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Research Reports

Hijkl-mnoP



TEACHER'S EXPERIENCES OF A GEOGEBRA ENRICHED INQUIRY MATHEMATICS TEACHING UNIT

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According to previous studies, inquiry mathematics enhances learning. However, teachers need support in implementing this kind of teaching. In this study, a high school teacher was given a short preplanned inquiry mathematics teaching unit which included activities with GeoGebra. The lessons were videotaped and the teacher was interviewed after every lesson. The interviews were analyzed to find out teacher's experiences of the teaching unit. It was found that the teacher felt that her ordinary teaching differed from the teaching unit. However, the teacher would like to develop her teaching toward inquiry mathematics. She felt that the teaching unit illustrated inquiry mathematics. The teacher also described several challenges in guiding the students.

INTRODUCTION

There is consensus that effective teaching methods are student centered, emphasize interaction and challenge students to investigate mathematical phenomenon in non-routine tasks. These kind of teaching approaches include, for example, inquiry mathematics (Cobb, Wood, Yackel, & McNeal, 1992), inquiry approach (Borasi, Fonzi, Smith, & Rose, 1999), open approach (Nohda, 2000) and problem-centered learning (Wood & Sellers, 1997). In this study, inquiry mathematics is used as an umbrella concept for these and other similar approaches. According to previous research, inquiry mathematics enhances students understanding and mathematical thinking (Fennema & al., 1996; Wood & Sellers, 1997) as well as creativity and problem solving skills (Kwon, Park & Park, 2006). This kind of learning is durable and the knowledge can be applied to new context (Francisco & Maher, 2005). Inquiry mathematics also develops positive attitudes and beliefs in students (Wood & Sellers, 1997). According to Sullivan, Mousley and Zevenberger (2006), inquiry mathematics enhances involvement of all kinds of students. Despite of the research based evidence of the benefits of inquiry mathematics, it is not applied very often in schools. Teachers should be supported more efficiently in implementing inquiry mathematics. One type of support is giving teachers preplanned teaching units with instructions.

In this case study, I study how a teacher experiences a preplanned teaching unit in relation to her ordinary teaching. Research questions are: How does the teacher experience the teaching unit differing from her ordinary teaching? What are teacher's attitudes toward the teaching unit, inquiry mathematics in general and GeoGebra? What challenges does the teacher experience in implementing the teaching unit?

Technology enriched inquiry mathematics

According to Stein, Engle, Smith and Hughes (2008), a typical inquiry mathematics lesson consists of three phases. In the launch phase, the teacher introduces the problems without revealing solution methods. During the exploration phase, students work in small groups as the teacher guides them. In the discussion and summary phase, students' solutions are discussed and the teacher summarizes the findings. This kind of lesson structure is common in Japan (Shimizu, 1999). However, in Finland, the typical lesson structure is Review-Lesson-Practice (Savola, 2008).

Dynamic geometry software support inquiry mathematics because students can, for example, drag a point in a figure and investigate how it affects the properties of the figure (e.g., Arzarello, Olivero, Paola & Robutti, 2002). In this study, the teaching unit utilized GeoGebra, which includes some aspects of computer algebra system in addition to dynamic geometry. I chose to use GeoGebra also because it is free and easy to use. Teachers can prepare GeoGebra applets beforehand so that students have to learn to use only few tools. Students can investigate, for example, how changing the parameters of a function affects the graph of the function. Thus, GeoGebra supports students in making connections between representations which is one of the main aims of learning (Hähkiöniemi, 2006). With GeoGebra students can try out multiple solution methods as well as to make conjectures and to test them.

Teacher change

According to Lloyd's (2002) synthesis, teachers often have difficulties to change their teaching toward inquiry mathematics. Teachers need support in applying inquiry mathematics. According to Lloyd (2002), offering innovative curriculum materials for teachers is one way to get them reflect on their mathematical views and change their teaching. Also Wood and Sellers' (1997) and Borasi's et al. (1999) studies have given evidence of the benefits of offering curriculum materials for teachers. Unlike in this study, the aforementioned research studied teacher change over a long period of time and teachers were also given other support such as workshops.

Based on previous studies, Kaasila, Hannula, Laine and Pehkonen (2008) constructed a model of the phases of teacher change: (1) problematising current beliefs and practices; (2) becoming aware of a new approach; (3) exploring and testing alternative beliefs and practices; (4) reflectively analyzing benefits; (5) changing one's views of mathematics and one's teaching.

Herbel-Eisenmann, Lubienski and Id-Deen (2006) emphasize that it is important to distinguish between two types of teacher change. According to them, global change means shifts in teachers' over-arching orientation toward teaching. Local change means shifts which depend on contextual conditions such as pressure and curriculum materials (ibid.). Herbel-Eisenmann et al. (2006) found that depending on the contextual conditions a teacher used inquiry mathematics in one course and traditional teaching methods in another course at the same time.

METHODS

The teaching unit of approximation of the area under a curve

The teaching unit consisted of three 75 minute lessons in grade 11 on approximating the area under a curve using rectangles, determining upper and lower sums, constructing the midpoint rule and investigating the limiting process inherent in the definite integral. The unit included all the materials and advices for the teacher.

In the first lesson, the students were asked to approximate the area under a given function using Rectangle-tool (<http://users.jyu.fi/~mahahkio/PME35>). Rectangle-tool shows the rectangle and the area when you choose the base by selecting two points from the x -axis and one point determining the height. I designed Rectangle-tool to guide students toward standard approximation methods so that they still have the freedom to develop their own method. After this, the students were asked to determine upper and lower sums so that using Rectangle-tool became more and more challenging. In the second lesson, the students were shown how to use the upper and lower sum commands of GeoGebra and they were asked to approximate a particular area so that they know the area to whole number accuracy for certain. In the third lesson, the students were asked, for example, to construct a formula for the midpoint rule and to use a GeoGebra-applet (<http://users.jyu.fi/~mahahkio/PME35>) to investigate the meaning of the formula $\lim_{n \rightarrow \infty} (f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x)$.

Data collection and analysis

The teaching unit was implemented by an experienced teacher who had taught the topic several times. The lessons were videotaped with two video cameras and screen capture software. I interviewed the teacher after each lesson. Also the interviews were videotaped. The first and second interviews lasted 20 and 24 minutes. The third interview was 83 minutes long. The format was semi-structured theme interview (Kvale, 1996). The themes which were discussed included the structure of the lesson, differences to ordinary lessons, benefits and challenges, how the lesson should be developed, and the students' work.

I began the analysis by writing a description of the lessons on the basis of the lesson videos. From the description I got an overall view of the lessons and was able to locate the points to which the teacher referred to in the interviews. The analysis of the interviews started as I watched the interviews and transcribed them. I read the transcripts several times. After this I did open coding based on how the teacher described the differences between her ordinary teaching and the teaching unit, what was her attitude toward the teaching unit, GeoGebra and inquiry mathematics, and what pros and cons she related to the teaching unit, GeoGebra and inquiry mathematics. Properties of each code were scrutinized by collecting together all the interview moments which were coded by the same code. This resulted the set of subcategories presented in Table 1. By studying the properties of and connections between the subcategories I grouped them to main categories (see Table 1).

RESULTS

Shortly described the structure of the lessons was following. The teacher introduced the activities shortly. She did not give examples or tell how the tasks are solved. Most of the lessons the students solved tasks in two or three person groups. The teacher circulated in the class and guided the students. At the end of the lessons, she led short whole class discussion about the solution methods. Table 1 presents the overview of the analysis of the interviews.

Main categories	Subcategories
Changes in the roles of the teacher and the students	Ordinary lessons are more teacher-centered
	In ordinary lessons teacher runs investigations
	In ordinary lessons reasoning happens in group under teachers control
	In the teaching unit students were more committed
Changes in the depth of the knowledge	The teaching unit enhances remembering
	The teaching unit enhances understanding
	Repetition was compensated by understanding in the teaching unit
	In ordinary teaching, learning is rarely checked before a test
Attitude toward the teaching unit, GeoGebra and inquiry mathematics	The teacher hopes that her teaching changes
	The teacher felt that the teaching unit illustrated inquiry mathematics
	The teacher liked that every student participated
	The teacher feels that GeoGebra is good aid in inquiry mathematics
	The teacher's attitude toward using inquiry mathematics is positive
Constraints for implementing inquiry mathematics	Inquiry mathematics takes time
	Courses have too much content
	National exams give pressure
	Developing teaching materials demands lot of work
	A small group size is needed in inquiry mathematics
	Going to computer lab causes trouble
Challenges in guiding the students	Advices are need for implementation in practice
	Guiding without giving too strong advices
	Activating interaction between students
	Understanding students' ideas
	Guiding students when they do not ask for advice
	Guiding students toward more mathematical solution methods
	Activating students to explain and justify
	Student's move too fast to the next task
	Students need advice even in basic calculation procedures
	Sometimes tasks has to be clarified
	Some students are faster, some slower

Table 1. Overview of the analysis of the interviews of the teacher.

In the following, the results are presented according to the main categories. Usually there were several interview moments related to a subcategory, but here only single quotes are presented to illustrate some of the subcategories.

Changes in the roles of the teacher and the students

The teacher explained that her ordinary lessons are teacher-centered and that she has not used inquiry mathematics. Although something is investigated in her lesson, it has been done in a teacher-led way:

“There might be that kind of approach that we investigate and wonder together, but that’s not really. I would not say that as inquiry learning, if I am the one who is running it.”

Thus, the teacher feels that essential part of inquiry mathematics is that students themselves are doing inquiry and reasoning. She emphasized that in the teaching unit the students were more committed and that all students participated reasoning:

“We can think also as a group. But when we are thinking as a group, it can easily happen that if you are tired or something else, then you can let yourself to think of other things so that you don’t have to participate. Here everyone has to think.”

Changes in the depth of the knowledge

The teacher told that in the teaching unit the students will remember what they learnt better than in ordinary lessons. She also felt that the teaching unit enhanced students’ understanding. For example, she explained that

“I felt that they really understood what it was about. It wasn’t that the teacher explained and well maybe I got it, but you could notice that they really understood.”

The teacher also told that in the teaching unit there was not as much repetition as in ordinary teaching. However, she felt that understanding compensates the lack of repetition:

“It becomes so familiar, that it is already understood. Thus, you don’t anymore need to repeat it with those tasks so much.”

As one difference to her ordinary lessons, the teacher told that in the ordinary teaching learning is hardly traced before the final exam:

“In traditional teaching, there’s lot of talk, but nobody is checking what part they got. Finally in the exam, it is checked whether you got it or not. And of course, whether you are able to do the homework, there you check a little.”

The teacher’s attitude toward the teaching unit, GeoGebra and inquiry mathematics

Although the teacher had not previously taught inquiry mathematics, her attitude toward the teaching unit and inquiry mathematics was very positive. She also told that she wishes that her teaching develops toward inquiry mathematics:

“This has been quite mind broadening to me. This has been quite interesting. Let’s see if some approach would change slightly toward something, then it would be quite nice.”

The teacher felt that the teaching unit illustrates the ideas of inquiry mathematics. She even told that she had previously been in inquiry learning course and that she had thought that this kind of teaching does not fit to mathematics:

"It lowered the gap very much and somehow I am starting to see in new ways these things. For example, how GeoGebra can be used for this. I remember that I have been in a course about inquiry learning. And I think that, it was some time ago, I had a kind of attitude that this does not fit at all to mathematics."

The teacher also expressed that she liked that in the teaching unit all the students participated in thinking. She also said that she would like to develop toward that:

"I think that it would be really wonderful to be able to go this direction. So that I could get the students themselves to participate more, somehow more personally."

Constraints for implementing inquiry mathematics

The teacher told about several factors which do not depend directly on teacher but which constrain possibilities to implement inquiry mathematics. According to her, inquiry mathematics takes more time than ordinary teaching, courses have too much content and matriculation examination gives teachers pressure to cover all the contents. Teacher commented the pressure given by the national matriculation examination at the end of the high school as follows:

"When there's the courses, and you think that they include all that stuff and then there's the matriculation waiting, and then you know that the students have to succeed there, then it easily happens that you only run through the content and sometimes you forget even yourself what it is that is the most important. So that you only try to teach all that."

Challenges in guiding the students

The teacher also specified several challenges which she faced in guiding the students during the exploration phase. For example, she thought that it was difficult to guide students when they did not ask for advice:

"After all, they didn't ask anything. I say that it was difficult, that how do you go there to guide. Particularly when you had warned, that I should not give too much advice. I felt it difficult to think how to guide. [...] It was not like a student would have asked from me. Instead, they did themselves, I guess. Then how do I get to guide them appropriately, it felt difficult. [...] And then the thing, that I went in the group and interrupted [...] It was not that a student would have asked for advice. I felt that difficult."

Sometimes students may solve tasks too superficially and the teacher should guide them toward more mathematical solution. The teacher felt this challenging:

"They had upper sum on this side and lower sum on this side, then I tried to force [stops the sentence]. They were satisfied with that. And I was not. Then how do I get them, so that they would start to investigate it so that they could use lower sum and upper sum. They were happy with rough estimate with eye, so that this must compensate this and like that. They were satisfied with quite vague."

In the above situation, the teacher wanted the students to develop method which does not include rough estimate with eye. This is essential for the aim of the teaching unit. However, it is challenging to guide and motivate the students to investigate this without only demanding different kind of solution method.

DISCUSSION

The preplanned teaching unit gave the teacher an opportunity to try GeoGebra enriched inquiry mathematics. The teaching unit gave her a chance to compare inquiry mathematics to her ordinary teaching. The teacher reflected several properties of inquiry mathematics and how they differed from her ordinary teaching. This kind of analysis of a new teaching approach is an important phase in teacher's change (Kaasila & al., 2008). This study does not allow claiming that the teacher's teaching changed also after the teaching unit. However, we can say that at least the preplanned teaching unit can be used to raise teachers' awareness of different teaching methods as well as their benefits and challenges. The teacher's attitude toward the teaching unit, GeoGebra and inquiry mathematics was very positive and she hoped that she could develop her teaching toward inquiry mathematics. Thus, this kind of local change could at least be a starting point for global change although a local change does not necessarily lead to global change (Herbel-Eisenmann & al., 2006).

Also studies by Borasi et al. (1999) and Wood and Sellers (1997) support the claim that preplanned teaching materials can help teachers to develop toward inquiry mathematics. The difference between the mentioned studies and this research is that, in this study, the teacher was not offered other in-service training and the teaching unit was very short. Thus, also very short preplanned teaching unit without other in-service training can support teachers to develop toward inquiry mathematics.

The teacher felt that the teaching unit concretized the ideas of inquiry mathematics. She even mentioned that courses where general ideas of inquiry learning are discussed do not necessarily help to see how the ideas can be applied in mathematics teaching in practice. Thus, preplanned teaching units may be used to help teachers to understand how the ideas of inquiry mathematics work in practice.

The teacher described several factors which constrain possibilities to apply inquiry mathematics. These can prevent a teacher from implementing inquiry mathematics even though he/she is interested and willing to try it as the teacher in this study. These constraints should be solved so that teachers could focus on teaching.

In this research, I found challenges that the teacher felt in guiding the students. These challenges are part of inquiry mathematics and presumably other teachers face similar challenges. Therefore, it is important that we are aware of these challenges and develop support for teachers to face them.

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THE STRUCTURE OF MATHEMATICS RELATED BELIEFS, ATTITUDES AND MOTIVATION AMONG FINNISH GRADE 4 AND GRADE 8 STUDENTS

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This study reports results from a Finnish survey for grade 4 and grade 8 students ($N = 927$). The survey measured students' mathematics related achievement goal orientation, attitude and beliefs. The results confirm the reliability of used instruments and they indicate that the structure of grade 8 student's view of mathematics is more coherent than on grade 4. Students' mastery goal orientation is primarily correlated to students' positive attitude and self-efficacy. Moreover, students' performance orientation is positively correlated with self-efficacy and this relationship is stronger among older students. On grade 8, performance orientation has a positive correlation also with attitude.

THE STATE AND TRAIT OF MATHEMATICS RELATED AFFECT

In this study, we explore the structure of mathematics related affect, especially the relationships between beliefs, attitudes and motivation of Finnish students. Moreover, we explore possible differences between grade 4 and grade 8 students' structures.

The affect towards mathematics has two temporal aspects. Firstly, there is the rapidly changing affective state. For example, in problem solving these emotions and beliefs may influence the critical choices that determine whether the problem will be solved or not. These affects are situational and contextual (Hannula, 2007). However, there is also a stabile pattern in how an individual feels and thinks in these different contexts and situations, i.e. an affective trait. For example, in any classroom situation high achievers are likely to have more positive expectations and affect than low achievers. This distinction has been reflected in different theoretical frameworks. In McLeod's (1992) conceptualisation, beliefs were considered more stable than attitudes, which were more stable than emotions. Also in Goldin's (2002) framework there is a distinction between local and global (more stable) affect.

We define attitudes as the trait aspect of emotions (Hannula, 2002). This is a narrow definition and we are aware that some researchers use this concept to cover also a wide range of beliefs and motivations. Unlike McLeod (1992) we see that beliefs have both a state and a trait aspect. While a student may have a belief trait that he is not very good with mathematical tasks, his belief state regarding a specific task evolves as he reads the task and begins to solve it. Both affective state and trait have three main categories: motivation, beliefs and emotion/attitude, and they all have several dimensions (Hannula, 2007). In this article, our focus is on affective traits and

more specifically, the relationships between the different types of affective traits: attitude, beliefs and motivation.

MATHEMATICS-RELATED BELIEFS AND ATTITUDES

McLeod (1992) identified three categories of mathematics-related beliefs: beliefs about mathematics education, beliefs about the social context and beliefs about oneself as a learner of mathematics. We focus on beliefs about self and mathematics.

The research project “Elementary teachers’ mathematics” developed an instrument to measure students’ “view of mathematics” (Hannula, Kaasila, Laine & Pehkonen, 2006; Rösken, Pehkonen, Hannula, Kaasila and Laine, submitted). This questionnaire instrument had primarily a focus on the systemic character of beliefs and they were interested in dimensions describing such a view of mathematics. The study (Rösken *et al.*, submitted) led to eight scales describing these students’ view of mathematics, and we will use in this study five of those dimensions. *Difficulty of mathematics* is a belief about mathematics, *ability* and *success* are beliefs about self. *Enjoyment of mathematics* is in our framework considered as attitude, a trait aspect of emotions. The last dimension, *effort* we see as a motivational trait.

Among Finnish teacher education students, a core view of mathematics was found to consist of three aspects. The first aspect focused on beliefs about own *ability*, the second aspect on beliefs about the *difficulty of mathematics*, and the third aspect on the person’s *enjoyment of mathematics*. These three dimensions had strong correlations with each other (Pearson correlations .71 - .78). The dimensions of the core had also strong connections with personal expectations for *success* (.50 - .68) and moderate correlations with *effort* (.32 - .46). The correlation between *success* and *effort* was non-significant. Although the correlations between some of the dimensions were high, there seemed to be important differences between them. This was reflected, for example, in clear gender differences in *ability* and *success* while there were no statistically significant gender differences in *enjoyment of mathematics* or *difficulty of mathematics*. (Hannula *et al.*, 2006)

Using the same instrument with Finnish Grade 11 students, Rösken *et al.* (submitted), found out that beliefs about *ability*, *difficulty of mathematics* and *success*, were closely related to each other (Pearson correlations .63 - .79). A strong connection was also found between these “core dimensions” and *enjoyment of mathematics* (.54 - .67), and weaker correlations with *effort* (.23 - .39). The correlation between *enjoyment of mathematics* and *effort* was moderate (.44).

MOTIVATIONAL ORIENTATIONS

Motivation research has several theoretical approaches and use of terminology is sometimes confusing (Murphy & Alexander, 2000). In this research we conceptualize motivation through achievement goal orientation, which is one of the main strands of motivation research (Pintrich & Schunk, 2002). Achievement goal theory is a sociocognitive theory, which focuses on student’s self set goals in achievement

situation and is interested in the student's reasons to engage with learning task (Middleton, Kaplan, & Midgley, 2004). Different terminologies are used for two main goal orientations: A student who focuses on learning the task, is said to have a mastery, task or learning orientation, while students whose primary interest is to impress others with their performance are said to have a performance, ego or ability orientation, but there are only nuanced differences between the different terminologies (Ames 1992; Järvelä, 1996; Pintrich & Schunk, 2002). In this research we will use terms mastery and performance goal orientation consistently.

Different researchers have found rather comparable positive relationships between *mastery goal orientation* and achievement (Friedel, Cortina, Turner, & Midgley 2007; Midgley *et al.*, 1998). Results concerning performance goal orientation and achievement have been less consistent. Some have identified negative learning behaviour, while other results indicate performance orientation to lead to positive learning behaviour and achievement (Freeman, 2004, 67; Midgley *et al.*, 1998.) This has led to a need to differentiate between *performance-approach* and *performance-avoidance goal orientations* (Elliot & Harackiewicz, 1996). More recent results have indicated that students may have several goal orientations influencing simultaneously, and the emphasis of each orientation is influenced by the situation (e.g. Vollmeyer & Rheinberg, 2000).

STRUCTURE OF STUDENTS' VIEW OF MATHEMATICS

Although it is generally assumed that there is a relationship between mathematics related motivation and beliefs, the theories of their relationships are fairly recent (Op 't Eynde, De Corte, & Verschaffel, 2006; Hannula, 2006). Research has identified a positive relation between mastery orientation and attitudes, effort, competence beliefs (Seo, 2000) and positive emotions (Kumar, Gheen, & Kaplan, 2002; Midgley *et al.*, 1998; Pekrun, Elliot, & Maier, 2006). On the other hand, the mathematics beliefs are seen to form an overall pattern, where positive beliefs are related to each other (Hannula *et al.*, 2006; Rösken *et al.*, submitted), and therefore we can assume a mastery oriented student to have high self efficacy in mathematics and to enjoy it.

Kaldo and Hannula (submitted) have tested empirically the connections between two main motivational orientations and a variety of mathematics related beliefs, including *effort* and competence. In their study of 970 Estonian university students they found *mastery goal orientation* and *performance approach motivation* to be only weakly correlated to each other (.228), while *mastery goal orientation* had moderate correlations with student competence (.489), and *effort* (.330).

THE SAMPLE AND THE SURVEY INSTRUMENTS

The data was originally collected for a study exploring the influence of an ICT learning environment on students' beliefs and motivation. The research was conducted among Finnish students in three municipalities with different access to a specific ICT-environment. Altogether 927 students (505 on grade 4 and 422 on grade 8) responded under teacher instruction through the web during spring 2009. (Laakso

& Hannula, 2010). As the influence of the ICT-environment was found to be weak, we assume it insignificant regarding affective structures.

Items for the survey instrument were selected from instruments that had been tested and verified reliable in earlier studies. In order to control the length of the instrument, no more than 5 items were selected for any scale. Items for the three achievement goal orientations and *avoiding novelty* were chosen from the instrument developed for The Patterns of Adaptive Learning Study (PALS) (Anderman & Midgley, 2002).

Mathematics related beliefs were measured using items from the view of mathematics indicator, which was developed in 2003 as part of the research project “Elementary teachers’ mathematics” (Hannula *et al.*, 2006). Most items relating to the beliefs about *ability* and *success* originate from the confidence subscale of the Fennema-Sherman mathematics attitude scales (Fennema & Sherman, 1976). From the view of mathematics indicator, we selected also items regarding students’ attitude (*enjoyment of mathematics*) and motivation (*effort*).

We constructed sum variables that correspond to previous studies (Table 1). The reliability of the scales was high (Table 2). The analysis was based on Pearson correlations between these sum variable on grades 4 and 8.

Sum variable	Number of items	Sample item
Mastery goal orientation	5	One of my goals in class is to learn as much as I can
Performance-approach goal orientation	5	One of my goals is to show others that I’m good at my class work
Performance-avoidance goal orientation	4	It’s important to me that I don’t look stupid in class.
Avoiding novelty	5	I don’t like to learn a lot of new concepts in class.
Success	4	I know I can do well in math.
Ability	4	I am no good in math.
Difficulty of mathematics	3	Mathematics is difficult.
Enjoyment of mathematics	5	Doing exercises has been pleasant.
Effort	4	I am hard-working by nature.

Table 1. Description of sum variables in the study.

RESULTS

The calculation of Pearson correlations (Table 2) reveals high level of interconnectivity between the different dimensions of mathematics related affect. The overall structure provides very few surprises, i.e. positive beliefs /orientations/attitude have positive correlation to other positive dimension and negative correlations with negative views. The few exceptions will be mentioned below. The belief *difficulty of mathematics* and self-efficacy beliefs *ability* and *success* are strongly correlated with each other. The attitude *enjoyment of mathematics* related this triad to *mastery goal orientation* and *effort*. *Avoiding novelty* was most strongly correlated to *enjoyment of mathematics* among grade 4 students, while among grade 8 students it correlated more strongly with *difficulty of mathematics*. *Performance approach orientation* and *performance avoidance orientation* were closely related to each other and less strongly connected to the other variables. However, both performance orientations correlate positively with positive self-efficacy and attitude and negatively with *difficulty of mathematics*. Therefore, we should consider both performance orientations to express a positive dimension of mathematics related affect.

We can see a general trend for the correlations to be stronger in the older student sample. This is most notable with regard to both performance orientation types. Among grade 4 students these performance orientations had rather weak correlations with other dimensions of affect. However, on grade 8 *performance approach motivation* was moderately correlated with *ability* and *success* as well as with *difficulty of mathematics* and *enjoyment of mathematics*. An exception to this trend is *avoiding novelty*, which correlated positively with performance orientations among grade 4 students, while the correlations were not significant among grade 8 students.

DISCUSSION

This study reconfirmed the high reliability of the instruments used, even if it used fewer items for some scales. The reliability of the affective scales was higher among the grade 8 students. This might indicate that the psychological constructs are more solid among older students, but it may also be simply a product of increased competence in comprehending the statements of the instrument.

The fact that correlations are higher among the older students indicates that the affective structure is more coherent in grade 8. This implies that the students are more clearly divided into those with a positive view of mathematics and to those who hold a negative view of mathematics. In grade 4, there was more room for inconsistencies, and a student with low self-efficacy was more likely to enjoy mathematics and to have mastery goal orientation.

Students' motivation seemed to be more coherent among grade 8 students, and all three motivational orientations formed an overall positive motivational trait. While *performance avoidance orientation* is often seen as a negative trait, now it correlated positively to *mastery goal orientation*, self-efficacy and *enjoyment of mathematics*.

Sum variable	Gr.	MG	Pap	Pav	AN	S	Ab	DoM	EoM	Eff.
Mastery goal orientation (MG)	4		.19	.18	-.12	.48	.19	-.21	.52	.57
	8		.41	.36	-.14	.48	.29	-.26	.48	.57
Performance-approach goal orientation (Pap)	4			.74	.27	.26	.17	-.12	.08	.12
	8			.78	-.09	.43	.35	-.36	.34	.28
Performance-avoidance goal orientation (Pav)	4				.30	.17	.08	-.04	.02	.08
	8				.07	.32	.25	-.19	.19	.20
Avoiding novelty (AN)	4					-.22	-.30	.33	-.42	-.22
	8					-.37	-.43	.50	-.53	-.23
Success (S)	4						.66	-.55	.54	.53
	8						.79	-.70	.64	.54
Ability (Ab)	4							-.70	.54	.47
	8							-.76	.68	.50
Difficulty of mathematics (DoM)	4								-.55	-.39
	8								-.68	-.41
Enjoyment of mathematics (EoM)	4									.57
	8									.57
Reliability full sample		.86	.88	.77	.78	.85	.88	.82	.86	.75
Reliability grade 4		.84	.86	.72	.78	.81	.85	.75	.85	.66
Reliability grade 8		.87	.91	.80	.78	.88	.90	.87	.87	.78

Table 2. Reliability (Cronbach α) of dimensions of mathematics related affect and Pearson correlations between them on grades 4 and 8. Gr. = grade, MG = Mastery goal orientation, Eff. = Effort. Correlations that are not statistically significant are written in italics. All other correlations are statistically significant ($p < .01$).

Regarding *ability*, *success*, *enjoyment of mathematics*, *difficulty of mathematics* and *effort*, it is possible to make a comparison to the previous results from grade 11 (Rösken *et al.*, submitted) and to the results from pre-service elementary teachers (Hannula *et al.*, 2006). However, it should be noted, that we used fewer items for all other scales except *difficulty of mathematics*. In all three studies, the same four components (*ability*, *difficulty of mathematics*, *success* and *enjoyment of mathematics*) are in the core of students' view of mathematics, and there is only minor variation regarding the correlations between these four. The trend of

strengthening correlations that were observed from grade 4 to grade 8 does not seem to continue to the older and more selective populations. On the contrary, the students' *effort* had lower correlations with belief dimensions in older populations. In other words, in grade 8 the variation in student *effort* was more strongly related to the students' overall positive view than among younger and older students.

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PROSPECTIVE TEACHERS' REASONING ON LINEAR AND QUADRATIC FUNCTIONS: A CASE STUDY

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The purpose of the study is to investigate how three prospective teachers at various stages of competence in mathematics reason about functions in connection with the relations $y=x+5$ and $y=\pi x^2$. As the performance of the prospective teachers in their studies of mathematics decreases, a process-based conception of the function concept becomes more prominent. The form of the relations influences the concepts that they are deemed to represent and prototypes seem to exist in relation to the function concept. The prospective teachers often use geometrical interpretations when describing various properties of a function and frequently lack a mathematical language to describe the properties of functions. None of the prospective teachers in the case study describe the concept of function as consistent with the definition of function.

INTRODUCTION AND THEORETICAL BACKGROUND

The concept of function is part of most areas in mathematics and is frequently considered a unifying concept that provides a framework for the study of mathematics. To become successful in dealing with the concept of function in their practice, it is important for mathematics teachers to have a well-developed conceptual knowledge of functions, including the concept's significance in mathematics and relationships to other concepts (e.g. Cooney & Wilson, 1993; Thomas, 2003).

Gaining knowledge of the relationships and being able to use functions in different contexts is a learning process that requires a longer period of time. However, Hansson (2010) show that it is common for prospective teachers to believe that the concept of function is a difficult one for compulsory students to learn and quite frequently, a concept that is largely relevant to high achieving students. Moreover, prospective teachers' perception of the presence of functions in school mathematics is often based on experiences of when the function concept is explicitly stated. This might imply that they do not give enough explicit attention to the functional aspects of linear and quadratic relations in their future teaching (e.g. Even & Tirosh, 2002).

Eisenberg (1992) describes what he calls "having a sense for functions" (p. 154) as a major goal in the curriculum, describing this notion as having insights about functions incorporating the integration of many skills. These skills are often taught in isolation where compartmentalization of knowledge risks occurring when a body of knowledge splits into a larger number of isolated bits. More concerns regarding knowledge compartmentalization are considered in students not being able to assimilate different forms of representations of functions (Mamona-Downs & Downs, 2002), with impact on understanding, facility in manipulation, mental imagery, etc.

The concept of function with its various sub-notions and contexts from which it can

be approached is a complex concept for students to grasp. Vinner and Dreyfus (1989) conducted a well-known study, showing that tertiary students during a course in calculus, even when the students were able to correctly account the definition of function, did not apply the definition of function successfully. Vinner (1983, 1992) describes a model using the notion of concept image consistent with these results. Conceptual development of function and the framework it provides is a long term process in which students are engaged in during their studies of mathematics.

Research questions

The purpose of this study is to investigate, by means of a case study, how the function concept is expressed in connection with $y=x+5$ and $y=\pi x^2$ for prospective teachers at various stages of performance in their studies of mathematics. What properties the prospective teachers identify and how relations between the function concept and other concepts are presented will in particular be examined.

METHOD AND PROCEDURE

This study is part of a larger study of prospective teachers' conceptions of functions. The prospective teacher who participated in the study are specializing in mathematics and science, grades 4 to 9. They were enrolled in the final mathematics courses of the educational program during the term the study was conducted. Data collection occurs primarily at the end of the term, after a calculus course when the function concept has been a central concept.

The choice of the relations $y=x+5$ and $y=\pi x^2$ enables the prospective teachers to relate to previously encountered concepts and areas in mathematics, as well as others encountered during the teaching program. The relations can possibly represent concepts such as a straight line, parabola, equation, formula, proportionality, function and others. In connection to the function concept (assuming that domain and codomain are also considered), the prospective teachers can identify various classes of functions related to $y=x+5$ and $y=\pi x^2$, e.g. linear, quadratic, along with continuous, differentiable or even. The prospective teachers may also indicate if the functions have inverse, extremes or other properties dealt with in a calculus course. The relations also give the prospective teachers an opportunity to relate to future teaching situations on the function concept.

To investigate the prospective teachers' views on the relations, a group of 25 prospective teachers were given a questionnaire before and after the calculus course. The questionnaire included open questions where the prospective teachers were free to make their own interpretation of the relations $y=x+5$ and $y=\pi x^2$. They further drew individual concept maps based on $y=x+5$ and $y=\pi x^2$ after the calculus course. Finally, 20 prospective teachers from the group were interviewed. Each interview was recorded on tape and transcribed. The interviews were based on the answers the prospective teachers had given on their two questionnaires where they had the opportunity to comment on and expand their answers, draw pictures, etc.

Three prospective teachers were selected to participate in a case study. They were drawn from a batch of data that was compiled on all prospective teachers in the group, and based on their performance in the calculus course using the three-grade scale: “high pass”, “pass” and “fail”. Their performances on other mathematics courses confirm their distinctly different levels of competency in mathematics. Their level of performance in mathematics can be arranged as follows in descending order: Anne, Bert, Carla.

RESULTS

Prototypes and general models

During the interviews, Anne and Bert both seem to use $y=x^2$ as a prototype when they draw the graph of $y=\pi x^2$. The concept map drawn by Bert for $y=\pi x^2$ (see figure 2) confirms his view of $y=x^2$ as a prototype in relation to $y=\pi x^2$, whereas Anne states on her map that π influences the shape of the graph, indicating her assumptions based on $y=x^2$. The concept images presented by Anne and Bert $y=x^2$ appear to dominate for a quadratic function and parabola, and an “ x^2 -curve”. However, it does not immediately lead them to draw a correct function graph of $y=\pi x^2$, as both Anne and Bert mistake making a vertical translation of the graph during the interview. Furthermore, the relation $y=x^2$ appears to have a strong influence on Carla’s concept image for quadratic functions. She identifies $y=x^2$ as a quadratic function and is uncertain whether $y=\pi x^2$ is a quadratic function, since the factor π is included.

In relation to $y=x+5$ and a straight line, the interviews and concept maps (see figure 1) reveal $y=kx+m$ to constitute a dominant feature in the prospective teachers’ related concept images, and they state that m and k are the coordinates’ y -value for the point of intersection with the y -axis and the slope of the line, respectively. It can be observed that in relation to $y=kx+m$, the prospective teachers seem to apply a general model in their reasoning without a specific example, in contrast to $y=\pi x^2$.

Function classes and properties of functions

Mathematicians often take note of various function classes and properties of functions when they draw concept maps for the concept of function (Williams, 1998). A comparison of prospective teachers in the case study reveals that they did not stress $y=x+5$ and $y=\pi x^2$ to represent different function classes. Nevertheless, $y=\pi x^2$ was a quadratic function for Anne, both in her interview and on her concept map. In this respect, Bert referred to classes of curves (x^2 -curves and quadratic curves) and did not explicitly mention that he was referring to function graphs. However, the concept map drawn by Bert for $y=\pi x^2$ contains a horizontal sequence of nodes and links from the term “function” to “ x^2 -curve” via “table” and “graph, thus indicating his intention. Carla seemed to associate quadratic function with $y=x^2$, but was uncertain if $y=\pi x^2$ was also a quadratic function. Furthermore, all stated that $y=x+5$ was a straight line (Anne stated that it was the equation of a straight line) and a function, though none mentioned that it was a linear function.

Anne and Bert leaned more towards a geometric (holistic) approach for $y=x+5$ and $y=\pi x^2$ during their interviews and on their concept maps, and to a greater extent than in their answers to the questionnaires. This is also true of Carla in the case of $y=x+5$, since she largely viewed $y=\pi x^2$ as a formula to calculate the area of a circle. When the students discussed properties in conjunction with the function concept, they primarily did so from a geometric perspective (in Bert's case this was often combined with a numerical approach where he refers to the values of the variables). A geometric approach is also reflected in the concept map drawn by Anne for $y=\pi x^2$, where she places "parabola" (referring to the function graph) in an underlying structure to "function". To this, she tied "min. point" and "symmetrical", among others. Anne further claimed during the interview that she now preferred to explain and illustrate problems with the aid of "images".

The relationship between function and equation

Another situation arises during the interview, when Anne spontaneously raises the relationship between equation and function. Anne seems to find it difficult to distinguish between the concepts function and equation and invents expressions, such as "the equation of the graph". The concept map for $y=x+5$ (with double links between the two concepts) and her reply of the second questionnaire (containing words such as "solution pair" in relation to function) also show this. During the interview, her understanding of the relations between "the equation of a straight line" and "function" appeared to, making it difficult for her to distinguish the concept of equation and the concept of function.

The form of a relation appears to determine which concepts the prospective teachers decide it represents, illustrating a semiotic function of the relations (Steinbring, 2005). During his interview, Bert says he "automatically" sees a " $y = x$ -expression" as a function. Bert evidently did not view $y=\pi x^2$ to represent an equation, since he stated in his interview that "you can write it as an equation" and considered gathering the variables on one side of the equal sign. He also said it would then become "extremely tricky" and did not appear to realize that the solution set of the equation coincided with the graph of the function. Furthermore, Anne did not state that $y=\pi x^2$ represented an equation, not mentioning the equation concept in the questionnaires, the concept map or the interview, as opposed to $y=x+5$. The relationships between the concepts represented in the concept maps also appear to have been influenced, e.g. function was strongly related to equation in the concept map drawn by Anne for $y=x+5$, whereas the concepts had no relations in the map she drew for $y=\pi x^2$, which had no equation concept.

Characterizations of function

The prospective teachers did not describe a function consistent with contemporary characterizations of function; that is, a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain). Nonetheless, Anne noticed an important component of the

function concept, the uniqueness criterion, when she wrote, “for each value of x , there is just one value of y ” in the first questionnaire. At the same time, she deviated from a contemporary definition of function by assuming that x and y are part of “an expression with two variables”. She commented on her answer in the questionnaire by stating, “... the theoretical stuff that one has learn, has been drilled into us...” and does not mention the uniqueness criterion in the second questionnaire, thereby showing signs of rote learning. In addition, Bert viewed the function concept as dependence between variables. Like Bert, Carla based her conceptual interpretation of function on a dependency relation. But in contrast to Bert, she specified a dependency that did not exclusively consist of variables, but also included constants analogous to numerical calculations in her explanation that “ y is dependent on π and x ”.

The prospective teachers give the impression of only having been exposed to problems that encourage little reflection of the function concept. Anne revealed this during the interview when she stated that she had not “come across” such situations. This was confirmed in the interview with Bert, who clearly could not recall the definition of function because he could not determine whether y was a function of x in the case $x=\pi y^2$, though he stated that a positive value of x results in two y -values $\pm\sqrt{x/\pi}$.

Concept maps

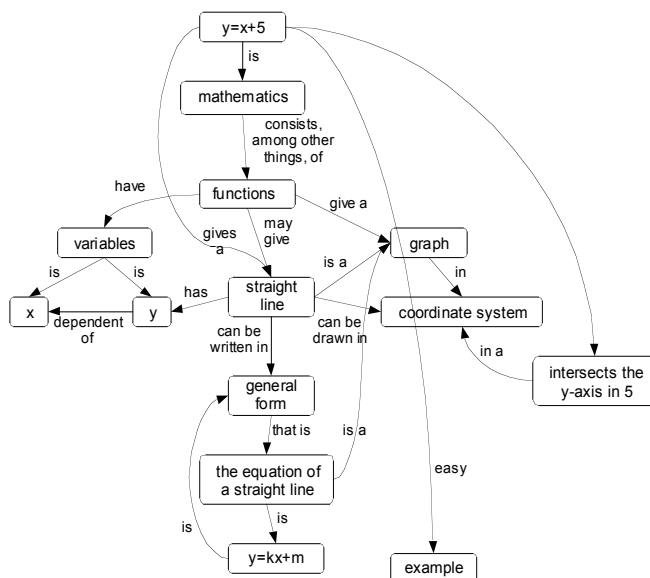


Figure 1. The concept map that Bert drew based on $y=x+5$.

It can be said that the maps drawn by the three prospective teachers are different in character. Anne and Bert drew maps that largely connected different sections of the

map. They generally contained more cross-links compared to Carla's two maps, neither of which contained any cross-links. This could imply that the knowledge representing the concepts are more compartmentalized for Carla than Anne and Bert. Furthermore, the maps drawn by Bert and Carla tended to contain more trivial elements (e.g. “draw”, “Greek letter”, “car romeo”), whereas the maps drawn by Anne were more subject-specific. This implies that Anne has knowledge with more meaningful relations to mathematical concepts.

The function concept is usually a more integrated part of the concept maps drawn by Anne and Bert than those drawn by Carla. In the map Anne drew for $y=x+5$, the function concept is placed in close relation to the equation concept. Of note, both function and equation concepts exist in all of the maps drawn by the prospective students for $y=x+5$ (all three also include $y=kx+m$, thus referring to the equation of a straight line). This is in contrast to the concept maps for $y=\pi x^2$; Carla is the only one who discusses the equation concept (by the link “remind me of” to the node “quadratic equation”). The prospective teachers appeared to see it as if the relationship between the concepts “function” and “equation” changed in relation to the two relations $y=x+5$ and $y=\pi x^2$.

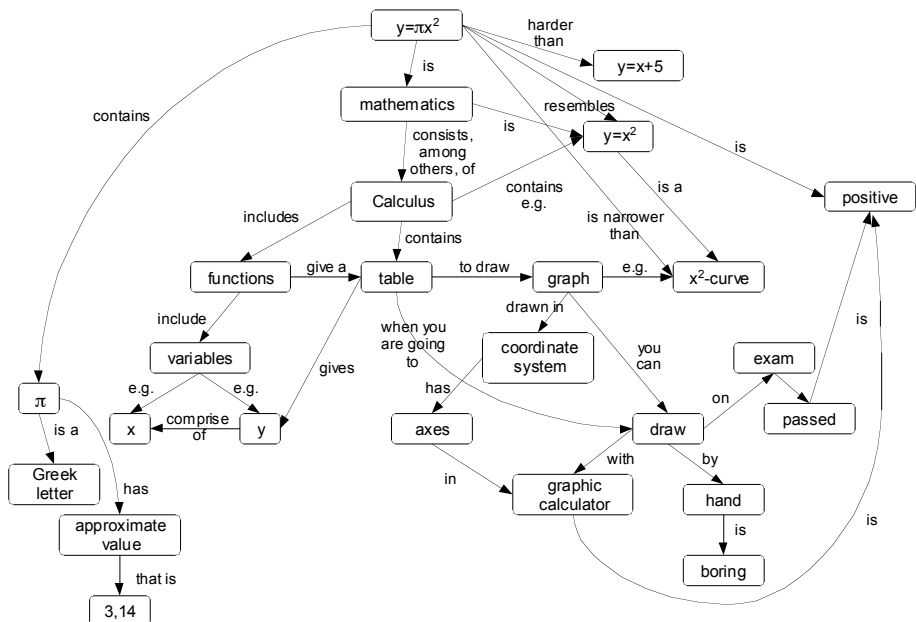


Figure 2. The concept map that Bert drew based on $y=\pi x^2$.

DISCUSSION

The prospective teachers in the case study were found to have a limited understanding of the function concept and still lack the skills an inservice teacher should possess. It is well known that mathematics students at tertiary level do not have a conceptual understanding of functions considered consistent with contemporary characterizations of function (e.g. Vinner & Dreyfus, 1989; Williams, 1998), as with the prospective teachers in this study. However, Even (1993) argues that prospective teachers ought to have a deeper understanding of the function concept, as teachers' pedagogical decisions, e.g. questions they ask, examples they put forward, activities they design, ideas they consider of value and students' suggestions they follow (Even & Tirosh, 2002), are partly based on their understanding of the topics in question. Hence, it is important that teachers have a well-developed concept image of function in their reasoning about functional relations.

The prospective teachers in the study seemingly need to be exposed to problem formulations concerning the relationships between mathematical concepts, including problems inviting reflection upon the definition of function. The prospective teachers do not appear to be experienced in working with such problems. A way to develop their understanding of the function concept could then be to stimulate a feedback of evoked concept image to the concept definition (Hansson, 2009). According to Vinner (1992), this type of reconnection is primarily possible with problems that are not of the standard variety. Moreover, if the prospective teachers' concept images of function were characterized by those examples of function with which they come in contact, and to a lesser degree by the formal definition of the concept, this should result in paying closer attention to those examples of functions prospective teachers encounter during mathematics courses. Furthermore, highlighting different properties of functions and their relations to other concepts could be one way to help the prospective teachers to create a more well-developed framework of reasoning in relation to the function concept.

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STUDENTS' USES OF TABLES IN LEARNING EQUATIONS OF PROPORTION: A CASE STUDY OF A SEVENTH GRADE CLASS

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The development of proportional reasoning has been discussed both theoretically and empirically. In this paper, I examine the proportional reasoning that is functioning in students' learning a new tool for proportion in the classroom. Observations of and interviews with three students we focused on in a seventh grade class show that they relied on tables, which they were more accustomed to, in their process of learning of equations ($y = ax$). Students applied personal meanings to these algebraic signs based on their ways of seeing tables of proportional relationships, the process of which was predominantly based on within measure space ratio reasoning. The students struggled with coordinating their views of tables because the equations impelled them to draw on between measure space ratio reasoning.

INTRODUCTION AND BACKGROUND

Studies about proportional reasoning have a long history that has revealed different strategies children use and their relationships with task variables. The development of proportional reasoning has also been discussed both theoretically and empirically and researchers have identified models and characteristics related to it (e.g., Hart, 1981; Kaput & West, 1994; Lo & Watanabe, 1997). However, the reality of students' development of proportional reasoning is still inadequate, as shown by students' overuse of proportionality (e.g., Van Dooren, et al., 2009) for example. In order to fully understand the situation in which students are positioned, it is necessary to draw on approaches from various perspectives and methodologies.

The purpose of this study is to examine the proportional reasoning that functions in students' learning new tools for investigating proportional relationships in the classroom. Mathematics learning is thought to be inextricably linked to semiotic activity, in which students' endeavor to create meanings and signs by reflecting on their interrelationship, and to adjust their creation of signs and meanings accordingly are constitutive elements (Van Oers, 2000). In the realm of proportional reasoning as well, several researchers began to borrow the approach of semiotic perspectives (e.g., Sáenz-Ludlow, 2003; Hino, 2008; Sook & Kwon, 2010). Following this line of thought, this study assumes that students' proportional reasoning is enhanced through learning different symbolic means, such as number lines, tables, equations, and graphs, which students encounter in classroom learning. It is thus essential to understand students' processes of acquiring these means in the classroom in conjunction with the impact they have on their proportional reasoning.

In this paper, I focus on students' acquisition of symbolic means in a seventh grade class. In Japan, proportion is taught both in the sixth and seventh grades. In the sixth

grade, students are introduced to the concept of proportion mainly through the use of tables. By using tables, proportional relationship is defined as the following: If two quantities \square and \circ change in such a way that \circ increases by a factor of 2, 3, ... as \square increases by a factor of 2, 3, ..., we say that \circ is proportional to \square . The relationship between \square and \circ is also formulated, such as in the equation $\circ \div \square = \text{fixed number}$. In this grade, a graph of a proportional relationship is introduced by plotting several points and observing their arrangement as a straight line that goes through the point where both quantities are 0. In the seventh grade, an equation with algebraic letters ($y = ax$) is introduced and proportional relationship is defined in the form of the equation. The extension of domains from positive to negative and the introduction to linear graphs in the Cartesian plane are also new topics.

The development in the curricular content from sixth to seventh grades are thought to provide opportunities for students to advance their proportional reasoning in at least two senses: First, there is a change in the definition of proportional relationship from the “within” to the “between” ratio. A ratio is classified as “within” or “internal” if its constituent magnitudes share the same measure space, and as “between” or “external” ratio if it is composed of magnitudes from different measure spaces (e.g., Freudenthal, 1983). In the sixth grade, the definition is based on the within ratio, whereas in the seventh grade, the between ratio comes to the front in the definition by equation. Second, the development allows students to change their major symbolic means for exploring proportional relationship from tables to equation with letters. Moreover, in the seventh grade, students deal with the connections among equations, graphs, and tables. On the basis of these circumstances, this paper aims to investigate students’ behaviors and thinking in real classroom situations to derive information on their development of proportional reasoning.

METHOD

This paper is based on the results of preliminary data collection that was conducted regarding students’ processes of learning proportions in lower secondary school. Data was collected from a seventh grade classroom of a public school in a rural area of East Japan. In order to get an initial idea of students’ learning processes, I used a qualitative case study approach (Merriam, 1998) to study the introduction of students to new symbolic means for investigating proportional relationships, looking closely at several students’ use of signs and the associated meanings they derived from them.

To do this, a graduate student and I visited the classroom and observed 15 lessons on a chapter involving proportion. We chose three students based mainly on their responses to a questionnaire that we gave them prior to the lesson regarding their proportional reasoning and their skills in using symbolic means. None of the three students showed high levels of proportional reasoning on the questionnaire nor had they yet developed skills for solving algebraic equations and linear graphs, which they would learn later in class. We gathered information on behaviors and thinking of the three students from each lesson, especially on how they reacted to newly introduced mathematical signs (e.g., $y = ax$ or linear graph). Our data included the

work they did in their notebooks and worksheets and the private and public discussions they held during the lessons. I also conducted interviews with them three times, asking them about their understanding of equations and graphs in progress.

Lesson	Topic
1-2	Introductory activity on proportional relationships
2-4	Introduction to x and y as variables; Introduction to equation ($y = ax$); Extension of x from positive to negative numbers
5-6	Extension of the parameter “ a ” in $y = ax$ from positive to negative numbers
7-10	Introduction to Cartesian coordinate system; Generating graph from equation ($y = ax$)
10-11	Relationship between graph and equation of proportion
12	Generating equation ($y = ax$) from linear graph of proportion
13	Introduction to variability domain
14-15	Problem solving and summary

Table 1: Topics dealt with in the 15 lessons

The content and sequence of the lessons basically followed those of the textbook (Table 1). In organizing the lessons, the teacher adopted the style of *structured problem solving* (Stigler & Hiebert, 1999) and gave students time to work freely by themselves, discuss the material with their friends, and present and compare the different solutions. In the analysis, I summarized each chosen student’s thinking activities prior to, during, and after the lessons. During the lessons, the students inscribed tables, equations, and graphs and other notations in their worksheets and notebooks. By examining how students generated such notations and used them in their thinking and talking, I tried to understand the conceptions and reasoning that were behind them and whether it changed as the lessons proceeded.

RESULTS

Among the students’ various notations, tables were observed most often. In this section, I will illustrate how one of the three students, whom I will refer to as Ryota, used tables in the early part of the lessons (Lessons 1–6), in which the equation $y = ax$ was taught.

Reading the Table Horizontally

In Lessons 1 and 2, students were engaged in the introductory activity of finding the relationship between the time of running water and the depth of water amassed in the bathtub. The major tool they used in the activity was a table. It was observed that the three students, including Ryota, read the table horizontally; for instance, “If time increased twice or three times, depth also increased twice or three times” (see horizontal arrows in Figure 1). In talking about co-varied values, the teacher attracted students’ attention to a particular pair of values from different measure spaces (time and depth) by using the Japanese word *moto* (its direct translation into English is *base*), such as in the statement “Time and depth increase 2 times or 3 times by letting

5 [minute] and 10 [centimeter] be *moto*. This also holds true when we let 10 and 20 be *moto*.”

Time (min.)	0	5	10	15	20	25
Depth (cm.)	0	10	20	30	40	50

Figure 1: Time-depth table used in the bathtub problem

The vertical reading of the time-depth correspondence in the class was also observed. One student's solution was based on the vertical relationship between two values, "Depth divided by time became 2 all the time" (see vertical arrows in Figure 1). However, at this stage, there was no notable observation of Ryota regarding a vertical reading as he had not yet become aware of this alternate way of reading the table.

In Lesson 2, the students were given an escalator problem that reads as follows: "There are two escalators, one up and one down, that meet at a point. Today we concentrate on the up escalator. The speed of the up escalator is 15 centimeters per second. (The height of the meeting point is considered as 0 cm.)" There were three questions to follow: The first question was "Write down your method of examination of the relationship between the time passed after the meeting point and the height of the up escalator." Students first thought about the question by themselves and then discussed it with their classmates. Ryota drew a picture of the situation and investigated it using a table. During his presentation, he explained his table as, "... when the time becomes 2 times, or 3 times, then the height becomes 2 times or 3 times, too. ... I used 1 and 15 as *moto*." He continued to look at the table horizontally. Here it is worth mentioning that he spontaneously announced the *moto* by noting one pair of values as the basis of his reasoning. Later in class, the teacher used a table and introduced the term *variable* together with the letters x and y by saying, "When we sum up all of the values that continue endlessly in the table as a whole and express them by using x and y , it is very convenient."

First Reaction to the Algebraic Signs x , y , and $y = 15x$

In Lesson 3, the remaining two questions were addressed. The second question was to examine the relationship between time and height in the up escalator by using a table. Ryota was observed to read the table horizontally in his individual work. During the discussion, the teacher stressed that the table was to be read vertically by pointing out the relationship of the escalators in terms of an *equation involving word*, such as *time* \times 15 = *height* or *height* \div *time* = 15.

時間 (x秒)	0	1	2	3	4	5	6	x
高さ (y cm)	0	15	30	45	60	75	90	15x

Figure 2: Ryota's first use of x and y

The third question was, "If we denote the time passed as x sec and the height as y cm, is it possible to express the relationship

between x and y ?" This was the first question that required the students to use letters in the context of expressing proportional relationships. For an answer, Ryota directly made arrows for x , y , and other letters in his table to express the horizontal relationships (Figure 2), showing that his first use of x and y directly within the structure of the table. Far from a sophisticated notation of $y = 15x$, Ryota's answer was only meaningful as a tabular expression of the relationship. It should be noted that as a premise for his use of x and y , he was using within ratio reasoning. Later in the discussion, some other students presented $y = 15x$ and $y/15 = x$. The teacher summarized that both equations express the same thing and that we proceed to use $y = 15x$ in lower secondary school.

In the interview, Ryota said that he knew that he should be reading the table vertically in Lesson 3. Based on this observation, it is likely that he was confronted with the need to convert his orientation of reading the table while he learned the new means of symbolizing the relationship, which was not an easy task for him. After their introduction to $y = 15x$, the students solved several exercises that involved reading the given tables and creating equations with x and y . The first table was about another bathtub situation in which 1, 2, 3 and 4 are written in the row of time and 2, 4, 6 and 8 are written in the row of depth of water. Here, the teacher observed that Ryota correctly wrote the equation $y=2x$ and asked him to present his solution.

Ryota: Let time be x and depth of water be y , and let 1 and 2 be *moto*, then in the cases other than this [1 and 2], too, if x becomes 2 times, y becomes 2 times, too, so if I try by $y = 2x$, it becomes the depth of water.

His explanation for the horizontal relationship in the table was clear. He mentioned important aspects such as *moto* or its applicability to other cases. Yet, how he created $y = 2x$ was hard to interpret because he did not state clearly where he got the "2" in $y = 2x$ from, which also suggests that he had not yet made sense of the connection between "a" in $y = ax$ and the vertical relationships of the values in the table.

Coordinating the Views of the Proportion Table

In Lessons 4 through 6, by using the extension of the domain and the constant of proportion in $y = ax$, Ryota was coordinating his views of the proportion table. In

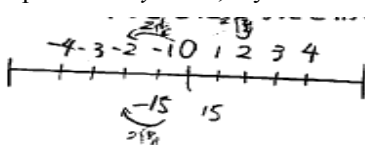


Figure 3: Ryota's number line

Lesson 4, the students were asked to examine the relationship between the time and the height of the up escalator before it meets the other escalator. In his individual work, Ryota expressed the relationship between time and height by using a number line (Figure 3).

Since no other students used a number line as the tool for exploration, his work attracted the attention of his teacher and classmates. The number line shows Ryota's consistent view of the proportional relationship that is based on his within ratio reasoning. He naturally extended the horizontal relationship, not in the table but in the number line, from the positive to the negative domain. His use of arrows and writing negative numbers 2 times shows his claim that the proportional relationship

remains in the negative domain. Here, he seems to have begun coordinating his view of the table to include the negative domain, in which he located the zero point as the center and lined numbers up in both directions to maintain a balance.

In Lessons 5 and 6, an extension of “a” in $y = ax$ from positive to negative numbers was made for the escalator problem; in this case, for the down escalator. In the same way as for the up escalator, the students were asked to examine the relationship by using tables and to make an equation using x (time) and y (height). During the individual work in Lesson 5, Ryota first made the table shown in Figure 4.

x	-3	-2	-1	0	1	2	3
y	45	30	15	0	-15	-30	-45

|← part c →|←part a→|← part b →|

Figure 4: Ryota’s x-y table for the down escalator

He wrote first two zeros (part a in Figure 4), then the positive domain (b), and finally the negative domain (c). Ryota’s table and his ways of generating it show that his view of the table is consistent with that of his number line. In class, many students made tables with only positive domains or that exclude zeros. When the class discussed their different tables, Ryota expressed that locating zeros is the advantage of his table. Though he did not express the reason for doing so, it seems that he continued to construct a coordinated view of the table in which zeros are important as the center.

During the Lessons, Ryota was also observed to develop an understanding of the connection between the constant of proportion (in this case “-15” in $y = -15x$) and the value of y when x is equal to 1 in the table. One piece of evidence of his understanding exists in the worksheet he wrote in accordance with the table (Figure 4), which reads: “When x is the time passed after [one] escalator met the other one, and y is the height, the constant of proportion is $-15 \div 1 = -15$, so it is -15 , and then $y = -15x$.” Here, he used 1 and -15 in the table to derive the constant. Another piece of evidence is that in the later part of Lesson 6, Ryota circled the y value when the x value is 1 in the table before making an equation when solving exercise problems. His connection between the equation and the table attached significant meaning. This was shown in his activity of thinking of the meaning of “-15” in $y = -15x$ in Lesson 6. In the discussion below, the teacher inquired of the students what their imagery of “-15” in the table was:

Teacher: Do many of you have the imagery of something like this? [She added a horizontal arrow and many students raised their hands suggesting the affirmative.]
Teacher: Who does not? Who instead imagined $-15 \div 1$, $-30 \div 2$? [Ryota raised his hand.]
Teacher: What did you write in your worksheet, Ryota?
Ryota: It goes down -15 centimeters from the base zero when it goes one second.

What Ryota expressed is that he looked at the table vertically. Moreover, in his response to the teacher, he mentioned a concrete meaning of “-15” as the distance the

escalator goes down when it passes 1 second. Here, it seems that he was creating a clearer meaning of *moto* (in this case, 1 and -15 in the table) based on his coordinated view of table, which served him in making meaning of constant of proportion.

DISCUSSION

In the previous section, I illustrated how Ryota used tables during the lessons in which new symbolic means of proportion ($y = ax$) were taught. Here, I shed light on the use of tables by comparing the results obtained by Ryota with the behaviors and thinking used by the other chosen students. All the three students made tables and used them to learn the equations. Their knowledge of tables played an important role in mediating their knowledge of the equations. For them, table was a familiar tool of thought they could rely until another tool was developed. Here, it is important to note that as a premise for their use of tables, their within ratio reasoning was predominantly functioning. The students noticed the ratio within the measure space, such as time, and applied the ratio to the other measure space such as depth or height. They actively drew on the horizontal relationship in the table to make an early interpretation of the equation.

In contrast, the teacher made use of the tables by making the students aware of the vertical relationship ($y \div x = \text{constant of proportion}$) to introduce the equation. The discrepancy between the students' within ratio and their between ratio reasoning in learning the equations was apparent. It was a challenge for the students to reorganize their views of the proportion tables while learning the equations. Previous studies have revealed students' difficulty in constructing sufficient meanings of the signs. For example, Sook & Kwon (2010) pointed out the breaking off of representation-meaning remarked on in their interviews. This paper, through reporting the very beginning of students' meaning construction to the new algebraic signs, argues that we need to pay careful attention to the shift in proportional reasoning required for students.

Compared with the other two students, in learning the equation, Ryota showed the coordination of his view of the proportion table more explicitly. As illustrated and interpreted in the previous section, his process of coordination seems to have several features: First, it took place at the same time that he made sense of the new knowledge he encountered. By extending his old view, he was able to make sense of the talks and writings in the equation. In the coordination process, his within ratio reasoning was still functioning. Rather than changing his view dramatically from the within ratio to the between ratio, he proceeded to utilize his familiar way of looking at the table. Here, it is curious that Ryota was paying attention to *moto*, which reflected his interest in the between ratio in the table. Even though his process of learning was not entirely clarified, I assume that *moto* functioned in different ways, such as extending the domain to negative numbers and making sense of the negative "a" as the y value when $x = 1$. The other chosen students were less attentive to the correspondence between the two quantities, even though they talked about the within ratios or about the difference of values within a measure space. In retrospect, Ryota

developed the most successful performance for using the equation among the chosen students. In the lessons on the graph as well, he showed less difficulty. I conjecture that his early attentiveness to the between ratio in the table enhanced his understanding of the connection between the table and equation and made it possible to interpret the graph meaningfully.

Since the results reported in this paper are based on a small number of students in a classroom, a suggestion of its generality needs to be noted. Keeping in mind this caveat, one of the future tasks would be to determine how to prepare students in the previous grades to enhance their shift from within to between ratio reasoning. Here, a clue could be gained from the construct of *particular intensive quantity* (Kaput & West, 1994) in the transition to rate conceptualization of intensive quantity, as suggested by Ryota's attention to *moto*. Another task would be to inquire into lessons about equations and graphs focusing on how students' proportional reasoning is elicited and how they appropriate new ways of reasoning.

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ACHIEVEMENTS OF SECONDARY MATHEMATICS TEACHERS INCORPORATING DIGITAL TECHNOLOGIES INTO CLASSROOM PRACTICE

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The work we are present here contributes to the issue of research on teacher learning on how to incorporate mathematics technology into their classroom practice, approaching relatively new in-service teacher training design and using a theoretical framework of analysis related to introducing innovation in school. The main data were obtained from videos of the participant teachers during their classroom practice. These teaching videos revealed five different ways the participant teachers realized how to use mathematics technology into their classrooms.

INTRODUCTION

According to Cobo (2009), there are many new digital artefacts that have the potential to be used in classrooms. From his point of view, the educational field has benefited from digital technologies irruption, especially with the emergence of the Web 2.0.

However, as the same author set out, the advent of new technologies generates the possibility for new student abilities and skills. Then, it makes teachers face real challenges: they have to be acquainted with new digital resources and should learn how to integrate these technological tools into their classrooms.

In this context it is plausible reconsider the following research questions.

- i) In reality, what has been the influence of digital technologies in school, especially regarding mathematics learning?
- ii) What do teachers think about using digital technologies in mathematics classrooms?
- iii) How do mathematics teachers achieve using digital technologies into their classrooms?

The work we are presenting here contributes to answering the third question, building upon previous work accomplished mainly by Zbieck and Hollebrands (2008), Ruthven (2007; 2002) and Ruthven and Hennessy (2002).

Specifically, we have already set up an exploratory study on the achievements of in-service secondary teachers, as they try to incorporate mathematics technology into their classroom practice.

Besides, we will only revise some of most relevant research results on the issues brought up in the two first questions formulated above.

BACKGROUND

Technological tools in school

In according with Olive et al. (2010), at the end of the 1980s and during the 1990s there were great expectations for the potential of new technologies to transform the way mathematics could be taught and learned (Howson & Kahane 1986, cited in Olive et al. 2010).

However, as Ruthven (2007) point out, in the opinion of revolutionary software designers (as Papert, the creator of LOGO), the uses of technology in education have often simply replaced paper with computer screens without changing tasks. Moreover, computers have been used to “simply transfer the traditional curriculum from print to computer screen” (Kaput 1992: 516) in ways that resemble traditional worksheets and structured learning environments, rather than working to transform learning (Tyack & Cuban 1995, cited in Ruthven 2007).

Papert (1997, cited in Ruthven 2007) observes that “the most insightful... teachers working in conventional schools understand what they are doing today... [as] not being the ideal they wish for.” And goes on to suggest that “as ideas multiply and as the ubiquitous computer presence solidifies, the prospects of deep change become more real.”

In particular, according to Ruthven (2007), Papert sees the day-to-day classroom work these teachers have with computers as the seed from which such change will grow.

In his analysis of the relationship between teachers, technologies and structures of schooling, Ruthven (2007) also mentions that indeed, contemporary theories of educational change, just like those of technological innovation, acknowledge how these processes are shaped by the sense-making of the agents involved (Spillane, Reiser & Reimer 2002, cited in Ruthven 2007).

Teacher conceptions on using digital technologies into mathematics classrooms

In this section we will refer exclusively to the work published in 2002 by Ruthven and Hennessy on teachers’ ideas about their own experience in successful classroom use of computer-based tools and resources. These authors obtained teacher accounts that were elicited through focus group interviews involving secondary school mathematics departments. They then analyzed these interviews qualitatively and quantitatively, so as to identify central themes and primary relationships.

They found that teachers thought technology can serve as a means of enhancing ambience of classroom activity, assisting tinkering of students, facilitating routine and highlighting properties and relations. These authors also obtained teacher

information on themes directly related to major teaching goals, and finally, on key learning topics connected with teaching uses and goals.

It is worth mentioning that Ruthven (2007) points out the fact that each of the constructs mentioned by the participant teachers represented a desirable state of affairs which teachers seek to bring about in the classroom, to which they see the use of technology as capable of contributing. But, as Ruthven says, that is only a model based on teachers' de-contextualized accounts of what they saw as successful practice, unsupported by examination of actual classroom events (ibid: 56).

THEORETICAL FRAMEWORK AND METHODOLOGY

Design of an exploratory online training course

From the work realized by Zbieck and Hollebrands (2008) that synthesizes ten years of research into incorporating mathematics technology in teachers' classroom practice, we extracted a structure to set up a six-month online training course (OTC) on mathematics and information technology topics for secondary mathematics in-service teachers. Zbieck and Hollebrands (2008) theoretical constructs, specifically their re-conceptualization of the PURIA model of Beaudin & Bowers (1997) allowed us to analyze the data obtained as those derived from the introduction of technological innovation.

The general context we set the OTC up in was within the official educational policies current in most of the world's countries. According with them, in-service teachers should learn how to incorporate mathematics technology into their classrooms (see for example, Assude et al. 2006). In Mexico too, official educational policies are supporting the integration of new technologies into teaching, with particular attention to the teaching of mathematics.

To train teachers in a way they could meet the task of incorporating mathematics technology into their classroom practice, we implemented an exploratory online training course (see <http://upn.sems.gob.mx>) of six months with 15 in-service secondary teachers that would allow the participants to learn to use technology and learn to do mathematics with technology.

Both of those aspects (learning to use technology and learning to do mathematics with technology) constitute important modes in the PURIA model of development along a learning continuum (Zbiek & Hollebrands 2008). In fact, this model implies that teachers experiment different modes or development states to advance toward successful incorporation of technology into classrooms: the Play, Use, Recommend, Incorporate and Assess modes.

As Zbiek and Hollebrands said (2008: 295), the growth during the P and U modes includes the transition of the technology as the developer's tool into the teacher's instrument for doing mathematics. Then, in the Incorporate and the Assess modes, the teacher's attention turns, implicitly or consciously, toward the use of technology as a pedagogical tool, including the development of instructional orchestrations

(Trouche 2000, cited in Zbieck & Hollebrands 2008) or elaborated plans regarding use of technology in the social dimensions of classrooms. Finally, the Recommends mode seems marked by a transition between mathematical and pedagogical aspects of the technology (Zbieck & Hollebrands, *ibid*: 295).

DATA COLLECTION

On the different technological resources displayed throughout the OTC

In order to describe how we obtained the data of what teachers had learned while on the course, we first will describe the OTC's components.

Computer programming topics were addressed, particularly an introduction to HTML and JavaScript programming, design of algorithms and their representations, algorithm development, flow charts, and codification, specifically with the purpose of teachers experimenting a change in the way they see or to approach to study mathematical algorithms.

