 2)$ , Kate ‘deviated from the agenda’<sup>1</sup> and made a new slide showing  $23 + 12$  as  $23 + 10 + 2$ . In the post lesson reflective interview I asked Kate why she had changed her demonstration in this way. Kate explained that when teaching two-digit addition, she normally used the strategy of keeping one number whole and partitioning the second number as children found this easier. When she realised that the slide was introducing a different addition strategy, she changed the slide to represent this method. Although she did not use the terms *I010* and *N10* (Beishuizen, 2001), Kate demonstrated that she was aware of these two different methods of addition (SCK), and was able to make decisions about the appropriateness of their use while teaching (KCT).

## DISCUSSION

Analysis of Kate’s mathematics teaching through the lens of the *foundation* dimension of the KQ revealed a number of issues in her propositional SMK and PCK over the four year period of the study. In the lesson observed in her training year Kate did not draw on knowledge about the stages in addition (KCL) or knowledge about the difference between column and quantity value (KCT) to inform her teaching. In her first year of teaching, observation and discussion suggested that Kate did not have a sufficiently secure understanding of the two structures of subtraction to support her teaching. However, there is evidence that participation in this developmental research study meant that Kate reflected on her teaching in a way that focused on her mathematical content knowledge and that this reflection supported the development of her knowledge. Data from the study indicate that focused reflection helped Kate to recognise incongruence between her propositional knowledge of the stages in addition and her practice, and to activate this knowledge in later teaching. Data also suggest that Kate’s reflection helped her to develop propositional knowledge relating to the appropriateness of using column value or quantity value in her teaching as well as to develop propositional knowledge of the structures of subtraction and to consider the implications of this knowledge for her teaching.

<sup>1</sup> A code from the contingency dimension of the KQ

There is evidence to suggest that, by the final year of the study, Kate had developed a more secure knowledge base on which to draw in her mathematics teaching, e.g. her knowledge of the different methods for addition. Her adaptation of the power point presentation suggested that Kate had become more confident that she could draw on her PCK to support children's learning. Kate's confidence in her mathematical knowledge for teaching was supported by a comment made during an individual interview in year 4:

In PE or DT I feel my knowledge is quite vague ... I don't really know whether we have learned what we are supposed to learn because I only just know what we were supposed to be learning anyway. I think I have a much better idea for numeracy of what we [Kate and her pupils] are supposed to be learning. (Kate, individual interview)

Kate felt increasingly confident about her mathematics teaching. There is evidence to suggest that her reflection, structured by the KQ to focus on the mathematical content of her teaching, helped to develop aspects of Kate's mathematical content knowledge and to increase her confidence that she would be able to draw on her SMK and PCK in her teaching.

The findings reported in this paper offer some evidence that reflection on practice, which focuses on mathematical content, helps early career elementary teachers to develop their mathematical content knowledge for teaching. There is also evidence that the KQ framework can be an effective tool in helping to focus reflection in this way. The KQ framework focused Kate's reflection on the mathematical content knowledge underpinning her teaching. This reflection was supported by discussions with me which were framed by the KQ, especially in the first two years of the study. Reflection on, and discussion of, situations revealed through the *foundation* dimension of the KQ appear to have supported development in aspects of Kate's propositional mathematical content knowledge for teaching.

It is uncertain whether independent use of the KQ framework, without input from a more knowledgeable other would support development of mathematical content knowledge in the same way. Further research is needed to establish whether this might be the case. Such research would help to establish whether/how the KQ might be used more widely by early career teachers and mentors to develop mathematical content knowledge for teaching. Further research might also suggest whether the KQ framework might usefully be employed by more experienced teachers to develop mathematical content knowledge for teaching.

## References

- Ball, D. L., Thames, M. H. and Phelps, G. (2008). 'Content knowledge for teaching: What makes it special?' *Journal of Teacher Education*, 59(5), 389-407.
- Beishuizen, M. (2001). 'Different approaches to mastering mental calculation strategies'. In J. Anghileri (Ed) *Principles and practices in arithmetic teaching*, pp. 119-130). Buckingham, Philadelphia: Open University Press.



- Carpenter, T.P. and Moser, J.M. (1984). 'The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15(3), 179-202.
- DfEE (1999). *The National Numeracy Strategy*. Department for Education and Employment Publications.
- Glaser, B. G. and Strauss, A.L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine de Gruyter.
- Haylock, D. (2006). *Mathematics explained for primary teachers*. 3rd Edition. London: Sage.
- Mason, J. and Spence, M. (1999). 'Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment'. *Educational Studies in Mathematics*, 38, pp. 135-161.
- Rowland, T. and Turner, F. (2009) Research Forum 1. In M. Tzekaki, M. Kaldrimidou and H. Sakonidis. *Proc. 33<sup>rd</sup> Conf. of the Int. Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 121-149).Thessaloniki, Greece: PME.
- Rowland, T., Martyn, S., Barber and Heal, C. (2001). 'Investigating the mathematics subject matter knowledge of pre-service elementary school teachers'. In M. van den Heuvel-Panhuizen (Ed.) *Proc. 25th Conf of the Int. Group for the Psychology of Mathematics Education*, (Vol.4, pp. 121-128). Utrecht, The Netherlands: PME.
- Rowland, T., Huckstep, P. and Thwaites, A. (2005). 'Elementary teachers' mathematics subject knowledge: The Knowledge Quartet and the case of Naomi'. *Journal of Mathematics Teacher Education*, 8, 255-281.
- Schön, D.A. (1983). *The Reflective Practitioner*. New York: Basic Books.
- Shulman, L. (1986). 'Those who understand: Knowledge growth in teaching'. *Educational Researcher*, 15 (2), 4-14.
- Shulman, L. (1987). 'Knowledge and teaching: Foundations of the new reform'. *Harvard Educational Review*, 57 (1), 1-22.
- Thompson, I. (2003). 'Place value: the English disease'. In Thompson (Ed.) *Enhancing Primary Mathematics Teaching*, pp. 181-190. Buckingham: Open University Press.
- Turner, F. (2008) 'Growth in teacher knowledge: Individual reflection and community participation'. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano and A. Sepúlveda (Eds) *Proc. 32nd Conf of the Int. Group for the Psychology of Mathematics Education*, (Vol.4, pp. 353-361).
- Turner, F. (2010). *Using the knowledge quartet to develop early career teachers' mathematical content knowledge: A longitudinal study*. Unpublished thesis, University of Cambridge.



# INSIGHTS INTO STUDENTS' MATHEMATICAL PROBLEM POSING PROCESSES

Xianwei Van Harpen and Norma Presmeg

Illinois State University

*Despite the importance of problem posing in mathematics education suggested by researchers and educators, research has shown that students in China are not as strong as those in the United States in the abilities of posing mathematical problems. This study focused on high school students with strong mathematics background in China and the United States and investigated their problem posing processes and their attitudes towards problem posing in mathematics.*

## INTRODUCTION

In the literature, problem posing is claimed to be an important aspect of problem solving that can be important in fostering students' creativity (Silver, 1997). In the United States, according to the National Council of Teachers of Mathematics (NCTM) (2000), students should be given opportunities to solve mathematical problems using multiple solution strategies and to formulate and create their own problems from given situations. In China, in a document entitled the Interpretation of Mathematics Curriculum (Trial Version) (Mathematics Curriculum Development Group of Basic Education of Education Department, 2002), it is pointed out that students' abilities in problem solving and problem posing should be emphasized and that students should learn to find problems and pose problems in and out of the context of mathematics. Recent comparative studies between the United States and China on mathematical problem posing have shown that Chinese students' problem posing is not stronger than that of otherwise comparable U.S. students (Cai & Hwang, 2002). At the same time, Chinese students' ability in posing mathematical problems has been a concern to mathematicians. For example, the Harvard professor Shing-Tung Yau (Sun, 2004), who is the only Chinese-born mathematician to have won the Fields Medal, compared graduate students who studied under him and commented that students from China are very good in basic skills but, in comparison with U.S. students, often lack mathematical creativity and ability in posing research questions.

The purpose of this study was to investigate U.S. and Chinese students' attitudes and abilities in posing mathematical problems. According to Usiskin's (2000) eight-tiered hierarchy of mathematical talent, students who are gifted and/or creative in mathematics have the potential of moving up into the professional realm with appropriate affective and instructional scaffolding as they progress beyond the K-12 realm into the university setting (Sriraman, 2005). Therefore, this study focuses on

high school students with a strong background in mathematics. The following three questions were addressed in this study:

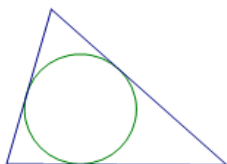
1. To what extent do the students pose mathematical problems in their mathematics learning?
2. What is the nature of the students' mathematical problem-posing abilities?
3. What are the students' perceptions of the role of problem posing in their mathematics learning?

## A PROBLEM POSING FRAMEWORK

In the present study, mathematical problem posing is defined as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and from these situations formulate meaningful mathematical problems (Stoyanova & Ellerton, 1996). In the framework proposed by Stoyanova and Ellerton (1996), they classified a problem-posing situation as free, semi-structured or structured. According to this framework, a problem-posing situation is free when students are asked to generate a problem from a given, contrived or naturalistic situation (see Task 1 below), semi-structured when students are given an open situation and are invited to explore the structure of that situation, and to complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences (see Task 2 below), and structured when problem-posing activities are based on a specific problem (see Task 3 below). The mathematical problem-posing test was developed based on Stoyanova's (1997) and Cai's (2000) research.

Task 1. There are 10 girls and 10 boys standing in a line. Make up as many problems as you can that use the information in some way.

Task 2. In the following drawing, there is a triangle and its inscribed circle. Make up as many problems as you can that are in some way related to this picture.



Task 3. Last night there was a party at your cousin's house and the doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang, three more guests arrived than had arrived on the previous ring.

- a. How many guests will enter on the 10th ring? Explain how you found your answer.
- b. Ask as many questions as you can that are in some way related to this problem. Also try to pose some other problems with a similar structure.

## METHODOLOGY

30 U.S. students and 55 Chinese students who were taking advanced mathematics in high school participated in this study. Two tests were administered to the students, namely, a mathematics content test and a mathematical problem-posing test. The purpose of the mathematics content test in this study was to measure the participants' basic mathematical knowledge and skills. This test was adapted from the National Assessment of Educational Progress (National Center for Educational Statistics, 2009) 12th grade Mathematics Assessment. The mathematics content test and the mathematical problem-posing test were translated into Chinese then translated back into English. Several pilot tests were conducted with U. S. students and Chinese students before their use in the research. This paper reports the results of interviews conducted with selected students to augment the data from their tests.

## ANALYSIS OF THE INTERVIEWS

Eight students in the Chinese group and twelve students in the U.S. group were interviewed. All of the 20 students achieved 40 or more points out of 50 in the mathematics content test. The interviews were audio taped and transcribed.

*Question 1: Have you posed problems in mathematics before (other than in this study)?*

Most of the students, including the U.S. group and the Chinese group said they had had limited or no experience in posing mathematical problems, as illustrated in the following excerpts. Among the U.S. students who were interviewed, six said that they had never before done problem posing in mathematics. For example,

Mulnon; No, I have never done anything like that before. That was the first time.

However, four students in the U.S. group mentioned their experience in posing mathematical problems for test reviews. For example,

Kyle: Sometimes when we are studying for finals..., I mean, for tests... I would just turn to the books and I just found the problems and I just changed the numbers.

There was one student in the U.S. group who reported some experience in mathematical problem posing.

Ramona: I have never been tested on it, but I think in eighth grade, we had to write an equation and give each other to make sure we knew it, like we could make up our own problems to solve them and understand. You know. We have done that but really I haven't had to do very much. I have never had to take a test on it.

The Chinese students also did not have much experience posing problems in mathematics. Among the eight Chinese students, four said they had never had experience posing mathematical problems. For example,

Ning: No. We have only solved mathematical problems.

One Chinese student, Yanan, said that she had some experience finding problems for other students in her class, but she did not pose problems on her own.

Yanan: No, our teachers never asked us to make up problems. What we have done is to look for problems. I remember in the second year of high school, our teacher asked us to take turns to give a problem to the whole class. We would go find problems in some books or tests. We were allowed to make up our own, but nobody did that, because it was hard.

There are three students in the Chinese group who said that they had some experiences posing problems in elementary or middle school, but not in high school. For example,

Shidong: I didn't do that in high school but I did in elementary school and middle school. There were times when we were taking a test, if we could pose a new problem, we would get extra points.

*Question 2: How did you pose these problems?*

U.S. students frequently used the terms "thinking outside of the box," "something fun," and "something interesting," while Chinese students focused more on the mathematical content. U.S. students tended to free their minds and let the ideas come to them, and when asked how they posed those problems, many of them did not know how to explain and said "I don't know" or "I am not sure." For Chinese students, unlike the case for the U.S. students, every one of the eight students knew how they were posing the problems. For example, Mulnon in the U.S. explained how he let the ideas come naturally to him without forcing himself to think hard.

Mulnon: Some of them (ideas) came and I kind of took them and sat back for a second just stared up the space and looked down again and then something else appeared.

Xiang, a Chinese student, explained how she started from a mathematical idea when she was posing the problems.

Xiang: I first started with a mathematical idea and then I tried to connect it to real life. For example, in the third task, the doorbell problem, it is obviously an arithmetic sequence, so I just made up a problem related to it.

Another Chinese student, Yanan, like student Xiang, explained very clearly her problem-posing process.

Yanan: It's like, when I saw the circle, I thought of radius, area, circumference, etc. Then when I saw the triangle, I immediately thought of area, perimeter, altitude, etc. Then I just tried to connect all of them to make the problems harder.

Most of the U.S. students seemed to have trouble explaining their thinking processes. For example,

Victor: I don't know. I just kind of thought of it. I don't really know.

Kyle: I was just putting down the first things that came into my mind. Things just popped into there. I don't know.

Among the 12 U.S. students, there was one student who gave a description of his problem-posing process that was similar to descriptions given by most of the Chinese students.

Kurt: Get started with a general idea of what you want to do and then from there you can be more specific. But on most of those, it's really general. I started with a main idea, I just kind of, kind of like when I write a story, you start with a main idea and... add different ideas on that. So it works for me to get more ideas.

*Question 3: Which of the problems that you posed do you think are creative?*

Two of the twelve U.S. students and two of the eight Chinese students said none of their problems were creative. The rest of the students had one or more than one problem that they thought were creative. But for the two groups of students, the reasons that made their problems creative were different. Seven out of the twelve U.S. students mentioned "rareness" as the reason that made the problems creative. For example, for the Doorbell problem, Mulnon thought the following problem he posed was creative (although it had nothing to do with the given information):

Problem: What is the probability someone ding dong ditches you?

Mulnon: I don't think many people would be sitting there thinking about that.

Only four out of the twelve students thought some problems were creative because of the mathematics involved. For example, Kurt posed a problem for the geometry task:

Problem: What is the perimeter of the triangle if the diameter of the circle is 1?

Kurt: [Creative] just because a lot of theorems are involved to get to the right answer.

Three out of the twelve U.S. students reported that some of the problems they posed were creative because the contexts were fun or interesting. For example, Daniel posed a problem involving the rainbow for the first task.

Problem: What color shirts would the last three children wear if they were told to all wear colors of the rainbow, ROYGBIV?

Daniel: It is not something you can answer...it is kind of different. So if we have got more information I would ...it would be something fun to try to solve.

Among the six Chinese students who reported having creative problems, four of them gave reasons involving mathematics. For example, Yanan posed several problems that she thought were creative for the geometry task.

Problem: If the two sides of the triangle are 3 and 6, find out the perimeter of the triangle when the area of the inscribed circle is the maximum.

Yanan: I think it is creative to involve the area of the circle and the perimeter of the triangle.

*Question 4: Do you think problem posing is important in your mathematics learning?*

No matter what the reasons were, U.S. students mostly said that they thought it is important to be able to pose problems in their mathematics learning. For example, Ramona said that problem posing helps her see the structure of problems if she could pose problems by herself. Scarlett explained that posing problems would help students see what is important in their mathematics learning. And Kurt related problem posing to his own experience in working backwards on his homework.

Kurt: Yes. It's... I think it's helpful because you can work backwards that way to find the answer and you see what works and what doesn't. That is what I started doing, like I said earlier too, that I like working from the answer and finding the values. A lot of the times, in my homework, if I don't understand it first, I'll look at the answer and work backwards and then I can rework again the other way and see what I was supposed to do. It helps me comprehend better I guess.

Two students explained that posing problems will help them learn how to read a problem fully when solving a problem. For example,

Iris: Yes, I think that ... like... you have to know both sides to have like a full understanding. Like if you only, like if you get so focused on finding the answer, you forget how to read the question completely.

When it came to Chinese students, however, the answers were not as consistent as the U.S. students. Among the eight Chinese students interviewed, two students said that it is not important in their mathematics learning in high school because of the college entrance examination. For example,

Yuan: I think it is in fact pretty important in mathematics learning, but it is not that important in the college entrance examination. I think it is important in mathematics learning because it can help you enhance your thinking ability, so it is very important. But the fact is that in high school, especially the senior year, it is more important to solve a lot of problems.

Two students said that posing problems would help them see the way the tests are designed so that they would be able to do better in solving problems. For example,

Yanan: I think it is very important. I think problem posing helps us see how our teachers make up tests problems to test us. It will help us think what they will put in the tests.

Three students said that problem posing helps them learn mathematics better. For example,

Xiang: I think it is pretty important. First of all you have to understand the methods of working on the problem before you can pose a problem yourself.

Ning: I think it is very important. It is an ability of summarizing and thinking. For example, if you are to pose a problem about an inscribed circle in a triangle, you have to know all about triangle and circle. That involves a great amount of thinking.

## CONCLUSIONS

In summary, although some students reported some experiences related or similar to the problem posing in this study, those experiences were usually not the same as mathematical problem posing in this study and those experiences were rare. The majority of the students reported that they had never had any experience in posing mathematical problems. In terms of the problem posing process, it seems that most of the U.S. students did not have a specific plan to pose problems and Chinese students explained clearly how they posed the problems. Those findings suggest that, despite the emphasis placed on this topic by the educators and governors in the United States and China (e.g., NCTM, 2000; Mathematics Curriculum Development Group, 2002), problem posing is not an established element in instruction in the classrooms or examinations yet. This lack might be due to the lack of teachers' abilities to integrate problem-posing activities in their instruction.

As to the factors that made the problems creative, U.S. students focused more on the contexts of the problems while Chinese students focused on the mathematics involved in the problems.

All of the U.S. students for various reasons said that problem posing is important in mathematics, although one of them said it is not very important. Two of the eight Chinese students said that problem posing is not important in high school learning because of the college entrance examination. That suggests that despite China's efforts in developing a new mathematics curriculum and assessment standards which try to incorporate the latest ideas on curriculum and assessment in western countries (An, 2004), summative assessment still have dominant influence on the Chinese classroom (Wong, 1998).

## REFERENCES

- An, S. (2004). *The middle path in math instruction: solutions for improving math education*. Lanham: Scarecrow Education.
- Cai, J. (2000). Mathematical thinking involved in U.S. and Chinese students' solving process-constrained and process-open problems. *Mathematical Thinking and Learning: An International Journal*, 2, 309-340.
- Cai, J., & Hwang, S. (2002). Generalized and generative thinking in U.S. and Chinese students' mathematical problem solving and problem posing. *Journal of Mathematical Behavior*, 21(4), 401-421.
- Mathematics Curriculum Development Group of Basic Education of Education Department [教育部基础教育司数学课程标准研制组]. (2002). *The interpretation of mathematics curriculum* (Trial Version).[数学课程标准（实验稿）解读]. Beijing: Beijing Normal University Press.

- National Center for Educational Statistics. (2009). *The National Assessment of Educational Progress Overview*. Retrieved August 28, 2009, from <http://nces.ed.gov/nationsreportcard/mathematics/>
- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. Retrieved, July 29, 2007, from <http://www.emis.de/journals/ZDM/zdm973a3.pdf>
- Sriraman, B. (2005). Are giftedness and creativity synonyms in mathematics? *Journal of Secondary Gifted Education*, 17(1), 20-36.
- Stoyanova, E. (1997). Extending and Exploring Students' Problem Solving via Problem Posing: A Study of Years 8 and 9 Students involved in Mathematics Challenge and Enrichment Stages of Euler Enrichment Program for Young Australians. Unpublished doctoral dissertation submitted to Edith Cowan University, Perth: Australia.
- Stoyanova, E., & Ellerton, N. F. (1996). A framework for research into students' problem posing in school mathematics. In P. C. Clarkson (Ed.), *Technology in mathematics education* (pp. 518-525). Mathematics Education Research Group of Australasia. The University of Melbourne.
- Sun, C. (2004). Severe critique from mathematician Shing-Tung Yau: Mathematics Olympiad in China kills children's curiosity [数学家丘成桐棒喝奥数热：奥数扼杀好奇心]. Yangcheng Evening. Retrieved May 7, 2006, from [http://news.xinhuanet.com/edu/2004-12/25/content\\_2379632.htm](http://news.xinhuanet.com/edu/2004-12/25/content_2379632.htm)
- Usiskin, Z. (2000). The development into the mathematically talented. *Journal of Secondary Gifted Education*, 11(3), 152-162.
- Wong, N. Y. (1998). In search of the 'CHC' learner: Smarter, works harder or something more? In H. S. Park, Y. H. Choe, H. Shin, & S. H. Kim (Eds.), *Proceedings of the ICMI-East Asia regional conference on mathematical education* (Vol. 1, pp. 85-98).



# LESSON STUDY: THE EFFECT ON TEACHERS' PROFESSIONAL DEVELOPMENT

Nellie C. Verhoef and David O. Tall and

University of Twente, University of Warwick

*This study combines elements of the Japanese Lesson Study approach and teachers' professional development. An explorative research design is conducted with three upper level high school teachers in the light of educational design research, whereby design activities will be cyclically evaluated. The Lesson Study team observed and evaluated two different research lesson cycles. The first one focused on the concept of the derivative. The second one deepened teachers' pilot experiences with regard to another mathematical concept. The Lesson Study revealed students' misconceptions with regard to the tangent line. Results of teachers' professional development are used to refine the Lesson Study observation instruments.*

## INTRODUCTION

Lesson Study is a teaching improvement and knowledge building process that has origins in Japanese elementary education (Lewis, 2002). A salient difference with European countries is that Asian students work to develop the understanding of mathematics so that their success is not only maintained, but improved (Tall, 2008b). European governments establish guidelines for teaching and learning approaches that are controlled in more or less directive ways. As a consequence teachers are focused on preparing for the exams. Individual teachers may reflect on and improve their practice in the isolation of their own classrooms. The complexity of their daily work rarely allows them to converse with colleagues about what they discover about teaching and learning (Cerbin & Kopp, 2006).

In this study the social system in which teachers learn, focuses on collaboration in a Community of Learners (CoL). This is a small research team, where participants are characterized as serious partners in the process of the development of knowledge and scientific research (Brown & Campione, 1996). The focus in a CoL is on specific subject matter based on the reality of everyday teaching. The participants share their experiences as they implement research activities and reflect on the results and the research methods. A necessary precondition is that the participants have adequate facilities in order to participate. Research reports findings that the working method explicitly prompts the participants of a CoL to mutual cooperation based on knowledge for teaching the content (Verhoef & Terlouw, 2007).