The training course also included reviewing sequences of activities on the use of interactive software (eg. Logo, GeoGebra, Aplusix, Excel, RecCon, FunDer) and exploration of a wide range of digital possibilities available on Internet, such as the library of virtual materials (eg. <http://nlvm.usu.edu/en/nav/vlibrary.html>) of Utah University (USA).

On the mathematical and pedagogical tasks along the OTC

The maths topics studied were introductions to the fundamental arithmetic theorem, Goldbach's conjecture, graph theory and calculating roots of polynomials (with the bisection and the *Regula Falsi* methods, the secant and Newton methods) purposed to review some of the algorithms important to secondary maths contexts, as well as the distinct possibilities of representing them in mathematics and computing.

Although we can observe that those topics are notably more advanced than those included in the standard high school math curriculum, they were planned so as to present the teachers with a challenge to their existing knowledge and to solve problems on topics they do not necessarily have a mastery of.

In relation to the pedagogical activities displayed throughout the OTC, four weeks of activities were included (two at the end of the first 10 weeks, and two at the end of the next 10 weeks) where teachers had to perform a series of tasks: (a) Choose one high school mathematics topic, along with the software, tools or digital materials they thought it would be useful to use in teaching it; b) orchestrate a classroom work session with their students in a convenient way for their chosen digital material; c) video-record that work session; d) upload a seven-minute version of that recording to YouTube; and finally e) upload to the training platform a descriptive report of the video's content together with its URL.

In fact, it was from this last segment of activities that we extracted the main data that generate the results we present here.

ANALYSIS AND RESULTS

From the OTC that we implemented, we obtained evidences that showed teachers displaying the PURIA modes of Play, Use, and Recommend, and also we could observe how they started to Incorporate mathematics technology into the classroom.

Description of the different ways teachers started to incorporate mathematics technology into their classrooms

There were nine teachers that accomplished all the tasks we required during the OTC, from a total of 15 teachers that participated in the group of teachers in observation. We summarize in the following table the teacher executions found.

Cas e	General data (a) Teacher's initials (b) City of residence	Topic and digital tool chosen & video URL	Way technology incorporated into classroom
1	(a) HA (b) Veracruz	- Solving equations - PowerPoint software http://www.youtube.com/watch?v=PILYsIO-Vh0	Teacher uses a LCD, laptop and software to explain or introduce a math topic
2	(a) AG (b) Baja California	- Relation between a function and its derivative - GeoGebra http://www.youtube.com/watch?v=Lk2yVHDjexA	Idem
3	(a) AM (b) Guanajuato	- Simplification of rational algebraic expressions - Java and HTML http://www.youtube.com/watch?gl=MX&hl=es-MX&v=N1FwbEo5KGI	Teacher adds to a classic way of teaching, questioning students on related maths topics
4	(a) HM (b) Baja California	- Calculation of the area of geometrical figures (2D) - GeoGebra http://www.clipshack.com/Clip.aspx?key=CDF72468862861A8	The teacher adds to a classic way of teaching an election of appropriate digital tools to justify or confirm complex calculations
5	(a) FM (b) Veracruz	- Graphics and equations of functions - GeoGebra http://www.youtube.com/watch?v=BXAE2b5U3M4	The teacher adds to a classic way of teaching an election of appropriate digital tools to justify or confirm complex calculations
6	(a) AL (b) Sinaloa	- Design of geometrical figures and calculation of areas - GeoGebra http://www.youtube.com/watch?gl=ES&hl=es&v=yhXs8BLMFIM	The teacher is able to orchestrate student computational work, driving student work by means of a work template
7	(a) SM (b) Colima	- Equation of a straight line - GeoGebra http://www.youtube.com/watch?gl=MX&hl=es-MX&v=X4c8IHEzQsM	The teacher is able to orchestrate student computational work, driving student work by means of a work template
8	(a) OV (b) Baja California	- Solving inequations - Aplusix http://mx.youtube.com/watch?v=gwGcPtyXYbs	Teacher's work orchestration uses both digital tools and paper and pencil to compare student executions and results
9	(a) FG (b) Hermosillo	- Properties of instruments in physics - PowerPoint software http://fcogurrola.blogspot.com	Teacher is capable of orchestrating student computational autonomous work, based on student project work and small group cooperation

Summarizing, the analysis of the teaching videos revealed five different teacher ways of starting to use mathematics technology into their classrooms:

(a) A pattern of incorporation of the technology probably derived from the classic approach to teaching (see cases 1 and 2 in the table). It is to say that the teacher uses a LCD, laptop and software to explain or introduce a math topic. One could name that way as a *classic pattern* of using mathematics technology in the classroom.

(b) A modified version of the *classic pattern* that added teacher interaction with the students, basically by teacher questioning the whole class (case 3), or the teacher adds to a *classic* way of teaching the election of appropriate digital tools to justify or confirm complex calculations (cases 4 and 5).

(c) An instrumental approach of the activity (Verillon and Rabardel, 1995; Assude et al., 2006) mainly directed by the use of a script or work template. In this case (cases 6 and 7) the teacher is able to orchestrate student computational work, and mainly drives student work by means of a work template.

(d) An orchestration (Trouche, 2004) of the activity using different instruments or artefacts, plus group negotiation of meaning. In the teacher's orchestration there are uses of both digital tools and paper and pencil to compare student executions and results (case 8).

(e) An organization of cooperative work centred on student appropriation of technology where the teacher is able to orchestrate student computational autonomous work, based in student project work and small group cooperation (case 9).

In order to see some images of each type found, readers may access the URLs specified in the table. We give here only a few examples.

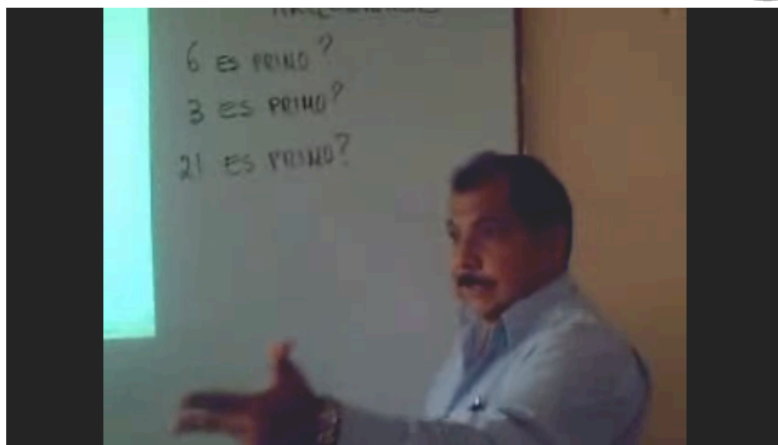


Fig.1. Classic approach to teaching, adding teacher questioning to the whole class (see case 3).



Fig.2. The teacher is driving an instrumentalization of the activity led by the use of a work template (see cases 6 and 7).

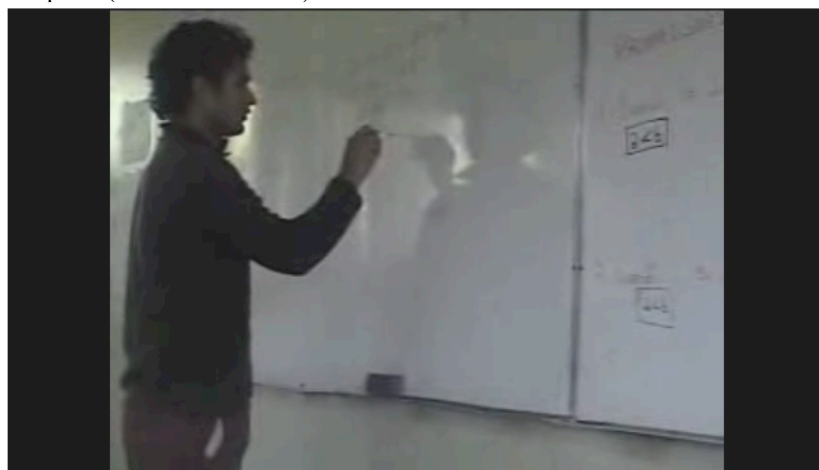


Fig.3. Orchestration of the activity using technological tools and making comparisons between advantages and disadvantages of using paper and pencil techniques (see case 8).

CONCLUSIONS AND WORK PROSPECTIVE

The different ways teachers displayed to integrate mathematics technology into their classroom instruction allows us to obtain qualitative appraisal of the state of the development of their craft knowledge (Ruthven, 2007; 2002) in relation with the incorporation of mathematics technology into their classrooms.

It becomes feasible to predict the progress of the teachers that participated in this exploratory study in relation with the process of learning to use technology for teaching mathematics, in accordance with the extended PURIA Model (Zbieck and Hollebrands, 2008). The next stage may be possible using pedagogical tools that allowed teachers to become involved in assessing or noticing their students' mathematical thinking (see for example Jacobs, Lamb & Philipp 2010; or Herbst 2010). It really would give account that participant teachers would have attained the last stage in the PURIA model (the Assessment mode).

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FOSTERING CHILDREN'S MATHEMATICAL UNDERSTANDING OF AREA MEASUREMENT

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This study explores the importance of 2-Dimensional geometry integrated with numerical calculations for area measurement using guided question-and-answer instruction as a way to develop fourth-grade children's conceptual understanding of area measurement and ability to solve area measurement problems. Furthermore, the enriched curriculum that integrated 2-Dimensional geometry with area measurement facilitates children's reasoning about the relationship between the perimeter and area of a figure.

INTRODUCTION

Children's conceptions and skills of measurement are mainly nurtured through school curriculum and instruction. Area measurement is an important domain of school mathematics (Lehrer, 2003). Unfortunately, middle- and upper-grade children's experiences in learning area and perimeter measurements are found to have been limited to rote learning and application of formulas (Tan, 1998; Bright & Hoeffner, 1993). Such limitations may lead to children's deficiency in understanding common measurement concepts and incompetence in solving area measurement problems (Tan, 1998). To enhance children's mathematical understanding of linear and area measurement, research-based curricula for mathematic education should be developed and their effects on children's learning should also be examined through classroom teaching practice.

Measuring perimeter and area involves relating numerical quantity with a geometric attribute (Sarama, Clements, Swaminathan, McMillen, & González Góme, 2003; Lehrer, 2003). However, the development of the relational knowledge seems difficult for elementary school children. Recent work suggests that constructing instructional activities that involve students in reflecting, explaining, reasoning, and communicating while performing measurement tasks may promote children's ability to integrate spatial and numerical structure of linear and area measurement (Battista, 2004; Schifter, Bastable, Russell, & Woleck, 2002). In accordance with the perspectives mentioned above, XXXX (year) argued for the importance of connecting geometry with area measurement and numerical strategies using guided question-and-answer instruction as a way of developing children's conceptual understanding of area measurement and ability to solve area measurement problems. The present study aims at exploring and extending the previous accounts of measurement thinking by investigating the effects of instructional treatments on children's mathematical reasoning in measuring area and perimeter.

THEORETICAL FRAMEWORK

Children learn area measurement and formulas for finding areas from middle to upper grades. Moreover, for conceptual understanding of the area formula of a rectangle “Area = Length x Width” ($A = L \times W$), which is the base for reasoning other formulas for area measurement, children need the ability to coordinate the three aspects of knowledge, spatial-geometric relationships, sense of numbers and numerical calculations, and structuring the rectangular arrays (Battista, 2003; Sarama et al., 2003). Although developing the relational knowledge of these three aspects seems difficult for elementary school children, mathematics researchers suggested that instructors and curricula play critical roles in facilitating children’s understanding and integrating these aspects of knowledge (Battista, 2003, 2004; Sarama et al., 2003).

Accordingly, for teaching materials of area measurement in school mathematics, mathematics educators and researchers propose that knowledge of 2-Dimensional (2-D) geometry and spatial structuring serves as a foundation for aiding children’s construction of space, which in turn facilitates their understanding of the structure of 2-D rectangular arrays (Battista, 2003, 2004), the area formulas and their relationships (Craine & Rubenstein, 1993; Burns & Brade, 2003). The knowledge of 2-D geometry includes the properties of basic shapes, congruence, and geometric motions (flips, turns, translations, and decomposition and composition); while spatial structuring is defined as the mental operations of constructing an organization for an object in space (Battista, 2003, 2004; Sarama et al., 2003).

In addition, mathematics researchers argued that heavy emphasis on and experiences with 2-D geometry explorations and geometric motions instead of numerical computations seem appropriate for aiding children’s learning of area measurement; in particular, their understanding of the rationale of area formulas (Bright & Hoeffner, 1993; Craine & Rubenstein, 1993; Burns & Brade, 2003). Hence, it is assumed that providing children with more experience with geometric reasoning and explorations regarding area formulas instead of concentrating on numerical computations will enhance their performance in solving area measurement problems.

With respect to curriculum development in terms of area measurement, XXXX (year) raised the importance of developing geometric concepts and exploring the area formulas through a series of problem-solving activities. These activities should be embedded with conceptual characteristics of area measurement for improving children’s conceptual understanding of area measurement and formulas. In their study, the teaching unit that involved only 2-D geometry and geometric motions was found to contribute little to children’s performance in solving area measurement problems. However, it is argued that the effect of the teaching unit was perhaps confounded by the types of problems utilized in the measures. There were more area measurement problems that required numerical calculations than mathematical justification and explanation problems that demanded higher-level understanding of area measurement. Furthermore, most of the problems that required numerical calculations contained figures given with grids. It was found that the children adopted

directly the count-and-add strategy to determine the areas of the given figures, which in turn refuted the effect of the teaching unit. Therefore, one of the purposes of the current study was to investigate whether a teaching unit that contained mathematical subject-matter elements with respect to 2-D geometry and geometric motions would enhance children's ability to solve problems that required mathematical reasoning and explanations for perimeter and area measurement.

In this study, it was hypothesized that children benefit more from receiving the enriched curriculum, which integrated 2-D geometry with area measurement (GMAM), and the geometric reasoning curriculum, which highlighted the 2-D geometry and geometric motions (GM), compared with receiving the area measurement curriculum, which stressed the numerical calculations for area measurement (AM). The current study aimed to address the following two main questions.

1. What is the influence of the three instructional treatments on children's ability to solve area measurement problems?
2. What is the influence of the three instructional treatments on children's reasoning toward relationship between the perimeter and area of a figure?

METHOD

This study used a quasi-experimental design to compare the effectiveness of the three instructional treatments, namely the GM, AM, and GMAM curricula, on children's ability to solve area measurement problems. In addition to the three instructional treatment groups, there was also a control group comprising children who received a regular mathematical unit, which was irrelevant to 2-D geometry and area measurement, during the instructional treatment period. Each curriculum was conducted in five class periods using guided question-and-answer instruction. Each class period lasted about 40 minutes. The children's learning was assessed prior to and after the instructional treatments by the pretest and posttest, respectively.

Participants

One hundred and thirty-six children, 74 boys and 62 girls, were recruited from four fourth-grade classes in a public elementary school that serves middle-class communities in Keelung city, Taiwan. The children were 9.78 years old ($M = 117.35$ months, $SD = 3.47$). All participating children had already received instruction in the basic concepts of area measurement, including the area formula $A = L \times W$. The four classes were randomly assigned to the three instructional treatments and the control group. An ANOVA test revealed no significant differences in pretest performance among the four classes, $F(3, 132) = .38, p = .77$.

Materials

The sets of teaching problems used in the three instructional treatments and the questions used in the pretest and posttest were adopted or revised from the materials developed by XXXX (year). Moreover, the problems regarding measuring perimeter

and size comparison were added to the materials. The mathematical concepts underlying the teaching problems included eight mathematical subject-matter elements in different combinations for the three sets of teaching problems. The subject matter elements were (A1) Decomposition and re-composition; (A2) Geometric properties, shapes (e.g., rectangles, parallelograms, and triangles), congruence, geometric motions, and superimposition; (B1) Meaning of perimeter and perimeter measurement; (B2) Meaning of area measurement; (B3) Discovery of other area formulas on the basis of the formula $A = L \times W$; (B4) Application of formulas for area measurement and numerical calculations; (C1) Size comparison; and (C2) Examples of changing a shape without changing the area.

The specific features of each curriculum were as follows: (a) The AM curriculum which stressed using area formulas and numerical calculations for area measurement comprised 28 teaching problems. The mathematical subject elements embedded in the teaching problems included A1, B1, B2, B3, B4, and C1. (b) The GMAM curriculum which was a spatial-and-area-measurement connection curriculum, involved 2-D geometry, geometric motions, area formulas, and numerical calculations for measuring areas. This curriculum also comprised 28 teaching problems embedded with the eight subject elements mentioned above. (c) The GM curriculum which emphasized conceptual understanding of the rationale underlying the formulas for area measurement involved extensive experiences with geometric reasoning, manipulations, and explorations of area formulas. The concepts such as congruence, translations, decomposition and composition, and examples of shape change without area change were emphasized as a problem-solving strategy for discovering area formulas and size comparison, whereas numerical operations for measuring areas were less stressed. This instructional treatment comprised 28 teaching problems embedded with mathematical subject elements A1, A2, B1, B3, C1, and C2.

When teaching the problems embedded the size comparison (C1) element, cutouts were utilized to reason the relationship between shape change and its size measurement by applying congruence and decomposition-and-composition skills in the GM treatment. Nevertheless, numerical calculations were less emphasized in the GM treatment. On the contrary, measuring operations by using a ruler and numerical calculations for area measurement were stressed in the AM treatment. Moreover, the approaches involved in both the GM and AM treatments were integrated in the GMAM treatment when teaching the problems embedded with C1 element.

Types of problems used in the three sets of teaching materials and assessments

There were three types of problems that required different levels of mathematical thinking and responses contained in the curricula and assessments. (a) Numerical Area Calculation (NAC) problems. These problems which demanded numerical calculations for determining the areas of the given figures could be solved by means of either counting the grids given with the figure or directly applying area formulas and calculations. According to Kennedy and Lindquist (2000), counting and

doing simple calculations require lower-level conceptual understanding. (b) Mathematical Judgment (MJ) problems. These problems required judgment for the accuracy of a given solution statement regarding area measurement. (c) Explanation (EXP) problems. These problems required a written explanation on the reason for justifying the judgement given to a corresponding MJ problem. The ability to make mathematical judgements and to explain the reasoning while solving problems represented mathematical thinking of a higher order (Kenney & Lindquist, 2000). Thus, both MJ and EXP problems can be applied to evaluating students' conceptual understanding.

All three types of problems were included in the pretest and posttest. In the current study, the MJ problems were combined with the EXP problems into MJ-EXP problem pairs. There were 11 problems embedded with area measurement concepts involved in the pretest and posttest, respectively. Each test contained five NAC problems and six MJ-EXP problem pairs.

The rubric schemes for the NAC, MJ, and EXP problems have been described by XXXX (year). Each correct arithmetical equation with a numerical answer to a NAC problem was scored from 0 to 5 points. When scoring the MJ-EXP problem pairs, two steps were taken. First, each judgment item was scored 0 or 2 points and each explanation item was scored from 0 to 2 points according to the correctness and completeness of the mathematical ideas. Next, summing up the scores for these two items yields the total score for each MJ-EXP problem pair. Hence, the maximum total score of the pretest was 49 points, and so was the maximum total score of the posttest. The reliability of the pretest and posttest was examined by 30 fourth-graders from a public school in Taipei. The results showed that the mean scores in the pretest and posttest were 25.02 ($SD = 14.03$) and 22.68 ($SD = 14.66$), respectively. The correlation between the pretest and posttest was .58, $p < .001$.

Moreover, 25 randomly selected pretest and posttest sets were independently scored by two raters. Pearson correlations displayed that the interrater agreement reached $r = .99$, $p < .001$. As to the reliability of the scoring and coding of the children's responses to the EXP problems, the results of Kappa analyses was at .89, $p < .001$.

In order to examine pre- to posttest changes with respect to children's reasoning of measurement, the children's performances in explaining the respective EXP items of the perimeter-and-area reasoning problems on the pretest and posttest, which required reasoning toward the relationship between the perimeter and area of a figure, were classified into four categories. They are (a) score gained: partial (or full) scores gained in both EXP items of the problems on the pretest and posttest; (b) score-gained- and no-score-gained: partial (or full) scores gained in the pretest but no scores gained in the posttest; (c) no-score-gained and score-gained: no scores gained in the pretest but partial (or full) scores gained in the posttest; (d) no-score gained: no scores gained in both EXP items of the problems on the pretest and posttest. The frequency and percentage of each category was calculated for each group.

RESULTS AND DISCUSSION

Table 1 exhibits the means and *SD* of the total scores of the pretest and posttest, and scores of the NAC problems and MJ-EXP problem pairs as a function of group. On the posttest as a whole, an ANCOVA with the pretest score as the covariate yielded a significant difference among the four groups, $F(3, 131) = 8.55, p < .001$, partial $\eta^2 = .16$. LSD follow-up tests showed that the GMAM group performed better than the GM group and the control group. Moreover, the AM group outperformed the GM group and control group.

Since there was no difference between the AM and GMAM groups on the total posttest scores, an additional analysis was conducted to analyze the children's posttest performances on the NAC problems and MJ-EXP problem pairs on the posttest. For the NAC problems, the ANCOVA with the NAC pretest score as covariate displayed a significant difference among the four groups, $F(3, 131) = 5.15, p < .01$, partial $\eta^2 = .11$. LSD follow-up tests showed that both the GMAM and AM groups performed better than the GM group and the control group. Moreover, the performance of the GMAM and AM groups were equally good.

Table 1. Means and SD of the pretest and posttest total scores, and scores of the NAC problems and MJ-EXP problem pairs by instructional treatment

Group	n	Pretest M (SD)	Posttest M (SD)	Adjusted M
Total Scores				
AM	34	22.93 (12.07)	30.94 (12.02)	32.27
GM	32	25.73 (12.49)	26.33 (11.56)	25.63
GMAM	35	25.19 (12.15)	33.14 (10.35)	32.84
Control	35	25.23 (10.78)	26.23 (12.23)	25.90
NAC problems				
AM	34	10.09 (7.41)	15.09 (7.73)	15.97
GM	32	11.80 (7.94)	11.67 (7.41)	11.50
GMAM	35	12.46 (7.97)	15.17 (6.98)	14.59
Control	35	11.74 (7.46)	11.73 (7.51)	11.59
MJ-EXP problem pairs				
AM	34	12.84 (5.86)	15.85 (5.30)	16.12
GM	32	13.94 (5.31)	14.66 (5.44)	14.20
GMAM	35	12.73 (5.26)	17.97 (4.59)	18.31
Control	35	13.49 (4.82)	14.50 (5.74)	14.34

For the MJ-EXP problem pairs, the result of the ANCOVA with the MJ-EXP

pretest score as covariate revealed a significant difference among the four groups, $F(3, 131) = 7.94, p < .001$, partial $\eta^2 = .15$. LSD follow-up tests showed that the GMAM group outperformed the other three groups.

As to the pre- to posttest changes in terms of children's reasoning of measurement, the McNemar test was performed to analyse the frequency change between the pretest and posttest for each group. The results revealed significant change in the GMAM group between the pretest and posttest, $\chi^2(1, N = 35) = 5.88, p < .05$. In contrast, the McNemar test results for the AM, GM, and control groups were $\chi^2(1, N = 34) = .36, p = .55$, $\chi^2(1, N = 32) = .00, p = 1.00$, and $\chi^2(1, N = 35) = .64, p = .42$, respectively. In the GMAM group, more children showed improvement on the posttest, whereas the children in the other three groups showed no significant changes between the pretest and posttest.

The results showed that on the posttest as a whole, both the enriched group, which received the spatial-and-area-measurement (GMAM) curriculum, and the AM group, which received the curriculum highlighting the numerical calculations needed for area measurement, outperformed the GM group and the control group. Moreover, since numerical calculations for measuring areas were taught in both the GMAM and AM instructional treatments, the two groups performed equally well in solving the NAC problems, which required numerical calculations. Although the GMAM group did not outperform the AM group on the global performance or on the performance of the NAC problems, the GMAM group was superior at solving problems that required mathematical justification and explanation. These findings are in line with the results previously obtained by XXXX (year) and Sarama et al. (2003).

Furthermore, the evidence that the GMAM group shows improvement in reasoning about the relationship between the perimeter and area of a figure also supports the perspective that a good understanding of linear and area measurement contributes to their reasoning in measuring perimeters and areas (Schifter et al., 2002). These findings suggest that both 2-D geometry and numerical calculations for area measurement have to be incorporated to foster children's conceptual understanding of area measurement and measurement thinking.

Compared with the control group, the group provided with the GM curriculum that highlighted 2-D geometry alone failed to achieve significant enhancement on the global performance or on the performances of the NAC problems and the MJ-EXP problem pairs. Children in the GM group were less likely to connect geometric reasoning with area measurement when solving problems. Such evidence implies that geometric concepts and geometric motions are necessary but not adequate for enhancing children's ability to solve advanced area measurement problems.

IMPLICATIONS FOR MATHEMATICS INSTRUCTION

The findings of this study suggest that integrating area measurement instruction with numerical strategies and geometric materials seems to be a promising approach to promoting children's conceptual understanding of area measurement and their

ability to explain geometric reasoning with measurement when solving problems. The geometric materials and explorations aid children to obtain more insight into the relationships between the properties of different shapes, which in turn facilitates their understanding of the relationships between the area formulas. Such understanding in turn promotes children's competence in solving problems that require higher-order mathematical thinking, such as justifications, reasoning, and communication. Furthermore, instructors should provide students with extensive geometric manipulations and discussion on the geometric properties of basic shapes and the relationships between them, combining the spatial-geometric relationships with numerical strategies for area measurement.

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ASSESSMENT PREFERENCES IN UNIVERSITY MATHEMATICS

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In undergraduate mathematics, there appear to be only a small number of forms of assessment in widespread use and these tend to be those most open to criticism about validity. This paper surveys students' preferences for assessment methods alongside their views about their lecturers' preferences of these methods. We uncover evidence of potential mismatches between how students want their achievement to be measured and what they perceive experts want. The paper shows that – with a few interesting exceptions – students tend to prefer traditional assessment methods over those often labelled 'innovative' and that where there are mismatches between students' and lecturers' perceived preferences, these can be accounted for by apparent ease of preparation.

INTRODUCTION

Mathematics lecturers are often exhorted to be more innovative with their assessment methods. Received opinion suggests that traditional closed book examinations do not capture as much about mathematical ability as more innovative methods such as projects and presentations (Burton and Haines, 1997; Challis, Houston and Stirling, 2004). Still older forms of assessment, such as oral examinations, while still commonplace in many countries, have long since disappeared from Anglo-American mathematics assessment traditions following doubts about fairness and validity (Stray, 2001). However, it is not clear what university students think about how they are assessed nor, in particular, how they wish to be assessed.

There is considerable evidence that, in general, students' perceptions of the skills required by assessment drive the form of learning they adopt. For example, Harlen and Crick (2003) suggest that students will tailor their learning according to their perception of what the form of assessment actually tests. For example, if students perceive one assessment method to test predominantly memory they will adopt surface learning approaches (Marton and Säljö, 1997). That noted, Marton and Säljö's work indicates that it is not enough simply to change assessment methods to change students' learning approaches – it is relatively straightforward to bias students' approaches towards surface learning through the design of assessment, but it is quite hard to bias them towards deeper approaches. For example, Birenbaum and Feldman (1998) suggest that low self-efficacy and poor learning skills are associated with a preference for multiple-choice examinations, while higher self-belief and better learning skills are associated with a preference for essay style examinations. Thus students with varying self-efficacy and learning skills can have their approaches

to learning influenced in quite different ways by the same modification to an assessment method. The relationship between approach and method is built of a complex mixture of learning approaches, assessment methods, self-efficacy, content and perceptions of assessment (Scouller, 1998).

Most of the work cited, however, has been the result of research across a range of university subjects, rather than having a specific focus on mathematics. In the most recent comprehensive review of student assessment preferences (Struyven, Dochy and Janssens, 2005) in which the authors synthesised the work of 19 different empirical research projects involving at least 7000 university students, not a single research participant appears to have been studying for a mathematics degree.

Indeed, despite a strong focus on mathematics education research at university level, little of the work has been focussed on assessment and, in the UK at least, assessment remains very traditional. The closed book examination is still the most dominant form of summative assessment, with some elements of coursework forming a very small component of formative or summative assessment in the first year. Some mathematics education literature has advocated the introduction of “novel” forms of assessment such as projects and presentations (Burton and Haines, 1997; Challis, Houston and Stirling, 2004). Reasons given for this include the opinion held by some mathematicians and mathematics educators that formal written examinations were disproportionately rewarding memory and calculation over other mathematical and employability skills (Berry and Houston, 1995; Houston and Lazenbatt, 1996; Steen, 2006).

Given the evidence from the general education literature of the complex relationship between perceptions, approaches and forms of assessment – which has come from research lacking any specific focus on mathematics – coupled with the paucity of empirical research behind the calls for innovation in university mathematics assessment, we felt there was a need to explore mathematics students’ views of how they are assessed. In particular, following van de Watering, Gijbels, Dochy and van der Rijt’s (2008) exploration of students’ preferences for assessment methods, we asked two key research questions:

1. What are undergraduate mathematics students’ preferred methods of assessment in their subject?
2. What do these students view as their lecturer’s preferred methods?

The answers to the first research question, when compared to the generalist higher education literature, will indicate whether there is something particular about mathematics students’ views. The answers to the second will indicate, in comparison to the answers to question 1, whether there is a mismatch between the way in which students want to be assessed and the way in which they think experts would want to assess them.

METHOD

Birenbaum (1994) developed a comprehensive *Assessment Preferences Inventory* (API): a Likert-scale questionnaire with 67 items measuring seven different areas of assessment (including preparation, cognitive processes and conative aspects). While comprehensive, the full API was too cumbersome for our purposes and we followed van de Watering et al. (2008) in reducing the API to focus only on assessment methods and to reduce and conflate the original 32 assessment method items to 8 which are either in current widespread use or which feature in the literature as alternative ways of assessing mathematics at university level. The 8 assessment methods are listed in Table 1.

Assessment method	Explanatory example
Multiple-choice examination	Test taken in an exam room, where for each question the student can select one response from five possible choices
Closed book exam	Test taken in an exam room, with a separate booklet in which the student writes solutions, but no support material is allowed
Open book exam	Test taken in an exam room, with a separate booklet in which the student writes solutions, but support material is allowed
Weekly examples sheets	Test completed in the students' own time over the course of a week
Projects	A piece of written work submitted in response to a question or problem, undertaken over the course of a number of weeks
Presentations	An oral presentation of the results of a project, undertaken in response to a set question or problem, after working on the project for a number of weeks
Oral examination	Working on a mathematical problem on a chalkboard or piece of paper with a tutor present who can provide suggestions or check errors as you work on it
Dissertation	A substantial piece of written work, on a set topic or problem, undertaken over the course of a long period, such as a term or two ⁴

Table 1: Taxonomy of assessment methods in mathematics

In order to address the two research questions, as part of a larger questionnaire on assessment in university mathematics, we asked participants to address two issues:

1. To what extent would you want your achievements in the course to be assessed by each of the following methods?

2. To what extent do you think your lecturers would want your achievements in the course to be assessed by each of the following methods?

For each question, participants were given the 8 assessment methods and asked to consider them in turn and respond on a five point Likert scale.

The questionnaire was completed by 48 first-year students on mathematics degrees at a high-ranking UK university. The students were randomly selected from the full year group of 152. They completed the questionnaire at the start of a seminar for a core first-year mathematics module.

Given the potential impact of experience on students' preference for assessment methods (Gijbels and Dochy 2006), the study was conducted in the middle of the first term, before students had any substantial experience with university assessment methods. This follows the general practice for empirical work in this field: in Struyven et al.'s (2005) review of the field, the modal sample group was undergraduate students in their first year of study. Clearly, however, all of our students had substantial experience of a variety of summative assessment methods at school (as would be the case for all similar research) and for all of our students that experience for mathematics would be near exclusively closed book examinations.

ANALYSIS

The first key question we asked was whether there was a preference for assessment methods across the students and, if so, what that preference was. A Friedman Test was conducted to determine if the students had differentially rank ordered their preferences for each assessment method. Results of the test indicated that there was a significant differential rank ordering, $\chi^2(7)=109.297$, $p<0.001$. A post-hoc Nemenyi procedure gave an indication of the significant differences between rankings, with a critical value 0.77. This allows us to provide a partial ordering on the students' preferences for assessment methods, as shown in Figure 1.

It is surprising to note that presentations, one of the innovative methods of assessment in use in universities in the UK, is ranked as significantly the least preferred assessment method. Moreover, given that it is unlikely that many students in our sample have any experience of open book examinations, it also seems surprising that this assessment method is ranked alongside the most common assessment methods (closed book examinations and weekly example sheets) as one of those most preferred by the students to measure their achievements.

Group 1	Group 2	Group 3	Group 4	Mean Rank
Open book				6.01
Closed book				5.96
Weekly example sheets	Weekly example sheets			5.33
	Projects			5.03
		Multiple choice		4.19
		Dissertations		3.63
		Oral examinations		3.60
			Presentations	2.16

Figure 1: Partial ordering of assessment methods according to students' preferences

The rank ordering also appears to be in stark contrast to the suggestions made in the mathematics education literature: traditional assessment methods appear much preferred by the students and innovative methods are not. This fits relatively well with the generalist education literature (such as van de Watering et al., 2008) in which students rank written tests highly, projects less highly and with oral tests ranked quite low.

In order to address the second question about students' views of their lecturers' preferences for assessment, a Friedman Test was conducted to determine if the students had differentially rank ordered their responses. Results of the test indicated that there was a differential rank ordering, $\chi^2(7)=110.564$, $p<0.001$ with Nemenyi critical value 0.77. The partial rank ordering is shown in Figure 2.

Unsurprisingly, the highest ranking methods which the students believe their lecturers prefer are precisely those methods which are most represented in the degree course they are taking (albeit that, at this stage, the students have no experience of any summative assessments at university). Only presentations, which the students in our sample will encounter informally in years 1 and 2 and formally in their final year, are both extant methods and poorly ranked.

Again, students do not appear to think that lecturers would prefer to assess using methods which are recommended as innovations in the mathematics education literature.

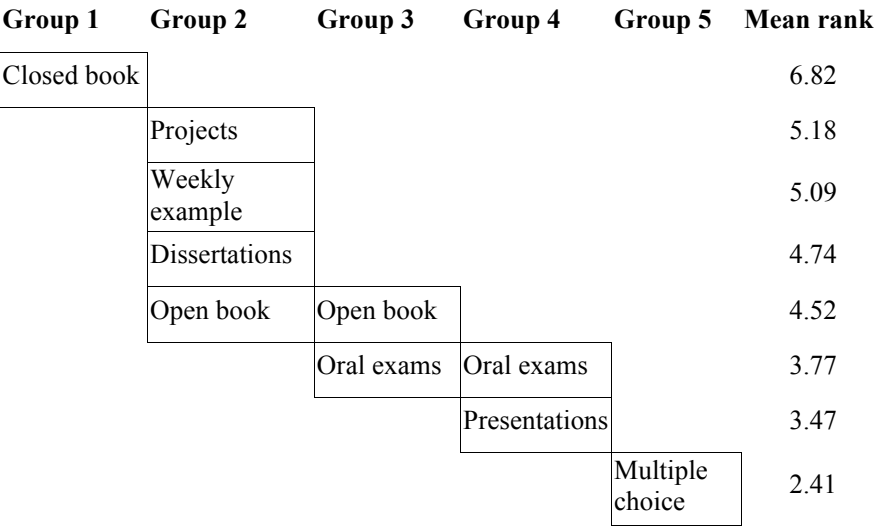


Figure 2: Partial ordering of students’ views of their lecturers preferred assessment methods

A Wilcoxon signed rank test was used to examine the relationship between students’ own preferences and their perceptions of their lecturers’ preferences for assessment methods. This tested for the extent to which their own preference dominated their perceptions of their lecturers’ preference. The results are summarised in Figure 3.

Direction	Assessment methods		
+	Multiple choice	Open book	
	$z=-3.691, p<0.001$	$z=-3.046, p<0.005$	
0	Weekly example sheets	Oral exams	Projects
	$z=0.428, p=0.669$	$z=1.319, p=0.187$	$z=1.509, p=0.131$
-	Closed book	Dissertations	Presentations
	$z=3.046, p<0.005$	$z=3.220, p<0.005$	$z=4.353, p<0.001$

Figure 3: Dominance of students’ own preference for an assessment method over their perception of their lecturers’ preference

Given the earlier conclusions, there is little surprising in this table. Students have a stronger preference for multiple choice examinations and open book than they

believe their lecturers do, and have less preference for closed book, dissertations and, particularly, presentations than they believe their lecturers do. One might account for these differences by the perception that students may hold about the ease of different assessment methods. Zeidner (1987) suggested that students prefer multiple choice examinations over essay style tests for many reasons, most related to ease (e.g. perceived difficulty, levels of anxiety engendered, apparent complexity). One might suggest that open book examinations would be preferred over many other assessment methods for similar reasons.

DISCUSSION

In this paper we have addressed two research questions: the first about students' preferences for different assessment methods and the second about their view of lecturer's preferences. In the first case, we found a general level of agreement with the generalist literature: that students prefer relatively traditional forms of assessment (such as closed book examinations, which remain by far the dominant assessment method in undergraduate mathematics). Surprisingly, however, we found that one non-traditional method, the open book examination, also ranked most highly amongst student preferences. In the second case, we found only minor mismatches between the students' preferences and those they perceive from their lecturers and these could be accounted for by a slight tendency for students to prefer what they think of as easier forms of assessment (or, symmetrically, they think their lecturers prefer harder forms of assessment).

However, the exhortations from what mathematics education literature exists on assessment at the university level seems starkly at odds with our findings about student preferences. For example, students do not appear to wish to present their work orally, nor do they think their lecturers would wish them to do so. This is despite the recommendations by, for example, Berry and Houston (1995). Innovations, then, may be the very last thing that students want.

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PRESERVICE MIDDLE SCHOOL TEACHERS' BELIEFS ABOUT THE PLACE OF PROOF IN SCHOOL MATHEMATICS

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The goal of this multiple-case study was to investigate how preservice middle school teachers' conceptions of proof developed throughout a geometry course for teachers. The results reported in this paper pertain to the development of participants' beliefs about the role and place of proof in school mathematics. Throughout the semester, the preservice teachers started valuing proofs and became aware of the explanatory power of proofs at least for themselves as teachers. However, they had limited knowledge about students' proving abilities and held low expectations in this matter.

INTRODUCTION

Proof is an “indispensable tool of mathematics” (Hanna, 1983, p. 89). It is not only a “sequence of steps each of which has the form of justifying one claim by invocation of another, to which the first claim is logically reduced” (Ball & Bass, 2000, p. 203), but also a “sequence of ideas and insights” (Yackel & Hanna, 2003, p. 228). Given the importance of proof in mathematics, it is also advocated as central to mathematics education at all levels (Carpenter, Franke & Levi, 2003; Hanna, 1995; National Council of Teachers of Mathematics [NCTM], 2000). Despite this advocacy, historically and traditionally, students' experiences with proof have been limited to Euclidean geometry in high school (Ball & Bass, 2003). Hence, middle school teachers have an important responsibility to incorporate proof into their teaching.

Integration of proof into mathematics lessons and using proofs to develop students' mathematical thinking depends heavily on teachers' beliefs about the role and place of proof in mathematics and mathematics education among other factors. However, there is little research concerning teachers' beliefs in these areas with the existing research focusing on elementary and secondary school teachers. Thus, the purpose of the present study was to gain an understanding of how preservice middle school teachers' conceptions of proof develop, which, in turn, might aid teacher educators in preparing these teachers for their future responsibilities. In this paper, I present the results regarding the development of participants' beliefs about the role and place of proof in school mathematics.

THEORETICAL PERSPECTIVE

There are several roles that proof plays in mathematics and mathematics education. De Villiers (1990) argued that the roles of proof in a school mathematics curriculum should reflect the roles of proof in the field of mathematics: verification, explanation, systematization, discovery, communication, and intellectual challenge. Mathematics

educators also emphasize the role of proof in promoting mathematical understanding (Ball & Bass, 2003; Hanna, 1995). De Villiers' framework provided an initial tool to analyze what the preservice teachers in my study said when they talked about how they envisioned the role of proof in their future classrooms.

Harel and Sowder (2007), based on a review of the literature, concluded that teachers "do not seem to understand other important roles of proof, most noticeably its explanatory role" (p. 48). In Knuth's (2002a) study with 16 inservice secondary school mathematics teachers, whereas all 16 teachers mentioned establishing the truth of a statement as a role of proof in mathematics, only three of the teachers mentioned the role of proof as explaining why something is true. Moreover, this was a procedural focus rather than one of promoting understanding. Mingus and Grassl (1999) found that, in their definitions of proof, the secondary education majors emphasized the explanatory power; however, the elementary education majors focused on the verification role of proofs.

In their discussion of the roles of proof in school mathematics, 13 teachers in Knuth's (2002b) study mentioned developing logical thinking skills, and four talked about displaying student thinking, which the teachers thought was beneficial for the student, the audience, and also helped them to assess student understanding. However, proof as a way of promoting understanding was not mentioned by these teachers. In addition, several teachers did not only dismiss proof as a central idea throughout secondary school but also believed that it was for advanced mathematics courses and for students who would study a mathematics-related area. They said that they would accept an empirically based argument as a valid argument from students in a lower level mathematics class. In a more recent study with 78 secondary mathematics teachers, Kotelawala (2009) found that "a majority of the teachers disagreed with the expectation of students being the ones proving" (p. 254).

METHODS

The course that provided the context of my study was a mathematics content course designed for preservice middle school teachers with a focus on geometry. Two of the objectives of the course as listed on the class website were "to strengthen the understanding of and the ability to explain why various procedures and formulas in mathematics work and to promote the exploration and explanation of mathematical phenomena." Although to construct proofs was not listed as a particular course objective, the instructor engaged the class in proving several formulas and theorems.

Examining how preservice middle school teachers' conceptions of proof evolved required rich descriptions of the participants' conceptions of and experiences with proof throughout the semester. Thus, a multiple-case study (Merriam, 1998) with each participant being a "unit of analysis" (Patton, 2002, p. 226) was an appropriate method as "a qualitative case study seeks to describe that unit in depth and detail" (p. 55). I used purposeful sampling (Patton, 2002) and selected six participants in order to represent a broad spectrum in terms of the variety of knowledge and beliefs about

proof they brought to this course. Five of the participants (Adam, Brenda, Kate, Nora, and Tammy) were undergraduates in the middle school education program. One participant (Casey) was a graduate student in the middle school master's program.

The main source of data for this study was three semi-structured interviews (Bernard, 2002, chapter 9) conducted with six participants at the beginning, middle, and end of the semester. All interviews included open-ended questions to elicit participants' conceptions of proof as well as statements for them to prove and arguments that went along with each statement to validate. These data were supplemented by surveys given to all students taking the class at the beginning and the end of the semester. After examining each participant as a case and focusing on the development of their conceptions of proof throughout the semester, I conducted a cross-case analysis to reveal patterns in the data.

RESULTS

In this section, I first present as a background participants' experiences with proof prior to this course and how their attitudes toward proof changed throughout the semester. Next, I discuss their beliefs about the role of proof in school mathematics.

Experience with and attitude toward proofs

All of the participants except Casey had limited experiences with proofs prior to this course. They either did not remember or vaguely remembered doing proofs in high school and rarely mentioned their college level mathematics classes when they talked about their previous experiences with proof. At the beginning of the semester, participants also had rather unfavorable beliefs about proofs. Kate started this course believing that proving is "an overwhelming task" and that proof is "not something that everyday people deal with" but "something that the mathematicians are working with" (Interview 1). Similarly, when I asked Tammy how we know that something is true in mathematics, she said, "Proofs and theorems" (Interview 1). However, she added, "I don't know why it's that way. I know that it is because it is, and that's what the math people say." Brenda also indicated that before taking this course, she would have said that she knows that something is true in mathematics "because someone else told me" (Interview 1). Nora said that thinking about proofs made her nervous because she had not worked much with them. What she remembered about proofs from her high school geometry class was "hating them" (Interview 1). Andy also disliked proofs in high school because he was "that person, like, if you tell me, you know, if a teacher is telling me it's pretty much true, and I believe it, you don't have to prove it to me" (Interview 1).

As the semester progressed, participants started developing positive attitudes toward proof. Instead of being this overwhelming task that is only for mathematicians to deal with, Kate started viewing proofs as something that she could do and feel successful about. In the third interview, she explicitly stated this change in her attitude toward proofs:

I used to think about proof as something that like I couldn't do. That it's like for mathematicians and like really, really smart people can do that, but I mean that's not something that I need to know. Um, but now I know that I can do a proof, and it's not as intimidating or scary as I thought it was going into it. (Interview 3)

Andy also had different feelings at the end of the semester: "after learning about proofs ... I actually like [mathematics] more, 'cause you are like, 'Oh now I can figure this out'" (Interview 3). Throughout the semester, he went from believing that a proof is "just a waste of time" to believing that proofs are "useful in math" (Interview 3). Nora talked about how much sense it made to be able to explain why something works as opposed to just learning procedures for doing things:

When I was in school like I was given, um, like this is how a lot of procedures like this is how you do things but it makes so much more sense when you, um, when you can explain like how and why that works every time. (Interview 2)

Andy, Kate, and Tammy emphasized that they learned in this course how important proofs were. The participants gave several reasons that proofs were important: communicating, convincing, building on knowledge, establishing truth, explaining why, and creating a common base of knowledge. In all, they mentioned all the roles that proof plays in mathematics outlined by deVilliers (1990).

Place of proof in school mathematics

When participants discussed the role and place of proof in mathematics education, they acknowledged several factors as to why it is important to learn proofs. They also developed an appreciation of proofs for themselves as teachers. Tammy, Kate, Nora and Casey believed that it was important for them as teachers to learn proofs and hence why things work because as teachers they would need to explain things to their students. Knowing proofs would provide them with the explanations that they would need to know to teach mathematics. For example, Nora said:

I think that it's important to, for the teacher to have a good understanding, a good basis of the proofs cause the students are gonna ask, you know, why is that work or why is that work every time, you know, what about this time. (Interview 1)

Kate also indicated that if a student says, "I don't understand why that's true" the teacher can prove it to them, in other words she or he "can explain it to them" because she or he has "the reasons behind it" (Interview 3). In addition, she believed that being able to prove things helps her to "know where something is derived from" and to develop a "deeper conceptual understanding" (Interview 3), which, in turn, help her as a teacher to be able to evaluate different student solutions and decide whether they are valid or not. In this way she would be able to "encourage all different kinds of thinking" (Interview 3) among students.

In the last interview; Tammy also talked about the importance of knowing proofs for herself as a teacher. She believed that as "the curriculum now is moving people more toward like discovering it and seeing why it works" (Interview 3) the teachers have to

know why. She believed that “if we can't even prove it, like, understand it ourselves, there is no way we can explain it to thirty kids and five classes” (Interview 3).

Casey emphasized the importance of being convinced of a statement's truth for her as a teacher so that she could teach that to her students. She said; “I'm not going to teach anything that I'm not convinced is true” (Interview 1). When Casey talked about the role of proofs in mathematics education, she stressed that she viewed them as being important more for the teacher. Also, with the level of understanding that she developed as a result of learning proofs, she believed that she would be “ready for any questions that students will have” (Interview 1).

Although the participants started viewing proofs as being important for themselves as teachers, they all had reservations about the place of proof in middle school. All participants definitely valued having students explore, investigate, and be able to reason for themselves; however, they either did not know whether middle school students would be able to construct proofs or believed that only more able students would be able to do so. For example, Andy, Kate, and Tammy mentioned that as they had not worked with middle school students yet, they did not know what to expect from them as far as proofs were concerned.

Andy said that because he did not take geometry in middle school he was not sure what middle school students could do as far as proofs are concerned. However, he also added that if that was one of the standards that he would need to teach then he would like the students “to be able to do that and understand why” (Interview 1). At the end of the semester, even though he indicated that he would expect to see proofs playing out in his classrooms more than during his own experiences as a student (Interview 3), he was not sure if middle school students would be able to come up with proofs by themselves. Kate expressed similar beliefs at the end of the semester as she thought that it would be “too hard” (Interview 3) for middle school students to come up with proofs by themselves. She believed that “it's good for the students to be able to, not necessarily do the proofs, but to know the ideas behind it so that they can generate formulas on their own and not just memorize stuff” (Interview 3). Toward this end, she would have them “try things out on their own, manipulate things, talk in their groups” (Interview 3) but not expect them “to be able to do like proving interior angles of a triangle are 180 degrees or proving the Pythagorean Theorem” (Interview 3). On the other hand, she acknowledged that she had not been in a middle school classroom and that she did not know what students' abilities were.

On the other hand, Brenda and Tammy, although Tammy had admitted that she did not know enough about middle school students, indicated that they would expect only gifted students or students who were more interested in mathematics to be able to come up with proofs as opposed to “regular” students. Casey also reserved her presentation of proofs for more advanced students who would question things.

Tammy did not think that middle school students could come up with proofs by themselves. Although she recognized her lack of experience with middle school students, she still did not think that she would expect proofs from regular students:

I haven't worked with enough middle school students ... but I don't think they're that advanced. Like if you had like a gifted math class maybe. But as for like a regular, just standard middle school student, I don't think that, unless they really liked math.
(Interview 1)

Although Brenda did not totally dismiss the idea of proof in middle school, she said that it would depend on the level of mathematics that she would teach. She would probably incorporate proof in a general mathematics class as an extra thing but would grade students on it in a higher level mathematics class. She also made a distinction between students based on their mathematical abilities in terms of what they can learn about or do with proofs:

This would apply to the, to the students that were a little higher up or that math came a little easier to and were into that reasoning kinda thing to prove to them, give them an actual proof from math I think would be good. I think the slower kids of the class wouldn't understand and it may confuse them and you may have to set it aside for them.
(Interview 1)

In the second interview, she also talked about how an argument that she did not believe would be accepted as a mathematical proof would be enough for middle school students. She further believed that the general argument proving that the exterior angles of a triangle add up to 360 degrees is something to be expected only from gifted students. She also admitted; however, that she did not know "how many proofs there are that would be, uh, that a middle schooler could actually understand" (Interview 3).

With respect to how she would use proofs in her classroom, Casey said:

I think we'll use a lot of explanations, um, I picture myself having my kids prove things a lot but not necessarily with formal proofs, um, I would ask them to do what I would call a more informal explanation that would be convincing. (Interview 1)

Although she wanted her students to provide explanations and to be able to reason through things, she would reserve more general questions and "actual" proofs for her more advanced students: "I think if you come across some more advanced students you could certainly start asking harder questions like well, 'How do you know this would always be true?'" (Interview 1) Furthermore she said, "I think it would be nice to be able to pull out a proof if I needed to with more advanced student that's really having questions" (Interview 1). In this way she would "challenge my [her] students further" (Interview 1).

DISCUSSION AND IMPLICATIONS

A promising result of this study was that preservice teachers started to think about proofs as arguments that explain why a statement works and to realize the importance

of proofs for themselves as teachers. However, another common theme was that most of the participants questioned the value of proofs for middle school students. Some of the participants indicated that they would expect proofs from gifted students or students who are more interested in mathematics, which is consistent with the findings of an earlier study (Knuth, 2002b). One difference between the participants in the present study and Knuth's participants was the amount of experience they had had. Knuth's participants were experienced inservice teachers, and participants in this study were preservice teachers. That raises the question of whether experience alone would influence the way these preservice teachers would start thinking about the role and place of proofs in middle school classrooms or whether they would need more explicit guidance.

Knuth (2002a) suggested that meaningful experiences with proof might eventually influence the way that teachers think about proofs in the classroom:

In short, teachers need, as students, to experience proof as a meaningful tool for studying and learning mathematics. Experiences of this nature may influence the conceptions of proofs that they develop as teachers, and these ideas, in turn, may influence the experiences with proof their students will encounter in secondary school mathematics classrooms. (p. 403)

However, my study suggests that Knuth's conclusion may not hold. The participants in the present study had considerable exposure to proof in the manner advocated by Knuth but did not see it as something middle school students can do. Furthermore, according to the results of a study by Wilcox, Schram, Lappan & Lanier (1991) investigating the influence of a teacher education program on prospective teachers' pedagogical content knowledge in mathematics, although the studied program "seemed to be a powerful influence on prospective teachers' thinking about mathematics for themselves, its impact did not seem to carry over to how they thought about mathematics for young children" (as cited in Borko & Putnam, 1996, p. 693). Hence, the participants in this study might have had difficulty transferring their experiences to similar experiences for students.

An important implication for teacher education is that as teacher educators we need to focus on the role and place of proof in school mathematics in methods classes. This focus could span a wide variety of topics ranging from the place of proof and reasoning in the NCTM standards, what research tells us about students' conceptions of proof, and what kinds of tasks and types of questions are likely to lead students to a better understanding of proofs. There is evidence that even young students can engage in proving activities (Ball & Bass, 2003; Carpenter et al., 2003), and thus it is important for prospective teachers to be equipped to reveal that potential.

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MATHEMATICAL PROCESSES AND U.S. SECONDARY TEACHERS' CONCEPTIONS OF INTEGRATED MATHEMATICS CURRICULA

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This study builds on previous work exploring U.S. teachers' conceptions of integrated mathematics curricula. In this study we used focus group and individual interviews to examine the mathematical processes teachers emphasized in their descriptions of implementing a new curriculum. Our results suggest that the mathematical processes teachers use in their classrooms play an important role in supporting the development of connections in an integrated curriculum and that a change in the organization of content may in fact provide new opportunities for teachers to consider and enact mathematical processes in their instruction.

Although integrated curricula are common internationally, Georgia is currently the only state in the U.S. that mandates an integrated secondary mathematics curriculum; in the rest of the states, most secondary schools organize into separate courses topics in algebra, geometry, elementary functions, and statistics and probability. This research is part of a larger study (de Araujo, Jacobson, Singletary et al., 2010) that characterized U.S. secondary teachers' conceptions of integrated mathematics curricula in the context of the statewide implementation in Georgia of the integrated ninth-grade course, Mathematics 1. The data collected to characterize conceptions of integrated mathematics also uncovered the mathematical processes that teachers described emphasizing in their instruction.

In this paper, we address the research question: How are teachers' conceptions of integrated mathematics related to the mathematical processes emphasized in their descriptions of implementing an integrated mathematics curriculum? We present three cases that illustrate different ways that participants' conceptions of integration interacted with the mathematical processes they described. This analysis illuminates earlier work that had identified correspondences between conceptions of integration and mathematical processes. We conclude by discussing the implications of our results for teacher development.

LITERATURE REVIEW

The phrase *integrated mathematics* may be interpreted in a variety of ways. The National Council of Teachers of Mathematics (NCTM) Task Force on Integrated Mathematics concluded, "Integrated mathematics seems to have many meanings and interpretations. [NCTM] found discussions about integrated mathematics to be problematic because participants may each have in mind their own, sometimes

different, definition or interpretation of what it means to ‘integrate’ mathematics” (Dickey et al., 1997, p. 3).

Usiskin (2003) described the importance of connections in mathematics as “one of the fundamental characteristics of mathematics” (p. 28) and proposed that integrated mathematics curricula organize mathematics so that the connections within mathematics and other content areas are emphasized. Usiskin also discussed a categorization of curriculum components based on size. From smallest to largest, the sizes of curriculum are the problem, the lesson, the unit, the course, the mathematics curriculum, and the school curriculum. In *A Mindful School*, Fogarty (1991) described 10 models of integrated curricula. Some models emphasize the connections within and between different school subjects, and one model represents a kind of anti-integration in which each subject of the school curriculum is taught in isolation with little internal coherence.

NCTM (2000) presented five process standards in the *Principles and Standards for School Mathematics*. These standards—communication, connections, multiple representations, reasoning and proof, and problem solving—are meant to play an important part in teaching and learning mathematics. The Georgia Performance Standards (GPS) also include similar process standards: technology, reasoning and proof, communication, connections, and multiple representations. The greater emphasis on mathematical processes in the new curriculum indicated “a shift towards applying mathematical concepts and skills in the context of authentic problems and for the student to understand concepts rather than merely follow a sequence of procedures” (Georgia Department of Education, 2006).

PRIOR RESULTS AND METHODOLOGY

The purpose of the larger study was to understand teachers’ conceptions of integrated mathematics curricula. We formulated our research design with the assumption that teachers’ conceptions might not be explicitly held. Therefore, we elicited detailed descriptions of teachers’ practice in order to infer their conceptions. Descriptions of practice also involved unsolicited descriptions of mathematical processes.

We recruited 27 Mathematics 1 teachers from 16 secondary schools in 9 school districts in northeastern Georgia. Of our participants, 9 were male; 7 had 3 or fewer years of teaching experience, and 10 had more than 10 years of experience. We employed a two-stage data-collection design. In the first stage, data from six focus group interviews of 4 to 6 teachers each yielded detailed descriptions of participants’ experiences implementing Mathematics 1. We used maximum variation sampling (Patton, 2002) for the second stage of inquiry, interviewing 9 of the initial participants who had the most diverse views on integrated mathematics curriculum. For analysis, we chunked data into episodes (defined topically or by turn-taking) and used coding that was iteratively refined through consensus among six researchers to identify conceptions of integration and other themes in the data.

Four conceptions of integrated mathematics curricula emerged from our analysis of the data generated from focus group and individual interviews. We developed a framework with two dimensions (Figure 1) to organize the four conceptions and to connect our work to the larger body of literature. The first dimension incorporated Fogarty's (1991) classification of connections between disciplines and those among ideas within a single discipline. The second dimension adopted Usiskin's (2003) sizes of curriculum to distinguish between participants who focused on integration that took place over a short period of time (such as in a problem or lesson) and those who saw integration over a long period of time (such as in a unit, course, or curriculum).

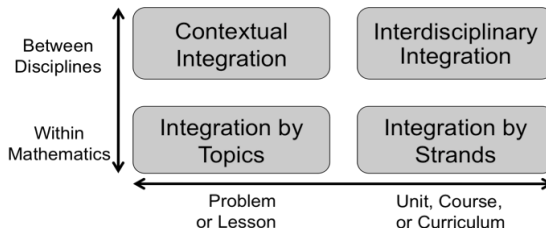


Figure 1. Integrated mathematics curriculum framework.

We tallied the number of times participants discussed a process theme and a conception of integration within a single data episode, and designated strong links as those supported by episodes from three or more focus groups and four or more individual interviews (Figure 2). At first, we were skeptical about the significance of these links because the GPS process standards follow NCTM's recommendations. However, mathematical processes were also emphasized when teachers talked about integrated mathematics they had experienced before Mathematics 1 and when they discussed their ideal vision of integrated mathematics. In April 2010, we presented preliminary results of this correspondence analysis at the NCTM Research Presession (de Araujo, Jacobson, Lowe, et al.). The major finding we reported was that teachers' conceptions of integration were associated with mathematical processes in the sense that when teachers talked about what was entailed by enacting an integrated mathematics curriculum they often discussed particular mathematical processes at the same time.

The correspondence analysis convinced us that mathematical processes were intimately linked to conceptions of integrated mathematics for the teachers we interviewed. For the purposes of this study, we reanalyzed all of the data that linked integration by context, integration by topics, and integration by strands with problem solving and connections. A close analysis of these episodes convinced us that we needed to look holistically at each individual participant. We then reanalyzed all of the data related to the nine individual interviewees, focusing on the relationships between these teachers' conceptions of integrated mathematics curriculum and the mathematical processes they emphasized in their descriptions of classroom

instruction. In the last phase of analysis, we developed descriptive narratives (Chase, 2007) for each interview participant. Each of us individually analyzed all of the data, and collaboratively we wrote the narratives.

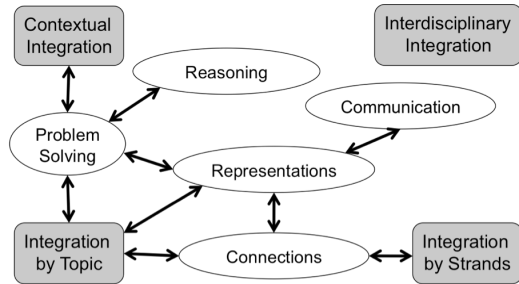


Figure 2. Strong links from the correspondence analysis.

RESULTS

We present three descriptive cases that illustrate the breadth of variation in how our participants' conceptions of integrated mathematics curricula were related to mathematical processes.

The Case of Jeremy—Integration by Topics

After teaching secondary mathematics for 3 years, Jeremy eagerly awaited the transition to the Mathematics 1 curriculum. He believed an integrated mathematics curriculum “would help the students make more connections ... and make more sense of all the branches of math” (Focus Group). However, after the first year of the statewide implementation, Jeremy reasoned, “The way that the state has thrown together Math 1, I don’t think about it as being an integrated curriculum, because they have specifically divided it into isolated subject units” (Interview). Rather, Jeremy believed an integrated mathematics curriculum should connect different strands of mathematics within a single problem or lesson:

I think a curriculum would be integrated if in one problem, or in one task, or in one day, you talk about geometry and then you talk about the algebra involved in that geometric situation. ... The students talk about ... how the algebra and geometry is related. And, then maybe they bring up a statistical discussion. ... I think you definitely have to see aspects of the algebra, geometry at the same time. (Interview)

Jeremy consistently discussed integration at smaller sizes of curriculum, perhaps because of his instructional emphasis on problem solving. For Jeremy, the connections across strands allowed students to “see how math is a unified whole ... and when they see that, I think they develop a better appreciation for it, and they become better problem solvers” (Interview). Mathematical tasks provided opportunities for his students to engage in problem solving using multiple representations.

I think a great problem is a problem that helps the student learn. ... I love problems that show connections in math ... where you can think about it graphically, and you can think about it algebraically, and that thinking both ways helps you solve the problem. I love problems that bring together lots of different pieces of math—problems that help students understand the problem-solving process. (Interview)

Jeremy found that integration increased student motivation and engagement because the students “had all this context to connect to what they were learning” (Interview). To achieve this result, Jeremy used tasks that allowed for student communication:

I make sure [students] talk to one another, to make sure they discuss these ideas with each other. And, if they have different solutions, different approaches, talk about who is right, and make sure they try to come to agreement on whether they both have good solutions or not. (Interview)

Jeremy reported that discussions helped his students notice and appreciate connections across multiple representations.

The Case of Allison—Integration by Topics

Allison, who had taught for 6 years at the time of the interview, volunteered to teach Mathematics 1 because she wanted to get acquainted with the mandated curriculum as soon as possible. Allison described the Mathematics 1 curriculum by saying:

It is integrated in the sense that there is more than one subject being taught during the year, but not so much, in that you don't take one problem and really dig in it and figure out what is going on with it in a lot of different areas.

Allison believed connections should occur at a smaller size of curriculum to achieve mathematical integration, and we classified her conception of integrated mathematics as integration by topics.

If you can talk to me about how to look at something algebraically, ... geometrically, or, graphically, ... that's really, to me, what integrated should be about. [It] is how? Where does this connect? Where else does it live in our world of math? (Interview)

Allison frequently discussed multiple representations, communication, and connections as critical mathematical processes that aid integration. For example, she said, “We talk about function notation, and then we talk about how it looks as an equation, how it looks graphically, and how it looks in a table” (Interview). Allison often expressed the idea that multiple representations support connections between concepts and enable students to achieve a more thorough and conceptual understanding of mathematical ideas. She said, “The integrated [quality] kind of pulls it [multiple representations and concepts] together and lets them see it as one big fell swoop, one chunk of information” (Interview). Describing instruction in the previous curriculum, she said, “When we split it up, ... we just did a whole bunch of tables, and [students] really didn't get that then. They just felt lost” (Interview). For Allison, rich classroom communication was necessary to achieve integration. She discussed the importance of questioning to help students make the connections: “A lot of it is questioning. Just asking them [students], you know, look at this picture, look at this

graph, what do you think? Talk to me about it” (Interview). The importance of multiple representations and communication for Allison’s implementation of integrated mathematics is clear.

The Case of Leandra—Contextual Integration

Leandra was in her 20th year of teaching and was “anxiously awaiting the rollout of Math 1 in the high school; this was my dream” (Focus Group). During both the focus group and individual interviews, Leandra characterized her practice as “a lot of group work” and “a lot of conversation”—features that had not changed much with the implementation of the new curriculum (Focus Group).

The connections between mathematical ideas and the role of context were important features of Leandra understanding of integrated mathematics. Based on Leandra’s experience teaching with *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002), she thought that the integration in Mathematics 1 fell short of the ideal by using problems that were applications instead of context-driven. She defined *context-driven tasks* saying, “The context is the base for the lesson, and then the mathematics comes out of it. ... You start with some really cool context, and then your mathematics just starts to come out of it, which the kids kind of naturally do” (Interview). One example she gave was the “Planning the Prom” task, which was integrated “because it did use statistics and algebra and geometry, not in a forced way; the problem just really lent itself to using all of that” (Interview). Applications, by contrast, used mathematics that students had already mastered and so the context did not drive learning in the same way.

Discussing Mathematics 1, Leandra commented that the GPS authors “did a good job of doing some multiple representations about [an] idea ... looking at algebraic, numeric, geometric representations of almost everything” (Interview). Asked about the mathematical activity of students using an integrated curriculum, Leandra described multiple representations, problem solving, and reasoning:

I would expect [students] to ... model the problem ... physically or with technology. I would expect there to be something in the problem that was changing, some parameter that they could change and see what happens, whether that be something geometric or algebraic. And then ... they could notice pattern, you know. ... And then they would have an opportunity to formalize that. (Interview)

Discussing tasks that are appropriate for an integrated curriculum, Leandra said, “I think mathematically they just need to be rich problems, ... something that you can stay in and talk about different approaches and different representations” (Interview). For Leandra, the mathematical processes of communication and multiple representations were vital for implementing an integrated mathematics curriculum.

Summary

Together, these cases illuminate the links between conceptions of integration and mathematical processes shown in Figure 2. All three cases demonstrate how

mathematical processes were employed by teachers to realize the connections afforded by the new integrated mathematics curriculum. The processes of multiple representations and communication are evidently necessary components of integration across both of the conceptions of integration illustrated in these cases. However, conceptions of integration are not uniquely tied to particular processes; Jeremy and Allison both expressed conceptions of integration by topics, but Jeremy emphasized problem solving, a mathematical process that Allison never addressed.

DISCUSSION

The three cases demonstrate that the mathematical processes advocated by NCTM are intertwined with teachers' conceptions of integrated mathematics and that the mathematical processes teachers use in their classrooms support the development of connections in an integrated mathematics curriculum. The goal of the instruction described by these three teachers is mathematics that is richly interrelated and deeply understood. These cases call to mind Skemp's (1978) description of relational understanding.

Our results suggest that a major curricular change ostensibly concerning content in fact provided teachers with opportunities to engage in new ways of thinking about mathematical processes and instruction. Certainly some, like Leandra, were committed to teaching mathematical processes before the curriculum change. For others, however, the new curriculum made new kinds of instruction possible. Vanessa, another participant, described the initial debate among teachers in her county to rearrange the Mathematics 1 algebra units to be concurrent, then remarked, "When we got all the way to May and saw the whole picture, we got it. It was like no; there's a purpose for ... the order" (Interview). Professional development that is focused on integration, even at the curricular size of a single task, might very well be leveraged to help more teachers develop sensibilities for relational understanding and promote the instruction of mathematical processes.

ENDNOTE

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UNIVERSITY STUDENTS' CONCEPT IMAGE AND RETENTION OF THE DEFINITE INTEGRAL

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This paper reports a part of a larger study researching the retention of key derivative and integration concepts months after the calculus course exam at a Croatian and Danish university. In this paper we focus on the students' long-term retention and concept image of the definite integral. 18 students in non-mathematics science study programmes were interviewed in pairs and presented with four tasks on the definite integral in order to expose their concept definition and concept image. Although it varied, no student had a coherent concept definition which caused problems solving the tasks. However, some students appeared to have a more coherent concept image, and solved some of the tasks, while others did not. We argue that the relation between the concept definition and the concept image varies from student to student.

INTRODUCTION

We have previously (Jukić & Dahl, 2010) investigated the relation between knowledge retention and exam results in differential calculus of university students in non-mathematics science study programmes. This survey showed that the students had forgotten a large part of the concepts and often those with the lowest grades had the better results two months later. Furthermore (Jukić & Dahl, 2011) the Danish students taught in a student-centred course statistically significantly outperformed the Croatian students on a teacher-centred course on the conceptual questions; vice versa for the procedural ones. In this paper we will further investigate the non-mathematics university students' retention and understanding of integrals by examining their concept image two months after the calculus exam.