## **THEORETICAL FRAMEWORK**

### **Lesson Study**

The typical small, but professionally scaled process of Lesson Study generates a collaborative research framework (Matoba & Sarkar Arani, 2006). The Lesson Study approach involves the design of the research lesson as part of an extended sequence of lessons to teach a particular topic, the implementation of the research lesson followed by evaluation and analysis, then refining of the lesson. Observation of the research lesson by colleagues and other interested persons is an essential part of this approach (Baba, 2007; Sowder, 2007). Having several pairs of eyes looking at the classroom activity gives a more comprehensive view of different aspects.

This approach culminates in at least two tangible products: (a) a detailed, usable lesson plan, and (b) an in-depth study of the lesson. The study investigates teaching and learning interactions, explaining how the students responded to instruction, and how instruction might be further modified based on evidence (Cerbin & Kopp, 2006).

The designing process is a process of learning, because the teachers consider how they will help students achieve the goals (Wiggins & Mc Tighe, 1998). In planning a research lesson, teachers predict how students are likely to respond to specific questions, problems and exercises and then analyse what actually happens.

The primary focus of Lesson Study is not only what students learn, but *how* they learn. The framework of long-term mathematical thinking will be used to categorize aspects of students' learning processes (Tall & Mejia-Ramos, 2009). In practice, the Lesson Study approach selects of a specific course, a well-chosen topic and goals for student learning followed by a research lesson that addresses academic learning goals (e.g., understanding specific concepts and subject matter) and broad goals (e.g., development of intellectual abilities, habits of mind and personal qualities).

### **Long-term mathematical thinking**

Skemp (1976) distinguished relational understanding in which relationships are constructed between concepts and instrumental understanding which involves learning how to perform mathematical operations. Various theories (e.g. Dubinsky & McDonald, 2001; Gray & Tall, 1994) suggest there are subtle processes occurring in learning in which operations that take place over time become thinkable concepts that exist outside of a particular time. This framework has been extended into what Tall (2008a) described as three mental worlds of mathematics:

- (i) the conceptual-embodied world (based on perception of and reflection on properties of objects);
- (ii) the proceptual-symbolic world that grows out of the embodied world through actions (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (called procepts); and

- (iii) the axiomatic-formal world (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.

### **Derivative**

Calculus in school is a blend of the world of embodiment (drawing graphs) and symbolism (manipulating formulae). The property of *local straightness* refers to the fact that, if one looks closely at a magnified portion of the curve where the function is differentiable, then the curve looks like a straight line, which, when extended gives the tangent line at this point. This conception can be encouraged by the use of technology to magnify graphs. Inglis, Mejia-Ramos and Simpson (2007) suggest that the derivative develops from embodiment to symbolism to formalism through definition and deduction. A ‘sensible approach’ to calculus proposes that a more natural approach to the calculus blends together the dynamic embodied visualisation of the changing slope as the eye traverses the curve and the corresponding symbolic calculation of the slope (Tall, 2010). This approach hypothesises is that it is more natural to build from an operation on an *object* (looking along the graph of the function) to build a new *object* (the graph of the slope function) than to encapsulate a *process* (calculating the slope at a point) to an *object* (the symbolic derivative). Focusing on the relationship between the initial stages of the student’s long-term thinking process the research question in this study is: *What is the effect of Lesson Study on teachers’ professional development?*

## **SETTING AND METHODOLOGY**

### **Participants**

Three upper level secondary school teachers from different school organizations and five staff members of the University of Twente participated in the Lesson Study team: two educational teacher trainers, a mathematician, a researcher and a PhD-candidate. The male school teachers tagged by capitals A, B and C indicated to be interested in professional development. A (age 56) attained a Bachelor’s degree in Mathematics and a Master’s degree in Mathematics Education. He worked as a mathematics teacher for 17 years with lower level to upper level high school students. B (age 48) attained a Bachelor’s degree in Mathematics and a Master’s degree in Mathematics Education. He worked as a mathematics teacher from 1988 mostly with upper level high school students. C (age 48) attained a Bachelor’s degree in Engineering and a Master’s degree in Mathematics Education. He worked as a staff member of the University of Twente for seven years. Since 2009 he works as a mathematics teacher with mostly upper level high school students.

### **Application of Lesson Study in a Community of Learners**

Each participant was given a research paper to study and to present the ideas to their colleagues in a seminar. Teacher A got ‘Student Perspectives on Equation: The Transition from School to University’ (Godfrey & Thomas, 2008). Teacher B got

‘Exploring the Role of Metonymy in Mathematical Understanding and Reasoning: The Concept of Derivative as an Example’ (Zandieh & Knapp, 2006). Teacher C got ‘The Transition to Formal Thinking in Mathematics’ (Tall, 2008a). In this first iteration of the project, teachers were encouraged to construct their own lessons based on their experience and their reading of the literature, operating as a true Community of Learning without directive guidance from research organizer.

### **Data gathering instruments**

The data gathering instruments (in the school year 2009-2010) consisted of:

- a pretest and a posttest - and additionally a (halfway the school year) posttest at the end of the first Lesson Study cycle - with regard to subject-specific elements;
- a pretest and a posttest with regard to topic-specific elements related to each Lesson Study cycle; an exit-interview focused on students’ understanding of the mathematical concept at the end of each Lesson Study cycle.

The pre- and posttest with regard to subject-specific elements consisted of priority lists on (a) goals of mathematics education, (b) the start of instruction to attain these goals. The pre- and posttest with regard to topic-specific elements consisted of aspects related to teaching the mathematical concepts.

*Subject-specific pre- and posttest.* The pretest was exactly the same as the posttest. Teachers were asked to prioritize statements of educational goals on a scale of 1 (high priority) to 12 (low priority). The statements represented conceptual understanding related to structures and mathematical proof e.g. ‘Structures as a basis for thinking’, in contrast with a focus on procedures to solve problems related to problem-solving skills e.g. ‘To be able to execute correctly’ (Thurston, 1990). The objectives of the teaching method *at the start* of the instruction were theoretically founded and supported (Schoenfeld, 2006). Teachers were asked to prioritize statements of teaching methods on a scale of 1 (high priority) to 8 (low priority). The statements represented a start with an abstract mathematical concept (in symbols) e.g. ‘A start with definitions’ or a start with situated examples e.g. ‘A start with practical worked examples’. The changes in pre- and posttest results showed teachers’ subject-specific professional growth.

*Topic-specific pre- and posttest.* The topic-specific pre- and posttest protocol focused on teachers’ free associated aspects related to teaching the mathematical concept with regard to students’ thinking and learning, e.g. ‘Turn almost into a solution’. The changed free associated aspects in pre- and posttest results showed teachers’ topic-specific professional growth.

*Exit-interview.* The exit-interview protocol focused on students’ understanding of the derivative. Teachers’ exit-interview statements about students’ understanding were related to e.g. ‘Students have their intuitions’ or ‘Students learn by doing’. The changed statements results showed teachers’ growth in students’ learning processes.

## Procedure

Firstly teacher A, followed by teacher B, ended by teacher C designed, implemented and evaluated two half year research lessons. The teachers designed observation and evaluation lists as well as criteria based on Skemp's (1976) instrumental and relational understanding and Tall's (2008a) embodied and symbolic mental worlds to analyse the data. Members of the Lesson Study team observed and evaluated each research lesson. Two independent assessors transcribed separately. In case of disagreement the assessors discussed and asked anew assessor. This resulted in a final agreement between the assessors.

## Data analysis

The data with regard to teacher's professional development were analysed using Skemp's characteristics of understanding and Tall's three mental worlds of thinking.

## RESULTS AND CONCLUSIONS

Teacher A continues to highlight understanding mathematical concepts as the goal of education. He moves from a start with different examples to a start with definitions. He accentuates instrumental understanding using symbols instead of embodiment. He is aware of students' thinking and learning, e.g. 'don't understand as you think'.

Teacher B moves to structures as a basis for thinking directly followed by learning problem-solving skills as the goal of education. He begins with examples and continually emphasizes relational understanding using embodiment. He prefers to develop students' problem-solving skills, believing they 'learn better without ICT'.

Teacher C focuses on understanding mathematical concepts rather than learning procedures as his goal of education. He switches from beginning with different examples to a focussing on a thinking model. He moves to relational understanding using embodiment. He is aware of students' thinking and learning e.g. '*seeing* before using a formula'.

All of the teachers consider general meanings of the derivative in the pretest. They consider velocity as an application of the derivative. None of them associates the concept of the derivative in a non-mathematical context. They all concentrate on the tangent line in the posttest.

The Lesson Study team discovered – after two executions of the research lesson – in an evaluating meeting at the university, that the uncovering of students' thinking processes failed. The plenary discussion was characteristic for these two executions. The Lesson Study team decided to change the written question lists into short written assignments with doing activities, asked in pairs. As a consequence the teachers learn to think about students thinking and learning processes as individuals during the Lesson Study period. The teachers highlight learning by doing to uncover students' learning processes, students' intuitions and students thinking processes as individuals. Teachers are known to mathematical vocabulary and possible misconceptions relating to students' interpretation of representations.

Extra remarks in the exit-interview accentuate practical tips from colleagues to improve lessons in general. The teachers indicate an increase enjoyment in teaching.

## **DISCUSSION**

The goal of the Lesson Study approach was to professionalize mathematics teachers by designing, observing, implementing and evaluating two research lessons. The lesson observations completing this study were focused on uncovering students' thinking processes as an effect of a research lesson and as an indication of a successful professional development. The refining of the Lesson Study instruments concentrated on teachers' observations by using worksheets to uncover students' thinking processes.

The teachers' professional development continued to be narrowly related to their classroom practices in spite of the recommended literature and the discussions in the Lesson Study team. As a consequence each teacher developed an individual knowledge base for teaching. For example, teacher A, with the longest time in school practice, used an applet with the intention to demonstrate local straightness as being most meaningful to understand the derivative. After his short introduction he concentrated on the ratio  $\Delta y/\Delta x$  with the intention to connect the lesson to the textbook. After the Lesson Study cycles he makes consistent choices with regard to definitions, symbols and students' teaching and learning. Teacher B, decided to focus on the concept of the tangent line before introducing the derivative. Each student was given a squared graph of  $y=x^2$  on squared paper and were asked to draw a tangent at a point that was not placed on a crossing of the grid line. As a consequence, the tangent lines they drew were slightly different and gave small differences in the numerical slope of the tangent. B's plenary discussion focused on the concept of the tangent, but also ended in the limit  $dy/dx$ , following the strict textbook guidelines. After the Lesson Study cycles, he focused on the learning of problems-solving skills to attain conceptual understanding relating to the standard approach. Teacher C, a former staff member of the university, kept the limit concept in mind throughout the lesson without actually naming it. The observers noted that the students were not amazed at all when their practical approach to the tangent produced different tangent lines with different slopes as compared with the graphic calculator that produced a single formula.

The teachers in this study were unable to design a research lesson based on the new theoretical framework in the first Lesson Study cycle, because of their desire to follow narrow textbook guidelines. The textbook assignments were built up step-by-step, without reflection. This approach seemed to hinder the development of 'thinkable concepts' (Tall & Mejia-Ramos, 2009). The experienced teachers tended to teach using familiar methods, as executed in their colleagues' groups who were teaching the course in the regular manner. External stimuli, like scientific literature, discussions in a lesson study team and reflection on classroom practices, made teachers aware of students' thinking and learning processes besides classroom

management. This study reveals the significance of the complex reality of school practice in reference with the powerful claim of curriculum guidelines, study guides based on textbooks, and the attaining of high exam results.

The Dutch curriculum can be typified as a realistic mathematics educational (RME) approach in lower level mathematics education. The higher level mathematics education is subdivided into a human and arts related program on the one hand and a science related program on the other hand. The last one, with a minimum of RME-characteristics, was seriously criticized because of a constant yearly grow of deficiency courses at the universities recently. Bad mathematics results at scientific studies were attributed to the use of realistic contexts in textbooks by the scientists. These realistic authentic contexts would impede the development of automated mathematics operations, helpful in science studies. As a consequence the science textbooks were influenced by traditional procedural fluency with hardly conceptual insight. The balance between procedural fluency and conceptual insight is missing.

More research in a Lesson Study approach in the context of complex school practices with the focus on compression of knowledge, requiring both conceptual insight and procedural fluency, is needed to investigate teachers' individual professional development. This will be the focus of the next iteration of the research project.

## References

- Baba, T. (2007). Japanese education and Lesson Study: An overview. In M. Isoda, M. Stephens, Y. Ohara & T. Miyakawa (Eds.), *Japanese Lesson Study in mathematics: It's impact, diversity and potential for educational improvement*. Singapore: World Scientific Publishing.
- Brown, A.L., & Campione, J.C. (1996) 'Psychological theory and the design of innovative learning environments. On procedures, principles, and systems'. In L. Schauble & R. Glaser (Eds.), *Innovation in learning: New environments for education* (pp. 289-325). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Cerbin, W., & Kopp, B. (2006). Lesson Study as a model for building pedagogical knowledge and improving teaching. *International Journal of Teaching and Learning in Higher Education*, 18(3), 250-257.
- Dubinsky, E., & McDonald, M.A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In Derek Holton et al. (Eds.), *The teaching and learning of mathematics at University level: an ICMI study* (pp. 273-280). The Netherlands: Kluwer.
- Godfrey, D., & Thomas, M.O.J. (2008). Student perspectives on equation: the transition from school to university. *Mathematics Education Research Journal*, Vol. 20, 2, 71-92.
- Gray, E. M., & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *Journal for Research in Mathematics Education*, 26(2), 115-141.
- Inglis, M., Mejia-Ramos, J.P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, 66(1), 3-21.



- Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional improvement*. Philadelphia: Research for Better Schools, Inc.
- Matoba, M., & Sarkar Arnani, M.R. (2006). Ethnography for teachers' development: Japanese approach of investigation on classroom activities. In N. Popov, C., Wolhutter, C. Heller & M. Kysilka (Eds.), *Comparative education and teacher training* (pp. 116-125). Sofia: Bulgarian Comparative Education Society.
- Schoenfeld, A.H. (2006). What doesn't work: the challenge and failure of the what works clearinghouse to conduct meaningful reviews of studies of mathematics curricula. *Educational Researcher*, 35(2), 13-21.
- Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20-26.
- Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Charlotte, NC: Information Age Publishing.
- Tall, D. O. (2008a). The Transition to Formal Thinking in Mathematics. *Mathematics Education Research Journal*, 20(2), 5-24.
- Tall, D. O. (2008b). Using Japanese Lesson Study in Teaching Mathematics. *Scottish Mathematical Council Journal*, 38, 45-50.
- Tall, D.O., & Mejia-Ramos, J.P. (2009). The Long-Term Cognitive Development of Different Types of Reasoning and Proof. In H.N. Jahnke & H. Pulte (Eds.), *Explanation and proof in mathematics: Philosophical and educational perspectives*. New York: Springer.
- Tall, D.O. (2010). A Sensible Approach to the Calculus. (Plenary at *The National and International Meeting on the Teaching of Calculus*. Puebla: Mexico).
- Thurston, W. P. (1990). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, 30(2), 161-177.
- Verhoef, N.C., & Terlouw, C. (2007). Training mathematics teachers in a Community of Learners (CoL), Volume 4 (pp. 256-264). *Proceedings of the 7<sup>th</sup> European Society for Research in Mathematics Education*. Cyprus: Greece.
- Wiggins, G., & McTighe, J. (1998). *Understanding by design*. Alexandria, VA: Association of Supervision and Curriculum Development.
- Zandieh, M., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning. The concept of the derivative as an example. *Journal of Mathematical Behavior*, 25, 1- 17.



# CRITICAL FEATURES OF FORMAL AND INFORMAL REASONING IN THE CASE OF THE CONCEPT OF DERIVATIVE

Antti Viholainen

Umeå Mathematics Education Research Center

Umeå University

*Mathematical reasoning can be categorised into formal and informal ones on the basis of the type of used arguments. Both types of reasoning are needed in mathematics, but students usually tend to restrict their reasoning either to formal or informal mode only. Three examples collected and re-analysed from earlier studies reveal some critical features of this kind of restricted reasoning: A formal reasoning may easily become dependent on memory or authoritative references or it may become superficial imitation of algorithms. Informal reasoning, for one, may easily be based on intuitive conceptions, and its validity may be difficult to evaluate.*

## ARGUMENTS AND MATHEMATICAL REASONING

Arguments are closely related to mathematical reasoning. In proving tasks, the aim of the reasoning process is to construct an argument, and, in that case, the argument can be considered as a product of the reasoning process. Also, in other kinds of tasks arguments are essential elements of reasoning, because all the attained results, if not purely guessed, are argued in some way. However, the same result or conclusion can often be reasoned by using different kinds of arguments. This makes it possible to categorise mathematical reasoning on the basis of the type of used arguments.

According to Toulmin's (2003) model of argumentation, the aim of argumentation is to construct an explanation (*a warrant*), for why the information concerning the initial state (*the data*), necessitates the statement which is argued (*the conclusion*). In some cases, also a justification for the authority of the warrant (*backing*) is also needed. The warrants of mathematical arguments in their final forms are usually required to be based on the elements of an axiomatic system, that is, on definitions, axioms and previously proven theorems. These kinds of *formal arguments* are needed in verifying and systematising mathematical knowledge (de Villiers, 1999). On the other hand, the warrants of mathematical arguments may be based on visual, physical or other concrete interpretations. These kinds of *informal arguments* have an important role in engendering personal understanding (Raman, 2003; Weber & Alcock, 2004). By applying this categorisation of arguments, mathematical reasoning can be categorised into formal and informal ones depending on the type of the used arguments.

In this paper, some examples about formal and informal reasoning concerning the concept of derivative are presented and discussed. The purpose is to catch critical features of formal and informal reasoning. Three earlier studies (Viholainen, 2006; 2007; 2008) are combined and re-analysed.

## STUDENTS' TENDENCY TO RESTRICTED REASONING

According to some earlier studies, it seems to be difficult for students to use formal and informal reasoning simultaneously. It has been found that, especially, in the area of analysis, students often avoid to use definitions (Pinto, 1998; Viholainen, 2007; Vinner, 1991), but they rather tend to use methods and arguments, which are not connected to a theoretical knowledge (Juter, 2005; Tsamir, Raslan & Dreyfus, 2006). On the other hand, Weber and Alcock (2004) have found cases where undergraduate students' reasoning was too much restricted around the formal definitions. In this case, the reasoning considered group theory and convergence of sequences.

In the case of the concept of derivative, students' tendency to avoid formal reasoning came out in the study presented in Viholainen (2007). In this study, the following task was given to 18 mathematics teacher students:

$$\text{"Problem 3: Let } f : R \rightarrow R, f(x) = \begin{cases} x^4 \cos(1/x^3), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Is the function  $f$  differentiable?"

It was noticed that majority of the students started by an informal method to solve this problem. Only a couple of students, who had a long and successful study history in university mathematics, started by a formal method. These students performed quite fast and without any problems a short calculation based on the definition of derivative, which led them to a conclusion that the function  $f$  was differentiable. In addition, they seemed not to doubt their reasoning. Instead, the students who had less experience in university mathematics or poorer study success were reluctant to start by a formal method. These observations refer to importance of experience in the formal reasoning, especially, in the selection between informal and formal methods.

## CRITICAL FEATURES OF FORMAL REASONING

In the previous task (the "Problem 3" above), the formal reasoning was based explicitly on the definition of derivative. However, quite often, formal reasoning is not explicitly based on axioms and definitions, but on earlier proven results. For example, in the case of derivative, the differentiating rules are frequently used results. In fact, any algorithm, which has been proved in a formal way, can be considered as an earlier proven theorem, and, thus, applying an algorithm has to be classified as formal reasoning.

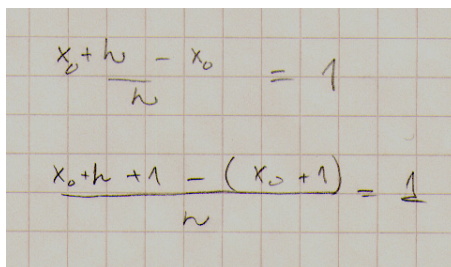
Lithner (2008) defines that in *algorithmic reasoning* the "strategy choice is to recall a solution algorithm" and that "the remaining reasoning parts of the strategy

implementation are trivial for the reasoner, only a careless mistake can prevent an answer from being reached” (p. 259). According to Lithner, in algorithmic reasoning the applied algorithm is chosen on the basis of surface features of the task. Lithner considers algorithmic reasoning as one type of *imitative reasoning*, which is based on imitation of ready-made models. However, if algorithms are used in formal reasoning, it is not enough only to select and apply an algorithm, but also to reason, why it is appropriate and justified in a given situation to apply the algorithm in question.

An illustrative example of careless use of formal results is presented in Viholainen (2008). In this case, Mark, a fifth-year mathematics teacher student, studied the differentiability of piecewise defined polynomial functions by differentiating both expressions used in the definition of the function and by checking if both the expressions obtained equal values at the point where the expression was changed. This method is correct, but it requires that the function in question is continuous. Mark did not take this condition into account. He studied differentiability of several functions, and his method seemed to work well. However, the following function led him to a conflict:

$$f_4(x) = \begin{cases} x, & x < 1, \\ x+1, & x \geq 1. \end{cases}$$

Following the rule described above, Mark differentiated both the expressions  $x$  and  $x+1$  and compared the values of derivatives at the point  $x=1$ . This led to a conclusion that this function is differentiable at the point  $x=1$ . However, Mark had a strong conception that differentiability requires continuity. After a while, Mark finally decided to examine the problem with the definition of derivative. He made a calculation presented in Figure 1.



$$\frac{x_0 + h - x_0}{h} = 1$$

$$\frac{x_0 + h + 1 - (x_0 + 1)}{h} = 1$$

Figure 1: Mark's calculation.

Mark calculated the difference quotients to both of the polynomials. This is a general way to calculate the derivative functions by using the definition of derivative. Mark concluded that, because the results were equal, the function was differentiable. Furthermore, he concluded that he remembered wrongly the relationship between continuity and differentiability: According to him, this example showed that a differentiable function is not needed to be continuous.

This example illustrates some dangers of formal reasoning. Mark did not take into account that continuity of a function was a prerequisite for the differentiating method he applied. Perhaps, Mark did not remember this condition. In addition, in this situation, he did not have any reference available to check the conditions of the method. This illustrates that formal reasoning based on earlier results is strongly dependent either on memory or on authoritative references: If the applied results are not rediscovered and reproved, all the details of them need to be memorized or warranted by some authority. On the other hand, students do not necessarily worry about the pre-requisites of methods, but their main attention is directed to carrying out the procedure. For example, Mark above seemed not even to think about prerequisites of the differentiating method he applied, but, despite of that, he had a strong confidence on it, and the observation that the method seemed to work very well increased this confidence (Viholainen, 2006; 2008). As well, Lithner (2000) has found that algorithmic reasoning based on surface features and superficial interpretations is common among university mathematics students. In this kind of careless use of results the applied piece of mathematical knowledge is not related to a wider structure, and, due to that, this kind of action refers to the *toolbox* (Törner, 1998) or *instrumentalist* (Ernest, 1989) view of mathematics, in which mathematics itself is not seen interesting, but as a tool/instrument to some other purpose.