THEORETICAL BACKGROUND

Concept image and concept definition

The term *concept image* includes all the non-verbal conceptions and associations that an individual has of a concept. It includes the mental pictures, properties, and processes related to the concept. The *concept definition* is the words used to specify the concept. To understand the formal concept definition, an individual creates his own personal interpretation of the definition. The personal concept definition may, or may not, be based on the formal definition (Tall & Vinner, 1981). Tall (2006) states that he and Vinner define a concept image differently. Tall regards a concept definition as part of the concept image whereas Vinner makes a distinction between them. Tall states that this difference had not had significant effect in the use of the

terms in mathematics education research. Through this study, we will also add to this discussion about the relation between the concepts.

Vinner (1991) argues that in the long-term process of concept formation, the relation between the concept image and the concept definition should be established in both directions. However, teachers usually see this as a one-way process assuming that the concept definition will form the concept image. However, students frequently do not use the concept definition in building the concept image (Vinner & Dreyfuss, 1989) and they do not rely on the concept definition but on the concept image when solving problems (Vinner, 1991; Rasslan & Tall, 2002; Rösken & Rolka, 2007). Furthermore, Rasslan & Tall (2002) showed that some English high school students had a concept image of the definite integral but only a small number of students knew and used the concept definition. Similarly, Rösken & Rolka (2007) found that the concept definitions played a marginal role in some German secondary school students' conceptual knowledge of integral calculus and the students mainly leaned on their concept images, which caused difficulties in their reasoning and problem solving.

The concept image is not necessarily coherent but can include contradictions to the concept definition (Vinner & Tall, 1981; Viholainen, 2008). In fact, Juter (2005) showed that university students can have contradictory conceptions about the limit value of a function. Although the students claimed that a function cannot attain its limit values, they considered it possible in problem solving. Viholainen (2008) stated the following conditions for a concept image to be coherent: the conception must be clear; all conceptions are connected to each other; there are no internal contradictions, and the concept image is not contradictory to the formal definition. A coherent concept image is part of the conceptual knowledge where one understands the connections between different concepts, while a coherent concept image refers to a single concept (Viholainen, 2008).

Research questions

What are the Croatian students' concept image and retention of the definite integral, particularly the geometric interpretation of it? How is the relation between their concept definition and concept image?

METHODOLOGY

Two months after the calculus exam, we examined the students' knowledge of integral calculus. Students were interviewed in pairs in order to better establish an atmosphere of confidence and the interviewees may 'fill in gaps' for each other (Arksey & Knight, 1999). Schoenfeld (1985) further states that this kind of interviewing helps alleviate the pressure the students might otherwise feel solving tasks individually. The tasks (Figure 1) were on the geometric interpretation and analytical definition of the definite integral and the calculation of areas. Tasks 3-4 were indirect (Vinner, 1991) in order to expose their concept images while Task 1-2 were directly about their concept definitions in order to see

the difference. Task 4 is from Mahir (2009). She studied first-year university students who passed a calculus course. Only 16 of 62 had solved this correctly. We find the task good at detecting if the students recognize the areas that constitute the whole area as the graph is both above and below the x -axis. We will compare our results with Mahir's.

Task 1. What is a geometric interpretation of the integral

$$\int_b^a f(x)dx?$$
 Give an illustration.

Task 2. What is the analytical definition of the definite integral?

Task 3. Calculate the area bounded by the curves $y = \frac{x^2}{2}$ and $y = \sqrt{2x}$ under the line $y = 2$.

Task 4. The graph of f is sketched to the right. Given that

$$\int_{-2}^5 f(x)dx = \frac{39}{8},$$
 find the value v .

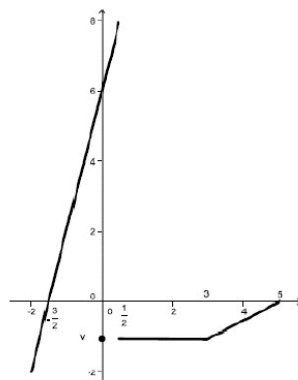


Figure 1: The four tasks (translated from Croatian).

The interviewed students were first-year non-mathematics students belonging to the civil engineering, electrical engineering, or physics study programmes. They were chosen randomly and paired according to study programme affiliation and therefore not strangers but acquaintances, hence we assume more comfortable in the interview where they had to re-create or remember their knowledge (Morgan, 1988). We interviewed nine pairs, eight females and ten males. Before the interview, they were informed that their identity will be kept safe and were acquainted with the purpose of the research. The students received a sheet with the tasks and plenty of empty space to write the answers. They were asked to think their solutions out aloud and if there was a long silence, they were asked what they were thinking at the moment. During the interview, students were also asked to elaborate their claims when they solved the tasks. The interviews were recorded and transcribed. The transcript pieces below are translated from Croatian.

RESULTS OF THE INTERVIEWS

Task 1: Geometric interpretation of the definite integral

All students defined the definite integral as a tool to calculate the area under the curve. However, their geometric interpretation stated only “area under the curve” and not the area between the curve and the x -axis. Their illustrations represented mainly the area above the x -axis on the interval $[a, b]$ except students J & P (Figure 2). When asked, J & P said they would put a negative sign in front of the integral to calculate this area. The figures drawn were very simple and noone addressed the possibility

that part of the area could be above the x -axis and another part below the x -axis. The answers showed that the students' concept definition is not coherent but strongly linked with the image of a certain function and their concept definition seems not to be based on the formal definition. This is similar to Rösken & Rolka (2007) who found, among 24 secondary school students, that the geometric interpretation of the definite integral is connected mainly with the image of the area above the x -axis.

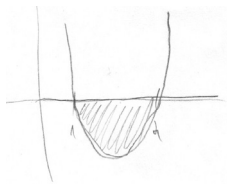


Figure 2: Area image drawn by students J & P.

Task 2: Analytical definition of the definite integral

All students knew that the area under the curve is approximated by rectangles and that the approximation becomes more accurate as the number of rectangles increases and they all used visual representations to elaborate their statement (Figure 3).



Figure 3: Approximating the area with rectangles, students M & D.

11 of 18 students had a coherent concept definition about the position of the rectangles – i.e. if they can be placed above or below the curve or mixed. This number emerged due to a conflict between students V & W who disputed if the rectangles can be positioned just above the curve or under the curve as well.

W: These are rectangles [draws a figure with rectangles about the curve]. So so small rectangles. One part is always sticking out. We strive to get as many rectangles as possible to divide the curve, i.e. the area under the curve, to minimize the loss.

I: Is it possible that the rectangles can be inscribed, under the curve?

W: No. I think that cannot be the case. Then you get a smaller area than what we are looking for.

V: And here you have larger one.

W: But here are minor losses.

V: How do you know? They can be equal.

W: So they taught us [laughs]. If this is the curve, and these are the rectangles [draws inscribed rectangles], there will be always some losses ...

V: But, if you draw very small rectangles, and they [losses] will be very small, so it will not affect much [draws two figures similar to Figure 3]. The area is bigger here [pointing to the first figure, to the circumscribed rectangles]... and here it is smaller [pointing to the second figure, to the inscribed rectangles]. In both cases, it is not the same as the area under the curve... but they are very close... So the rectangles can be above and under the curve.

W: Hm... I do not know... I do not remember it that way.

The students knew the constitutive connections in the analytic definition but most were unable to define it adequately. For instance the students knew that they had to divide it into rectangles but they did not know what to do next. Just one student gave an appropriate definition using formal mathematics expressions. This suggests that the students' concept definitions are weakly related to the formal analytic definition.

Task 3: Calculation of the area

Six pairs had serious difficulties with the shape of the curves. To overcome this, they calculated a table of x,y values or squared the expression and used “the new” curve. They knew that the area was to be calculated using an integral but they had problems defining the limits for the integration. The shape of the area caused difficulties since both the curve $y = 2\sqrt{x}$ and the line $y = 2$ represented the top of the area. In the calculus course, students learnt to “subtract” the upper and lower curves. The combination of two curves both being the upper bound for the area made a conflict in their concept image of the geometric interpretation of the definite integral as the area for seven pairs. One pair, V & W, wanted to calculate the area above $y = 2$. However, two pairs avoided the problem with the two upper bounds by switching variable for integration. They rotated the figure and posed the problem in terms of y . This way they had only one upper curve and one lower curve (Figure 4). Hence, their concept image may be more coherent since they had a more flexible use of the concept.

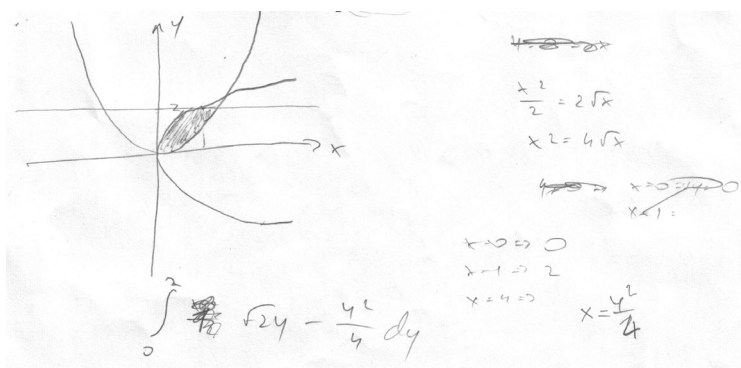


Figure 4: Solution to Tasks 3 by students M & D integrating by y .

Task 4: Using areas to determine the desired value

The students had problems determining the limits of the areas and six pairs marked the area from $x = -3/2$ to $x = 0$ not bounded from the bottom (an example of the six pairs is seen in Figure 5). The second difficulty arose between $x = 0$ and $x = 1/2$ where some marked the area under the x -axis, not bounded for the bottom. A seventh pair, D & M, claimed that the task was not well-defined due to the part under the x -axis.

M: I do not see which areas are included in the whole area.

D: And this under the x -axis between 0 and $1/2$?

I: No

M: How come this under the x -axis from $-3/2$ to -2 is included and from 0 to $1/2$ is not? Also, M & D wanted to include the part between the y -axis and the line $y = 4x + 6$ on the interval $[0, 1/2]$ (Figure 6). Their confusion made them give up the task. This is interesting as M & D made a creative solution to Task 3 (Figure 4), but we now see that this might have been their way of coping with a “hole” in their concept image. Task 4 exposed the hole and left no way around it. Only two pairs marked the area correctly.

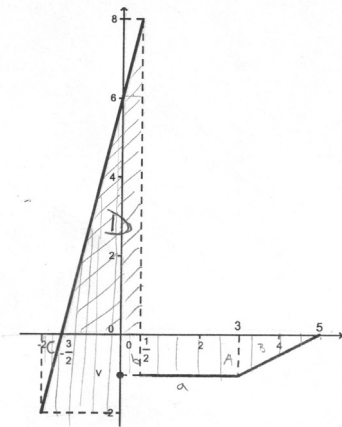


Figure 5: Areas marked by J & P.

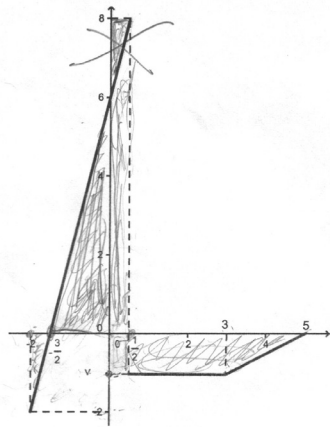


Figure 6: Areas marked by M & D.

It seems that the students' concept definition of the integral influenced their ability to see, or not to see, which areas constitute the whole area. This was seen when the students were told by the interviewer that they had marked the wrong area parts. Six students repeated the geometric definition of the integral as the area under the curve and gave this as an argument for their wrong answer. The students did not discuss the fact that the area must be bounded both from above and below, although they used this fact in Task 3, and therefore had difficulty understanding why they should mark the area only above the x -axis from $x = -3/2$ to $x = 1/2$. Some pointed with fingers to the figure and argued that the line stops at $y = -2$, and “therefore this part under is also included”.

I: What was confusing for you?

V: This part... the line [$y = 4x + 6$]... when it goes under the x -axis.

Furthermore, all students claimed that the task was too difficult to solve even though it included only lines and not other types of curves. All had trouble with calculating the value v and 14 students did not by themselves mark the appropriate area, but they had less trouble in Task 3 than with Task 4. Hence, facing the area problem from a different perspective exposed a conflict within their concept image. They were accustomed to tasks like Task 3, which suggests that their reasoning sequence follows the following pattern “given function - figure to draw - area to be calculated - integral as tool for calculating the area”. Changing the order exposed deficiencies in the students’ concept image.

CONCLUSIONS

From the first two tasks it appeared that no student had a coherent concept definition of the geometric interpretation of the definite integral, although some students’ definitions were more coherent than others. Tasks 3-4 aimed at exposing their concept image. In Task 3, all students had problems coping with two upper boundaries although two pairs found it by integrating by y . In Task 4, all but two pairs made erroneous conclusions concerning what constitutes the given area. The students’ incoherence of the area concept image might, on the one hand, have been foreseen from their answers to Task 1, which asked for the concept definitions and where all students only drew the area on one side of the x -axis. On the other hand, as discussed above, the concept definitions may not always be part of the concept images. The study therefore also adds to the Tall-Vinner discussion about if the concept definition is part of the concept image or different from it. We saw that six students in Task 4 referred to the concept definition when the interviewer told them that the area marked was wrong. At least for these students, their concept definition was part of their concept image. On the other hand, four students marked the appropriate area in Task 4, hence exposed a coherent concept image without having shown a coherent concept definition in Tasks 1-2. Hence, we argue that the relation between concept definition and concept image varies from student to student. It also reveals that the long-term retention of the concept image of the definite integral is weak even though the students two months earlier had passed the calculus course.

Task 4 was from Mahir (2009) where 16/62 students solved it correctly. Mahir did not focus on the concept image but we assume that the 16 students had a coherent concept image since they could not otherwise have solved it. These were mathematics students, hence they usually develop different mathematical concepts than non-mathematics students (Maull & Berry, 2001; Bingolbali et al., 2007). The two samples are not representative, hence not comparable, however the correct understanding rate was quite alike since 4/18 of the non-mathematics students in our study marked the correct area. Hence, the task was difficult to both student groups.

Acknowledgement

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TEACHERS PARTICIPATING IN A RESEARCH PROJECT ON LEARNING: THE SPONTANEOUS SHAPING OF RESEARCHER-DESIGNED RESOURCES WITHIN CLASSROOM TEACHING PRACTICE

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Teachers participating in a research project on the learning of algebra with CAS technology spontaneously adapted the resources designed specifically by the researchers for the project. Analysis of the classroom-based observations of teaching practice showed that adaptive shaping occurred with respect to all three key features (the mathematics, the students, and the technology) of our researcher-designed resources, whether our intentions with respect to those features were explicitly stated or implicitly suggested. Using the framework of the “documentational approach of didactics,” the results highlight the differential role that the same resources can play vis-à-vis the dialectical processes of ‘documentational genesis’ whereby resources are viewed as both shaping and being shaped by individual teaching practice.

INTRODUCTION AND BRIEF LITERATURE REVIEW

Mathematics education research has, over the years, yielded numerous resources. But little is known about the ways in which teachers take on such research-based resources and adapt them to their own needs. In 2000, Adler proposed that, “mathematics teacher education needs to focus more attention on resources, on what they are and how they work as an extension of the teacher in school mathematics practice” (Adler, 2000, p. 205). However, much of the resource-related research has focused more on the mathematical design of the resources (e.g., Ainley & Pratt, 2005) or on their impact with respect to student learning (e.g., Hershkowitz et al., 2002), rather than on the ways in which the resources are used by teachers.

The manner in which teachers adapt researcher-designed resources – as opposed to commercially-based resources – is a new area of research, but one that fits into the recently emerging frame of the *documentational approach of didactics* (Gueudet & Trouche, 2009, in press). Within this frame, documents are considered central to didactic phenomena – but so too is the teacher, as indicated by the pivotal construct of *documentational genesis* with its dialectical processes involving both the teacher’s shaping of the resource and her practice being shaped by it. Building on a distinction introduced by Rabardel (1995), Gueudet and Trouche (in press) emphasize that not only does the teacher guide the way the resource is used, but also that the affordances and constraints of the resource influence the teacher’s activity. As they point out, “design and enacting are intertwined.” However, within this framework, little research has of yet used the design characteristics of given resources as a focal lens

for studying teachers' enactive shaping of them. Remillard (2005), in her review of the research literature on teachers' use of mathematical curricula, argues that features of the curriculum matter to curriculum use as much as characteristics of the teacher and that such research is rather unexplored terrain. In the spirit of Remillard, this report uses the main features of the researcher-designed resources as a backdrop for analyzing their adaptation by participating teachers within their teaching practice.

METHODOLOGICAL CONSIDERATIONS

Participants

When our research team began to create task-sequences for the technical and theoretical development of algebra learners, we also contacted several mathematics teachers. In a workshop setting, the volunteer teachers gave us feedback regarding the nature of the tasks that we were conceptualizing. The week-long workshop included discussions regarding the main mathematics-related and technology-related intentions of the researcher-designers. After modifying the task-sequences in the light of the teachers' feedback, we observed their integration of the task-sequences into their regular mathematics teaching of Grade 10 classes over a five-month period.

The three teachers who are featured in this report all taught in the same city and thus shared a common curricular experience. They are named T1, T2, and T3 (and we use the masculine gender to refer to each of them). T1, whose undergraduate degree was in economics, had been teaching mathematics for five years. He considered his class of students to be of medium-high mathematical ability. T2, who was the most mathematically qualified of the three teachers, had previously taught college-level mathematics during 8 years, before teaching at the secondary school for another 8 years. T2's students were in the top mathematics class. T3, whose undergraduate degree was in the teaching of high school mathematics, had five years of experience at the secondary level. T3's students were weaker in algebra than the other students.

Three key features of the researcher-designed resources

This study is part of a larger program of research whose first phase was oriented toward student learning (see, e.g., Kieran & Drijvers, 2006). The second phase, oriented toward teaching practice, included secondary analyses of the first-phase video-data; these provide the foundation for this report. The analysis centres on teachers' adaptations to the three key features of the researcher-designed resources – the mathematics, the students, and the technology – with a particular focus on whether the adaptations related to an explicit or an implicit aspect of the resources. The student versions of the task-sequences, presented in the form of Activity packets, constitute a central component of the researcher-designed resources; however, the resources also include the accompanying teacher guides, the particular CAS tool that was used (along with its guide), and the discussions that were held during the workshop sessions regarding the spirit embedded within the textual materials, as well as any ad hoc conversations that occurred with the teachers after each of their lessons.

Mathematics-wise, all of the task-sequences involved a dialectic between technique and theory within a predominantly exploratory approach, with many open-ended questions. In brief, the intended mathematical emphases included: i) coordinating the technical and theoretical aspects, ii) pattern seeking, inductive and deductive reasoning, and developing technique, and iii) conjecture making, testing, and proving. Student-wise, we built into the task-sequences questions where the students would be encouraged to reflect on their mathematics, and also indicated moments where they would be expected to talk about their mathematical thinking during whole-class discussions. Technology-wise, all of the task-sequences involved technical activity with either the CAS, with paper and pencil, or with both. We viewed the CAS as a mathematical tool that, through the task, stimulates reflection and generates results that are to be coordinated with paper-and-pencil work. The CAS served thus as a confirmation-verification tool and/or a surprise generator (producing results that would likely not be expected by the students). Additional expected technologies included the CAS view-screen and the blackboard.

The issue of explicit versus implicit researcher-designer intentions

The teacher guides included many specifics that were directed to the teacher alone. Firstly, they offered explicit suggestions as to the precise mathematical content that might be addressed within the collective discussions. Secondly, they presented a few examples that illustrated, pedagogically-speaking, how a particular topic might be further explained at the blackboard. But, in general, the teacher guides did not elaborate on the student-related or technology-related intentions of the researcher-designers. For example, the teacher guides did not specify how to conduct the collective discussions – how to encourage reflection, how to inquire into student thinking, how to have students share their thinking with their classmates during the collective sessions, how to use the blackboard to help students coordinate their CAS and paper-and-pencil techniques, or how to orchestrate theoretical discussions.

The specificity of the students' written task-questions was intended, in a sense, to help fill in some of the gaps regarding that which was not communicated explicitly. Thus, the teacher guides, which included a copy of the student task-questions, were a blend of the implicit and the explicit. Explicit within the structure of the task-sequences were the mathematical aims, the issues on which students were expected to reflect, and the ways in which the CAS and paper-and-pencil technologies were to be used. Implicit was the fact that all three of these were to be combined and coordinated, as well as a manner for doing so, within the collective discussions.

A few theoretical remarks regarding both the implicit and its adaptation are in order. In all reading of text, the reader has a part to play. This notion is discussed in many theoretical writings, including Otte's (1986) complementarist position on the dialectic between textual structure and human activity, as well as Remillard's (in press) view that "the form of a curriculum resource includes, but goes beyond, what is seen." Nevertheless, as argued by Helgesson (2002, p. 34): "What is implicit, and thus

unstated, is not necessarily less clear (or obvious) or less direct than what is explicitly stated; in other words, that an assumption is implicit does not mean that it is hidden and hard to find, or realized to be there only after some reflection.” Helgesson, who defines *implicit* as that which is implied, understood, or inferable – tacitly contained but not expressed – points out that the tone and style in which the text is written may also say something about what it is intended to communicate. In keeping with Helgesson, we consider as implicit those unwritten and unspoken aspects of the researcher-designed resources that can be inferred from what was explicitly stated, those aspects that could be said to be in the spirit of what was communicated directly. Also, like Helgesson, we would argue that the implicit does not necessarily require any additional reflective interpretation than that which is called upon for the explicit.

TEACHERS’ CLASSROOM ADAPTATIONS OF THE RESOURCES

The two task-sequences that are the focus of this report are Activities 6 and 7 (for the full set of task-sequences: <http://www.math.uqam.ca/apte/indexA.html>). Activity 6 was related to the factoring of $x^n - 1$, for integral values of n (a task-sequence inspired by Mounier & Aldon, 1996). Activity 7 dealt with the use of factoring to solve equations with radicals. The extracts analyzed from Activity 6 will bear on adaptations made to the *implicit* aspects of the design, with examples drawn from the practice of T1 and T2. Activity 7 will focus on adaptations related to changing or reorganizing an *explicit* aspect of the design, with examples from T3’s practice.

Examples of Adaptations Observed During the Unfolding of Activity 6

This analysis begins with the adaptations made to the implicit, unwritten and unspoken, aspects of the researcher-designed resources. For our first of two examples drawn from Activity 6, we examine the beginning of the first collective discussion within the activity, where T2 conveyed his particular style for dealing with mathematical issues of a technical and theoretical sort (see Figure 1). The context was Question 2d: *How do you explain the fact that the following products $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ result in a binomial?*

T2: [while writing at the board] When you expand this $(x-1)(x+1)$, and add all your terms you get (x^2-1) . Agree? And for the other one $(x-1)(x^2+x+1)$ the same idea, I multiply the -1 throughout, getting $-x^2-x-1$, and that is going to give you x^3-1 . What do you notice about the middle parts?

Ss (several students, all at once): They cancel out.

T2: They cancel out, because the x just elevates the degree of everything, and when you bring the -1 , all the middle terms will cancel. You are going to have your x^3 because you elevated the degree, but you are going to have your -1 at the end as well, and everything in the middle will cancel out. That is why without doing any algebraic manipulations, if I did $(x-1)(x^3+x^2+x+1)$, I notice that these (x^3+x^2+x+1) are just a decreasing degree of x , so without doing any distributing, you figure out the results.

Figure 1. Extract from discussion surrounding Question 2d in T2’s class

The technique and the theory of the mathematics are being talked about. But notice that T2 is not drawing these aspects from the students, but is rather presenting them himself. If one could say that our general intention about coordination between technique and theory has not been disregarded, our implicit intention with respect to fostering personal mathematical reflection in students, and on inquiring into their thinking, is clearly set aside by T2's intervention. This is in contrast with T1's style of orchestrating a whole-class discussion, as is seen with the next example involving elements from the subsequent task of Activity 6.

For the factoring of $x^4 - 1$, the CAS had not yielded what the students had expected: not $(x-1)(x^3 + x^2 + x + 1)$, but rather $(x-1)(x+1)(x^2 + 1)$. In T1's class, the following discussion ensued (see Figure 2).

T1: What does it turn out is the case?
 S1: Sometimes they like factor even more.
 T1: What we did initially is not wrong. It's just not complete. ... So for $x^4 - 1$, it's what?
 S1: $(x-1)(x+1)(x^2 + 1)$ [teacher writes at the board: $x^4 - 1 = (x-1)(x+1)(x^2 + 1)$]
 T1: So let's look at this one. How can we go about getting that without the calculator?
 S2: Use the rule.
 T1: Is that right (as the teacher writes at the board: $(x-1)(x^3 + x^2 + x + 1)$)
 Class: Yeah.
 T1: And what do you do from there?
 S2: Group it.
 T1: And how do you group it?
 S2: [student explains how she would group the second factor, as the teacher writes at the board that which she dictates]
 T1: That's one way of doing it. Bob [S3]?
 S3: [the student Bob then describes how he would factor $x^4 - 1$ by first breaking the x^4 part into two equal halves]
 T1: What concept have you used?
 S3: Difference of squares [the student continues his explanation of the technique, which the teacher writes at the board as per S3's dictation]
 T1: So both ways reconcile the differences, coming in from different points of view.

Figure 2. Extract from the discussion on the factoring of $x^4 - 1$ in T1's class

The extract provided in Figure 2 illustrates the ways in which T1 adapted the researcher-designed resources by filling in some of the unstated gaps in the teacher guide. He inquired into students' thinking and used this as a basis for discussing some of the different approaches to factoring completely $x^4 - 1$. This was done with the stated aim of reconciling the differences between the unexpected result produced by the CAS and the paper-and-pencil result yielded by the general rule. T1 also displayed on the blackboard the various factoring approaches offered by the students, which thereby presented a public record of their different techniques.

Examples of Adaptations Observed During the Unfolding of Activity 7

Our analysis continues, this time bearing on the adaptations made to explicitly-stated aspects of the researcher-designed resources, with examples drawn from the practice of T3. Mathematics-wise, our primary intention in Activity 7 was to make students aware of the possible loss of solutions when they simplify an equation by dividing both sides by some factor. Students were thereby to be directed towards the more reliable solving method of isolating terms on one side and using the zero-product theorem, that is, “a product is zero iff either one of the factors is zero”. The teacher’s guide suggested a way of handling the class discussion related to lost solutions and their verification with the CAS. In brief, the central explicit components related to the first three tasks of Activity 7 concerned, in this order: (a) a focus on the meta-level aspects of solving a particular equation containing common factors with radicals, (b) the actual solving of a related equation having a similar pattern of common factors (but without radicals) and which could induce a loss of solutions, and (c) the verification by CAS of the paper-and-pencil solutions which would lead for many students to a required reconciliation of the two sets of solutions.

T3 carried out several adaptations to these explicit aspects of the task-sequence. For Equation 1, $5(\sqrt{x-4})^3 + 11\sqrt{x-4} = (2x+1)\sqrt{x-4}$, a meta-level reflection question asked students how they would approach the solving of this equation. Immediately afterward, the written task-sequence directed them toward the simpler equation $(y-2)^3 - 10(y-2) = y(y-2)$ (Equation 2), which was the one to be actually solved. But from the start, while reading aloud and rewording the instructions, before anything had been done by his students, T3 suggested replacing $\sqrt{x-4}$ by a .

This adaptation interfered with our intention of having students recognize by themselves in what facet Equations 1 and 2 have the same structure, and to what extent the solving steps they were asked to sketch for Equation 1 could be put to the test by actually solving Equation 2. As well, we note that T3 did not follow the explicitly-given sequence of holding off on the class discussion until after the students had worked on both Equations 1 and 2 and had tested the solutions of Equation 2 with the CAS. Following his too early and wordy discourse on Equation 1, T3 had students work briefly on Equation 1, but in fact never asked them how they viewed it at a meta level.

T3 then wrote a transitional equation on the board, $5(a-3) + 2(a-3)^2 = 3(a-3)^3$, and proceeded to illustrate his recommended substitution technique by replacing $(a-3)$ by x , suggesting that students apply this technique to the solving of Equation 2. This was an adaptation that not only further confounded our initial intentions with respect to students’ seeing structural similarities between the two equations, but also presented an added mathematical difficulty for the students: Equation 2 conveying a term in both y and $y-2$, the substitution of x for $y-2$ gives either a two-variable

equation or a term in x and $x+2$. The transitional equation introduced by T3 did not involve such a hindrance. As the students began working on Equation 2, one did complain that the substitution of x for $y-2$ gave him an xy term, which got him stuck.

A further adaptation by T3 concerned his use of the CAS technology. For the third question, which asked students to check their solutions with those produced by the CAS, T3 chose to eliminate it. Having introduced Equation 2 with a view-screen display of the three solutions yielded by the CAS, he subsequently asked the students to find the same three solutions with paper and pencil. Thus, the surprise realization that there might be three solutions, and how it came to be that one of them had been lost through their paper-and-pencil techniques, was never provoked in T3's class.

DISCUSSION

Our analysis of the spontaneous shaping of the researcher-designed resources indicated adaptive activity not only in all three of the key features of the task-based resources (the mathematics, the students, and the technology) but also in their coordination. Researchers (e.g., Freeman & Porter, 1989) have argued that, if teachers' guides were more explicit and less ambiguous, the degree of closeness between teaching practice with these resources and the intentions of the resource designers could be greater. Our findings are in disagreement with such arguments that suggest that greater detail will necessarily lead to a closer following of curriculum materials. No matter how explicitly expressed the researcher-designers' intentions may be, adaptation of the resources will take place. Our analysis of the nature of the adaptations that were forged with respect to both the implicitly-suggested and explicitly-expressed intentions of the researcher-designers showed that, in both intentional domains, teachers will adapt the resources that they use. Clearly, teachers' past experiences guided the ways they used the resources; however, discussion of this issue is beyond the scope of the present report (for the complete analysis from which this abridged research report is drawn, see Kieran, Tanguay, & Solares, in press).

In closing, our findings regarding the various ways in which teachers adapted the researcher-designed resources cast light on a particular aspect of the theoretical frame of the *documentational approach of didactics*, namely the differential role that the same resources can play within that process of *documentational genesis* whereby resources occasion the shaping of individual teaching practice. The implicit and explicit aspects of the researcher-designed resources served as both affordances and constraints that influenced teachers' activity. Resources are not neutral; they speak to different teachers in different ways – even to teachers using the same resources and sharing the same goal of participating in a research project aimed at developing the technical and theoretical knowledge of algebra students within a CAS-supported environment. It is important to note, however, that the different ways in which the same resources were shaped were by no means irrelevant or insignificant in nature; they either promoted or impeded the emergence of different techniques and theoretical-conceptual elements in students. But that is a whole other story.

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HOW DO STUDENTS GENERALIZE A CONJECTURE THROUGH PROVING?: THE IMPORTANCE OF BOUNDARY CASES BETWEEN EXAMPLE AND COUNTEREXAMPLE

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A “discovery” function of proof indicates that one can generalize a conjecture from an analysis of its associated proof. This function is important in mathematics education because it would enable us to achieve more productive learning of proof and proving. In order to deepen our understanding of students’ behavior, this paper analyzes an experiment with a pair of 9th graders, and explores how they generalized their conjecture through proving. In the experiment, after they proved their conjecture and faced its counterexamples, they applied their proof to a boundary case between examples and counterexamples of their conjecture. This application was crucial for their generalization which was also true in the case of the counterexamples. This paper further discusses a prospect to construct a more sophisticated framework for proofs and refutations in mathematical learning.

INTRODUCTION

Proof and proving have played an important role in mathematical research. For example, de Villiers (1990) pointed out several functions of proof in mathematics: verification/conviction, explanation, systematization, discovery and communication. In order to achieve mathematical learning that is intellectually honest to a discipline of mathematics (Stylianides, 2007), these functions should be also emphasized in mathematics education.

Among various functions of proof, this paper focuses on “discovery”, meaning that one can discover or invent new results through proving; and in particular, one can generalize a conjecture from an analysis of its associated proof (de Villiers, 1990, 1998). If this discovery function is incorporated into mathematics classrooms, students could regard proving as valuable not only for the verification of given statements but also for the invention of new conjectures.

While many researchers have explored students’ behavior on “explanation” seemingly regarded as a main function of proof in mathematics education (Hanna, 1995), there have been only a few empirical studies on the discovery function. In one, for example, Miyazaki (2000) indicated essential conditions for students to discover new statements from analysis of proof, and illustrated these conditions with 8th graders’ activities. In addition to such normative research, it is also important to carry out descriptive studies in which we can deepen our understanding on students’ behavior. This paper therefore explores how students generalize their conjectures through proving.

THEORETICAL FRAMEWORK

A goal of this study is to achieve mathematical learning based on ‘fallibilism’, an approach advocated by Lakatos who insisted that “informal, quasi-empirical, mathematics ... grow(s) ... through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations” (Lakatos, 1976, p. 5). He further described, with an illustration of Descartes-Euler conjecture on polyhedron, the process in which, after imaginary students and their teacher proved a conjecture, they refined the conjecture and proof variously through coping with counterexamples. Toward a mathematical learning that mirrors such mathematical process, this paper analyzes students’ activities from a Lakatosian perspective.

Though Lakatos (1976) indicated various methods to cope with counterexamples (for example, ‘monster-barring’), this paper focuses on ‘lemma-incorporation’ and ‘increasing contents by deductive guessing’ or simply ‘deductive guessing’. Suppose that one proves a conjecture and then faces its counterexamples, which Lakatos (1976) named ‘global counterexamples’. In lemma-incorporation, one analyzes the proof first, and then discovers a certain part of the proof refuted by the counterexamples (this part was called the ‘guilty-lemma’). After that, one restricts the domain of the conjecture by incorporating this guilty-lemma to a condition of the conjecture.

Lakatos’ second method (deductive guessing) is to invent a more general conjecture which also holds true in the case of the counterexamples of the primitive conjecture. It seems that this generalization has not been enough deliberated in mathematics education research (for example, Larsen & Zandieh, 2008). However, Lakatos formulated the deductive guessing as the fifth heuristics rule, and stated “if you have counterexamples of any type, try to find, by deductive guessing, a deeper theorem to which they are counterexamples no longer” (Lakatos, 1976, p. 76). In addition, Lakatos illustrated this generalization on Descartes-Euler conjecture, in which a certain polyhedron (‘picture-frame’) that had been a global counterexample of the primitive conjecture was changed to an example of a more general conjecture.

Thus, this paper addresses the process from lemma-incorporation to generalization. Of course, any investigation by students may be mathematically inappropriate because they are not usually familiar with lemma-incorporation and generalization. This paper therefore uses students’ standpoints to decide whether students engage in these activities, not from a purely mathematical standpoint.

METHOD

This paper analyzes an experiment which the author carried out with a pair of Japanese 9th graders, Sakura and Yuna (pseudonyms). They attended a public junior high school, where in Japan students begin to learn geometric proof from the 8th grade. Before this experiment, the author also carried out a questionnaire on basic contents of proof among all of 9th graders in the school. Sakura’s and Yuna’s results on the questionnaire were good, and this suggests that they were strong in proof and proving.

The task in the experiment was related to equilateral triangle (Figure 1). The meaning of the task was ambiguous for Sakura and Yuna, and as a result its ambiguousness brought their interesting activities. In the experiment, the students had access to worksheets and manipulative objects representing equilateral triangles. The author observed their activities and sometimes asked questions (for example, raised a counterexample) in order to facilitate their activities. The experiment was recorded and transcribed; the transcripts were analyzed with the focus on the process in which the students generalized their conjecture through proving. The video record and what they wrote on the worksheets were also utilized for data analysis.

Question 1. On two equilateral triangles, a smaller triangle is placed on top of a bigger one as Fig. 1-1. Then, if we join each vertex of two triangles by a segment, we gain two segments which are equal in length (thick lines in Fig. 1-2). When we rotate one of two triangles around point O, how will relation on length of these two segments change?

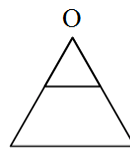


Fig. 1-1

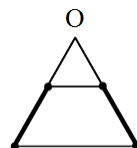


Fig. 1-2

Question 2. Prove your conjecture in Question 1.

Fig. 1: The task in the experiment

RESULTS

Conjecture and its Proof

It took some time for Sakura and Yuna to understand the meaning of this task; for example, they were confused about which parts meant ‘two segments’ (thick lines in Fig. 1-2) when they rotated one of two equilateral triangles. The author therefore showed, with manipulative objects representing the two triangles, the case in which they overlapped each other (hereafter, overlapping case), and illustrated the meaning of two segments in the overlapping case (thick lines in Fig. 2).

This illustration seemed to have a significant influence on Sakura and Yuna; they consistently examined the overlapping case. Yuna then conjectured “(are two segments) always equal in length?”, and Sakura agreed with her conjecture. After that, they wrote its proof as Fig. 3.

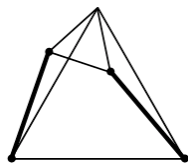


Fig. 2: Two segments

In $\triangle OAC$ and $\triangle OBD$
 Because $\triangle OAB$ is equilateral, $OA=OB \dots ①$
 Because $\triangle OCD$ is equilateral, $CO=DO \dots ②$
 $\angle AOC = \angle AOB - \angle BOC$
 $\angle BOD = \angle COD - \angle BOC$
 Therefore, $\angle AOC = \angle BOD \dots ③$
 Because of ①②③, two pairs of sides and the included angles are equal, and therefore
 $\triangle OAC \cong \triangle OBD$
 Because corresponding sides of congruent figures are equal in length, $AC=BD$

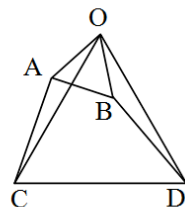


Fig. 3: Students' proof

Global Counterexample and Lemma-Incorporation

After completing the proof, Sakura and Yuna began to think the case where two equilateral triangles did not overlap each other (non-overlapping case). Sakura rotated the manipulative object of the smaller equilateral triangle, and said “(two segments are) equal even if we rotate continuously”. Yuna answered “it will be, for sure”. However, Sakura stated “wait, how about the case where this (smaller triangle) runs off (bigger one)?”, and placed the manipulative objects shown as Fig. 4-1. Then, she considered the ‘two segments’ as thick lines in Fig. 4-1, and noticed that the two segments did not seem to be equal in length; she said “(two segments are) not absolutely equal”. That is, they recognized this non-overlapping case as a global counterexample which refuted their conjecture.

Of course, if one grasps appropriately the correspondence between vertices of two equilateral triangles, ‘two segments’ always stay equal in length (as shown by the thick lines in Fig. 4-2). However, Sakura and Yuna misunderstood this correspondence in the non-overlapping case.

Then, the students attempted to understand why their conjecture became false in this non-overlapping case. They placed manipulative objects of equilateral triangles as Fig. 4-3, and Sakura remarked “these (sides of smaller triangles) are equal, and these (sides of bigger ones) are also equal, but these angles (two dots in Fig. 4-3) are a little strange”. As this remark showed, Sakura analyzed their proof in Fig. 3, and found that the congruency of the included angles in their proof was refuted in the non-overlapping case. Of course, the pair of triangles which they examined from the viewpoint of congruency was discontinuous between Fig. 3 and Fig. 4-3: ‘triangle OAC and OBD’ in the former, and ‘triangle OAD and OBC’ (if each vertex had been named) in the latter. However, Sakura and Yuna did not notice this mathematical difference, and for the students the congruency of the included angles became ‘guilty-lemma’ in terminology of Lakatos (1976).

Yuna also said “well, (are two segments) equal only in overlapping case?”. Thus, she found that it was in non-overlapping case that the congruency of the included angles in their proof became guilty-lemma. She then stated “had we better write this (i.e. our conjecture is true) in the case only where figures overlap?”, and incorporated this overlapping as a condition of their conjecture to restrict its domain.



Fig. 4-1



Fig. 4-2

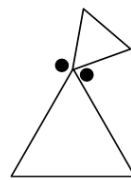


Fig. 4-3

Fig. 4: Non-overlapping case

Examination of Boundary between Example and Counterexample

Afterward, Sakura and Yuna began to examine the border between overlapping and non-overlapping cases; this was a boundary between examples and counterexamples of their conjecture that two segments were equal in length. Yuna said “how about this case?” and placed the manipulative objects of equilateral triangles as Fig. 5-1. At first, Sakura grasped ‘two segments’ as thick lines in Fig. 5-1, and thought that the lines were not equal in length. But, Yuna answered “you connect wrongly”. Yuna then checked the place of point A and C on the manipulative objects, and regarded ‘two segments’ as thick lines in Fig. 5-2.

The students further conjectured that the two segments were also equal in the boundary case, and tried to prove it. For example, Yuna said “(do the two segments) seem to be equal (in the boundary case)?”, but she was not able to prove her conjecture. On the other hand, Sakura rotated the manipulative object of smaller equilateral triangle slightly to overlap two triangles, and verified again that their conjecture was true in the overlapping case. After that, Yuna re-examined the boundary case, and named each vertex of the manipulative objects from point A to D again (Fig. 5-3). She then found that their proof in Fig. 3 was also applicable to the boundary case because triangle OAC and OBD were congruent in the case too; two pairs of sides were equal in length ($OA = OB$ and $CO = DO$), and both of the included angles ($\angle AOC$ and $\angle BOD$) were 60 degrees because they were interior angles of equilateral triangles.

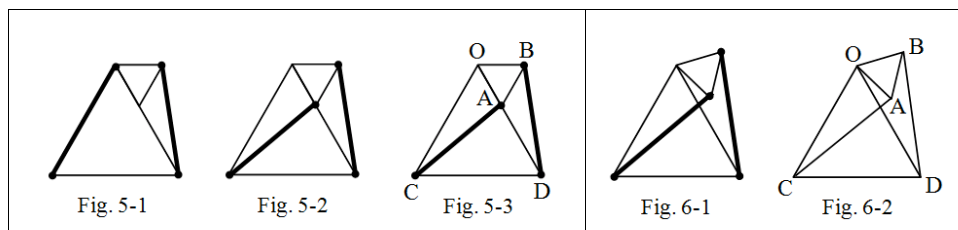


Fig. 5: Boundary case

Fig. 6: Non-overlapping case revisited

Generalization

Next, they investigated again the non-overlapping case. Yuna rotated consecutively the manipulative object of smaller equilateral triangle from Fig. 5-3 to Fig. 6-1, and Sakura identified the ‘two segments’ as thick lines shown in Fig. 6-1. Yuna also drew a diagram similar to Fig. 6-1. She then named each vertex from point A to D, and drew segment AC and BD (Fig. 6-2). The students further noticed that triangle OAC and OBD which they had addressed in Fig. 3 seemed to be congruent in Fig. 6-2 too, and Sakura said “Is it (the conjecture that two segments are equal in length) true in any case?”. Thus, by utilizing their proof in Fig. 3, they generalized their conjecture which had been restricted only to the overlapping case, to the one which also included the non-overlapping case.

After that, Sakura and Yuna attempted to make sure whether their proof in Fig. 3 was really applicable to Fig. 6-2, in particular whether triangle OAC and OBD were congruent in Fig. 6-2. Firstly they confirmed $OA = OB$ and $CO = DO$, but they considered that the included angles ($\angle AOC$ and $\angle BOD$) became equal because the angles were still ' $60 - \angle AOD$ '. Therefore the author asked them to re-examine the angles, and Yuna noticed that the angles were not ' $60 - \angle AOD$ ' but ' $60 + \angle AOD$ '.

DISCUSSION

The Importance of Boundary between Example and Counterexample

At first, Sakura and Yuna judged that Fig. 4-1 was a global counterexample which refuted their conjecture (two segments were equal in length), because they assumed the 'two segments' were the thick lines shown in Fig. 4-1. They further analyzed their proof in Fig. 3, and thought that the congruency of the included angles became 'guilty-lemma' in the non-overlapping case. Yuna then incorporated this overlapping as a condition of their conjecture to restrict its domain. Next, the students investigated the border between overlapping and non-overlapping cases, as a boundary case between examples and counterexamples of their conjecture; they found that their conjecture was true in the boundary. After that, they re-examined the non-overlapping case, and then could identify the 'two segments' as thick lines in Fig. 6-1. The students eventually realized that their conjecture, which had been restricted only to the overlapping case, could be generalized to the one which also included the non-overlapping case.

The boundary case between examples and counterexamples of their conjecture played a crucial role in the process from lemma-incorporation to generalization. In fact, it led Yuna to rotate consecutively the manipulative object of smaller equilateral triangle from its boundary position (Fig. 5-3). In this way, the two students were able to grasp the 'two segments' as thick lines as in Fig. 6-1, not as in Fig. 4-1. Having shifted the meaning (denotation) of 'two segments' from Fig. 4-1 to Fig. 6-1, the two students were then in a position to generalize their conjecture.

Moreover, when the students investigated this boundary, they applied their proof on the overlapping case (Fig. 3) to the boundary. For example, the students had at first identified the two segments as thick lines shown in Fig. 5-1. However, they were able to revise their identification (Fig. 5-2) because Yuna grasped the place of point A and C which they had named in Fig. 3. When the students further attempted to prove that their conjecture held in the boundary case, Sakura also overlapped two triangles by her rotation of the manipulative object, and confirmed again that their conjecture was true in the overlapping case. Their subsequent proof on the boundary corresponded to their proof on the overlapping case too; Yuna deduced the congruency of triangle OAC and OBD according to $OA = OB$, $CO = DO$, and $\angle AOC = \angle BOD$.

In summary, it was essential for this generalization process that the students applied their previous proof (Fig. 3) to the boundary between examples and counterexamples of their conjecture. This conclusion complements the results which mathematics

educators have obtained. For example, Nunokawa (1997) reported that when a graduate student proved a statement, it became a turning point that the student looked into a border case between examples and non-examples of the statement. This paper similarly shows that it is productive to investigate a boundary between examples and counterexamples of a conjecture, because such investigation may lead students to invent a more general conjecture which also holds in the case of the counterexamples. In a related paper, Komatsu (2010) analyzed a similar generalization process of 5th graders. Komatsu there argued that when one faced counterexamples of a conjecture which had been already proved, it became important for such generalization that one attempted to find the part of the proof which was applicable to the counterexamples. The result of this study indicates the significance of such applicability not only to counterexamples but also to the boundary between examples and counterexamples.

Toward a Framework for Proofs and Refutations in Mathematical Learning

This paper introduced the notion of deductive guessing (Lakatos, 1976) for analysis of students' activities. The deductive guessing was to invent a more general conjecture which was also true in counterexamples of the primitive conjecture. Though some researchers (for example, Davis & Hersh, 1983; Larsen & Zandieh, 2008) have utilized Lakatos' research in order to analyze students' activities, they may have not attended to deductive guessing enough. For example, according to Lakatos (1976), Larsen & Zandieh (2008) constructed a framework for understanding mathematical activities in classrooms, and argued that the framework was useful for instructional design too. However, the framework referred to only three types of activity: monster-barring, exception-barring, and proof-analysis (lemma-incorporation).

It is valuable in both normative and descriptive senses that deductive guessing is integrated to a framework for proofs and refutations in mathematical learning. In a normative sense, Lakatos (1976) formulated deductive guessing as the fifth heuristic rule, implying that Lakatos regarded it as an important method in mathematical research. Thus, the addition of deductive guessing to a normative framework opens up valuable mathematical learning. On the other hand, without deductive guessing, this paper could not understand adequately the generalization that Sakura and Yuna engaged in; Komatsu (2010) also dealt with a similar process. Therefore, deductive guessing is essential in a descriptive sense for mathematics educators and teachers to understand proofs and refutations in mathematical learning more sufficiently.

CONCLUDING REMARKS

The students in this study faced counterexamples of their conjecture after they proved it. They then applied their proof to the boundary case between examples and counterexamples of their conjecture, and this application led to their generalization which was also true in the case of the counterexamples. The results of this study therefore suggest that in order to facilitate students' generalization, teachers should

provide their students with opportunities in which they examine similar boundary cases through analysis of their proof.

This paper further discussed a possibility of a more sophisticated framework for proofs and refutations in mathematical learning. While the conclusions of this paper are based on an experiment with two students who seemed to be strong in proof and proving, it is necessary to re-examine these conclusions by refining the framework and observing other students' activities. It is also important to develop effective instruction to achieve mathematical learning where students can engage in proofs and refutations.

Footnotes

The task in the experiment (Fig. 1) and students' dialog (including their proof in Fig. 3) were translated from Japanese to English by the author. Parentheses in the students' dialog were added by the author too.

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CONSTRUCTING THE CONCEPT OF APPROXIMATION

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Students have a tendency to see integral calculus as a series of procedures with associated algorithms and many do not develop a conceptual grasp giving them the desirable versatility of thought. In designing a curriculum that supports an improved cognitive base for a flexible proceptual understanding of the integral and integration, approximation is, in our opinion, an excellent candidate to start with. In this paper, we focus on the following questions: What is the structure of the approximation concept in terms of elements of knowledge? What are operational definitions for these elements that allow the researcher to identify concept construction? How does the process of knowledge construction occur during an instructional intervention?

INTRODUCTION

The research study presented in this paper is situated in the wider context of a study of learning integral calculus in school. The literature (Bagni, 1999; Thomas & Hong, 1996; Thompson, 1994) shows that in current mathematics curricula many students fail to acquire certain desirable conceptual understandings. The large number of students who were unable to make any attempt at solving meaningful integral problems for which there is no obvious procedure shows that their experiences to date have left important gaps in their conceptual understanding. They have a tendency to see integral calculus as a series of procedures with associated algorithms and do not develop a grasp of concepts that allows for versatility of thought. Thus, instead of having a proceptual view of the symbols in integration they have, at best, a process-oriented view. However, it is not surprising that many students find concepts such as the integral difficult when they are unable to experience these processes directly in many classrooms. It should be possible to design curriculum materials that develop an improved cognitive base for a flexible proceptual understanding of the integral and integration. Constructing the integral concept on the basis of the idea of accumulation has been proposed (Thompson, 1994; Kouropatov & Dreyfus, 2009). Approximation is a necessary component of accumulation. It is a complex, multi-faceted issue. It does relate to the definite integral, but it also relates many other notions in mathematics such as differentiation, estimation of large numbers, probability and measurement. Hence, approximation is, in our opinion, an excellent candidate to start with. There are at least three reasons for this claim:

- In common school mathematics situations regarding integral calculus (positive continuous functions defined on a closed interval), the definite integral concept is closely connected to the concept of approximation;
- The process-nature of approximation (e.g., calculating the area of a circle by using circumscribed polygons with progressively more sides) and the

concept (object)-nature of approximation (e.g., an area estimate for the circle) seem to be intuitively clear for students;

- The idea of approximation allows for developing didactical tools, with which a student's work may become observable.

In light of the above, this paper focuses on the following questions:

Question 1. What is the structure of the approximation concept in terms of elements of knowledge and what are suitable operational definitions for these elements that allow the researcher to identify concept construction?

Question 2. How does the process of knowledge construction occur during an instructional intervention?

Knowledge constructing and Abstraction in Context

The main aim of this paper is to analyze the knowledge that is constructed as a result of an appropriate instructional intervention. Researchers interested in studying knowledge construction have adopted various theoretical approaches and methodologies. For this study we have adopted Abstraction in Context (AiC), a theoretical framework, which allows describing the construction of abstract knowledge during the learning process (Hershkowitz, Schwarz and Dreyfus, 2001). AiC takes abstraction to be an observable activity of vertical reorganization of previous mathematical constructs in order to arrive at a new (to the learner) construct. The activity is interpreted in terms of epistemic actions performed by the learner or by a group of learners for a specific purpose, in a particular context. The context includes the social environment as well as the learner's personal background. The previous mathematical constructs result from previous abstractions. Reorganization includes establishing new connections between such constructs, making mathematical generalizations, and discovering new strategies for solving problems. "Vertical" implies building a new level of abstraction above a previous level. An essential component of AiC is a model of three epistemic actions for describing and analyzing at the micro-level the process by which learners construct new knowledge:

- R The learner *recognizes* a previous mathematical construct as relevant in the present situation.
- B The learner *builds-with* the recognized constructs to achieve a local goal such as solving a problem or justifying a claim.
- C The learner uses B-actions to assemble and integrate previous constructs so that a new (to the learner) *construct emerges* by vertical mathematization.

These epistemic actions have been chosen because they are relevant for processes of abstraction and observable. In processes of abstraction, R-actions are nested in B-actions, and B-actions are nested in C-actions. Nesting sometimes makes it difficult to fully differentiate between them. For example, a sequence of R- and B-actions can constitute a C-action or another B-action, depending on the context, in particular on the student's previous actions. However, no matter how the epistemic actions are interpreted, the researchers claim that the methodology helps making processes of

knowledge constructing observable. This claim is based on studies by themselves and other researchers, reviewed in Schwarz, Dreyfus and Hershkowitz (2009).

The elements of knowledge and operational definitions

One of the tasks of the researcher when using the RBC model is to decide which knowledge elements to focus on, and to describe them as combined of constituent knowledge elements and links between the constituent elements. For this purpose, we carried out an a priori analysis of the content domain, and of the activities proposed to the students. Hence, we used theoretical as well as didactical considerations. The theoretical considerations include that approximation is a kind of meta-knowledge that should be clarified (or defined) specifically for any concrete concern; for us the proceptual nature of approximation is particularly important. The didactical considerations are that, for the purpose of this study, we see approximation as a procedure, and the result of this procedure, of calculating (as accurately as required) some unknown value (length, area or volume) by using known values. A well-known example for this view on approximation is the ancient Greek proof of the formulae of the area of the circle. This approach to approximation notion immediately leads to the following conclusions:

- We should know (even if only intuitively) that the measure of a given object (length of a line, area of a shape or volume of a solid) has an exact value, otherwise the issue of value calculation becomes meaningless;
- We should know how to calculate the values of the measures of some basic objects (e.g. the length of line segments, the area of polygons), otherwise we can't start the approximation procedure;
- We should know how to replace an "unknown object" by an aggregate of basic objects and how to refine this process in such a way that the replacement becomes more accurate.

Approximation may be made by measurement, by geometrical consideration (known formulas) or by algebraic consideration (analyzing some algebraic term). In light of these considerations, we have chosen to focus on the following knowledge elements:

AP_G "General approximation": The size of a given object can be approximated by replacing the given object with known objects (the known objects are objects whose size can be found based on existing elements of knowledge).

AP_R "Refined approximation": The approximate size of the given object can be made more precise by decreasing the size of the replacing objects and increasing their number.

AP_L "Approximation limit": The size of a given object can be found exactly by continued refinement; here an informal intuitive understanding of limit may be used: the value is a limit if we can get as close to it as we want.

The following operational definitions will be used to assess whether these knowledge elements have been constructed:

AP_G We will say that students have constructed this element if they explicitly (verbally or/and graphically) replace a given object with known objects.

AP_R We will say that students have constructed this element if they explicitly (verbally or/and graphically) refine the replacement by decreasing the size of the replacing objects and increasing their number.

AP_L We will say that students have constructed this element if they explicitly (verbally or/and graphically) clarify that some value is the exact size of a given object if this value is a limit of some calculations or measurements.

The elements of knowledge that we assume to exist on the basis of previous learning experiences are:

SM "Segment measurement" – Any line segment can be measured.

PA_C "Perimeter / Area calculation" – The Perimeter and Area of a triangle, rectangle, trapezoid and circle can be calculated by using the appropriate known formulas.

AD_G "Geometrical additive property" – All geometrical sizes (length, area, volume) are additive in the Euclidian sense.

Description of the research

The integral unit is a ten-session unit that has been implemented with five groups of students. For the purpose of this paper, we report on a group of three female advanced-level mathematics high school student volunteers and focus on one activity of the unit. We worked with groups rather than single students in order to observe them discussing the problems together, thus making the knowledge constructing process more observable. When analyzing the transcript we encountered a question: Should we examine the knowledge construction process in each of the three students, or consider them as a group? We decided to do the latter. This was done for several reasons. First of all, the ways in which the students discussed the problems was very succinct and simple. It seems that the social mathematical norms in their class at school did not require them to go into depth when describing their opinions and arguments. Secondly, it seems that the students were at fairly different mathematical levels (subsequently, one of them moved from the advanced mathematics level to the regular one). Lastly, they took on different roles during their group work: Student A acted as generator of ideas, B examined whether the generator's ideas were valid, and C recorded their findings. Occasionally, they switched roles. Because of this, the knowledge constructing process was a group more than an individual process. It is also important to note that what the students worked on during the interviews was quite independent of what they were learning at school.

findings

Here we describe and analyze the students' learning processes of approximation through the lens of the RBC model. Then we show how the epistemic actions in the knowledge construction process can be indicative of knowledge constructing.

The students were asked to calculate the length of four given lines: a segment, a semicircle, and two irregular curved lines. They started with the segment, using SM. It was interesting to note that even in this (very common for students) situation of segment measurement, they still experienced some degree of difficulty deciding what the exact value is. They continued with the semicircle. At the beginning, they intended to use the approach that worked in the case of the interval: they tried to measure the length of the semicircle by using a protractor. It seems that at this stage it was much easier for them to adopt the measurement approach (it worked once!) than to think about another idea. But they soon realized that this is inappropriate. With a little help from the researcher they were able to make a switch. Our intention had been for the students to find a measurable line that gives a lower approximation for the length of the semicircle, but this is not what happened. The students were drawn to using a known formula according to PA_C ("Perimeter / Area calculation").

Next, they moved to one of the irregular curved lines, which looked like part of a parabola, soon using the their constructs for SM (we could identify it by a student's explicit suggestion to draw and to measure the segments: "If...maybe...it's pretty much, let's say doing the perimeter of a triangle, maybe?") and AD_G ("Additive property") (we could identify it by a student's appropriate drawing). Recognizing these previous constructs was followed by building-with them (Figure 1, lines 141-150). In the figure, R, B and C stand for recognizing, building-with and constructing of the construct(s) listed in the respective columns.

Teaching Interview		Epistemic Actions		
		R	B	C
141	Student A: Ah right, so we can do the perimeter of the...and measure to stretch lines like more or less.	PA _C SM		AP _G
142	Student B: But then it won't be exact.		SM	
143	Student A: But we said approximately, it doesn't have to be that...		AD _G	
144	Researcher: Exactly, exactly. That's a start, right? So let's try that. What did we see there?			
145	Student B: To stretch...	AD _G		
146	Student A: To stretch lines from the ends.			
148	Student B: To the point.			
150	Student A: That'll be pretty much a triangle we can calculate.	PA _C		

Figure 1: Constructing AP_G .

We claim that in the course of the excerpt reported in Figure 1, the students, as a group, constructed AP_G ("general approximation"). We can confirm it by one

lengths (using PA_C with AD_G), which precisely corresponds to the operational definition of AP_R .

Discussion

In the study presented here, we tried to find an answer to the question how processes of knowledge construction occur during an instructional intervention in the form of a teaching interview. The data gathered allows us to answer this question. The first constructing process relates to AP_G ("general approximation"). During the teaching interview it was possible to identify how from the existing constructs PA_C and SM and then through construct AD_G , the students arrived at a new calculation: the PA_C of another object. This whole process encapsulates the construction of AP_G . Schematically this is represented in Figure 3a.

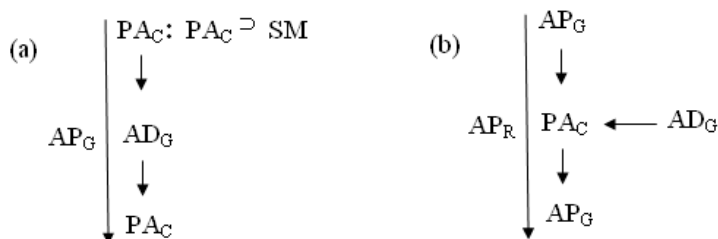


Figure 3: Schematic representation of the constructing processes of AP_G (a) and AP_R (b)

The second constructing process relates to AP_R ("refined approximation"). Using the new construct AP_G and the previous constructs PA_C and AD_G , the students arrived at a new, refined approximation. This whole process encapsulates the construction of AP_R . It is similarly represented schematically in Figure 3(b).

The main conclusion we draw from this study is as follows: It seems that the proposed didactical tool (designed teaching interview) allows us to support students' knowledge constructing processes, and the adopted research methodology allows us to observe these constructing processes very closely. One of the important practical considerations of such a conclusion is that it allows educators to identify the problems in the design of an educational intervention. We give an example of the kind of problem we discovered: The presented teaching interview (as a didactical tool) was not sufficiently developed to allow us to support and to observe the constructing process of AP_L ("Approximation limit"), which is a crucial component of approximation. Taking this into account the design of the activity for another group of students has been refined in a way that on the one hand supports the constructing process of AP_L and on the other hand made this process observable to the researchers. We hope to present data from that other group and their analysis at the conference.

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DEVELOPMENT OF PRE-SERVICE TEACHERS' KNOWLEDGE RELATED TO BIG IDEAS IN MATHEMATICS

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Designing rich learning opportunities in the mathematics classroom can be facilitated by professional teacher knowledge related to Big Ideas in mathematics and mathematics education. Hence, such professional knowledge should be developed. However, empirical research on such knowledge and the possibility of fostering it by professional development (PD) activities is scarce. The two studies we report on in this paper focus on corresponding findings, based on data of more than 100 resp. 30 German pre-service teachers. The results indicate that many pre-service teachers were initially unable to discern Big Ideas behind mathematical concepts and techniques and to link elements of content matter according to these Big Ideas, but also that their professional knowledge in this area can be improved through PD courses.

INTRODUCTION

The awareness of overarching concepts or “Big Ideas” related to mathematics for designing rich learning opportunities should include both the areas of content knowledge (CK) and pedagogical content knowledge (PCK; Shulman, 1986). However, there is little empirical research on professional knowledge connected with mathematics-related Big Ideas and on the development of this knowledge.

Responding to this research need, this study concentrates on indicators for components of professional knowledge linked to Big Ideas. A new test and questionnaire instrument developed in the EU-funded project ABCmaths (“Awareness of Big Ideas in Mathematics Classrooms”) has been used to assess the prior knowledge of 117 pre-service teachers (study A) and to gather evidence for describing possible developments in the professional knowledge of 32 pre-service teachers (study B). The results of study A show that—prior to the course—the pre-service teachers were often unable to link mathematical contents according to selected Big Ideas and to communicate these ideas. However, the results of study B suggest that such knowledge can be acquired through professional development courses.

The first section casts light on key elements of the theoretical background by referring to a working definition of Big Ideas and a theoretical model of professional knowledge components we used in this study. The second section explains the research questions. Based on information about research design and sample given in the third section of this paper, results are reported in the fourth section. The fifth section contains a short discussion of the evidence and conclusions.

1 THEORETICAL BACKGROUND

Big Ideas related to mathematics and mathematics instruction

The role of overarching concepts or fundamental ideas in mathematics and its teaching and learning for creating conceptually rich learning opportunities has been highlighted by many researchers (e.g. Schweiger, 2006; Bishop, 1988). Overarching concepts are also used to describe mathematical competency in many national standards (e.g. Office of Qualif. and Examinations Regulation, 2002; KMK, 2003; NCTM, 2000). Improving the awareness of such “Big Ideas” is the aim of the teacher education project ABCmaths (www.abcmaths.net). According to the working definition of ABCmaths, *Big Ideas associated with mathematics in the classroom* anchor, link and constitute mathematical knowledge in contexts (within or beyond mathematics and the curriculum) and foster making sense of and communicating this knowledge in a more general way. Following a pragmatic approach, which aims primarily at encouraging teachers’ reflection on overarching concepts in mathematics and on their potential for learning, four key aspects have been collected that may help to identify Big Ideas (ABCmaths team, in preparation). Big Ideas can be characterised as:

- Ideas that should have a high mathematics-related potential of encouraging understanding of conceptual knowledge (including orientation, linking and anchoring of knowledge),
- Ideas that should have a high relevance for building up meta-knowledge about mathematics as a science (adapted to the target group of learners) including knowledge necessary for interdisciplinary comparisons,
- Ideas that should encourage meaningful communication about mathematics and develop mathematics-related reasoning,
- Ideas that should also encourage the reflection processes in teachers connected with designing rich and cognitively activating learning opportunities and supporting learning processes of students.

These aspects can be seen as a pragmatic answer to a partly divergent discussion of multiple approaches in the area of Big Ideas: In the German-speaking discussion for example, the diversity of the notions of “fundamental ideas” (e.g. Schweiger, 2006), “central ideas” (Schreiber, 1983), “universal ideas” (Schreiber, 1983), “leading ideas” (e.g. KMK, 2003) and “basic ideas/basic conceptions” (“Grundvorstellungen”, e.g. v. Hofe, 1995) give a heterogeneous picture and call for an integrated and sufficiently open perspective as suggested by the criteria above. Three examples of Big Ideas are the following:

- *Using multiple representations*: This Big Idea reflects strategies used and required in many mathematical domains related to the use of different ways of representing mathematical facts or concepts as well as with changing representations and linking them.
- *Giving arguments/proving*: Mathematics as a science can be characterised by the forms of argumentation and proof used in this discipline (cf. Heinze

& Reiss, 2003). As a consequence, argumentation plays an important role in all domains of mathematics.

- *Dealing with infinity*: This Big Idea refers to exploring phenomena linked with infinity in mathematics, to strategies for establishing generality in order to include a potentially infinite number of cases, or to the idea of patterns and structures that are inherently infinite.

In the project work of ABCmaths we consider also *Big Ideas associated with pedagogical content knowledge (PCK)* (ABCmaths team, in preparation). For example, the Big Ideas “using multiple representations” and “giving arguments/proving” appear both in the domains of CK, i.e. as mathematics-related Big Ideas, and PCK.

Professional knowledge related to Big Ideas

Professional knowledge related to Big Ideas is not only “horizon knowledge” (Ball, Thames, & Phelps, 2008), but it is central to different areas of CK and PCK (Shulman, 1986). For example, the idea of dealing with infinity can not only enhance understanding of mathematical concepts and strategies, but it can also explain potential difficulties of learners; for example, with general statements (holding for an infinite number of cases) or formal descriptions of (potentially infinite) patterns. Moreover, awareness of Big Ideas can link different components of mathematics-related knowledge. For this reason, the teachers’ awareness of Big Ideas in mathematics and mathematics instruction appears to be crucial for designing learning opportunities and also for their ongoing professional learning.

As a theoretical framework for components of professional knowledge, the study refers to the model shown in Figure 1 (Kuntze, in press).

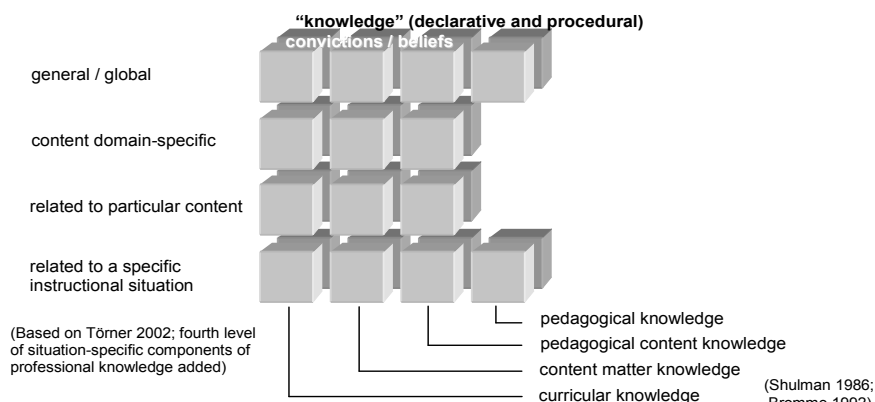


Figure 1: Model for components of professional knowledge (Kuntze, in press)

This model takes into account the spectrum between knowledge on the one hand and convictions and beliefs of mathematics teachers on the other, as a dichotomous theoretical distinction between knowledge and beliefs is impossible (cf. Pajares, 1992). Hence, both are considered to be part of the notion of professional knowledge. The

distinction between different domains of professional knowledge as suggested by Shulman (1986) appears in the vertical columns (see Ball, Thames, & Phelps, 2008, for the possibility of further refinement into domains). Taking into account that individual professional knowledge is often organised in an episodic structure (Leinhardt & Greeno, 1986; Bromme, 1992), different levels of globality (Törner, 2002; Lerman, 1990; Kuntze & Reiss, 2005) are distinguished on the horizontal layers. Professional knowledge related to Big Ideas is relevant for different levels of globality, as these ideas are important for mathematics and mathematics instruction as a whole but also relevant for many specific content and instructional situations.