Omitting of details comes also out in Mark's way to apply the definition of derivative. He did not take into account that when the left-hand limit of difference quotient is calculated, the value  $f_4(1)$  cannot be calculated from the function  $x$ , but from the function  $x + 1$ . Therefore, this illustrates that a careful consideration of details is important not only in applying earlier proven results, but also when reasoning is explicitly based on definitions. Mark used the term  $x_0$  instead of  $-1$  in his calculation, and, thus, he calculated the expressions for the derivative functions of the functions  $x$  and  $x+1$ . This, in fact, refers to algorithmic reasoning defined according to Lithner above, because this is a commonly used way to calculate derivative functions for polynomials with the definition of derivative. Mark did not reflect what the results obtained with this method mean, except that he noted them to be equal.

## CRITICAL FEATURES OF INFORMAL REASONING

### Example 1

In an interview described also in Viholainen (2006), Theresa, a mathematics teacher student having studied four years, remembered that there existed a theorem concerning the relationship between continuity and differentiability, but she was not sure what it said. She tried to investigate this relationship visually on the basis of graphs of different functions. She drew a graph with a corner, and, by drawing several possible “tangents” to the corner, she explained that there was no unambiguous tangent at this point, and, thus, this function was not differentiable even though it was continuous. In this reasoning, her criterion for differentiability was the existence of an unambiguous tangent line. However, by using the same criterion, she

also reasoned that a discontinuous function may be differentiable. She drew a parabola with a hole (see Figure 2), and, also a tangent line for the parabola to the point where the hole lied. On the basis of that she reasoned that this function was differentiable.

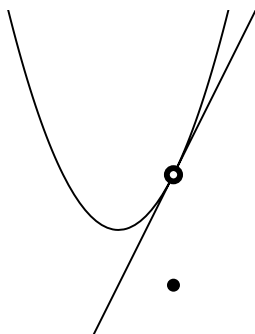


Figure 2: A parabola with a hole and a tangent line drawn to that point.

Interviewer: You think that this is discontinuous but differentiable?

Theresa: Yes, because it is possible to draw a tangent!

These two conclusions led Theresa to a conflict, because she thought she remembered that there existed a theorem concerning the relationship between continuity and differentiability. However, after rechecking her reasoning, she concluded that her memory was wrong.

Theresa's reasoning concerning the parabola with a hole reveals one fundamental danger of informal reasoning: It may be based on intuitive conceptions, which can be incomplete and possibly in contradiction with the formal definitions. In this case, reasoning was based on an intuitive conception about the tangent. The tangent drawn like in Figure 1 may -of course, misleadingly- seem to fulfil two beliefs which have been found to be strong among students: A curve and its tangent have one and only one common point, and, a tangent keeps the whole curve in the same semi-plane (Biza, Christou & Zachariades, 2008; Biza, Nardi & Zachariades, 2009). Theresa did not reflect the essence of tangent enough, but, because the line drawn to the parabola at the point with a hole looked like a tangent, she reasoned, without any hesitation, that the tangent existed.

## Example 2

William was the only student who succeeded to solve the "Problem 3" presented above in an informal way. He started by sketching the graphs of the functions  $x^4$ ,  $\cos$  and  $1/x^3$ . First, he reasoned on the basis of these graphs that because the values of  $1/x^3$  increases when  $x$  approaches zero from the right<sup>1</sup>, the function  $\cos(1/x^3)$

---

<sup>1</sup> William did not consider the left side of zero.

oscillates with a very high frequency near zero. He recognized that the oscillation happens inside the interval  $[-1,1]$ . On the basis of that, he reasoned that the product of the functions  $x^4$  and  $\cos(1/x^3)$  cannot get bigger values than the function  $x^4$ . Thus, he reasoned that the graph of the function  $x^4 \cos(1/x^3)$  is bounded by the graph of the function  $x^4$  and that it oscillates with an increasing frequency near zero. Even though William did not draw the graph of the function  $-x^4$ , which works as a lower boundary for the graph, he drew the graph symmetrically with respect to x-axis. Williams sketch is presented in Figure 3. On the basis of the graph, it was easy for William to see that the values of the function approached zero when  $x$  approached zero, and, thus, the function was continuous.

William did not see any reason to doubt differentiability in other points than in zero. In addition, he concluded that a tangent line drawn to origin had to go along x-axis. He reasoned this by drawing a line (he spoke about a tangent) to the graph of the function  $x^4$  going through origin (see Figure 3) and explaining that it pivots to a horizontal line (x-axis) when the other intersection point approaches origin.

William: The maximum value of this function follows this  $x^4$ . It, in any case, approaches zero. Then, it means that the steepest possible tangent line on some given interval (draws the line and shows an interval on x-axis), its slope approaches zero... if we follow only the steepest possible one on the interval. And, if the interval shortens, it turns to this direction (shows x-axis).

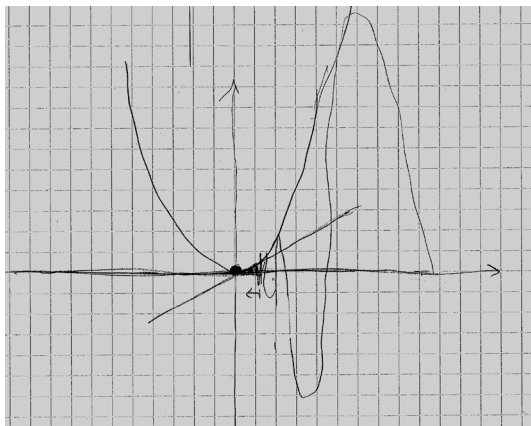


Figure 3: William's sketch for the graph.

Williams' reasoning resulted in a correct conclusion, but it is ambiguous: On the basis of the available data, it is not clear whether William speaks about a secant, whose other intersection point with the graph is in origin, or about a tangent drawn to the graph at a point on outside origin. If he means a secant, the main idea of his reasoning is valid: The secant is in its steepest possible position when the other intersection point lies on the graph of the border function  $x^4$ , and, when the interval in

the neighborhood of origin diminishes, all the secants, whose other intersection point is origin and the other inside this interval, approaches x-axis. Williams' reasoning is interpreted in this way in Viholainen (2007). According to this interpretation, the only inconsistency in Williams reasoning is the use of word tangent instead of secant. However, neither on the basis of the excerpt of Williams' explanation presented above, nor on the basis of the drawing presented in Figure 3<sup>2</sup>, nor on the basis of any other elements of the data it is possible to refuse the interpretation that he really speaks about a tangent drawn to a point at the graph outside origin. If that is the case, William's conclusion about the steepest possible tangent is erroneous. This example illustrates the ambiguity of informal argumentation: The validity of the argument can be entirely dependent on how the expressions and drawings are interpreted.

## CONCLUSION

The above-presented examples illustrate some critical properties of formal and informal reasoning: The example with Mark illustrates the importance of details in formal reasoning, which makes formal reasoning dependent either on memory or authoritative references. It also shows that formal reasoning easily becomes algorithmic based on imitation of models instead of deep structural/conceptual thinking. Theresa's reasoning with the tangent in a hole reveals that an informal reasoning may easily be based on intuitive conceptions, which are not necessarily in accordance with the formal theory. Williams' case, for one, illustrates that evaluation and judging the validity of informal reasoning may require interpretation of used expressions and drawings, and, therefore, it is not necessarily unambiguous.

Both formal and informal reasoning have their own purposes, and, therefore, in the design and implementation of teaching, they both should be taken into account so that students could get opportunities to train both of them. In that, it is valuable to be aware of the observed critical features. Perhaps the most important aspect is to emphasise connections between formal and informal reasoning and simultaneous use of them.

## References

- Biza, I., Christou, C. & Zachariades, T. (2008). Students' perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education*, 10, 53-70.
- Biza, I., Nardi.E. & Zachariades, T. (2009). Do images disprove but do not prove? Teachers' beliefs about visualization. In F. L. Lin, F. J. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education*, Vol.1 (pp.59-64). Taipei, Taiwan: The Department of Mathematics, National Taiwan Normal University.

---

<sup>2</sup> It is impossible to say, whether the straight line in Figure 3 is a secant line going through origin or a tangent line drawn to a point outside origin.

- De Villiers, M. (1999). The role and function of proof with Sketchpad. In M. de Villiers (Ed.), *Rethinking Proof with Sketchpad* (pp. 3-10). Emeryville, CA, USA: Key Curriculum Press.
- Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249-253). New York: Falmer.
- Juter, K. (2005). Limits of functions: traces of students' concept images. *Nordic Studies in Mathematics Education*, 3-4, 65-82.
- Lithner, J. (2000). Mathematical reasoning and familiar procedures. *International Journal of Mathematical Education in Science and Technology*, 31, 83-95.
- Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67, 255-276.
- Pinto, M. M. F. (1998). *Students' understanding of real analysis*. University of Warwick, UK.
- Raman, M. (2003). Key ideas: what are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52, 319-325.
- Toulmin, S. E. (2003). *The uses of argument*. Cambridge, U. K.: Cambridge University Press.
- Tsamir, P., Rasslan, S. & Dreyfus, T. (2006). Prospective teachers' reactions to Right-or-Wrong tasks: the case of derivatives of absolute value functions. *Journal of Mathematical Behavior*, 25, 240-251.
- Törner, G. (1998). Self-estimating teachers' views on mathematics teaching – modifying Dionne's approach. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood & L. Stiff (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 627-634). Columbus, OH, USA.
- Viholainen, A. (2006). Why is a discontinuous function differentiable? In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of The 30<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 329-336). Prague, Czech Republic: Faculty of Education, Charles University.
- Viholainen, A. (2007). Students' choices between informal and formal reasoning in a task concerning differentiability. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of The Fifth Congress of the European Society for Research in Mathematics Education* (pp. 2369-2378). Cyprus: Department of Education, University of Cyprus.
- Viholainen, A. (2008). Incoherence of a concept image and erroneous conclusions in the case of differentiability. *The Montana Mathematics Enthusiast*, 5, 231-248.
- Vinner, S. (1991). The role of definitions in teaching and learning. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.
- Weber, K. & Alcock, L. (2004). Semantic and syntactic proof production, *Educational Studies in Mathematics*, 56, 209-234.



# DISCURSIVE PRACTICES OF TWO UNIVERSITY TEACHERS ON THE CONCEPT OF 'LINEAR TRANSFORMATION'

Olov Viirman

Karlstad University & University of Gävle

*This paper reports on an ongoing study focusing on the teaching of functions in undergraduate courses in mathematics at three Swedish universities. In this paper two excerpts of lectures on linear transformations are analysed, using commognitive theory. The two teachers' discursive uses of the terms 'linearity' and 'linear transformation' are described, and potential consequences for student learning are presented. One lecture displays a strong mathematical content while lacking contextualization, while the other is well grounded in everyday experience, but with a lack of attention to mathematical detail potentially detrimental to student learning.*

## INTRODUCTION

Interest in research on the teaching of mathematics in higher education has increased in the last decade. However, there are still not that many studies done on the actual teaching practices of university mathematics teachers. This is even more the case looking at Swedish research. My ongoing thesis project is an attempt at a contribution in this area. The project, a first report of which was presented at CERME 7 (Viirman, 2011), aims at investigating how mathematics teachers at the university level work with the function concept, and how the discursive practices of the teachers help promote learning of the function concept in their students.

## THEORETICAL FRAMEWORK

The commognitive theory (Sfard, 2008) is a participationist (Sfard, 2006) theory of thinking and learning, drawing on ideas from Vygotsky and Wittgenstein. Starting from the assumption “that *patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual.*” (Sfard, 2008, p. 78, *emph. in original*), Sfard defines thinking as “an individualized version of (interpersonal) communicating” (ibid, p. 81). To capture both inter- and intrapersonal communication, she coins the term 'commognition'. Different types of communication are called discourses, and these discourses are in constant development. Within the commognitive framework, learning can be viewed as individualizing discourse, becoming more capable at communicating within the discourse, with others as well as with oneself (Sfard, 2006, p. 162). The unit of commognitive analysis is the discursive activity, the “patterned, collective doings” (ibid, p. 157). Hence, the subject of my research is the discourse of function, as it is manifested in the communicative practices of the teachers (and students). Sfard presents four characteristics which can be used to distinguish discourses (Sfard, 2008,

p. 133ff): *word use*, *visual mediators*, *narratives* (sequences of utterances speaking of objects, or relations or processes involving them, subject to endorsement or rejection within the discourse) and *routines* (repetitive patterns characteristic of the discourse).

The commognitive theory of mathematical discourse is many-faceted, and much could be said about it, but this would go beyond the scope of this paper. For my present purposes, however, some clarification of the notion of 'concept' within commognitive theory is needed. Sfard defines the term 'concept' as a symbol together with its discursive uses (ibid, p. 111). Closely related to this is the notion of 'mathematical object'. Since mathematics is a discursive activity, these are discursive objects. Sfard speaks of signifiers having different realizations. For instance, realizations of a certain function could include its formula and graph. However, "the distinction between signifier and realization is relative" (ibid, p. 164). Whether a word or symbol should be seen as a signifier or a realization depends on the use it is being put to within the discourse at that point. In these terms, a discursive object can be defined as a signifier together with all the objects signified by its realizations (ibid, p. 166). This is a subtle and intricate matter (see ibid, ch. 6), but the above brief treatment will do for the purposes of this paper. The point I wish to make is that in order to study a mathematical concept within a discourse, we need to study the discursive uses of it.

Given this, the question this study aims to answer is: How is the concept of *linear transformation* presented in the discursive practices of the two teachers, and what consequences for their students' learning could this have?

## PREVIOUS RESEARCH

The topic of mathematics teaching in higher education has seen an increase in research activity over the last decade. A prolific researcher in this field is Nardi (e.g. Nardi, 2008; Nardi, Jaworski & Hegedus, 2005), investigating university mathematicians' views about the teaching of mathematics. There is also an anthology (Holton, 2001), covering many aspects of university mathematics teaching. The chapter by Dorier & Sierpiska (2001) on the teaching and learning of linear algebra is of particular relevance for this paper, although this is not a study of the teaching of linear algebra *per se*. This is however an active research subject in its own right (see e.g. Dorier, 2000). As for the everyday practice of mathematics teaching at the university level, it is not that well-researched, although there are a number of studies, including some (e.g. Weber, 2004; Wood, Joyce, Petocz & Rodd, 2007) focusing on "traditional" mathematics instruction. In Sweden, research on university mathematics teaching is rare, but one example is Bergsten (2007), discussing ways of investigating the quality of lectures in mathematics, building on a case study of one calculus lecture on limits of functions.

Since Sfard's commognitive theory is relatively recent, and still under development, not that many studies have been reported using this framework. Those that do exist tend to focus on the mathematical learning of younger children (e.g. Sfard, 2001,

2007; Sfard & Lavie, 2005) or on elementary mathematics, like arithmetic (Ben-Yehuda, Lavy, Linchevski & Sfard, 2005). However, very little has been published on university mathematics education from a commognitive perspective. One example is the work of Ryve (2006), which makes use of, but also critiques, an early version of the theory, as presented in for instance Sfard (2001), in order to investigate student interaction in problem-solving. As far as I know, there is yet no published research using commognitive theory to study university mathematics *teaching*.

## METHOD

The empirical data in my study consists mainly of videotaped lectures and lessons given by teachers in freshman year mathematics courses at three Swedish universities, chosen for diversity – one old, large university, one more recently established, and one smaller, regional university. The teachers were then selected amongst those willing to participate, and giving freshman courses on relevant topics during the time available for data collection. At the large university, where the number of possible participants was large enough to allow further choice, I again aimed for diversity, both in topics covered and in teaching experience. One thing all teachers in the study have in common, however, is an active interest in teaching.

In the present paper, the focus is on two 40-minute excerpts from lectures given by two different teachers. The collection and transcription of data is ongoing, and these excerpts were chosen from the ones available in transcribed form at the time of writing. They were chosen since they covered very much the same topic, thus enabling me to contrast the discursive activities of the teachers. The excerpts are from lectures in linear algebra, on the topic of linear transformations. The first excerpt is from a course given by a male teacher (A) in his thirties, having recently gotten his first position following some years of post-doctoral work. The second excerpt is from a course given by a male teacher (B) in his fifties, educated as a teacher. He does not have a doctoral degree, but has taught at the university for about 20 years. The first excerpt is from the large university, and the second from the regional one. The students in both courses were first semester engineering and computer science students. The excerpts were transcribed verbatim, speech as well as the writing on the board. The transcribed lectures were then analyzed, trying to distinguish the discursive patterns characterizing the teachers' respective discourses of functions, paying special attention to the discursive uses of central terms and symbols, particularly that of *linear transformation*. I first analyzed each lecture separately, and then looked at the two together, searching for differences and similarities. Since the unit of commognitive analysis is the discursive activity, I have intentionally chosen an outsider perspective, trying to view the enfolding discourse in as unbiased a way as possible. At the same time, I am of course aware of, and also making use of, the fact that my mathematical knowledge makes me an insider to the discourse. However, I have specifically tried to avoid making references to what is *not* present in the discourse, except in contrasting the two teachers' discursive activities.

## RESULTS

I will here attempt to describe how the concept of *linear transformation* is presented in the discursive practices of the two teachers. They approach this topic in quite different ways. Teacher A begins the lecture by defining the concept of linear transformation algebraically/geometrically, as maps on vectors respecting addition and scalar multiplication, and then proceeds to tell his students about these new objects, providing examples outlining what characterises linear transformations in specific situations, and what distinguishes them from general functions. He also states and proves a theorem on the relation between linear transformations and matrices. This relation is discussed on several occasions, pointing out their close relationship, as here, where we also can see the relativity of signifier and realization:

Teacher A: so if you understood that example then I can calm you down by saying that that is all you'll need to understand regarding linear transformations, 'cause it turns out, and we will see this later, that every linear transformation can be given by a matrix, or defines a matrix, and every matrix a linear transformation, so they are more or less the same

But he still takes pains to distinguish between the two. Indeed, one of his first examples consists of showing that the transformation given by multiplication by a matrix really is linear. Besides algebraic realizations through matrices, geometric realizations also feature prominently, with both rotations and projections shown to be linear. Matrix realizations of these transformations are not introduced initially, also indicating the distinction made between matrix and transformation. When the matrix realization of a given linear transformation is introduced, its dependence of the choice of basis is emphasised, pointing out the relationship between the algebraic and geometric realizations. In the following excerpt, this is also used to show the special character of linear transformations. A calculation involving iterated application of the definition of linearity has led to the following:

On blackboard:  $F(v) = v_1 F(e_1) + \dots + v_n F(e_n)$

Teacher A: and what does this mean? Well, it means something extraordinary, although it doesn't show in the formula, and now I'm going to tell you why

It means that if we know the values of  $F$  on these  $n$  vectors, then we know the whole of  $F$ , then we know what  $F$  does on all other vectors, so that it is enough to know the values of this function on a finite number of vectors to know it on all other vectors. This is seldom the case with functions

He then goes on to show how this most definitely isn't the case for a general continuous function on  $R$ .

All in all, teacher A presents a mathematical story, told in a precise language, paying attention to detail. What he doesn't do, however, is discuss the use of all this math. He states a couple of times that linear transformations are important, but never why. For teacher B, on the other hand, this is a central theme of the lecture. He very

consciously aims at connecting the mathematics to the everyday experiences of the students:

Teacher B: Everyone has been doing Photoshop or something like that

What happens if you shall decrease your image in breadth? All of a sudden we don't want to look as fat as we are

then you do like this, that you pull- you, to somehow get that feeling you multiply all breadths by 0.8, and then you get taller, or you get as tall but you look thinner, right?

He builds on a number of examples like this, and on the previous mathematical experiences of the students, of for instance straight lines as graphs of functions, to describe the concept of linear transformation, with the definition given near the end of the lecture. As we shall see, however, he is not as careful about the details as teacher A, something that has consequences for the understandability of the presentation.

At the beginning of the lecture he uses the word 'linear' for the first time:

Teacher B: What does this look like? [writes  $y = 2x + 1$  on the board] What kind of thing is this? It is a linear function, right? It is a straight line. Everyone of you knows exactly how it goes

Although using this straight line, and its corresponding equation, as a realization of the concept of linear function would make perfect sense within the discourse of functions in calculus, in the context of linear algebra it becomes problematic. Indeed, after having taken some time discussing the function concept in general, teacher B returns to the concept of linearity:

Teacher B: Now we're only going to deal with linear functions. In linear functions we must only have

without this [erases  $+1$  from  $y = 2x + 1$ ], we can only have this

we can't have any plus, because then we get a lateral movement

in that case we should have that lateral movement with another function, another calculation, giving two different functions

Suddenly, a function explicitly presented as linear, isn't linear any more. The only explanation given for rejecting addition by constants is that we must only do one operation at a time. But this, together with the statement about performing the lateral movement with another function, suggests that such a translation would also be linear, which isn't the case. He goes on saying that

Teacher B: What we will be looking at is something which looks

a vector  $y$  which really is the same thing as a matrix,  $A$ , times another vector  $x$  [writes  $y = Ax$ ]

Here, he identifies the linear transformation with its realization as a matrix, without any distinction of the kind teacher A makes.

Teacher B then presents an example, giving a specific matrix  $A$  and examining how this behaves viewed as a linear transformation. This time it isn't the word 'linear' which causes problems; it is the word 'transformation'. Following are a number of statements given during this example. In English translation they don't appear very problematic, but in Swedish they are, and I will attempt to explain why below.

Teacher B: Where will it map itself?

Teacher B:  $A$  times  $v_1$  will be my new image  $y_1$

Teacher B: so the image of  $v_1$  will be this

Teacher B: if I map  $A$  on  $v_2$ , what will happen?

The underlined words are all translations of the same Swedish word, either as a noun or a verb: *avbildning* or *avbilda*. The Swedish term for linear transformation is *linjär avbildning*. The problem is that, although perfectly correct everyday Swedish, using *avbildning* to denote what is here translated as *image* is incorrect in mathematical Swedish, the correct term being *bild*. Actually, the last statement is incorrect in both Swedish and English: the map isn't mapped, it is applied. In conclusion, in this example teacher B uses the same word to denote both the linear transformation itself and its image, as well as its application.

Finally, let's take a look at what happens when teacher B introduces the actual definition of linearity. He begins by noting the following property of linear transformations:

Teacher B: The demand on a linear transformation is of course that the origin is mapped to the origin

because if the origin isn't mapped to the origin, then we had that with, well, plus  $m$  which we had before, and then I said that it wasn't anything linear because we had an addition

As we see, this is formulated in a way suggesting that this is a consequence of wanting to avoid scalar addition. He then goes on to introduce the two rules of linearity – respecting addition and scalar multiplication. However he never states this as a definition. An explicit statement of what linearity means is never given.

## DISCUSSION

As already noted, there are very obvious differences between the two lecture excerpts, but there are also similarities. Hillel (2000) speaks of three languages of linear algebra: 'geometric', 'algebraic' and 'abstract'. Both lectures studied in this paper are conducted solely in algebraic and geometric terms. However, Hillel also speaks of teachers' tendencies to shift between these modes of presentation in an

implicit way. Here we can note that teacher A is more explicit in his discussion of the two modes, whereas teacher B does not acknowledge the difference at all.

As for the possible consequences of the two teachers' discursive practices, the impreciseness of teacher B can be expected to be detrimental to student learning. Indeed, neither the concepts of linear transformation, or more fundamentally, linearity, are used in a consistent way. Linearity is discussed in a way which makes it highly unclear whether its properties are the product of definition or whether they depend on some other, unexplained property. Also, 'linear transformation' is used in ways making it mean at least three different things, and at the same time the distinction between the object of linear transformation and its realisation as a matrix is not made clear. Indeed, you almost get the impression that they are different names for the same thing. Dorier & Sierpinska (2000, p. 270) speak of linear algebra as an “explosive compound” of languages and representations, which needs careful handling. The consequences of carelessness are clearly seen in my data. But there are strengths to be found in teacher B's excerpt as well. His conscious attempts at grounding the mathematical theory in students' everyday experiences are likely to be beneficial. Teacher A, on the other hand, provides little or no motivation for the need of studying linear transformations beyond verbally stating their importance. Since the lecture is part of a course aimed at engineering students, this might be problematic. But as for the mathematical content, he successfully handles the “explosive compound” without having it blow up in his face.

The conclusion would be that a combination of the strengths of the two approaches, presenting the material starting from the students' practical experiences, but paying more attention to mathematical detail, should be a reasonable approach. This may seem like an obvious statement, but implementing it in practice is perhaps not so trivial.

## References

- Ben-Yehuda, M., Lavy, I., Linchevski, L. & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetical discourses. *Journal for Research in Mathematics Education*, 36(3), 176-247.
- Bergsten, C. (2007). Investigating Quality of Undergraduate Mathematics Lectures. *Mathematics Education Research Journal*, 19(3), 48-72.
- Dorier, J.-L. (Ed.) (2000). *On the Teaching of Linear Algebra*. Dordrecht: Kluwer.
- Dorier, J.-L. & Sierpinska, A. (2001). Research into the Teaching and Learning of Linear Algebra. In Holton, D. A. (Ed.), *The teaching and learning of mathematics at university level: An ICMI Study* (255-273). Dordrecht: Kluwer.
- Hillel, J. (2000). Modes of Description and the Problem of Representation in Linear Algebra. In J.L. Dorier (Ed.), *On the Teaching of Linear Algebra* (191-207). Dordrecht: Kluwer.
- Holton, D. A. (Ed.) (2001). *The teaching and learning of mathematics at university level: An ICMI Study*. Dordrecht: Kluwer.

- Nardi, E. (2008). *Amongst Mathematicians. Teaching and Learning Mathematics at University Level*. New York: Springer.
- Nardi, E., Jaworski, B. & Hegedus, S (2005). A spectrum of pedagogical awareness for undergraduate mathematics: From 'tricks' to 'techniques'. *Journal for Research in Mathematics Education*, 36, 284-316.
- Ryve, A. (2006). *Approaching mathematical discourse: two analytical frameworks and their relation to problem solving interactions*. Department of Mathematics and Physics. Mälardalen University.
- Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, 46, 13-57.
- Sfard, A. (2006). Participationist Discourse on Mathematics Learning. In J. Maasz & W. Schloeglmann (Eds.), *New Mathematics Education Research and Practice* (153-170). Rotterdam: Sense.
- Sfard, A. (2007). When the Rules of Discourse Change, but Nobody Tells You: Making Sense of Mathematics Learning From a Commognitive Standpoint. *The Journal of the Learning Sciences*, 16, 565-613.
- Sfard, A. (2008). *Thinking and Communicating. Human Development, the Growth of Discourses, and Mathematizing*. Cambridge, U.K.: Cambridge University Press.
- Sfard, A. & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction*, 23, 237-309.
- Viirman, O. (2011). Discourses of Functions – University Mathematics Teaching Through a Commognitive Lens. Paper presented at CERME 7, Rzeszow, Poland. Retrieved from [www.cerme7.univ.rzeszow.pl/index.php?id=wg14](http://www.cerme7.univ.rzeszow.pl/index.php?id=wg14)
- Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, 23, 115-133.
- Wood, L. N., Joyce, S., Petocz, P. & Rodd, M. (2007). Learning in lectures: multiple representations. *International Journal of Mathematical Education in Science and Technology*, 38, 907-915.



# ON THE CLASSIFICATION OF PERSONAL MEANING: THEORY-GOVERNED TYPOLOGY VS. EMPIRICISM-BASED CLUSTERS

Maïke Vollstedt

Leibniz-Institute for Science and Mathematics Education (IPN), Kiel, Germany

*Personal meaning understood as personal relevance of an object or action is important for students when learning mathematics. In an interview study conducted in Germany and Hong Kong, 17 different kinds of personal meaning could be reconstructed. They were further developed into a theory-governed typology depending on the level of their relatedness to mathematics and the self. In addition, the empirical data is analysed with the help of cluster analyses to discover groups of persons preferring certain kinds of personal meaning. The paper discusses in how far the theory-governed typology and the empiricism-based clusters complement one another in addition to the results from the comparison of students from Germany and Hong Kong.*

## INTRODUCTION

The question for the meaning of learning mathematics is posed time and again by students. Therefore, the demand for meaning in (mathematics) education has been detected for many years, and meaningful learning has been identified as one of the major goals of education (Vinner, 2007, p. 10). Hence, to find convincing answers to the questions for meaning is one of the challenges posed for mathematics education.

There is no commonly accepted interpretation of the term *meaning* in the field of mathematics education. The diversity concepts is due to a mixture of philosophical and non-philosophical interpretations, as Kilpatrick, Hoyles, and Skovsmose (2005, p. 14) point out: “Even if students have constructed a certain meaning of a concept, that concept may still not yet be ‘meaningful’ for him or her in the sense of relevance to his/her life in general”. Two very different aspects of meaning are generally focussed on. Howson (2005, p. 18) distinguishes them as

those relating to relevance and personal significance (e.g., ‘What is the point of this for me?’) and those referring to the objective sense intended (i.e., signification and referents). These two aspects are distinct and must be treated as such.

In line with Howson’s distinction, the terms *personal meaning* and *objective meaning* can be distinguished denoting the personal relevance and the collectively shared meaning of an object or action respectively (cf. Vollstedt, 2010, 2011).

## PERSONAL MEANING

The concept *personal meaning* understood as personal relevance of an object or action was developed in the context of the Graduate Research Group on Educational

Experience and Learner Development at the University of Hamburg, Germany. The main focus of research in this field is on the learners' perspective of their own educational process. Therefore, the study reported on in this paper also takes the students' perspective when investigating personal meaning.

Mathematical contents and learning mathematics seen from the perspective of the students contain a productive range of aspects being personally relevant for the students. The following example given by Kilpatrick, Hoyles, and Skovsmose (2005, p. 9) shows that – among others – aspects such as purpose or usefulness, interest and motivation, the need for social relatedness and friendship as well as the students' performance and the outlook on their lives can become relevant for the construction of personal meaning in school context:

Some students find it pointless to do their mathematics homework; some like to do trigonometry, or enjoy discussions about mathematics in their classrooms; [...] other students are told that because of their weakness in mathematics they cannot join the academic stream.

The understanding of personal meaning as personal relevance of an object or action keeps this productive range of aspects. Therefore, this criterion was chosen to be the basis of our model of personal meaning consisting of a relational framework of different concepts from mathematics didactics, the didactics of Educational Experience and Learner Development, and educational psychology. These concepts are understood to have an impact on the construction of personal meaning as they denote aspects which are personally relevant for the students.

Let us assume we have a context in which an individual is dealing with a certain situation. In school, this might be a learning context in which a student deals with a mathematical problem. This student, let us call him Toby, has certain individual characteristics (i.e. characteristics that can be influenced over time) and a certain personal background (i.e. characteristics that cannot be influenced over time) respectively. Aspects of the personal background are for instance Toby's cultural and family background, whereas his individual characteristics consist of personal traits and concepts which are topic of research in mathematics didactics, the didactics of Educational Experience and Learner Development, and educational psychology. Toby might construct a different kind of personal meaning depending e.g. on his mathematical beliefs (Op 't Eynde, Corte, & Verschaffel, 2002), on his interpretation of the developmental tasks he is dealing with at that point in time (Havighurst, 1972), and on different aspects of learning motivation (Wild, Hofer, & Pekrun, 2001). In addition, Toby's academic self concept (Möller & Köller, 2004) and the three basic needs for autonomy, competence, and relatedness (Ryan & Deci, 2002) are believed to play a decisive role for the process of constructing personal meaning.

## **THE STUDY**

The study is based on 34 guided interviews conducted in Germany and Hong Kong with students of the lower secondary level (about 15-16 years old). Seventeen stu-

dents participated from each country; all attended the highest school type in the respective educational system. In Hong Kong, I collaborated with schools using English as medium of instruction so as to conduct the interviews in English.

The guided interviews lasted for about 35 to 45 minutes and began with a sequence of stimulated recall (Gass & Mackey, 2000). For this, the students were shown a five to ten minutes video abstract of the last lesson they attended. Their task was to verbalize and reflect on the thoughts they had during the lesson. The interviews then tackled various topics inspired by the relational framework of personal meaning (see above) to come as close as possible to the aspects related to learning mathematics which are personally relevant for the students in a school context. Students were for instance asked about their associations of the words *mathematics* and *mathematics lesson*, they were interrogated about their beliefs with relation to mathematics, mathematics lessons and their learning of mathematics as well as about their feelings, their learning strategies, their goals etc.

The data was coded in the style of Grounded Theory (Strauss & Corbin, 1996) by developing concepts directly from the interview material as well as theory-governed taking elements from the theoretical framework of the study as sensitising concepts. The coding was done partly in teamwork, partly independently but together with a team member so that the results could be discussed afterwards, and – after having received consistent results – partly on my own. By comparison and by using a coding paradigm, relations between concepts were disclosed so that core categories were developed denoting 17 different kinds of personal meaning.

Depending on the level of their orientation towards mathematics and the self, the kinds of personal meaning were grouped in seven different types following a theoretical point of view. The level of orientation towards the self (LOTS) distinguishes the different kinds as to their focus on the individual and their intraindividual relations (high LOTS), as to their relation towards mathematics lessons with less focus on emotional or cognitive aspects of the individual (intermediate LOTS), and as to their relation towards societal demands (low LOTS). The level of orientation towards mathematics (LOTM) makes a distinction as to the relation towards concrete mathematical contents. On a high LOTM, they are directly in focus, on an intermediate LOTM, the relation towards contents is rather oblique as they are dealt with as means to an end, whereas on a low LOTM, no concrete relation to mathematics is found so that social interaction becomes more central.

Strictly speaking, the model of personal meaning as presented above is also a result from the study as the concept and the relational framework were specified and developed further throughout the research process. To assure validity, the results as well as aspects lacking clarity from a Western perspective were discussed with professors of mathematics education and mathematics teachers from Hong Kong.

After reconstructing the kinds and types of personal meaning, they were analysed with respect to the students' cultural background. Bound to the reconstructive frame-

work, I opted not to generate hypotheses concerning cultural differences or similarities on theoretical basis with relation to the relevant literature. Instead, I conducted exploratory statistical analyses using the software SPSS (version 15) to generate culture-specific hypotheses from the data. Hence, with the help of the unspecific and undirected hypotheses used in the *t*-tests of my study I checked whether differences exist between the two places. On this basis I generated hypotheses about similarities and differences between the two places from empirical data. Statistics therefore acted as a means to conduct exploratory analyses instead of making general statements about Germany and Hong Kong.

To obtain a measure of the students' personal preferences, the relative frequency of every kind of personal meaning was calculated for every student. In order to achieve this, the codings of every kind of personal meaning were counted. Then, the percentage of codings of one kind of personal meaning was calculated from all codings of personal meaning of this student. These relative frequencies formed the basis for the calculation of *t*-tests. From these results, empirically grounded hypotheses were generated to the relation between different kinds of personal meaning and the students' cultural background. The results are rather tentative as each sample only includes 17 students. Therefore, to support these results with a non-parametric test, the Mann-Whitney-*U*-test was calculated. This paper only includes the results of the *t*-tests, because the results from the two tests were comparable (cf. Vollstedt, 2011).

In addition, a cluster analysis was carried out in order to find patterns between students. At first, outliers were detected by using the Single Linkage Method. The results of the coefficients indicating the distance between clusters suggested excluding one person from further calculations. Hence, another cluster analysis was calculated with the remaining 33 persons using Ward's Method. A decision was made for three clusters as the differences between the coefficients between clusters tree and four increased the most. Each of the three empiricism-based clusters consists of eleven students. The means of the clusters were finally compared with the help of Mann-Whitney-*U*-test applying Bonferroni correction (i.e. the critical value for significance *p* is lowered to .05/3 as three tests were calculated).

## RESULTS OF THE STUDY

As mentioned above, 17 different kinds of personal meaning could be reconstructed from the interview data. Depending on the level of their orientation towards mathematics (LOTM) and the self (LOTS), they could be grouped in seven theory-governed types. The following sections give a short overview of the types and underlying kinds of personal meaning as well as the results from cultural comparison and cluster analyses. The different kinds of personal meaning necessary for the description of the clusters or significant differences between Germany and Hong Kong are described in the discussion.

## Kinds and types of personal meaning

The type *fulfilment of societal demands* embraces all kinds of personal meaning with low LOTS: *vocational prerequisite*, *examinations*, *positive impression*, and *duty*. Societal demands are in the focus of all four kinds of personal meaning and fulfilling them is personally relevant for the interviewees.

The types *active practice of mathematics* and *efficient and supportive lesson design* show an intermediate LOTS as they embrace kinds of personal meaning dealing with actions which primarily focus on lesson elements. The first type consists of the kind *practice mathematics actively* (high LOTM), the second encompasses *efficiency* and *teacher's support* (intermediate or low LOTM).

The remaining types have varying LOTM but high LOTS as they focus on emotional and cognitive aspects personally relevant for the students. The type *cognitive self-development* embraces the kinds *purism of mathematics*, *cognitive challenge*, *experience autonomy*, and *self-perfection* with high or intermediate LOTM. The type *relevance of applications* is formed by the kind *application in life* having intermediate LOTM like the kinds *experience competence* and *marks* forming the type *well-being due to own performance*. The last type, *emotional-affective development*, consists of the kinds *relaxation*, *experience social relatedness*, and *emotional-affective relationship to teacher* (low LOTM).

## Results from cultural comparison

As the types consist of different kinds of personal meaning, differences between the students from Germany and Hong Kong blur. Therefore, significant differences only resulted from the comparison of the kinds *practice mathematics actively* ( $M_{\text{Ger}} = 1.53$ ,  $SE_{\text{Ger}} = 1.59$ ,  $M_{\text{HK}} = 5.94$ ,  $SE_{\text{HK}} = 3.87$ ;  $t(21.28) = -4.35$ ,  $p = .00$ ,  $d = -1.49$ ), *relaxation* ( $M_{\text{Ger}} = 3.60$ ,  $SE_{\text{Ger}} = 2.76$ ,  $M_{\text{HK}} = 8.88$ ,  $SE_{\text{HK}} = 6.33$ ;  $t(21.88) = -3.15$ ,  $p = .01$ ,  $d = 1.08$ ), *positive impression* ( $M_{\text{Ger}} = 3.57$ ,  $SE_{\text{Ger}} = 2.54$ ,  $M_{\text{HK}} = .78$ ,  $SE_{\text{HK}} = 1.80$ ;  $t(32) = 3.7$ ,  $p = .00$ ,  $d = 1.27$ ), *examinations* ( $M_{\text{Ger}} = .47$ ,  $SE_{\text{Ger}} = .90$ ,  $M_{\text{HK}} = 3.71$ ,  $SE_{\text{HK}} = 3.16$ ;  $t(18.60) = -4.06$ ,  $p = .00$ ,  $d = -1.39$ ), and *marks* ( $M_{\text{Ger}} = 4.99$ ,  $SE_{\text{Ger}} = 2.83$ ,  $M_{\text{HK}} = 2.13$ ,  $SE_{\text{HK}} = 2.30$ ;  $t(32) = 3.24$ ,  $p = .00$ ,  $d = 1.11$ ).

Judging from the effect size given in Cohen's  $d$ , to *practice mathematics actively*, *relaxation*, and *examinations* are more important for Hong Kong students, whereas giving a *positive impression* and getting good *marks* reflecting own performance is more important for German students. As mentioned above, these differences are only tentative due to the small sample size and the exploratory use of statistical methods.

## Clusters of persons and underlying patterns of personal meaning

The results from empiricism-based cluster analyses suggest three clusters for the data of this study. All clusters contain eleven students, yet with different shares from the two places: Cluster 1 is mixed (6 GER, 5 HK), Cluster 2 is primarily German (9 GER, 2 HK), and Cluster 3 is primarily a Hong Kong cluster (1 GER, 10 HK).

When comparing the three clusters with the Mann-Whitney-*U*-test, significant differences can be found. The results are presented in the following table:

Kind of personal meaning	Cluster	Median		<i>U</i>	<i>z</i>	<i>p</i>	Effect size $r = \frac{z}{\sqrt{N}}$
		First Cluster	Second Cluster				
Efficiency	1 vs. 2	10	6.67	30.5	-1.97	.049	-0,42
	1 vs. 3	10	5.08	20	-2.66	.008*	-0,57
	2 vs. 3	6.67	5.08	41.5	-1.25	.212	-0,27
Experience competence	1 vs. 2	6.25	12.25	17	-2.86	.004*	-0,61
	1 vs. 3	6.25	6.52	55.5	-.33	.743	-0,07
	2 vs. 3	12.25	6.52	14	-3.06	.002*	-0,65
Positive Impression	1 vs. 2	1.56	3.70	37.5	-1.55	.122	-0,33
	1 vs. 3	1.56	0	43	-1.26	.270	-0,27
	2 vs. 3	3.70	0	24.5	-2.48	.013*	-0,53
Relaxation	1 vs. 2	2.38	3.44	29.5	-2.04	.042	-0,43
	1 vs. 3	2.38	11.86	0	-3.98	.000*	-0,85
	2 vs. 3	3.44	11.86	4	-3.71	.000*	-0,79
Teacher's support	1 vs. 2	18.75	9.78	13	-3.12	.002*	-0,67
	1 vs. 3	18.75	10.87	14	-3.05	.002*	-0,65
	2 vs. 3	9.78	10.87	57	-.23	.818	-0,05

Table 1: Significant results from Mann-Whitney-*U*-test comparing three clusters.  
(\* Result is significant at .0165 level; *N* = 22 (number of students in two clusters)).

According to these results, *efficiency* and *teacher's support* are very much personally relevant for the students in Cluster 1. The students in Cluster 2 put great emphasis on giving a *good impression* and the *experience of competence*, whereas *relaxation* is important for the students in Cluster 3.

DISCUSSION

With reference to their different level of orientation towards mathematics (LOTM) and the self (LOTS), the 17 different kinds of personal meaning could be group into seven types. In contrast, with the help of cluster analyses it was possible to group persons with comparatively similar relative frequencies concerning each kind of personal meaning into three clusters. In a second step, underlying patterns of personal meaning could be discovered between the three clusters, which are significantly different.

When comparing the results of these two approaches, it turns out that in the cluster analyses significant differences could only be found for kinds of personal meaning with an intermediate or low LOTM. At a closer look, it turns out that both kinds of personal meaning of the type *efficient and supportive lesson design (efficiency and*

*teacher's support*) are very important for Cluster 1. The other types were not reflected in the results of the cluster analyses.

The kinds of personal meaning which are especially relevant for the students in Cluster 2 and 3 can be reflected on from a cultural perspective as they are primarily a German and a Hong Kong cluster. The results from the Mann-Whitney-*U*-test (comparing clusters) and the *t*-test (comparing the students from Hong Kong and Germany) do not totally coincide but show great overlap. For Cluster 2, two kinds of personal meaning turned out to be especially relevant: *positive impression* and *experience competence*. Giving a positive impression in front of their teachers is particularly relevant for German students as active oral participation comprises a great share in the overall mark students get for a subject. Therefore, this kind of personal meaning is significantly more important for the German than for the Hong Kong students of this study (see above). Yet, to experience competence e.g. after getting a right solution did not turn out to be significantly more important for the German or the Hong Kong students although it is significantly more important for the primarily German Cluster 2 in comparison to both, the mixed Cluster 1 and the Hong Kong Cluster 3.

The students of Cluster 3 show a preference of *relaxation*, i.e. it is personally relevant for them that they can relax when they are dealing with mathematics (e.g. doing Sudokus in their free time) and that there are situations in the mathematics lesson in which they can take a cognitive break. This kind of personal meaning is significantly more important for the Hong Kong students than for the German ones (see above). A reason might be that getting good results in the *Hong Kong Certificate of Education Examination* (HKCEE) is a great burden for Hong Kong students. Therefore, they value every situation in lessons in which they can relax.

Interestingly, *examinations* did not turn out to be significantly relevant for Cluster 3 nor did *marks* for Cluster 2 although there were significant differences between the students from Germany and Hong Kong for these kinds of personal meaning in the results of the *t*-tests (see above).

## CONCLUSION

Students are in the need for meaning when dealing with mathematics. In this study, seventeen different kinds of personal meaning could be reconstructed from guided interviews with students from Germany and Hong Kong. The relative frequencies of the different kinds of personal meaning coded for every student were – with the necessary caution – the basis for exploratory statistical analyses such as *t*-tests and cluster analyses. Due to the small sample size, the results are only tentative. Significant differences could be found between the students from Germany and Hong Kong as well as between the three clusters. A comparison of the results from the different analyses shows that they do not coincide totally. Yet, the interpretations enrich each other so that a clearer picture of the structure of personal meaning can be drawn.

## REFERENCES

- Gass, S. M., & Mackey, A. (2000). Stimulated Recall Methodology in Second Language Research. Second language acquisition research. Mahwah, NJ: Lawrence Erlbaum.
- Havighurst, R. J. (1972). Developmental Tasks and Education (3rd ed.). New York: Longman.
- Howson, A. G. (2005). "Meaning" and School Mathematics. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in Mathematics Education* (pp. 17–38). New York, NY: Springer.
- Kilpatrick, J., Hoyles, C., & Skovsmose, O. (2005). Meanings of 'Meaning of Mathematics'. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in Mathematics Education* (pp. 9–16). New York, NY: Springer.
- Möller, J., & Köller, O. (2004). Die Genese akademischer Selbstkonzepte: Effekte dimensionaler und sozialer Vergleiche. *Psychologische Rundschau*, 55(1), 19–27.
- Op 't Eynde, P., Corte, E. de, & Verschaffel, L. (2002). Framing student's mathematics-related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs. A hidden variable in mathematics education?* (pp. 13–37). Dordrecht, Boston: Kluwer.
- Ryan, R. M., & Deci, E. L. (2002). Overview of self-determination theory: An organismic dialectical perspective. In E. L. Deci & R. M. Ryan (Eds.), *Handbook of self-determination research* (pp. 3–33). Rochester, NY: University of Rochester Press.
- Strauss, A. L., & Corbin, J. (1996). *Grounded Theory: Grundlagen qualitativer Sozialforschung*. Weinheim: Beltz.
- Vinner, S. (2007). Mathematics education: Procedures, rituals and man's search for meaning. *Journal of Mathematical Behaviour*, 26(1), pp. 1–10.
- Vollstedt, M. (2010). „After I do more exercise I won't feel scared anymore”: An example of personal meaning from Hong Kong. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. January 28th - February 1st 2009, Lyon (France) (pp. 131–140). Lyon: Institut National de Recherche Pédagogique.
- Vollstedt, M. (2011). *Sinnkonstruktion und Mathematiklernen in Deutschland und Hongkong: Eine rekonstruktiv-empirische Studie*. Wiesbaden: Vieweg+Teubner.
- Wild, E., Hofer, M., & Pekrun, R. (2001). Psychologie des Lerner. In A. Krapp & B. Weidenmann (Eds.), *Pädagogische Psychologie. Ein Lehrbuch* (pp. 207–270). Weinheim: Beltz Psychologie Verlags Union.