2 RESEARCH QUESTIONS

As empirical evidence related to professional knowledge about Big Ideas is scarce, this paper concentrates on a corresponding analysis of needs study (study A) and an evaluation study (study B) with pre-service teachers. More explicitly, the studies focus on professional knowledge related to linking examples of subject matter by Big Ideas, as well as analysing examples against the background of specific Big Ideas. We focus on the three mathematics-related Big Ideas introduced above, namely “using multiple representations”, “giving arguments/ proving”, and “dealing with infinity”. Hence, this study aims to provide evidence for the following research questions: *(A) What professional knowledge related to the Big Ideas “using multiple representations”, “giving arguments/proving” and “dealing with infinity” do German pre-service teachers have? (B) Can such professional knowledge be developed in professional development courses?*

3 SAMPLE AND METHODS

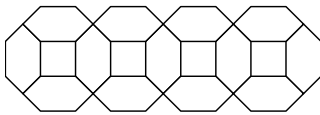
Regarding research question A, a test was administered to 117 German pre-service teachers (78 female, 35 male, 4 without data) before the beginning of a university course. The pre-service teachers had a mean age of 22.33 years ($SD = 3.56$ years) and had been studying on average for 2.19 semesters ($SD = 1.12$). 61 pre-service teachers were training to teach in primary schools, 35 in secondary schools for lower-attaining students, and 15 for working in schools for students with special needs.

For research question B, 32 pre-service teachers, (22 female, 8 male, 2 without data) were asked to complete the test before and after a professional development course. Comparable to the larger sample of study A, these pre-service teachers had a mean age of 21.67 years ($SD = 2.91$ years) and had been studying on average for 2.10 semesters ($SD = 0.91$). 17 pre-service teachers were training to teach in primary schools, 9 in secondary schools for lower-attaining students, and 3 for working in schools for students with special needs (3 without data).

Corresponding to the aspects emphasised in the previous section, the test focused especially on analysing, and perceiving links between, mathematical concepts according to Big Ideas. The test instrument of study A concentrated on the ideas “dealing with infinity”, “giving arguments/proving”, and “using multiple representations”,

study B focused on “dealing with infinity”, using a short version of the test instrument.

A sample task related to the idea “dealing with infinity” is shown in Figure 2. In this task, the pre-service teachers were given an example figure, asked about learning opportunities related to the idea of infinity (“local part” of the question) and asked to give other mathematical examples related to the idea of infinity (“linkage part” of the question).



This figure can be used to explore infinity with younger students.
Explain what students can learn about phenomena of infinity here. Can you think of other mathematical contents which are suitable for students' learning about infinity?

Figure 2: Sample task related to the idea “dealing with infinity” (task 1)

The answers of the pre-service teachers were collected in an open format. There were two tasks related to the Big Idea of “dealing with infinity”, and (in study A) three tasks for each of the ideas “using multiple representations” and “giving arguments/proving”. The pre-service teachers were given as much time as they required to complete the test.

In order to gain an overview of the quality of the answers from the pre-service teachers, a top-down coding method was used. The coding categories concentrated on the aspects of existence of a codable answer, the quality of the answer in the “local part” of the question, the quality and transfer level of the examples in the “linkage part”, and the embedding of these examples. For clarity, more details about the codes are reported together with the corresponding results in the following section.

4 RESULTS

Study A focused on the prior knowledge of the pre-service teachers. For an overview of the number of answered tasks which could be coded, the categories “no answer given to the task”, “irrelevant answer given, i.e. no detectable semantic relationship between answer and the task” and “codable answer with respect to quality codes” were assigned to the answers in an initial coding. The frequencies of codes are displayed in Figure 3. It can be observed that the frequencies of codable answers were low: For almost all tasks, between one third and two thirds of the pre-service teachers could not give an answer at all (cf. also Kuntze et al., accepted).

Against the background of these findings indicating relatively low answering rates, study B concentrated on the question of whether professional development courses could help pre-service teachers to improve their knowledge related to Big Ideas. As it was difficult to reach the participants of the course after its completion, the results are restricted to the Big Idea “dealing with infinity”. Figure 4 shows that after the course, non-codable answers were an exception. The frequencies of the quality codes for the “local” part of the questions are displayed in Figure 5. Even though there were more

answers after the course relating the example to the Big Idea, answers with an adequate in-depth analysis of the example regarding the Big Idea were infrequent.

The results of an overview coding of the “linkage” part of the questions are given in Figure 6. Particularly for task 1, there is a visible shift towards the ability to give adequate examples of other content related to the Big Idea “dealing with infinity”.

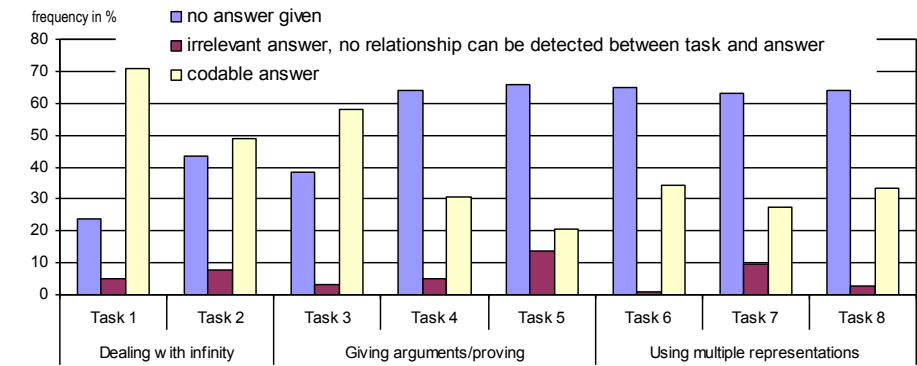


Figure 3: Frequencies of answers to tasks (cf. Kuntze et al., accepted)

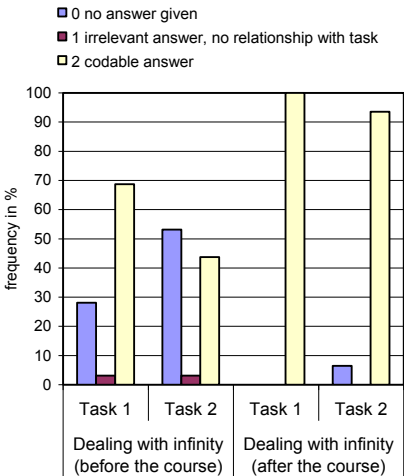


Figure 4: Answers to tasks (global code)

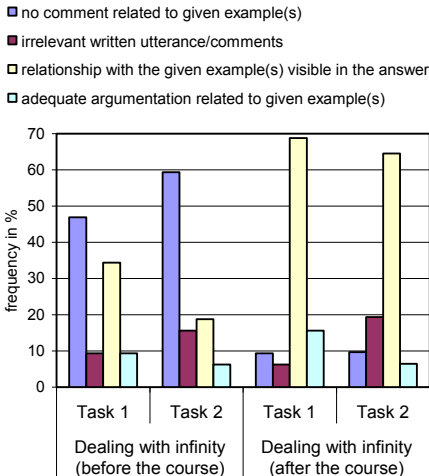


Figure 5: Quality of “local” answers

5 DISCUSSION AND CONCLUSIONS

The results of study A suggest that pre-service teachers were frequently unable to give examples of mathematical content linked to Big Ideas. As answers based on relatively simple mathematical concepts were possible, this can be interpreted as a lack of awareness of Big Ideas in the professional knowledge of the pre-service

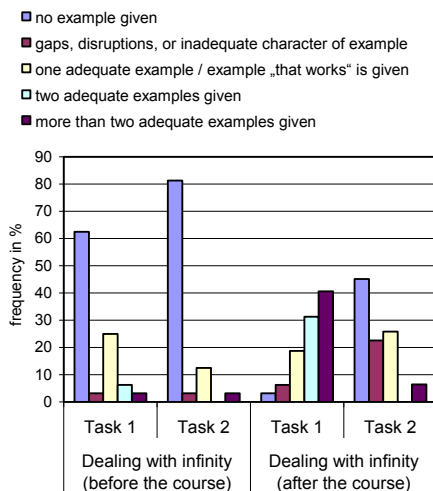


Figure 6: Quality of “linkage” answers

overview findings given in the Figures above could be interpreted as a rise in the awareness of Big Ideas, and, at the same time, as ongoing difficulties the pre-service teacher still might have when asked to reflect on mathematics according to a big idea like “dealing with infinity”. We would like to conclude that, very probably, learning to reflect on and to communicate about mathematics-related big ideas is a longer process – but a process that can be started through professional development activities. Accordingly, further attention should be devoted to research into the knowledge development of in-service teachers related to Big Ideas, into possible impacts of professional development programs, into extensions to other Big Ideas, and into the role of school culture.

Acknowledgements

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WORKING WITH TEACHERS: INNOVATIVE SOFTWARE AT THE BOUNDARY BETWEEN RESEARCH AND CLASSROOM

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This paper is based upon an experiment around research designed software. A first assumption was that dissemination of outcomes is possible through communal activity involving researchers and teachers. Another assumption was that all teachers are not to be considered at the same level. □First-adopters□ are teachers associated with the project development. □Mid-adopters□ are teachers interested by using innovative software in the classroom, but who will choose to do it only when they see real added value. The expected outcome was classroom scenarios intended to □late adopters□ The notions of activity and praxeology help to analyse this experiment.

INTRODUCTION AND FRAMEWORK

Innovative tool design and diffusion is an important dimension in Mathematics Education research. The European project ReMath (IST4-26751) worked in this dimension, developing six ‘Dynamic Digital Artefacts’ (DDAs) and offering associated materials for teachers. Although a first start, this one way communication is not sufficient and an assumption is that diffusion of research outcomes is better achieved through communal work involving researchers and teachers. The aim of this paper is to analyse a communal work around one of the ReMath DDAs: Casyopée. The conceptual framework involves the notion of ‘boundary object’. We follow Fuglestad, Healy, Kynigos and Monaghan (2009)’s idea of using this notion to make sense of the collaboration between researchers and teachers in the integration of technology. Another notion is ‘activity’. According to Rogalski (2004) activity both affects situation and subject through the results on the objects of the situation and through the effects on the subject. The idea is then that, beyond the task of scenario creation, communal activity will affect researchers and teachers, and that studying the effects will improve the understanding of how innovation provided by research can disseminate towards classrooms. While boundary objects and activity help for the methodology, the notion of praxeology (tasks, techniques, theories) is useful for analysing this experiment. Lagrange (2000) borrowed this notion from the Anthropological Theory of Didactics in order to make sense of the impact of computer based tools on school mathematics: these tools introduce new techniques, competing with existing, and then most often require reconsidering praxeologies. To give an idea of what a praxeology is, consider the signs of real functions, an example that will be analysed below in the communal work. A type of task is to find intervals of positivity of a given function or class of functions. There are techniques to conjecture, exploring graphs or tables. There are also different techniques to prove: solving inequations, or factoring the formula, or using a property of monotony associated to the zeros. These techniques are based upon specific theoretical

elements: understanding of graphs, of inequalities equivalence, algebraic or calculus theorems... In learning topics like the signs of functions, beyond acquiring techniques the goal is for students to approach and use the theoretical elements. In this sense praxeologies are crucial tools in teachers' hands to organize students' learning.

CASYOPÉE AND CAS: CHOICES AND CONSTRAINTS

Casyopée has been developed for ten years in a project involving researchers and teachers after they experimented with 'standard' Computer Algebra Systems (CAS). They were concerned that while technology is able to offer multi-representational and symbolic manipulative capabilities very effective for solving problems and learning about functions, no tool really adapted presently exists for students' use. CAS are designed for more advanced users. Dynamic Geometry (DG) offers means for constructing and exploring dynamic figures but exploration is limited to numerical values. Students are neither encouraged nor helped to use algebraic notation and to work on algebraic models. Consistent with the focus of ReMath upon computer representations of mathematical objects, we took the opportunity to extend Casyopée's representational capabilities, in order to consider functions as models of non-algebraic dependencies. The choice has been to consider 2D geometry as a field for functional modelling. The result is that Casyopée has now two main windows. In the symbolic window the fundamental objects are functions. Casyopée helps students to perform operations on functions like: algebraic manipulations; analytic calculations; graphical representations; support for proof... A new window offers usual DG capabilities and also distinctive features: geometrical objects can depend on algebraic objects and it is possible to export geometrical dependencies into the symbolic window in order to build algebraic models. On the one hand, these epistemological choices are consistent with the notion of function as considered at secondary level and thus they should facilitate Casyopée's integration. On the other hand these choices introduce non trivial constraints and differences with software tools generally proposed for classroom use at this level. First, offering computer algebra has been decided in order to help students to identify the different stages in a solution of a problem; this decision also implies that Casyopée relies on computer algebra algorithms which are not always deterministic: for instance, it happens that two expressions are mathematically equivalent but also that none of the algorithms implemented in the computer algebra kernel can recognize the equivalence. More generally, Elbaz-Vincent (2005) points out the "decidability limitation" inherent to computer algebra, and concludes that an 'intelligent usage' of CAS in mathematical courses is not obvious. Second, DG in Casyopée is designed in order that students use a figure as a support to model algebraically geometrical dependencies. This implies constraints that will be analysed below.

disseminating Casyopée: goal and method

In the process of diffusion of an innovation, potential actors and their cultural context can be identified according to their relation with the innovation and its use in

classrooms. Teachers that have been associated with the project development can be considered as “first-adopters”, or “experts”. “Mid-adopters” are teachers that can be interested by using innovative software in the classroom, but will choose to use a software like Casyiopée only when they will be convinced that it brings added value to their teaching. All other teachers potentially users of Casyopée (that is to say, every mathematics teacher at upper secondary level) would make a third group. The idea is then to use classroom scenarios as means for communicating between these groups: the elaboration and experimentation of scenarios would be first a way of collaborating between experts and mid-adopters: mid-adopters would propose uses corresponding to their needs and ask the experts for their advice and support. The scenarios would be built in order to be proposed to all potential users and thus designed as a way to communicate between the second and third group. The goal of the research was then to investigate what particular approaches those “mid adopters” take when working at these interfaces with the first and the third group, shedding light on how teachers appropriate innovative software and on how they grasp its didactical potential and constraints. A hypothesis was that the mid adopters would adapt the use of the software to their needs and sensibility. It was expected that these teachers would be primarily interested in easy-to-achieve and close-to-curriculum applications of Casyopée, and sensible to problems and constraints related with the time required by implementing technologies in their classes, with curriculum requirements, with training needs, etc. Thus their production would provide useful material for an easy integration by teachers in the third layer. For this study, the first-adopters or “experts” were two teachers that had been involved in Casyopée’s design. They contributed to specifying functionalities and experimented successive versions in their classes. Crucial steps in the project were undertaken as a consequence of dissatisfactions they expressed. The decision to develop a software environment around a computer algebra kernel for classroom use of symbolic computation derived from the difficulty they felt when using standard CAS. The decision to append a DG window was taken after a long term experiment about modelling geometrical dependencies in their tenth grade classes. In the same region where the experts teach, a group of six teachers had experimented the use of the Interactive White Board (IWB) during two years using software packages (DG and CAS) on the IWB and they were keen to enter a new project. They were good candidates to be “mid adopters”: they were convinced that technology can support mathematical teaching and learning; they were relatively experienced in the classroom use of technology but not involved in Casyopée’s history. Researchers, the two experts and the six mid-adopters met 14 times along 2 years in 3 hour sessions. Eight different scenarios were elaborated and each was experimented in several classes, a teacher offering his class and the rest of acting as observers. We could use a professional platform to communicate between the meetings, but also for communicating with all mathematics teachers in the region. This third group involving the mathematics teachers at upper secondary level in the region, around 500 teachers, constituted the group of potential ‘late adopters’.

Data analysis

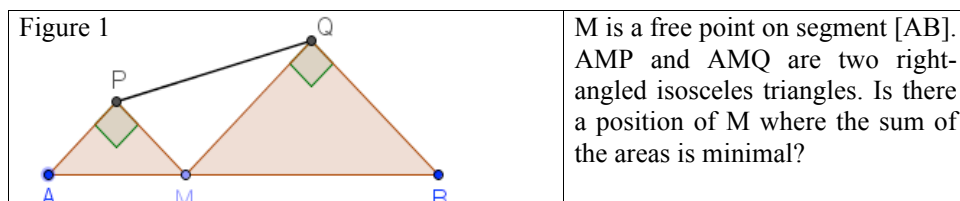
Consistent with a focus on activity, the choice was to analyse objective elements accounting for the communal work especially the video recording of the meetings. The recordings were coded in a video analysis database and split into about 90 clips varying from ten to thirty minutes. The recording of experimentations, the messages and files exchanged on the platform and the scenarios proposed and experimented were used as complementary data. The clips were classified in non exclusive categories. Three categories reflect directly discussion about Casyopée itself. We distinguish between technical aspects, i.e. details of the implementation, and fundamental options, one category relating to the Casyopée's 'CAS nature' and the other to how modelling is conceived in Casyopée. Two other categories deal with classroom situations of use, one related to a priori analyses conducted during the meetings, and the other with scenarios of use experimented and published as a contribution to Casyopée's diffusion. We added three more categories reflecting topics appearing recurrently in the meeting: other software, differences between experts and mid-adopters, and general math education. Due to limited length, this paper restricts to a presentation and a qualitative analysis of the categories, and excludes technical aspects: these aspects were very present in discussions but their implication has been widely discussed by Lagrange (to appear) and will be considered here in the other categories with which they interfered. Note that Lagrange (to appear) also analyses two cases of mid-adopters' personal evolution.

Casyopée's 'CAS nature': Mid-adopters thought that a CAS would be particularly useful for problems with complicated algebraic calculations. That is why they proposed situations resulting in very complex expressions that the computer algebra kernel and other modules of Casyopée had difficulties to handle. It took time for these teachers to realise how CAS' decidability limitation stressed above, influences Casyopée's operation. For instance in the third meeting, a member expressed her concern that, for two functions with equivalent expressions, Casyopée was not directly able to recognize their equality, and that only one of the symbolic calculations provided by Casyopée returned zero for the difference. This concern was linked with the ambition to offer students realistic problems, and views that Casyopée should be used to scaffold students' weakness in algebraic computation in these problems. More often computations could not be entirely achieved by Casyopée. Situations had to be carefully adapted in order to articulate CAS and by hand calculation. Then mid-adopters realized the actual potential of computer algebra in Casyopée: it does not everything by miracle, but it can help when one understands its potentials and limits, and takes advantage of these for specific praxeologies.

Modelling in Casyopée: It was mentioned above that Casyopée offers opportunities for modelling but also that it brings specific constraints. This is an example showing how mid-adopters were attracted by the opportunity, while encountering obstacles deriving from the constraints. In the second meeting a teacher protested that she could not implement Casyopée for an optimisation problem she used to propose her

students (figure 1). She considered two free points in the plane A and B and a free point M on the segment $[AB]$. The exploration was done geometrically and on numerical values like in other DG systems. After that, she wanted to take advantage of Casyopée for modelling the dependency into a function in order to solve the problem algebraically. She chose AM as the independent variable and the calculation $\frac{AP^2 + MQ^2}{2}$ as dependent variable. But Casyopée replied that the calculation depended of more than a free point and modelling was unsuccessful. The teacher expected a function like $x \rightarrow \frac{x^2}{4} + \frac{(AB-x)^2}{4}$. Actually, Casyopée handles functions of one variable.

It is also possible to create parameters and then to consider families of functions. Thus any situation in which several variables are involved has to be modelled selecting which variable has to play the role of the independent variable and which variables have to play the role of parameters. In the above situation, a consistent approach is to define the two points A and B as coordinated points, depending on a parameter representing the distance AB . For instance A might be at the origin and B might be $(0; a)$. This represents a didactical potential, but it is difficult to grasp for a teacher. The same mid-adopter met again this issue a year later when preparing a scenario about an area under the parabola between two points. She understood the necessity of having a single free point, but also insisted that in this scenario, modelling by a one variable function brings some limitations.



Classroom situations: Many situations proposed by mid-adopters in the first meetings could not be directly implemented bringing evidence of misunderstanding of choices and constraints referred in the first two categories, as well as to technical aspects. The situations and the software had to be adapted together: situations had to take into account the fundamental choices and constraints, while software development could correct technical limitations or less central choices. In the second year, the mid-adopters' appropriation of Casyopée progressed notably especially with regard to the relationship between the algebraic and the geometric windows. This appropriation was done through the preparation and experimentation of situations but also seemed inseparable from the discussion on the software itself. Why is it designed like that? Could other options be decided? Actually, the efforts devoted to discuss these issues can be related not only to teachers' appropriation of the software, but also to their quest of its didactical potential.

Scenarios and communication with other teachers: Preparing the diffusion of Casyopée to mathematics teachers in the region was an objective of the project, and

the experts often insisted on. Despite this, there was no outside communication towards the third layer of teachers in the first year. The mid-adopters were reluctant to publish material not really achieved. They insisted on publishing reports on successful sessions with a deep didactical analysis, rather than “raw scenarios”. They also stressed that they could not promote Casyopée without fully understanding its potential and operation. In the second year, experts and mid-adopters decided to publish 8 high quality “mini web sites”¹ i.e. detailed scenarios with precise objectives and account of the advantages brought by Casyopée. The mid-adopters’ concern for a strong didactical added value clearly appears in these productions. It implied deep reflection after the experiments and could be achieved only by the end of the second year. Thus two ways communication with “ordinary teachers” expected in the project, could not occur. This seems to reflect the specific experience that mid-adopters had within the group. Struggling to reconcile their expectations with the real use of the software made them aware of the difficulties that their colleagues might have. The need of communicating led them to a deeper analysis of the scenarios

Other software: In many clips, especially those in the category “classroom situations”, Casyopée is compared to other software. TiInspire was used by some mid-adopters, but abandoned because it is not free and considered too complex. This experience nevertheless brought questions about the validity of CAS for the classroom that mid-adopters tried to elucidate when discussing Casyopée use. Geogebra is popular among French teachers as a free DG, but mid-adopters saw limitations and were looking for Casyopée as a more mathematical alternative –in the sense that it allows coordinating exploration and algebraic proof by offering consistent objects for the two activities. Nevertheless they often mentioned Geogebra as easier to use.

Differences experts-mid-adopters: These differences appear in many clips in conjunction with the topics referred by the above categories, giving evidence of how Casyopée’s characteristics had been integrated in experts’ school practice while they participated to the design. The evolving state of Casyopée was not an obstacle when working with expert teachers who understood well the difference between fundamental choices and minor defects, adapted the scenarios accordingly, and accepted the risk of experimenting provisional versions. More or less explicitly it was expected that some of the mid-adopters would progressively adopt this flexible attitude, but it was not the case: it seems that these teachers thought more interesting to keep a mid-adopter attitude, even from the point of view of Casyopée’ diffusion.

General math education: from time to time in the meetings the discussion broke away from Casyopée and the experiments to tackle some general math education theme. Often the issue of consistency with the curriculum was raised. It happened also that situations proposed for Casyopée use, especially linked with its symbolic characteristics, conflicted with existing praxeologies. In the introduction, finding

¹ These web sites can be accessed publicly via <http://code.google.com/p/casyopee/wiki/Activites>

intervals of positivity of a given function or class of functions was offered as an example of a praxeology. It happened that an expert proposed to use Casyopée's symbolic window in order to help students conjecture and prove the interval of positivity of a linear function. He proposed to conjecture graphically and then to use the monotony of a linear function, a property about which Casyopée brings help and that allows an easy proof also supported by Casyopée. Mid-adopters agreed about making students conjecture this property on graphs, but, relatively to the algebraic proof, each of them wanted to expose his (her) personal praxeology whose conservation he/she found very important and difficult to implement with Casyopée. Eventually mid-adopters remarked that monotony was not in the curriculum at this grade, a fact the expert overlooked because he considered first Casyopée's potential. Mid-adopters's praxeologies based upon inequations (not considered in Casyopée) could not be implemented. Casyopée could support praxeologies based upon a factorisation, a fact that mid-adopters considered interesting, but the implementation was not obvious and brought long discussions.

Discussion and Conclusion

This experiment of a communal work about a “boundary” piece of software between researchers, experts and “mid adopters” was surprising in many aspects. The appropriation of Casyopée by “mid adopters” was much slower than expected. As a difference with what was expected these teachers did not focus on easy-to-achieve and close-to-curriculum applications and very slowly engaged in communication with other teachers. They wanted first to have a clear appreciation of Casyopée's potential and to implement situations with a strong added-value, consistent with this potential. This can be seen in the light of current praxeologies in the French upper secondary schools. The curriculum insists on using technology in problem solving situations, but because current technology (spreadsheet, dynamic geometry) is mainly numerical, teachers in France saw its benefits for exploring and conjecturing properties. Teachers in France are also much attached to formal proof that numerical technology cannot support. That is why current technological praxeologies separate two phases: one is the numerical exploration of a situation, generally ending by a conjecture and the second is the formal proof, generally done in paper pencil but sometimes aided by CAS. These praxeologies are relatively easily accepted by most teachers because they want to encourage students' exploratory activity while preserving formal proof practices that they see at the core of mathematics. Many examples are provided by textbooks and other professional resources. The “mid adopters” were teachers who really integrated these praxeologies into their practises but they were interested in going beyond, taking advantage of CAS potential.

From the beginning, the ambition of the Casyopée project was to associate closely exploration and elements of proof in an authentic mathematical activity thanks to computer algebra. Classroom situations and software design were conducted together by the designers (experts and researchers). Because Casyopée was a small project, this design developed around a restricted set of problems, especially geometrical

optimisation like in fig. 1. Another limitation is that although some efforts had been done in ReMath, designers did not see means for numerical exploration as central in the project. As a consequence, the specific set of praxeologies, emerging in relation to Casyopée design, resulted too limited in respect to the mid-adopters' expectation. It seems that designers were not fully aware of these limitations, thinking that, thanks to mid-adopters, the range of situations would rapidly expand, and that the power of computer algebra would compensate for limited numerical means. They were also not aware that fundamental choices in Casyopée could be seen by mid-adopters as conflicting with exploration and deeply impacting their proof related practices. While the expected scheme was roughly that mid-adopters would propose "narrow" praxeologies, and that they would enhance these praxeologies, progressively understanding Casyopée's design, the reality was that this design appeared relatively constricted with respect to mid-adopter's expectations. Extending the range of situations actually implied to build new praxeologies, both with regard to current technological praxeologies and to existing Casyopée praxeologies, *and* to adapt software design. This experiment sheds lights on how mid adopters' activity implies a deep reflection on their expectations with regard to the software and a reconsideration of the tools' potential. In this communal activity, the designers became aware of a not obvious conceptual and practical work necessary for adapting a research prototype into a "real life" classroom tool. Fuglestad, Healy, Kynigos and Monaghan (2009) conceptualise a type of work between researchers and teachers, using the term of "half baked" microworld. Casyopée was not conceived as a "half baked" microworld, but certainly one has to think of innovative software as a never finished product, evolving through communal work with users.

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MEANINGS ABOUT DYNAMIC ASPECTS OF ANGLE WHILE CHANGING PERSPECTIVES IN A SIMULATED 3D SPACE

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We report findings from design-based research aiming at shedding light on the meanings about angle in 3D space generated by 12 year olds while changing virtual perspectives of their angular constructions with a specially designed Turtle Geometry with dynamic manipulation microworld. The results show that the various perspectives taken of the constructed graphical objects as well as the fact that this could be achieved through various means mediated the notion of angle in quite distinct ways with interesting influences on meaning construction.

THEORETICAL BACKGROUND

Even though angle (along with length and distance) is one of the most important mathematical tools for describing and analysing physical space and present in a wide variety of physical situations, these are not easily correlated or connected in relation to angle concepts by children at the end of primary (Freudenthal, 1983, Mitchelmore & White, 2000). According to Henderson and Taimina (2005) angle can be defined from at least three different perspectives: (a) angle as a geometric shape, i.e. formed between two geometrical objects embodying directionality which can be either segments or 2d geometrical figures (b) angle as a dynamic notion, indicating a change of one direction both as a turn and as the result of a turn; and (c) angle as a measure represented by a number. In typical school education angle is basically approached as a static geometric shape while the notion of angle as turn is usually underrepresented although it is considered the most natural, the most instinctive aspect of angle (Freudenthal, 1983). Even in cases where angle is approached as turn this is done only through static 2d representations, which, no matter how cleverly designed, may delay the development of dynamic aspects of the concept and their integration with the static ones (Clements et al, 1996). Digital media seem to provide the potential to re-address the use of dynamic perspectives for students to form meanings about angle. Its contribution to the teaching and learning of geometry in general is perceived to be strongly linked with interactivity, multiple interlinked representations including symbolic ones, dynamic manipulations and dynamic visualisations (Laborde et al., 2006). Here we report how students' intuitions and ideas concerning angle as a spatial visualisation concept were challenged as they worked with a set of activities we designed adopting a constructionist theoretical perspective (Kafai & Resnick, 1996, Kynigos, 2007). The students used a digital medium called MaLT which integrates a 3d Logo Based Turtle Geometry with a) dynamic manipulation of procedure parameter values and the resulting constructed figures and b) dynamic manipulation of the students' viewpoint inside the simulated

3d space containing the constructed objects. Turtle geometry is based on a different geometrical system to those usually associated with the learning of geometry and it has been characterised as differential by Papert (1980) and as intrinsic by Abelson & diSessa (1981). It's considered as differential since a given geometrical state of the turtle is fully defined by its relation to the turtle's immediately previous state. In a similar vein it is characterised as intrinsic in the sense that there is no need to refer to places outside the turtle's immediate vicinity when deciding on an input to a procedure to change turtle's state. Researchers seem to conclude that carefully designed Logo-based microworlds are an effective medium in offering rich mathematical experiences and encouraging the construction of meaning in relation to the notion of angle as turn in 2d through the turtle metaphor (Clements & Sarama, 1997, Kynigos, 1997). However extending such microworlds to 3d space raises new issues related to the way the turtle metaphor may be put to use and the way deeply rooted intuitions about experiencing space and locomotion can be exploited so as to make sense of angle (Latsi & Kynigos, 2010). In viewing the images in the computer screen as 'signifiers' mediated by the system in which they are created and acted upon (Morgan et al., 2009), we wanted to investigate how observing an object from various perspectives would interact with the actual process of constructing meanings about angle as a determinant of its spatial properties (Kynigos, 1997).

TOOLS METHOD AND TASKS

MaLT is a constructionist microworld environment that extends 'Turtleworlds' to 3d geometrical space. 'Turtleworlds' blends Logo based Turtle Geometry with tools to dynamically manipulate procedure variables and observe the resulting 'continuous' change to the respective figural constructions (Kynigos et al, 1997). In MaLT, we used a well established method to extend Turtle Geometry to 3d by adding two kinds of turn commands (Reggini, 1985): 'UPPITCH/DOWNPITCH n degrees' ('up/dp n') which pitches the turtle's nose up and down on a plane perpendicular to the one defined by right-left turns and 'LEFTROLL/RIGHTROLL n degrees' ('lr/rr n') which moves the turtle around its own axis. A second feature of MaLT is that we kept the 'Turtleworlds' feature of variation tools. These tools recognise the procedure responsible for any figural construction and afford dynamic manipulation of variable values resulting in DGS-style change in the figures. A third feature also affords dynamic manipulation but this time what is changed is the users' viewpoint of the Turtle Geometry space a) by a toggle fashion by using buttons to pick among 3 default views (front, side, top-down) (Fig. 3) and b) by dragging a specially designed vector tool, which we called 'the active vector', where the user can define the camera's direction or position.

The work reported in this paper is part of a design experiment in the sense that Cobb et al. (2003) have described it. The research took place in a classroom with 12 year olds in a public school in Greece. The class had totally 16 teaching sessions with the experimenting teacher over two months. The activity sequence was divided in two phases with a strand of two tasks each. In task 1 (phase 1) the students were asked to

navigate the turtle in such a way so as to simulate the take-off and the landing of an aircraft. In task 2 (phase 1) the students were asked to construct rectangles and to position them in at least two different planes. In the second phase students experimented with two half-baked microwords (Kynigos, 2007). In each case they were given a ‘buggy’ procedure and were asked to experiment, figure out what was wrong or superfluous in the code and correct it. In particular, in task 3 (phase 2) students were asked to use the variation tool to control and experiment with the three variables of the procedure ‘movedoor’ (Fig. 1) that corresponded to different turtle turns so as to create the simulation of door opening and closing. The procedure was designed to have on purpose more than the variables needed. Variables a and c need to have a constant value of 90 while variable b needs to remain a variable. The students were not told of this of course. They first had to decide which was the role of each variable and which values could be given to it. Then they had to make changes to ‘movedoor’ so as to develop a procedure that only creates the simulation of a door opening and closing with the least possible variables. In task 4 (phase 2) the students were asked to use the variation tool to control the four variables in procedure ‘revolving’ (Fig. 1) which corresponded to turtle turns so as to create the simulation of a revolving door with four rectangular flaps. The procedure was again designed to have more than the variables needed. Students had firstly to work out the role of each variable and which values could be given to them. As before, they had to make changes to ‘revolving’ so as to develop a procedure that creates the simulation of a revolving door with the least possible variables. Finally the students were asked to extend the procedure of the revolving door in order to create a simulation of the fan of a watermill.

<pre> to movedoor :a :b :c uppitch(:a) leftroll(:b) repeat 2 [forward(3) right(:c) forward(2) right(:c)] end </pre>	<pre> to revolving :a :b :c :d up(:a) lr(:b) repeat 4 [repeat 2 [fd(7) rt(:c) fd(4) rt(:c)] lr(:d)] end </pre>
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Figure 1: The Logo code of the two half-baked microworlds

In order to describe pupils’ learning trajectories as they happened in real time we adopted a participant observation methodology while the main corpus of data included video-recorded observational data, researchers’ observational notes as well as the sorting and archiving of pupil’s work on and off computer. Data were categorized in clusters of specific critical episodes that do not represent some quantifiable entity but are chosen to represent clearly the kind of activity that was going on in a specific time in the classroom. The results presented here are based on the work of one focus group.

CONSTRUCTING MEANINGS ABOUT ANGLES WHILE CHANGING PERSPECTIVES

The analysis that follows shows that the various perspectives taken of the constructed graphical objects as well as the fact that this could be achieved through various means mediated the notion of angle in quite distinct ways with respective consequences on meaning construction.

Perspective taking through the use of microworld’s cameras

The students seemed to have initially focused on angle as a directed turn trying to syntonize their embodied motional experiences with turtle’s motion as its is evident both from the view of the 3d space which they preferred as well as from the kind of turn commands they used (Fig. 2). Flying the turtle along the z axis by maintaining a frontal view the orientation of the vehicle of motion coincided both with the orientation of students’ bodies in the lived in 3d space and with the standard way of referring to the orientation of information on the computer screen. Students’ comments corroborate this result. When asked why they preferred this kind of ‘flight’ they replied: *‘If we wanted to turn turtle right or left, we could see from our hands. If we wanted to turn it right, let’s say, we would think where our hand is and we would send it to the right’*. Moreover in this view the turn commands given to the 3d turtle was in accordance with our earthly experiences where the directions of ‘up’ and ‘down’ are fixed as a result of the gravitational effect. This was rather critical in particular as far as the use of the commands `uppitch/downpitch` are concerned, as other relative researches have underlined students’ difficulty in applying the above mentioned set of commands in an intrinsic way (Kynigos et al., 2009) and not in relation to the standard up/down directions that are automatically applied to 3d space,

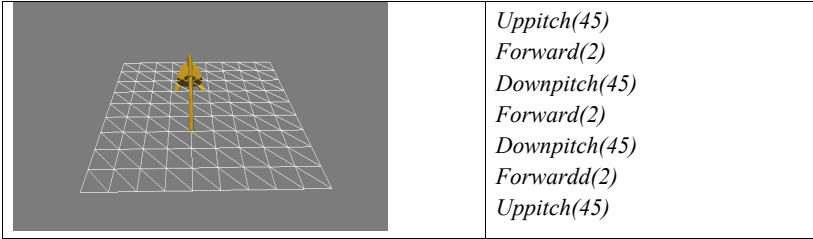


Figure 2: ‘Flying’ the turtle along the Z axis and the respective Logo code

Progressively however, the students were not so concerned about body-syntonicity and they used various views of the simulated 3d space when they wanted to focus on the graphical results of specific turtle’s turns. For instance during task 4 students had extra difficulties in finding out the role of the `:d` variable, which determined the measure of turtle’s turning and respective position in the 3d space before drawing each successive door of the revolving door model. In the following episode students conjectured about the number of the visible rectangles (doors) if the value given to `d` is 720. However they did not find the front default view convenient and after testing

all the available default views they choose to continue working with the top-down view active, where the doors/rectangles created by the turtle was more clearly visible.

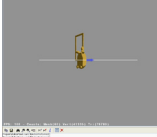
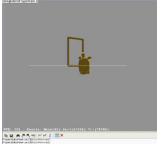
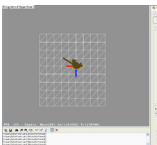
<p>S1: Lets see how many doors there are if the value is Only one? This perspective is not convenient, I will change it (He activates successively all the 3 default views and opts for the top –down one).</p>		<p>Front view</p>	
<p>S2 Yes, exactly like in the case of 360. It turns two rounds.</p>			<p>Side view</p>
<p>S1: Yes, it collects all of them in one. When we move it, the doors are changing position. They are sticking together or they are unsticking.</p>			
<p>S2: We can't say that. With d we determine their place. Look, If it is 90o they are turning and they are forming a cross, they form right angles, yes right angles, with 360o or 720o they are placed together in the same line.</p>			

Figure 3: Episode 1 and the 3 default views of the simulated space

It should be noticed that with the default views students came in contact with simplified 2d views of the simulated 3d space which possibly helped them focus on specific aspects of their construction as a result of turning commands. In the above episode trying to explain screen phenomenology in relation to the measure of a leftroll turn command students opted for the top – down view where they could easily observe turtle's rolling and the number of drawn rectangles. In this view the dihedral angle between the parallelograms is rather more easily discernable as it looks more like the 2d geometrical figures that they are accustomed to. Thus students can in a way coordinate turtle's turning – one round for 360o and two rounds for 720o- with the static geometrical figure of 360 and 720 angle where the position of the two rays that form the angle coincide.

Changing perspective of the constructed 3d object through the use of the variation tool.

During the 3rd and 4th task the simulated 3d object's position and orientation could be dynamically arranged through the combined use of Logo programming and the variation tool. The sequential change of the values of variable a and b in both respective procedures created a film-like succession of the different instances of the 3d model that gave the impression of rotation. For instance the dynamic manipulation of the values of variable a helped students view their constructions from different perspectives while not changing perspective as observers of the 3d space. In the following abstract students are trying to change the orientation of the revolving door model in relation to the ground plane so as to create the fan of a watermill. As a result

of their previous experimentation they can easily discern that it is the same construction with a different orientation in 3d space (*we should lie it down*), while their construction in a front 3d view should look like viewing it from the top. They also instantly recognise that they should change the values of variable a and they start experimenting with it through the variation tool.

S2: Now we should lie it down.
 (First print screen on the right)
 S1: It's the same as before.
 S2: Yes, it's like viewing it from the top. (Second print screen on the right)
 S1: Move variable a...yes 360, it makes a whole circle and lies down
 S2: or 720, 1080 etc'

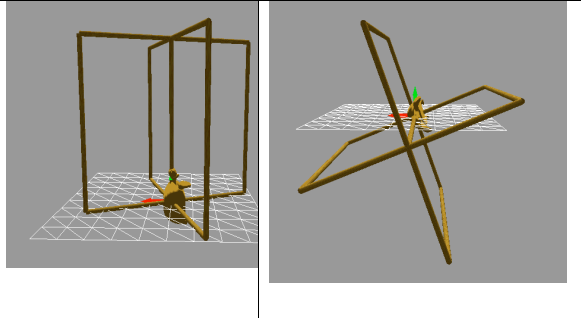


Figure 4: Episode 2

Changing the values of variable a sequentially, they observed the turtle and the whole 3d construction rolling around the x axis and having the desired orientation in 3d space for multiples of 360 values. Dragging the variation tool didn't provide thus just a kinaesthetic sense of dynamic manipulation and animation of mathematical objects changing only their visual characteristics. It provided an action/notation context that fostered experimentation and rendered the various turn commands descriptors of evolving geometrical objects' place and orientation in 3d space.

Imaginary perspective taking through the turtle metaphor

In the end of Task 2 there was some free time available and students spontaneously decided to try to construct a closed figure building upon their experimentation during Task 1 where they simulated the flight of an aircraft (Fig 5). Each take-off and landing of the turtle was used as the building block of a 'peculiar' figure that came as result of four repeats of the initial turtle's journey while turning turtle 90 degrees before each re-execution. It is also interesting (Fig. 5) that students adopted a more analytic strategy, visualising the whole turtle's journey and explaining it to each other before entering commands. Moreover they adjusted the view of the 3d space with the active variation tool so as to have a clear 3d view of the simulated space and they kept it fixed throughout their construction. It seems that as students got progressively more accustomed to the turtle's motion and the software's representational infrastructure the crucial point was not so much body syntonicity with the turtle but the coordination of two different view points: the view- point of the turtle which must be moved in an appropriate way so as to draw a figure and the view point of an external observer who looks at the figural results of turtle's movement.

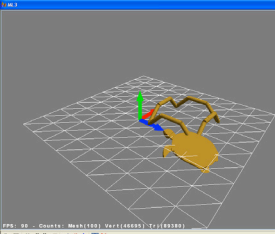
<p><i>S1: Let's make a square. So not right 45 but right 90, so as to go this way and then again 90</i></p> <p><i>S2: Yes, and again 90 and we will come back. (So far they were talking to each other and now they return to the microworld inserting the commands)</i></p>		<p><i>Uppitch(45)</i></p> <p><i>Forward(2)</i></p> <p><i>Downpitch(45)</i></p> <p><i>Forward(2)</i></p> <p><i>Downpitch(45)</i></p> <p><i>Forward(2)</i></p> <p><i>Uppitch(45)</i></p> <p><i>Right(90)</i></p>
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Figure 5: Episode 3 and the respective Logo and graphical result

Constructing the simulation of a 3d object while viewing the simulated space in perspective was not only more realistic or more familiar but allowed students to reconceptualise 3d objects in terms of Logo commands while paying attention not only to turtle's immediately previous state but to the whole 3d space. In the above episode students executed four times the command right (90) so as to *come back*, to use students' words. Changing turtle's perspective in 3d space while having a fixed 3d view of the simulated space students seem to have intuitively articulated the total turn trip theorem (Papert, 1980): walking all the way round a polygon and returning to its initial position with the same orientation the turtle makes a full turn.

CONCLUSIONS

The above analysis addressed some of the meanings generated by the 12 year olds as they interacted with a microworld that afforded opportunities to act on multiple interlinked representations of angular concepts in 3d space (Morgan et al., 2009). The students used the simplified 2d views of the simulated 3d space through the manipulation of cameras to coordinate the figural results of the turtle's turning in 3d space. They addressed angle as a directed turn in using the variation tool to experiment with figural changes resulting from changes in variable values in the context of noticing and understanding 3d objects' spatial and geometrical properties. They seemed to see sense in moving from a turtle metaphor to that of a vehicle of motion metaphor (e.g. flying the turtle) in order to coordinate the intrinsic perspective of a moving entity with the perspective of an external observer of the 3d figural constructions as a whole and thus to use angle as a spatial visualisation concept. The students' mathematical expressions may have been more or less divergent from institutionalised mathematics. However their constructionist activity seemed to offer them rich mathematical and phenomenological experiences which we suggest could scaffold meaning construction later on when formal teaching of 3d space takes place (Papert, 1980, Freudenthal, 1983). This research addressed meaning generation in the context of using dynamic and symbolic representations of 3d space in constructionist tasks. However, further research is needed in order to investigate the way angular concepts can be integrated with spatial navigation and orientation as well as with the use of metaphor in virtual 3d environments. For instance the didactical notion of building upon intuitive embodied metaphors of locomotion (Clements & Sarama,

1997) is highly questionable in 3d space as it is not obvious how those metaphors can be associated with the different kinds of turn in 3d space, taking into account that our earthly experiences are confined to a two dimensional (ground) plane.

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TEACHER STUDENTS' IMPROVEMENTS IN CALCULATION SKILLS AND UNDERSTANDING IN THE CASE OF DIVISION

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Abstract: In many studies in Finland, it has become out that elementary teacher students have severe lacks in mastering of division and understanding its functional principle. In this paper, it will be presented a teaching method where students via independent pondering could get an insight on the principles of division, and thus their calculation skills will be developed. To teacher students at the University of Lapland were offered two possibilities to accomplish the basic course in mathematics: an examination or writing accounts. Results dealt with here are from four years experimenting (N = 220). According to the tests in the beginning and at the end of the course, the traditional teaching method with examination seemed to be ineffective, whereas the pondering method with account writing produced clearly better results.

One of the primary goals of the Finnish comprehensive school curriculum in mathematics is to make pupils understand and master basic calculations (NBE 2004). Division is the most complex operation children have to learn in elementary school mathematics, as it requires good skills in other basic operations. But research has shown that Finnish students have problems understanding division in upper secondary school (e.g. Hellinen & Pehkonen 2008), and even in university-level of elementary teacher education programs (e.g. Kaasila & al. 2010).

For elementary teacher education it is a challenge, since prospective teachers should have the skills and knowledge expected in the curriculum (NBE 2004), in order to be successful in their teaching. A minimum requirement is fluent calculation skills, but teachers should be ready to explain in teaching situations the principles of calculation operations, if needed. This demand expects the mastering of algorithms on the level of understanding (Leinonen & Pehkonen 2009). Here we present a teaching method that activates students' own thinking, and thus helps them to develop reasoned knowledge on division algorithm, in order to understand it properly.

THEORETICAL FRAMEWORK

One goal for mathematics teaching in the comprehensive school is to develop pupils' mathematical thinking (NBE 2004). Other central goals in the curriculum are good calculation skills, adapting mathematical concepts and understanding.

Understanding

Mathematical understanding can be characterized as a continuous process that is connected with a certain person, a mathematical content domain and a special environment (Hiebert & Carpenter 1992). Mathematical understanding answers the question "Why?" and entails, among other factors, the skills required to analyze mathematical statements. Within the last twenty years, researchers have developed

theories of mathematical understanding as a dynamic process, how an individual's mathematical understanding develops (e.g. Pirie & Kieren 1994). In the Pirie & Kieren model, understanding is seen as a process where an individual can progress from one level of understanding to the next. The progress from one level to another is not necessarily linear: an individual may regress in his/her understanding.

If we go deeper into the concept of understanding, it seems to be more complicated. For example, Leinonen (2011) introduces the four modes of understanding: conceptual knowledge, grasping meaning, comprehension and accommodation. The function of those modes is to give the background and conceptual instruments for thinking, to interpret the information, to synthesize the knowledge, to integrate the message into permanent memory, and to reorganize the cognitive structure. These ideas he has discussed in detail in the publication (Leinonen 2011).

The definition of understanding is difficult also therefore, since its meaning depends on the view point selected. Understanding has linguistic, epistemological, cognitive and social dimension that are totally involved in learning. In the paper at hand understanding is seen as a cognitive process that results structured knowledge (cf. Hiebert & Carpenter 1992; Kilpatrick 2009).

Polarisation of knowledge

In learning theories, mathematics is often described as polarized knowledge: procedural and conceptual knowledge (e.g. Hiebert & Lefevre 1986) or operational and structural knowledge (e.g. Grey & Tall 1993, Sfard 1994). It has been seen that such a rough two-division leaves a gap between different kinds of knowledge, and it could form a pedagogical problem. In the literature, it is often shown like Sfard (1994) and Skemp (1976) that procedural knowledge lacks a personal grip and reasoning that belong to conceptual knowledge: it is "rules without reasons". Procedural knowledge offers readiness to routine performance, and thus mathematics can be seen by an individual only as manipulation of number symbols.

On the pedagogical viewpoint studying mathematics can be seen instead of dualistic viewpoint as a complementary event where procedural and conceptual knowledge develop together accomplishing each other (Hiebert & Lefevre 1986). One possibility to reduce this gap is individual pondering that is typical for mathematics (cf. "adaptive reasoning" by Kilpatrick 2009) as a contrary to remembering memory rules ("rules without reasons").

Division

Division is an important but complex arithmetical operation in the mathematical education of future elementary teachers. The long division algorithm begins from the left, unlike other basic operations, and it requires estimation skills at every step. Furthermore, division by a decimal number requires an ability to expand the fraction given by the division task, and to determine the place of the decimal point. Earlier studies have shown that students in education consistently display weakness in

division (e.g. Simon 1993, Campbell 1996, Merenluoto & Pehkonen 2002, Leinonen & Pehkonen, 2009; Pehkonen & Kaasila, 2009, Kaasila & al. 2010).

Division is an essential arithmetical operation to consider in teacher education because many prior studies show that a part of elementary teacher students have lacks of understanding division (e.g. Tirosh & Graeber 1990, Simon 1993). One reason for the shortage of understanding seems to be primitive models of division (e.g. Simon 1993). For example, students use only the strategy of partitive division in their calculation. Other important components for insufficient understanding of division are 1) weak understanding of remainder, 2) insufficient understanding of the connections between mathematical operations, 3) difficulties in explicating and giving reasons for strategies used in reasoning and 4) staying on the integer level (Kaasila & al. 2010).

Focus of the paper

The purpose of this paper is to inquire whether activating teaching methods produce better skills in division than traditional teaching. Students were encouraged to reflect on the principles of the long division algorithm and to write about this process of reasoning in their accounts to the instructor during their mathematics course.

IMPLEMENTATION OF THE STUDY

In the LOMA project (Elementary teachers' mathematics), financed by the Academy of Finland (project number #8201695), data was gathered with a test measuring students' calculation skills and understanding (cf. Kaasila & al. 2008). The same test was used in this study.

Participants

The participants in the study were 268 students of elementary education at the University of Lapland (Rovaniemi, Finland) from four different courses in 2007–2010. Approximately one third of them had studied advanced mathematics in upper secondary school, whereas two thirds had taken the general mathematics course.

Indicators

The test used in the LOMA project was also administered here to gather data from a group of 268 elementary teacher students at the University of Lapland in 2007–2010. Additional empirical data offered also accounts written by a smaller group of students that wanted to accomplish the course without an examination.

The students were tested in division both in the beginning and at the end of the course. The students were given 45 minutes for each test, and they were not informed about the tests beforehand. Three division tasks were selected for a closer study. The purpose of the testing was to analyze how much the students' division skills had improved during the course. The students were not allowed to use calculators during the test, and all calculations had to be written down.

Data analysis

Altogether 220 students participated in both tests (the start and end tests). Here we focus only on the papers of these students. The tasks in the test were scored on the scale 0, 1, 2 in such a way that 0 point was given for a totally wrong answer, and 2 points for a totally correct answer. One point was given for a partly correct answer.

For the points of the tasks, mean values were calculated. Students' development in

<u>The start test:</u>	<u>The end test:</u>
Task 1. Calculate with a long division $3159 : 13$.	Task 1. Calculate with a long division $27408 : 12$.
Task 2. Reduce as long as possible $96/504$.	Task 2. Reduce as long as possible $144/1584$.
Task 3. Calculate with a long division $1.488 : 0.24$.	Task 3. Calculate with a long division $2.618 : 0.14$.

calculation was evaluated with the changes in the mean values for each task. Here we concentrate on students' performance in following three tasks, and consider them more closely.

The papers were checked by the course teacher, and in addition by another mathematics teacher, too. The accounts were interpreted by two researchers and a mathematics teacher. After discussions they came to an agreement. The differences in calculation tasks between different groups were tested with the U-test.

The influential factor

The students participated in a basic, six-week course of mathematics at the University of Lapland from four different years in 2007–2010. Each week they had a 90-minute lecture and a follow-up 90-minute exercise in small groups. Both multiplication and division with rational numbers were topics in the lecture and exercise of one week. These operations were taught in the case of integers and decimal numbers. The lecturer used the models of both quotitive and partitive division. He gave the students several examples to demonstrate the principles of the long division algorithm.

In accomplishing the basic course in mathematics of the elementary teacher program at the University of Lapland, it was offered in each course in 2007–2010 two alternatives: the examination after the course or writing accounts (of a couple of pages) during the course on the topics dealt with. All students were expected to participate actively in the lectures and small groups. The topics of the course were i.a., as follows: basic calculations, percentages, number systems. Two double-hour units were used for division, one for a lecture and the other one for a small group.

The students received from their accounts regularly a written feedback and more instructions for accounts. Messages between the teacher and the students were sent by e-mail. Additionally, there were a couple of personal discussions. All students had in daily use a personal computer that the university has offered them.

In the beginning of the course, some ideas were given for encouraging the students to independent thinking and writing accounts:

The most important is wondering. Nothing is automatically clear. There are no wrong why-questions, only answers could be wrong. Try to remember what was difficult in mathematics or what kind of problems you struggled with during your school mathematics. Ponder the reasons for using calculation rules used, and write your pondering into the account.

For division there were some more accurate hints:

Think by yourself, and then answer the following questions: Why is it important to estimate the result of the division before beginning to divide? Why should division begin always from the left hand side? Ponder the meaning of separate phases in division. How will you begin when a decimal number should be divided by a decimal number? What is the place of the decimal point in the quotient, and how can you reason the place? What is a difference in division, if you will use another number system?

RESULTS

In the years 2007–2010 at the University of Lapland, there were altogether 268 elementary teacher students, and of them 220 participated in the start and end tests.

Table 1. The number of teacher students in different years.

	2007	2008	2009	2010	Altogether
The account group	12	13	20	12	57
The examination group	36	46	30	51	163

U-test showed that there were no statistical significant differences between different years. Therefore, the student total for four years has been dealt with together.

Almost all participated students passed the course during the same year. Thus we consider two groups of teacher students who were accomplishing the course with different methods: the account group ($N = 57$) and the examination group ($N = 163$).

Solution frequencies

According to the tests (Figure 1), the differences between the groups in starting situation were only a couple percentage units. In the three tasks, the differences were deviating. In the first one (division of natural numbers), the start points of the students were almost in the top, but in the third one (division of decimal numbers) the corresponding mean value was under the half of the maximum value. In the second task (reducing), the start points were between the other ones.

In all three tasks, the rise of the mean value in the test points was bigger in the account group than in the examination group (Figure 1). According to the t-test, the difference of mean values was only in the results of the third task statistically significant (level .05; $t = 2.051$; $df = 218$).

The most common mistake in the first test was that in the third task, the quotient obtained was divided by the expander. This mistake was very rare in the second test. In the second test (division of decimal numbers), some students multiplied the divisor and the dividend with different numbers before the division algorithm. For example, in the task $2.618 : 0.14$ the dividend was multiplied by 1000 and the divisor by 100. Furthermore, some students had difficulties in multiplication tables, in some papers were also the remainder divided by the divisor.

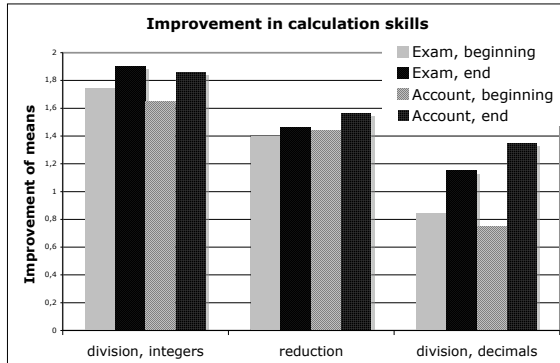


Figure 1. The mean values of the test scores in the start and end tests in 2007–2010 (the account group $N = 57$ and the examination group $N = 163$).

Comments on accounts

In writing accounts, students were first very careful, and discussed more mathematics teaching in school than the topics dealt with in the course. They described their own relationship to mathematics during the school time, and most of them had negative memory pictures. They remembered learning-by-heart and fear before examination. Therefore, accounts offered to some of them a released way to study, since they were not in need to fear the examination. School received also much critics for constant hurry and performance-centeredness.

Most of the accounts could be grouped into one of David Tall's three worlds of mathematics (cf. Gray & Tall 1993), as one can read in the following quotations:

- (1) The embodied world:
"I figure minus numbers always with a thermometer" (Paul).
"The fourths are always easy to think with a pizza model" (Julia).
- (2) The symbolic world:
"Mathematics is full of rules, and every now and then I have a feeling that there is no logic. However, they evidently have, but their understanding will demand a deeper diving into the world of mathematics" (Julia).
- (3) The formal world:
"Somebody asked why in dividing fractions one should change the place of the numerator and the denominator in the latter fraction" (Josh).

In several accounts, it can be read that peaceful pondering and writing accounts has cleared up the principles of the division algorithm. For example, it became clear to Julia and Henry that the quotient is not allowed to be divided by the expander:

"For me it cleared up, among others, that if one divides with a decimal number, the divider and the dividend can be expanded to integers, and that after division one needs not to change the quotient. When I have once seen it, the rule seems to be self-evident" (Julia).

"Also for myself it was important to notice that after division one should not divide the quotient with the expander" (Henry).

In many comments it became clear that the writing of accounts encouraged students to own thinking and discussions. Additionally, the writing of accounts was a welcome alternative for an examination. For example, Anna wrote, as follows:

"I must say that the course has given more than I dared to expect ... Accounts offered for our group of girls more opportunities to stop and think" (Anna).

DISCUSSIONS

It seems that, based on the results of the starting tests, the goals of the comprehensive school curriculum (NBE 2004) have not been fulfilled in the case of good calculation skills even in all of the best of the students. And there are severe gaps of calculation skills of students in the elementary teacher education (e.g. Kaasila & al. 2010).

An explanation for the modest effect of traditional university teaching might be the passive way of studying: Students are present in the lectures and small groups, and they work on given tasks, but they are not initiatively trying to understand mathematical rules, relations, principles and new perspectives. It seems that the traditional teaching method (lectures, small groups, examination), at least in the case of calculation skills, is wasting resources of students and teacher educators. From the results of the activating teaching, one can reason that writing accounts will commit students to ponder the principles of calculation algorithms. At the same time, students' mathematical thinking and calculation skills will develop.

In summary one can state that an activating teaching method seems to have many advantages. The pondering of principles and the writing on them will produce more effectively calculation skills than the traditional teaching with examinations. Division as a complex basic calculation needs for routine learning much memory capacity. Perhaps, therefore, a conceptual generalization and structural thinking (cf. Sfard 1994; Gray & Tall 1993) have an opportunity to show their power compared to routine learning, especially in division. Through developing elementary teacher students' mathematical thinking, one can indirectly influence the quality of mathematics teaching in the comprehensive school. Such a teaching method, based on pondering, can be applied already in the elementary school, and thus promote pupils' mathematical thinking and calculation skills, as Ji-Eun (2007) has done.

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THE POVERTY OF MOTIVATION: A STUDY OF DISAFFECTION WITH SCHOOL MATHEMATICS FROM A REVERSAL THEORY PERSPECTIVE

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Disaffection is a complex phenomenon that can be understood from the perspective of motivation and emotion. It is easy to present disaffected students as lacking in motivation. This study presents evidence that students who are disaffected with school mathematics are in fact, highly motivated, but their motivational needs are frustrated and unsatisfied by their experiences of school mathematics. Reversal Theory is presented as a viable theoretical framework for understanding the complexity and dynamic richness of motivational and emotional phenomena. Qualitative data is presented from a phenomenological, student-centred and interpretative perspective.

DISAFFECTION AS A MOTIVATIONAL ISSUE

Disaffection with mathematics education has been a significant concern for a long time and has recently been expressed, not only in the research and educational communities, but also in reports and policy documents by government, industry and other public bodies. In the PME context, there has been some consideration of negative affect. So, for instance, (Pantziara & Philippou, 2009) have shown that motivational constructs are an integral part of students learning, and (R. Zan & Di Martino, 2009) have related learned helplessness to low perceived competence. (Underwood, 2009) in a review of research literature on low and under-achievement, says, 'In the UK, the problem is crystallised in a cohort of 16- to 18-year-olds that are not in education, employment or training (NEET). This group contains a preponderance of individuals who have failed to acquire even the basic skills that will allow them to participate as full citizens in tomorrow's world.'(Underwood, 2009)

Disaffection is not an easy construct to characterise in research terms, and characterisations of disaffection in educational research have focused on truanting and bad behaviour. (Nardi & Steward, 2003) have sought to widen the definition to include 'quiet disaffection', meaning low engagement and perceived lack of relevance.

Disaffection is not the same as innumeracy or low achievement, but it does mean that significant numbers of students are 'lost' to mathematics post-school, with the consequent social and economic cost. It is an emotionally and motivationally based phenomenon mediated by reflected cognition and judgement. It impacts on attitudes, interests, beliefs (including beliefs about self), perceived utility and other aspects of

affect. It involves a falling away over time – it has a level of stability, and is not purely situational, of-the-moment. It is interwoven with social, political and cultural issues. For the purposes of this study we will take a common-sense working definition of disaffection with school mathematics as :

a negative orientation to school mathematics, involving aspects of motivation, attitude, beliefs and emotions that inhibit or prevent engagement, the development of competence and achievement.

This study adopts a number of views based on a study of the relevant literature. The first of these is that the study of disaffection can be best served by a shift of focus from attitudes and beliefs to motivation and emotion. There is a growing consensus that it is necessary to move beyond the simple dichotomies of positive/negative attitude, and a narrow range of emotions such as fear and anxiety (Hannula, Pantziara, Waege, & Schloglmann, 2010). Secondly, we hold with (R. Zan & Di Martino, 2007) and others, that there needs to be a focus on more qualitative research, and a variety of methods to access the complexity of motivational and emotional phenomena. Finally, there is an opportunity to introduce a theoretical framework new to research in mathematics education.

To investigate disaffection more fully, it is proposed that a focus on motivation and emotion, and a widening of the methodologies used in the context of a robust theoretical framework, will provide a deeper, richer and more dynamic picture of the landscape of disaffection, and thus create potentially new insights.

THEORETICAL FRAMEWORK

In this study Reversal Theory is used as an interpretive framework for the qualitative data. Reversal Theory is a theory of personality, motivation and emotion which has its origins in a phenomenological perspective on the troublesome and problematic behaviour of young people. Over the last 30 years it has been applied to a whole range of human behaviour as reported in (Apter, 2001). Its potential utility in educational research has been evidenced in (Apter, 2001; Mallows, 2007). However, it has yet to be applied in systematic research, and not at all in relation to mathematics education.

Space prevents a comprehensive description of the theory, and its potential relevance to educational research, but we will seek to identify how the eight motivational states posited by the theory are expressed and instantiated in relation to school mathematics, and their relevance to understanding the phenomenon of disaffection.

The motivational states and their characteristics are shown below.

State	Core value	Desired feeling	Way of experiencing
Telic	Achievement	High significance	serious
Paratelic	Fun	Low significance	playful

Conformist	Fitting in	Low negativism	conforming
Negativistic	Freedom	High negativism	Challenging/rebellious
Mastery	Power/control	High toughness	competitive
Sympathy	Love	Low toughness	affectionate
Autic	individuation	Low identification	Self-oriented
alloic	transcendence	High identification	Other-oriented

Table 1 Defining characteristics of the eight motivational states

These states comprise four pairs of polar opposites (serious-playful, conforming-rebellious, mastery-sympathy, self-other oriented). A person will be in one state from each pair (ie. four states) at any one time. However, one of these four states will be focal, which is to say at the centre of our phenomenal field, and the most salient to our felt experience at that moment. The dynamic complexity of motivation arises because we can switch from any state to its opposite (called a psychological reversal), and do so frequently. When we are in a motivational state, we may gain the satisfaction associated with that state, or we may not, depending on our interpretation of events. This alignment or discrepancy gives rise to the primary emotions.

This study presents the results of a number of exploratory qualitative interviews with disaffected students. Five semi-structured interviews were conducted, involving six young volunteers (17 to 19 years old), at a Connexions office in the East Midlands of the UK. This organisation exists to help young people with the transition from school to employment, training or education. As such, they encounter many young people who fit the NEET (not in education, employment or training) category, and who are likely to be significantly disaffected with school mathematics as they have tended to perform quite poorly in national examinations.

The interviews reflect a constructionist approach, which characterises research from a participatory point of view, and privileges the point of view of the individual and how they make sense of their learning experiences (Hodkinson & Macleod, 2010). In this way, motivation and emotion become salient features of the research agenda. Thus, insights were sought into the depth and complexity of the phenomena under consideration from the subjective perspective of the young people involved. The focus of the interviews was on motivationally and emotionally significant experiences of, and dispositions to school mathematics.

Since many young people in this NEET category are not highly articulate in terms of their own emotional landscape, some projective techniques were used to elicit the more emotional and non-rational responses to aspects of their experience. This was done by presenting the interviewee with photographs of typical classroom situations, to which they then make up the story. Initial evaluation suggests that this approach elicits powerful and positive results. Interviewees were also asked to choose from

lists of emotionally and value-laden words those which applied most to them, and to describe their importance and relevance.

A REVERSAL THEORY PERSPECTIVE ON DISAFFECTION

In order to put structure to an analysis of the motivational landscape of these young people, the eight motivational states of the Reversal Theory framework will be used to examine the motivational and emotional experiences reported. The analysis will focus on the narrative from Sam, but will be supplemented by narrative comments the other interviewees. Although Sam enjoyed and was able to do mathematics at primary school, he lost his way at secondary school and performed badly. He can be considered as disaffected during his latter compulsory school years.

The **serious state** provides a focus on goals, outcomes and future consequences. The motivational need here is for a sense of felt significance or purpose, and our need is to make progress towards satisfactory outcomes. These students appear to have internalised the importance of mathematics, but in many instances they are unable to sustain this sense of significance in individual tasks or even whole topics in mathematics. This perceived pointlessness is expressed in many ways: ‘long division, what’s the point?’ (Scott)

There is substantial evidence (as one would expect) of the serious state being operative in the data. When combined with the **conforming** state, (serious-conforming) a person will want to do what is expected and required of them, and that will include complying with the socio-cultural norms and expectations. Despite this, we see in the evidence many statements of an inability to perform or achieve what is expected. This causes the felt arousal (the controlling variable in somatic states) to increase and the associated negative emotions such as anxiety, panic and fear to appear. This is evident in many of the accounts: ‘You feel put on the spot....nervous.’ (Mel). ‘When I get asked questions I freeze.’ (Kelly). ‘I’m scared to put my hand up.’ (Mel). ‘I panic and cry in maths exams.’ (Mel)

On the other hand, there is a motivational payoff when people achieve what they wanted to : ‘(I got)...satisfaction that I’d passed it.’ (Scott). ‘[I enjoy maths most]... once I’ve got something correct.’ (Scott)

Sam has a number of sources of felt significance in relation to mathematics. He is quite articulate about the importance of mathematics in life in general: ‘Everything in life is controlled by some sort of equation.’ ‘Maths is the control within the chaos.’ He seems to have an almost philosophical notion that mathematics is a part of understanding the bigger picture of life (‘if you don’t possess that skill....you are dumb....you’re not as human as you could be.’). However, this is in contrast to his inability to do mathematics, other than basic calculations. In his account of his worst experience of maths he recalls an episode when he is invited up to the board and he freezes with panic. ‘The teacher says it’s simple but I just don’t understand it...I don’t think I need to understand it. From that point on I didn’t stand up any more.’ He

holds practical maths in high regard as the utility makes sense to him ('I can manage my bills, taxes, loans...that's power isn't it?'). There are a number of factors involved in his disaffection, but partly it is down to being put into a top stream, and introduced to (as he sees it) abstract mathematics for which he didn't see the point.

The **playful** state also features in all of the interviews. In this state, our attention is on the enjoyment and satisfaction of the immediate present. We are pleasantly immersed in our current activity without focus on future outcomes or consequences. This state is associated with fun, excitement, risky behaviour. It can be brought about by cognitive stimulus such as intrigue, fascination and so on. In the playful state we will be arousal seeking, as high arousal is experienced as pleasurable (excitement). When combined with low arousal, it will be experienced negatively as boredom or sullenness. There are many references in the narratives to fun. Young people will spend a large part of their lives in the playful state, and in this state, their motivational needs are for enjoyment in the moment. This need is expressed widely: 'I want to make it fun.' (Mitchell). '[the teacher] will try to make it as fun as possible.' (Mitchell). '[Primary school] was more fun, and you understood.' (Kelly).

The paratelic pleasure is one that seems not often available to these students, but there are occasional glimpses: 'Once you got the hang of it I was wizzin'(sic) through it.'(Mel). '[games]...that's what made it fun.'(Mel). 'It's exciting isn't it...money.....the risk and everything' (Sam).

On the other hand, when this is not available, arousal is low and boredom sets in: '[on photo 2]....she's bored – because of those books.' (Braidon). 'Most of it is bookwork – boring.' (Mel).