# EXPLORING YOUNG CHILDREN'S FUNCTIONAL THINKING

Elizabeth Warren<sup>(1)</sup>

Jodie Miller<sup>(1)</sup>

Thomas J. Cooper<sup>(2)</sup>

Australian Catholic University<sup>(1)</sup>

Queensland University of Technology<sup>(2)</sup>

*This paper reports on a section of the Early Years Generalising Project (EYGP) involving Australian Years 1-4 (age 5-9) students and investigates how young students grasp and express generalisations. This paper focuses on the data collected from six Year 1 students in an exploratory study within a clinical interview setting that required students to identify function rules. The preliminary findings suggest that the use of gestures (both by students and interviewers), self-talk (by students), and concrete acting out, assisted students to reach generalisations and to begin to express these generalisations in everyday language. It also appears that as students become more aware of the structure of functions, their use of gestures and self-talk tended to decrease.*

## BACKGROUND FOR THE STUDY

EYGP is a three year longitudinal project that is studying a cohort of students from Year 1 to Year 4. The aim of the project is to build theories with regard to young students' ability to generalise mathematical structures. The cohort of students is a representation of a wide range of abilities across the first four years of school. Initially six students from each year level participate in a one-on-one clinical interview. From the results of these interviews conjectures are posed, and tested in a one-on-one semi-structured interview conducted with a further cohort of 20 students for each year level. This paper reports on one aspect of this project, an exploration of how 5 year old students generalise the function rule. (Note: EYGP is funded by ARC Discovery grant DP0987737).

The concept of a function is fundamental to virtually every aspect of mathematics and every branch of quantitative science. Presently, this type of thinking is corralled at the Secondary level, and yet it has many benefits for deepening our understanding of arithmetic. This is particularly so in the way that operations can be considered as "changing" and how functions explicitly illustrate the way in which addition and subtraction (and multiplication and division) are inverse operations, with each 'undoing' the other. While Blanton and Kaput (2005) suggest that students can engage in co-variational thinking as early as Kindergarten, are able to describe the relationship between quantities as early as Year 1, little is known about if they can reach generalisations, that is, identify the function rule and use this to predict other pairs of elements that conform to this rule, and even less is known about how they do this. As Lannin (2005), Kaput (1999), and Mason (1996) argue: "Statements of generality and discovering generality are at the very core of mathematical activity. Thus, the focus of this exploratory study is on what Radford (2006) calls the perceptual act of noticing generalities from specifics, and how this occurs as five year old children explore the concept of a function.

Mathematics is an intrinsic symbolic activity that is accomplished through communicating using oral, bodily, written, and other signs (Radford, 2006). Semiotics lends itself to the exploration of teaching and learning activities in mathematics, as this discipline is considered as abstract and is heavily based on perceivable signs. Semiotics assists us to understand mathematical processes of thought, symbolisation, and communication as the teaching and learning of mathematics draws on a variety of representations and resources. Two activities of particular importance to this communication is the use of body, the activity of interacting with artefacts, and activity with signs (Sabena, 2008). The use of the body, social, and cultural experiences is seen as strongly related to cognition (Lakoff & Núñez, 2000). It is this theoretical framework that guided our research.

In the last few years, research on generalisation (focusing on upper primary to upper secondary school years) has begun to identify different approaches. Harel (2001) has proposed two different forms of generalisation for mathematics induction: Results generalisation (developing a generality from a few examples, usually by trial and error) and process generalisation (developing and justifying a generality in terms that show progression across many steps). The distinction between Harel's forms appear to be very similar to Radford's (2006) distinction between naïve induction and generalisation and Lannin's (2005) distinction between empirical justification and generic examples, both of which were found from their studies of geometric growth patterns. Cooper and Warren (2008) also showed that 9 year old students can generalise both patterning and equivalence contexts. Incorporated in many of these theories are the notions of gesture, embodiment, and communication (including language and symbols), all considered as signs. Radford considers gesture as a type of sign, and has identified semiotic nodes as those 'pieces of the students' semiotic activity where action, gesture and word work together to achieve knowledge objectification' (Radford, 2006, p 144).

## METHOD

Piagetian clinical interviews were conducted by two of the researchers with Year 1 students (n=6), three female and three male with an average age of 5 years. The students were from a middle socio-economic elementary school in the outer suburbs of a major city. They were representative of a range of academic abilities and cultural backgrounds. Interviews were approximately 20 minutes in length and consisted of 5 tasks; with two having a language focus, two having a geometry (shape) focus, and one having a number focus. The aim of the tasks was to probe students' understanding of functions. All questions were posed to the students in a flexible manner, and the tasks were set up as play-like activities starting from unnumbered situations and moving to the numbered situations. All interviews were video-taped. Table 1 presents the five tasks, each tasks function rule, and an example of the input and output values for each rule presented to each student in the interview.

	Unnumbered situations								Numbered situations	
Task	Language (1)		Language (2)		Shape (3)		Shape (4)		Number (5)	
Rule	Add 'ip'		Add 'ap'		Make it thinner		Make it thinner and smaller		Add two	
Example	Input	output	Input	output	Input	output	Input	output	Input	output
	T	Tip	M	Map	Red, large, thick triangle	Red, large, thin triangle	Blue, large, thick square	Blue, small, thin square	5	7

Table 1: Description of tasks used in the interviews

Each task was presented using a physical function machine (Rosie) made from a cardboard box. The input and output values were presented on cards or as physical shapes. The student was introduced to the function machine, Rosie, and the interview began with the simple language task (Language (1)). The student was shown a letter which they placed in Rosie's ear (input) and the researcher then produced the output card for the student, out of the opposite ear. The student was given further examples of the rule and was asked if they could describe the rule. The questions posed were contingent on the responses given by the student. Depending on their responses, students were either given further examples or were asked to predict output cards for given input cards, and then to predict input cards for given output cards. The researcher asked students to justify their answers and express the rule and its inverse in general terms. It also should be noted that these students had not engaged in formal experiences with the operations of addition and subtraction prior to the interview.

Each videotape was transcribed with students' thinking processes represented by their verbal responses and their manipulation of the concrete materials. All data was analysed by at least two researchers and member checks were performed. This was particularly important with regard to identifying gestures and actions students' used as they articulated their responses. Of particular interest to this paper was their use of gestures and self-talk as they reached generalisations and discussed their conjectures with the interviewer. In this context gestures are defined as all those movements [hands, arms, eyes] that subjects perform during their mathematical activities (Sabena, 2008; McNeill, 1992).

## FINDINGS AND DISCUSSION

For reporting purposes each student was allocated a code, namely, S1, S2, S3, S4, S5, and S6. The students were identified by the classroom teacher as being high achievers in mathematics (S1 & S4), medium achievers in mathematics (S5 & S6), and low achievers in mathematics (S2 & S3). The data associated with each task was

organised into three categories, namely, the student’s ability to correctly predict: (a) output values from given input values, (b) input values from given output values, and (c) the particular function rule relating to that task. Table 2 presents the tasks together with the students who were successful in each category.

Tasks	Language tasks		Shape tasks		Number task
	(1)	(2)	(3)	(4)	(5)
Rule	Add ‘ip’	Add ‘ap’	Make it thinner	Make it smaller and thinner	Adding Two
Predict output	S1,S4,S5, S6	S1,S4,S5, S6	S1,S2,S3, S4,S5,S6	S1,S3,S4, S5,S6	S1,S2,S4, S6
Predict input	S2,S3,S4, S5	S1,S4,S5	S2,S3,S4, S5,S6	S3,S4,S5, S6	S1,S2,S4, S6
Identify rule	S4	S1,S4,S5	S1,S2,S3, S4,S5,S6	S1,S3,S4, S5,S6	S1,S2,S4, S5,S6

Table 2: Student’s success on the five tasks

Some preliminary findings were: (a) students’ ability to identify the rule increased as they moved across the tasks, and (b) students experienced most difficulty with the language tasks as compared with the shape tasks and number task. Whether this is related to the tasks themselves or whether it is the result of learning as students moved across the tasks needs further investigation. It should also be noted that S4 was the only student who successfully answered all aspects of the interview.

Language played an important role in assisting students to justify their responses. For example, during Language Task 1 most students had difficulties verbalising Rosie’s rule. While many could identify that Rosie was changing the initial letter into a word they experienced difficulty in describing specifically what was happening, and this inability impacted on their capacity to predict the output. With the shape tasks, one difficulty that the majority of students experienced was their lack of appropriate geometric language that would assist them to justify their answers. It seemed as students progressed across the tasks the sophistication of the language they used to identify the rules and to justify their predictions increased.

An example of this was the responses by S1 to the 5 tasks. This student’s ability to identify the rule became more specific as she progressed through the tasks. She was also asked a sixth task (a number task where the rule was subtracting two). Below is an example of how this student’s language became more specific in identifying the rule for each task after being asked ‘What is Rosie’s rule?’

S1: (Response to identifying the rule for Language 1): nip, sip, tip, dip.

Interviewer: What does she (Rosie) do to these words? [*Researcher covers up the 'ip' of the word tip.*]

S1: She turns them into a sentence because I put the letters in and new letters came out. She turned the letters into words [*S1 points to the cards*]. She is doing the rules. You put this in and get it on the other side. It's magic. [*S1 is constantly looking at the box. Her eyes are tracking from left to right along the function machine.*]

S1: (Response to identifying the rule for Language 2): She is making 'ap' words.

S1: (Response to identifying the rule for Shape 3): She (Rosie) turns them all flat. [*During this task the researcher gestured along the front of the box from left to right. S1 manipulated the shapes in her hands and clearly described the attributes of the shape before it was placed into Rosie.*]

S1: (Response to identifying the rule for Shape 4): Two differences now. It turns it (red, big, thick triangle) into a little small one and flat again. [*Once again the researcher gestured along the front of the box and the student was manipulating the shapes using a rich description.*]

S1: (Response to identifying the rule for Number 5): It is skipping one. Before I had six and it turned to eight so it skipped seven. Now I have 15 and it skipped 16 to get 17. We are skipping numbers. [*S1 is now looking only at the function box and tracking her eyes. S1 is also self-talking (utterances) while determining the predictions in this task.*]

S1: (Response to identifying the rule for Number 6): It is skipping two backwards. [*S1 is looking at the box and tracking with her eyes, looking at card, and using self-talk to count backwards*].

In each of these cases this student not only improved in her ability to determine the generalisation for each of the tasks, but she also increased her gesturing and use of self talk.

Another student that showed similar increases in gesture and self-talk as the interview progressed was S2, a low achieving student. Initially the gestures made by S2 were subtle; however the dynamics of these gestures and self-talk increased as she moved through the interview. S2 also exhibited a marked improvement in her ability to identify the rule as she moved through the tasks.

S2: (Response to identifying the rule for Language 1): Changing it. [*S2 looking at the box. Eyes follow the box across from left to right when the examples are given. S2 does not look at the box when predicting or when identifying the rule. S2 cannot see all the output cards as the interviewer has taken them*].

S2: (Response to identifying the rule for Language 2): She changes the little letter. She keeps a 'p' on the end. [*Interviewer's hand is only moving along the back of the box. S2 predicts the input looking at the card. Student looking from left to right, turning head. Possibly reading is an issue for this student.*]

S2: (Response to identifying the rule for Shape 3): It got flatter. *[S2 feels the shape as she puts in. As she states 'It got flatter' her eyes follow the box and look to Rosie's output ear. She uses her hands to describe the shape becoming flatter and to describe the triangle.]*

S2: (Response to identifying the rule for Number 5): It has to go to a higher number by 2. *[S2 gesturing increases in this task. S2 moves body along the front of the box and self-talks when predicting.]*

Interviewer: So if I put in 3 what am I going to get?

S2: 5

Interviewer: What did you do to work that out?

S2: 3 plus 2 equals 5 *[demonstrates using fingers]*

Interviewer: Okay, you were right. Now this is for clever kids: What if I got 16 out? What do you think I might have put in?

S2: 14 *[S2 self talking whilst solving function]*

Interviewer: How did you work that out?

S2: I was using my toes and my hands. You take down 16 and 15 and you make it to 14 *[demonstrates using fingers and toes]*.

A tentative conclusion is that students who used self-talk and gestures, in conjunction with their thinking, seem to demonstrate improved performance as they moved through the tasks. In the case of S4 this use of self-talk and gestures did not seem as imperative. S4 already seemed to have a clear understanding of the notion of function from the commencement of the interview, as indicated by his correct identification of the rule for Language Task (1).

## DISCUSSION AND CONCLUSIONS

The preliminary results have identified four themes that appear to be important in assisting young students to identify the function rule, and to share their thinking with regard to how they identified the rule.

First, in all tasks students who had the output cards placed in front of them were more able to identify the rule, and therefore provide a generalisation for the task, than those who did not. This was particularly important for the language tasks. However, the students who could generalise (S1, S4, S5) were all high/middle achievers.

Second, gestures/body actions from both students and researcher are an important dimension of the generalisation process. Students performed better when the researcher was gesturing along the front of the box from the input to the output. With the exception of S4, students who gestured less and did not interact with the box experienced difficulties in answering the questions posed. They also took a longer period of time to predict outputs and explain how the function operated. In addition, these students exhibited inconsistencies in answering the questions that required prediction and appeared to be utilising a random guess approach. By contrast,

students who utilised gesturing as they engaged in the tasks experienced the most growth during the interview and were able to identify the function rule. The use of gesturing was not necessarily related to perceived cognitive ability. An example of this was S2. For her, gesturing increased from eye movement to full body movement as the interview progressed. Accompanying this was her increased ability to identify the rule and articulate the generalisation.

Third, the inclusion of concrete tactile items seemed to allow the students to explain more fully what was occurring during the functional change. This is possibly because tactile items have more tangible attributes that can be described rather than just a number or letter on a card. This was particularly evident in the geometry activities (shape tasks). Initially when the students were given the attribute blocks they knew the shape names. Most students needed prompting during their discussion of the shapes to include attributes such as colour and size. These discussions certainly appeared to help the students identify the change rule, and the geometry tasks were the two function activities that most students experienced success on.

Fourth, probing the students' initial hypothesised function rule to assist them to refine their thinking was achieved by directing students to test their hypothesis with other examples (provided by the interviewer and the student) or asking for further clarification using questioning techniques such as 'How does Rosie do this?'. The students who did not initially identify the rule with success were assisted by the researcher. This was achieved by encouraging them to revisit the initial examples given for the input values and to regenerate the corresponding output values.

The results of this research begin to align with Radford's notion of semiotic nodes, an idea he conjectured from research with older students. Meaning-making requires students to coordinate a range of signs as they objectify their understandings, and fundamental to this process, at the beginning stages, is the incorporation of body movement and self-talk. Even with very young students, the use of the body (pointing, hand movement, and eye movement), and engagement with signs (the cards, the function box, and the geometric blocks) all assisted these students' (the interpretant) cognitive development (Sabena, 2008). Interestingly these dimensions appear to be most important as students begin to understand the task, or concept. The results of this exploratory study suggest that this need seems to diminish as the structure of problem context becomes more apparent. Thus, as students objectify their understanding, the need for an array of signs, particularly concrete and iconic signs, seem to be of less importance. The question is, while S4 did not rely on gesturing and self-talk as he progressed through the tasks, how important were the other signs in assisting him to grasp the mathematical structure?, and is there a hierarchy of signs that assist young students to reach understanding or is it the continual mapping across signs and the bundling of signs that assist them to engage with the core understanding?

The results also begin to resonate with our other results from this study, young students' ability to pattern. We are beginning to hypothesise the act of grasping is complex and entails two aspects, namely the growth of an underlying understanding of the pattern (or function in this instance), and translation of this to a process that efficiently reaches accurate answers, a relationship between structure and efficient completion. This begins to appear in the data from an examination of S3 responses. By Shape tasks (3) and (4) she is beginning to see the structure of the pattern but the Number task (5) required her to also be efficient in her understanding of number, a dimension of mathematics that she exhibits weaknesses in and hence her classification by her teacher as a low achiever. These ideas need further exploration.

## References

- Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36, 412-446.
- Cooper, T.J., & Warren, E. (2008). Generalising mathematical structure in Years 3-4: A case study of equivalence of expression. In Figueras, O. and Cortina, J. L. and Alatorre, S. and Rojano, T. and Sepulveda, A., Eds. *Proceedings of the 32<sup>nd</sup> Conference of the International Group for the Psychology of Mathematics Education*. Volume 2, pp. 369-376. Morelia: Mexico.
- Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell, & R. Zazkis (Eds.), *Learning and teaching Number Theory. Journal of Mathematical Behavior* (pp. 185-212). New Jersey, Ablex Publishing Corporation.
- Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romnberg (Eds.), *Mathematics Classrooms that Promote Understanding* (pp. 133-155). Mahway, NJ: Lawrence Erlbaum.
- Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.
- Lannin, J. (2005). Generalization and Justification: The Challenge of Introducing Algebraic Reasoning Through Patterning Activities. *Mathematical Thinking and Learning*, 7(3), 231-258.
- Mason, J. (1996). *Expressing generality and roots of algebra*. Dordrecht: Kluwer Academic Publishers.
- McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.
- Radford, L. (2006). Algebraic Thinking and the Generalization of Patterns: A Semiotic Perspective. In J.L.C.S Alatorre, M. Saiz, A. Mendez, *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education*. Volume 1, pp.2-21. Mexico: Merida
- Sabena, C. (2008). *On the semiotics of gestures*. In Radford, L., Schubring, G., & Seeger, F. (Eds). *Semiotics in Mathematics Education: Epistemology, History, Classroom and Culture*, pp. 19-38. Rotterdam: Sense publishers.



# ACTIVATION OF INNER MATHEMATICAL DISCOURSES OF STUDENTS ABOUT FRACTIONS WITH THE HELP OF IMAGINARY DIALOGUES: A CASE STUDY

Annika M. Wille

*A case study is presented concerning the question of how an inner mathematical discourse of students about fractions can be activated in order to develop their own mathematical ideas. Within the theoretical framework of commognition by Sfard (2008), thinking is seen as the process of communicating between a person and herself. In the approach presented in this article the mathematical discourse of a single 11 years old student will be analysed against the commognitive background. Her discourse can be traced in an imaginary dialogue she wrote about fractions. The main research questions are whether imaginary dialogues can activate an inner mathematical discourse and whether students develop mathematical ideas while they are writing imaginary dialogues.*

## THEORETICAL FRAMEWORK

### Commognition

Anna Sfard (2008) defines thinking as “individualization of interpersonal communication”, and furthermore as “the process of communicating between a person and herself, one that does not have to be verbal” (p. 302). For Sfard, the process of thinking is closely connected to communication. Hence, she uses for both of them the term *commognition*. Within this framework mathematics is seen as a discourse. Concerning mathematical thinking, mathematical self-communication is as Sfard states “difficult to observe” (ibid., p. 276), since thinking is invisible whereas the interpersonal channels as in a dialogue can be perceived. Sfard identifies linguistic communication “as the primary source of sustainable, accumulable changes in human forms of doing” (ibid., p. 123) and points out the unbounded recursivity of human linguistic communication. An example of this recursivity is communicating-on-communicating as “utterances in which one reports on what somebody else has said, remarks on her own thoughts, or reflects on other interlocutors and their communicating actions” (ibid., p. 103).

### Mathematical writing

In mathematics education, writing has been used and studied in different forms. Advantages of journal writing have been explored by Borasi & Rose (1989). Gallin & Ruf (1998) investigated journal writing where an initial task has parts that everyone can undertake but also parts that can be a “ramp” (ibid.) for the more advanced students. They point out that similar processes go on while writing as while speaking. However, the writing processes are slower, enabling students to be more aware of them (ibid.). Two categories of mathematical writing, journal writing and expository writing are differentiated by Shield and Galbraith (1998). Clarke, Waywood & Stephens (1993) distinguish between three modes of mathematical

writing which they call Recount, Summary and Dialogue. They understand Dialogue as an internal dialogue and describe that the students who wrote in Dialogue mode began to focus on the “ideas” being presented and that they were able to identify, analyse their difficulties and suggest reasons why they were thinking in a certain way (ibid.). Another form of written dialogues, namely *imaginary dialogues*, was studied by the author (Wille 2008). Students wrote their own imaginary dialogues between protagonists who discuss different mathematical tasks or questions.

## METHOD

Within the school year 2008 to 2009 a project was conducted by the author with 30 students in a class of grade five of a Grammar School (Gymnasium) in Bremen, Germany in order to study, amongst other topics, their appreciation of fractions. In this article the *case study of a single student* of this class, whom we will call Emma, is presented.

Since in the commognitive framework the mathematical discourse is the principal unit to be analysed, and since thinking, seen as self-communication, is a part of the mathematical discourse of a student, the question is how to activate an inner mathematical discourse and how to explore it from the view of a researcher. Since self-communication is, as Sfard (2008) notes, difficult to observe, the approach presented here is to activate a form of *written self-communication*. The idea is that writing an *imaginary dialogue* could invoke a mathematical self-communication in order to develop mathematical ideas, to revise them and reflect on one’s own mathematical thoughts. Imaginary dialogues are a form of communicating-on-communicating, because they consist of utterances in which one reports on what a protagonist could have said, but also own thoughts flow into the dialogues.

The students wrote five imaginary dialogues in which two protagonists talk about mathematical questions. Every imaginary dialogue started with an initial dialogue that was to be continued by the students. The initial dialogue was given to ease the start of the writing process. On three occasions a *stimulated recall* was conducted with four to seven students after they had written the imaginary dialogues. In the stimulated recall the students saw a video taken of their writing process. The video was often stopped by an interviewer to ask what the students had been thinking about at that moment, why they had written what they did, and when they had had the different ideas.

The research questions were:

- Can imaginary dialogues activate an inner mathematical discourse in order to develop own mathematical ideas about fractions?
- Do students develop mathematical ideas while they are writing?
- Can traces of their mathematical discourse be seen in the imaginary dialogues?
- How can learning obstacles be detected in imaginary dialogues without the help of a consecutive stimulated recall?

Here, the research questions will be investigated using one imaginary dialogue and the corresponding stimulated recall of the 11 year old student Emma.

## LEARNING ENVIRONMENT

Klaus Lies, a teacher in Bremen, Germany, developed a *ladder model* for his classes (2002 to 2006). Here the number line is arranged vertically and possesses ladder steps. Stefan Halverscheid also used and explored this model (Halverscheid, Henseleit & Lies 2006). In the *ladder model for fractions*, new ladder steps are added. For example, a ladder for halves has as additional ladder steps  $1/2$ ,  $3/2$ ,  $5/2$ , and so forth. Before the students get the task below they are given paper ladders for different fractions. The following topics had already been taught: whole numbers, greatest common divisor, least common multiple, comparing and adding fractions.

“Two students are talking with each other:<sup>1</sup>

S1: Imagine I add 1 and  $1/2$ .

S2: Okay. That is  $3/2$ .

S1: Now add to the result  $1/4$  and to the next result  $1/8$  and to the next  $1/16$ , then  $1/32$  and so forth.

S2: You mean something like this?

*S2 is writing on a paper:*

$1+1/2+1/4+1/8+1/16+1/32+\dots$

S1: Exactly! That is getting quite big, or not?

*S2 thinks...*

S2: I am not so sure. I believe, it won't be that big.

S1: Maybe pictures are helping us with it or we calculate everything one after another.

S2: Good idea! Let's begin! Maybe we'll find out how large that will be.

Continue the dialogue. Write at least one page.”

## The mathematics behind the task

Behind the task stands the geometric series

$$\sum_{i=0}^n q^i$$

with  $q=1/2$ . Of course geometric series are not a topic of this class of grade five. However, this special series offers possibilities for the students to practice fraction addition, to imagine fractions and to discover several properties of the fractions. If we look at the infinite geometric series, the sum is  $1/(1-q)$ . Thus for  $q=1/2$  the result is 2. However, also without the help of the formula we can establish that any finite sum of

---

<sup>1</sup> Throughout this paper fractions are written horizontally, like  $1/2$ , in order that all spaces between the rows have the same size. In contrast, in the project all fractions were written vertically, like  $\frac{1}{2}$ .

this form never reaches 2. If we add 1 and  $1/2$ , we get  $3/2$ . So, there is a difference of  $1/2$  to 2. Now, we add  $1/4$ . Therefore  $1/4$  is missing to get 2. But since we add  $1/8$  in the next step, still  $1/8$  is missing. Thus by induction, after each addition the difference between the sum and 2 is the same as the amount added in the previous step. (We omit a formal proof at this point.) Some intermediate sums can also be illustrated in the ladder model or in any other model for fractions.

Every student of the class is at least able to work on some sums and try to illustrate them. More advanced students can consider between which whole numbers the result could be located, as explained above, or they can discover various properties. Two of them we will see in Emma's imaginary dialogue.

## FINDINGS

Let us look at the imaginary dialogue that Emma wrote originally in German. The dialogue was translated by the author.

- 1    "S1:    If I add  $3/2$  and  $1/4$  then that should yield  $7/4$ .
- 2    S2:    Why?
- 3    S1:    Because if you want to add two fractions, you convert both of them so that they have a common denominator.
- 4    S2:    Okay, go on.
- 5    S1:     $7/4 + 1/8$  should be  $15/8$ .  $15/8 + 1/16$  yields  $31/16$ , and  $31/16 + 1/32$  is  $62/32$ . If you divide this number by the half, in this case 16, then the result is 2. Then you know how much  $62/32$  is in halves.
- 6    S2:    I see. And that should be, wait a moment, I must just calculate it.

The calculation of S2:

- 7    S1:    But 62 can't be divided by 16.
- 8    S2:    Right. But I noticed something else, while you explained the addition.
- 9    S1:    What did you notice?
- 10   S2:    Every time when you said for example  $7/4 + 1/8$  or  $15/8 + 1/16$ , if you calculate crosswise, then the numerator is the result. For example:  
 $7/4 + 1/8$ ,  $7+8$  is 15, and that was the numerator from the actual result, thus  $15/8$ . With  $15/8 + 1/16$  it was the same.  $15+16$  is 31 and that is the numerator of  $31/16$ , the result of  $15/8 + 1/16$ .
- 11   S1:    You are right. I did not notice that.

- 12 S2: Moreover I noticed that with, for example  $7/4+1/8$ , every time you add the both numerators, then the resulting number is the denominator of the second fraction. For example:



A photograph of a piece of lined paper with the equation  $\frac{7}{4} + \frac{1}{8} = 8.$  written in blue ink. The numbers are written in a slightly messy, handwritten style.

- 13 S1: I didn't notice much else.
- 14 S2: Well, at least that's something.
- 15 S1: But to get back to our first question: we don't know yet, if the number is big or not.
- 16 S2: No, we do know it.
- 17 S1: How?
- 18 S2:  $32/32$  would be on ladder step 1. The double of 32 is 64. That would be the ladder step 2. But  $62/32$  is not yet on ladder step 2. It is therefore not even  $2/1$ .
- 19 S1: Well, you have a good memory today.
- 20 S2: Thank you!

### Emma's mathematical ideas

We can divide Emma's imaginary dialogue into five blocks: row 1 to 4, 5 to 8, 8 to 11, 12 to 14 and 15 to 20. In the first block (row 1 to 4) Emma discusses the addition of  $3/2$  and  $1/4$ . In the stimulated recall she states that she first calculated it mentally and then wrote it down.

In the second block (row 5 to 8), first she calculates sums up to  $31/16+1/32$ . In the last step she has a little arithmetical error by writing  $62/32$  instead of  $63/32$  as the result, but this is not that important for the following dialogue. Emma's first idea appears in row 5. It deals with the question of how large the result could be. Emma wants to convert  $62/32$  into a fraction with 2 as the denominator. In the stimulated recall she says:

Emma            To see how big that is, I wanted to come back to halves.

We can also see that Emma did not have this idea in the beginning of her writing, since in the stimulated recall the interviewer asks within the context of this first idea by pointing to the first row:

Interviewer    Did you notice something at that point?

And Emma answers:

Emma            Not really.

Thereafter, Emma's idea leads to a learning obstacle, as we can see in row 7, when she writes "But 62 can't be divided by 16."

In the third block (row 8 to 11) Emma discusses something different that she noticed. She saw that in each addition,  $7/4 + 1/8 = 15/8$  and  $15/8 + 1/16 = 31/16$ , if one adds the numerator of the first summand with the denominator of the second summand, the result will be the numerator of the result (row 10)<sup>2</sup>. In the stimulated recall Emma states:

Emma            Here (she points at row 1) I did not notice it, but while I was writing this (she points at row 5), I looked at this carefully one more time and then I noticed it.

Thus, Emma had this idea while she wrote her imaginary dialogue.

In the forth block (row 12 to 14) Emma notices something similar. She sees that for  $7/4 + 1/8$  and  $15/8 + 1/16$ , the addition of the two numerators, respectively, yield the denominator of the second summand (row 12)<sup>3</sup>. In the stimulated recall she explains what she did before she began to write the forth block:

Emma            Then I looked, to see if I could notice something else.

Thus again, Emma did not notice this before she wrote the imaginary dialogue.

Finally, in block 5 (row 15 to 20) Emma comes back to the starting question, how large the result could be. She identifies  $32/32$  as the ladder step 1 and  $64/32$  as the ladder step 2 in the ladder model for fractions. So, she sees that her resulting number  $62/32$  is smaller than  $2/1$  (row 18). In the stimulated recall the interviewer asks at which point Emma thought about this. And she answers:

Emma            While I was writing.

### **The activated inner mathematical discourse**

Let us recall the mathematical ideas Emma had and focus on the question of when these ideas occurred to her.

In the first block we can assume that Emma reflects on something she already knows, since she says in the stimulated recall:

Emma            I did not first write the calculation down, but I thought about it first mentally.

Later on, Emma does not only reflect on her previous mathematical knowledge, but while writing the blocks 2 to 5 she has new mathematical ideas which she lets her protagonists discuss.

Hence, we see that in Emma's case:

- The process of writing the imaginary dialogue activated an inner mathematical discourse.

---

<sup>2</sup> This property can be shown by induction.

<sup>3</sup> Also this property can be shown by induction.

- Moreover, traces of this inner mathematical discourse can be read in the statements of her protagonists.

### Indications of learning obstacles

In block 2, Emma hits a learning obstacle, namely that she cannot convert  $62/32$  to a fraction with the denominator 2 in order to know how large this number is. Later, in block 5, she gets over this obstacle by directly comparing fractions to whole numbers.

When reading imaginary dialogues as a researcher, learning obstacles often seem to appear. However, the question is whether we can identify them even without a stimulated recall. Is it possible to know something about students' learning obstacles just by reading their imaginary dialogues? The problem is that we read something the protagonists say and not directly a statement of the student writing the dialogue. This raises the question of when we can assume a learning obstacle for the student if a protagonist hits one.

By looking at all imaginary dialogues and stimulated recalls that were made within this project, some points became more clear. Emma's dialogue helps as an example. Until now we can say that:

- We can assume a learning obstacle is encountered by the student writing the dialogue, if *both protagonists* hit this learning obstacle.

This is the case in block 2 of Emma's dialogue. At that point she writes "But 62 can't be divided by 16" and the other protagonist answers "Right.". Both protagonists seem to be clueless at that point. Furthermore, in the stimulated recall she says about this:

Emma            For the time being it did not work.

- On the other hand, if a student has previously overcome a learning obstacle, has only observed it in other students or simply thought it up him- or herself, it seems probable, that *only one protagonist* will hit the learning obstacle while the other will take on more of a teaching role.

We can see this in the beginning of Emma's imaginary dialogue where in row 2 the protagonist S2 asks "Why?'" and S1 explains. In the stimulated recall Emma confirms the assumption by saying that she first calculated  $3/2 + 1/4 = 7/4$  and then wrote it down. Furthermore, she states:

Emma            There I thought about what S2 could say. So I chose 'Why?'.

This observation could be important in cases where the researcher wants to detect learning obstacles without it being possible to do stimulated recalls after the writing of imaginary dialogues.

### The infinite sum

Finally, it should be mentioned that Emma did not discuss the infinite sum in her imaginary dialogue. In the stimulated recall she explains that she could not add

endlessly, so she stopped at  $31/16+1/32$ . Just to see how another student handled the starting question of the task, here is what Lukas wrote in his imaginary dialogue:

- S1:           While you were telling me that, it became clear to me that we do not have to calculate it. Since the numbers are becoming smaller, it is clear that the resulting number cannot be particularly big.

## SUMMARY

Finally, we can revisit the research questions from above and answer them regarding Emma's discourse:

- For Emma the writing of her imaginary dialogue was suitable to activate an inner mathematical discourse.
- Emma developed mathematical ideas while she was writing her imaginary dialogue.
- Traces of her mathematical discourse can be seen in the imaginary dialogue.
- An indication for a learning obstacle of the student writing the dialogue can be, if both protagonists hit the learning obstacle within the imaginary dialogue.

## References

- Borasi, R., & Rose, B. J. (1989). Journal writing and mathematics instruction. *Educational Studies in Mathematics*, 20(4), 347-365.
- Clarke, D. J., Waywood, A., & Stephens, M. (1993). Probing the structure of mathematical writing. *Educational Studies in Mathematics*, 25(3), 235-250.
- Gallin, P., & Ruf, U. (1998). *Dialogisches Lernen im Mathematikunterricht, Band 2: Spuren legen – Spuren lesen*. Seelze: Kallmeyer.
- Halverscheid, S., Henseleit, M. & Lies, K. (2006). Rational numbers after elementary school: realizing models for fractions on the real line. In J. Novotná, H. Moraová, M. Krátká, N. Stehlíková (Eds.) *Proc. 30<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 255-232). Prague: PME.
- Shield, M., & Galbraith, P. (1998). The analysis of students expository writing in mathematics. *Educational Studies in Mathematics*, 36(1), 29-52.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, USA: Cambridge University Press.
- Wille, A. M. (2008). Aspects of the concept of a variable in imaginary dialogues written by students. In O. Figueras, J. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Proc. 32<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 417-424). Cinvestav-UMSNH, Mexico: PME.



# RELATIONSHIPS BETWEEN ELEMENTS OF COGNITIVE, SOCIAL, AND OPTIMISTIC MATHEMATICAL PROBLEM SOLVING ACTIVITY

Gaye Williams

Deakin University; The University of Melbourne

*The mathematical problem solving activity of an elementary school student (Lenny) was studied to find relationships between cognitive, social and psychological activity. Lenny was selected because he changed from non-optimistic to optimistic orientation providing opportunity to contrast activities. Four cameras captured activity during six tasks over a two-year period. 'Two up' video images of group discussion, and reports to the class stimulated student reconstruction of lesson activity in individual post-lesson interviews. Data collection techniques enabled retrospective analyses of videos and interviews to study activity of students who changed in optimistic orientation. Findings illuminate the simultaneity of some cognitive, social, and optimistic activity raising awareness of the need for further study of optimism.*

## INTRODUCTION

Focus on deep learning of mathematical concepts rather than rules and procedures is emphasised by Sawyer (2008) as valued by the “knowledge economy” (p. 48):

... deep conceptual understanding of complex concepts, and the ability to work with them creatively to generate new ideas, new theories, new products, and new knowledge [is needed]. ... [people] need to learn integrated and useable knowledge, rather than the sets of compartmentalised and decontextualised facts” (Sawyer, 2008, p. 49).

Mathematics education research has found many aspects of problem solving activity associated with deep learning including: ‘cognitive’ and ‘social’ elements of the process of abstracting new knowledge (e.g., Dreyfus, Hershkowitz, & Schwarz, 2001), subcategories of these processes during ‘spontaneous abstracting’ (Williams, 2007a), and high positive affect associated with such processes (e.g., Liljedahl, 2002; Williams, 2010). This study extends research into links between cognitive, social, and psychological activity during problem solving. Constructs are elaborated later.

## THEORETICAL FRAMEWORK

‘Problem solving’, for the purposes of this study, is activity of groups and / or individuals working on unfamiliar problems (problems for which they do not know a rule or procedure). The creative development of new knowledge (spontaneous abstracting, Williams, 2007a) involves high positive affect during ‘flow’ situations (Csikszentmihalyi, 1992). Conditions for flow during mathematical problem solving (Williams, 2010) include a spontaneous self-set or group-set challenge that can be overcome by developing new skills and conceptual knowledge. In other words, the

student/s are not solving routine problems but rather finding ways to answer a question they have posed for themselves. To do so, they need to ‘move into unknown territory’ by working with mathematics in unfamiliar ways. During flow, students lose all sense of time, self, and the world around as all their energy is focused on the task at hand. Epstein, Schorr, Goldin, and colleagues (2007) identified interactions during group work that they linked to students’ ways of ‘surviving’ in an urban culture. These types of interactions: “Don’t Disrespect Me” (p. 651) and “Stay Out of Trouble” (p. 653) inhibited student engagement with mathematical ideas because the focus was on ‘not losing face’ rather than on mathematics as the authority in the task at hand. In other words, students’ sense of self inhibited their problem solving activity. Had these students entered a flow situation, sense of self would not have been operating. This may be what happened during the other type of behaviour identified by Epstein and colleagues: “Check This Out” (p. 653) where the interactions resulted from interest and curiosity and students were inclined to try to work out something they did not know. This happened in the case in this study.

Seligman’s (1995) construct ‘optimism’ is employed as an analysis tool to study psychological aspects of problem solving in this study because students who creatively develop mathematical ideas have been found to be optimistic and enact optimism during problem solving (Williams, 2010). Optimistic people perceive successes (**S**) as permanent, pervasive, and personal (**SPnl**), and failures (**F**) as temporary (**FTemp**), specific (**FSpec**), and external (Seligman, 1995). Success during problem solving, for this study, is ‘knowing’ (finding something out), and failure is ‘not knowing’. Solving unfamiliar problems requires optimism because not knowing (failure) is perceived as temporary and able to be overcome (success) through personal effort by looking in to the situation and working out what can be changed (failure as specific) / cannot be changed (as failure externally controlled).

The construct of ‘cognitive elements of the process of spontaneous abstracting’ involves abstracting without mathematical input of an expert other *during* the interval of spontaneous abstracting. The RBC Model, Recognizing, Building-with, and Constructing (Dreyfus, Hershkowitz, and Schwarz, 2001) (DHS), and Krutetskii’s (1978) ‘mental activities’ were used in combination to develop the thinking framework employed (Williams, 2007a). Cognitive elements of problem solving processes are made visible through discussion. In this case, Lenny’s group work in class was studied for the first task, and his discussion in the post lesson interview for the second task. The thinking framework includes the following elements. 1) Recognizing (**R**) which includes realizing certain mathematical constructs were applicable to the task at hand (DHS, 2001). 2) Building-with (**B**) (DHS, 2001) previously known rules and procedures in familiar and unfamiliar ways has been sub-categorised into (a) Simple analysis (**Bsa**): identifying and using rules and procedures in routine ways possibly in new sequences and or combinations; (b) Element-analysis (**Ba**): isolating parts of something unfamiliar and examining them one by one; (c) Synthetic-analysis (**Bs-a**): simultaneously examining and building with several

elements; and (d) Evaluative-analysis (**Be-a**): synthetic-analysis for purposes of judgment. 3) Constructing (C) (DHS, 2001): the process of developing insight or seeing something mathematically profound that the student was not previously aware of has been subcategorised into (a) Synthesis (**Csyn**): insight illuminating mathematics not previously known which could occur through identification of generality; and (b) Evaluation (**Cev**): reflection on mathematics progressively developed and the results obtained for the purpose of checking internal and external consistency and considering the usefulness of the insight for other purposes.

Social elements of the process of abstraction (DHS, 2001) are ‘control’ (**Co**), ‘elaboration’ (**El**), ‘explanation’ (**Ex**), ‘query’ (**Qu**), ‘agreement’ (**Ag**), and ‘attention’ (**At**). During spontaneous abstracting, these elements are generally internal to those taking part in the abstracting process. If the source is external and spontaneous abstracting continues, mathematical input into the abstracting process was not given (see Williams, 2004). This sometimes happens with external queries and attention. Meanings of social elements are elaborated in interpreting the data.

Research Question: What empirical evidence is there of links between cognitive and social elements of the process of spontaneous abstracting and optimistic activity?

## RESEARCH DESIGN

This data was drawn from a broader study on the role of optimism in collaborative problemsolving. Lenny was a Grade 4 (2009), Grade 5 (2010) student in an Australian elementary school who was selected as a case for this study because he reported changing from not perseverant, to perseverant in his problemsolving over the two-year period. In his interview after Task 3, 2010 Lenny reflected on changes to his problem solving activity over time: “Instead of just going (pause) ‘I don’t know’, I (pause) [now] sit there and I *really really* (pause) think about it- even if it is [only] a sum or an angle [learned procedure] ...”. When asked how that happened, Lenny said: “I don’t really know ... it just made me think more (pause) It was probably actually *doing* the [research] tasks ...”. Lenny’s contrasting activities were expected to help elaborate links between optimistic, cognitive, and social problem solving activity. Lenny performed at average level in mathematics classes.

Classroom video and video stimulated post-lesson student interviews were employed to identify links between cognitive and social elements of problem solving activity, and the enactment of optimism. Four videos were used to capture these three types of activity in class, and individual video-stimulated student interviews were undertaken after each lesson to identify where such activity occurred, and gain further information about it. Students were questioned about what they had learnt, and asked to find the parts of the lesson that were important to them and their learning. The video images they viewed were a ‘two up’ showing the activity of their group, and the group reports to the class. Students were also asked additional questions about how they thought they were going in maths, whether they enjoyed maths, and to describe a good lesson for them. These questions elicited indicators of optimism or

lack thereof. The data collection methods allowed the retrospective analysis of the behaviour of any student identified (like Lenny) to have changed in their optimistic orientation.

The researcher (author) was the primary implementer of the problem solving activities and the interviewer. The classroom teacher participated as the class worked in groups with the tasks, and groups reported at intervals to the class as a whole. For more detailed descriptions of the pedagogy employed see Williams, (2007b). Three tasks were undertaken each year (Task 1, 3 x 80 Min, Task 2, 2 x 80 Min, and Task 3, 1 x 80 Min. session). T1, 2009 and T3, 2010 are briefly described:

Task 1, 2009: Using all 14 tiles (each time), make as many different flat ‘filled’ rectangles as you can (using top surface of tiles). Repeat using 12 tiles. Have you found all possibilities? Make an argument that justifies that you have them all. Do more tiles make more rectangles? Why or why not? Select a number of tiles between 16 and 45 to make as many rectangles as possible. Explain your thought process.

Task 3, 2010: Design an advertising slogan by constructing a Blue Smartie Promise to attract lovers of blue Smarties to buy. Remember broken promises are not good for the company. Each group starts with a small-unopened box of Smarties (coloured candy), predicts the number of blue Smarties in their box giving reasons for their predictions, opens the box, counts, and discuss their findings. This is followed by reports to the class with the reporter adding a tally to the board (See Figure 1). The procedure is repeated with each student opening a box after predicting. Each student adds their data to Figure 1. Groups analyse the data, then report to the class on their Blue Smartie Promise. The feasibility of keeping each promise is discussed.

## **ANALYSIS AND RESULTS**

Lenny’s activity during Task 1 2009 fitted his descriptions. Where previous knowledge could be easily be applied, he participated. When his group worked with the 12-tile case, Lenny quickly realised the two remainder (reported by another group in 14-tile case) could be removed giving a three by four rectangle. He took time to make sure the group report included this. As his group did not use factors in the 12, 14, or 16-tile parts (even though some other group reports mentioned factors) they did not have an easy procedure to apply to find solutions for the final task part. Lenny did not ‘step’ into this unfamiliar territory to try to work this out. Instead, he spent considerable time off task—sometimes distracting others by poking them playfully.

During Task 3 2010, the following diagram (Figure 1) was generated on the board by students adding tally marks to record their findings. For example, if a box had 6 blue Smarties, a tally mark was added beside the box with 6 in it [top of Box Column 2]. The five tally marks beside this box show five boxes with six blue Smarties in each.

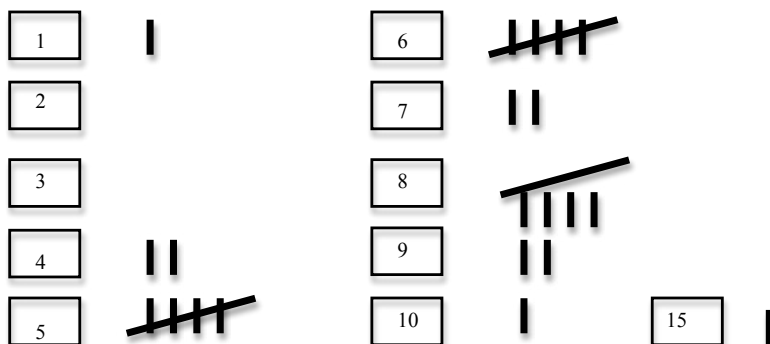


Figure 1: Diagram on board: tallies of numbers of blue Smarties in boxes.

Lenny considered the diagram as he puzzled about how to answer his spontaneously posed question. He continued to puzzle about this after the group had begun writing their promise thus displaying his interest: “*Yeah I didn't really put that much into our ... promise because I was [soft laugh] trying to figure out the average*”.

The following excerpt of Lenny's interview shows what led to his spontaneous challenge and how he began to overcome it. It includes interpretations of lines of transcript (cognitive, social, and optimistic activity), and short notes to support understanding. The transcript is to be considered in conjunction with Figure 1.