In Sam's case, there is evidence in his account of the need for paratelic enjoyment. But perhaps the most convincing expression is his fascination with the more speculative notions he has about the importance of mathematics in history and culture. There is excitement and relish in his discussion on these matters.

Mitchell is also an excellent example of the expression of the playful state. Much of his conversation is focussed on the notion of fun or making lessons more enjoyable, suggesting that his motivational style is dominated by the playful state. He also talks about boredom, again suggesting he is in the playful state with the need for excitement unsatisfied. It was interesting to observe that he also conducted a good deal of the interview itself in the playful state – treating it as a game, and occasionally saying things to shock in a playful or mischievous way (playful-rebelliousness).

Self-mastery is the desire for power and control over our immediate world whether this be physical (eg a skill like driving a car), or in relationships. There are at least two aspects of self-mastery relevant in the context of a mathematics classroom – the cognitive ('I understand') and the performative ('I can do it'). For many of these students, there is little separation – to understand means to be able to do it. References to needing to understand (or not understanding) possibly outnumber any

other category of motivational statement in this body of evidence. Thus: ‘[I like doing]...something easy, that I can do.’ (Braidon). ‘[the best thing about maths is]..the satisfaction if you know how to do it.’ (Scott). ‘When you understand it you can enjoy it.’ (Mel). ‘I’ve triumphed it...I can do it again and again.’ (Braidon).

However, disaffected students appear to experience self-mastery substantially in ‘lose’ rather than ‘gain’ mode. ‘How on earth do I do this? (Mel). ‘[I freeze when]..I just don’t know it.’ (Kelly). ‘I can’t do it.’ (Mitchell).

Many of the stories and examples recounted relate to the impending sense of humiliation of being seen not to be able to do what is required in front of other people. This fear of public embarrassment is very strong, and subjects report feeling ‘stupid’. Not being able to understand (or do it) is a major source of pain for many students, and repeated experiences are clearly a major factor in disaffection. Emotions related to self-mastery (losing) reported here include humiliation and embarrassment. It also accounts for the many mentions of powerlessness. Emotions reported relating to self-mastery (winning) are pride and ‘triumphing’.

Self-mastery can also be expressed in terms of dominance over others or competitiveness, and for disaffected students this is usually expressed in ‘losing’ mode. When Sam is recounting his experiences of school, his expressions of self-mastery are clearly related to lacking in power and control. When discussing the competitive nature of the classroom he says: ‘It made me feel dumb...it ruined my self-confidence.’ In response to a photograph of an apparently confused student he recounts a motivational pathway or sequence that seems to be familiar with disaffected students: He can’t do what is expected (self-mastery, losing). Arousal increases and he then becomes anxious (serious-conforming). This turns to frustration, anger (reversal to rebellious state) and misbehaviour. This leads to him feeling isolated (self-sympathy, losing). However, he is later able to assert his self-mastery in his perception of himself in practical matters: ‘I know how the monetary system works.’

The **Self-sympathy** combination is the need for nurturing, affection and love, together with an awareness of one’s own emotional disposition. In the context of this study, this is often expressed as needing help and support. Self-sympathetic emotions range from gratitude when the need is satisfied, to resentment at feeling uncared for. There is some evidence here that pupils satisfy this need for each other by supporting and helping, when they are allowed to do so. In contrast to this is the **other-sympathy** combination— where we respond to the needs of others for care and nurturing. Other- orientation is also the source of our identification with others in relation to individuals, groups and institutions. It is expressed in the unsatisfied sense by Sam who feels excluded from the elite, and in the positive sense in his description of how mathematics classes should be – cooperating and working together and helping each other. In this data, the lack of the ability to cooperate and work in teams

is lamented by a number of the subjects. The climate in some of the classrooms reported does not appear to have a strong nurturing or cooperative element.

Scott conducts the whole interview alternating between self-mastery (losing) – ‘I struggled’, ‘I can’t do this.’ and self-sympathy (losing) – ‘no one’s helping me’, combined with the serious state. This indicates an important pattern in his individual motivational style.

The most frequent evidence of the **rebellious** state were expressions of anger (serious-rebellious, with high arousal), and this is evident in Braidon and Mitchell’s accounts. There is perhaps more evidence of sullenness (playful-rebelliousness, low arousal), as in the classic ‘whatever’ stance of teenagers.

The range of life histories and experiences of these young people is interesting, and defies attempts to classify disaffection in any unitary way. The narratives also demonstrate that motivational style is quite individual to each student.

Sam’s narrative can be seen as a desire to find significance and purpose, and the tension between his desire to gain mastery through practical attainment and his frustration at not being able to achieve this in a school context. Mel and Kelly see little value or interest in mathematics. One of the key factors that seems to drive their disaffection is a frustrated and unsatisfied need to understand. Braidon doesn’t like mathematics, although he enjoyed it until year 7. Since then however, overwhelmingly, his experience is of not being able to understand or do it (self-mastery, losing), together with feelings of humiliation and pointlessness. Scott’s narrative is dominated by his self-sympathetic perspective, and confirms his resignation and negativity in relation to school mathematics. He mentions the word struggle 17 times. He has a very limited and narrow view of mathematics, and frankly, just didn’t see the point. He is the personification of what Nardi & Steward termed ‘quiet disaffection’.

Mitchell was a self-confessed ‘problem pupil’ and a classic disaffected one. Perhaps more than the others (apart from Sam), he can articulate a range of rationales for why school mathematics is important, from life-skill (‘so you don’t get scammed on your wages’) to the vocational (plastering calculations) and other potential aspirations (‘you need your maths to get into higher things – like science.’). In fact, as he says : ‘It’s maths, maths, maths all your life.’ One of his dominant moods is boredom, and he is strongly driven by the need for mathematics to be fun, and he describes a number of ways he believes this can be achieved.

DISCUSSION AND CONCLUSIONS

In summary, on the basis of this data we can find quite substantial evidence of all of the motivational states described in Reversal Theory operating for these young people in mathematics classrooms. This enables us understand that there is often a mismatch between the motivational needs of the students and the teaching style, methods and pedagogy of teachers. When the need to inculcate procedural knowledge

and achievement of results outweighs the focus on understanding, many pupils will feel excluded. The students in this study give an account of mathematics classrooms that are motivationally impoverished. It helps us to understand on the other hand what a motivationally rich mathematics classroom might look like.

This data has provided further evidence of how motivational states are experienced and expressed in a mathematics education context. It is precisely because the motivational needs are so strong, that when they are unsatisfied, negative feelings and emotions arise.

It is clear from these different accounts that the patterns and styles of motivation are highly individual. They are likely to relate to learning style and the ways in which individuals prefer to engage with learning. This pilot study has confirmed the potential utility of Reversal Theory as an interpretative framework for mapping the motivational and emotional landscape of young disaffected students.

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PRIMARY STUDENTS' PERFORMANCE ON MAP TASKS: THE ROLE OF CONTEXT

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This study investigated the longitudinal performance of 583 students on six map items that were represented in various graphic forms. Specifically, this study compared the performance of 7- 9-year-olds (across Grades 2 and 3) from metropolitan and non-metropolitan locations. The results of the study revealed significant performance differences in favour of metropolitan students on two of six map tasks. A second phase of the study analysed the difficulties non- metropolitan students (n=48) had when interpreting these two tasks. Implications include the need for teachers in non-metropolitan locations to ensure that their students do not overly fixate on landmarks represented on maps but rather consider the arrangement of all elements encompassed within the graphic.

INTRODUCTION

Being numerate in today's society has created increased demand on our capacity to represent, manipulate and decode information in various graphical forms (e.g., maps, graphs). New technologies allow data to be transformed into detailed, complex and dynamic graphic displays (e.g., Google Earth) and consequently, there is greater need for students to become proficient in decoding maps. At the same time, the school mathematics tasks students are required to solve are becoming more authentic and realistic (van den Heuvel-Panhuizen, 2005)—and as a result the broad everyday experiences students possess are increasingly useful in decoding graphics. The purpose of this paper is to investigate the effect that students' lived experiences (in terms of geographic locality) have on their ability to decode maps. In particular, we examine the performance differences of students living in different contexts (metropolitan and non- metropolitan areas). This research builds on a body of work which highlights the influence of contexts and contextual understanding in mathematics meaning making (de Corte, Verschaffel, & Greer, 2000) and the transferability of real-world knowledge to school mathematics (Boaler, 1993).

UNDERSTANDING AND INTERPRETING MAPS

Information in maps is encoded through location of fixed attributes (marks and symbols) in a particular spatial location (Mackinlay, 1999). The attributes include landmarks, simple icons and coordinates. The spatial formats include landmark and coordinate maps. Although maps are embedded in the school curricula from the first years of school, many primary students experience difficulty interpreting relatively simple maps. Diezmann and Lowrie (2008) found that students were distracted by

different foci on the map; and that information critical to understanding was often overlooked. The ability to interpret or decode maps involves the student analysing: locations (through position and placement) and attributes (what is actually represented); and understanding map representations are small scale depictions of real world place or spaces (Wiegand, 2006).

Map decoding occurs at three levels of sophistication (Wiegand, 2006). The initial stage involves *extracting information from a map* and generally reading names and attributes. In this phase, the user records or recognises visual stimuli and is able to recognise and identify specific elements (or icons) that are contained within maps. The subsequent phase involves *ordering and sequencing information*. This could include monitoring, comparing, sequencing and even manipulating information or data. These two levels of sophistication are contained in the tasks in the present study however the final level, *the application of information*, is not.

Since maps are representations of location, it seems plausible that students' knowledge of maps will be influenced by their experiences in their home locations. From an encoding perspective, previous research has found that students living in metropolitan locations generally *encode* maps (when drawing their own maps) in a grid-like structure whereas students in non-metropolitan areas often encode using landmark and "mud map" techniques (Lowrie, Francis, & Rogers, 2000). The present study investigates this home location difference further by seeking to determine whether students from different geographical locations *decode* maps with different proficiency.

DESIGN AND METHODS

This study is part of a longitudinal investigation of primary students' ability to interpret information graphics including maps. Two research questions are explored:

1. *Are there performance differences between students' from metropolitan and non-metropolitan locations on Map items?*
2. *What difficulties do students from different localities experience on Map items?*

The Instrument and Items

The same six map tasks (T1 – T6) from the Early Primary [EP-GLIM] Test and the Graphical Languages in Mathematics [GLIM] Test (see Diezmann & Lowrie, 2009 for a description of the Instrument) were used in this study. The multiple-choice instruments were developed to assess students' ability to interpret items from six graphical languages including maps. The six map items varied in complexity, required substantial levels of graphical interpretation and conformed to reliability and validity measures (Lowrie & Diezmann, 2005). The map items were administered to students in a mass-testing situation annually when students were in Grades 2 and 3. The Appendix presents the six map items.

Participants

There were two cohorts of participants in this study. Cohort 1 comprised 583 students ($M=271$; $F=313$) from two geographically-distinct locations (Metropolitan=206 and Non-Metropolitan=377) in Australia. This cohort completed the six map items from the EP-GLIM tests for each of two years (Grades 2 & 3, aged 7-9 years). Cohort 2 was 48 Grade 2 (aged 7 or 8 years) students from Non-Metropolitan location. This cohort participated in individual interviews on the six map tasks first selecting a multiple choice response and then justifying their selection.

RESULTS

The first aim of the study was to investigate performance differences for mathematics tasks by location. A multivariate analysis of variance (MANOVA) was used to analyse mean scores across Grade and Location dependent variables. The MANOVA revealed statistically significant differences between the scores of students across both *Grade* [$F(1, 6)=7.13, p<.001$] and *Location* [$F(1, 6)=3.09, p<.01$] variables. There was no interaction (*Grade x Location*) effect [$F(1, 6)=0.38, p>.05$]. Table 1 presents the means (and standard deviations) for grade and location over the two-year period.

Grade	Location	T1	T2*	T3	T4	T5	T6*
2	Met	.77	.76	.56	.58	.32	.53
	n=206	.42	.43	.49	.49	.47	.50
	N-Met	.80	.67	.52	.60	.32	.43
	n=377	.40	.47	.50	.49	.47	.49
3	Met	.87	.84	.50	.66	.41	.65
	n=206	.39	.37	.47	.47	.49	.47
	N-Met	.83	.77	.67	.67	.42	.48
	n=377	.37	.42	.47	.47	.49	.50

* Statistically significant at $p<.007$

Table 1: Means (and *Standard Deviations*) of Student Scores by Grade and Location

A Bonferroni correction method was calculated in order to determine where differences were with respect to tasks across location (probability value $p = .05 \div 7 = .007$). Subsequent ANOVA's revealed statistically significant differences in the performance of students on two tasks across the two locations of the study, namely, Task 2 (*The Bike Task*) [$F(1, 873)=7.66, p<.007$], a coordinate map, and Task 6 (*The Playground Task*) [$F(1, 873)=9.45, p<.007$], a landmark map.

The second aim of the study was to determine the difficulties that non-metropolitan students ($n=48$) experienced when solving these two map items. Table 2 and 3 outline the type of solution approach employed by students on Task 2, *The Bike Task*, and Task 6, *The Playground Task*, respectively. For *The Bike Task*, the most common correct solution approach (42%) involved students using a visual cue (the key) and eliminating the other answer options in a systematic manner. Other correct solution responses were spread evenly across three other appropriate solution approaches (see Table 2).

Response type	Solution approach	No. of responses
Correct	Used visual cue and eliminated	8
	Worked from the Key	4
	Process of elimination	3
	Used and understood coordinates	4
<i>Total correct</i>		19
Incorrect	Misinterpreted diagram	8
	Incorrect counting	3
	Guessed/unable to verbalise	16
<i>Total incorrect</i>		27

Table 2: Solution Approaches (n=46) for *The Bike Task*.

The majority of incorrect solutions (59%) were associated with students' lack of understanding about what the task entailed. These students were unable to verbalise their approach or there was evidence they had guessed their solution. For example, Joseph's response gave no real insight into his thinking or how he approached the task.

Well I couldn't actually figure it [out] I just took a guess...Yeah it says A5, B5, A4, and B4 and there's no B4, A4 or anything [on the map].

It could be that these students were not ready to verbalise their thinking, and hence, were only able to give a vague indication of how they solved the task. Other incorrect students misinterpreted the diagram (30%), focussing on landmarks that were not relevant to the task. As Lily's explanation highlights, these students had some understanding of the map elements however they were not able to apply this knowledge to the task (i.e., managing the relationship between the coordinate points and the overlay of the bike track representation).

Well, she can't go through there, she can't go through the playhouse, she can't go through there, she can't go through the toilets, she can't through the picnic tables, she can't go through the water bubblers... [eliminating items - pointing to items on the map]. She could go through the swings ...but, I didn't really want to pick that. So I picked the sandpit - that's what she could've gone through.

Consequently, these students were unfamiliar with the representation of a map in a coordinate structure—potentially because they had not encountered such a graphic in either in-school or out-of-school contexts.

For Task 6, *The Playground Task*, 96% of students who correctly solved the task monitored the number of times they crossed the track as they located the landmarks. By contrast, the majority of students (72%) who answered incorrectly were not able to monitor the number of times they crossed the track as they followed the route. Instead of counting the number of track crossings on the map, they counted the landmarks presented in the text stimulus. For example, Jackson showed confusion counting landmarks instead of the track crossings

Jackson: My answer was four. I wrote down all the things he went past and it came up as four (counted the number of landmarks).

Jackson's response is indicative of students who misread the intent of the task and concentrated on the landmarks as opposed the sequential movement between them while crossing the track.

Response type	Solution approach	No. of responses
Correct	Described how the track was crossed	22
	Counted landmarks	1
<i>Total correct</i>		23
Incorrect	Misinterpreted diagram	3
	Incorrect counting	1
	Vague	2
	Counted landmarks	18
	Read map incorrectly	1
<i>Total incorrect</i>		25

Table 3: Solution Approaches (n=48) for The Playground Task.

DISCUSSION AND CONCLUSIONS

Our study examined the effect that home geographic location had on the performance of primary-aged students' capacity to decode items rich in graphics. The only items where metropolitan and non-metropolitan students differed were on a coordinate-map (Task 2) and a landmark-map task (Task 6). The source of the difference between the performances of students on the coordinate task (The Bike Task), in favour of the metropolitan students, could be experience related. It is possibly the case that metropolitan students are more likely to be exposed to coordinate map systems than those students in non-metropolitan areas—especially in out-of-school contexts as they interpret train timetables, navigate bus routes and have a lived experience in the grid-like structure of a city.

More surprisingly is the fact that the non-metropolitan students' results were comparatively low on the landmark-based task (Task 6) given previous research on encoding or drawing maps. Lowrie et al, (2000) found that metropolitan students drew more grid-like maps while non-metropolitan students relied more on landmarks. Most of the non-metropolitan students were unable to move beyond the first level of map decoding of locating landmarks and following directions. This level of processing was required for the four tasks where no performance differences occurred (Tasks 1, 3, 4, & 5). Conversely, the two tasks (Tasks 2 & 6) where differences were noted required a second level of processing which required the students to monitor movement beyond the location of landmarks (Wiegand, 2006). Students needed to decode information and interpret the map beyond direct instructions. For example, on Task 2 identifying where Deb would *not* ride her bike and on Task 6 considering how many times a track was crossed on a route rather than counting the landmarks. It appears the additional requirement for students to locate information besides what was provided in the direct instructions proved challenging for non-metropolitan students.

IMPLICATIONS

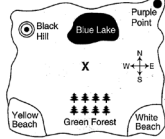
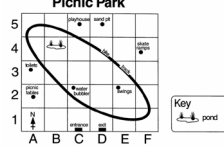

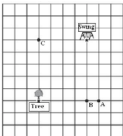
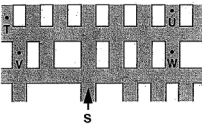

Two implications emerged from the study. First, it is important that students of this age group are exposed to tasks which require the processing of information at Wiegand's (2006) second level of decoding. This research shows that non-metropolitan students have not acquired such experiences and we propose that it is less likely that these students will gain an awareness of such understandings in out-of-school contexts. Second, students need to better understand the relationship between map structure and the elements embedded within a map. In this way, students who are unfamiliar with particular map representations (e.g., coordinate maps) can still utilise general map principles to decode the graphic elements. The findings of this study indicate that further research needs to be undertaken on how different geographic locations can impact on learning—especially on map tasks that require higher levels of map decoding and embedded map structures.

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APPENDIX

<div></div> <p>This is a map of Colour Island. Pete the Pirate is standing at X. From there, he walks south and then west. Where could Pete the Pirate be now?</p>	<div><p>Picnic Park</p></div> <p>Deb rides her bike along the bike track. What part of the Park won't she ride through?</p>	<div><p>Danni is playing a game.</p></div> <p>Danni begins on the Start box and moves as far to the left as she can. She then moves to the diagonally opposite corner. Which shape will she finish on?</p>
<p>Task 1. EAA (2006)</p> <div></div> <p>Some children want to hide some treasure 5 units from the tree and 4 units from the swing. At which labelled location can the children hide the treasure?</p>	<p>Task 2. QSA (2001)</p> <div></div> <p>Omar is at S moving in the direction of the arrow. If he follows these directions: take the second turn to the right, then take the third turn to the left, he will be at point?</p>	<p>Task 3. ETC NSW (2001)</p> <div></div> <p>Ben went from the gate to the tap, then to the shed, then to the rubbish bins. How many times did he cross the track?</p>
<p>Task 4. MDE (2006)</p>	<p>Task 5. ACER (n.d.)</p>	<p>Task 6. QSA (2002)</p>

SCHOOL STARTERS' EARLY STRUCTURE SENSE

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Low and high achieving children's competences regarding pattern and structure at the beginning of formal schooling are comparatively analysed in order to evaluate the range of school starters' early structure sense. The results suggest overall high pre-instructional competences which, however, differ strongly between the mathematical high and low achievers. Cognitive milestones for the development of a sound early structure sense are named.

INTRODUCTION AND THEORETICAL FRAMEWORK

From the first day at school children have to deal with mathematical patterns and structures. In patterning activities they encounter repeating patterns in order to identify regularity, recognize relations, abstract rules, build sequences or make predictions (see e.g. Economopoulos, 1998; Threlfall, 1999). Spatial patterns are often used as (standard) number presentations to visualize numerical structures in a specific geometrical way. In doing so particular characteristics of numbers can be illustrated and are used to develop mental representations of numbers. The perception of pattern and the ability of structuring also are the basis for subitizing. (see e.g. Mulligan, Prescott, Papic & Mitchelmore, 2006) Pattern and structure are an important part of mathematics lessons at the beginning of primary school. Therefore the ability to perceive and use pattern and structure – in short: having an early structure sense – is a significant precondition.

Thus said, it immediately arises the question if children already have a structure sense at the beginning of formal schooling or if it has to be developed instructionally. Over the last couple of years there have been a few studies addressing this topic. Mulligan & Mitchelmore (2009) tested 103 Australian Grade 1 students (5.5 to 6.7 years) on 39 pattern and structure items. Responses were categorised according to the degree of structure and four stages of structural development could be identified. According to these results patterning competences evolve from a 'Pre-structural stage' where "representations lack any evidence of numerical or spatial structure" to an 'Emergent', then to a 'Partial structural stage', and finally to the 'Stage of structural development', where children's "representations correctly integrate numerical and spatial structural features" (p.41). Mulligan & Mitchelmore also believe that children from an early age onward have an – as they call it – "Awareness of Mathematical Pattern and Structure" (p.44). Van Nes (2009) interviewed 38 Dutch Kindergarteners (4 to 6 years) on tasks about counting, subitizing, repeating and spatial structure patterns. She also identified four phases in the development of spatial structuring ability. In the lowest 'Unitary Phase' a child is not able to *recognize* spatial

structures. This competence characterises the next ‘Recognition Phase’. In the ‘Usage Phase’ a child furthermore *uses* and in the highest ‘Application Phase’ even *applies* spatial structure.

All existing studies describe rather general characteristics of young children’s structural development. To enable teachers to understand and support this development more specific and clear descriptions of abilities regarding pattern and structure are needed. In an initial attempt I described early structure sense as a collection of abilities which includes recognizing a configuration as a familiar structure or pattern (e.g. dots on dice, finger pattern), in particular recognizing a familiar structure both in its simplest form and as part of a more complex pattern. Further abilities are dividing a pattern into sub-structures, recognizing mutual connections and relationships between sub-structures (e.g. find regularity, detect similarities and differences ...) and integrating sub-structures to see a pattern as an entity (e.g. in order to determine its quantity, extend ...) (Lüken, 2010). To become more specific this paper addresses the following two questions: How is the range of patterning competences, in particular what are the competences of low compared to high achieving children at the beginning of school? What are the cognitive milestones in the development of a sound early structure sense?

METHODOLOGY

Data Collection

My data comes from a 2-year longitudinal research conducted in two state primary schools in a large German city. The sample comprised 74 children, 38 girls and 36 boys, ranging from 5.8 to 7.2 years of age at the study’s beginning. The children in both primary schools came from low to high socioeconomic families, with 31% having a migrant background (at least one parent not born in Germany).

In two out of three assessments the children’s mathematical competences were measured with standardised tests. For the first part of the assessment, that took place in Kindergarten two to three months before school enrolment, I used the German version of ‘The Utrecht Early Numeracy Test’ (van Luit, van de Rijt, & Pennings, 1994). For re-assessing the mathematical competences after two years of schooling (third part of assessment) I employed the standard German mathematics test DEMAT 2+ (Krajewski, Liehm, & Schneider, 2004). The data which is in the main focus of this paper comes from the second part of the assessment where I assessed school starters’ competences in patterning and structure-perception. The interviews took place during the seven weeks after school enrolment. Pattern & Structure Tasks were developed based on a theoretical framework and tested in a pre-study. Six task-categories with each comprising several items were designed to explore the children’s ability to conceive, reproduce, copy from memory, use, extend and create repeating and spatial patterns. The Pattern & Structure Tasks were administered as semi-structured individual interviews. The children were asked to think aloud; concrete objects were used in every task. All interviews were video recorded.

Data Analysis

In a first step quantitative analyses were conducted which showed a significant correlation between school starters' early structure sense and their mathematical competences. These results are subject of a former paper (Lüken, 2010). This article focuses on the qualitative analyses which I conducted in a second step on the basis of the quantitative outcome. To assess the range of patterning competences and in particular compare the structure sense of mathematically low and high achieving children the sample was divided in quartiles according to the children's results in 'The Utrecht Early Numeracy Test' and the Pattern & Structure Tasks. Twenty interviews of children who scored in the same quartile for both tests were analysed. The interviews were evenly distributed among the four quartiles. The data analyses of the interviews followed a grounded theory framework, called "Thematisches Kodieren" (thematical coding; Flick, 1999). This method compares groups that are established in advance. Thus it meets my interest to describe the similarities and differences in the structure sense of children with different mathematical abilities. An analysis template was set up for each task and all students' responses (the videotapes had been fully transcribed) were categorized accordingly. Strategies were then empirically verified for the four quartiles and the relevant categories/strategies for the discrimination of the quartiles further analysed.

RESULTS AND DISCUSSIONS

This paper focuses on the comparison of low and high achieving children's strategies while dealing with the Pattern & Structure Tasks (first vs. fourth quartile). The analyses of the second and third quartile are left out here but can be found in Lüken (2011). The presentation of the results is structured into the following significant pattern and structure abilities: 'Pattern recognition', 'Grasping structure', 'Making use of structure', 'Ability of spatial structuring' and 'Awareness of/attention to mathematical pattern and structure'.

Pattern recognition

Recognizing a pattern is shown in two different ways. On the one hand one can recognize a familiar pattern in the sense of a well known "picture" (most common with spatial structure patterns) on the other hand one can detect *regularity* (e.g. in a repeating pattern). The competences of low and high achieving children vary strongly according to either meaning of pattern recognition. Both aspects are addressed here.

The children were shown several flash cards with spatial dot patterns, asked to determine the quantity and to reproduce the pattern from memory. The following transcript¹ is taken from the interview with Moskan, a low achieving child, while dealing with the pattern of seven dots (shown below).

¹ The transcripts have been translated from German to English for the purpose of this paper. Names are anonymized.

Interviewer: How many counters did you see?

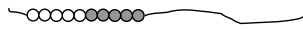


Moskan: Mhm, there was the six. And one is missing.

Moskan seems to recognize the die-pattern of the six in this pattern, although it is not quite clear if she attends to eight dots with one missing as *the* six (8-1) or six dots with one extra at the top (6+1). Anyway Moskan is one of the few (but still there are some!) low achieving children that are able to recognize familiar patterns both in their simplest form and as part of a more complex pattern. High achieving children are all able to do this.

Furthermore high achievers, in contrast to the low achievers, can recognize regularity. Lukas, a high achiever, was asked to extend a repeating pattern out of five red and five blue pearls and explain his action.

Lukas: Every time I make five.



The high achieving children show a sound understanding of pattern as unit of repeat and are thus able to extend a repeating pattern according to the given regularity; sometimes they can even explain the rule. No regularity was found in the low achieving children's extensions.

Grasping structure

For grasping structure there are always relations to be discovered or established. The two following transcripts show the way, low and high achievers differently grasp the structure of the Twenty Field shown below. Celina and Joshua were asked to explain the picture.

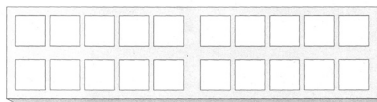
Celina: *(counts silently the ten squares on the right side while pointing a finger on each)* These are seven, here. *(points somewhere in the middle of the ten squares on the right side)* One, two, three, four, five, six, seven, eight, nine, ten and here there are ten. *(counts the ten squares on the left side of the Twenty Field)*

Interviewer: Something else?

Celina: *(Looks at the backside of the Twenty Field)* One can also, well, I see seven counters.

Interviewer: Mhm, and something else?

Celina: And ten counters.



Joshua: *(counts the squares on the left top silently and without pointing)* Five whites in this corner *(describes with his finger a circle around the five squares on the left top)* five whites in the other *(describes with his finger a circle around the five squares on the right top)* I assume, and thus five in every corner.

Celina, like other low achieving children who decompose the Twenty Field in two parts, is only able to count up to around ten. She *has* to structure and count the sub-structures in order to answer the question of the interviewer. The bigger vertical gap serves her as external, visual-geometrical stimulus for the decomposition. During the interviews I often watched children stressing the vertical or horizontal gap of the Twenty Field by putting a hand on it. Joshua, as a high achieving child, shows his shift of focus by stressing the groups of five instead of the gap. His attention as a high achieving child lies on the sub-structures which he relates to each other. He is sure of the equal cardinality of his established sub-structures; he does not even have to count the squares in more than one of his sub-structures. Celina decomposes and determines the quantity of each established sub-structure by counting but does not relate them to each other. She does not perceive that her sub-structures are equal and consequently her counting (with two different subsets) cannot be correct.

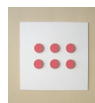
To sum the competence of grasping structure up: Low achieving children are able to grasp the structure of a pattern by the help of Gestalt principles (see Goldstein, 2002). They structure a pattern mentally by decomposing it rather subconsciously along geometric clues for grouping, like different colours, similarity or (like in the Twenty Field above) bigger and smaller distances between the objects. In that way the focus of their perception lies on the figural aspects and the visual impression of the pattern. The established sub-structures stay unrelated from each other and the external grouping is not related to numerical aspects. In contrast, high achieving children structure a pattern consciously by decomposing it flexibly along external, geometric clues for grouping into sub-structures that can be perceived simultaneously. They relate the sub-structures to each other while linking the structural with the numerical aspects. The focus of high achievers lies on numerical aspects of the sub-structures rather than on figural features.

Making use of pattern and structure

Lukas: Six.

Interviewer: How did you see it so fast?

Lukas: Three plus three!



High achievers use the spatial structure of an arrangement as well as familiar patterns explicitly to abbreviate numerical procedures (see also van Nes, 2009). To determine a quantity they decompose complex patterns flexibly and consciously and relate them either with a familiar partition of number or identify the quantity through relating and comparing. Low achieving children do not possess these abilities.

Ability of spatial structuring

To assess children's ability to apply structure to an unordered amount of objects I asked them to put five counters on the table in such a way that a hand puppet (which I presented) could easily see how many there are. The following transcripts show three examples of arrangements with the child's respective explanation.

- Helene: (*arranges all counters in a horizontal line, touches each one with a finger, counts, thinks, nods*) Okay.
- Lion: (*arranges the counters as the die-pattern*) Because a five looks like this.
- Lukas: (*arranges the counters as the die-pattern*) Because you can see it immediately that here (*tips the four outer counters*) are four on each side and, here are two and, here, four (*tips the top two and then the bottom two counters*) and in the middle is the fifth.

Helene shows a very common strategy for the low achieving children – the arrangement in a line for convenient counting. Except only one child that arranged the counters randomly on the table and seemed to have no idea at all that counters can be organized, all the low achievers did structure their counters spatially. Their arrangement, however, reflect their mathematical abilities. They have learned that objects that are ordered in a row can be more easily counted than a random arrangement. They are not aware of criteria's for a quick and easy number perception, only for easy counting.

The most common strategy for all quartiles except the first was to arrange the five counters into a die-pattern, like Lion and Lukas did. What makes the difference in the structuring ability is the explanation of this configuration. The children in the quartiles between the high and low achievers (e.g. Lion) interpret the die-five as a *number* ("the" five); the high achievers (e.g. Lukas) interpret it as a *partition* of a number ($4+1$; $2+2+1$). High achievers have an awareness of the spatial structure and function of particular configurations. Sometimes they even hold metacognitively criteria for an easy and quick number perception. Their already developed mathematical abilities reflect their structural ability. They have an idea of numbers as appropriately grouped quantities and know that the spatial structure of a pattern represents its mathematical structure.

Awareness of and attention to mathematical pattern and structure

The study addressed no special tasks for assessing children's *awareness* of mathematical pattern and structure. This point derives from the complete analyses regarding the tasks altogether. Low achieving children tend to a preassigned perspective on and way of structuring patterns. They have difficulties to relate sub-structures. Number presentations are more often seen with a daily framework than from a mathematical perspective. While perceiving structure external characteristics, spatial dimension und figural aspects are most important. On the contrary high achieving children have an insight into the convenience of structure for determining, comparing and operating with small quantities. They are able to flexibly structure a pattern and shift their focus on different aspects of pattern and structure. They relate established sub-structures in more than one way. Number representations are also seen with a mathematical framework. They understand that some configurations support numerical procedures. External aspects are less important, the focus lies on the decomposing and the established sub-structures. Figural aspects and arithmetical

knowledge, mathematical abilities and structuring competences are integrated with high achieving children and are used naturally while solving problems.

CONCLUDING REMARKS

Children at the beginning of school show high pre-instructional competences regarding pattern and structure. The qualitative analyses, however, reveal that these competences differ strongly between the lowest and the highest achievers. This fact has consequences for further mathematics learning especially for the low achieving children. Quantitative analyses found that the 25% school starters with the lowest *patterning* competences after two years of schooling also belonged to the 25% children with the lowest *mathematical* competences (Lüken, 2011). Despite the overall high patterning competences some children obviously need stimulating and supporting instruction to further develop their abilities regarding pattern and structure. To put it in Luis Radford's words: some children need support in "the domestication of their eye" (Radford, 2010). Radford describes the domestication of the eye as "a lengthy process in the course of which we come to see and recognize things according to 'efficient' cultural means" (p. 4). Transferred to patterning activities and perception of structure at the beginning of school the "cultural means" refers to the mathematical perspective. A learner has to organize the perception of things in a particular, mathematical way, for instance learn to relate geometric clues to numerical matters. Perceiving the different colours and succession in a repeating pattern or the visual gaps that groups a spatial pattern is not the problem with low achieving children. Relating the figural, external aspects with mathematical aspects is the step they obviously cannot take alone but have to be instructionally supported with.

At the end of this paper some cognitive milestones in the development of a sound structure sense are described, drawn from the comparison of low and high achieving children's patterning strategies.

Recognizing a configuration as a familiar pattern both in its simplest form and as part of a more complex pattern is something a lot of the low achievers are able to accomplish. To connect the pattern's spatial structure with its numerical structure, however, is a milestone. Similarly almost all low achieving children can divide a pattern into sub-structures, the difficulty lies in recognizing and establishing mutual connections and relationships between the sub-structures and in integrating the sub-structures for example to abbreviate numerical procedures. The milestone in the work with repeating pattern constitutes the perception of regularity, to "see" the unit of repeat. A milestone only very few children have accomplished at the beginning of formal schooling is the ability to flexibly decompose and relate sub-structures, to intentionally reframe the structures of a pattern. This competence seems not to be the least important although it is the least developed. It is assumed that special instructional support is needed to develop a *flexible* pattern perception.

At this point further research is needed to develop well-founded instructional actions which can help to support the development of a child's structure sense that consequently might also lead to improved mathematical competences.

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HIGH SCHOOL STUDENTS' PROBLEMS WITH INFINITY

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In this study we examine high school students' thinking about infinity. Students were given tasks based on basic set theoretical principles and calculations with cardinals and thereafter they had to solve the Ping-Pong Ball Conundrum. Previous research showed that when people are introduced to the notion of infinity, they tend to typify this as a process, rather than as an object. In our research students' work revealed they weren't able to use the newly learned concepts adequately for solving the Ping-Pong Ball Conundrum. They couldn't switch from the process notion to the object notion of infinity. Their lack of mathematical confidence hindered them in using counterintuitive properties of infinity. Students' adherence to properties of finite sets impeded them to solve these infinite set problems.

INTRODUCTION

Infinity is one of the concepts that mark a difference between basic and advanced mathematics. Therefore it is useful to know how students in upper high school think about infinity and how they solve problems referring to infinity. Hazzan (1999) reported that students tend to reduce the level of abstraction of a new problem to a lower level of abstraction which they are already comfortable with. Mamolo and Zazkis (2008) showed that students – using the notion of infinity – try to reduce the level of abstraction, since they extend the properties they know about finite sets to solve problems about infinite sets. Tirosh and Tsamir (1996) reported that intuition plays a role when students use infinity. Since many properties of infinite sets are counterintuitive, this results in confusion.

In this research we use two conundrums to extend previous studies. First we focus on students' thinking processes with regard to Hilbert's Hotel. Second we highlight the thinking processes with regard to the Ping-Pong Ball Conundrum. The Ping-Pong Ball solution needs the use of infinity as an object, while Hilbert's Hotel is solvable with a more procedural notion of infinity.

Mamolo and Zazkis (2008) used the Ping-Pong Ball Conundrum to describe students' understanding of infinity. Their research aimed at upper level high school mathematics teachers in a graduate program and undergraduate students in liberal arts and social sciences. We extend their study with a specification on the question how the knowledge of calculations with cardinals affects students' approaches in solving the Ping-Pong Ball Conundrum. The expectation is that in the beginning students will characterise infinity as an ongoing process in time. For example, you add one number to a set and eventually this will result in infinity, instead of applying manipulations to a set of infinitely many numbers. When students know how to use cardinals, we

expect this will give them some grip on the more abstract notion of infinity needed for solving the Ping-Pong Ball Conundrum.

THEORATICAL FRAMEWORK

We used several theories to describe students' thinking about infinity. These were the strands of mathematical proficiency, the process or object nature of mathematical conceptions and the mental process of reducing the level of abstraction. Moreover we looked at intuitive notions of infinity as well as the three mental worlds of mathematical thinking (Fischbein, Tirosh & Hess, 1979; Fischbein, 2001; Hazzan, 1999; Kilpatrick, Swafford & Findell, 2001; Sfard, 1991; Tall, 2008).

According to Kilpatrick, Swafford and Findell (2001) five intertwined strands (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition) are needed to develop mathematical proficiency. In our study students were introduced to infinity and set theoretical properties for the first time. So we expected that these strands were not yet developed with respect to infinity. Sfard (1991) argued that when people are introduced to a new mathematical notion, they commonly use this notion as a process rather than as an object. So when students examined infinity as a process, that is: as an ongoing process in time, there was a lack of conceptual understanding. This could result in the conviction that infinity is synonymous with eternity (Mamolo & Zazkis, 2008).

When students were introduced to a new mathematical concept they tried to reduce the level of abstraction (Hazzan, 1999). Since a process view was on a lower level of abstraction than an object view of a mathematical concept, this was consistent with Sfard (1991). The level of abstraction in problems with infinite sets could be reduced by carrying over properties of finite sets to infinite sets. Reducing abstraction could be effective to solve problems with new abstract notions. However, since the properties of infinite sets differed much from those of finite sets, this would not yield correct solutions.

Another difficulty here is that the properties of infinite sets can be very counterintuitive, for example because you have to admit that the whole may be equivalent to one of its parts (Fischbein et al., 1979). These counterintuitive notions make it hard for students to perceive infinity as an object, and hence attain conceptual understanding (Kilpatrick et al., 2001).

In addition Tall (2008) argued as follows: each counting number is followed by another, and another, resulting in 'potential infinity'. By categorising the collection of numbers and giving it a name, or the symbol N , we can conceive of an 'actual infinity' of numbers as a single entity. Thus repetition and categorisation can result in the notion of actual infinity. Here potential infinity corresponded to the process notion of infinity, but there still is lack of conceptual understanding. When trying to solve problems using actual infinity, the conceptual understanding could come in and infinity could become an object.

Based on these theoretical backgrounds our research questions were twofold: 1) What problems arise when high school students solve problems on infinity? 2) What kind of approaches typify the Ping-Pong Ball Conundrum solution when calculations with cardinals are known?

METHODOLOGY

We examined three students tagged as S1, S2 and S3, approximately 17 years old, from grade 11 (one year before final exams) in the science stream at pre-university level. The students were selected because of their high mathematical level, as they followed an additional elective mathematics course, which is known as ‘Mathematics D’ This elective subject is a recent addition to the Dutch curriculum, in which students are exposed to advanced mathematical topics. The three students had never before received an introduction to infinity or to set theoretical notions. The research took place during one hour, in which we gave the students two tasks. The teacher typified the students as relying on more procedural knowledge than conceptual knowledge in the sense of Hallet, Nunes and Bryant (2010). While individually working on the two tasks, the students wrote down their answers and ideas on their own paper. Afterwards they compared their answers in a discussion. This discussion was audio taped. So we collected handwritten personal ideas and the audio taped group discussion.

The first task consisted of set theoretical assignments. This task started with an introduction to Hilbert’s Hotel, followed by five assignments in which the students became familiar with the notion of cardinality and the use of a one-to-one correspondence to show the equal cardinality of two sets. The second task was the Ping-Pong Ball Conundrum.

Hilbert’s Hotel is the well-know problem of a hotel with infinite many rooms: suppose there is a guest in each room and a finite number of new guests arrive. Is it possible to give each new guest a room? What happens if an infinite number of new guests arrive? In this way we introduced infinite large sets and showed that fundamental arithmetic of finite sets cannot always be extended to arithmetic of infinite sets. Exercises 2, 3 and 4 are showed in Figure 1.

Ex. 2. Assume set A consists of all integers greater than zero, so $A = \{1,2,3,\dots\}$. Set B consists of all integers greater or equal to zero, so $B = \{0,1,2,\dots\}$. Which set has more elements? Can you let the elements of set A correspond to the elements of set B?

Ex. 3. Take the set O of all odd numbers greater than zero and set E of all even numbers greater than zero. Do these two sets have the same cardinality?

Ex. 4. Take the set A of all integers greater than zero and the set E of all even numbers greater than zero. Do these sets have equal size?

Figure 1: The set theoretical tasks

We introduced the Ping-Pong Ball Conundrum as follows (Berger & Starbird, 2000): imagine you have an infinite set of ping-pong balls numbered 1, 2, 3... and a very large barrel. You are about to embark on an experiment. The experiment will last for exactly one minute. Your task is to place the first 10 balls into the barrel and then remove ball number 1 within 30 seconds. In half of the remaining time, you place the balls with numbers 11–20 into the barrel, and remove ball number 2. Next, in half of the remaining time (and working more and more quickly), place balls 21–30 into the barrel, and remove ball number 3. Continue this task ad infinitum. After 60 seconds, at the end of the experiment, how many ping-pong balls remain in the barrel?

There were three types of infinity involved in this conundrum: the number of in-going balls, the number of out-going balls, and the infinite repetition of halving the remaining time. This could result in confusion. When students typified infinity as a process, it would be hard to imagine infinitely many actions in a finite time. Also, at every split moment in the minute there would be nine times as many balls moved into the barrel as moved out. Yet, when the minute ended, all the balls would be taken out and the number of balls in the barrel would be zero. This resulted in a one-to-one correspondence between the in-going and out-going balls, leading to the conclusion that there were as many in-going as out-going balls. This thinking process was supported through our choice of set theoretical tasks on one-to-one correspondences.

We constructed a list of possible answers such that we could label each answer in three categories: process oriented, object oriented or reducing abstraction oriented. The labelling process occurred stepwise. First, we labelled independently. Second, we discussed the results up to unanimous agreement. Also we looked what strands of mathematical proficiency were developed and where intuition played a role. In this way we classified students' written answers and their arguments in the discussion.

RESULTS AND ANALYSIS

As prior research suggested, the students tended to typify infinity as an ongoing process instead of an object. Below we discuss for each task students' answers and we analyse the problems encountered by the students.

The set theoretical task

While working on the set theoretical task students mostly reduced the abstraction of the infinite sets to familiar notions of finite sets. They considered infinity as an endless process or failed to use a one-to-one correspondence. Also their intuition hindered them in attaining correct answers, for example on exercise two, comparing the set of all non-negative integers to the set of all integers greater than zero. S3 didn't use a one-to-one correspondence to show that the two sets were equal. He answered:

(S3) The set of integers larger than one and the set of integers larger than zero are of equal size. The first set has one element more but one plus infinity is infinity, so the two sets are of equal size.

Here he was self-contradictory, stating that the first set had one element more and simultaneously that they were of equal size. He recognized the rule ‘infinity plus one is infinity’, but showed a lack of conceptual understanding by not connecting this rule to a one-to-one correspondence.

While S1 used a correspondence, he wasn’t convinced that the sets were equal in size. He answered:

(S1) You can make the correspondence, but B always has one element more.

The use of a one-to-one correspondence to show that two sets have the same cardinality was a new phenomenon for the students. They would use the definition but they were not sure whether it was a good definition. As S2 said:

(S2) If the existence of a one-to-one correspondence is the definition for two sets to be of equal size then it is true [there are as many even numbers as natural numbers].

Here both students doubted the use of a one-to-one correspondence. They were able to recognize the process of correspondence, but they didn’t believe that this proved that the cardinalities were equal. So there was procedural fluency, but they reduced the level of abstraction to familiar notions of finite sets. They acknowledged that if you added one element to the finite set, the set became larger. The fact that this was not true for infinite sets, was too counterintuitive to accept (Fischbein et al., 1979).

Student S3 had a clever way to avoid the construction of the correspondence:

(S3) Suppose you can make a one-to-one correspondence, then you need to point out the end of the ability of making a correspondence. Since there are infinitely many numbers there is no end, hence a correspondence must exist and the sets must be of the same cardinality.

Here we saw that S3 used the strand of adaptive reasoning. He still saw infinity as an ongoing process since he mentioned the end of the action. So adaptive reasoning and a process view of infinity was sufficient to solve this problem correctly.

Student S2 reduced the level of abstraction in a different way. Similar to her answer to the question whether the set E of even numbers had the same cardinality as the set N of all natural numbers, she showed:

(S2) Let every number correspond to an animal. Then all the elements that occur in E will also occur in N. Suppose 3 is a goat, then we don’t have a goat in E, but all animals in E will be in N, so N is larger.

And during the group discussion she said:

(S2) I don’t know if you can make a definition for infinity that I can accept.

This intuitive argument was consistent with Fischbein et al. (1979). They reported that when accepting the existence of infinity one has to admit that the whole may be equivalent to one of its parts. Clearly student S2 couldn’t accept this counterintuitive notion. This was also a good example of the confusion that arises when one reduces the level of abstraction. Thinking about a group of animals was on a lower level of

abstraction than manipulations with infinitely many numbers. In this case this resulted in an incorrect conclusion.

The Ping-Pong Ball Conundrum

The students had difficulties with the fact that the Conundrum was a mind game and not something you could do in practice. S2 and S1 respectively answered as follows:

(S2) The minute will never end because you keep putting balls into the barrel.

(S1) For the one who is doing the task the minute will never end, because he keeps repeating the action endlessly.

Fischbein et al. (1979) reported that almost half of the students in their study didn't accept the infinite divisibility of an interval. The answers of students S1 and S2 in our study confirmed this observation. For these students infinity was equal to eternity.

Student S3 had a combination of two answers.

(S3) The minute will not end. Also you can say that you put infinitely many balls into the barrel but also get infinitely many balls out. But every time you put in more than you get out, so close to the 60 seconds there will be infinitely many balls in the barrel.

Student S3 didn't see that this was a contradiction, because only when the minute has ended you've put infinitely many balls into the barrel.

During the group discussions students' views changed. They accepted that the conundrum was a mind game and that it was possible to think about 'after 60 seconds'. They concluded unanimously that there should be infinitely many balls in the barrel, since at each step you put nine times as many balls into the barrel as you take out, and 'nine times infinity is infinity'.

Here all the students used the reduction of abstraction (Hazzan, 1999) since at every moment before the end of the minute, when you have done the action finitely many times, their answer was indeed true, and there were nine times as many balls inside as outside the barrel. They used this more comfortable finite calculation and they extended this to the case where one has carried out infinitely many actions.

At the end of the session, when we asked them if they could name the number written on the balls in the barrel, S3 reacted with:

(S3) Oh no, there are no balls in the barrel...

CONCLUSIONS

The set theoretical tasks showed that S1 and S2 had identical problems with the notion of infinity. Even though their adaptive reasoning was well-developed, they couldn't accept the counterintuitive properties of infinite sets. Therefore they reduced the level of abstraction and carried over the properties of finite sets to solve the assignments on infinite sets. This hindered them from attaining correct solutions.

The adaptive reasoning and procedural fluency of S3 were sufficient to find the correct answers, but still there was a lack of conceptual understanding, since he kept typifying infinity as an ongoing process rather than an object.

For the second task, the Ping-Pong Ball Conundrum, we obtained roughly three possible answers. 1) The minute will never end. 2) Each time you put more balls into the barrel than you get out, so at the end there are infinitely many balls in the barrel. 3) Each ball that you put into the barrel you will also get out, so the barrel is empty. Students' thoughts are gathered in Figure 2. The second row represents the three possible answers. The second column shows the students' initial answers and the third column shows the responses after discussion. If there are two answers marked, this means that the student was in doubt which of those two was the correct answer.

	<i>First answer</i>			<i>Answer after discussion</i>		
	The minute never ends	Infinite balls in the barrel	The barrel is empty	The minute never ends	Infinite balls in the barrel	The barrel is empty
S1	x				x	
S2	x				x	
S3	x	x			x	

Figure 2: Answers on the Ping-Pong Ball Conundrum.

Figure 2 shows that the students typified infinity as an ongoing process that continues eternally, which resulted in their conclusion that the minute never ended. Their thoughts after the discussion showed that they reduced the level of abstraction.

None of the students used the concepts from set theoretical tasks to solve the Ping-Pong Ball Conundrum.

DISCUSSION

It is remarkable that the three students in our study used the same methods and came to the same conclusions as the Canadian graduate students from the research by Mamolo and Zazkis (2008). In their study, a formal introduction to set theory didn't make much difference to students' answers, but there were some who at least tried to use a one-to-one correspondence. In our research none of the students tried to use a one-to-one correspondence to solve the Ping-Pong Ball Conundrum. The fact that our students are somewhat younger, and hence less mature in mathematics, could be an explanation for this. Additionally, students' use of the newly learned one-to-one correspondence in the set theoretical context could conflict with students' preference of procedural knowledge. Another explanation for the lack of transfer from the set theoretical tasks to the Ping-Pong Ball Conundrum could be interpreted in terms of Kilpatrick et al.'s theory (2001): a well-developed procedural fluency goes hand in hand with conceptual understanding, and the latter is necessary for a productive disposition and confidence in your own mathematical reasoning. Since the students hadn't developed these strands well, their lack of a productive disposition could be a

reason for not using the instructions from the set theoretical tasks in ensuing assignments.

The intuition plays an important role as well. As Fischbein et al. (1979) argued, the concept of infinity may develop itself by the instructional process, while the intuition of infinity may remain unchanged. Hence we can offer students instructions on infinity, but they need much more time and practice before they can use the learned concepts instead of following their intuition.

Another remarkable point is that in both research findings none of the students was able to solve the Ping-Pong Ball Conundrum properly. This casts doubt on the use of this conundrum as an instrument to describe students' notion of infinity. It is reasonable to conclude that this conundrum is just too counterintuitive for the target population. Students at this level still have too little mathematical confidence to trust their mathematical ability more than their intuition.

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RELATIONSHIPS AMONG FEATURES OF PRE-SERVICE TEACHERS' ALGEBRAIC THINKING

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In this study we examined 18 pre-service middle school teachers' ability to use algebraic thinking to solve problems. The data revealed that pre-service teachers' AT abilities varied across different features of the algebraic habit of mind Building Rules to Represent Functions, and that significant correlations existed between 8 pairs of the features. The ability to justify a rule was the weakest of seven AT features, and it was correlated with the ability to predict patterns and the ability to chunk information. Implications for mathematics teacher education are discussed.

BACKGROUND

Over the last three decades the mathematics education community has engaged in discussions about the role and the nature of school algebra in the mathematics curriculum. While most mathematics educators advocate for the inclusion of algebra-based topics at the K-8 level, they are by no means calling for elementary and middle school students to be taught algebra in the traditional way. Traditional algebra focuses on issues related to skills such as manipulating algebraic expressions and solving equations. In contrast, early algebra instruction aims to advance students' conceptual knowledge and skills by shifting attention away from symbolic manipulations and equation solving toward analyzing and generalizing patterns using multiple representations (NCTM, 2000; Silver, 1997; Kieran, 1996; Carpenter & Levi, 2000). Ideally, algebraic experiences in the elementary and middle grades are designed to allow students to see algebra as a network of knowledge and skills rather than as a muddle of isolated concepts. Done in this way, early algebra instruction is much more likely to prepare students for a smooth transition from arithmetic to more formal algebra (Carpenter & Levi, 2000; Silver, 1997; Kieran, 1996; Kaput, 1998).

Algebraic Thinking

The term algebraic thinking has various connotations (Swafford & Langrall, 2000; Kieran & Chalouh, 1993; Kieran, 1996). For some, algebraic thinking closely relates to what Cuoco, Goldberg, and Mark (1996) defined as habits of mind: useful ways of thinking about mathematical content. For example, Driscoll (1999) used the term algebraic thinking to signify thinking about quantitative situations in ways that make the relationships between variables obvious. He conceptualized algebraic thinking as including three habits of mind, one of which is *Building Rules to Represent Functions*. According to Driscoll, Building Rules to Represent Functions is a habit of mind that enables thinking processes that, for example, include recognizing and analyzing patterns, investigating and representing relationships, generalizing beyond

specific examples, analyzing how processes or relationships change, or seeking arguments for how and why rules and procedures work.

Building Rules to Represent Functions

For the research reported here, we used Driscoll's (1999, 2001) taxonomy of algebraic habits of mind as a framework, and focused our investigation on the Building Rules to Represent Functions habit of mind. Thus, unless otherwise specified, throughout this paper we interpret *algebraic thinking* in terms of thinking processes characteristic to Building Rules to Represent Functions. Our operational definition of algebraic thinking is based on Driscoll's identification of features that characterize the algebraic habit of mind Building Rules to Represent Functions: (1) Organizing Information, (2) Predicting Patterns, (3) Chunking Information, (4) Different Representations, (5) Describing a Rule, (6) Describing Change, and (7) Justifying a Rule.

Teacher Knowledge

The call for the early introduction of algebraic ideas has many inherent challenges. For example, teachers' own experiences with traditional school algebra often strongly influence and limit their views of algebraic thinking and, in turn, counter their efforts at mathematics education reform. Both practicing and pre-service teachers' understanding of algebraic topics often consists of the fragmented knowledge of a disconnected system of symbols and procedures (Ball, 1990; Ma, 1999). Teaching that is informed by such limited knowledge short circuits the algebraic-thinking (AT) goals of early algebra instruction.

It is commonly accepted by mathematics educators that elementary and middle school teachers must understand the ideas behind algebraic thinking in order to take advantage of opportunities to engage students in algebraic thinking. Effective early algebra instruction requires a more adequate preparation of elementary and middle school teachers than currently exists (National Mathematics Advisory Panel, 2008; Greenberg & Walsh, 2008). Teachers' knowledge has been identified as an important factor that influences the outcome of their practice (Borko & Putman, 1996; Mewborn, 2003; Sowder & Schappelle, 1995; Hill, Rowan, & Ball, 2005).

Despite Ball's (1990) stated concerns about pre-service teachers' limited and procedural knowledge of the K-12 mathematics curriculum, few research efforts have focused on understanding the breadth and depth of pre-service teachers' algebraic thinking (AT) ability. An in-depth understanding of the relationships among various features of pre-service teachers' algebraic thinking would contribute to our ability to successfully design programs that prepare teachers to introduce early algebra concepts and foster algebraic thinking in their K-8 students.

Research Question

Guided by the need for an in-depth understanding of pre-service teachers' AT ability, we designed this study to answer the following question: Which features of pre-

service teachers' own algebraic thinking support and strengthen one another, and which features develop independently?

METHOD

Participants

Participants in this study included 18 undergraduate pre-service teachers (grades 1-8 teaching certification candidates) at a large private Midwestern university. All were enrolled in a mathematics content course designed to help pre-service teachers develop the ability to interpret, compare, connect, and generalize across multiple algebra topics within the middle school mathematics curriculum. In the course, the pre-service teachers engaged in activities that solicited multiple solutions and representations of algebra-based tasks, and encouraged sharing, explaining, comparing, and making interpretations of various representations and reasoning.

Data Sources and Data Collection

To investigate the pre-service teachers' AT ability, we collected the pre-service teachers' written solutions to 125 AT tasks, which they completed for homework, during class, and on performance assessments. The tasks were designed to elicit the pre-service teachers' algebraic thinking.

Data Analysis

The three authors independently coded the data. We rated the pre-service teachers' demonstrated use of a feature of algebraic thinking as (3) proficient, (2) emerging, or (1) not evident. In order to establish validity and reliability, we compared the three sets of independent results and cited specific examples to clarify the coding schemes and negotiate coding agreement to 100%.

First, we identified which features of algebraic thinking are encouraged by each of the 125 AT tasks. For each task we rated the pre-service teachers' written solutions, assessing how well the pre-service teachers' demonstrated the use of each identified feature. Then we produced seven feature scores for each pre-service teacher by averaging all of his/her ratings on each given feature across the collection of tasks. A pre-service teacher's feature score quantified his/her ability to use the given AT

feature. We also calculated an AT proficiency score for each pre-service teacher by computing the mean of his/her seven feature scores. Finally, we used correlation analysis to investigate the associations among the features of the pre-service teachers' algebraic thinking.

RESULTS

In this section, we provide a quantitative summary of the pre-service teachers' performance on each of the AT features and overall. Then, we compare these performances to infer relationships among the features of the pre-service teachers' algebraic thinking.

Each pre-service teacher submitted solutions to 125 AT tasks. Some tasks solicited all seven features of algebraic thinking, but most did not. As described above, we computed seven feature scores for each pre-service teacher. Each assessed the pre-service teacher’s ability to use one of the features of algebraic thinking. The means of the pre-service teachers’ feature scores are presented in Table 1. Of all the means, the pre-service teachers’ mean on the Justifying a Rule feature (2.20) was the smallest. A repeated measures ANOVA revealed that there were significant differences among the seven means ($F(6,102)=4.89$; $p < 0.01$). Bonferroni-adjusted pairwise comparisons of the means confirmed significant differences between the Justifying a Rule mean and three other means: Organizing Information ($p < 0.01$), Predicting a Pattern ($p < 0.01$), and Describing a Rule ($p < 0.01$). The other differences were not significant.

Feature	Frequency	N^*	\bar{M}	SD
1. Organizing Information	78 (62%)	1404	2.56	0.30
2. Predicting Patterns	58 (46%)	1044	2.54	0.29
3. Chunking Information	38 (30%)	684	2.39	0.39
4. Different Representations	27 (22%)	486	2.50	0.42
5. Describing a Rule	66 (53%)	1188	2.58	0.22
6. Describing Change	71 (57%)	1278	2.46	0.31
7. Justifying a Rule	40 (32%)	720	2.20	0.41

*Number of scores for a given feature across all soliciting tasks and all 18 pre-service teachers.

Table 1: Pre-service Teachers’ Mean Feature Scores

We assigned an (overall) AT proficiency score to each pre-service teacher by averaging all seven of his/her feature scores. The AT proficiency score estimates the overall strength of a pre-service teacher’s AT ability. The average of all 18 mean feature scores was $\bar{M} = 2.46$ (max 3); $SD = 0.36$, which is an estimate of the overall strength of the algebraic thinking of the pre-service teachers in our study. These finding suggests that the cohort of pre-service teachers in our study had a relatively high overall ability to solve a variety of algebra-based tasks ($n=125$), but their ability to justify a rule lags behind the others.

Associations Among the Features of Algebraic Thinking

We used correlation analysis to examine the strengths of the pairwise associations among the pre-service teachers’ feature scores. The analysis showed that eight of the pairwise correlations were significant (Table 2). Figure 1 illustrates the relative strengths of the eight significant correlations. The heavier weights of four segments in the diagram illustrate that those four significant pairwise correlations are greater ($0.72 < r \leq 0.91$) than the other four significant correlations ($0.48 < r < 0.54$).

Feature	1	2	3	4	5	6	7
1 Organizing Information	–						
2 Predicting Patterns	0.72**	–					
3 Chunking Information	0.54*	0.91**	–				
4 Different Representations	0.39	0.47	0.40	–			
5 Describing a Rule	0.51*	0.77**	0.73**	0.28	–		
6 Describing Change	0.46	0.34	0.32	0.12	0.17	–	
7 Justifying a Rule	0.44	0.54*	0.48*	0.32	0.36	0.38	–

* $p < 0.05$, ** $p < 0.01$

Table 2: Correlations Between the Pre-Service Teachers' Feature Scores

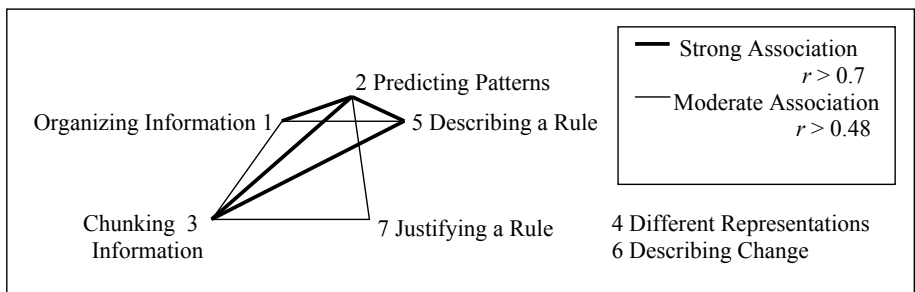


Figure 1: Pairwise Correlations Among the Seven Features of Algebraic Thinking

Significant ($p < 0.05$) pairwise correlations existed among the proficiency scores for features 1 (Organizing Information), 2 (Predicting Patterns), 3 (Chunking Information), and 5 (Describing a Rule). Feature 7 was correlated with features 2 and 3. Neither feature 4 (Different Representations) nor feature 6 (Describing Change) was correlated with any of the other 5 features. The pattern of these correlations indicates that features 1, 2, 3, 5 (and to some extent 7) are associated, perhaps because they develop concurrently, supporting and strengthening one another. On the other hand, since neither of features 4 (Different Representations) or 6 (Describing Change) was associated with the ability to use any of the remaining five features of algebraic thinking, these two features may develop and be used independently of the other five features.

DISCUSSION AND IMPLICATIONS

We found that the pre-service teachers in our cohort were able to competently use many features of algebraic thinking to solve algebra-based tasks. Although promising overall, we found that the pre-service teachers' ability to justify a rule or procedure was weak in comparison to their ability to employ other features of the algebraic habit of mind Building Rules to Represent Functions. This result is consistent with Castro (2004), who also found that pre-service teachers lacked sufficient ability to justify why algebra-based algorithms and procedures work.

Another significant result that our analyses revealed was the complex nature of the algebraic thinking identified in the pre-service teachers' work. Our research revealed correlations between different features of our pre-service teachers' algebraic thinking. Taken together, these correlations suggest that the abilities to organize information, predict patterns, chunk information and describe a rule may develop and support one

another in a mutual, symbiotic, and holistic way. However, our research also suggests that the ability to justify a rule may be related somewhat to predicting patterns and chunking information, but not organizing information or describing a rule. These correlations among the features of algebraic thinking might suggest that helping teachers to become competent algebraic thinkers may be accomplished by targeting learning activities at groups of AT features. For example, rather than implementing

learning activities designed solely at improving pre-service teachers' ability to justify a rule, it may be more effective to implement learning activities aimed at improving their ability to employ the group of three features: Predicting Patterns, Chunking Information, and Justifying a Rule.

Our study, which was motivated by the need to prepare pre-service teachers for the challenges of early algebra instruction, helps fill a gap in the existing body of teacher preparation literature. Algebraic thinking is at the heart of teaching and learning algebra at the K-8 level. Building pre-service teachers' algebraic thinking ability should be an important goal for elementary and middle school mathematics teacher education programs. The research reported here has investigated pre-service teachers' ability to use features of algebraic thinking to solve problems. It provides a window into the complexity of the relationship between different features of pre-service teachers' AT abilities. Although this is only a single step in the quest to determine effective ways of preparing pre-service teachers to make algebraic thinking a focus of their K-8 instruction, our results can help mathematics teacher educators and mathematics education researchers design programs sensitive to important issues related to early algebra instruction.

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PROOF IN CLASSROOM SOCIAL PRACTICE

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How does the construction of proof relate to the social practice developed in the mathematics classroom? This report addresses the role of diagrams in order to focus the complementarity of participation and reification in the process of constructing a proof and negotiating its meaning. The discussion is based on the analysis of the mathematical practice developed by a group of four 9th grade students and is inspired by the social theory of learning.

INTRODUCTION

The present study dealt with the problem of proof in school mathematics (Rodrigues, 2008). Its main goal was to identify the ways in which students validate their mathematical results, relating them to the social practice developed in the classroom. The questions posed were: 1) what is the nature of proof in a school context?, 2) what is the role of proof in students' mathematical activity?, and 3) how does the construction of proof relate to the social practice developed in the mathematics classroom? We will present in this paper just some results related with the third question.

The study's framework is rooted in the theoretical frame of social practice in the line of Wenger (1998). Mathematics learning is seen as a situated and a social phenomenon (Lave, 1997; Matos, 2010). As a social participation, it is the process of being an active participant in the practice of social communities and constructing identities in relation to those communities (Wenger, 1998). "Such participation shapes not only what we do, but also who we are and how we interpret what we do" (Wenger, 1998, p. 4). The social theory of learning includes components that are interrelated and characterize the social participation as a process of learning and of knowing: (a) community, (b) identity, (c) practice, and (d) meaning.

The construct of *community of practice* is a central one in this theory. The basic structure of a community of practice is composed of three elements: (1) the *domain* of knowledge that defines the area or the set of shared topics; (2) the *community* of people, concerned with the domain, creating relationships and a sense of belonging; and (3) the shared *practice* developed by people to deal with the domain, consisting of the body of shared knowledge and resources that enables the community to proceed efficiently (Wenger, McDermott, & Snyder, 2002). The relation, by which practice is the source of coherence of a community, has three dimensions: (1) the mutual engagement, (2) the joint enterprise, and (3) the shared repertoire (Wenger, 1998). The communities of practice may not be spontaneous and the introduction of a

member may not be voluntary. Nevertheless, the maintenance of a community depends on the energy produced by the proper community and not by an external mandate.

In this paper we also analyse the role of a diagram in proof construction through the concept of *reification* — “the process of giving form to our experience by producing objects that congeal this experience into ‘thingness’” (Wenger, 1998, p. 58). This concept refers both to a process and its product. In fact, they always imply each other.

PROOF: THEORETICAL ISSUES

Proof is inherent to the nature of mathematics as a science (Jahnke, 2010). The notion of proof has evolved throughout the history of mathematics and it is nowadays the subject of debate among mathematicians. Yet, proof maintains a central role in mathematics (Hanna & Jahnke, 1996; Thurston, 1995). Discussing the epistemological status of proof, we have to examine issues, in the philosophy of mathematics, such as (a) the nature of mathematical objects; (b) the relationship between the experimental reality, the natural and human world and mathematics; and (c) the issue of truth. We assume mathematics as a human and social construction, but non-arbitrary. It is this non-arbitrary nature that explains the parallelism between the physical reality and the mathematical one (Hersh, 1997). Mathematical knowledge develops through conjectures and refutations (Lakatos, 1991) and relies on linguistic knowledge, conventions and rules.

We also need to look at the curriculum in general terms and specifically the mathematics curricula, regarding how proof should be integrated. Many mathematics educators attach great importance to proof in the curriculum. Two essential reasons justify the relevance of the teaching of proof: (a) a more comprehensive vision of the nature of mathematics (de Villiers, 2004; Hanna, 2000; Jahnke, 2010), and (b) the promotion of mathematical understanding through the primordial function of proof in mathematics education, the explaining function (Hanna, 2000; Hersh, 1997). The more recent curricular documents, in Portugal and in other countries, have attached major importance to proof, advocating that from elementary to upper level there should be a gradual and continuous transition from justification and explanation activities to the proof itself. This curricular perspective regards proof as a process that evolves along all the school years. Counterexample proof is a particular method that can be introduced very early as a way of proving the falsehood of a statement or conjecture. According to Harel and Sowder (2007), upper elementary school children can deal with proof if they are taught appropriately.

However, internationally, studies in mathematics education provide empirical evidence that students reveal a great difficulty in understanding the need for proof (Rodrigues, 2008), understanding the functions of proof (Harel & Sowder, 2007) and constructing proofs (Healy & Hoyles, 2000). The majority of students of various levels (from the more basic to the first years of university level) use specific cases to

establish the truth of conjectures they make (Chazan, 1993; Harel & Sowder, 2007; Healy & Hoyles, 2000; Rodrigues, 2000; 2008).

The discussion of mathematical ideas, developed within a small group of students orchestrated by the teacher within the class, plays a decisive role (a) in the emergence of proof meaning, (b) in the motivation to prove mathematical statements, and (c) in changing the spontaneous attitude of students towards construction of proof.