**Transcript Key:** ‘I’: interviewer; ‘L’ Lenny; ‘(pause)’ pause; ‘/’ cut across statement of another; ‘*italics*’ emphasis; ‘[text]’ researcher comments, ‘...’ part of transcript line not essential to meaning omitted; ‘---’ transcript lines not essential to meaning omitted. Interviewer's soft word or two to encourage further discussion, omitted without the inclusion of ‘---’ is evident from line numbering. Cognitive, social, and optimistic activity symbols used were defined in the theoretical framework.

1. L ... [a] group had (pause) one (pause) ... I found that really *really* surprising (pause) ... even the *four* (pause) because that is *half* (pause) what I thought it would be (pause) ---
2. I And why do you think that happens? [External **Qu**]
3. L It is probably (pause) when they were putting them in the boxes it is just (pause) *random* [**R**, possibly **B**, commencing **C**, as considers meaning of his term]
4. I ... Had you thought that out at that time? Or (pause)? [**Qu**]
5. L [intense] Yeah I was trying to think (pause) what the (pause) *average* was [**R**; **FTemp**, **SPnl**, **FSpec**, inclined to shift into ‘unknown territory’]
6. L And I think I did it wrong but- I added all of them up [**Bsa**] so I counted the fifteen as one [**Be-a**, **At** has identified a problem, **FSpec** looks into situation]
8. L I added it all up and I think it added to twenty four and then ... I forget what we- *I* was supposed to do then so I just counted all the ones that had (pause) ... the ah numbers next to them and then I think there was nine

and then I divided it by nine and then it was like **[Bsa]**, partially correct: need to divide *totals*; total smarties/boxes found incorrectly, **[FSpec]** ---

12. I Oh it was *two* (pause) was it? [surprise] **[Qu]**
13. L Yeah (pause) two and a bit **[Ag, El]**
14. I Oh did you *add* the numbers? **[Qu]**
15. L *Yeah I added all of them up like* (pause) so I added this one three sss- eight [no. of tallies not no. of Smarties represented by tallies] **[Bsa, El, FSpec]**
17. L So I did it wrong [confident voice no **Qu**] **[El]**, Perceived 'not knowing' **(F)**
18. I And so when you are trying to average it you were trying to work out how many you are sharing amongst/
19. L *No!* h- what the average of *blue* was in each packet (pause) so like in between twenty *no!* (pause) four- five packets (pause) yeah **[Ex]** ---
22. I ... do you ... know something was the matter with what you did do you? **[Qu]**
23. L *Yeah* but I can't remember how (pause) to do it properly [Trying to recall rules and procedures for finding an average rather than making meaning]
24. I How did you know there was something the matter with what you did? **(Qu)**
25. L Because I knew the (pause) it's there's if there was eight (pause) six and five each (pause) *mmore* of them are over five so how is it under two? **[R]** (average partially correct) **Be-a** partially correct; perceived **S]**
27. L And I knew probably it would be around five six because the *one* would bring it down a fair bit ... **[R]** (average partially correct) **Be-a**; perceived **S]**
35. L ... But I-I like di- I just went *one* (pause) *two* and stuff but I didn't count like (pause) ... yeah I didn't count all *of them as 15* (pause) I just counted them as *one* each **[Be-a, Qu, At, El]** (Line 15), **[FSpec]**, identified **F]**
36. I ... so I wonder how many blue Smarties are there altogether? [External **Qu]**
38. L [soft exclamation] *I don't know-* its two (pause) three (pause) oh eight (pause) mmm thirteen (pause) what's that- is that two or three (pause) on the ssseven --- **[Bsa]** still counting tallies only (partially correct); **El; FSpec]**
41. I So are you counting the number of *Smarties* there or are you counting the number of boxes? [External **Qu**, External **At]**
42. L I am counting the number of (pause) how many *lines* **[Bsa; El]** shows not **Bs-a** yet]
43. I ... I wonder what those lines stand for (pause) whether they stand fo/? **[Qu]**
44. L ... [excited] /They stan- that st- that one stands for *one* [one tally beside 1-box] and that one stands for (pause) *four* [two tallies beside 4-box] **[Bs-a]** commenced, probably **Csyn** commenced; **El; FSpec; S** perceived (*beginning* to realise need tallies and number in box to count no. blue Smarties so Constructing an understanding of nature of frequency tables]
45. I Four what [soft questioning/wondering voice]? [External **Q** requesting further **El]**
46. L Four (pause) blue Smarties [confident] **[El]**, perceived **S]**
47. I Okay (pause) so to find how many blue smarties (pause)? **[Qu]** eliciting **El]**
48. L I'd have to count *one* ... add *four* [then corrected] ... add *eight* which would be nine and then I'd have to add five fives ... **[Ex, Bs-a, S]** perceived] ---

## DISCUSSION AND CONCLUSIONS

Lenny changed from ‘giving up’ (2009) to looking to see what he could find out (developed Failure as Specific) which was integral to achieving successes within his problem solving as he persevered (Failure as Temporary, Success as Personal). He displayed this behaviour when class findings surprised him [see Line 1 (L1)] and he spontaneously asked “What’s the average?” recognizing the relevance of this construct (**R, At, FSpec**) [L5]. He first *tried* just adding the tally marks to find the number of blue Smarties [L6], and counting the number of boxes with tallies beside them to find the number of boxes and dividing these [L8] (**Bsa, FSpec**). He considered whether his answer was reasonable [Line 25, 27] displaying some understanding of average as he ‘justified’ that it was not (**Be-a, El, FSpec**). He considered that he did not yet know—**F**, so looked further to see if he could identify what was the matter, and realised he had counted (for example) 15 Smarties as one [L6, 35] (**Be-a, Qu, FSpec**). He had realised what needed to change to find the number of Smarties (**At**) but not how to change it [L38, 42]. The interviewer’s queries [L36, 43] led to his realising how to simultaneously consider two ‘columns’ of Figure 1 (boxes and tallies). He demonstrated this [L44, 48] (**Bs-a, At, El, FSpec**).

Lenny’s activity showed the evaluative-analysis subcategory of the cognitive element building-with, and social elements query and attention, occurred simultaneously with the enactment of the optimistic activity Failure as Specific. The ability to identify failures and successes through enactment of these activities guided the problem solving. Perseverance (perceiving Failure as Temporary, Success as Personal) was not sufficient to achieve problem-solving successes. Perception of Failure as Specific was also required to identify failures to be reconsidered, find what could change, make changes, consider the results, and recognise when success was achieved. This study highlights optimism as a productive area for study to further enhance student problem solving. Studying more cases should elaborate and extend identified links and inform the process of partially correct constructs becoming more correct.

## Acknowledgements

Fellowship hosted by ICCR. Study funded by Australian Research Council (DP0986955).

## References

- Csikszentmihalyi, M. (1992). Introduction. In M. Csikszentmihalyi & I. Csikszentmihalyi (Eds.), *Optimal experience: Psychological studies of flow in consciousness* (pp. 3-14). New York: Cambridge University Press.
- Dreyfus, T., Herschkowitz, R., & Schwarz, B. (2001). The construction of abstract knowledge in interaction. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 377-384). Utrecht, The Netherlands: PME.
- Epstein, Y., Schorr, R., Goldin, et al. (2007). Studying the affective/social dimension of an inner-city mathematics class. *Proc. 29th Annual Meeting of the North American Chapter*

- of the International Group for the Psychology of Mathematics Education*, (p. 649-656). Stateline (Lake Tahoe), NV, University of Nevada, Reno.
- Krutetskii, V. (1976). *Psychology of mathematical abilities in schoolchildren*. (J. Kilpatrick, & I. Wirzup (Eds.), J. Teller, Trans.). Chicago: University of Chicago Press. (Original work published in 1968).
- Liljedahl, P. (2002). The 'AHAmoment': Students insights in learning mathematics. *Proceedings of the 24th International Conference for Psychology of Mathematics Education—North American Chapter* (Vol. 2, pp. 887–896). Atlanta, GA.
- Sawyer, K. (2008). Optimising learning implications of learning sciences research. In *Innovating to learn, learning to innovate: Organisation for Economic Co-operation and Development*.
- Seligman, M. (1995). *The Optimistic Child*. Adelaide: Griffin Press.
- Williams, G. (2004). The nature of spontaneity in high quality learning situations. In M. Hoines & A. Berit Fuglestad (Eds.), *Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 433-440). Bergen, Norway: PME.
- Williams, G. (2007a). Abstracting in the context of spontaneous learning. In M. Mitchelmore & P. White (Eds.). *Abstraction, Special Edition, MERJ*, 19(2), 69-88.
- Williams, G. (2007b). Classroom teaching experiment: Eliciting creative mathematical thinking. In J. Woo, H. Lew, K. Park, & D. Seo (Eds.). *Proceedings of the 31st conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 257-364). Seoul, Korea: PME.
- Williams, G. (2010). Symbiosis between creative mathematical thinking accompanied by high positive affect, and optimism. In Pinto, M., & Kawasaki, T. (Eds.). *Proceedings of 34th conference of the International Group for the Psychology of Mathematics Education*, (Vol. 5, pp. 297-304). Belo Horizonte, Brazil: PME.



# PRE-SERVICE TEACHER ACTION RESEARCH USING BIBLIOTHERAPY TO ADDRESS MATHEMATICS ANXIETY.

Sue Wilson

Australian Catholic University

*High level of mathematics anxiety in pre-service primary teachers affects not only their current study but also their self-efficacy, and their future teaching of mathematics, and hence the attitudes and performance of their future students. Bibliotherapy, incorporated into a pre-service teacher's action research cycle during her final practicum, increased her understanding of the impact of previous experiences on her identity as a learner and teacher of mathematics, and produced insights into strategies that could address her mathematics anxiety. With each cycle of her action research, supported by the bibliotherapy process, the pre-service teacher was able to develop greater insight, eventually leading to a more robust projection into her future as a teacher.*

## **Introduction**

Recent directions in mathematics education are based on the premise that all students are capable of learning mathematics. This contrasts to the traditional view, where only a few students were expected to succeed. However, beliefs that success in mathematics relates to participants' inherent worth still dominate thinking (Gates and Jorgensen (Zevenbergen), 2009, p. 164). Failure in mathematics can have a powerful emotional impact that may extend far beyond the mathematics classroom (Boaler, 1997). The impacts of mathematics instruction produce for many an enduring state of mathematics anxiety.

This paper is part of a larger project investigating the use of bibliotherapy, within a framework of action research, as a tool for addressing primary pre-service teachers' (PST) affective responses to mathematics. This will add to existing frameworks for the study of affect in mathematics education (Hannula et al, 2004).

## **Theoretical framework and literature review**

This research is located at the intersection of the literature on the impacts of mathematics anxiety on primary teacher mathematics education, and bibliotherapy.

## **Mathematics anxiety**

Mathematics anxiety is a learned emotional response, characterised by a feeling that mathematics cannot make sense, of helplessness, tension, and lack of control over one's learning. Mathematics anxiety has been associated with inappropriate teaching practices, and a prevalent belief that success in mathematics is determined by ability rather than effort. Ma's (1999) meta-analysis of studies of elementary and secondary students found significant relationship between anxiety towards mathematics and achievement in mathematics.

Baroody and Costlick (1998) suggested that children who develop mathematics anxiety tend to fall into a self-defeating, self-perpetuating cycle, describing a mathematics anxiety model that illustrates how their beliefs can lead to anxiety, reinforcing unreasonable beliefs. Theoretical models of mathematics anxiety have multidimensional forms which incorporate attitudinal, cognitive and emotional aspects, (Hart, 1989; Wigfield & Meece, 1988).

The impact of teachers' beliefs about mathematics can be far-reaching in promoting positive outcomes for students, and remains an important focus for educational research, (Leder, 2007). Many pre-service primary or early childhood teachers have a fear of mathematics, and see themselves as unable to learn effectively. A great deal of research has been done in this area, but it is outside the scope of this paper. Ambrose (2004) reports mechanisms that have potential for changing beliefs are those providing emotion-packed, vivid experiences, becoming immersed in a community, and promoting reflection on beliefs.

Many students come to tertiary teacher education with only basic mathematics understandings, and a pattern of avoidance and anxiety. Researchers of primary (elementary) PST report high levels of mathematics anxiety, low confidence levels to teach mathematics and low mathematics teacher efficacy. For a more detailed discussion of these issues see Wilson (2009).

Bandura's theory of self-efficacy indicates the significance of teachers' beliefs in their own capabilities on student learning and achievement. Bandura (1994) defines self-efficacy as "people's beliefs about their capabilities to produce designated levels of performance that exercise influence over events that affect their lives . . . Self-efficacy beliefs determine how people feel, think, motivate themselves and behave" (p. 71). People need a strong sense of efficacy before they try to apply what they have learnt or try to learn new things. Teachers' beliefs about their own ability are a significant factor in their approach to teaching mathematics and even militate against their willingness to teach upper primary classes (Wilson, 2009). High teacher efficacy improves student performance learning and achievement (Ashton, 1984; Gibson & Dembo, 1984; Allinder, 1995; Madison, 1997).

Many researchers conclude that high levels of teacher mathematics anxiety can be perpetuated in classrooms (for example, Furner & Berman, 2005). When students are marginalised and do not identify themselves as confident learners of mathematics, they are unlikely to map mathematics into their future identities in a positive way (Boaler, 1997). This is not only a cognitive experience, but also an emotional one. The way individuals perceive themselves is integral to their continued learning of mathematics and to their teaching. Previous research (Wilson & Thornton, 2008; Wilson, 2007) found that many PST identified an interaction as students, where they experienced a loss of confidence and started to identify themselves as persons who couldn't learn mathematics, impacting on the professional identity as future teachers of mathematics that they constructed.

Identity brings together affective qualities and cognitive dimensions. Walshaw, (2004, p. 557), argues that “teacher education must engage the identities of pre-service students”, and describes the journey of a pre-service secondary teacher, Helen, who, “through a process of formation and transformation, finally at the end of the year, understood who she might become” (p. 563).

In summary, PST with mathematics anxiety are less likely to engage with mathematics, and have low confidence and low self-efficacy, impacting on their identity as teachers of mathematics. It is for these reasons that teacher education has become a crucial site for further research.

### **Bibliotherapy**

Bibliotherapy is a technique that was developed in psychology and library science. It involves guided reading of written materials used in gaining understanding or solving problems. The procedure is based on reading about the dilemmas of a third person, enabling the person to identify with the protagonist in the story, followed by individual or group discussion in a non-threatening environment. The reader is an active participant in the process, but feels safe because they are not the one experiencing the crisis; so they are able to interpret through the lens of their own experiences. Bibliotherapy has been used in preparing pre-service teachers to teach students with emotional and behavioural disorders, and students with special needs (Morawski, 1997).

The first four stages of bibliotherapy (Hebert & Furner, 1997) are, the reader:

*identification* - identifies with and relates to the protagonist.

*catharsis* - is emotionally involved and releases pent-up tension.

*insight* - learns through the experiences of the character and becomes aware that their problems might also be addressed or solved.

*universalisation* – recognises that we are not alone in having these problems .

Wilson and Thornton identified a fifth stage in their study of pre-service teachers (2008), relating it to projective identity (Gee, 2001).

*projection* – the reader can envisage a different future identity.

Ricoeur (1994) suggests that people make sense of their own personal identities in a similar way to their understanding of the identity of characters in stories. Identities are mobile, and remain open to revision. The potential of bibliotherapy is that it is a stimulus for this revision, and the planning cycle of action research. Previous research used bibliotherapy during mathematics units for pre-service teachers to examine their attitudes towards themselves as learners and teachers of mathematics (Wilson & Thorton, 2008; Wilson, 2009). The significance of the changes in response to the bibliotherapy process was that they contributed to the understanding of aspects that drive the development of their mathematical identity. Themes

identified through the analysis of previous research strongly suggest the importance of insight as a major factor in bringing about a positive projective identity.

### ***Methods***

This paper reports an action research project by a primary PST that examined how she might address the impact of her mathematics anxiety on her mathematics teaching practices during the final practicum of the course. Action research was chosen as has been identified as a powerful process for reconstructing and transforming practice (Somekh, 2005). Bibliotherapy was used within the framework of action research, as a tool for addressing her affective responses to mathematics. The goal was to understand what had influenced her in the development of her teaching practice, examining the relationship between her beliefs and her classroom practice, and their impact on her professional identity.

Three instruments were used in the action research cycle:

Initially, the PST completed a short questionnaire about her self-perceptions as a learner and teacher of mathematics, and past experiences that contributed to these. This was repeated at the end of the project.

The second procedure was a cycle of written reflections on lessons. This involved pre- and post- self-assessments of each mathematics lesson. These comprised a survey and short questions completed before and after each lesson, (including feelings, preparedness and teaching success, rated on scale of 1- 10; and notes on level of confidence, what went well, what would be changed in future).

The third instrument was a journal of written reflections. Previous papers about mathematics anxiety were provided as part of the process of action research. The readings formed the stimulus for the written reflections, as one of the means of incorporating bibliotherapy into the action research process. The reflections were shared with the researcher and fellow PST undertaking action research projects, during and in a presentation and discussion after the practicum.

The reflections were triangulated with the answers to the questions and lesson assessments, and the conclusions were then reviewed in the light of student feedback, and related to the outcomes of previous research using the bibliotherapy framework.

### ***Results and Discussion***

Using readings to clarify understanding of learning is central to the bibliotherapy technique. An important part of the initial answers revolved around the view of mathematics the PST they had developed during her schooling. "I've always been able to 'keep up' but not necessarily understand what I was doing".

The PST's preliminary comments gave voice to the concern of researchers to ensure that negative learning experiences will not reinforce negative beliefs and feelings about mathematics in the future students PST will teach, and echo the concerns of teacher educators who identify this as an issue. A major concern was that she would

“inadvertently pass on my fear and anxiety of maths to my students. I don’t want them having the same negative experiences that I have had”. During her presentation and discussion with peers, she emphasised the strength of the concern she felt at the start of her practicum. “I was concerned that I would instill [sic] in students the same feelings about maths as what I have”. This echoes previous research findings, (Wilson, 2009) where teachers’ comments reflect a concern for their students that negative learning experiences will not reinforce negative beliefs and feelings about mathematics.

The reflections on readings showed a strong identification: “I think this perfectly describes how I feel about maths – especially the tension”. Identification is one of the stages of the bibliotherapy process. The PST also commented that the findings of the readings were interesting and relevant, for example the reports of mathematics anxiety starting in primary school related to her school experiences.

The reflections on individual lessons indicated that her assessment of her feelings before the lessons stayed in the range from 6 to 8 ½, but the that level after the lessons had a much broader range, from 2 after the first lesson, rising to 8 ½, plummeting to 4 and then rising back to 8. The PST commented that when she became flustered in lessons, the “lesson focus would change dramatically” and this lowered her assessment of her feelings after the lesson. She related this to her attitude. “If I felt confident before the lesson started, I most often felt good about it afterwards, however if I went into the lesson with a negative attitude, then I most often had negative feelings about that lesson afterwards”. Her positive experiences increased her confidence that she would be able to decrease her level of anxiety. “I may even be able to change my negative attitude of this subject over time”.

With each cycle of her action research, supported by the bibliotherapy process, the pre-service teacher was able to develop greater insight, eventually leading to a more robust projection into her future as a teacher. “I found that acting confident in maths actually made me feel more confident and I was then able to more clearly convey the concepts”.

It might take more time for some students to go through the stages of bibliotherapy, although it is important to realise that everyone is unique and there is no schedule for the process. The PST reflected, “I know my anxiety about teaching maths has not disappeared”. The positive impact of the experience is shown by her motivation to continue with more readings and reflections, as she completes her course and begins teaching.

The final answers when the initial questions were repeated provided evidence that the PST had shown an emotional response to the readings, had reflected on her own experiences and had engaged in some stages of the bibliotherapy process. Her reflection on their own experiences was followed by a consideration of what it could mean for the future and the implications of her insights for her teaching. Her assessment of her increased confidence was authenticated for her by feedback from

the class, which corresponded to her feelings about the lessons. In lessons didn't go well, she felt the class "was struggling to understand what I was talking about". However, as the action research cycle progressed and she was able to demonstrate more confidence, "the majority of the class said they felt better about the maths when I felt better about teaching it".

The final answers and reflections demonstrate the potential of bibliotherapy to change the way PST feel, as she summed up her experience by saying: "I believe my self-esteem has risen dramatically".

### **Conclusions**

This research connecting bibliotherapy to cycles of action research is innovative as it brings together analysis of reflections of PST with a study of the beliefs, attitudes and insights that shape their mathematical identities. Negotiating this issue has the potential to transform learning and teaching beyond that of the PST to the future students. These results have implications for the way the bibliotherapy process could be incorporated into teacher education courses.

The juxtaposition of bibliotherapy with action research is potentially a powerful strategy in addressing mathematics anxiety in PST. Bibliotherapy, used as part of the process of action research, is able to address Ambrose's (2004) criteria for changing beliefs, as it can provide emotion-packed experiences, encourage PST to become immersed in a reflective community and connect beliefs and emotions and teacher practice.

Bibliotherapy allows PST to reconstruct their own experiences, and re-evaluate their identities as learners and teachers of mathematics, potentially affecting not only their current study but also their future teaching of mathematics and hence the attitudes of their future students. The special feature of the bibliotherapy approach of eliciting PST reflections stems from its ability to call forth cognitive responses paralleled by emotional responses. In comparison to other reflective practices, the potential of bibliotherapy lies in opportunity to change the way pre-service teachers feel.

Bibliotherapy, allied with action research, provides a new framework that has much to offer. It provides teacher educators with a shared language to talk about cognitive and emotional responses in terms of the processes of identification, catharsis, insight, universalisation and projection. Hence, it provides teacher educators and researchers with a framework and language for communicating research outcomes. Further research would also investigate the conditions under which the process would have the most impact.

Finally, the potential exists for teachers who have gained insights through this process and, an understanding of the process during their training, to use their experience to help their students address and overcome their mathematics anxiety.

## References

- Allinder, R.M. (1995). An examination of the relationship between teacher efficacy and curriculum-based measurement and student achievement. *Remedial and Special Education*, 16 (4), 247-254.
- Ambrose, R. (2004). Initiating change in prospective elementary school teachers' orientations to mathematics teaching by building on beliefs. *Journal of Mathematics Teacher Education*, 7, 91-119.
- Ashton, P. (1984). Teacher efficacy: A motivational paradigm for effective teacher education. *Journal of Teacher Education*, 35 (5), 28-32.
- Bandura, A. (1994). Self Efficacy. In V. S. Ramachaudran (Ed.), *Encyclopedia of human behavior*. Vol. 4. New York, NY: Academic Press.
- Baroody, A.J. & Costlick, R.T. (1998). *Fostering children's mathematical power. An investigative approach to K-8 mathematics instruction*. New Jersey: Lawrence Erlbaum Associates, Inc.
- Benz, C.R., Bradley, L., Alderman, M.K., & Flowers, M.A. (1992). Personal teaching efficacy: Developmental relationships in education. *Journal of Educational Research*, 85 (5), 274-285.
- Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting*. Buckingham: Open University Press.
- Furner, J. & Berman, B. (2005). Confidence in their ability to do mathematics: The need to eradicate math anxiety so our future students can successfully compete in a high- tech globally competitive world. *Dimensions in Mathematics*, 18(1), 28-29-31.
- Gates, P. & Jorgensen (Zevenbergen), R. (2009) Foregrounding social justice in mathematics teacher education. *Journal of Mathematics teacher Education* 12: 61-170
- Gee, J. (2001). Identity as an analytic lens for research in education. *Review of Research in Education*, 25, 99-125.
- Gibson, S. & Dembo, M. H. (1984). Teacher efficacy: A construct validation. *Journal of Educational Psychology*, 76(4), 569-582.
- Hannula, M., Evans, J., Philippou, G. & Zan, R. (2004). Affect in mathematics education – Exploring theoretical frameworks. *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education*, (Vol 1, pp. 107-136) Bergen, Norway: PME.
- Hart, L. E. (1989). Describing the affective domain: Saying what we mean. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp.37-45). New York: Springer
- Hebert, T. & Furner, J. (1997). Helping high ability students overcome maths anxiety through bibliotherapy. *Journal of Secondary Gifted Education*, 8(4), 164-179.
- Leder, G. (2007). Beliefs: What lies behind the mirror? *Montana Mathematics Enthusiast*, Monograph 3, pp.39-50 The Montana Council of Teachers of Mathematics

- Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal for Research in Mathematics Education*, 30(5), 520-540
- Madison, S.K. (1997). Preparing efficacious elementary teachers in science and mathematics: The influence of methods courses. *Journal of Science Teacher Education*, 8 (2), 107-126.
- Morawski, C (1997). A role for bibliotherapy in teacher education. *Reading Horizons*, 37(3), 243-260.
- Ricouer, P. (1994) *Oneself as other* Chicago, University of Chicago Press. Translated by Kathleen Blamey
- Somekh, B. (2005). *Action research: A methodology for change and development*. Open University Press: Milton Keynes
- Walshaw, M. ( 2004 ). Becoming knowledgeable in practice: The constitution of secondary teaching identity. In I. Putt, R. Faragher & M. McLean (Eds), *Mathematics education for the third millennium: Towards 2010*. Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, (pp. 557-563) Townsville, Australia
- Wigfield, A., & Meece, J. L. (1988). Math anxiety in elementary and secondary school students. *Journal of Educational Psychology*, 80, 210-216.
- Wilson, S. (2007). My struggle with maths may not have been a lonely one: Bibliotherapy in a teacher education number theory unit. In J. Watson & K. Beswick (Eds) *Mathematics: Essential research, essential practice*. Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, (pp. 815-823) Hobart, Australia
- Wilson, S. (2009). “Better you than me”: Mathematics anxiety and bibliotherapy in primary teacher professional learning. In R. Hunter, B. Bicknell, & T. Burgess (eds), *Crossing Divides*: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia (pp. 603-610). Palmerston North, NZ.
- Wilson, S. & Thornton, S. (2008). “The factor that makes us more effective teachers”: Two pre-service teachers’ experience of bibliotherapy. *Mathematics Teacher Education and Development* (pp. 22-35)



# EXPLORING MATHEMATICS TEACHERS' MOTIVATION TO CHANGE: A CASE STUDY

Zulfiye Zeybek

Indiana University

Enrique Galindo

Indiana University

*We investigated teachers' motivation to attend professional development sessions. In particular, this study explores why teachers engage in professional development as well as how personal and organizational factors influence teachers' professional development experiences. Six teachers from grades K-6 and their math coach were interviewed at three different schools in a rural Midwestern County in the US. In addition, field observations were conducted. Data were analyzed qualitatively through three major categories—motivators, learning activities, and work context. We conclude that professional development activities form complex inter-relationships with teacher motivation consisting of intrinsic and extrinsic motivators, a range of learning activities, and teacher work context. There is bidirectional interaction between each contributor and professional development.*

## PURPOSE OF THE STUDY AND RESEARCH QUESTIONS

The purpose of this study is to examine the relationship between teachers' motivations and their work context. By professional development, we broadly refer to learning activities that are intended to enhance teachers' professional growth, such as teacher networks, workshops, conferences, or in-service activities. Specifically, this study explores why and how teachers engage in professional development, as well as investigate what they seek to gain at the conclusion of their professional development training. The study also addresses how teachers' work contexts influence their professional development experiences. By work context, we refer to teachers' school environments and all aspects of teachers' work, including their recognized work such as classroom teaching, school and district obligations, or formal professional development, and also, their unpaid work such as after school activities. To understand the relationship between teacher development and their motivation to engage in professional development, we address the following research questions:

1. What are the factors that motivate teachers to engage in professional development?
2. How do teachers regulate their own professional learning?
3. According to teachers, how does the context of work influence teacher professional development?

## LITERATURE REVIEW

School improvement has been a central focus of decision-makers at the national, state, district, and school levels in the U.S. since the publication of *A Nation at Risk* (USDE, 1983). There is agreement that mathematics education in the U.S. needs

improvement. For example, in international studies American students usually exhibit lower academic achievement in mathematics than their counterparts in other nations of the world (Stigler & Hiebert, 1999). As education reform continues to gain significance and recognition, the process of teacher change has also received increased attention; given that successful educational reform demands an understanding of the process by which teachers change.

The current education reform requires teachers to master new skills, to take on new responsibilities, and to change their practices (Corcoran, 1995; Schifter & Fosnot, 1993; Ball, 1996). Teachers are being asked to implement a new kind of teaching, despite the fact that teachers have never experienced such teaching (Schifter & Fosnot, 1993; Ball, 1996; Llinares & Krainer, 2006). To meet these challenges, teachers not only need to deepen their knowledge of their subject matter, but they also need to implement this knowledge in their classrooms (Corcoran, 1995).

It is clear that teachers play a vital role in the successful implementation of varying and demanding reform efforts. Furthermore, teachers will not implement reform movements unless they understand and believe in them (Battista, 1994). Teachers' prior beliefs affect their learning to teach process and practices (Ball, 1996). However, as Thompson claimed, it is hard to change beliefs (as cited in Philip, 2007). Therefore, teachers need support to understand the importance of reform movements in fostering students' thinking and their understanding of mathematics, so that teachers can implement reforms successfully in their classrooms.

There is agreement in the research literature on the features of effective professional development sessions that prove successful in changing teachers' approaches. Effective professional development programs should include learning opportunities in which teachers can work collaboratively on a specific job-related problem, a problem that they identified with their colleagues. In such programs teachers concerns are understood and they are provided appropriate support and opportunity to implement new teaching practices (Collins, 1999). Other features of promising professional development sessions include providing opportunities for teachers to develop further expertise in subject content, teaching strategies, and uses of technology; providing active engagement and accessibility to every teacher and demonstrating respect for them as the professionals they are (Corcoran 1995; Hill 2004; Garet et al. 2001).

High quality professional development seems to be the central component of education reform; however, the majority of professional development programs have failed because they do not take into account two important issues: the factors that motivate teachers to engage in professional development sessions, and the way in which teacher change takes place (Guskey, 1986). Thus, exploring factors that both motivate and deter teachers from participating in continued education should be analyzed, the attention given to research on affectivity appears to have decreased in

recent years (Leder & Forgasz, 2006), and little attention has been paid to what motivates professionals to learn in their field (Scribner, 1998).

Motivation theories are built on a set of assumptions to explain why a behavior has occurred (Deci & Ryan, 1985). Motivations are based on the needs of individuals (Csikszentmihalyi, 1990), and people are motivated by five basic groups of human needs that emerge in a hierarchy of importance (Maslow, 1970). A national study revealed that the main reasons behind teachers' motivation to pursue continued learning are student learning, 77%; improving teaching skills, 5%; increasing their own knowledge, 34%; career advancement, 7%; financial reward, 5%; and maintaining professional certification, 5% (Renyi, 1996). Given that two main forms of motivation-intrinsic and extrinsic- induce people to invest energy in learning, this review suggests a need for studies that focus specifically on intrinsic and extrinsic motivational factors that affect teachers' willingness to attend professional development. Moreover, a critical analysis of teachers' work contexts and their reasons to attend professional development is necessary.

## **RESEARCH METHODOLOGY**

In this study we used using qualitative research methods to identify factors that influence teachers' engagement in professional development. The data comes from a Mathematics Partnership (MP), a collaborative endeavor among five school corporations in one rural Midwestern County, several mathematics and mathematics education professors from a local university, and a math coach, to improve mathematics achievement in grades K-6.

### **Sample**

Teachers, the primary unit of analysis, were selected based on two criteria: being a participant of the MP grant and teaching different grade levels. Being a participant of the MP grant was an important criterion due to the reason that those teachers had already committed to a two-year professional development. Teaching different grade levels, on the other hand, might provide different perspectives on teachers' motivations to attend professional development. The participants consisted of six elementary teachers from different grade levels and their math coach who was working with them on a weekly basis.

### **Data Collection and Analysis.**

We relied on interviews as the main source of data. However, observations were used to further our understanding of the topic of interest throughout the study. The one-time semi- structured interview was tape-recorded, included open ended interview questions, and took 30-45 minutes. Observations were conducted during monthly professional development sessions and planning meetings. In order to analyze the data, handwritten interview notes were typed, coded, and synthesized. The data analysis started with a reading of all the data sources (interviews and observation notes). Then answers were coded into different categories according to the three

research questions. All answers for each research question were read again to develop initial categories related to each of the broader themes (three research questions), namely: what motivates teachers to learn, how do they learn, and the ways their work context stimulates or inhibits their learning. Initial categories were developed within these three themes by grouping similar answers into a broader category. The next step was testing each category by going back and matching all the answers. If not all answers could be matched with the initial category, a new category was developed. Sometimes a quote matched several of the three themes. In this case, the quote was copied several times and put into the categories of each theme in which it fitted.

FINDINGS

We now discuss findings for each of the research questions.

Teacher Motivation.

Five intrinsic motivators and two extrinsic motivators were found among these teachers (See Table 1). The data suggest that the teachers in this study favored activities focusing on pedagogy over other intrinsic motivators. Pedagogical skills ranged from teaching specific content- math, to using different teaching techniques, to implementing standards in their classrooms. For various reasons, teachers often felt challenged when trying to devise multiple strategies to connect with their diverse students. As teachers’ comments suggest, pedagogical knowledge was attractive to teachers when the knowledge acquired was directly applicable to their classrooms.

Intrinsic Motivation	Extrinsic Motivation
<ul style="list-style-type: none"><li>• Pedagogical knowledge</li><li>• Content knowledge</li><li>• Classroom management</li><li>• Students academic achievement</li><li>• Personal Interest</li></ul>	<ul style="list-style-type: none"><li>• Licensure requirements</li><li>• Salary advancement</li></ul>

Table 1: Motivators to engage in professional development.

According to the interview data, another major catalyst that affected teacher participation in professional development was their interest in content. These teachers favored activities focusing on math content knowledge. Some teachers were motivated by the opportunity to deepen their math knowledge, while the others saw professional learning opportunities as a chance to extend their knowledge into other content knowledge. On the other hand, getting lots of content knowledge, in this case mathematics was mentioned as something that might discourage teachers and cause them to drop out of the professional development program.

Teachers, especially novice teachers, were most concerned about behavior issues. Another property of intrinsic motivators expressed by teachers is the capacity of professional development to address areas of interest to teachers in a larger context beyond work. Student academic achievement and personal interest appeared to be a

motivator in that some teachers found themselves more motivated to participate in activities they were interested in and dealt with material that they were passionate about. In addition to intrinsic motivators, two major extrinsic motivators influenced teachers' engagement in professional development: licensure renewal, and salary advancement.

### Professional Learning

One of the most frequently cited ways of learning was teacher interaction. Teachers reported that they learn in (mostly) informal, unplanned interaction with a colleague. By collaborating with other teachers, especially with teachers in their schools and grade level band, these teachers found a level of credibility in responses to their questions not always found in other forms of professional learning. Teacher learning activities fell into three subcategories: Focus, learning context, and applicability (See Table 2). Focus refers to the purpose of teacher learning, and is therefore closely related to intrinsic motivators. The predominant focus characteristics mentioned by teachers are: (1) content, (2) pedagogy, and (3) classroom management. Learning Context describes the ways teachers learn and is organized according to following attributes: (1) group, (2) individual, (3) organizational, and (4) formal or informal. Finally, Applicability or usefulness depends upon teachers' perspectives according to three characteristics: (1) Content relevance, (2) Method relevance, and (3) Classroom management relevance.

---

#### Learning Activities:

Interactions [interaction with colleagues, students, and in meetings]

Readings, University courses, and Internet

---

Focus	Learning Context	Applicability
<ul style="list-style-type: none"> <li>• Content</li> <li>• Pedagogy</li> <li>• Classroom Management</li> </ul>	<ul style="list-style-type: none"> <li>• Group/Individual</li> <li>• Formal/Informal</li> </ul>	<ul style="list-style-type: none"> <li>• Content Relevance</li> <li>• Method Relevance</li> <li>• Classroom management relevance</li> </ul>

---

Table 2: Teachers ways of learning.

The most common point of focus mentioned by teachers was content. Teachers explained that the focus of their learning was either to develop or to broaden out their content knowledge. Teachers valued collaboration with colleagues as a learning activity to answer questions of practice, especially issues of pedagogy and classroom management. In addition, these teachers also described teacher collaboration as a motivator that encouraged them to engage in continuing education. Learning Context refers to how teachers learn. The most preferred way of learning mentioned by teachers was group learning or cooperative learning, which refers to working together to accomplish shared goals. Within group learning, teachers sought outcomes that were beneficial to themselves and all other group members. All of the participant

teachers said that they would go to their colleagues as their first choice when they faced problems in their classrooms. Applicability or usefulness is something that all of the participant teachers mentioned as important during their interviews. Teachers’ descriptions of high quality professional development programs suggest that they highly value activities that are applicable to their practices.

**Teacher Work Context**

Two subcategories emerged within Work Context (See Table 3). Teachers in this study perceived district policies as playing an influential role in their learning experiences. Most participants cited license renewal as a reason for attending professional development programs. However, some of the teachers mentioned that incentives for continuing education were uneven across career stages. Another district policy reported was AYP: Adequate Yearly Progress (AYP), a tool that is required by No Child Left Behind (NCLB) to determine which school districts and schools are making adequate academic progress.

District/State Level	School Level
<ul style="list-style-type: none"><li>• Licensure requirements</li><li>• District professional development requirement</li></ul>	<ul style="list-style-type: none"><li>• Leadership</li><li>• Structure: (1) time (2) funding</li><li>• Competing demands</li></ul>

Table 3: Effects of work context.

**CONCLUSION**

Renyi’s (1996) broad survey cited time, teachers’ role in developing professional development, community organizations, and funding as issues that impact the quality of professional development. Results from this study align with such findings. However, this study also reveals that the context of work has an influence on professional development. By documenting the influence of teachers’ work context on their learning, this study adds to Cochran-Smith and Lytle three conceptions of how teachers learn (1999) and provides additional evidence to findings from a related study (Llinares & Krainer, 2006). Teachers in our study perceived professional development as a phenomenon with broad application that is situated at the intersection of three contributors: teachers’ motivation to learn, their ways of learning, and their work context. Also, there is bidirectional interaction between each contributor and professional development. For instance, despite the fact that teachers engage in professional development activities in response to intrinsic and extrinsic motivators, effective professional development can stimulate their motivation to learn. Furthermore, the phenomenon of professional development is embedded in teacher work context, which influences teachers’ behavior and choices regarding activities by privileging some over others. Conversely, professional development can change teachers’ work context in schools. Therefore, it is not enough to mention the effects of professional development activities on teacher learning, motivation, and

work context. We should also recognize the effects of each contributor to professional development and the work context in order to provide effective programs for teachers. Figure 1 below shows how major categories related to the phenomenon of professional development are interrelated.

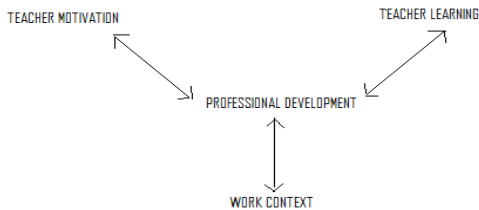


Figure 1: Professional development schematic.

### Implications.

This study underscored the complexity of teacher professional development and showed that district, state, and school level policies affect it and are closely linked to teacher practice. Therefore, professional development at the school level should certainly be examined and evaluated in order to implement reforms successfully. Furthermore, this study yields several implications for schools, districts and states.

First, if professional development programs are to improve teacher practice and thereby student achievement, districts and schools need to critically evaluate their own policies to help schools reshape their cultures into lifelong learning organizations. Second, state, district, and school policy makers need to ensure that a rich variety of learning activities are incorporated into teachers' professional development activities so that they help teachers meet the challenges they face. Third, school administrators need to create and support an environment in which teachers can meaningfully explore individual practices and organizational objectives. Also, principals should work individually with teachers to develop plans that address the unique needs of teachers based on past experiences and current classroom.

### REFERENCES

- Ball, D.L. (1996). Teacher learning and mathematics reforms: What we think we know and what we need to learn. *Phi Delta Kappan*, 77 (7), 500-517.
- Battista, M. (1994). Teacher beliefs and the reform movement in mathematics education. *Phi Delta Kappan*, 75(6), 462-470.
- Borko, H. (2004). Professional development and teacher learning: Mapping the Terrain. *American Educational Research Association*, 33, 3-15.
- Cochran-Smith, M., & Lyte, S. L. (1999). Relationships of knowledge and practice: Teacher learning in communities. *Review of Research in Education*, 24, 249-305.
- Collins, D. (1999). *Achieving your vision of professional development: How to assess your needs and get what you want. Hot topic*. Washington, DC: Office of Educational Research and Improvement. Retrieved from the ERIC Database. (ED446060)

- Corcoran, T. C. (1995). *Transforming professional development for teachers: A guide for state policy makers*. Washington, DC: National Governor's Association.
- Corcoran, T. B. (1995). *Helping teachers teach well: Transforming professional development*. CPRE policy briefs. Washington, DC: Office of Educational Research and Improvement. Retrieved from the ERIC Database. (ED388619)
- Csikszentmihalyi, M. (1990). Litaracy and intrinsic motivation. *The MIT Press*, 119, 115-140.
- Deci, E. L., & Ryan, R. M. (1985). *Intrinsic motivation and self-determination in human behavior*. New York, NY: Plenum Press.
- Garet, M. S., Porter, A. C., Desiomone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Education Research Journal*, 38, 915-945.
- Gutskey, T. R. (1986). Staff development and the process of teacher change. *American Educational Research Association*, 5, 5-12.
- Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Reseracher*, 31, 3-15.
- Hill, C.H. (2004). Professional development standards and practices in elementary school mathematics. *The University of Chicago Press*, 104, 215-231.
- Leder, G. C. & Forgasz, H. J. (2006). Affect and mathematics education. In A. Gutierrez, P. Boero (Eds), *Handbook of Research on the Psychology of Mathematics Education*, 403-427. The Netherlands: Sense Publishers
- Llinares, S. & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutierrez, P. Boero (Eds), *Handbook of Research on the Psychology of Mathematics Education*, 429-459. The Netherlands: Sense Publishers
- Maslow, A. H. (1954). *Motivation and personality*. New York, NY: Harper & Row, Publishers, Inc.
- Renyi, J. (1996). *Teachers take charge of their learning: Transforming professional development for student success*. Washington, DC: National Foundation for the Improvement of Education. Retrieved from the ERIC database. (ED401251).
- Philipp, R. (2007). Mathematics teachers' beliefs and affect. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, 257-318. Charlotte, NC: Information Age Publishing.
- Shifter, D., & Fosnot, C. T. (1993). *Reconstructing mathematics education. Stories of teachers meeting the challenge of reform*. New York and London: Teachers College Press.
- Stigler J.W., & Hiebert J., (1999). *The teaching gap*. New York: The Free Press.
- United States Department of Education. (1983). *A nation at risk*. Washington, DC: Author.



**Author Index  
Research Reports  
Volume 4**





**A**

Andra', Chiara	4-113
Assimakis, Nicholas	4-273

**B**

Barchfeld, Petra	4-33
------------------	------

**C**

Carrillo, José	4-41
Chumachenko, Dmitry	4-169
Cooper, Tom J.	4-329

**D**

David, Maria Manuela	4-257
Delice, Ali	4-153

**G**

Galindo, Enrique	4-361
Garcia, Mercedes	4-105
González-Martín, Alejandro	S 4-137
Grootenboer, Peter	4-225

**H**

Heinze, Aiso	4-9
Hofmann, Riikka	4-81

**J**

Jared, Libby	4-73
Jorgensen, Robyn	4-225

**K**

Kaufman, Julia	4-217
Kaur, Harpreet	4-193

**L**

Lee, Yong Hah	4-57
Lindmeier, Anke	4-33

**M**

Mercer, Neil	4-81
Miller, Jodie	4-329

**O**

Oliveira, Hélia Margarida	4-1
---------------------------	-----

**P**

Ponte, João Pedro	4-185
Presmeg, Norma	4-289

**Q**

Quintas, Sandra Maria	4-1
-----------------------	-----

**R**

Rach, Stefanie	4-9
Radford, Luis	4-17
Rangnes, Toril Eskeland	4-25
Reiss, Kristina M.	4-33
Ribeiro, C Miguel	4-41
Rivera, Ferdinand	4-49
Roh, Kyeong Hah	4-57
Rott, Benjamin	4-65
Rowland, Tim	4-73
Ruthven, Kenneth	4-81

**S**

Sack, Jacqueline	4-89
Saldanha, Luis A.	4-97
Sanchez, Victoria	4-105
Santi, Giorgio	4-113
Schubring, Gert	4-121
Seah, Wee Tiong	4-129
Seffah, Rachid	4-137
Selden, Annie	4-145
Selden, John	4-145
Sevimli, Eyup	4-153
Shin, Bomi	4-161
Shvarts, Anna	4-169
Silfverberg, Harry Untamo	4-177
Silvestre, Ana Isabel	4-185
Sinclair, Nathalie	4-193
Sodian, Beate	4-33
Spinillo, Alina Galvao	4-201
Spyrou, Panagiotis	4-209
Stein, Mary Kay	4-217
Sullivan, Peter	4-225
Swidan, Osama	4-233

## **T**

Tall, David O.	4-297
Tekkumru-Kisa, Miray	4-217
Thomas, Michael O. J	4-241
Thwaites, Anne	4-73
Toluk Ucar, Zulbiye	4-249
Tomás Ferreira, Rosa Antónia	4-1
Tomaz, Vanessa Sena	4-257
Triantafillou, Chrissavgi	4-265, 273
Turner, Fay Alison	4-281

## **U**

Ufer, Stefan	4-33
--------------	------

## **V**

Van Harpen, Xianwei Yuan	4-289
Vazquez, Irma	4-89
Verhoef, Nellie C.	4-297
Viholainen, Antti	4-305
Viirman, Olov	4-313
Vollstedt, Maike	4-321
Vourvoulakis, John	4-273

## **W**

Warren, Elizabeth Anne	4-329
Wille, Annika	4-337
Williams, Gaye	4-345
Wilson, Sue	4-353

## **Y**

Yavuz, Birseli	4-249
Yoon, Caroline	4-241

## **Z**

Zagorianakos, Andonis	4-209
Zeybek, Zulfiye	4-361