There is empirical evidence that a classroom environment rich in social interactions among students and between the teacher and the students can foster the development of the actual *proof schemes* of students. “A person’s (or a community’s) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)” (Harel & Sowder, 2007, p. 809). However, according to Balacheff (1991), there are situations of social interaction that do not guarantee effective involvement in mathematical discussion and a construction of a proof at the end.

Our point is that in some circumstances social interaction might become an obstacle, when students are eager to succeed, or when they are not able to coordinate their different points of view, or when they are not able to overcome their conflict on a scientific basis. In particular these situations can favour naïve empiricism, or they can justify the use of crucial experiments in order to obtain an agreement instead of proofs at a higher level. (Balacheff, 1991, p. 188)

All the efforts of children in elaborating their arguments should be valued but the teacher should insist on the need to improve them to become successively more general. The teacher must also give back to the students the responsibility of validating their statements (Balacheff, 1991; Harel & Sowder, 2007).

METHODOLOGY OF THE STUDY

The methodology adopted has an interpretative nature because it is suited to the aims of the study. It focused on the participants’ meanings. The unit of analysis was proof scheme of students. Through the analysis of school mathematics practice, we tried to understand how students reason within this practice, how the meaning of the proof is negotiated, and how the process of proving evolves over time, studying the phenomenon in its natural setting — the mathematics classroom. For that reason, we paid attention to all aspects concerned with students practice: their utterances, their acting, their facial expressions and the mediating resources.

Data was collected in a state school in a class of the 9th grade, over one school year. A group of four students was selected to be videotaped and audiotaped. The researcher played the role of participant observer, having observed and participated in all the mathematical activities of the class during the 16 lessons in which inquiry tasks were carried out. To collect data we used (a) video recording of mathematical activities of students, (b) audio recording of students’ dialogues, (c) field notes made by the researcher, (d) video recording of students and teacher semi-structured

interviews, and (e) documental analysis of the work done by students and of video and audio recordings.

DISCUSSION OF SOME RESULTS

In this section we present results related to the carrying out of a single task: “*What is the relationship between the bisectors of supplementary adjacent angles?*” The task included a small note suggesting the drawing up of a diagram with the angles and their bisectors.

Proof by insight

Within the group, this was Ricardo, who suddenly, by insight, discovered the problem solution. He read the question of the task attentively and then he exclaimed: “Yeah!! I know! Bisectors of supplementary angles because added together they give an angle of 90° .” This is the transcript of the following moment:

- 1 Ricardo: Well, I'll explain. These are two supplementary angles. It gives
- 2 180. (...). When we divide, it's this and this. (*drawing the bisectors*)
- 3 Sara: It is the bisector.
- 4 Ricardo: Hang on!... We can add these two, it is half of 180 outside. So, this is
- 5 a right angle. So, this is 90° .
- 6 Sara: I don't understand. It's what? We've got a mathematician here!
- 7 Seriously, I don't understand.
- 8 Ricardo: I know you don't... Neither do I.
- 9 Sara: Ah! You don't understand! Good!
- 10 Ricardo: I'm sure that it is correct. But now I don't understand...

The discovery of the solution problem was made by a narrative and informal proof, constructed individually by Ricardo, without relation to the social interactions in the group. His fast process of solving the problem includes conceiving a proof: a general and a deductive argument. The Figure 1 illustrates the structure of Ricardo's argument, using Toulmin (1969) model.

All Ricardo's efforts in sharing the proof with his classmates came up against the communication difficulties presented by the prematurity of the moment: his classmates hadn't yet assimilated the task sense yet. When Ricardo says that he didn't understand (lines 8 and 10) this is because he had difficulty in communicating his thinking. Therefore it is his understanding that gives him the certainty “that it is correct” (line 10). But his thinking is regarded by him as a *tourbillon*, in a syncretic phase, as something that needs to be dissected so that all his classmates can understand clearly what he saw and knew to be correct.

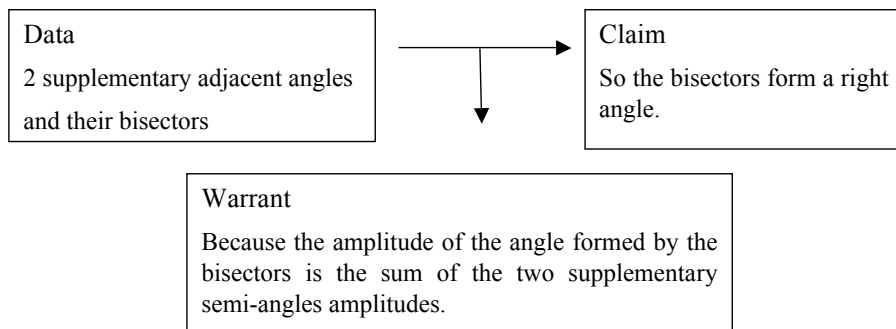


Figure 1: Structure of Ricardo's argument

Despite the fact that Ricardo was the single owner of the original meaning of the proof, he never concealed it from his classmates. On the contrary, he used various resources, including the specialization and the diagram, to make transparent to them his individual process of reasoning:

- 11 Ricardo: (*looking at Sara and pointing to the diagram*) For example, for
 12 example, I'll use crazy numbers, well... Here it is 60, here it is 120. The
 13 bisector making this, it gives here 30 and here 60. 30+60, 90. Right angle.
 14 Have you got it now?

The diagram drawn by Ricardo is a *structuring resource* (Lave, 1997) since it shaped the process of constructing and communicating the proof. A structuring resource is something — concept, object, people, activity — that supports a situation giving it structural form. Probably Ricardo visualized the diagram in his insight and then he used it for the purpose of communication. First, drawing the diagram had a social motive of explaining, that is to say, of making Ricardo's thinking intelligible to others, without any relationship with the task suggestion of a diagram. Later, the student group drew up a diagram, in reply to what was asked in the task:



Figure 2: Diagram drawn up by student group

The bisectors were drawn with a ruler reifying a product in such a way the students consider appropriate to give to their teacher. The angles were seen in their generality and the diagram highlights theoretical properties such as the notion that a bisector creates two congruent angles. So the students did not care if it was drawn exactly in half and they did not measure angles. It was a conceptual congruency and the

diagram was a support to thinking. Even when Ricardo refers to specific cases of angles (line 12), they were used to illustrate the general properties. So the specialization functioned as a communicative resource.

There is an equilibrium of power between Ricardo and Sara and communication benefits from this equilibrium. Ricardo speaks almost exclusively to Sara. It is her understanding that concerns him. Both Bernardo and Maria withhold their incomprehension, thus seeming to build up *identities of non-participation* (Wenger, 1998) within the group, maintaining a marginal position since their participation is restricted by non-participation.

The algebraic proof

When the teacher came up to the group, Ricardo said:

Teacher, it gives an angle of 90° . The relationship is that it forms an angle of 90° . Now, I can't explain it to the others. The others...

The teacher did not validate Ricardo's statement and Ricardo did not want any validation either. He was certain of his deductive conclusion and his single worry was the difficulty of communication. Then the teacher negotiated the use of Greek letters to label the angles in the diagram and the construction of a formal and algebraic proof:

$$\begin{cases} 2\alpha + 2\beta = 180^\circ \\ \frac{2\alpha}{2} + \frac{2\beta}{2} = \frac{180^\circ}{2} \end{cases}$$

$$\begin{cases} 2\alpha + 2\beta = 180^\circ \\ \alpha + \beta = 90^\circ \end{cases}$$

R: The relationship between the bisectors is that they form an angle of 90° . (Rodrigues, 2008, p. 678)

The written group report does not include any mark of the narrative and informal proof in the terms used by Ricardo to communicate it. The algebraic proof is the result of teacher negotiation leading to its reification.

This proof is strongly linked to the diagram since it translates the relations observed in the diagram. The symbolic notation by assigning equal letters to congruent angles led the students to concentrate on the essential, distinguishing the angles of interest (the angles inside the bisectors) and ignoring those which were not of interest (the angles outside the bisectors). The students, implicitly, treated a geometric situation as algebraic, assuming angles as quantities. The succinctness of the diagram focuses the negotiation of meaning produced in the process of constructing the proof. And in this sense, the communicative ability of this artifact depended on how negotiating meaning of the proof was distributed between reification of a diagram and participation in the carrying out of the task by each student.

Ricardo had a leading role in the entire construction of the algebraic proof. It was Ricardo that dictated the final answer. The other members' group drew up their own diagrams but they needed to look at Ricardo's and they copied the algebraic proof which Ricardo wrote. Because Ricardo's writing was untidy, it was always his classmates that wrote on the final sheet to be given to the teacher. The mutual engagement of the team members is characterized by complementary contributions. The whole process of constructing the proof, anchored by the drawing up of the diagram, increased ownership of meaning for all the members of the team in different degrees.

CONCLUDING REMARKS

The group of students can be seen as a community of practice since all members share a concern with their mathematical tasks, in the classroom, create relationships and a sense of belonging to the team, and develop a shared practice to deal with the tasks set. In this paper it was illustrated that this community of practice does not entail homogeneity. The mutual engagement is characterized by diversity and it is inherently partial. The members assume different roles depending on their competence. It is a community of practice where people help each other. Ricardo shared his original proof meaning and through the process of sharing, all the members of the team increased their ownership of meaning in different degrees depending on the degree of participation. Participation is both a kind of action and a mode of belonging. The degree of ownership of meaning refers to the degree to which anyone can make use of or assert as his or her own the meanings that negotiate. The process of Ricardo communicating his deductive reasoning gives him the opportunity of clarifying his mathematical thinking. As stated by Wenger (1998), the resulting relations of a shared practice are diverse, reflecting the complexity of doing things together, and they are not reducible to a single principle such as power or collaboration.

Regarding the roles played by different elements in the social practice, we can focus on the teacher and the diagram. The intervention of the teacher led students to express the written proof in a formal and algebraic format. The diagram played an important role in the process of sharing and increasing the ownership of meaning of proof by highlighting the relevant properties.

Finally, we must pay attention if there is a split between production and adoption of meaning within a group of students because this split compromises learning, reflecting an enduring patterns of engagement among members that can result in non-participation and marginality.

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ARE USELESS BRACKETS USEFUL TOOLS FOR TEACHING?

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In this paper two populations of different cultural settings (Italy and Greece) are examined. The aim is to explore whether the presence of brackets affects the pupils' performance on simple equality number sentences as well as whether brackets influence their understanding of the symmetric property of equality. The findings in both countries showed that when pupils work with brackets they get better scores in the success rates and give evidence of relational thinking.

INTRODUCTION

Hoch and Dreyfus (2004) working with students in the context of solving equations found that students react differently in the presence or absence of brackets. They found that the presence of brackets by giving a clue where to look and by focusing the students' attention affected positively the use of structure sense, acting more or less as a supporting prosthesis. Thus, the students dealing with a compound term as a single entity, and through an appropriate substitution, recognise a familiar structure in a more complex form and this is a fundamental ingredient for obtaining structure sense (Hoch & Dreyfus, 2010). This is in accordance with Radford (2000) who following previous studies found that instead of seeing signs as the reflection mirrors of internal cognitive processes they rather are considered as tools or prostheses of the mind to accomplish actions as required by contextual activities in which the individuals engage. So, the practice of brackets as “external components” could be an appropriate way for pointing out this aspect as far as the “internal components” are concerned. In other words, brackets as “external components” facilitate the recognition of internal components. Moreover since brackets are a typical feature of the first steps in algebraic thinking, it is a question whether their presence affects also the *relational thinking* (Molina, Castro & Mason, 2008) (pupils' thinking makes use of relations between the elements in the sentence and sentences are considered as wholes instead of as processes to carry out step by step).

However, even though brackets are a kind of signs used very frequently in mathematics, researchers are not yet able to find reports or results on how the brackets affect pupils' success in solving equations (Hoch & Dreyfus, 2004). Given that there is a lack of such a research in primary school we wanted to explore whether and in which direction the presence of brackets affects the performance of children when working on simple examples of equality relations (equations), in grades 2 and

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3, i.e., before an explicit use of brackets in their first steps towards algebraic learning. Additionally we would like to record instances showing the presence of relational thinking in the pupils’ work and whether this is connected to the presence of brackets. The study took place in two different cultural settings (Italy and Greece). The innovative nature of this study relies on considering lower primary school grades (instead of ones in secondary or high school) so as to see whether the aforesaid cognitive aspect is still present in these early school years and in what extent this aspect could be developed properly even from the very beginning of mathematical learning.

DESCRIPTION OF THE STUDY

This study is only a facet of a wider and multipurpose investigation about equality. The questionnaire we used is a modified version of the one proposed by Parslow-William and Cockburn (2008) and it is comprised of two parts that include number sentences as ‘equations’ with one-digit numbers. The first part (Level 2a) deals with addition whereas the second (Level 2s) deals with subtraction. Perhaps this choice seems limitative, but a careful reading of the questions shows that the way the questions are formulated requires simultaneously an awareness of both addition and subtraction. From the original questionnaire we obtained a new version adding useless brackets (Table 1), in order to detect whether in lower school grades the presence of brackets affect the results (possibly in a positive manner).

	Level 2a	Level 2a with brackets	Level 2s	Level 2s with brackets
a)	$7+2 = \square$	$(7+2) = \square$	$9-3 = \square$	$(9-3) = \square$
b)	$5+ \square = 8$	$(5+ \square) = 8$	$7- \square = 2$	$(7- \square) = 2$
c)	$\square+4 = 9$	$(\square+4) = 9$	$\square-4 = 5$	$(\square-4) = 5$
d)	$\square = 3+4$	$\square = (3+4)$	$\square = 4-1$	$\square =(4-1)$
e)	$5 = \square+1$	$5 = (\square+1)$	$5 = \square-1$	$5 = (\square-1)$
f)	$8 = 5+\square$	$8 = (5+\square)$	$5 = 7-\square$	$5 = (7-\square)$
g)	$5+4 = \square+8$	$(5+4) = (\square+8)$	$5-4 = \square-8$	$(5-4)=(\square-8)$
h)	$6+2 = 3+\square$	$(6+2) = (3+\square)$	$6-2 = 9-\square$	$(6-2) = (9-\square)$
i)	$1+\square = 6+2$	$(1+\square) = (6+2)$	$7-\square = 8-2$	$(6-\square) = (8-2)$
j)	$\square+3 = 7+2$	$(\square+3) = (6+2)$	$\square-3 = 7-5$	$(\square-3) = (7-5)$
k)	$5+\square = \square+7$	$(5+\square) = (\square+7)$	$6-\square = 8-\square$	$(6-\square) = (8-\square)$
l)	$9 = \square$	$9 = \square$	$5-4=7-\square = \square$	$(5-4) = (7-\square) = \square$
m)	$5+4 = \square+6 = \square$	$(5+4) = (\square+6) = \square$	$8-5 = 5-\square = 6-\square=\square$	$(8-5) = (5-\square) = (6-\square)= \square$
n)	$4+3 = 2+\square = \square+1 = \square$	$(4+3) = (2+\square) = (\square+1) = \square$		

Table 1: A unified version of the two levels of the questionnaire

During school year 2009/2010 we administered the questionnaire to 292 pupils including 140 of grade 2 and 152 of grade 3. The classes involved were 7 for grade 2 and 7 for grade 3. All the Greek participants were pupils of two primary schools in Thessaloniki whereas the Italian participants attended primary school in various villages near Parma. The worksheets were given to the pupils in two sessions in different days. During the first week the 2nd graders completed firstly the test of addition and then that of subtraction; the duration of each session was about 15 minutes. In the next week 3rd graders followed in the same order and the duration of each session was about 10 minutes. Both the length of the questionnaire and the time given for each sheet performance were chosen deliberately for our research aims, since we assume that shortage of time might reveal in a better way the child's deep beliefs when engaging in a problem solving activity.

Each of the participating class was divided into two parts roughly equivalent. During the session one of the two parts worked on the without brackets version whereas the other one worked on the version of the same test, with brackets. This ensured the comparison of the results of the different tests. In this way, we get for Level 2a, 283 protocols (141 without brackets and 142 with brackets; for Level 2s, 276 protocols (142 without brackets and 134 with brackets). It is not strange that there are different numbers for the different levels since the sessions took place in different days and so it was not possible for various reasons to keep the participants' number constant.

Analysis of the questionnaire.

A main element of the current questionnaire is the usage of an empty square for denoting the unknown. Considering the available ways of denoting the unknown and according to the existed literature we resulted to the widespread usage of an empty square, known also as the 'scaffolding style' (Hejný & Slezáková, 2007). We expected that this choice might engender possible misunderstandings: the expectation that the pupils could interpret every time the same sign in the same way in case they had an empty square more than one times in the same expression. For example, in items 2a.k, 2a.m and 2a.n and 2s.k, 2s.l and 2s.m there exist, at least, two empty squares in the same expression. This is not something new in mathematics. There is an implicit rule for the pupils that whenever a sign is present a lot of times in a series of tasks then the certain way of interpreting the sign in the first task is the rule for interpreting the same sign in all the tasks. But, despite that the issue of using the multiple presence of the unknown in the same number sentence has been the focus for various research studies (Sáenz-Ludlow & Walgamuth, 1998; Radford, 2000; Hejný & Slezáková, 2007) the case of the identical interpretation of the symbols for the unknown has not been considered yet.

From the morphological point of view we can easily classify the questions based on different criteria: i) the number of addition/subtraction signs, ii) the number of equality signs and, iii) the number of empty boxes. Another feature that plays a crucial role is the relative place of the box(es) in the sentences in relation to the signs

of +, -, and =. All the aforementioned morphological features can give hints about: the combinatorial ideas that inspired the questionnaire; the way the items were organized so as to examine whether the position of the empty square in the number sentence and more generally the types of sentence structure could influence pupils; the gradual increase in the items level of difficulty - the first six questions in each part are at the same time the simpler ones from the cognitive point of view and the shortest ones (except 2a.1) and they deserve a particular ‘attention’- these six questions play an introductory role to familiarize pupils with the forthcoming increase in difficulty of the next questions.

Due to the limitation in the number of pages we restrict ourselves to present just one possible classification of the tasks that is relative to the operation(s) needed to solve them (Table 2).

Level	one addition	one subtraction	first then subtraction	addition	first subtraction then addition	two subtractions
2a	a), d),	b), c), e), f),	g), h), i), j), m). n)			
2s	c), e)	a), b) , d), f) ,			g), j),	h), i)*, l), m)

Table 2: Operation algorithm(s) involved in the standard solution of the tasks.

Items in bold, when seen as equations, can be solved if the unknown (empty square) is interchanged with the number that is in the other side of the equal sign. This algorithm can be followed since the empty square is at the right of the minus sign. The items k) are not included in this Table (for both Levels) since this equation is actually a Diophantine equation. Item 2a.1 is not included in Table 2 since it is about an equation presenting at the same time its solution. For items 2s.i and 2s.i-bra it could be mentioned that actually the algorithm is similar to the rest of the items in bold font.

We do not restrict our expectations to these known algorithms since we know that young students are able very often to surprise us by choosing other ways for solving the tasks that are completely irrelevant to the standard algorithms, obtaining thus even quickly the correct answer. Some of them could be the Subtraction by complement (for example, items 2a.b, 2a.c, 2a.e, 2a.f, 2a.g, 2a.h, 2a.i, 2a.j, 2a.m, 2s.b, 2s.f) and the Invariantive property of addition or subtraction (balance) (for example 2a.g, 2a.h, 2a.i, 2a.j, 2a.k, 2a.m, 2s.g, 2s.h, 2s.i, 2s.j, 2s.k, 2s.l, 2s.m).

Moreover, we expect our findings to give some evidence concerning pupils’ early relational thinking. We will focus on the symmetry of equality relation. It is necessary to make a distinction between the pair 2.a.b and 2.a.f ($5 + \bar{y} = 8$ and $8 = 5 + \bar{y}$ respectively) and the remaining five pairs (2a.a-2a.d, 2s.a-2s.d, 2s.b-2s.f, 2a.c-2a.e, 2s.c-2s.e). In the first pair both items have as their solution the number 3. Therefore,

the role of symmetry is obvious since the involved numbers are identical ('strict' symmetry). In the remaining five pairs we can speak for symmetry 'at large' but only for the structure of the number sentences and not for the numbers involved. For example, a solver of the $7 - \ddot{y} = 2$ who has trouble with $5 = 7 - \ddot{y}$, can think of this second task in terms of $7 - \ddot{y} = 5$ to find the right answer. These tasks give us the chance to detect which pupils are 'blinded to the symmetric property of the equality', i.e., incomplete understanding of the *symmetric* property of the equality (Attorps & Tossavainen, 2007) either in its 'strict' or 'at large' meaning: a (correct) answer at only one question of these pairs may suggest a lack of an 'at large' application of the symmetry in the specific pair at the trial (or success) level.

RESULTS-DISCUSSION

The collected protocols showed two significant types of data: we call them *trial* and *success*. When a child tries to cope with a problematic situation in mathematics then there are two options: either to get round the task or to attempt finding a solution. Both reactions belong to the realm of the behaviour and this is why it is not an easy task to investigate the reasons that make a pupil to give up from being engaged in a problem solving process. There are a lot of potential explanations: tiredness, inadequate mastery of skills relevant to the certain mathematical topic, lack of available time for completing a task, lack of motivation, state of health and so on. The presence of answers in the test's items (no matter whether it is about a correct or wrong answer) is indicative of the pupil's positive disposition towards the task, of a familiarity, self-reliance, self-confidence. We adopted the term 'trial' as a measure of the positive disposition towards the task and we denote with T its percentage. 'Success' refers to the right answers to the test's items and we denote with S its corresponding percentage.

The answers collected can give some evidence about the learners' awareness as far as the specific mathematical topic is concerned. Obviously, it is not possible to know in a dogmatic way whether a wrong answer is an evidence of incomplete mastery of certain skills or knowledge. When a statistical elaboration takes place, then, it may happen some times to consider lack of answer and wrong answer as identical events, otherwise a distinguish between them has to be made.

For the evaluation purpose we adopted a twofold kind of characterization. The first one is related to trial rate (T) whereas the second to success rate (S). Thus, in relation to the first kind, a task is characterized as *adequate* if $T \geq 60\%$, and *inadequate* in the remaining case. For the second, a task is characterized as very *hard* if $S \leq 30\%$, hard if $30\% < S \leq 60\%$, *easy* if $60\% < S \leq 80\%$ and very *easy* if $80\% < S$.

According to these kinds of evaluation our data showed that in a global level the questionnaire was adequate for both Levels: Level 2a ($T = 91.44\%$) and Level 2s ($T = 88.88\%$), but it was hard: Level 2a ($S = 56.99\%$) and Level 2s ($S = 52.26\%$). Given that our focus was on the potential contribution of brackets we split the data in two groups: brackets and no-brackets versions. The corresponding percentages obtained

were as in Table 3. It can be seen that the test was again considered as adequate and hard for each of its versions no matter the cultural setting. With a χ^2 -test we obtained a statistically significant difference only between the two versions of Level 2a (=0.04%).

	T(rial) whole sample	S(uccess) whole sample
2a no bracket	91.19%	54.20%
2a bracket	91.70%	59.76%
2s no bracket	89.06%	51.14%
2s bracket	88.69%	53.44%

Table 3. Items’ global evaluation concerning trial and success

The presence of brackets seemed to be useful for 22 of the items in both levels. There were items such as 2a.a., 2a.l., 2s.c., 2s.h. and 2s.i. that got better success rate in the no-bracket version. In more details the success for the entire questionnaire is presented below in Table 4:

	very hard	hard	easy	very easy
2a no bracket	<i>g), k), m), n)</i>	<i>h), i), j), l)</i>	<i>d),</i>	<i>a), b), c), e), f)</i>
2a bracket	<i>m), n)</i>	<i>g), h), i), j), k), l)</i>		<i>a), b), c), d), e), f),</i>
2s no bracket	<i>g), j), l), m)</i>	<i>e), h), i), k)</i>	<i>c), d), f)</i>	<i>a), b)</i>
2s bracket	<i>g), j), m)</i>	<i>h), i), k), l)</i>	<i>c), e)</i>	<i>a), b), d), f)</i>

Table 4. Items’ global evaluation in relevance to success for Levels 2a and 2s

Item in *italics* indicate that for this item the obtained success rate was the highest among the two versions (bracket and no-bracket).

From the overall results we have evidence that the use of (useless) brackets supported better performance in terms of success rate. It seems that brackets’ function was to give ‘unity’ to different terms connected by a sign of operation. It can be said that they are useful for understanding correctly the equality sign since this can be easily misunderstood as another operation sign. The case of item 2.a.c ($\square + 4 = 9$) is a suitable example. In the version without brackets we found pupils from both grades (2nd and 3rd) who proposed number 13 as the solution for the empty square. Obviously this was the result of the operation $4+9=13$. However, no one in the no-brackets version ($((\square+4) = 9)$) suggested 13 as the solution of the task.

There were also examples with lower scores in the brackets version rather than in the no-brackets one. As an example we present item 2.s.i: $7-\square = 8-2$ (no brackets

version) and $(6-\square) = (8-2)$ (brackets version). For both versions the commonest wrong answer was 6, with relative frequency of $^{25}_{65}$ (38.46%) for the bracket version and $^{20}_{63}$ (31.74%) for the other one. The fact that the different numbers were used in each version possibly was the reason for getting lower scores in the bracket version. This is connected to the known fact that handling of zero (as a solution) is a great obstacle for younger pupils. Therefore, instead of using zero as the wanted number in the expression $(6-\square)$ they were lead to the 6 by a mistaken interpretation of item 2.s.i working only on a part of it: $\square = 8-2$.

Another surprising finding was that exactly the same task, in the same place in both versions got different success rate. The item 2.a.l was presented in the form $9 = \square$ in both versions. A potential explanation for the wrong answers in the brackets version (success rate 45.77% vs 56.74% for the no-bracket version) could be that this task was the only one without brackets in a worksheet where all the other items included brackets. The commonest wrong answer was 10 and this could be the result of reading the equality sign as a ‘separator’ along the line number.

For the relational thinking we mentioned earlier that for both items 2a.b and 2a.f the correct solution is 3. In case there was a correct answer only for one of the two items this could be attributed to an incomplete mastery of the formal property of equality. For the remaining pairs we presume that a right answer to one question of the pair and the firm awareness of the symmetry of equality can suggest a good strategy for solving the other question of the couple, even if the numbers are not identical. The simplicity of both tasks (2a.b and 2a.f) had as a consequence great success rate: 95% (99%) for 2a.b and 84% (88%) for 2a.f; percentages in brackets show the rate that corresponds to items in the bracket version. The percentages for those who did not give identical solutions to these two items were 18% and 13% for the no-brackets and brackets versions respectively. These numbers mean that there exists blindness concerning ‘strict’ symmetry. The accumulated rates for the other five pairs (that show blindness concerning ‘symmetry at large’) for the whole levels 2a and 2s were respectively 24% (16%) and 34% (30%). These percentages strengthen the thesis that the usage of useless brackets contribute to the reduction of the phenomenon of ‘blindness’ the symmetric property of equality.

CONCLUSIONS

It is widely accepted that an understanding of equality is crucial in developing pupils’ algebraic thinking. A typical feature for the first steps in this development is the usage of brackets in numerical expressions. In this study we tried to use ‘useless’ brackets aiming to highlight their potential contribution to pupils’ performance as well as to detect presence of relational thinking. Our findings showed that in general the presence of brackets increased the success rates of the involved pupils showing at the same time a reduction in phenomena of ‘blindness’ concerning the symmetric property of equality. The findings were more or less similar for both countries something that gives an extra support to our hypothesis that the usage of brackets

could affect positively pupils no matter the cultural setting. However, it must be said that after a comparison that was made between the official curriculums in both countries and from the cognitive point of view, the situation found to be similar so much for the Greek as well as for the Italian pupils.

Our origin was the view that brackets help pupils see structure but the major research findings mainly concern older students. However, in this study we tried to add and demonstrate another aspect for using brackets and at the same time to legalize this ‘useless’ way of applying brackets when a teacher is engaged in fostering early algebraic thinking to very young students. Such a perspective constitutes useless brackets valuable teaching tools in the teaching practice and make teachers able to respond the their students’ needs for an early understanding of equality either in terms of performance or acquiring relational thinking.

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TASKS FOR TEACHERS IN MATHEMATICS LABORATORY ACTIVITIES: A CASE STUDY

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This paper presents and discusses some examples of tasks for teachers carried out in the MMLab-ER Project. It will be shown how, in the teacher education program, the laboratory approach with artifacts (i.e. ruler and compasses) and specific tasks can shed light on crucial aspects of mathematical activity, and teachers succeeded in implementing this experience in their classrooms.

INTRODUCTION

This paper refers to activities carried out during the Project “Laboratory of Mathematical Machines for Emilia-Romagna” (MMLab-ER) coordinated by the Mathematical Machine Laboratory of Modena (MMLab - www.mmlab.unimore.it). This Project aimed at facilitating the implementation of a laboratory approach in the teaching and learning of mathematics through an education program that involved one hundred primary and secondary school in-service teachers (Bartolini Bussi et al., 2011). During the Project teachers were engaged in laboratory activities with the mathematical machines (reconstructions of tools belonging to the historical phenomenology of mathematics from ancient Greece to 20th century, i.e. curve drawers and pantographs); therefore, they designed and carried out several teaching experiments. In order to document their experiences, teachers filled logbooks and final reports in which they described the teaching experiments and collected materials (protocols, photos, videos), reflecting on the successes and difficulties encountered. The Project results and all documentation, written in Italian, were published in Martignone (2010) and on the MMLab website (<http://www.mmlab.unimore.it/online/Home/ProgettoRegionaleEmiliaRomagna/RisultatidelProgetto.html>).

The MMLab-ER Project activities were based on this assumption: a laboratory approach, which makes use of tools belonging to the history of mathematics, together with the development of fitting tasks, can provide a suitable learning context for the activation of important processes (such as exploration processes and production of argumentations) and for the analysis of cultural, educational and cognitive aspects involved (Bartolini Bussi, 2009).

This paper, after an overview of the MMLab-ER Project background, presents and discusses tasks for teachers developed during the MMLab-ER teacher education program, classroom tasks, and some teachers’ reflections on teaching experiments.

BACKGROUND

MMLab-ER Project was grounded on a laboratory idea that is well expressed by this metaphor: “We can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing” (Bartolini Bussi et al., 2004; p. 2- <http://www.icme-organisers.dk/dg20/italians.pdf>). The theoretical framework, for describing and interpreting the different phases of laboratory activities with artifacts, is the construct of semiotic mediation introduced by Bartolini & Mariotti (2008). In this framework

the teacher's main roles are the following: to construct suitable tasks; to create the condition for polyphony, eliciting the polysemic feature of the artifact; to guide the transformation of situated “texts” (signs) into mathematical “texts”. In this way the teacher mediates mathematical meanings, using the artefact as a tool of semiotic mediation (Bartolini Bussi, 2009, p.125).

From a cognitive prospective, Martignone & Antonini (2009) have been studying the interaction between a subject and a mathematical machine using Rabardel theory (Rabardel, 1995). This study sheds light on the importance of analyzing the exploration processes linked to the study of the machine artifact components (*What does the machine consist of?*), the genesis and development of its utilization schemes (*What does the machine do?*) and the analysis of the mathematical meanings embodied in the machine (*Why does it do that?*).

The MMLab-ER Project has taken into account the international researches on teacher education, in particular those collected in *The International Handbook of Mathematics Teacher Education* (Wood, 2008) and in the PME Proceedings (many of which are summarized and analyzed in Llinares & Krainer (2008)).

The peculiarities of our teacher education program are the laboratory approach and tasks focusing on the analysis of processes and on the cultural aspects involved – CAC - Cultural Analysis of Content (Boero & Guala, 2008).

Conjecturing and proving in mathematics is recognized today as a major source of educational challenges all over the world. It is one of the characterizing features of mathematical culture [...] the impact on teachers of CAC education in this area [conjecturing and proving] might be: first, to let them experience these aspects of mathematical activities; second, to induce them to distinguish between the development of productive processes, on one side, and the elaboration of their products (according to cultural constraints), on the other, as different sides of mathematical competency. (Boero & Guala, 2008, p. 231)

Teachers faced tasks (tasks for teacher - “to include the mathematical prompts, many of which may be classroom tasks, that are used as part of teacher learning” (Watson and Sullivan, 2008; p. 109)) that require them to engage in known mathematics in a new way and, therefore, aimed at improving their Specialized Content Knowledge: “the mathematical knowledge and skill unique to teaching” (Ball et. al, 2008; p. 400).

This paper presents some tasks for teachers that focus on important aspects of mathematical activity as the *adaptive reasoning*: “the capacity for logical thought and for reflection on, explanation of, and justification of mathematical arguments” (Kilpatrick, 2001, p. 107). Teachers, working with their colleagues, faced these tasks and reflected on what they were doing and on which way they could make something similar for their students (classroom tasks).

MMLAB-ER TEACHER EDUCATION PROGRAM

Aims and methodology

The MMLab-ER education program aimed at: exploring and using special hands-on tools coming from the history of mathematics, such as the mathematical machines; rediscovering mathematical meanings embodied in these machines; practicing typical formats of mathematical thinking (problem solving, conjecturing and arguing); reflecting on the processes involved and on the role of instruments; comparing with colleagues.

In the teacher education program teachers were involved in laboratory activities. We can say that during the workshops teachers were placed, with the obvious differences, in learning situations “acting as students”: they were divided in small working groups and joined the discussions orchestrated by a teacher educator. Teachers were asked to explain their geometrical constructions, discussing roots, motivations and development of their chains of reasoning. The different possible solution strategies were described and discussed with all participants who belonged to different schools and grade levels. They reflected on the analysis of one’s and others’ problem solving processes, not only related to the mathematical contents involved, but also to the use of tools.

Tasks for teachers: instrument exploration and geometrical constructions

The first mathematical machine, used in the MMLab-ER teacher education program, is the most known and used through all school grades: the compasses. We wanted to reassess the importance of ruler and compass constructions in mathematics teaching-learning: the compasses, in fact, although widely used for practical purposes (e.g. in technical education), is not often analyzed in its foundational role in the mathematics culture (we just think to Euclid’s *Elements*). Teachers explored the compasses as an artifact and as an instrument (Martignone & Antonini, 2009) following the *key questions*: “What does the machine consist of? What does the machine do? Why does it do that?”. The analysis of the instrument ended with a problem solving activity guided by the open question: “What would happen if it were to change...?”. Teachers explored the possible changes of the compasses (e.g. with equal or different rots and with extensions) and the existing different types of compasses (e.g. plane compasses and the blackboard compasses). We added this last task to the key questions because, as argued by Watson & Sullivan (2008), “The *What if?* template is useful for open-ended and mathematically-focused investigative tasks” (p.130).

After this instrument analysis, teachers faced a task that focus the attention on the development of *adaptive reasoning*: “Construct with ruler and compass an isosceles triangle; present your construction explaining the construction steps and arguing your choices; find similarities and differences among the different constructions made by your colleagues trying to understand their motivations”. This task prompted teachers to analyze the following aspects: how and why the same final product (in this case, the isosceles triangle) can be the result of different constructions; the importance of analyzing the theoretical and practical reasons grounding the different choices; the role of artifact components and utilization schemes analysis in the planning and development of the resolutions.

The discussion of teachers’ solutions

Teachers faced the task of isosceles triangle construction individually and discussed their solutions in small groups and then collectively. Table 1 presents the teachers’ constructions.

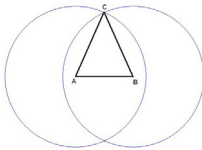
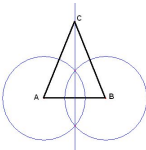
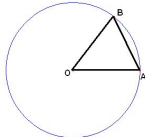
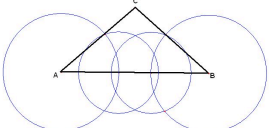
<p>A</p> 	<p>B</p> 
<p>C</p> 	<p>D</p> 

Table 1: Different isosceles triangle constructions made by teachers: A) Given three sides when two sides are equal; B) Using the segment axis; C) Using the circumference rays; D) Given one of the equal angles.

The laboratory approach gives room for discussion and comparison of solution strategies among peers and experts. The importance of verbalization and explanation of choices and procedures is underlined. The teacher educator orchestrated the collective discussion using different techniques: calling teacher to play his/her construction, asking to dictate the procedure in detail, comparing different constructions, and giving suggestions for other possible constructions (Martignone, 2011). From the point of view of mathematical contents, teachers realized that even starting from the same definition of isosceles triangle there are different constructions and cognitive processes involved: i.e. solutions A and C (Tab.1). During the collective discussion it emerged that more than half of the teachers made

constructions like in solutions B and A, few like solution C, and almost none like solution D. Teachers thought about these choices and argued that the first two strategies (A and B) are more frequent because they are usually made in technical drawing or when you draw on squared paper; while solution D was a challenge for teachers who did not remember the construction. This can be considered a simulation of “classroom” activities where students either apply known strategies or use a trial and error approach, and discuss and collaborate with colleagues arguing their choices.

Although this task deals with simple mathematical content, together with a laboratory approach, it supports teachers in thinking on their problem solving processes and in sharing their knowledge and reflections with colleagues and teacher educators.

Tasks for teachers: guessing construction strategies

After the isosceles triangle construction, teachers faced a task that presented a hypothetical student’s construction in which it is shown the first step and the final product. The processes that led to the final construction were not explained: “Below find two geometric constructions (carried out with ruler and compasses), which represent the first step and the last step of a student’s construction of two parallel straight lines. What is the link between the first and the second figure? Why are the two lines parallel? Try to apply the student’s construction to solve the following problem: given a straight line r , construct the parallel line through a point P not on r . Explain all the steps carried out”.

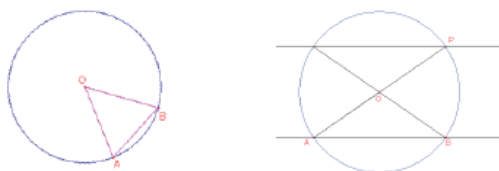


Fig 1: A student’s construction first and last step.

To answer this task teachers had to guess the construction processes, to find the reasons and then to apply this procedure to another task. This activity fostered a strong attention on the analysis of possible solutions knowing only the starting and the final steps and on the difficulties of applying unknown procedures to other situation.

As in the previous example, the focus was on the analysis and the comparison of solution strategies, with the difference though that the object of this analysis was the possible students’ solution processes. The task reproduced a situation that teachers usually face: to interpret and understand students’ works. Teachers discussed the potential of the problem-solving processes analysis (a priori and a posteriori) to understand students’ errors, different solutions, and possible chains of reasoning.

FROM TEACHER EDUCATION TO CLASSROOM IMPLEMENTATION

Classroom tasks

In all teaching experiments the teachers designed and carried out laboratory activities aimed at fostering students' individual production and critical observations, always asking students to verbalize answers, discuss and share knowledge. The classroom tasks differed according to school grades and educational goals. For example in a grade 4 class (Ferrari, in Martignone, 2010), the compasses is used as a semiotic mediator (Bartolini & Mariotti, 2008) to analyze the definition of circumference, by means of the instrument exploration guided by the *key questions* (see above). In the higher grades, the laboratory activities were similar to the teacher education workshops: shedding light on the mathematical contents and cultural aspects involved in ruler and compass constructions, fostering the development of *adaptive reasoning*, verbalization of exploration and argumentation processes, and constructive dialogue among peers and experts. Teachers introduced ruler and compass activities highlighting the historical background linked to the Euclid's Elements. They set up laboratory sessions in which students explored the compass according to the *key questions* and faced geometric constructions (e.g. the isosceles triangle construction, but also the perpendicular bisector of a line segment, the angle bisector, the parallel straight lines construction, etc.).

From the teaching experiments final reports: teachers' reflections

This paragraph presents an excerpt of a teaching experiment final report (Buonuomo et al., in Martignone, 2010) in which we can see teachers' comments and final reflections about their classroom activities. This teaching experiment was made by a group of secondary school teachers in three classes (grade 6). These teachers collaborated in designing and developing the teaching experiment and also in writing the final report in which they reflected on and analyzed the results. The students faced different tasks working in groups and discussing their solutions. Teachers orchestrated the discussions analyzing with students the different procedures, their argumentations and often pointing out particular solutions like the one presented in Fig. 2: a solution made by some groups in order to construct the straight line parallel to r through a point P not on r .

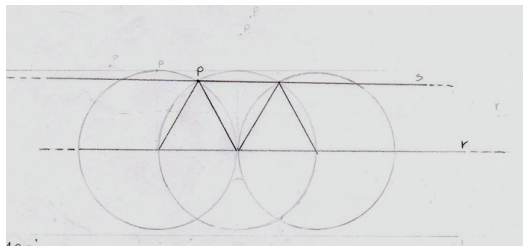


Fig 2: A parallel straight line construction

This solution was a good starting point for teachers to analyze and discuss a non-standard construction with the whole class, as they stated in their report:

the understanding of this procedure, based on the recognition of the properties of isosceles triangles, was not immediate for all the students and therefore required a discussion orchestrated by the teacher. [...] The students have enthusiastically faced the tasks, strengthening their knowledge of circumference and circle and recognizing the compasses not only as a useful tool to carry out the work of technology, but as a machine with mathematical properties that justified the geometric constructions produced.[...] The mathematical language has been improved and the students have organized their knowledge more clearly, realizing, during the verbalization of the results, the importance of using a language that is correct and appropriate to the subject. The team spirit and the cooperation among students have been strengthened. (Buonuomo et al.; p. 120-121)

Teachers underlined the role of the compasses in the construction and in consolidation of knowledge, and the value of verbalization and production of the argumentative chains fostered by the laboratory approach.

CONCLUDING REMARKS

The MMLab-ER teacher education program was based on the assumptions that the teachers' learning is promoted by: activities in which they learn in the same way as they were expected to teach it; meaningful tasks; and a reflective engagement with such activities. We are in accordance with many PME researches:

“These in-service programmes for teachers implemented a problem-centred approach and encouraged reflection on their experiences. The goal was to improve teachers' knowledge of the process of doing mathematics (Murray, Olivier & Human, 1995)” (Llinares & Krainer, 2008, p. 440)

In particular, consistently with the results of the researches on epistemological, didactical and cognitive aspects involved in laboratory activities with artifacts belonging to the historical phenomenology of mathematics (Bartolini & Mariotti, 2008; Martignone & Antonini, 2009), we argue that the laboratory approach and specific tasks, foster the development of *adaptive reasoning* (Kilpatrick, 2001), the explanation and comparison of solution strategies, the exploration and argumentation processes, and reflections.

Even if the analysis of Project results is only at the beginning, because the project started in 2008 and ended in 2010, we have already observed from the reports that teachers implemented what they have experienced during the teacher learning program: as matter of fact, they introduced the laboratory approach in their teaching practice, organizing working group sessions, asking their students to verbalize their chains of reasoning, and giving a room for students' critical observations and reflections. Moreover, the classroom tasks aimed at fostering the development of exploration and argumentation processes. We can say that the Project goals are satisfied. Further research will analyze more in depth the different aspects involved and the relations between task for teachers and classroom tasks.

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INSTRUMENTAL GENESES IN MATHEMATICS LABORATORY

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In this paper, we consider the first three steps of a teaching experiment in primary school (4th and 5th grade course) concerning the introduction of a mathematical machine for arithmetic (Zero+1) to reflect upon arithmetical operations and algorithms. The analysis presented here is carried out crossing the theoretical framework of semiotic mediation and the instrumental approach in mathematics education. The focus is students' instrumental geneeses and systems of instruments on one hand, their management by teacher on the other hand.

INTRODUCTION

The use of tools, old and new technology in mathematics education, raises questions concerning not only mathematical contents, but also their appropriations by students and their didactical management by teachers (Bartolini & Borba, 2010). In this paper, we consider the introduction of a cultural artefact concerning arithmetic (Zero+1, Fig. 1), as a tool of semiotic mediation (Bartolini Bussi & Mariotti, 2008). This tool is proposed to students according to the methodology of mathematics laboratory (Maschietto & Trouche, 2010), defined as follow: “a structured set of activities aimed to the construction of meanings for mathematical objects. As such, it involves people (students and teachers), structures (classrooms, instruments,...) and ideas (plans for didactical activities)” (AA.VV.UMI, 2004, English version p.60). The analysis of the first three steps of the teaching experiment allows contributing to the discussion about instrumental geneeses and systems of instruments. From a theoretical viewpoint, this paper tries to relate the theoretical frameworks of semiotic mediation and instrumental approach (Trouche, 2004; Trouche & Drijvers, 2010).

THE ARTEFACT ZERO+1

The arithmetical machine Zero+1 (Fig. 1) is an artefact inspired by the mechanical calculator (*pascaline*) designed by B.Pascal in 1642. It is a small plastic tool with a gear train (five wheels). Each wheel can be rotated, pushing a tooth with a finger. When the wheel A has turned a complete rotation, the wheel D makes the wheel B to go one step ahead. The same happens when the wheel B has turned a complete rotation (D and E are auxiliary wheels to transmit motion). Digits from 0 to 9 are written on the lower wheels. The number is read over the red small triangles under wheels A, B and C, which play the function of units, tens and hundreds. Two procedures can be performed to write numbers (based on decimal position system) starting from 000 position: iteration (to repeat the operation of pushing one-step clockwise the wheel A on the right until you reach the number to write) and decomposition (to push clockwise the units wheel, the tens wheel and the hundreds

wheel as many steps as need). To make calculations, similar procedures can be performed, where addition is related to clockwise, while subtraction to anticlockwise. A fundamental feature is the automation of the number to be carried (by D and E).

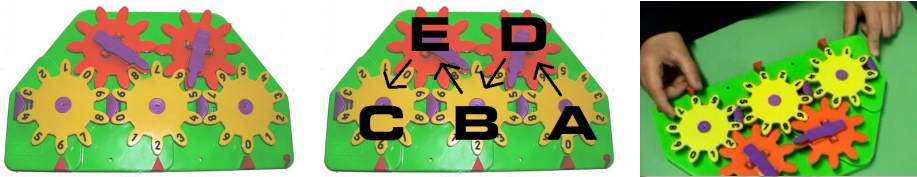


Figure 1. Zero+1 (called “pascaline” by students and teachers)

THEORETICAL BACKGROUND

Instrumental approach

Instrumental approach is developed by Rabardel (1995) and considered afterwards in mathematics education (Trouche, 2004, Trouche & Drijvers, 2010). It is based on the distinction between artefact (a material or abstract object, already produced by human activity) and instrument (a psychological construct, built from the artefact and utilisation schemes), and on processes (called instrumental geneses) leading to the construction of instruments from the artefact. The instrumental genesis is composed of two kinds of processes: instrumentation and instrumentalisation. The former is relative to the emergence and evolution of utilisation schemes; the latter concerns the emergence (as a first level) and evolution of artefact components of the instrument. Among utilisation schemes, Rabardel distinguishes: usage schemes (Us.Sch), that are related to the management of characteristics and specific properties related to the artefact; instrument-mediated action schemes (Im.A.Sch), oriented to carry out specific tasks, incorporate usage schemes as constituents. The characteristic of a scheme to be an Us.Sch or an Im.A.Sch does not refer to a property of the scheme, but to its status in subject's activity. Utilisation schemes have both a private and a social dimension: the private dimension is specific to each individual, while the social dimension (it is shared by many members of a social group) results by the development of schemes during a process involving individuals who are not isolated. Other users, as well as the artefact's designers, contribute to the elaboration of schemes (Beguín & Rabardel, 2000). Rabardel specifies that instruments are not isolated for a subject, but they form systems that are constituted by tools available for him, as a function of tasks and contexts. He also adds that *“The introduction of a new artifact must, at the didactic level, be equally managed in its impact on previously built instrument systems”* (quoted in Trouche 2004, p.304). According to this framework, several mathematics education studies, especially carried out in technology-rich environments, show the complexity of instrumental geneses and the necessity to teacher assistance of students' geneses. For this reason, the notion of instrumental orchestration is introduced (Trouche, 2004) and defined by three components: didactical configuration (*“the layout of the artifacts available in the*

environment”, Trouche, 2004 p.296), exploitation modes of these configurations, didactical performance (that “*involves the ad hoc decisions taken while teaching*”, Trouche & Drijvers, p.676). Trouche also regards instrumental orchestrations can act at several levels, from the artefact as technical instrument to reflection upon the relationship of a subject with an instrument.

Artefacts in the construction of mathematical meanings

The mathematics laboratory methodology, considered in this paper, is founded on the theoretical construct of semiotic mediation, which has been elaborated and applied to mathematics education by Bartolini Bussi and Mariotti (2008). It concerns the mediation of cultural artefacts to construct mathematical meanings, from a Vygotskian perspective. In their study, Bartolini Bussi and Mariotti pay attention to the following elements, with the assumption that artefacts are not transparent with respect mathematical meanings rooted in them. Firstly, the analysis of semiotic potential of an artefact (the double semiotic relationship between the use of artefact and mathematical meanings on one hand, the use of artefact and personal meanings on the other hand), taking partially into account instrumental approach, allows to underline signs that artefact offers, its components and mathematical meanings it mediates. In the case of Zero+1, the generation of the sequence of natural numbers by the operator ‘+1’, which is consistent to Peano’s axiomatic system for arithmetic, is suggested by the iteration procedure. Numbers are written in decimal position system (wheels with ten teeth). The direction of wheels, clockwise for addition and anticlockwise for subtraction, links these two operations. Zero+1 is a “rich semiotic” artefact, because it contains mathematical signs (e.g., Arabic numeral and point) but also other signs, like triangles and arrows, to be interpreted. Secondly, teacher plays a complex role when he wants to use an artefact as a tool of semiotic mediation. Beside the choice of an artefact related to specific mathematical meanings, teacher has to plan tasks with artefact for students in order to foster the production of signs and to guide the evolution of those signs (“situated texts”) toward mathematical texts (representative of the meanings embedded in the artefact). The process of semiotic mediation is grounded in the structure (called didactical cycle) of activities: activities with artefacts, individual production of signs and collective production of signs (including mathematical discussion).

By the analysis of the first steps of the teaching experiment, we intend to discuss some relationships between these two theoretical frameworks, from a didactical and cognitive viewpoints. We put forward the hypothesis that instrumental orchestration is interlinked with the didactical cycle. From a cognitive viewpoint, we want to deep the analysis and interpretation of the complexity of an instrumented action (Mariotti, 2006) for students involving in a semiotic mediation process. We put forward the hypothesis that a system of instruments, related to specific mathematical meanings (such as positional notation and decimal grouping, arithmetical operations) begins to be brought back by students working with new artefacts when constructed utilisation schemes allow them to compare instruments. Another hypothesis is that an

instrument is “cognitively interesting” if there is a “distance” with respect to other instruments of existing systems.

THE ANALYSIS OF THE TEACHING EXPERIMENT

The teaching experiment, started at the end of the 4th grade course and pursued as a part of the mathematical curriculum of the 5th grade course, is composed of nine steps, including a final evaluation (Maschietto & Ferri, 2007; Maschietto & Trouche, 2010). The activities are structured according to the definition of didactical cycle. As a tool of semiotic mediation, Zero+1 is introduced in the classroom to reflect upon arithmetical operations (in particular, the distinction between unary and binary operations) and algorithms. The different steps allow the teacher to guide the evolution of signs in the class; according to the theoretical framework, the passage from the work with the artefact to mathematical text (concerning mediated meanings) is taken into account. In final evaluation, the students are asked to write the two procedures to make additions (decomposition and iteration) in mathematical terms. The analysis of the first three steps is carried out in terms of both didactical management of the activities by teacher (by the lens of instrumental orchestrations) and students’ processes (by the lens of instrumental geneses). It is based on students’ productions (texts and drawings) and video-recordings of all the steps.

Step 1. Exploration of the Zero+1

Orchestration. Didactical configuration: a Zero+1 for each pair of students. Exploitation modes: introduction to the activities and oral description of the required work, group work for exploration Zero+1, collective phase to share results. In the oral description, two tasks can be distinguished to foster the appropriation of that artefact and the construction of utilisation schemes: 1) “*Observe, try, experiment, try to use it*”, which are suggestions to stimulate the mobilisation of Us.Sch, related to the structure of Zero+1 (e.g., wheels can be turned) on the one hand and the beginning of instrumentalisation process as emerging of artefact components on the other hand; 2) “*Make calculations, in particular addition*”, corresponding to a request that fosters the construction of Im.A.Sch for arithmetical operations. An element is left implicit in that task: the utilisation scheme to write and read numbers, which is fundamental to make calculations. But during the final collective phase, the teacher proposes a specific task on writing numbers; this decision could be interpreted in terms of didactical performance. This orchestration aims to support social dimension of utilisation schemes (according to the semiotic mediation framework).

Instrumental geneses. Basing on videotapes, the following elements are identified. Firstly, Us.Schs emerge and seem to be evoked by the structure of Zero+1. In excerpt #13, the student turns wheels (Us.Sch.R) and explains their linked movement (cfr. Fig. 1), using deictic gestures. This movement between wheels A and D is related to the “number to be carried” (mathematical meaning), when operations are considered. Students look for an Us.Sch for the auxiliary wheels D and E. These actions can be interpreted in term of “semiotic richness” of the artefact, where each

component has to be directly used. This seems to be strengthened by the transparency of Zero+1, i.e. it is not a black box. Students also notice the red triangles at the bottom of Zero+1, which is a component of utilisation schemes to read numbers.

- 13 Fede Rotation because, before everything these ... these [*he points wheels*] are attached (...), if, for instance, I turn this one [*wheel D*], it starts to turn also the wheel next to [*wheel A*] and... everything starts to work [*he turns wheel A again and again*] and, then, if, instead, I turn this one [*wheel E*], the other [*he points wheel B*] turns.

An Im.A.Sch for addition emerges (decomposition procedure, #2), that requires the scheme to write numbers and another scheme, that takes into account the relationship between direction of rotation and addition/subtraction (#45). These schemes can be interpreted as Us.Sch with respect to the principal task to make calculations. They became Im.A.Schs when the teacher asks to write numbers during the collective moment. In students' speech, position of wheel is associated to decimal position system by gestures (#2) or in an explicit way (#45). In Giac explanation (#2), the Us.Sch corresponding to turn one tooth at a time is implicitly performed (Us.Sch related to +1-iteration). Another component of this Im.A.Sch for calculation is implicit at this step: the number to add has to be kept in memory.

2. Giac Addition..., (...) I make 239 [*he writes this number*] + 24. There are not hundreds, so it remains like this [*he points wheel C*], then there are 2... tens and hence we add them [*he turns wheel B*] and 3 becomes 5 and then to 9 we add 4 [*he shows 4 digits*] and [*he turns wheel A*] [there is a] change and 5 [*he points wheel B*] becomes 6, because a ten is added.
- 45 Chiar (...) the bottom wheel [*she points wheel A*] is the units, the middle one [*she points wheel B*] is the tens and the top one [*she points wheel C*] is the hundreds(...) Clockwise like addition and, instead, subtraction works anticlockwise (...).

In general, students show different levels of instrumental geneses, related to a kind of hierarchy of utilisation schemes. Some students seem to have constructed only Us.Sch.R (attracted by wheels and colours), most of students show utilisation schemes for addition and multiplication (that are strongly related), while very few students present utilisation schemes for subtraction and division. Moreover, these schemes have utilisation schemes for writing and reading of numbers as components. This difference can be also interpreted in terms of handling of the Zero+1: clockwise movement by forefinger is quite "natural" (cfr. Fig.1). The Im.A.Sch for addition by iteration (the second number is added to unit wheels) appears in the intervention of one student, which can be attributed to the big size of numbers chosen by the students (#45). With respect to instrumentalisation, some students identify the bounds of Zero+1 ("just, it cannot go beyond 999"), but also the possibility to overcome them by adding other wheels. A student also detects the point for decimal numbers. Some students' difficulties in exploring Zero+1 concern the interpretation of the task, while others can be gone back to their knowledge. Following Vergnaud (1990), the knowledge of binary operation seems to block the construction of Im.A.Sch for operation based on the structure of Zero+1, because it is associated to different

schemes (for instance, to write the two numbers to add on wheels C and B, and to wait for the result on wheel A). This kind of difficulties not only pays attention to the interlacement between the detection of Zero+1 components (instrumentalisation) and utilisation schemes, but also seems to confirm a hierarchy among them (writing and reading before making calculations).

In this step, instrumental geneses starts and different instruments begin to be constructed. The analysis shows how the utilisation schemes are related to knowledge; from this viewpoint, we claim that semiotic mediation process aims to explain the knowledge arose from the work with the artefact.

Step 2. Report of the exploration of Zero+1

Orchestration. Didactical configuration: a Zero+1 for each pair of students, paper for texts and drawings. Exploitation modes: introduction and oral description of the activity, group work. During the collective phase before working group, teacher proposes some elements for the report of the exploration of Zero+1 that the students are supposed to prepare. So, the teacher forces the first level of the instrumentalisation process. For this reason, drawings are also accepted in the reports.

Instrumental geneses. Writing text allows the students to remember their work with the Zero+1 and, therefore, to revise their instrumentation and instrumentalisation processes. For instance, the importance of the structure of Zero+1 to understand its functioning is stressed by Orlando: *“Because, if we don’t know how it [Zero+1] is made, how we can use it!”*. This second step solicits the production of personal signs, which can replace gestural signs arisen from the activity with the artefact. Some signs (“accounts” in Trouche 2004, p.296) in situated texts can be interpreted in terms of visible part of utilisation schemes. Figure 2 shows two kinds of drawings.

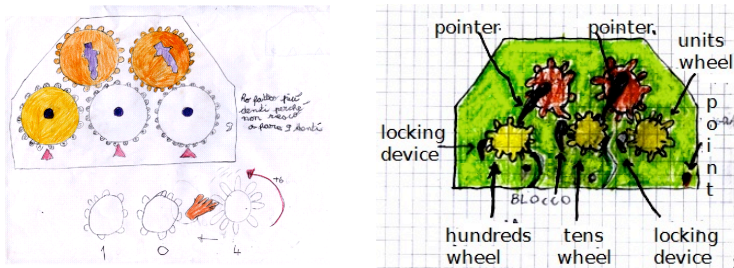


Figure 2. Drawings with movement (on the left) and components (on the right)

On the left, the student represents Zero+1 as artefact and, lower down, an utilisation scheme for $4+6$ is represented, where the attention is paid on the action to add 6 units to number 4 (already written). The Us.Sch.R is represented by the arrow and the hand to turn the wheel, even if the direction is incorrect. For Im.A.Sch for addition, there are two steps: in the first step, the student acts on unit wheel with a hand and, in the second step, the result appears (on the two wheels on the left). On the right, another student shows a first level of instrumentalisation (all the components are drawn and

named). In his text, he brings back the Im.A.Sch for operation concerning rotation direction.

Step 3. Operation with Zero+1 and discussion about instruments for calculations

Orchestration. Didactical configuration: a Zero+1 for each pair of students, blackboard, notebooks. Exploitation modes: introduction to the activity and tasks on blackboard, collective work, group work. The teacher proposes 4 calculations (task with Zero+1), whose the first one is collectively made. Then, a discussion about tools for calculating follows (which can concern the level of didactical performances).

Instrumental geneses. The proposed task allows students to consolidate their utilisation schemes, making Zero+1 an instrument for calculation. After that, the students begin a spontaneous discussion, recalling other tools like abacus and pocket calculator. This discussion can be interpreted as the emergence of students' systems of instruments, which is partly built in class (the abacus was introduced in a previous year linked to decimal notation system within the framework of semiotic mediation). The pocket calculator, on the contrary, is an individual instrument for students. The analysis of that discussion seems to confirm the hypothesis the new instrument (Zero+1) is related to other instruments (or it joins a systems already built) when the instrumental genesis process allows the construction of robust utilisation schemes. Zero+1 and calculator are mainly compared. The abacus is only considered for *“the number to be carried out”*. Since it is not further recalled in the discussion, we interpret that it is not “distance enough” from Zero+1 in terms of making calculations with respect to calculator. Some elements of the comparison are: structure of the artefacts and U.Sch (push a button, turn a teeth of the wheel); utilisation schemes to make calculations (e.g., *“Pocket calculator can be uses in only one way; instead, the pascaline does not have a strict way: you can make from right to left or from left to right”*); transparency and usefulness of instruments in learning mathematics.

CONCLUDING REMARKS

In this paper, we have analysed three steps of a teaching experiment concerning the use of tools in mathematics education from the viewpoint on instrumental geneses (concerning students) and their management in terms of orchestrations (concerning teacher). The analysis of these steps shows the construction (emergence and evolution) of utilisation schemes to make operations by Zero+1, which contains other schemes (i.e., to write and read numbers) as components. It emphasises that the status of a scheme as Us.Sch or Im.A.Sch depends on proposed tasks. Nevertheless, a hierarchy of utilisation schemes is confirmed. Instrumental geneses have a strong social dimension (later, students wrote the instructions for use of Zero+1 and to the comparisons with other instructions). In this way, the support of social instrumental geneses allows to emerge the mathematical meanings embedded in the artefact and reach didactical aims. During the third step, systems of instruments appear and are suggested by mathematical meanings and utilisation schemes.

With respect to the orchestration, the analysis shows similar configurations and exploitation modes; the third component needs further elements. It confirms the hypothesis on relationship between orchestration and didactical cycle, in particular exploitation modes contains elements of didactical cycle. In addition, didactical orchestrations seem to present different levels, according to Trouche (2004), because they do not only concern artefact, but also the relationship among instruments.

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PLOTTING THE USE OF ORAL LANGUAGE AND RICH MATHEMATICAL REPRESENTATIONS OF TEACHERS IN THE EARLY YEARS OF FORMAL SCHOOLING

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Many Indigenous Australian students begin school with limited experience using Standard Australian English, the discourse of schooling. This paper analyses self-reported teacher data collected at the outset of a longitudinal study in Queensland, Australia. The data suggest that there is a mismatch between what is considered effective practice and what actually transpires in the classroom. Teachers in Indigenous and multi-cultural schools appear to rely heavily upon a literacy approach to mathematics instruction, rather than a focus upon using rich mathematical representations to model concepts.

FOCUS OF THE PAPER

This paper examines the nature of oral language and representations used by teachers as they instruct young Australian students in mathematics at the beginning of formal schooling. The nature of mathematics instruction is based upon structured-play activities focusing upon the development of key mathematical understandings. In particular, the use of Standard Australian English (SAE) and the interplay with mathematical representations during classroom instruction are analysed. The data are drawn from forty teachers at the initial stage of a large, longitudinal study, with the particular aim of identifying effective pedagogical practices to ensure foundational mathematical understandings are developed by the students. This paper seeks to analyse and describe the pedagogical practices of these teachers at the commencement of the project, and proposes ways in which these practices may be enhanced. The findings indicate that the pedagogical practices of these teachers continue to reflect a strong emphasis on literacy acquisition rather than mathematical understanding.

THEORETICAL FRAMEWORK

A grounded theory approach was used to analyse the data obtained from the initial teacher interviews. *Open coding* was used to break down the data into distinct units of meaning. This process commenced with a full transcription of the audio-recording of the initial teacher interviews, after which the text was analysed line by line for each question in an attempt to identify key words or phrases which connected the teacher's self-reported practices to the experience under investigation. A fundamental feature of grounded theory is the application of the *constant comparative* method which involves comparing like with like, to look for emerging patterns and themes.

This process facilitates the identification of concepts, that is, a progression from merely describing what is happening in the data to explaining the relationship between and across incidents.

REFERENCES TO RELATED LITERATURE

A basic definition of oral language is ‘communicating with other people’. However *communication* is not a simple concept; it involves thinking, knowledge, and skills, and requires practice and training. Teachers who believe that oral language acquisition is a natural process for students, requiring little effort and occurring long before attendance at school, assume that the primary learning tasks for students in school are reading and writing (Zhang & Alex, 1995). When this is the case, oral language development is often neglected rather than forming a crucial component of teaching and learning. Furthermore, oral language in these classrooms is generally used more by the teacher than the students. It is used to generate *initiate-respond-evaluate* (IRE) sequences and to direct and transition activities; seldom does it function as a means for students to gain knowledge and to explore ideas. In this paper, oral language is defined as communication between the teacher and students which is characterised by open as well as closed questions, affirmations, negotiating and verifying meaning, and the use of gestures and facial expressions.

It has long been acknowledged that oral language is crucial to a student’s literacy development (Aldridge, 2005), and more recently to emergent mathematical development (Krause, Bochner, Duchesne, & McMaugh, 2010). In the early years of formal schooling, an oral language approach to teaching and learning is appropriate due to the limitations of reading and writing skills of these students. When the language of schooling is Standard Australian English (SAE) the mathematical register used by teachers consists of words that come from two primary sources: (i) everyday English, and (ii) mathematics. The words from everyday English may have the same meaning when used in the mathematics register (e.g., *increase*), may have a different meaning (e.g. *table*), or may have a subtly different nuance (e.g., *between*). Words sourced from the discipline of mathematics seem to only have meaning in mathematics, such as *pronumeral*. In order to be positioned to engage with school mathematics, teaching and assessing using SAE, students need to possess an adequate linguistic repertoire (Meaney, Fairhill, & Trinick, 2008).

In this paper, the proposition is that *effective* mathematics teaching and learning in the early years of formal schooling will take place when an oral language approach in *conjunction* with rich mathematical representations is used to develop a mathematical register. This focused approach requires teachers to: actively engage students in learning processes, scaffold learning experiences, and encourage students to explore and speak about their own thinking and make schematic connections. Furthermore, the ‘bundling’ of oral language and rich mathematical representations is characterised by movement among and between the representations, with the oral language being the conduit for the movement. We contend that ‘good mathematics

teaching' occurs when this bundling is frequent and of a consistently high quality. It is within this construct that the teacher and the students create a social constructivist learning environment with oral language being the primary tool for meaning making (Bikner-Ahsbabs, 2006).

Developing an understanding of mathematical concepts requires engagement in a variety of models and representations; the depth of understanding is inextricably linked to the richness of these representations. Models are ways of thinking about abstract concepts (e.g., counters to represent numbers) and representations are various forms of the models (e.g., placing the counters on a number line or placing the counters on a grid). Mathematical ideas are *experienced externally* (materials, pictures, diagrams, spoken words, and written symbols) and *comprehended internally* (mental models and cognitive representations) as connections to existing schemas or modifications of these are made. From this perspective, mathematical understanding is determined by the number and strength of connections in the students' internal network of representations (Hiebert & Carpenter, 1992) and effective teachers arguably are those who are able to facilitate the development and strengthening of these networks.

METHODOLOGY

The initial stage of a 2010 – 2014 longitudinal project (*Representations, Oral Language and Engagement in Mathematics: RoleM*) funded by the Australian Department of Education, Employment and Workplace Relations (DEEWR) involved telephone interviews of 40 teachers of students in the first two years of formal schooling (Prep and Year 1). The teachers were located in a variety of locations and represented urban, rural, and remote schools, and those ranging from no Indigenous students to all Indigenous student cohorts. In this context 'Indigenous' refers to students from either Australian Aboriginal or Torres Strait Islander background. The teacher arm of the RoleM project was based upon a model which incorporated several phases. The phases are outlined as follows:

- Pre-testing (conducted by researchers at each school site in each classroom).
- Three professional development days held at four different locations throughout the school year.
- Three telephone interviews following the professional development days.
- Three follow-up school visits occurring two to three weeks after each professional development day.
- Post-testing (same conditions as pre-testing).

Prior to the interview, the questions were sent to all participants, allowing them time to prepare their responses. Due to the distance between the participants and the interviewers, all interviews occurred by telephone at a time that was convenient to the participants. The interviews were conducted by two research assistants, especially assigned and trained for this task. Each had a copy of the interview questions and

considered the types of probes that would be appropriate to ask in order to gather a fuller understanding of each response. All interviews were audio-taped for transcriptions. The interview consisted of 20 questions, three relating specifically to this paper. These three questions were: (1) What is your understanding of the term 'oral language'? (2) What role does oral language play in your current practices in mathematics instruction? and (3) How do you model the appropriate use of language to your students?

The transcripts of the audio recordings of the interviews were analysed by each researcher independently. This process exhibited the characteristics of grounded data analysis. In the first instance the researchers independently read each transcript and identified the themes in each, coded the categories, and sorted the data into categories, constantly comparing the data across interviews. They then concurred with regard to the nature of each category, giving supporting evidence from the transcripts. In the cases of disagreement, each researcher returned to the original data gathering excerpts to support particular stances until final agreement occurred. In most instances this entailed at least five iterations through the raw data by each of the researchers.

The themes relating to mathematical *representations* resulted in three clear categories: (i) no use of mathematical representations, (ii) some use of representations, and (iii) rich use of representations. Teachers identified in category (i) made *no* mention of mathematical representations in their responses to the interview questions; while those who were identified in category (ii) mentioned either a vague use (e.g., *I use hands-on materials*) or mentioned a specific representation such as attribute blocks for patterning. The third category of rich mathematical representations was used to locate teachers who discussed a variety of representations incorporating hands-on materials, pictures, photos, diagrams, and symbols.

The nature of oral language approaches used by the teachers was described using four categories: (i) Speaking, (ii) Speaking-Linguistic, (iii) Speaking-Understanding, and (iv) Communicating. The *Speaking* category referred to teachers who defined 'oral language' solely in terms of spoken language, in particular, the teacher speaking to the students in Standard Australian English (SAE). It involved a focus upon the teacher use and student acquisition of mathematical vocabulary and terminology. The *Speaking-Linguistic* category was used to describe teachers who specified speaking and listening, student mirroring of language, by repeating what they had heard (including teacher correction), and translating between SAE and student 'home' language. The third category of *Speaking-Understanding* referred to teachers who focused upon students understanding the mathematical concept being taught and then verified this understanding by questioning in order to probe. The *Communicating* category referred to teachers who described speaking, listening, using gestures and facial expressions, and checking for understanding by using particular question types

(e.g., *how do you know that this is the next [element] in this pattern?*) as central to their pedagogical practices.

SAMPLE DATA AND RESULTS

This process resulted in a two-way plot which was then used to situate the forty teachers. The horizontal axis of the plot indicates the teacher's use of mathematical representations, and the vertical axis indicates the nature of oral language used by the teacher. Figure 1 shows the plot with the 40 teachers located in placement grid squares.

Figure 1. Plot mapping oral language use and degree of mathematical representation for all teacher participants (n = 40).

Language	Communicating (C)	Nil	35 (2.5%)	13 (2.5%)
	Speaking - Understanding (SU)	1, 11, 34 (7.5%)	20, 26, 28, 30, 32, 36 (15%)	14, 31 (5%)
	Speaking - Linguistic (SL)	7, 10, 15, 16, 22, 23, 24, 27, 33, 37 (25%)	2, 4, 8, 9, 12, 17, 18, 19, 38, 39 (25%)	Nil
	Speaking (S)	5, 29, 40 (7.5%)	3, 6, 21, 25 (10%)	Nil
		No representations (NR)	Some representations (SR)	Rich representations (RR)
Mathematics				

Fifty percent (shaded) of the 40 teachers were located in the *Speaking-Linguistic* and *No representations* (SL/NR) or *Speaking-Linguistic* and *Some representations* (SL/SR) grids, suggesting that the preferred practices of these teachers involved a focus on speaking, listening, repeating, and translating (between SAE and home language). The data in this plot suggest that teachers who use rich mathematical representations do so in conjunction with questioning, gestures, and facial expressions (teachers 13, 14, & 31). Also of note are the nil entries for the *Communication* and *No representations* (C/NR) location, suggesting that teachers who are concerned about students understanding and demonstrating this understanding appreciate those mathematical representations need to be used in conjunction with the oral language. Research on both Indigenous students' learning of mathematics as well as favoured pedagogical practices for teaching in the early years of formal schooling, indicates that grids Communication-Rich representations

(C/RR) and Speaking-Understanding-Rich representations (SU/RR) are optimum combinations for teachers to pursue.

To further investigate the data, two separate plots were developed: (1) teachers in schools with a population of all Indigenous students, and (2) teachers in multi-cultural schools. In this context ‘multi-cultural’ refers to classrooms where students were drawn predominantly from Asian and Pacific Island backgrounds, as well as some from Indigenous and African backgrounds. These students may have little or no SAE. However, they may have some proficiency in their home language. Figure 2 shows the plot of the responses of teacher participants who were situated in schools with a population of all Indigenous students.

Figure 2. Plot mapping oral language use and degree of mathematical representation – teachers at schools with all Indigenous student population ($n = 23$).

Language	Communication (C)	Nil	Nil	Nil
	Speaking Understanding (SU)	1, 11 (9%)	20, 26 (9%)	Nil
	Speaking Linguistic (SL)	7, 10, 16, 22, 27 (22%)	2, 4, 8, 9, 12, 17, 18, 19, 38, 39 (43%)	Nil
	Speaking (S)	5 (4%)	3, 6, 21 (13%)	Nil
		No representations (NR)	Some representations (SR)	Rich representations (RR)
Mathematics				

Forty-three percent (shaded) of these 23 teachers lie in the grid described as *Speaking-Linguistic* with *Some mathematical representations* (SL/SR). This suggests that teachers in this context favour a focus on speaking, listening, repeating, and translating (between SAE and home language) in conjunction with the use of some mathematical representations. No teachers self-reported the use of *Rich mathematical representations*, and no teachers were identified in the *Communication* code. Figure 3 shows the plot of the teachers in multi-cultural schools.

Figure 3. Plot mapping oral language use and degree of mathematical representation – teachers at multi-cultural schools ($n = 17$).

Language	Communication (C)	Nil	35 (6%)	13 (6%)
	Speaking Understanding (SU)	34 (6%)	28, 30, 32, 36 (24%)	14, 31 (12%)
	Speaking Linguistic (SL)	15, 23, 24, 33, 37 (29%)	Nil	Nil
	Speaking (S)	29, 40 (12%)	25 (6%)	Nil
		No representations (NR)	Some representations (SR)	Rich representations (RR)
		Mathematics		

Interestingly, 53% of the 17 teachers in the Figure 3 plot are located in grids which the researchers considered to represent effective to highly-effective pedagogical practices for the development of mathematical concepts. This plot also clearly indicates that *Communication* does not occur with *No representations*, and the converse, that *Rich mathematical representations* are not employed with *Speaking or Speaking-Linguistic* approaches. When the plots of Figures 2 and 3 are compared the following observations can be made:

- 8.6% of teachers in all Indigenous student schools are located in the upper-right grids (i.e. C/SR, C/RR, SU/SR, and SU/RR) while 50% of the teachers in multi-cultural schools are located in the same grids,
- 79% of teachers in all Indigenous student schools are coded as *Speaking-Linguistic* (SL) while only 29% of the teachers in multi-cultural schools are coded as SL, and
- both plots indicate that C/NR, SL/RR, and S/RR are not viable combinations of practice.

DISCUSSION

Whereas a literacy program approach to English as a second language (ESL), for Indigenous students and those from diverse cultural backgrounds, may be helpful in engaging with early mathematical understandings of the language of mathematics, it must be remembered that mathematics is much more than just expressing. It is apparent that an ESL approach, incorporating speaking-listening-translating, was employed by the majority of teachers in schools with high proportions of Indigenous

students. The danger of focusing on an ESL approach is that the interaction soon becomes a *linguistic* exercise rather than a means to develop an understanding of mathematical concepts (Howard, 1997). In addition, separating the language from the context can lead to misconceptions which may result in delayed or little acquisition of mathematics in later years of schooling. It appears that an ESL approach was *not* as prevalent in the multi-cultural schools in this project. We propose two reasons as to why this may be the case: (i) these multi-cultural schools are situated in schools which are able to access support for their ESL students from trained ESL teachers and/or (ii) it is not possible or reasonable that a class teacher is able to translate between SAE and the multitude of languages that may be used in their classroom. So every effort is made to ensure *understanding* of mathematical concepts occurs which entails more than merely developing the facility to use mathematics terms. The challenge is to improve the quality of teaching mathematics in the early years of formal schooling by encouraging teachers to attend to a combination of oral language communication and rich mathematical representations, rather than an approach which favours repetition, acquisition, and transmission of vocabulary. As this longitudinal study progresses, investigations will be made to determine if the practices of the teachers involved in the project have moved to the desired state of communication with rich mathematical representations as a result of the ensuing professional development days and follow-up at their school sites, and the relationship between different approaches and students acquisition of mathematical understanding.

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YOUNG CHILDREN'S SPONTANEOUS FOCUSING ON QUANTITATIVE ASPECTS AND VERBALIZATIONS OF THEIR QUANTITATIVE REASONING

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This paper presents a cross-sectional study of young children's Spontaneous Focusing on quantitative Relations (SFOR), Spontaneous Focusing on Numerosity (SFON) and verbalizations of their quantitative reasoning. Two tasks were presented to 4.5 to 8 year old children (N=86) during two separate sessions, first in order to measure their SFOR and SFON tendencies and second in order to measure their verbalizations of their quantitative reasoning through stimulated response questions. Children were found to differ in their SFOR and SFON tendencies. Children's SFOR tendency increased with age. Children were more likely to perform the task based on SFOR or SFON than they were likely to verbalize about their quantitative reasoning during the stimulated recall session.

INTRODUCTION

Young children have a variety of skills for reasoning about quantitative relations already before school age (Boyer, Levine, & Huttenlocher, 2008; Sophian, 2000). Reasoning about quantitative relations, such as proportional relations or relations between numerosities, can be seen as the informal understanding of ratios and fractions and has been described as a "quantitative form of analogous reasoning" (Boyer et al., 2008, pg. 1479). Young children have been found to have well-developed reasoning abilities about a concept or skill long before they are able to verbalize these abilities (Becker, 1993; Sophian, 2000). Children's verbalizations about quantitative relations may provide a key link in the development from these informal concepts to formal mathematical concepts such as fractions.

Sophian (2000) examined young children's ability to verbalize their reasoning about proportional relations and found that this ability seemed to appear after they were already able to reason proportionally. In her study, 4- and 5-year-old children were asked to compare a model pair of circles and find the matching proportional pair of circles from two pairs of choices. After a set of 48 test trials, the children were asked to explain their choices on 6 post-test trials, being asked, "Why did you pick that one?" The 4-year-olds were unable to verbalize their reasoning, while still performing above chance on the task. 5-year-olds proved somewhat more able to verbalize their reasoning, while still performing above chance; though only 4 children (out of 20 5-year-olds) were able to explicitly relate size of the circles to each other.

Similarly, Becker (1993) found that 4- to 5.5-year-old children were unable to verbalize the correct answer to a many-to-one correspondence problem, but were able

to distribute the correct amounts. A majority of the participants gave incorrect answers when asked how many items would be needed in total if x number of dolls needed 2 or 3 items each. However, when asked to distribute the number of items they had verbalized, almost all of the participants who gave incorrect answers were able to parcel out the items in correct amounts (e.g. 2 items for 1 doll), even though they would run out of items before giving all dolls their share.

There are substantial individual differences in how often young children focus their attention on exact numbers of items or incidents in their natural surroundings (Hannula & Lehtinen, 2001; 2005). A child's frequent Spontaneous Focusing On Numerosity (SFON) leads to increased practice with enumeration skills, which is crucial for the development of numerical skills and number concept (Hannula & Lehtinen, 2001; 2005; Hannula, Lepola, & Lehtinen, 2010). Focusing attention on more complex mathematical aspects of a task may be needed in order to use more advanced numerical skills beyond basic number recognition. Based on this notion, in the current study, we propose that there may exist differences in how often children spontaneously focus their attention on quantitative relations. Thus, Spontaneous Focusing On quantitative Relations (SFOR) is defined as the spontaneous (i.e. undirected) focusing of attention on quantitative relations and the use of these relations in reasoning (McMullen, Hannula-Sormunen & Lehtinen, in preparation).

By combining stimulated recall methods (cf. Sophian, 2000) with methods used to measure spontaneous focusing tendencies, the present study can provide insight into the development of young children's quantitative reasoning. Two tasks were presented to children during two separate sessions, first in order to measure their SFOR and SFON tendencies and second in order to measure verbalizations of their quantitative reasoning. In the first session, children were undirected to the mathematical nature of the task and completed the imitation task based on the aspect of the activity which they found most relevant, be it quantitative relations, numerosity, or non-mathematical aspects, without feedback. The tasks were reintroduced to the child in a second session with stimulated recall questions.

Research Questions

In the present cross-sectional study of three age groups, we aim to investigate children's SFOR and SFON tendencies and their abilities to verbalize about their quantitative reasoning. To this end, we ask two questions: 1) Do children's SFOR and SFON tendencies differ from their verbalizations of their mathematical reasoning during stimulated recall interviews? 2) Are there differences in children's verbalizations of their reasoning about quantitative relations, numerosities, or non-mathematical aspects?

METHODS

Video-recordings of 86 Finnish-speaking children (50% female), with no diagnosed learning impairments, were collected during two 30-minute sessions in a quiet room at the child's day-care or school by a trained male researcher. Children were between

the ages of 4y; 5m and 8y; 4m ($M = 6y; 8m$; $SD = 1.0$ years) at the time of their testing. Children were separated into three age-groups based on their placement in kindergarten ($n = 31$; $M_{AGE} = 5y; 6m$), pre-school ($n = 27$; $M_{AGE} = 6y; 9m$), or first grade ($n = 28$; $M_{AGE} = 7y; 9m$). Two tasks from the first session and stimulated recall interviews of the two tasks from the second session will be reported in this paper.

Bread Task

Two identical stuffed-dogs named “Nassu and Tassu”, 20 cm in height, were used as characters that were fed bread. The breads were circular foam pieces 6.5cm in diameter, which were a different color for each trial. The whole breads, originally the same size, were cut into different proportions (halves, thirds, quarters or sixths), and disarranged on the plate in order to prevent the direct mapping of the area of the two sets of bread (Figure 1). The child was told that the bread “*have broken...but, Nassu and Tassu don’t mind.*”

In the first session, Nassu and Tassu were introduced as being two friends who always do the same (e.g. run or jump the same, eat the same), and want the same for a snack (cf., Imitation task, Hannula & Lehtinen, 2005). Two plates of the breads were then placed on the table in front the child and in front of the researcher. The child was told “*Watch carefully what I give Nassu, and then you give Tassu exactly the same.*” The experimenter gave the bread from his plate to Nassu, one at a time, turned over his plate, and said, “*Now you give Tassu exactly the same.*” The child then gave from the plate in front of them to Tassu. The child got no feedback about his or her task performance. There were altogether 4 trials in the task (see Table 1).



Figure 1. Sets of bread, trials 1-4, child’s plate on the left in each trial

For the stimulated recall task, the dogs were reintroduced and the child was told that they would be giving bread again and that they would be asked some questions. The instructions to “*watch carefully*” were repeated and the experimenter gave the same amounts as in the first session. However, the child was stopped before giving their bread and asked “*How do you know, from this bread (points to child’s bread) what to give Tassu?*” To check their memory, the child was then asked, “*How many pieces of bread did I give Nassu?*” Finally, the child was allowed to give the bread to Tassu. Again no feedback was given to the child about his or her task performance.

Rice Task

Two stuffed-monkeys “Miina” and “Pate” were fed using two pairs of spoons (Figure 2), each pair proportional in height. Set A were plastic cylinders, the small spoon 3cm in diameter and 3cm high and the larger spoon (twice the size) 3cm in diameter and 6cm high. Set B were metal rectangular prisms, the small spoon was 2.5cm x 2.5cm x 2.66cm, and the big spoon (three times the size) was 2.5cm x 2.5cm x 8cm.

Trial	Researcher	Participant	Response Type		
			Relation	Numerosity	Non-math
Bread					
1	2 Halves- Give 1	4 Fourths	2 pieces	1 piece	3-4 pieces
2	6 Sixths- Give 2	3 Thirds	1	2	3-6
3	6 Sixths- Give 3	4 Fourths	2	3	1, 4-6
4	3 Thirds- Give 2	6 Sixths	4	2	1, 3, 5-6
Rice		Size of spoon			
1	Set A Big- Give 1	Small A	2 spoons	1 spoon	3+ spoons
2	Set B Big- Give 1	Small B	3	1	2, 4+
3	Set B Small- Give 3	Big B	1	3	2, 4+
4	Set A Big- Give 2	Small A	4	2	1, 3, 5+

Table 1. Task trials. Researcher and Participants materials and possible responses.

In the first session, the monkeys were said to be brother and sister who like to have the same for lunch. A bowl of rice was placed on the table in front of each the monkeys and an empty bowl was placed in front of each these bowls. The plastic spoons were held up for possible comparison and the child was told that “*we will use these spoons*” and that “*Pate and Miina always want full spoonfuls*”. For the first trial, the smaller spoon was placed in the bowl in front of the child, the experimenter held the larger spoon, and then said “*watch carefully what I give to Pate, and you give Miina exactly the same.*” The experimenter placed one spoon of rice in the bowl and then asked the child to “*give exactly the same.*” No feedback was given to the child. In total there were four trials (see Table 1).

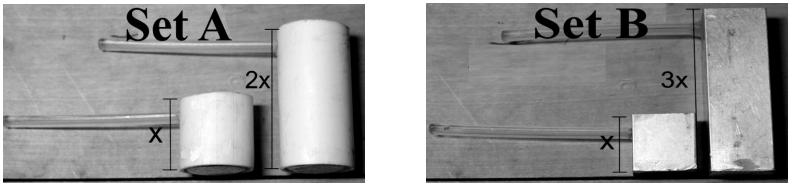


Figure 2. Rice Task spoons. See Table 1 for more detail.

For the stimulated recall task, a method similar to the bread interview was used. The child was told that they would “*give Pate and Miina rice again,*” and the child would be asked a couple of questions. The child was told to “*watch carefully,*” and the experimenter gave the same amounts as in the first session. The child was asked to wait before giving rice to Miina and asked, “*How do you know what to give Miina now?*” and “*How many spoonfuls did I give Pate?*” The child was then allowed to give rice to Miina. No feedback was given about the child’s task performance.

Analysis

For the tasks in the first session, task performance was analyzed for each trial based on the most complex aspect on which the child focused, with quantitative relations (SFOR) being the most complex, then numerosity (SFON), and finally non-mathematical aspects. A child’s scores were based on, a) the number of pieces or spoonfuls given (see Table 1) or b) any utterances, classified as relational (e.g., “You

put 2 of those big ones, so I should put 4 small ones”), numerical (“Now just one!”), or non-mathematical. The maximum total score for each task was 4. Two independent raters analyzed 21% of the participants’ responses and agreement was found on 98% of the trials for the bread task, and 97% of the trials on the rice task.

For the stimulated recall tasks, the child’s response to the first question, “*How do you know...what to give...?*” was analyzed to determine if the child’s answer was based on either a) quantitative relations (e.g. “*because two pieces like this are formed by one of those*”), b) numerosity (e.g., “*One bread, because you gave one as well.*”), or c) non-mathematical aspects, including no explanation (e.g., “*This piece, because you gave from the same location.*”, “*I don’t know*”) Again, the response was coded based on the highest level that the child discussed in this answer. 21% of the trials were analyzed by two independent raters and 94% agreement was found on both tasks.

RESULTS

Table 2 and Figure 2 display evidence that children differ in their SFOR and SFON tendencies during their performance on these tasks, as well as in their verbalizations of their reasoning during the stimulated recall tasks. Intraclass correlations calculated for all trials across the bread and rice tasks for the responses to the task performance, ICC=0.85, and for the stimulated recall answers, ICC=0.88, were found to be sufficiently high enough to warrant combining both the bread and rice tasks in analysis.

Task	Relational	Numerosity	Non-Math
Task Performance			
Bread	17.2 %	61.9 %	20.9 %
Rice	14.8 %	66.3 %	18.9 %
Total	16.0 %	64.1 %	19.9 %
Stimulated Recall			
Bread	18.3 %	20.9 %	60.8 %
Rice	15.4 %	23.8 %	60.8 %
Total	16.9 %	22.3 %	60.8 %

Table 2. Percentage of responses by reasoning level. ($N_{\text{bread \& rice}}=344$; $N_{\text{total}}=688$)

A 2 x 2 x 3 ANOVA [(Task Performance, Stimulated Recall) x Reasoning Level (Relational, Numerosity) x Age Group (Kindergarten, Pre-school, First Grade)] was run for children’s responses. Main effects of task ($F(1, 83)=103.37$, $p<0.001$), reasoning level ($F(1, 83)=38.01$, $p<0.001$), and age group ($F(2, 83)=6.58$, $p<0.01$) were significant. Children were a) more likely to spontaneously focus on mathematical aspects in their task performance than verbalize about their quantitative reasoning, b) more likely to spontaneously focus on or verbalize about numerosity than quantitative relations, and c) more likely to perform the task based on SFOR and SFON and verbalize about their mathematical reasoning the older they were. Interaction effects of Reasoning Level x Age Group ($F(2, 83)=6.73$, $p<0.01$) and Task x Reasoning Level ($F(2, 83)=63.45$, $p<0.001$) were also significant. First

graders were more likely to reason about quantitative relations than younger children and the difference between children’s relational and numerical reasoning was larger in their task performance than in their verbalizations.

Planned pair-wise comparisons were run in order to compare children’s SFOR and SFON scores with their verbalizations of their mathematical reasoning. Children displayed SFOR in their task performance with the same frequency as they verbalized about quantitative relations, $F(1, 83)=0.16, p=ns$. However, children displayed SFON in their task performance significantly more than they verbalized about their numerical reasoning, $F(1, 83)=98.22, p<0.001$. This difference was seen in all age groups separately (all $p<0.001$).

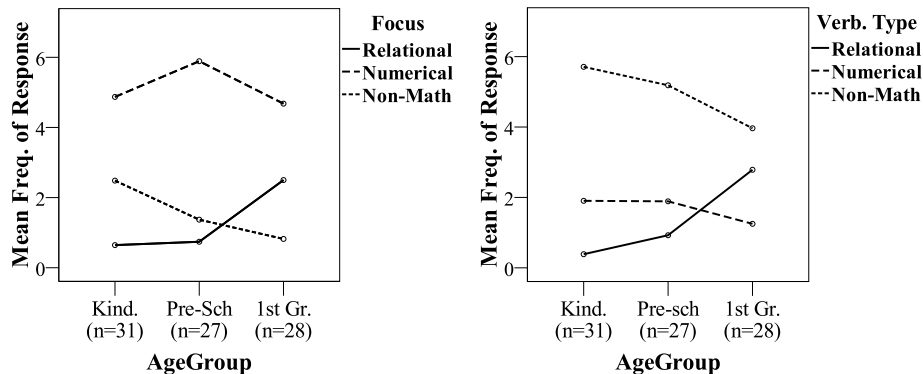


Figure 2 - Mean frequencies of spontaneous focus in the task performance (left), verbalizations (right) by Age

Children’s non-mathematical verbalizations were categorized (Table 3). For the non-mathematical responses children verbalized about a) the act or way of giving (“*I have to give this rice because you gave that rice.*”), b) location (“*This one, because you gave from the same spot.*”), c) shape or size (“*That one, because it’s shaped like a party hat...*”), d) the nature of the bread or rice (“*Bread*”, “*Rice*”, “*Breakfast*”), or e) no explanation (“*I don’t know*”).

Task	Giving	Location	Size/Shape	Bread/Rice	No Response
Bread	20.6 %	10.5 %	7.2%	18.6%	43.1%
Rice	33.5%	3.8%	3.8%	32.5%	26.4%
Total	27.0%	7.1%	5.5%	25.6%	34.8%

Table 3. Non-mathematical verbalizations by category. (N=418 responses)

DISCUSSION

Previous studies have found that there are substantial individual differences in children’s SFON tendency (Hannula & Lehtinen, 2001; 2005; Hannula et al., 2010). The present study indicates that there are also differences in children’s SFOR tendency in tasks that can be attended to based on both mathematical and non-

mathematical aspects. Children were also found to increase in their SFOR tendency with age. The novel methods of this study seem promising for the isolation of SFOR tendency in children's task performance. These findings suggest that spontaneous attentional processes may be relevant for more areas of mathematical development, including the recognition and reasoning about quantitative relations.

Prior studies have found that children's ability to verbalize about their mathematical reasoning lags behind their ability to display this reasoning in their task performance (Becker, 1993; Sophian, 2000). The results reported in this study show a similar pattern, with children being less likely to verbalize about their quantitative reasoning than they display spontaneous focusing on mathematical aspects in their task performance. The majority of children displayed SFON in the completion of these tasks, but a majority were then unable or unwilling to verbalize about their quantitative reasoning when asked to do so on a subsequent occasion. However, this discrepancy between spontaneous focusing and verbalizations of reasoning does not hold true for all levels of mathematical reasoning. The frequency of children's task performance based on SFOR did not significantly differ from the frequency of verbalizations based on quantitative relations. This discrepancy can be a result of several reasons.

One interpretation of these findings is that the wording of the stimulated recall question may have influenced children's responses, as linguistic cues have been found to influence children's answers (Wagner & Carey, 2003). The stimulated recall question specifically directed the child toward his/her own bread or rice, asking "*How do you know, from this bread, what to give...*" Those who were more likely to spontaneously focus on numerosity in these tasks may have found this question confusing, as they may have focused only on the researcher's portion of the task (e.g. the number of breads the researcher gave), whereas those children who were more likely to spontaneously focus on quantitative relations during these tasks would have needed to focus on both sets of bread or spoons of rice to determine what to give. This confusion among those who were more likely to focus on number may have led to the relatively low frequency of numerical verbal responses during the stimulated recall session. The sizable number of responses not providing any substantive reasoning (categories d and e, of non-mathematical responses) could be a result of this confusion.

One alternative explanation for the large disparity between children's SFON tendency in the task performance and numerical verbalizations in the stimulated recall is that numerical aspects of a situation may be more automatically perceived than quantitative relational aspects, which may require more active processing in order to be perceived. SFOR requires both the recognition of numerosity in the two sets of items and relating of the sizes and numerosities involved with respect to each other. It is plausible that this requires more deliberate reasoning than pure recognition of numerosity. More studies are needed to fully understand these preliminary findings. It is also possible that those children who have a well-developed

understanding of quantitative relations may be more likely spontaneously focus on quantitative relations in completing these tasks and, as well, be more able to verbalize their reasoning about relations. This could be a consequence of stronger general cognitive abilities in children with high SFOR tendency or possibly the influence of schooling, as the first graders are more likely to have had to develop the ability to verbalize their reasoning in the classroom.

The findings of this cross-sectional study provide a better understanding of the role of spontaneous attentional processes and verbalizations in children's mathematical reasoning. Children are found to have differed in their SFOR tendency in these tasks and this SFOR tendency seems to be related to the ability to verbalize about quantitative relations. However, conclusions regarding true developmental patterns require a longitudinal study of these skills. Nonetheless, unguided imitation tasks seem to be potential measures for the isolation of spontaneous attentional processes, such as SFON and SFOR. Finally, the combination of these tasks with carefully constructed, open-ended, stimulated recall tasks may provide an even more useful tool for the investigation of children's mathematical reasoning.

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COLLABORATIVE RESEARCH AS A STRATEGY OF PROFESSIONAL DEVELOPMENT OF TEACHERS

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This article reports a study conducted with three Portuguese primary school teachers who participated for two years, along with a professor in a collaborative research project focused on the analysis of their communicative practices in mathematics. The study shows that teachers develop professionally, manifest developments in their forms of collaboration in the group from providing aid and assistance to joint work, and in parallel deepen their teaching knowledge and professional practices.

INTRODUCTION

In Portugal, the primary school teachers (who teach in the first four years of schooling) develop their professional practice in an environment characterized by individualism, more salient here than in teachers from other grade levels. In addition, except in recent years, offers of training in mathematics for these teachers were almost nonexistent. As a consequence, the professional development of these teachers was very conditioned and marked by the initial training.

In this context, it seemed to me important to study the professional development of primary teachers in a collaborative project, focused on their classroom practices in mathematics. Among these practices, because of its importance in the classroom activity and the interest that was manifested by the teachers, the collaborative project was focused on the research of their communicative practices. Therefore, we defined the following problem: How does the teacher development occur in the context of a research project with a collaborative nature, focused on mathematical communication? To address this problem, some questions were pointed out, among which I highlight in this article the role of collaboration in teacher professional development.

The idea of collaboration between teachers is central in this study for two reasons. The first, is that we recognize the collaboration as a dimension of the professional development process (Krainer, 2001). The second is of methodological nature and concerns the need to reach out to teachers, interacting with them over a considerable period of time in order to study a complex process as professional development. So we constructed a work context in which the teachers and the researcher could establish a collaborative professional relationship

THEORETICAL FRAMEWORK

The professional development of teachers and collaboration are key concepts in this study. The professional development is a dynamic process that occurs over the life of the teacher involving diverse types of learning. These learning opportunities arise

both from informal opportunities lived at the school and from formal provided by training programs (Krainer, 2001; Liberman, 1994; Ponte, 1996, 2009). Assuming the complex and dynamic nature of the professional development, raises the question of what develops in this process. Hargreaves and Fullan (1992) argue that the development of teachers can result in: (i) improving knowledge and skills, (ii) self-understanding of his person, and (iii) change in context. Liberman (1994), for his part, says that professional development is a learning process, in which the critical reflection of practices has a decisive role. Deepening the understanding of this process, Krainer (2001) identifies four key dimensions of the professional development of teachers, organized into two binomials: action / reflection and autonomy / collaboration. The professional knowledge is present in both action and reflection, and also emerging from the interaction between them. The professional knowledge that the teacher mobilizes to prepare, execute and evaluate their classes is designated didactic (Ponte, 1996), which covers diverse aspects, such as knowledge of mathematics, learning processes, curriculum and instructional activity. The concept of collaboration is widely used in Education. However, Christiansen et al. (1997) warn that collaboration is a process largely undefined and only partially understood. Stewart (1997) emphasizes that this is a relation in which people with various career paths are engaged in a common work. The author points out its essential elements: the interdependence and a give and take attitude; solutions that emerge as a result of a work of mutual construction that takes advantage from differences; partners who sought to question the stereotypes and find with others new directions; joint ownership of decisions, collective responsibility for the destiny of the work, emergent process through negotiation with the interactions, and the rules for future interactions being constantly updated (Stewart, 1997).

The professional collaboration can be materialized in several ways, depending on the objectives. Harris and Anthony (2001) distinguish two forms. In one, the collegial interactions essentially help to create an environment of emotional support. Teachers present their difficulties, making a kind of catharsis, but there is no real interest in approach them intellectually, analyzing them and studying them. In the other, the interactions contribute to a significant professional development of the teachers involved, which is implied in a job-sharing.

The forms of collaboration among teachers are also examined by Little (1990), which proposes four broad categories: (i) Storytelling and scanning for ideas, (ii) Aid and assistance, (iii) sharing and (iv) Joint work..

In the *storytelling and scanning for ideas*, interactions among teachers are “opportunistic”, since the basic intention is to collect new ideas, swapping brief stories, informal and sporadic. In *aid and assistance* teachers expect more from other colleagues - to help resolve a difficult case, through a precise technique. This form of collaboration is clearly asymmetric and uni-directional, fully preserving freedom of choice of the teacher. The *sharing* takes place through the exchange of materials, methods, ideas and opinions. In this case, there is already some exposure of the

teacher to the rest group, representing a more public form of collaboration. Finally, *work on joint ownership* stems from meetings between teachers signed shared responsibility for the work of teaching (interdependence), the idea of collective autonomy, support for the initiatives and leadership of teachers in relation to professional practice and in membership of the group, founded in professional work (Little, 1990). In all categories, is the most demanding for teachers in terms of responsibility, commitment and time-consuming.

METHODOLOGY

The research methodology follows a qualitative interpretative paradigm, based on case studies of teachers (this article presents some data of them). Over nearly two years, the three teachers in the study (Matilde, Jorge and Ana) participated in a collaborative project with a professor (author of this article), reflecting about and investigating their communicative practices in mathematics classes (this work included sessions of group, observation and reflection of lessons of each teacher, publishing articles and participating in seminars).

In order to understand their professional development in this context, data were collected through observation of 10 lessons for each teacher and written records and audio from 28 group sessions of the project, three individual interviews with each teacher, teacher diaries, field notes and written documents (individual and collective). Data analysis followed the data collection, allowing for the ongoing project to identify tensions, problems and issues that were fundamental to the progress of the study. After the data collection, subsequent analysis allowed the de-construction and re-construction of information, leading to the establishment of formal categories.

PRESENTATION AND DISCUSSION OF RESULTS

The presentation of the results is organized into two parts. The first presents data on the negotiation and development of the collaborative project. In the second, I analyze the forms of collaboration experienced by teachers, relating them to other dimensions of professional development.

Negotiation of the project: Thematic and working methods

In the first meeting of the entire team, seeing as the three teachers did not know each other, I tried to ensure that the presentations were not the mere circumstance not too much to bare the individuality of each. Creating an environment of trust, essential in professional relationships of this kind, started from the beginning. Ana, the more experienced teacher, but who had shown less warm in the first approach, was the one that set the tone at the first session of the project, speaking of herself as a professional and as a person - a fact which created a favorable environment for the project. With the aim of establishing a compromise among the participants was distributed in this first session of the group a document with the form of an open and flexible protocol negotiable. The introductory text of this document indicated the emphasis of the proposal:

The document is a simple proposal to discuss, change, add, and not just a program to be applied to teachers. For this reason, I expected of each team member a critical perspective, in order to adjust and improve. The discussion of the proposed project is not a task that it is finished now, but it is something that continually builds. (Protocol, January. 2002)

It was also clarified that in view of the importance of the subject and the little attention that it has deserved, the project's theme would be the mathematical communication. One sign that the project was not closed and there will be opportunity to negotiate subjects and procedures was the relief that, coupled with the mathematical communication, took the problem solving. The protocol was structured around several aspects: What is proposed? General theme of the research activities to develop, time and resources, benefits for members of the team. Concerning the first aspect, it proposed to:

- Reflect on the professional work they undertake;
- Work in collaboration with other colleagues;
- Develop research work, focused on their lessons, with the support of the entire group (...) (Protocol, January 2002)

In regards to the work philosophy, it seemed important to clarify in this document some guidelines: "Nobody is the sole owner of reason; It is the group that solves problems and make decisions; There is no single leader" (Protocol, January 2002).

At the first meeting, teachers had difficulty making decisions. On the second, a week later, some decisions had already been taken, including the duration of the project which could eventually be extended and the type of activities to be undertaken: a discussion of current topics that are related to mathematics education in the early years, discussion of cases classes, discussion of problems, observation and recording of lessons, preparation of articles, and participation in meetings.

In the course of the project, we can find three phases. The first phase corresponds to the first three months with the team meetings to occur weekly. This was a phase of mutual understanding, essential to creating a relationship of confidence, which were basically two aspects that were more closely related: the discussion of texts about mathematical communication and the reflection on incidents of classroom teachers. This reflection allowed, firstly, build a shared discourse on mathematical communication and, secondly, to identify and formulate problems in their classroom practices, which could serve as a starting point to research work.

The second phase of the project, with about 10 months, developed around the creation of a collaborative research work on teachers' practices. The previous phase helped to identify a set of questions where was applicant the concern of teachers, especially the young ones: What influence do the tasks set out in the process of problem solving? This concern led the group to devise a means of research, which began by building a set of mathematical tasks (problems), that were developed in the

classes of each teacher.. Associated with these tasks, we defined a set of tools for data collection, which went by the diaries, audio and video records and written materials of students. Data collected were analyzed and presented in group sessions of the project. At this stage, there are two moments in the affirmation of the group and represent high levels of exposure, both individually and collectively: (i) the dynamics of a group discussion about the ongoing work at a seminar of teachers of mathematics, and (ii) the construction and publication of an article together.

The third phase of the project, with about 10 months, aimed to promote the professional autonomy of each teacher. During this period, the involvement of teachers was different. Ana, a teacher with more years of service (about 30 years), continued to be strongly committed to the understanding of her practice through research of mathematical communication. The other two younger teachers (Matilde and Jorge, about 5 years in the profession, each) although enthusiastic about the project, have decreased their involvement, replacing the previous research by simply reflection.

Forms of teacher collaboration in group

Throughout the project, the teachers engaged in diverse forms of collaboration, both in terms of what it was intended as the time they occurred. We identified three patterns of collaboration among group members: aid and assistance, sharing and joint work. The aid and assistance has the result of a relationship where there was a clear asymmetry in the participation of the teachers. This form of collaboration was observed in the younger teacher, Matilde, seeking answers to her professional problems through interaction with others. So, she did not attribute significant value to their own ideas for the evolution of the group's thoughts: "Sometimes I feel so [pause] I feel that my participation in the project is [pause] less important"(1.st Matilde interview, February 2002). The emergence of this way of collaboration exclusively in the youngest teacher, is due mainly to the strong feeling of insecurity in the pursuit of teaching: "I think some of my difficulties and insecurities had to do with the fact that I do not feel prepared to discuss some aspects of Mathematics in primary. It was not the Mathematics (...) but in order to make the students acquire this knowledge. "(2.nd Matilde interview, July 2002). About her conception of knowledge, Matilde believed in the existence of a didactic knowledge into practice, in the form of norms. Since she was the only one who had no course specialization for primary school, she believed that her colleagues were more able to gain access to this knowledge

Ana: This idea is important, here in the text, is important, even that. Teaching is really a very complex thing there are no ready-made recipes

Investigator:- The teaching provides us with a set of guidelines that serve as a kind of repertoire that the teacher, in every moment of class, depending on events of the class, will reach for.

Matilde: It's funny that we're waiting for answers to our problems, to solve them ...
(Group session, March 2002)

The evolution of this concept - which has undergone an enhancement of practice through reflection and research as a source of knowledge - coupled with a growing appreciation of some areas of their didactic knowledge, including knowledge of mathematics - a fact that helped her feel more confident in her skills - were responsible for a new way of working, more interactive and transactional, sharing: "Anyway, I feel safer in my work, I feel more competent professionally. I think it goes through a better understanding of the situation and greater attention to what goes on ". (3.rd Matilde interview, December 2002). The sharing was the form of collaboration that the other two teachers, Jorge and Ana, went in first phase of the project. Apart from reflecting on episodes of their classes, the sharing was being enriched, extending the project to various situations, such as mathematical tasks, articles, books, films or texts. The sharing took on a form of collaboration that was halfway between modes of work of individual nature and other modes involving more commitment and time-consuming. These characteristics justify the choices of these two teachers by sharing; therefore they allowed a smooth transition from individual work to another distinct in which there was greater openness and exposure group.

The first phase of the project favored the initiation of collaborative research work through questioning of classes episodes. This form of collaboration that was developed in second phase - joint work - represented an improvement over the shares, because the teachers were involved, as equals, in this work. Associated with collaborative research practices, in this second phase of the project, the group was engaged in two joint ventures that gave greater identity - on one hand, the participation in a group discussion at a meeting of teachers of mathematics and on the other hand, the publication of an article. The joint work, in a different way of the sharing, implied a greater commitment of each teacher with the group, through the division of tasks and seeking common goals to which everyone contributed. Therefore, this phase was the most demanding in terms of the work and time spent in it. In the third phase, and unlike of previous, the forms of teacher collaboration diverged, as shown in Table 1, referring to the entire project:

Phase of collaboraive project				
	1. st phase		2. nd phaase	3. rd phase
Matilde	Aid and Assistance	Sharing	Joint Work	Sharing
Jorge	Sharing		Joint Work	Sharing
Ana	Sharing		Joint work	

Table 1: Evolution of forms of teacher collaboration

In the final phase, while Ana remained co-ownership, the other two teachers returned to sharing. Ana continued research work around the mathematical communication, involving the parents of his students from 1st grade:

During the research study "Parents and mathematical communication, " in several stages, Ana tried to establish a collaborative relationship with me which was based on negotiation, a process of constant adjustments. The meetings with parents was first suggested by Ana, who defended her usefulness - these sessions they became important in how the work came to the course. (Note field, June 2003)

The other two teachers decreased their collaborative involvement, continuing to attend group meetings, reflecting on cases of their classes on various aspects of communication, but without a clear focus. In addition to the group sessions, there were other meetings between Ana and myself, in the development of research "Parents and mathematical communication. ". The divergence of forms of collaboration of teachers in the last phase results of many factors, some structural and other of the conjuncture. The last ones relate to some logistic issues, because in the second year of the project the younger teachers changed to schools farther away from the place of meeting, and had in their classrooms students from more than one year of schooling simultaneously. These reasons, while important, do not seem strong enough to explain the way in advancing forms of teacher collaboration, and therefore need to call to explain other reasons of a structural nature. The continuation of Ana in the co-ownership is based on how she conceived the role of collaboration in their professional development, including the possibility of bringing forward research projects on their practice. In this context the project "Parents and mathematical communication, " through which developed a study aiming to understand how evolve the ability of students to communicate mathematically.

FINAL REMARKS

This study shows that teachers develop their ability to collaborate professionally if they have an appropriate and challenging contexts. The collaborative research seems to be a good framework to progress this capacity and is capable of having their involvement, despite being a little remarkable feature of the culture of primary teachers, where individualism still prevails. To get the forms of collaboration such as joint work appears to be essential to go through other less demanding, based on shared reflections on episodes of classes, usually in a narrative way. The collaborative research is well placed to promote the professional development of teachers because it is based on a genuine search for answers and not the handling of questions for which answers some know in advance - as happens often in other training programs. By deepening the collaboration, the teachers seem to develop also a new sense of professional identity. The collaboration reinforces this idea, insofar as this process becomes an interactive brand teacher, something that gives them more strength as a professional group. The participation of project members in meetings with teachers or the production of articles contributing to the strengthening of professional identity, making them feel as mathematics teachers.

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CONSTRUCT VALIDITY OF A DEVELOPMENTAL ASSESMENT ON GEOMETRICAL FIGURE UNDERSTANDING: A RASCH MODEL ANALYSIS¹

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The aim of this study was twofold, firstly to validate the construction of a test measuring middle and high school students' geometrical figure understanding and secondly to identify possible levels of geometrical figure understanding. The use of Rasch model allowed the creation of a scale, which was shown to be invariant for the four groups of students examined. The analysis of the data revealed that the instrument has satisfactory psychometric properties. Three Levels of geometrical figure understanding were also identified.

THEORETICAL FRAMEWORK

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry (Mesquita, 1998). Parzysz (1991) mentions that figures can illustrate definitions, sum up a complex set of information and help in conjecture and proof. Geometrical figures are simultaneously concepts and spatial representations. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Furthermore, a geometrical figure may have the nature of an *object* when it is possible to infer geometrical relationships that may be used in geometrical reasoning and proof and when the visual perception of the figure is consistent with the verbal statements of the problem. On the contrary, when the external representation has the nature of an *illustration*, it is then impossible to directly extract a geometrical relationship from the construction of the figure, the figure seems to 'mislead' and the visual perception of the figure is in contradiction within the verbal statements (Mesquita, 1998).

Duval (1995, 1999) distinguishes four apprehensions for a geometrical figure: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, *perceptual apprehension* refers to the recognition of a shape in a plane or in depth. In fact, one's perception about what the figure shows is determined by

¹ This paper is a part of the research project "Ability to use multiple representations in functions and geometry: the

figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. *Sequential apprehension* is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. *Discursive apprehension* is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). The epistemological function of the discursive apprehension is the proof. However, it is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. *Operative apprehension* depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one makes the figure larger or narrower, or slant, while the place way refers to its position or orientation variation.

In this context the main purpose of this research study was to develop an assessment tool for measuring middle and high school students' geometrical figure understanding and to examine the construct validity of the test. The study was not restricted to the construction of a valid tool for assessing students' geometrical understanding, but also aimed to identify levels of geometrical thinking that could be useful for diagnostic teaching of geometry at middle and high school.

Test development – method

For the construction of the test we took into account Duval's (1995, 1999) apprehensions for a geometrical figure. In order to align the test with the mathematics curriculum concerning geometry of grades 9 to 11, a prior examination of the curriculum and of the geometry textbooks was conducted. A content analysis of the geometry textbooks used for the teaching of Geometry in the Cypriot schools for Grades 9 to 11 was performed (Michael, Gagatsis, Deliyianni, Monoyiou & Philippou, 2009), in which the types of geometrical figure apprehension required and used in the exercises and examples were identified. Thus, we developed a test involving four groups of tasks, each one corresponding to the four different types of geometrical figure apprehensions proposed by Duval (1995, 1999). For each type of apprehension we tried to form several subcategories. Concerning discursive apprehension we also took into consideration Harada, Gallou-Dumiel and Nohda's (2000) conceptualization, who indicated that the hypothetical-deductive proof is produced by this kind of apprehension. In fact, the discursive apprehension is produced by inferences based on definitions and valid procedures of proof. In specific, the test was comprised of the following four groups of tasks:

1. The first group of tasks examined students' *perceptual apprehension* (PE) and included tasks examining the ability to discriminate and recognize (r) several subfigures in a perceived figure (PEr1, PEr2, PEr3) and a task that tested the ability to identify and name (n) the squares in a complex figure (PEn1).
2. The second group of tasks examined students' *operative apprehension* (OP). It included tasks requiring a mereologic (me) modification of a geometrical figure (OPme1, OPme2, OPme3), an optic (op) way of modification (OPop1, OPop2, OPop3) and a place way (pw) of modification (OPpw1, OPpw2, OPpw3).
3. The third group consisted of tasks examining students' *sequential figure apprehension* (SE). It comprised of tasks testing students' ability to construct (c) a figure (SEc1, SEc2, SEc3) and to describe (d) its construction (SEd1, SEd2).
4. The fourth group of tasks concerned students' *discursive apprehension* and included verbal problems (VE). The solution of tasks VEde1, VEde2 required inferences based on definitions (de). The solution of tasks VEpo1, VEpo2, VEpi1, VEpi2, VEpi3 and VEps3 required inferences based on procedures (p) for proof. The figure in tasks VEpo1 and VEpo2 had the nature of an object (o), whereas in the tasks VEpi1 and VEpi2 the figure had the nature of an illustration (i). In tasks VEpi3 and VEps3 students had to discriminate a formal proof (s3) from an empirical and a semi – empirical proof.

The final version of the test was administered to 881 students, aged 15 to 17, of middle (Grade 9) and high (Grade 10, Grade 11) schools in Cyprus (312 in Grade 9, 304 in Grade 10, 125 in Grade 11 – basic mathematics and 140 in Grade 11 – advanced mathematics. In Grade 9 and 10 the students are all attending the same mathematics course while in Grade 11 they should make a choice between basic and advanced mathematics. The results concerning students' answers to the tasks were codified with PE, OP, SE and VE corresponding to perceptual, operative, sequential and discursive (verbal) apprehension respectively.

The Rasch model was considered to be more appropriate than other item response theory models for the validation of this test. Firstly, the raw scores of the persons were considered to be a sufficient statistic for the estimation of their underlying ability. In addition, we did not wish to award more points for correct or partly correct responses to more difficult questions and did not penalize the persons for incorrect or partly-correct responses to easier questions (if we had intended to do this, a two parameter model might be more appropriate). Finally, the nature of the open-ended questions did not encourage guessing (so a three-parameter model was not a choice). Overall, models that had weighted scores as sufficient statistics or incorporated pseudo-guessing parameters were not appropriate for our data. Three statistics were selected for this study in order to evaluate model-data fit for individual items. The first two are the Infit Mean Square and the Outfit Mean Square (Wright & Stone, 1979), while the third fit statistic we used was Andersen's Likelihood Ratio Test (LRT) (Andersen, 1973). For the estimation of the Rasch models, the Analysis

(Lamprianou, 2008) and the R (R Development Core Team, 2010) software were used. More specifically, the eRm package (Mair & Hatzinger, 2007) was used which operates on the R platform. In order to investigate whether the psychological construct manifests itself in the same way across the four groups, the Partial Credit Rasch model was fit on the data for each group independently. We evaluated the global model-data fit using the LRT. We also investigated the fit of each individual item using the three item-fit statistics mentioned before. For each age group we tried to improve the model-data fit by removing items which appeared not to be aligned with the rest of the test. We then compared the items' hierarchy of the remaining items. In addition to the above, we also used all the data in one single calibration of the items and utilized the LRT to investigate for between-group fit (between the four groups). Scatterplots with confidence intervals were used in order to compare the items' hierarchy. Finally, we reached to a conclusion regarding a single sub-set of items which seemed to behave psychometrically in the same way across the four groups, we computed an ability estimate for each person and then compared the performance of the four groups.

Results

The initial Rasch analysis on Grade 9 data showed that a number of items indicated high fit statistics to at least one of the three item-fit statistics we used. However, for the Opop1 item there was an agreement between all three fit statistics that the item was misfitting the Rasch model. Other items indicated some overfit which could be an indication that these items were redundant, but we did not feel at this stage that we needed to remove any of the items in order to reduce the length of the instrument. The Opop1 item was removed and the model was re-fit on the data. The rest of the items did not show extreme misfit. For the Grade 10 group, the results of the Rasch model showed that the question Opop1 was, as for Grade 9, a badly fitting item. It was subsequently removed from the data and the model was estimated again. Two other questions, the Penl and the Vep1s3 were also candidates for removal because the Outfit MNSQR and the chi-square tests indicated misfit. The Infit MNSQR for the two questions was within the acceptable limits. Vep1s3 and Penl were removed and the model was re-estimated. The remaining questions had a satisfactory model-data fit. For the Grade 11 – basic mathematics group, when the question fit statistics were inspected, Penl was the most misfitting question. It was removed, the model was re-estimated and the item fit statistics were investigated. The rest of the items had satisfactory statistics. For the Grade 11 – advanced mathematics group, the Opme2 question was the most misfitting and was removed from the analysis. Then Opop2, Opop1, Opme3 were iteratively removed from the analysis because of extremely high fit statistics.

Overall, the single sub-set of items which seemed to behave in the same way across the four groups and the statistics about the test items fit for the whole sample are presented in Table 1. The sub-set of items comprise of 20 tasks, corresponding to the

four types of geometrical figure understanding: OPme1,OPop3, OPpw1, OPpw2, OPpw3, Per1, PEr2, Per3, SEc1, SEc2, SEc3, SEd1, SEd2, Vedel, VEde2, VEp2s3, VEpi1, VEpi2, VEpo1 and VEpo2. As we can observe, the fit statistics for the test items are satisfactory for the purpose of this study, since most of the Infit mean squares and the Outfit mean squares are approximately 1. The difficulty of the test items can be considered invariant for the four groups of students. Thus, we were able to create a scale with sample – free item difficulties.

	Chisq	Df	p-value	Outfit MSQ	Infit MSQ	Outfit t	Infit t
OPme1	822.291	878	0.910	0.935	0.948	-2.23	-2.74
OPop3	911.756	878	0.209	1.037	0.957	0.52	-0.92
OPpw1	853.957	878	0.713	0.972	0.965	-0.70	-1.42
OPpw2	764.995	878	0.997	0.870	0.895	-4.49	-5.46
OPpw3	702.740	878	1.000	0.799	0.914	-2.88	-1.61
bcPer3	770.445	878	0.996	0.877	0.970	-2.34	-3.07
PEr2	861.129	878	0.651	0.980	0.902	-0.83	-1.72
Per1	801.287	878	0.969	0.912	0.904	-2.42	-2.86
SEc1	739.693	878	1.000	0.842	0.895	-2.97	-2.85
SEc2	634.958	878	1.000	0.722	0.900	-2.52	-1.75
SEc3	905.890	878	0.250	1.031	0.942	0.35	-0.98
SEd1	587.956	878	1.000	0.669	0.850	-4.32	-3.22
SEd2	503.063	878	1.000	0.572	0.869	-3.38	-1.58
VEde2	884.272	878	0.434	0.827	1.011	0.25	0.64
Vedel	726.694	878	1.000	1.006	0.944	-1.25	-1.24
VEp2s3	910.285	878	0.219	1.036	1.041	0.64	-5.68
VEpi1	873.42	878	0.537	0.994	0.944	-0.08	-1.61
VEpi2	555.719	878	1.000	0.632	0.813	-6.05	-7.14
VEpo1	48.564	878	0.999	0.852	0.896	-5.50	-5.02
VEpo2	658.031	878	1.000	0.749	0.835	-6.85	1.23

Table 1: Items fit Statistics for the whole sample

Figure 1 illustrates the scale for the remaining items of the test with item difficulties and the whole group of students calibrated on the same scale. Three levels of geometrical thinking can be identified. In specific, students of Level I (below -1 logits) are able to recognize subfigures in a geometrical figure (Per1, Per3) and to

perform modifications concerning its size (Opop3), position (OPpw1) and orientation (OPpw3). In Level II (between -1 and 1 logits) students are able to handle a reconfiguration of a geometrical figure (OPme1) and to achieve a combination of a position and orientation modification on a figure (OPpw2). Furthermore, students of this level are solving proof tasks, whose solutions are either based on definitions (VEde2) or on procedures of proof (VEpo1, VEpo2, who are accompanied by a figure as an “object”). In Level III (above 1 logits) students are able to construct a figure (SEc1, SEc2, SEc3) and describe its construction (SEd1, SEd2). Students of this Level seem to have reached a higher level of proof understanding, since they are able to discriminate a formal proof from an empirical and a semi – empirical proof (VEp2s3) and to give a proof even in the case that the geometrical figure accompanying the task is not supporting this, as it has the nature of an “illustration” (VEpi1, VEpi2). Focussing on the persons’ distribution, we observe that the biggest proportion of our sample is situated in Level II, whereas the rest of the sample seems to be almost equally distributed to Level I and Level III respectively.

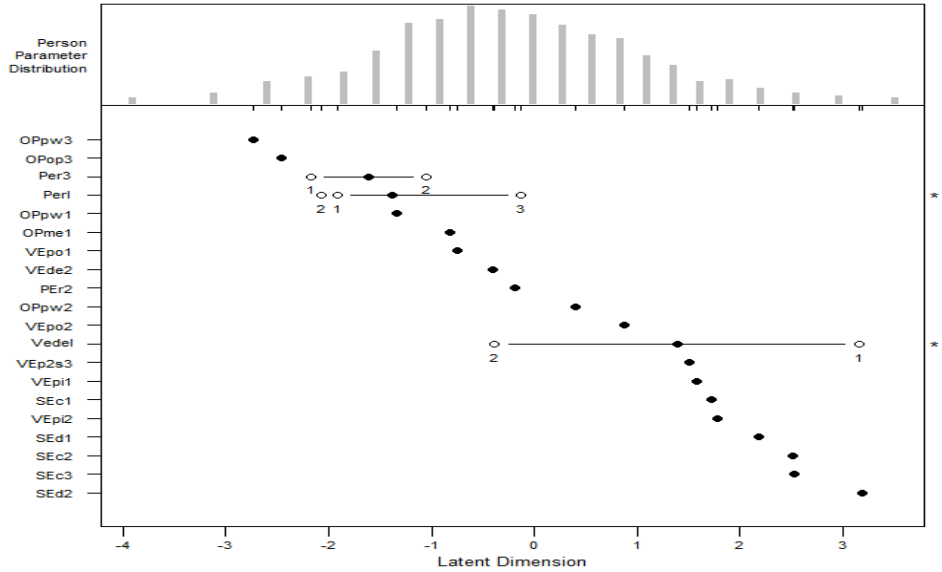


Figure 1: Item – ability map

Conclusions

The purpose of this study was the development of a valid geometrical figure understanding measurement tool for middle and high school students and the identification of possible levels of geometrical thinking. The Rasch model allowed the investigation of the validity of the geometrical figure understanding test and the creation of a good interval level measure for the middle and high school students’

geometrical figure understanding. The findings showed that the Rasch analysis supports the conceptual design of the test, since a scale of items measuring students' geometrical figure understanding is indicated. The items psychometrically behaviour is invariant in the four age groups examined. The hierarchy of the items is reasonable. However, there are no similar previous studies in order to test the agreement with them.

The findings also suggested three levels of middle and high school students' geometrical understanding, which can be related to the three geometrical paradigms as proposed by Houdement and Kuzniak (2003). According to the results, Level I may corresponds to Geometry I (Natural Geometry), in which intuition is often linked to immediate perception and enriched by experiment. Students in Level II have characteristics of Geometry II (Natural Axiomatic Geometry), in which a system of axioms is necessary but the axioms are as close as possible to the intuition of the space around us. Level III can be related to Geometry III (Formalist Axiomatic Geometry), in which axioms are not any more based on the sensitive and can be without any relation to reality. The results also showed that most of the students are located in the Level II, while the rest of them are almost equally separated into the other two levels.

It is important to mention that for students of Level I the geometrical figure is mainly an object of study and of validation, while for Level II students the geometrical figure is supportive for reasoning (Houdement & Kuzniak, 2003) and operates as a “figural concept” (Fischbein, 1993). Proving ability of these students seems to be based on the geometrical figure, since the items concerning proof tasks including a geometrical figure having the nature of an object are situated in this level. Houdement and Kuzniak (2003) suggest that the axiom system can be uncompleted in this Level, but the demonstrations inside the system are necessary requested for progress and for reaching certainty. For students in Level III, the geometrical figure functions as a heuristic tool and as a schema of a theoretical “object” (Houdement & Kuzniak, 2003). Therefore, these students appear to be able to overcome the negative influence of a geometrical figure having a nature of an “illustration” in proof tasks.

The construction of the geometrical figure understanding test and its Rasch scale can function as a tool for teachers in order to examine their students' geometrical understanding and identify their learning needs in relation to the three levels of geometrical understanding proposed in this study. Thus teachers may be helped in order to organise their teaching in a way that will fulfil their students' needs and correspond to their abilities. If students have the proper guidance and instruction, they might be able to move more easily into a higher level of geometrical thinking and understanding. However, we shall keep in mind that even though students are located in the same level, it is not ensured that their abilities are the same. Consequently, the distribution of the students into different levels of geometrical understanding should not be approached superficially. Nevertheless, further research about students' geometrical figure understanding is needed in order to validate the

geometrical figure understanding test. An improvement of this test may also be needed in order to achieve a better targeting of the items concerning the geometrical figure understanding. Longitudinal studies can also be performed in order to examine the invariance of the geometrical figure understanding structure and the transition of students from one level to a higher one.

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SOUTH AFRICAN TEACHERS' COMMON CONTENT KNOWLEDGE OF GRADIENT

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This study looked at the common content knowledge (Ball, Thames, and Phelps, 2008) of South African secondary mathematics teachers regarding the concept of gradient (slope). Results are reported from nine free response items on a paper and pencil test administered to 251 practicing teachers pursuing qualifications to teach Grades 10-12 mathematics. Findings suggest that teachers' understanding of gradient varies greatly and that many of the teachers lack even a basic understanding of this important concept.

LITERATURE REVIEW

Research provides evidence of students' weak covariational reasoning, including difficulties with the concepts of slope (or gradient as the concept is referred to in South Africa) and rate of change (Barr, 1981; Carlson, Jacobs, Coe, Larson, & Hsu, 2002; Orton, 1984; Stump, 2001; Teuscher & Reys, 2010). There has been particular evidence of students' inability to make connections between various representations of these concepts. Stump (2001) found that students held varying views of gradient as an angle, a formula, rise over run, or steepness, and that students did not make connections between rate of change and gradient.

Some studies (Stump, 1999; Moore-Russo, Conner, & Rugg, 2011) have helped identify and analyze the many different ways that the concept of gradient can be conceptualized, specifically considering how teachers in the United States understand this key concept. Stump (1999) found that teachers expressed concern with their students' understanding of gradient; however, they focused on student difficulties in the procedures for determining gradient rather than conceptual notions of the concept. This prompted Stump's (1999, p. 142) suggestion that, "...both pre-service and in-service mathematics teachers need opportunities to examine the concept of slope ... [and] to construct connections among its various representations." While more work should still be done to address teachers' understanding of gradient in the U.S., the mathematics education research community should also look to better comprehend teachers' understanding of gradient in countries besides the United States.

THEORETICAL FRAMEWORK

The Mathematical Knowledge for Teaching framework recently introduced by Ball, Thames, and Phelps (2008), a refinement of Shulman's (1986) work, served as the underpinning theory for the study reported here. While Shulman suggested three categories of teacher knowledge (content knowledge, pedagogical content

knowledge, and curricular knowledge), Ball, Thames, and Phelps (2008) divide Mathematical Knowledge for Teaching into 1) subject matter knowledge (SMK) and 2) pedagogical content knowledge (PCK).

Shulman's (1986) seminal work in PCK has been a rich, persistent theory in educational research. In contrast, SMK is what Ball, Thames, and Phelps (2008) label the "relatively uncharted arena of mathematical knowledge necessary for teaching that is not intertwined with the knowledge of pedagogy, students, curriculum, or other non-content domains" (2008, p. 402). They highlight the fundamental components of SMK by further dividing it into common content knowledge (CCK), specialized content knowledge, and horizon content knowledge.

CCK is outlined as the knowledge and skills held by educated adults that are essential and used in a wide variety of settings. Specialized content knowledge represents the deep, flexible, nuanced understanding of mathematics that is uniquely related to teaching including knowing how concepts are represented, related, developed, and validated. It allows teachers to "mediate students' ideas, make choices about representations of content, modify curriculum materials, and the like" (Ball and Bass 2000, p. 97). Horizon content knowledge relates to understanding how mathematical knowledge is related and connected to more advanced mathematical concepts.

CCK of a concept is necessary before either specialized or horizon content knowledge can be developed. The research reported in this study looks at the common content knowledge of South African secondary school mathematics teachers regarding the concept of gradient. The study was driven by the following research question. How well do practicing teachers pursuing qualifications to teach mathematics in Grades 10-12, especially those who teach previously disadvantaged student populations of South Africa, understand the concept of gradient?

METHODOLOGY

Setting

Recent curricular changes have been made in mathematics in South Africa (South Africa Department of Education, 2003). This major curriculum change was "driven by a need for social, economic, and political transformation" (Parker, 2006, p. 59). Even prior to such changes, de Villiers (1997) called for major revisions to South African teacher education programs since even "qualified" secondary mathematics teachers hardly knew more geometry, the content band under which gradient falls in the South African curriculum, than their students.

In South Africa, education is compulsory for all youth through Grade 9. Grades 10 to 12 are optional, and the completion of Grade 12 brings numerous economic opportunities. The adoption of new curricula in mathematics brought with it changes in the qualifications needed to teach Grades 10-12 mathematics. These changes in regulations, along with the closing of all Apartheid-created teacher colleges and the transfer of responsibility for teacher education to the universities, unintentionally has

made it difficult for teachers of previously disadvantaged populations to earn the qualifications needed to teach Grades 10-12 mathematics. Many of the teachers in these areas that serve disadvantaged populations have had little or no opportunities to develop either their subject matter or pedagogical content knowledge. These changes have had less impact on traditionally white, often private, schools where mathematics teachers often attended universities in the past and have had various opportunities to attend professional development courses sponsored by the Department of Education. This resulting shortage of qualified mathematics teachers, in public and particularly in rural schools, has been labeled “critical” by Adler and Davis (2006).

In order to help address this situation, an Advanced Certificate of Education (ACE) was created. The ACE program provides an alternate means for practicing teachers, especially those in high needs areas, to obtain minimal qualifications to teach mathematics in Grades 10-12. This intervention was specifically created in order to help prepare under- and unqualified teachers currently teaching mathematics as well as teachers in other subject areas meet the minimal qualifications for teaching mathematics. The ACE programs offer flexible delivery at multiple learning centers across a single South African province to reach some of the most remote, rural, and disadvantaged populations. Enrollment in the ACE program at the University of KwaZulu-Natal has no cost implications for teachers since it receives funding from the regional Department of Education with additional funding provided by the United States Agency for International Development through the non-governmental organization Higher Education for Development.

Participants and Data Collection

Data was collected from 251 practicing teachers from a single region of South Africa during the summer of 2010. The data come from teachers’ responses to a paper and pencil pre-test. The pre-test was administered on the first day of a Geometry module; the module in which the concept of gradient is addressed. The module is part of the coursework that leads to the ACE previously described.

All of the 251 teachers in the module would have been classified as Black, Indian, or Coloured under the Apartheid policy. Some of these respondents were not teaching mathematics and were using this program as a means to retrain into becoming mathematics teachers. Those who were teaching mathematics were in schools whose populations are considered disadvantaged and whose students would not have been classified “White” during Apartheid. Some were teaching Grades 10-12 mathematics without the appropriate qualifications; others were teaching mathematics at Grade 9 and below often at schools not offering any mathematics above Grade 9.

Nine of the 27 items on the instrument were free response and addressed gradient; only these items were analysed for this study. The teachers were instructed to show all work and provide an explanation at how they arrived at each answer. The first item asked teachers to write an equation that modeled a given real-life linear situation (the total cost of a text-only advertisement in the newspaper if it costs R10.00 to run

an advertisement plus an additional R1.00 for each word in the advertisement). The second and third items used the same situation and asked the teachers to create a graph for the situation and to determine the gradient respectively. The fourth, fifth, sixth, and seventh items displayed graphs of an increasing line with a negative y -intercept, a decreasing line with the origin as its y -intercept, a horizontal line with a positive y -intercept, and a concave up parabola with a positive y -intercept respectively. All were shown on separate xy -axes without any demarcations except for the two axes' labels. For each graph the following question was posed, "Which could possibly have a gradient of two?" The eighth item showed three distinct linear graphs on a single grid: $y = x - 3$, $y = (1/3)x$, and $y = -3x$. The three were shown without their equations on xy -axes complete with a labeled integer grid that could be used to determine the coordinates of points. It also posed the question, "Which of the labeled graphs could possibly have a gradient of two?" The ninth item showed the graph of an increasing line with the origin as its y -intercept. The graph was on xy -axes without any demarcations except the labels on the axes and 30° marking the measure of the angle between the positive x -axis and the section of the line with a positive domain.

Data Analysis

The unit of analysis was a response to a single pretest item. Since 251 teachers answered nine items, there were exactly 2259 responses (304 of which were blank) that were analyzed. Data analysis began with creating task-specific rubrics for each item. The rubrics were all based on the following 0 to 2 general scale. For each item, rubrics were to be applied not only to the final answer but were to take into account the entire response including all inscriptions (e.g., teacher-produced writing, equations, tables, and drawings) related to the item as is shown in Table 1.

Score	Teacher's response showed...
0	No evidence of understanding (no response, any response with incorrect reasoning)
1	Some evidence of understanding (any response that was supported with partially correct reasoning or reasoning that was correct yet incomplete for item addressed)
2	Strong evidence of understanding (any response that was supported with correct, complete reasoning)

Table 1: General scoring rubric for teacher responses to all items.

Both members of the research team analyzed all teacher responses to each item independently. Proportion agreement for each of the nine items was above 0.97. The Cohen's kappa for each item was well above 0.80, which is considered "Almost Perfect" agreement (Landis & Koch, 1977). The two independent scorers reached consensus by means of discussion for each response they had coded differently. After the items were analyzed, the research team revisited the teachers' responses together.

FINDINGS

Teachers' scores ranged from the minimum possible score, zero, to the maximum possible score, eighteen, on the nine items. As shown in Figure 1, there was a wide distribution of scores with a mean score per teacher of 9.66 (SD 5.16). Teachers' responses provided evidence that they held misconceptions. The most common misconception found in 29 responses was that a line that goes through the origin must have a gradient of zero. The second most common error occurred in 26 responses; it was found exclusively on the eighth item where teachers ignored the units marked on the graph and assigned a gradient of two to an increasing linear function whose gradient was one. A related error occurred in 22 responses to various problems when teachers created markings with specific values on axes, when none were given, leading them to various incorrect assumptions. A misconception that occurred in 20 responses was that horizontal lines have no gradient. In 19 responses nonconventional labeling of units was used on the axes such that the axes were made to cross at a point other than (0, 0). Fifteen teachers' responses confused the gradient with the y -intercept in a linear equation. All other errors occurred in fewer than 15 responses. None of the teachers used the mnemonic phrase "rise over run" that has been suggested as contributing to an instrumental understanding of gradient (Walter & Gerson, 2007) in any of their responses. This is most likely due to the fact that this phrase is generally not used in South African schools.

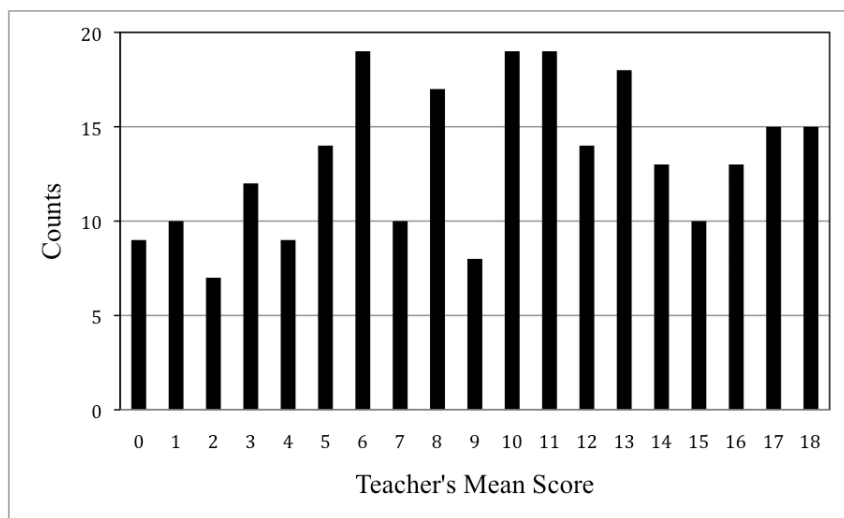


Figure 1: Histogram showing distribution of teachers' mean scores

Table 2 displays the counts of scores that were assigned to the teachers' responses to the nine gradient items. The mean score for all items was 1.07. The item with the

highest mean (1.26) was the first item; the item with the lowest mean (0.73) was the second item. Since both items used the same verbal situation, it suggests that moving from a situation to an equation was easier for the teachers than moving from a situation to a graph. In contrast to other’s findings (Stump, 1999; Moore-Russo, Conner, & Rugg, 2011) with U.S. teachers, these teachers’ responses to item nine suggest that their trigonometric conceptualizations of gradient were not less developed than their other conceptualizations of the concept. Of the 11 conceptualizations of gradient offered by Moore-Russo and colleagues, the three most evidenced in the teachers’ responses were 1) the parametric coefficient conceptualization (in responses from 153 different teachers), 2) the behavior indicator conceptualization (in responses from 135 different teachers), and 3) the trigonometric conceptualization (in responses from 113 different teachers).

Item	Score Counts		
	Score of 0	Score of 1	Score of 2
1	75	35	141
2	132	56	63
3	105	61	85
4	68	59	124
5	90	36	125
6	58	29	164
7	101	36	114
8	114	64	73
9	75	90	86

Table 2: Counts for scores assigned to teachers’ responses to items.

CONCLUSIONS AND IMPLICATIONS

Gradient is a topic in the secondary mathematics curriculum of most countries. Paradoxically, this important concept is “well known but not well understood” (Moore-Russo, Conner, & Rugg, 2011, p. 3). This study adds to the research literature by providing additional insight into common errors that occur in gradient and linear function problems that require 1) moving from one representation to another and 2) working with graphs on labeled and unlabeled Cartesian coordinate systems. Findings from this study suggest that those teachers participating in the study varied greatly in their understanding of the gradient. In particular, the current study sheds light on the dire situation that faces those in some of the underserved areas of South Africa where secondary mathematics teachers are unable to respond correctly to even basic questions regarding gradient.

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LEARNING CREATIVELY WITH GIANT TRIANGLES

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In a mathematics methods course for pre-service teachers, carefully designed activities made deep and interconnected mathematics quickly available requiring no pre-requisite content knowledge. Using brightly colored 1-meter edge length equilateral triangles that can be quickly assembled and reassembled into a wide range of polyhedra by small groups of students, high levels of student engagement and collaboration were achieved. The van Hiele (1976) Model of Geometric Thought and NCTM (2000) process standards were explicitly referenced in the course.

INTRODUCTION

The challenge of developing deep content knowledge and pedagogical content knowledge (e.g., Shulman, 1986; Ball & Bass, 2000) for both in-service teachers and pre-service teacher candidates has been expressed for some time. We have achieved this, to new levels, with a unique inquiry experience to develop visualization and deep, analytical content knowledge as well as pedagogical knowledge within an urban pre-teacher mathematics methods course. The triangles that facilitated this are equilateral, 1-meter edge length, resilient, lightweight and brightly colored, and can be quickly assembled and reassembled into a wide range of polyhedra.

THEORETICAL FRAMEWORKS

Mathematics embodied in the manipulatives

The hidden mathematics framework suggested by Abramovich and Brouwer (2006) entails finding, creating and working with mathematical problems that connect across the mathematics curriculum. This helps prospective teachers make relevant connections between their undergraduate mathematics courses and the K-12 school curriculum. Their research entails integration of rigorous mathematics activity with technology-assisted learning in expert-novice, socially mediated classroom settings (Vygotsky, 1986). We claim that a similar learning environment can be fostered through carefully applied manipulatives such as the giant triangles. Furthermore, the triangles relate physically to learners through their kinesthetic character and aesthetic appeal. The giant triangles, as constructed, have powerful inbuilt mathematics that has to emerge when learners interact with them using very carefully designed activities. The learners' use of the materials changes as their knowledge develops observably shifting from concrete to abstract through interaction with the materials (M. L. Connell, personal communication, December 6, 2010).

van Hiele Model of Geometric Thought

The van Hiele Model of Geometric Thought (van Hiele, 1976), in differentiating increasingly complex levels of geometric thinking, is a useful framework to describe

the evolving shape-building activities in this study. Additionally, its understanding was an explicit pedagogical goal for the pre-service teachers in the course. Reflection on the activities contributed to learners' pedagogical appreciation of the model.

Our activities are introduced at the Visual Level (What do you notice?) but quickly move learners to the Descriptive/Analytical Level (What properties can you specify? For example, how many vertices, edges and faces does this particular figure have?) and Relational Level (Which figure has this many faces? Or, How is this structure like that structure?), in which learners begin to abstract and generalize. No prior knowledge or experience is needed to engage in these activities, which reveal and develop deep mathematics without interference from learners' prior mis- or pre-conceptions. Learners collaborate and cooperate spontaneously in building their figures, facilitated by the size and construction of the triangles.

METHOD AND CONTEXT

University and course context

The university, one of the most ethnically diverse liberal arts institutions in the mid-south-western United States, is a federally designated Minority Serving Institution. It provides 4-year degree programs and has an open enrolment policy. A large percentage of its undergraduate students are the first college attendees in their families, and work full-time while attending college. The teacher certification/degree program requires students to take at least two mathematics content courses for teaching prior to their mathematics methods course, which is typically offered during their third year of study. Many students take these content courses at collaborating community colleges and then transfer to the university to complete their bachelors' degrees. The content courses tend to be factual and non-exploratory in structure.

The pre-service teacher methods courses are limited to no more than 30 participants and meet face-to-face for 2.5 hours per week over 14-15 weeks. While the curriculum for the methods courses spans all mathematics content strands, the geometry strand focuses on understanding the van Hiele model through interactive experience, integration of the National Council of Teachers of Mathematics (NCTM, 2000) process standards, and connections among the mathematics content strands.

Research method

The study is guided by the following research questions:

- 1) How do the triangle activities impact learners?
- 2) How do learners relate the triangle experiences to the van Hiele model?

Data consist of instructor field notes, students' online discussion comments, and photographs. A narrative approach is utilized to illuminate the findings.

LEARNING TRAJECTORY

We share a learning trajectory that was successfully enacted in ten mathematics methods classes for elementary and for middle-grades pre-service teachers in single 160-minute class sessions during two successive semesters.

Deepening the mathematics through pyramid construction

The trajectory began with a simple visual level activity: How many triangles can one fit around a given point laying flat on the floor? Learners predict and then build the figure. What happens if successive triangles are removed and the newly exposed sides are connected (see Figure 1)?



Figure 1. Triangles around a point and the three folded pyramids

This activity integrated a review of naming conventions for polygons and properties of the particular polygons and pyramids that emerged from this construction. Learners considered which pyramids might have the greatest and least volumes given that their base areas increased while their altitudes decreased with base side number (from 3 to 5). From a pedagogical perspective this problem was pointed out to be an alternate learning trajectory that could be enacted at this point in the lesson.

From pyramids to Platonic solids – Visual to descriptive level transition

Learners' next task was to attach additional triangles to the pyramids to form figures with the same number of regular triangular faces at each vertex. Figure 2a shows the three regular polyhedra that can be constructed using equilateral triangles. Initially, group members paid close attention to the number of faces at particular vertices but did not notice incongruence occurring at other vertices as they added triangles to the growing figure. Figure 2b shows some typical intermediate figures that learners constructed as they worked toward building the regular icosahedron. As the incongruent vertices were pointed out, learners began to attend more closely to properties of the figure, i.e., to the number of faces at every single vertex, rather than to its global appearance. In this way, the manipulative itself carried the mathematical development initially at the visual level toward the descriptive level.

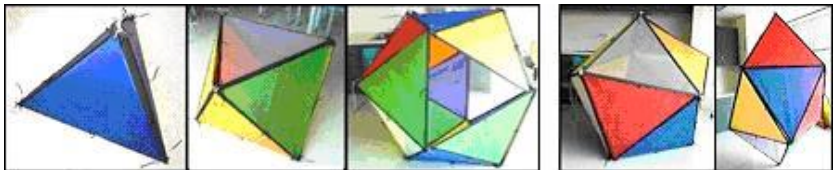


Figure 2a. Three regular polyhedra

Figure 2b. Intermediate figures

Learners shared their insights facilitated by key questions dealing with comparing and contrasting the three regular polyhedra. These included visualizing the figures from different perspectives, such as when lying on a face versus standing upright on a vertex (Figure 3). The light weight of the manipulative allows one to lift such very large figures into the air so that one can actually stand inside them. A triangle was deliberately removed from the icosahedron (Figure 2a) so that each person could experience being inside it. This is a unique mathematical and aesthetic experience that can rarely be made available with other manipulatives.



Figure 3. Lying on a face versus standing upright on a vertex

Symmetry and enumeration – Descriptive to relational level transition

Learners used the symmetries in the upright octahedron and icosahedron to justify their enumeration of faces. Enumeration of vertices and edges also evolved from symmetry considerations.

Generalization using other regular polygons – Relational level development

To further develop the entire visualization and enumeration process, the instructor asked the class to consider what would happen if only 2 triangles were connected. Then, what can be constructed using other regular polygons, such as squares, pentagons, hexagons, heptagons, etc. Only two new regular polyhedra, the cube and the dodecahedron, could be constructed.

Number patterns embodied in the Platonic solids

Using Polydrons™, individual learners constructed the cube and dodecahedron and then the whole class worked to complete the enumeration table as shown in Table 1. Symmetry again played an important role enumerating faces, vertices and edges for the new figures. Field notes from one session indicated that three particular patterns were discovered:

- 1) As listed in Table 1, as the number of faces increases, the number of vertices and edges also tend to increase;
- 2) the numbers are all even;
- 3) the numbers of vertices and faces switch for the cube and octahedron, and for the dodecahedron and icosahedron, while their edge numbers are the same; and,
- 4) the tetrahedron does not have a switch partner since it has the same number of faces as vertices.

These relationships were typical of those emerging across all classes. The concept of duality was revisited geometrically later in the lesson.

Name	Vertices	Edges	Faces
Tetrahedron	4	6	4
Hexahedron (Cube)	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Table 1. Enumeration of vertices, edges and faces

Doubling the tetrahedron – Descriptive and relational levels

The next challenge involved building a tetrahedron with doubled edge lengths. They noticed that a useful net for building a smaller, unit-sized tetrahedron was a larger, doubled edge length triangle consisting of 4 unit triangles. Therefore they would need four of these for the larger tetrahedron, 16 unit triangles in all. Scaling the lengths by a factor of 2 implies a scale factor of 2^2 for a “doubled” figure’s surface area. For this figure, the net itself provided an opportunity to examine the sum-of-odd-numbers series shown in Figure 4.

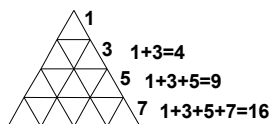


Figure 4. Doubling the edge lengths of the basic tetrahedron

Next, they were asked to predict how many unit-size tetrahedra would fill the larger tetrahedron. Prediction activities have been documented to enhance mental visualization (e.g., Battista, 1999). Learners predicted 4, 5, 6, or 7 unit-size tetrahedra would be needed to solve this problem, the majority choice being 5. The durability and construction of the triangles provides opportunities to examine these types of figure in novel ways, such as from the inside, that most other manipulatives cannot provide. Learners initially inserted four smaller tetrahedra into the spaces by the vertices of the larger figure. Then, they attempted to fill the center region with additional tetrahedra. Eventually someone realized that tetrahedra cannot tessellate this space. Some classes built the figure that fills the space by inserting loose triangles and connecting them together in place. Others recognized that the triangular base on the floor was rotated 180 degrees relative to the triangular base at the figure’s midlevel, a property of the octahedron. The octahedral skeleton is visually evident in the photograph in Figure 4. At this point, the instructor referred back to the scale factor concept, that doubling the edge lengths should result in an 8-fold scaling of volume. Thus, the larger tetrahedron consisted of 4 smaller tetrahedra and 1 octahedron, which itself occupies a volume equivalent to 4 smaller tetrahedra. Referring back to the earlier problem of comparing pyramid volumes, the octahedron’s volume is equivalent to the combined volumes of 2 square pyramids

(see Figure 3). Therefore, the square pyramid's volume is equivalent to the combined volumes of 2 smaller tetrahedra. A different learning trajectory can be enacted to confirm this finding. Interconnections across the mathematics curriculum were constantly becoming apparent as this trajectory progressed.

Platonic solids duality exposed – Relational level

The final activity involved attaching small tetrahedra to the central triangles on the doubled-edge length tetrahedron's faces. When stood upright on one of its vertices this figure looked like two intersecting large tetrahedra, one pointing upward and one pointing downward. Learners used masking tape to connect the vertices of this stellated figure (see Figure 5). To their surprise, the tape formed the edges of a cube.



Figure 5. Stellated octahedron

Knowing that the interior of this figure was the octahedron (from the previous filling activity), the class was able to see that the centers of each face of the cube were the vertices of the interior octahedron. Referring back to Table 1, the switching of the vertices and faces of the cube-octahedron duals now made sense.

The class then viewed a very short animated video clip on the stellated octahedron that exemplifies the duality properties of the cube and octahedron.

Learner reflections

A selection of unedited learner quotations, from online reflection discussions, shows a high level of learner engagement, collaboration, deeper thinking and understanding:

I really enjoyed that the class worked together to figure the lesson out. It was not a teacher lecturing and the students un-engaged.

It had the whole class involved. It seemed to get more input from individuals whereas if we were all sitting down just a few people would have responded.

I got to walk inside the icosahedron! I think children and young adults would love this experience because it is basically playing while learning.

The value of this activity is not only that it is hands-on but that it also reaches beyond the surface of just looking and playing, the class could explore in depth.

I actually got the concept we were learning from the big triangles. I'm not sure if it is because it was more hands on or because I was working with my peers, and they helped me understand.

On the day this lesson took place I had no idea how many aspects of education it was going to cover. I was also very surprised on how large this project was, this was not a bad thing it played an important role in what was being taught and how. Not only was this project informative on how to teach our future students but it was full of application of how to teach in large groups. So looking back and taking in what this lesson was about it was about math, but it was about so much more.

... it shows how if you have a flat shape and you take away a triangle it can turn it in to a polyhedron. It was amazing just by manipulating a few triangles could completely change the properties of a shape.

This project also can help create a learning community because this cannot be done without team work. All of the students have to pitch in or else the lesson will not reach its full potential.

Many students mentioned that the lesson was fun. This suggests that the hands on play aspect of the giant triangles and collaboration they engender have an affective advantage associated with successful learning in carefully planned activities.

CONCLUSIONS

Developing deep content knowledge and pedagogical content knowledge for pre-service teacher preparation can be challenging. Several important points about how we achieved this development to new levels with this trajectory include:

1) Substantial, deep, and interconnected mathematics, as described by Abramovich and Brouwer (2006) is made available quickly and effectively using the triangles. These activities reinforce the NCTM (2000) process standards of communication, problem solving, connections, reasoning and justification, and representation while interconnecting with other content strands (measurement and algebraic thinking).

2) No entry-level content knowledge is required and transfer from prior content courses has generally not been observed. In attempting to bring highly interactive and interconnected mathematical experiences to our methods classes, a big challenge is to overcome learners' attitudes about this intense form of teaching since most learned to "do math" in very traditional "copy the model and practice" ways. Thus, weakly conceptualized mathematical knowledge may intrude on the learner's openness toward deeper mathematical understanding and pedagogy. However, the triangles immediately engage learners, who remain open-minded to the mathematics and to the methods as new activities are introduced.

3) High levels of student engagement and collaboration are achieved associated with hands on play and figuring out activities, in a positive affective social context. For example learners did not give up or express frustration or discouragement from making 'intermediate figures' that needed correcting as in Figure 2b. Rather, they enjoyed the teamwork required to complete the activity. This shows how the use of the triangles facilitates social mediation of mathematical thinking by requiring negotiation of co-operative actions though visualization. Such negotiation promotes communication at deeper levels of geometric thought, even though the learning objectives were relatively simple (e.g., Add more triangles to the figure so that each vertex contains the same number of triangular faces.)

4) Use of these manipulatives may avoid some of the affective pitfalls that occur when introducing challenging mathematical problems. Researchers have noted the importance of 'the struggle' and often-associated 'perplexity' when extending one's

mathematical knowledge through difficult problems (Hiebert & Grouws 2007). With this trajectory, we believe that the aesthetic and size appeal of the triangles, and the personal commitment to the collaborative constructions, enabled learners to persist, working in groups, until achieving success. At no point did they give up. Even during the enumeration activity, relying on mental imagery, all learners participated with enthusiasm transferring the knowledge just gained through the hands-on experience.

Future work

These activities will be placed earlier in the course to help open learners' minds to deep mathematics and conceptual learning methods. Future learners will write additional reflections on how the different van Hiele levels were addressed across all course-based geometry activities. Also, work has begun with a relatively new teacher in a struggling urban middle school. Her 6th graders have mainly had skill-driven mathematics experiences aimed to raise test scores. The triangles have brought them enthusiasm through an initial activity, to 'build something interesting and beautiful'. We will continue to develop their mathematical knowledge and dispositions toward doing mathematics, while co-teaching with their teacher to develop her pedagogy.

Acknowledgement

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ENGINEERING STUDENTS' VISUALIZATION AND REASONING PROCESSES WHILE INTERACTING WITH A 3D DIGITAL ENVIRONMENT

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In this paper we report on a research conducted to study engineering students' visualization processes as they interacted with a digital environment designed to support mathematically driven explorations in 3d space. We particularly focused on identifying connections between the visualization processes in which the students engaged, the use of the digital environment's tools and functionalities and the unfolding of their mathematical reasoning activity about concepts related to 3d space visualization and orientation. Since the development of spatial and visualization abilities is considered to be fundamental for engineering students' education, we looked for instances in which students' visualization, construction and mathematical reasoning processes may have contributed to the enhancement of those abilities.

SPATIAL AND VISUALIZATION ABILITIES IN MATHEMATICS AND ENGINEERING EDUCATION

Spatial and visualization abilities have long been considered as being connected to mathematical learning and aspects of it, such as geometrical thinking and problem solving (Presmeg & Balderas-Cañas, 2002). Apart from basic mathematics education, however, the development of spatial and visualization abilities have also been recognised as fundamental for engineering students. Several tertiary institutions urge engineering freshmen to take Spatial Ability and Mental Rotation Tests, so as to define if they possess these abilities and offer courses to accentuate them. Based on a perception of visualization as the process of creating, retrieving from memory and manipulating mental images, with the minimal interference of external representations (including technology), most of these tests are designed to determine the level of accuracy and speed in which engineering students may manipulate internal representations of complex 3d objects (Sorby & Baartmans, 2000).

Under this perspective, the use of computer environments has mainly been viewed as an alternative to paper and pencil means for students' spatial instruction in creating engineering drawings and has been restricted to the use of professional software, such as CAD packages. In this kind of software, however, procedures that seem to be connected with visualization in 3d space (e.g. changing an object's scale or rotating it around an axis) are just a matter of pressing the correct virtual button. As the mathematics behind such buttons become invisible and crystallized in black boxes inside these sophisticated 3d professional environments (Kent & Noss, 2001; Straesser, 2000), visualization processes in 3d space result in being more and more

obscure for engineering students.

In mathematics education however, visualization has been perceived as the ability to represent, transform, generate, communicate, document, and reflect on visual information (Hershkowitz, 1989, p. 75), processes not independent of the use of external representations (for a discussion see Gutierrez, 1996). Environments that hold the potential for the manipulation of dynamic images (dynamic visualization), are thought to contribute to the development of the students' spatial and visualization abilities (Christou et al., 2007), as these often appear to be interwoven with the software's semantics and functionalities (Kynigos & Latsi, 2007). These kind of environments are considered not only to empower students to work with a 3-d frame of reference, building and observing 3d structures in it, but also to transform those constructions in real time (Arcavi & Hadas, 2000), through dynamic manipulation.

Bringing the mathematics educators' experience in developing digital environments meant for mathematically driven explorations in 3d space (e.g. MaLT), we developed in the context of engineering education the "3d Modelling & Cutting" microworld (Fig. 1). In this paper we report on a research conducted to study engineering students' visualization processes as they represented and manipulated 3d mechanical engineering components in the "3d Modelling & Cutting" microworld's virtual space and generated Logo programs for simulating their shaping and cutting procedures. In analysing the students' discourse, we specifically tried to identify links between the visualization processes in which they engaged, the use of the digital environment's tools and functionalities and the course of their mathematical reasoning, as they attempted to provide explanations for their actions. Taking into account Duval's model for geometrical reasoning (1998) that recognises connections between a) visualization processes, b) construction processes (using tools) and c) reasoning processes, we attempted to define how the students' interactions with this digital environment may have contributed to the enhancement of their conceptualisations of 3d space (affecting, thus, their spatial and visualization abilities) as well as to the advancement of their mathematical reasoning.

THE DIGITAL ENVIRONMENT

The MachineLab Turtleworlds (MaLT) environment

MaLT is a Logo programmable environment that allows the creation, exploration and dynamic manipulation of 3d geometrical objects. The objects visualized inside MaLT are either *constructed* in a wire-frame form when navigating a 3d Logo Turtle, or *inserted* in a 3d solid form, when selecting them from a library that contains numerous ready-made stereometric objects, such as cuboids, cylinders and cones. Inheriting elements from a previous version of the software, "E-Slate 2d Turtleworlds", MaLT integrates Logo-mathematics symbolic notation with dynamic manipulation of 3d geometrical objects. This is done through the use of specially designed Variation Tools with which the student varies procedure parameters to view dynamic changes in the generated figures (Kynigos & Psycharis, 2003). Additionally,

a 3d Camera Controller gives students the opportunity to navigate around, inside and through their 3d constructions. In our research, we wanted to study the extent to and the way in which dynamic manipulation of the 3d objects themselves, along with the dynamic manipulation of the camera, can prove to be powerful tools for understanding mathematical concepts, especially ones related to stereometry. We were particularly interested to understand how working with such media may offer new means for solving mathematical problems by fostering not only visual but also diverse ways of mathematical thinking (Presmeg & Balderas-Cañas, 2002).

The “3d Modelling & Cutting” microworld

Using MaLT as a platform, we developed the “3d Modelling & Cutting” microworld, in which the Turtle is replaced with a cylinder representing a milling machine’s

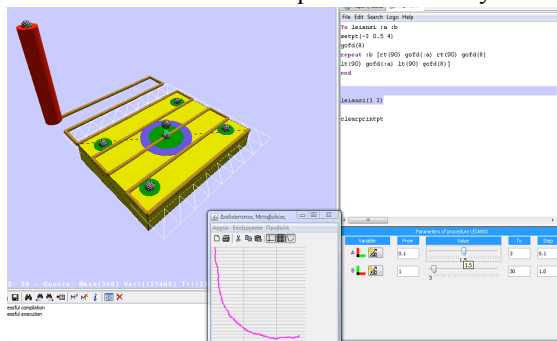


Fig. 1: Machining an object’s surface using a tool parametric programming of machining procedures.

RESEARCH DESIGN AND METHODOLOGY

Our research approach was based on the idea of studying learning in authentic settings through “design experiments” (Cobb et al. 2003), in an attempt to shed light on the relationships between the artefacts designed for the experiment and the learning processes within a specific context of implementation.

The experiment took place for 15 hours in a Secondary Vocational Education school in a small industrial town near Athens with three 12th grade students studying mechanical engineering and having a specialization in Programming Computer Numerical Control (CNC) Machines. These students had taken courses in using Computer Aided Design (CAD) software and operating CNC machines, while all of them were working at that time at mechanical engineering workplaces. In order to recreate a situation that could be experientially real to them (Gravemeijer et al., 2000) and close to their professional life, we asked them to work with the digital environment as if they were in their workplace and all together had to explore and understand its functionalities, so as to represent in it objects to be consequently cut in CNC machines. In this perspective, and adopting a “participant observation” methodology, the participant researcher (also a Mechanical Engineering teacher), did

not intervene to provide instructions or answers, but chose to pose meaningful questions at certain time points, so as to encourage students to continue their explorations, collaborate, share and discuss their ideas. In analysing the data, we searched for verbal exchanges and interactions with the digital environment indicating that the visualization processes in which the students engaged as they utilized the environment's tool connected to their mathematical reasoning, allowing them in the way to enhance elements of their spatial and visualization abilities.

TASKS

For the first phase of the experimentations, we developed a microworld that consisted of just one solid, a rectangular parallelepiped sized $5 \times 5 \times 1$, which, with the Top View being activated on the environment, looked like a 2d parallelogram sized 5×5 . To induce the students to explore the digital environment's virtual space in all 3 dimensions, we asked them to resize it and consequently move it. Possible visual mismatches between their established perceptions of the 2d and 3d space and the environment's feedback could serve as starting points for new conceptualizations.

For the second phase, we asked the students to represent in the environment's 3d space an engineering component for which we only provided a 2d drawing showing its Top View (Fig. 3). Since the cuboid of the previous phase was a main part of the component, we expected students to try to combine its position with the positions of other stereometric objects (e.g. cylinders) used to represent the rest of its parts. This process could foster the need to form specific spatial relationships between different 3d shapes, possibly changing viewpoints to inspect the assembled component.

For the third phase, we gave students a half-baked microworld (Kynigos, 2007) depicting, as we told them, a familiar to them machining procedure. However, the tool was programmed to move inside 3d space in unexpected ways (Fig. 5). Half-baked microworlds, being incomplete by design, intrigue students to explore their functionalities, deconstruct them and built on their parts. The "debugging" of a faulty machining procedure could engage students in visualisation processes which may entail making sense of an object's position and displacements in 3d space.

RESULTS

Tracking an unfamiliar 3d space

The students were originally given a $5 \times 5 \times 1$ cuboid placed in a random position inside the environment's 3d space and were asked to turn it to $3 \times 4 \times 1$ and move it to $(0,0,0)$. Although it was quite clear that this was a 3d object to be cut in a CNC machine, the students chose to represent it on paper using a 2d (XY) frame of reference (Fig. 2), giving it a shape that corresponded to the static 2d orthographic view activated at that time on their computer screen (i.e. a rectangle 3×4).

S2: $Z = 0$ is here [*points vertically to the XY surface on the paper*]. Now we can't see it. I guess that's because we look at it from above

S1: Z is... comes from above [*points vertically to the screen*]

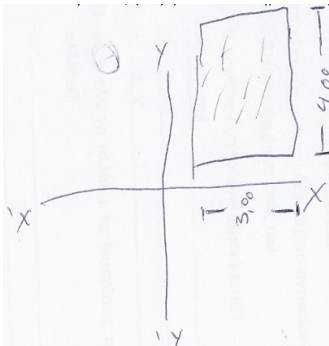


Fig. 2. The 2d reference system the students first came up with.

This led them to devise an egocentric XY frame of reference that corresponded to the way they were looking at the object on the screen from their own viewpoint, disregarding the fact that this was a 3d object with its position defined in the environment by a triple of coordinates (X,Y,Z). However, shifting from paper and pencil to a digital tool that allowed them to dynamically manipulate the object and observe the effect of their actions, seemed to trigger new visualization processes. These new visualization processes incorporated mathematical interpretations of the object's changes of position in 3d space, guiding them to specify a new frame of reference to which they attributed the mathematical properties of being 3d and "rotated" with regard to the one they had first come up with.

Representing a 3d object using a 2d drawing

When we asked the students to represent a common engineering component in the environment's 3d space, we deliberately gave them a 2d drawing showing only its Top View (Fig. 3). Although it was once again quite obvious that this was a 3d component to be cut in CNC machines and that the "3d Modelling & Cutting" was a digital environment inside which 3d objects could be represented as such, the fact that we gave them a 2d drawing seemed to disorient and confuse the students.

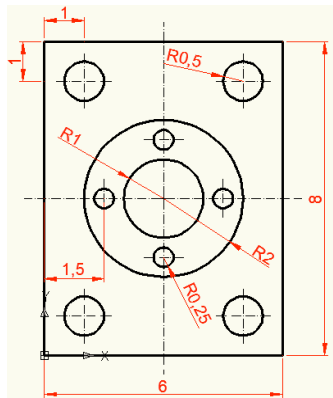


Fig. 3: The 3d component's 2d drawing (Top View)

S3: [S2 moves the cuboid to $Z=0$]. What happened???

R: How did it move?

S1: Upwards!

S2: At the Y axis's direction! At its positive direction! I'll move it again. Let's make $X=1$[the object moves to the right]. That's normal for the X axis.

S1: I'd say this goes the other way around. That's the XZ there. Not the XY!

S2: Make $Z = -2$!.... [after the visual feedback] That's rotated! Z defines upwards-downwards!

The students seemed to originally engage in a visualization process that was more or less connected to a heuristic exploration of a unfamiliar to them geometrical situation using only paper and pencil.

S3: Now, to represent the rest of the component's parts, these drawings [pointing at the circles of Fig 3]....

R: What are these?

S3: Circles

R: What they would be in the real component?

S3: Holes

S1: Holes

S3: We want to represent them, right? What will we do? S1: Use circles!

With the Top View being activated on the screen and looking at the 2d drawing, the students represented the holes of the component not as cylinders (having a specific depth up to which the drilling tool would reach), but as 2d circular disks. The fact that the 2d drawing didn't include information about the 3rd dimension (width/depth), seems to have tricked students into transferring mathematical properties of the 2d orthogonal projection into the 3d representation of the component (Parzys, 1988).

Taking into account that a real client would examine the component in detail before giving his approval for CNC machine production, the students moved the Camera.

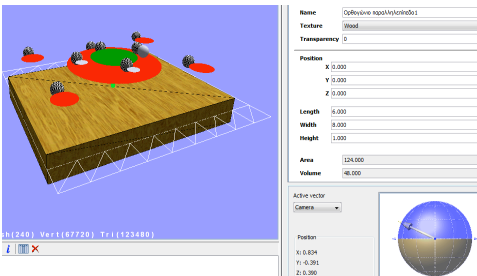


Fig. 4: The hidden view

S1: [Fig 4 reveals] What happened? The client won't be satisfied! [laughing]... This shows us everything. All the views we need, the front, the top, the side.

R: What about the parts we inserted?

S1: We have to state that these are holes

R: What kind of objects we'll use for this?

S1: Solids, 3 dimensional. The circle is not one of them. Look how we got misled!!

R: What can we do?

S1: Let's give the circle some height!

Initially, the students seemed to perceive the figures and shapes represented in the 2d drawing and the environment's 3d space in a purely iconic way instead of a mathematical one. Using a tool that allowed them to dynamically manipulate their viewpoint (the Camera), enabled them to expand and move forward their mathematical reasoning. Explaining the outcome of their actions, the students came to realise that they had been "misled" by the static 2d drawing and needed to use 3d geometrical objects -instead of 2d ones- and specify spatial relationships among the component's parts that would not differentiate as they changed viewpoints.

Using Cartesian and Turtle geometry to explain movements in 3d space

At the third phase of the experimentations, the students were given a half-baked

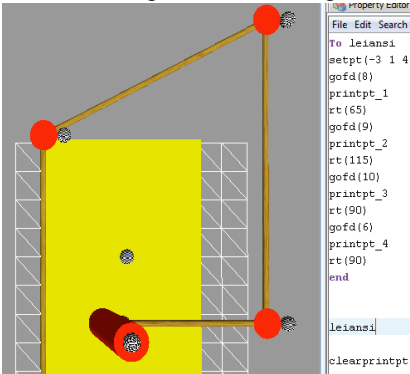


Fig 5: The graphical result of the half-baked machining procedure

microworld representing, as we told them, a familiar to them machining procedure. However, the tool, after executing correctly the fist cutting move, started moving around in a random way (Fig.5).

S2: WHAT IS THAT? [Laughter]

R: It was supposed to plan all four edges.

S2: At least THERE ARE 4 edges! [laughter]

S3: We'll have to start from the beginning!

S2: We should CORRECT it right from the start!

As they attempt to correct the Logo procedure the students start talking about Cartesian geometry coordinates and Turtle geometry commands:

S2: $X = -3$, $Z = 4$. Here

R: What happens next?

S2: Go forward 8.

S1: It goes to $X - 3$ and $Z...$ *[takes a moment to add 8]...12....?* It should be minus 4!!!

S2: It goes here *[points at the upper left corner]*. That's $X = -3$ $Y = 0.5$ $Z = 12$. It's another way to say that from the zero point *[the previous one]* it moved 8 forwards.

S3: Yes, but didn't it move in the upwards direction? There is where the negative Z s are. Measure it from the centre. Minus 4. 8 is the whole piece.

Deconstructing the Logo program to generate a “debugged” machining procedure, the students attempted to translate the Turtle Geometry commands and their inputs into Cartesian coordinates so as to specify the tool's positions as it moved inside 3d space. Working with a Logo microworld, where moving an object in 3d space is not matter of just pressing the correct virtual button as it is the fact for professional software, the students engaged in visualization processes that allowed them to expand their mathematical knowledge about working with a single frame of reference and come to combine intrinsic with Cartesian geometry. Reasoning about the validity of their conceptualizations of movement in 3d space, the students ended in recognizing two frames of reference, a global one (“ $X = -3$, $Y = 0.5$ and $Z = 12$ ”) and a local one (“8 in the forward direction from zero point”) that they used interchangeably in the processes of composing the “correct” cutting procedure for their component.

DISCUSSION

Mathematics educators have for decades been developing digital environments that employ external representations as means for exploring and understanding mathematical concepts related to visualisation and spatial orientation. Visualization and spatial thinking in engineering education, however, are viewed as processes independent of the use of external representations and when professional software are used in this context, the black-boxes they contain hinder authentic mathematically driven explorations in 3d space. Taking into account Duval's model about geometrical reasoning we attempted to identify connections between visualization processes, construction processes and mathematical reasoning as engineering students interacted with a digital environment that incorporates mathematics educators' view of visualization. Our findings indicate that these environments seem to provide a richer context for the engineering students, as they allow them to enhance elements of their spatial and visualization abilities. Thus, the mathematics educators experience may prove to be valuable in the context of engineering education, especially in informing the didactical design for engineering environments meant for genuine mathematically driven explorations in 3d space. Analytical tools, such as Duval's, shaped and specialised to fit the context of engineering education, may offer new means for transitioning knowledge between the two contexts, allowing us to fine-tune our understanding about engineering students' visualization processes and mathematical reasoning. Further research may enlighten this field and bring forth new methods and tools in this direction.

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AN INVESTIGATION OF THE THINKING STYLES DEVELOPMENT OF UNIVERSITY MATHEMATICS STUDENTS

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In this paper, we focus on the 'thinking styles' development of the students following an undergraduate mathematics degree. A cross-sectional quantitative study ($N = 366$) was conducted in two Greek universities, identifying the thinking styles development of undergraduate mathematics students and of students following an undergraduate degree in preschool education (a non-mathematically specialised degree). The analysis revealed that the more experienced students of both departments showed a stronger preference for non-prioritised thinking. However, the mathematics students seemed to combine this preference with a weaker preference for conformity and implementing rules and instructions. A discussion on these findings and on the factors that may account for them is presented.

INTRODUCTION

Various researchers argue for the existence of general cognitive preferences in the human's thinking, usually described by the construct of *cognitive style*, which is "an individual's characteristic and consistent approach to organizing and processing information" (Tennant, 1988, p. 89). Nevertheless, little research has been conducted into the interaction between these cognitive styles and studying mathematics (Duffin & Simpson, 2002; Pitta-Pantazi & Christou, 2009). In this paper, we adopt one conceptualisation of the general construct of cognitive style, Sternberg's (1999) *thinking style* defined as the "preferred way of using the ability one has" (p. 8). By concentrating on the students' thinking styles, we look for the medium-to-long term interactions between university education and the student's way of thinking. Such a study is crucial both for the mathematical community and for the general community, since it will help us in finding out more about the outcomes of the existing university mathematics educational system on the thinking of the students who form the 'pool' from which future mathematics teachers and mathematicians derive.

In a previous study, we focussed on the transition from university mathematics to a taught postgraduate degree in mathematics (Moutsios-Rentzos & Simpson, 2005). The postgraduates were found to have a weaker preference for conformity, details and the concrete in their thinking than the undergraduates. That finding may be attributed to studying university mathematics, which could be linked to the abstract and often counter-intuitive nature of mathematics. Nevertheless, the effect of general university education may also account for this difference. Thus, in order to gain further understanding about that finding, in this study the focus is on the development of the thinking styles of university mathematics students and of students following a

non-mathematically specialised degree such as a bachelor's degree in preschool education: *What are the differences and similarities in thinking style development between undergraduate mathematics students and the students who follow an undergraduate degree in preschool education?*

THEORETICAL FRAMEWORK

Thinking preferences and mathematics

Entwistle, McCune and Walker (2001) noted that there can be “identified both stable and less stable patterns in students’ reactions to tasks and contexts” (p. 114). By adopting Sternberg’s conceptualisation of thinking styles, we focus on relatively stable patterns in the students’ thinking. This decision is in line with evidence suggesting that learners have some stable patterns when thinking about mathematics; including mathematicians (Burton, 2001), graduate research students (Duffin & Simpson, 2006) and elementary school students (Gray & Pitta-Pantazi, 2006).

Furthermore, the mathematical thinking preferences identified by mathematics educators can be linked with more general psychological or educational research. For example, Rayner and Riding (1997) identified two cognitive style dimensions: a) verbal vs. image-based, and b) wholistic vs. analytic. The first dimension is associated with the preferred *mode of information representation*, whilst the second is related to the preferred *mode of information processing*. Burton’s (2001) work with research mathematicians can be linked with the dimension of the preferred mode of information representation. She identified three styles of thinking about mathematics: “Style A: Visual (or thinking in pictures, often dynamic); Style B: Analytic (or thinking symbolically, formalistically); and Style C: Conceptual (thinking in ideas, classifying)” (p. 593). Borromeo Ferri and Kaiser (2003) in a study with 15-16 year old students partially reconstructed these styles. Duffin and Simpson’s (2006) four types of mathematical learners can be linked with the dimension of preferred mode of information processing: *alien* (preference for absorbing new information without trying to link it with the existing knowledge), *natural* (preference for integrating new and old knowledge in a coherent ‘global’ structure), *coherence* (preference for finding a ‘local’ structure in the new knowledge) and *flexible* (preference for adopting different ways of thinking depending on the situation).

Consequently, the construct of cognitive style can be conceptually related to existing mathematics education research findings about consistencies in the students’ thinking about mathematics and, therefore, may be useful to further our understanding about the effect of university mathematics education in the students’ thinking.

Thinking styles and the theory of mental self government

Sternberg’s (1999) *Theory of Mental Self-Government (MSG)* is based on a metaphor between the way that the individuals organise their thinking and the way that society is governed (Sternberg, 1999). Thirteen thinking styles are identified and structured in five dimensions: *function, forms, levels, leanings* and *scope* of MSG (see Table 1).

Dimensions	Thinking Styles (description; sample items from TSI)
Functions	<p>Legislative (preference for creativity; “I like problems, where I can try my own way of solving them”)</p> <p>Executive (preference for implementing rules and instructions; “I like projects that have a clear structure and a set plan and goal”)</p> <p>Judicial (preference for judging; “I like to check and rate opposing points of view or conflicting ideas”)</p>
Forms	<p>Monarchic (preference for focussing on only one goal; “I like to concentrate on one task at a time”)</p> <p>Hierarchic (preference for having multiple prioritized objectives; “I like to set priorities for the things I need to do before I start doing them”)</p> <p>Oligarchic (preference for having multiple equally important targets; “Usually when I have many things to do, I split my time and attention equally among them”)</p> <p>Anarchic (preference for flexibility; “When there are many important things to do, I try to do as many as I can in whatever time I have”)</p>
Levels	<p>Global (preference for the general and the abstract; “I tend to emphasise the general aspect of issues or the overall effect of a project”)</p> <p>Local (preference for details and the concrete; “I pay more attention to the parts of a task than its overall effect or significance”)</p>
Leanings	<p>Liberal (preference for novelty and originality; “I enjoy working on projects that allow me to try novel ways of doing things”)</p> <p>Conservative (preference for conformity; “I like to do things in ways that have been used in the past”)</p>
Scope	<p>Internal (preference for working alone; “When faced with a problem, I like to work it out by myself”)</p> <p>External (preference for working in a group; “I like to participate in activities where I can interact with others as part of a team”)</p>

Table 1: Dimensions of thinking styles and sample items from TSI (Sternberg, 1999).

Considering Rayner and Riding’s (1997) cognitive style dimensions, by adopting Sternberg’s conceptualisation, we focus on the students’ general thinking preferences from the perspective of the mode of information processing. Moreover, after considering conceptual and empirical evidence, Zhang and Sternberg (2006) identified three *types* of thinking styles: a) *Type I* (preferences for originality and freedom in thinking, low degrees of structure, high levels of freedom and more complex information processing), b) *Type II* (preferences for conformity, structured tasks, authority and straightforward information processing), and c) *Type III* (styles that can be linked with characteristics of either Type I or Type II styles, “depending on the stylistic demands of a specific task”; p. 115).

METHOD

Sample and procedures

This study was conducted with 366 mathematics students in two Greek universities (see Table 2). There is a distinct contrast in the mathematical training and knowledge of the students belonging to each population. The mathematics undergraduates were following a 4-year BSc mathematics degree mostly taught through lectures, whereas the students following the 4-year BA in preschool education form the pool from which kindergarten school teachers derive and their mathematical knowledge is of high-school level, as they are not explicitly trained in ‘advanced’ mathematics as part of their degree. Mathematics (mostly elements of set theory, number theory, logic and combinatorics) is only taught to them as part of mathematics education related courses. Furthermore, the mathematics students radically differ from the preschool education students in the proportion of male and female students (as shown by the official records of the enrolled students for each year): the mathematics undergraduates are almost equally divided (55% male – 45 % female), whilst almost all the pre-school education students are female (less than 4% male), which is also evident in our sample (see Table 2). Finally, different year groups were included, since this implies different levels of university experience and enculturation in the respective scientific communities. This was assumed to allow us to investigate possible stylistic developments as the students progress through university.

	Mathematics			Preschool education	
	2 nd year	3 rd year	4 th year	1 st year	4 th year
Male	45	38	21	0	0
Female	54	72	21	80	75
Total	99	70	42	80	75

Table 2: The participants of this study.

We asked each student to complete a translated to Greek version of ‘TSI’ – the Sternberg-Wagner Thinking Styles Inventory (Sternberg & Wagner, 1991). The TSI is a self-report, paper-and-pencil test consisting of 104 items (see Table 1). In order to identify the students’ thinking styles, their ‘raw score’ for each scale is computed. Subsequently, the raw scores for each style are compared against the norms provided (which vary according to the participants’ gender and educational level) to determine the participants’ thinking style: ‘Very High’ (Top 1%-10%), ‘High’ (Top 11%-25%), ‘High Middle’ (Top 26%-50%), ‘Low Middle’ (Top 51%-75%), ‘Low’ (Top 76%-90%), ‘Very Low’ (Top 91%-100%).

The identification of the students’ thinking styles

Though the TSI has had its validity and its reliability verified across various studies and countries (Zhang & Sternberg, 2001), the absence of research implementing this instrument in a Greek university population urged caution in its implementation. Therefore, to administer TSI in this study, it was first independently translated and

back translated from English to Greek and subsequently it was piloted to refine the language of the items. Moreover, TSI, is a norm referenced test and this is particularly important since it appears that there is no published norm for Greek university students. Hence, following Moutsios-Rentzos and Simpson (2005), we decided that the participants' scores would be labelled both according to Sternberg's norm ('Sternberg's labels') and according to norms ('adjusted labels') produced by the data of each population (mathematics and preschool education) following Sternberg's process. The latter norms serve as a 'tighter lens', which helps in focussing on each population and spotting intra-population stylistic differences and developments. The data analysis of our previous study showed the usefulness of this lens (by spotting a difference in the 'local' thinking style that the wider lens failed to identify). Following these, for this paper, the analysis of intra-population differences will be based on adjusted norms (different for each population).

RESULTS

A note on the validity and reliability of the translated to Greek TSI

The translated to Greek TSI demonstrated good cross-cultural validity and reliability, as shown by its internal consistency, construct validity and by comparison against theory and previous studies (Zhang & Sternberg, 2001). For example, 'internal' and 'external' were negatively correlated ($r_s = -.192, p < .001$), whilst principal axis factoring (oblimin with Kaiser normalisation) led to a 3-factor solution (accounting for the 66.7% of variance), which, in general, was in line with the Type I, Type II and Type III thinking styles (Zhang & Sternberg, 2006). Nevertheless, though 'hierarchical' is considered to be a Type I style (ibid), prioritised thinking ('hierarchical' and 'monarchic') was assigned to the factor related with the Type II styles. Finally, the reported results are based on the adjusted norms, since no contradictions were found when comparing these results against Sternberg's norms.

The students' thinking styles

In order to identify the stylistic differences between mathematics undergraduates and pre-school education undergraduates, we first considered the mathematics undergraduates. Since the undergraduate mathematics sample consisted of three year groups, we conducted the Kruskal-Wallis test to detect any statistically significant stylistic changes as the students progress through undergraduate study. Four significant changes were found: the more university experienced the students are the less 'executive' ($H(2) = 10.683, p < .01$), less 'hierarchical' ($H(2) = 9.353, p < .01$), less 'global' ($H(2) = 8.591, p < .05$) and less 'conservative' ($H(2) = 11.125, p < .01$) they are. We conducted Mann-Whitney tests, with Bonferroni corrections, to follow up these findings: a) the 2nd year students against the 3rd year students, and b) the 3rd year students against the 4th year ones. The 2nd year students were found *not* to be significantly different in their thinking styles from the 3rd year students, whilst the 4th years were found to be less 'executive' ($U = 954.5, p < .05, r = -.30$), less 'hierarchical' ($U = 1088, p < .05, r = -.22$) and less 'conservative' ($U = 972, p < .05, r = -.29$) than

the 3rd years. Hence, it appears that the thinking styles profiles of the students who have just completed the first two years in university are not significantly different. On the other hand, there seem to be significant changes in the students' thinking styles profile as they enter the fourth year of their mathematics degree.

Moreover, considering the differences in gender between the mathematics and the preschool education undergraduates, we rerun the analyses focussed only on female mathematics undergraduates, in order to detect any potential gender effect in the aforementioned differences. The results of these analyses were similar to the above. The Kruskal-Wallis test showed that the more university experienced the female students are the less 'executive' ($H(2) = 9.347, p < .01$), less 'hierarchic' ($H(2) = 12.143, p < .01$), less 'global' ($H(2) = 12.057, p < .05$) and less 'conservative' ($H(2) = 6.720, p < .01$) they are. Furthermore, the Mann-Whitney tests, with Bonferroni corrections, suggested that the 2nd year female students were found not to be significantly different from the 3rd year female students in any of the thinking styles dimensions. Moreover, the 4th years were found to be less 'executive' ($U = 203.5, p < .05, r = -.34$) and less 'hierarchic' ($U = 196, p < .01, r = -.36$) than the 3rd years. Contrary to the findings when both genders were included, the 3rd year and the 4th year female mathematics undergraduates do *not* significantly differ in their 'conservative' thinking preferences ($U = 241, ns, r = -.24$). It seems that though the female students become significantly less 'conservative' in their thinking as they progress through university, this stylistic development is not 'steep' enough to be detected in female students belonging to consecutive year groups.

Subsequently, we focussed on the pre-school education undergraduates. We used the Mann-Whitney test to compare the thinking styles profiles of the 1st year students with the thinking styles profiles of the 4th year students. The 4th year students were found to have a stronger than the 1st year students preference for 'oligarchic' ($U = 2119, p < .001, r = -.26$) and 'anarchic' ($U = 2320.5, p < .05, r = -.20$) thinking.

DISCUSSION

These findings suggest that as the students of both departments progress through university, they develop a stronger thinking preference for non-prioritised thinking ('anarchic' and 'oligarchic') and/or weaker preference for prioritised thinking ('monarchic' and 'hierarchic'). Therefore, these stylistic developments might be viewed as a general university education outcome. Notice that, in our study, prioritised thinking has been linked with Type II thinking styles, whereas non-prioritised thinking has been related with Type I thinking styles.

Nevertheless, only in the mathematics students did we find statistically significant stylistic development away from conformity ('conservative'; a Type II style) in the students' thinking. This stylistic development is in accordance with the findings of a previous study (Moutsios-Rentzos & Simpson, 2005). Though this development was still noted when considering only the female undergraduates, it wasn't statistically significant when consecutive year groups were considered. Thus, it appears that this

stylistic development is not ‘steep’ enough in the female undergraduates. The preschool education students showed a similar stylistic change towards novelty and originality in their thinking (‘liberal’), albeit not statistically significant ($U = 2477$, ns , $r = -.15$), which means that more than three years of university experience did not seem to affect the students’ thinking towards that direction.

Moreover, university mathematics students seem to develop thinking preferences away from implementing rules and instructions (‘executive’; a Type II style), which is also evident when considering only the female students. Nevertheless, such a stylistic development was not spotted in the preschool education students.

Following these findings, it appears that university education has a general effect on the students’ thinking away from Type II styles (norm-favouring styles; Zhang & Sternberg, 2006) and/or towards Type I styles (creativity-generating styles; *ibid*). We hypothesise the existence of two aspects in this development: a common aspect identified even in students following radically different degrees in terms of the content taught (such as weaker prioritised thinking and/or stronger non-prioritised thinking) *and* an aspect that may be linked to the special characteristics of each degree (such as weaker ‘executive’ and ‘conservative’). Nevertheless, research should be conducted to investigate this hypothesis, since the common and non-common aspects of the students’ stylistic development will further our understanding of the outcomes of university education (general and mathematics).

Furthermore, since in both departments the main way of teaching is through lectures, we posit that the non-common aspect of the stylistic development identified in the mathematics undergraduates (towards a weaker preference away from conformity and the implementation of rules and instructions) may be linked with the nature of the mathematical content taught. We wish to identify two content-related factors that may account for these differences. First, the abstract and often counter-intuitive nature of the mathematics taught in a mathematics department may affect the students’ thinking away from conformity and from implementation of rules and instructions. Moreover, Biggs (2001) reported that the students taught using problem-based learning develop a stronger *deep orientation to learning* (focussing on the meaning and ideas conveyed in the task), which has been linked with Type I styles (Zhang & Sternberg, 2001). Following this and Weber’s (2005) view of proof construction as a problem-solving activity, we posit that the mathematics undergraduates’ extensive training in proof constructions (since the notion of proof is at the crux of undergraduate mathematics) may account for the identified stylistic development away from Type II styles. We conjecture that the first, the second or a mixture of both factors may account for the identified differences.

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COGNITIVE EMPATHY AND MATHEMATICS TEACHING

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This study examines the nature of cognitive empathy among prospective primary teachers, specifically in its relation to teaching of mathematics. Cognitive empathy is defined as the ability for deliberate intellectual perspective taking of another person in a teaching-learning situation. Using case studies, we identified prospective primary teachers in a methods course using a 2-by-2 matrix based on their cognitive empathy and mathematical knowledge for teaching. We found that although in general prospective teachers are empathetic in nature, most of them are able to relate to struggling students from an affective aspect, and rarely from a cognitive stance. Empathy is a trait that can be developed and deep mathematical understanding is a requisite in developing cognitive empathy.

INTRODUCTION

The concept of empathy has been developed significantly in the field of psychotherapy and counselling psychology. Much of the focus of empathy in the educational psychology circle, however, has been on the emotional level in negative social situations such as empathy toward students who are victimized in schools (Craig, Henderson, & Murphy, 2000). Zhou et al. (2003) stress that, “empathy motivates helping others and the desire for helping others, as well as inhibits aggression, facilitates people’s social competence for interacting with others, and provides a sense of connection among people” (p. 269). Although the goal of empathy is to help other, the targeted outcome is rarely cognitive, but mainly social or affective. For instance, Tettegah and Anderson (2007) define teacher empathy as “the ability to express concern and take the perspective of a student, and it involves cognitive and affective domains of empathy” (p. 50) and they contend that a teacher should be able to perceive the victim’s feelings and take on the perspective of the victim through a conscious and unconscious process, which may involve an awareness of a student’s feelings and the ability to put herself in the student’s place and feel what the student feels.

Although empathy has been documented as an important disposition for educators to possess in order to facilitate positive interactions among students (Good & Brophy, 2000; Noddings, 1988) there is a lack of research on the role of empathy in teaching and learning. Therefore, this study aims at filling the gap in our understanding of the role of empathy, specifically cognitive empathy, in the teaching and learning process.

Cognitive empathy in this study is defined as the ability for deliberate intellectual perspective taking of another person in a teaching-learning situation (Gladstein, 1983; Tettegah & Anderson, 2007). Gage and Berliner (1998) contend that

intellectual empathy involves the manner in which teachers pay special attention to the way children define, describe, and interpret phenomena and problem-solving situations in their daily experiences and can begin to understand these experiences from the unique perspectives of children. More specifically in the context of teaching and learning mathematics, intellectual empathy is the ability to place oneself into the students' emerging and incomplete thinking in order to anticipate cognitive hurdle the students might have in learning a specific topic. As an example of cognitive empathy, consider van Hiele's (1958/1985) well-known theory of geometric thinking:

Two people who reason at two different levels cannot understand each other. This is what often happens between teacher and student. Neither of them can manage to follow the thought process of the other and their dialogue can only proceed if the teacher tries to form for himself an idea of the students' thinking and to conform to it. Some teachers make a presentation at their own level while asking students to reply to their questions. In fact, it is nothing but a monologue, for the teacher is inclined to consider all the answers which do not belong to his system of relations as stupid or misplaced. A true dialogue must be established at the level of the students. For this to happen, the teacher must often, after class, ask himself about the responses of his students and strive to understand their meaning (van Hiele, 1958/1985, p. 63)

THEORETICAL BACKGROUND

Sociocultural theory holds that human development is founded upon social interaction in cultural practices that are mediated by the use of cultural artefacts, tools and signs, which include the use of language. Rogoff and Lave (1984) describe social context as a place in which cognitive activity occurs through interaction with other people. Furthermore, Vygotsky (1978, 1986) stresses the role of human relationships in learning and highlights the separation of affect and cognition as a key failing of psychology. Affect is central to communication and the formation of relationships between people. In an educational setting in particular, empathetic relationships have immediate effects such as produce conversation, communication, personal exchange, esteem building, friendship, emotional links, and understanding and also result in consolidated effect such as transformative on teaching and learning and on classroom climate (Cooper, 2010). Students develop greater self-worth and a sense of security and trust. The creation of trust lead to improved learning and risk-taking and better behaviour.

Cooper (2010) identifies four different but interlinked types of teacher empathy: *fundamental*, *profound*, *functional*, and *feigned*. *Fundamental empathy* is concerned with the characteristics necessary to initiate relationships. This form of empathy involves "non-judgemental approach ... [by] being accepting and open in ... beliefs and attitudes, paying close attention to others' feelings, listening carefully, showing signs of interest and being very positive in verbal and non-verbal communication" (Cooper, 2010, p. 86). In the sense of cognitive empathy, this means being attentive to students' thinking or cognitive models. When teachers understand the cognitive

level at which their students operate, they are able to follow their thought processes and conform to that level (van Hiele, 1958/1985).

Profound empathy results from frequent one-on-one interaction where deeper understanding and higher quality relationships occur. This form of empathy is characterized by understanding the emotional (and cognitive) barriers to learning where the “teachers create rich mental models of individuals, which they can relate to closely, both emotionally and cognitively” (Cooper, 2010, p. 88). *Functional or relative empathy* is an adaptation of empathy for dealing with large groups where teachers treat the whole class as one entity during interaction, using mental representation of the whole group, which can support group cohesion and a sense of belonging. In the sense of cognitive empathy, teachers need to create shared mental model of students’ cognition based on their knowledge of the students at that developmental stage. Teachers have to cope with such shared model, which tends to result in stereotyping and the more diverse the ability of the students, the less the model will work. *Feigned empathy* represents deceptive behaviour when people exhibit superficial signs of empathy but eventually reveal quite the opposite.

AIMS OF THE STUDY

The aim of this study is to investigate the nature of empathy in pre-service elementary teachers, specifically their cognitive empathy and how it is related to their mathematical knowledge for teaching (MKT).

METHOD

Participants of the study consisted of 52 preservice elementary teachers in their third year of the program enrolled in a mathematics methods course. Two instruments were administered to them at the beginning of the course. The Mathematical Knowledge for Teaching (MKT) measures, which measures participants’ knowledge of mathematics (LMT, 2006), and the Interpersonal Reactivity Index (IRI) Perspective Taking subscale (Davis, 1983). Based on results from the MKT and IRI pre-test scores six pre-service teachers are selected using a 2-by-2 matrix based on whether they score high or low on IRI and high, or low on MKT (two H-MKT/H-IRI; two L-MKT/H-IRI; one H-MKT/L-IRI; and one L-MKT/L-IRI). Semi-structured interviews, lasting on average half an hour, were conducted with these preservice teachers at the end of the semester to investigate how preservice elementary teachers perceived their relationships with students and their understanding of the role of empathy, specifically cognitive empathy, in teaching mathematics. The purpose of these interviews was to elicit the interactions between mathematical knowledge for teaching and cognitive empathy and to explore the nature of cognitive empathy in preservice elementary teachers. They were presented with situations where a student was struggling with a specific mathematics concept (such as place value or fractions) or failed in a test and was frustrated. They were then asked how they would respond

to that student and what they would do in those situations. These interviews were audio recorded, transcribed, and analysed.

RESULTS

In general, the preservice teachers consider themselves to be empathetic in the sense of taking on others' perspectives (IRI mean score 26.03 with a standard deviation of 4.66 from a possible range score of 7 to 35). During the interview several themes emerged that differentiate the quality of these preservice teachers' empathy, specifically when it pertains to cognitive empathy.

Personal Empathy and Teacher Empathy

All the six preservice teachers claimed that they were empathetic by nature, however, two preservice teachers indicated that they were not generally, but became one when face with the task of teaching students. Because of the demand of teaching students with diverse background and ability, being empathetic is a requisite for teachers and resulting in a possibility of *feigned empathy* (Cooper, 2010).

John: I guess it depends with what. As a teacher, yes. Maybe in my personal life, not so much. With teaching, I think I consider myself very empathetic. If I see a student having a problem, I don't just immediately be like, "That's just a troublemaker student." I always assume there's something more to it. I need to figure out something else because I figure that, you know the principle that there's a reaction to every action, and I think that the kids are... if they're acting out a certain way, it's because of something. It's not just because that that kid... I hope I never fall into that where I'm like, "Oh, he's just a bad child. Or he just doesn't want to listen." Because I do think that there's something different and if you just take that little extra time to find out what's going on with the student...

Personal Experiences Contribute to Teacher Empathy

The preservice teachers attributed their empathetic behaviours to personal struggles, either in mathematics, other subjects or other areas of life. Those who had not struggled in mathematics were able to transfer their experience struggling with other areas and become more empathetic toward students.

Kathy: I haven't really struggled a lot with math. Maybe I don't relate in that way, but I've definitely struggled in a lot of things. And I can be able to know what that feels like when everyone else seems to get it and you feel like you're just lost and you're behind. Just being able to think about and understand how they are feeling. I feel like that's a strength that I do have, of being able to see when other people are suffering or struggling and being able to try to relate that to myself. I think, "How would I feel in that situation?" if I was sitting down trying to take a math test, and it's on basic writing numbers, and I just couldn't understand how to do that, what would that feel like to me? Thinking in that way has helped me to relate to those students better. It helps me to feel more confident as a teacher,

too—to feel that I can understand where they’re coming from. It’s really neat to be able to relate to the children, especially the students who are struggling. It makes a huge difference.

These results suggest that providing the necessary experience can facilitate the development of fundamental empathy among preservice teachers. This is especially important for teachers whose personal learning of mathematics have been easy and may be difficult to relate to students who struggle in their understanding of mathematical concepts:

Kathy: When I first started tutoring, it was really hard because I knew the answer. It was really easy for me, and I knew he could understand it ... I tutored a boy in high school but ... he gave up or he just really couldn’t do it. It was so hard for me in the beginning when I started tutoring to just feel like, “OK, I feel your pain. I understand that it’s hard. Let’s work through it slower or at your pace.” It was hard for me ... After that tutoring experience in high school it’s so much easier for me to relate ... So empathy-wise, I used to not be so empathetic.

Many of the students expressed that during their practicum they had the opportunities to work one-on-one with students and began to develop profound empathy as they developed relationship with these students in a close interaction. They are able to create a mental model of the students’ cognition and provide intervention to help them.

Kathy: I have a kid in my [practicum] class who struggles. Everyone else is on his add 9s or whatever in addition, and he is on his add 3s. And he just really can’t do it ... He’s at that point where he can’t visualize just numbers. Numbers don’t make much sense to him ... And so we give him cubes to work with ... We’ll have the numbers and he’ll connect these cubes to these numbers, and then count ... And then we say, “OK now that you’ve used your fingers and you’ve used the cubes. Now let’s find another way to do this.” A lot of times that’s where he struggles. It’s just he can’t focus on the numbers and the cubes or the numbers and the fingers. For him, he has a lot of home problems. And so once something is hard, he gives up ... And he doesn’t use his cubes ... because he’s embarrassed. He doesn’t want other kids to look at him.

Mathematical Knowledge Contributes to Cognitive Empathy

The cognitive aspect of empathy is what distinguishes among the preservice teachers. Although, all of the preservice teachers interviewed expressed that they were empathetic towards students who are struggling in mathematics, those who scored low on the MKT focused mainly on the affective aspect of empathy. They could relate easily with students who struggled with mathematics, but failed to identify cognitive hurdles the students might have, and rather attended to behavioural or affective aspects such as boosting the students’ self confidence.

Sam: [Students struggle in math because] they just don't pay attention. They're not looking at the board. They're talking to their neighbours, playing in their desk. They don't do their homework ... [Other things] like I remember that one day the room got really hot. Everyone started to zone out. And if it's really noisy or if it's a nice day or if it rains. There can be tons of things ...[that] distracts them and then they can't focus.

Laura: I would tell them that it's ok. It doesn't make them dumb. They might feel that way sometimes because they see their peers getting it and understanding the concept. But it doesn't mean that there's anything wrong with them ... Trying to be sensitive to the fact that they might not understand it because, you know, a lot of students, sometimes it's hard at that age when their self-esteem might come from school. And why allow them to fail or feel like failure when they're so young, when they still have so much of life still ahead of them.

And despite their attention to emotional empathy, when it comes to helping struggling students, their remediation focused more on what the teacher needed to do and not understanding the students' cognitive model and what specifically the students need to progress in their thinking.

On the other hands, preservice teachers with more robust understanding of mathematics were able to focus on the students' thinking and plausible cognitive hurdles.

Kathy: And you ask him thinking questions. "OK, what is the first thing you are thinking of? What do you think we should do? Or what does this problem say to you?" Trying to get him to think out loud so I can understand what's going on in his head. Because some of the kids I can't understand how they got where they got or what they understood. And we do it... it's a lot easier.

Chris: You have to take a couple of steps back and *see* where their understanding is with math, and to try to build up the skills that they need to be able to learn new concepts.

However, having a robust mathematical understanding does not guarantee being able to relate and help students who struggle in mathematics as exemplified by the following excerpt of a preservice teacher with high mathematical knowledge but low empathy:

Anita: [If my student is struggling with a math concept] ... a lot of review games ... So the students who already knew the concept were getting a lot of practice in, and they were still entertained. It wasn't boring because it wasn't just like a paper or whatever. And then with the struggling that weren't like so far... I mean they got the concept but still... you know ... at that point you might just have to start from the basics kind of. Obviously tell them, "No, this is stuff that first graders know." I mean don't make it sound like they can't do it, but just say, "You can do it

because you're in this grade and that's what's expected of you, and we know you can do it."

DISCUSSION

Although empathy has been considered to be an important disposition that teachers need to possess, there is a lack of conscious effort to develop it among preservice teachers (Cooper, 1997). In general, those who aspire to become teachers are empathetic by nature as supported by the results of this study. However, there is a lack of emphasis on cognitive empathy. Preservice teachers tend to pay attention to the affective aspect of empathy when dealing with students who struggle with mathematics. They fail to attend to cognitive aspects important in the learning process such as creating mental models of student's mathematical thinking and understand cognitive hurdles they experience when encountering certain mathematical concepts.

Cooper (2010) found that teachers consider empathy to be central to successful teaching and vital to demonstrating care, however, different contexts, more specifically poor people-teacher ratios and lack of interaction over time, constrain the ability of teachers to demonstrate empathy. Preservice teachers have to learn to cope with working in a large class size and often with diverse groups of students. Focusing on the affective aspect of empathy may be detrimental for student learning as one preservice teacher commented, "It could be a weakness in a way because I care so much about the students and I'm so empathetic towards any problem that they bring to me ... I just feel like I'd rather work one-on-one with them." Teacher education programs should direct prospective teachers' attention to different cognitive models to enable them to work with functional cognitive empathy in the context of teaching a large group of students.

Prospective teachers need to develop their content knowledge during their preparation program, but embedding such content knowledge by examining them in the context of different children's cognitive models may be beneficial to develop cognitive empathy. Thus, implications of this study include the importance for teacher education programs to pay attention to the development of cognitive empathy in preservice teachers by creating opportunities for them to experience cognitive hurdles in their learning and focusing on children's mathematical thinking to be familiar with different cognitive models.

This study is limited in its scope because of its exploratory nature. We hypothesize on the essential role of cognitive empathy especially when it pertains to teaching and learning mathematics at the primary level. To further understand the nature and importance of cognitive empathy, we intend to examine practicing teachers in classroom settings by identifying empathetic teachers, observe classroom episodes of these teachers in their interactions with students, and explicate factors that contribute to student learning.

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FIFTH GRADERS' ARGUMENTS FOSTERED IN THE LEARNING OF INCLUSION RELATIONS BETWEEN GEOMETRIC FIGURES

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This study explores the process of transition from elementary to secondary geometry through design experiments. We aim at clarifying the types and characteristics of fifth graders' mathematical arguments, which can be fostered in the learning of inclusion relations between geometric figures. In our analysis, we identified three types of arguments: similarity or difference between properties of figures, suggestion of general-special relations, and consistency among relations and conviction of others. Our examination of their characteristics may imply a process in which each argument connects with each other based on some dynamic views and leads eventually to the last argument.

INTRODUCTION

This paper is part of a study that explores the theory and practice that enable students to move from elementary to secondary geometry by design experiments. Quadrilateral inclusion relations are regarded as an introductory process into deductive geometry (van Hiele, 1986; Clements and Battista, 1992), and we investigate the types of mathematical arguments of fifth graders fostered in the course of learning about these inclusion relationships.

Previous studies have shown that a majority of students find it difficult to understand concepts in secondary geometry even after learning proofs (Senk, 1989; Okazaki and Fujita, 2007). Students naturally develop strong but limiting tacit models in their mind that they have abstracted from the typical special examples continually provided by their teachers (e.g. Hershkowitz, 1990). However, recent research indicates that a dynamic environment improves their understanding of the concepts and moving beyond these tacit models (Leung, 2008; Okazaki, 2009). In particular, we have observed that students can develop their own mathematical arguments in the course of that learning. In this paper, "mathematical argument" means "a connected sequence of assertions intended to verify or refute a mathematical claim" (Stylianides, 2006). We pay particular attention to what foundations students use in their arguments.

We have conducted our design experiment aimed at clarifying how fifth-grade students can develop their arguments regarding secondary geometry. It consists of three stages: fostering geometric dynamic views, exploring quadrilateral inclusion relations, and constructing geometric definitions. Okazaki and Kageyama (2010) examined the first stage and clarified the types of dynamic views in students. This paper analyzes the second stage of the experiment and explores the arguments fifth graders can develop in the learning of quadrilateral inclusion relations and their characteristics.

THEORETICAL BACKGROUND

Harel and Sowder (2007) distinguished three types of schemes in students' proof and proving: external conviction (authoritarian, ritual, non-referential symbolic), empirical (inductive, perceptual), and deductive (transformational, axiomatic). They suggest that the empirical schemes are inevitable because "the construction of new knowledge does not take place in a vacuum but is shaped by existing knowledge" and "natural, everyday thinking utilizes examples so much." The question then is how students can move from their empirical schemes towards deductive schemes.

Furthermore, Harel and Sowder (2007) indicated a cognitive and epistemological dependency between the nature of the entities considered and the nature of proving applied, saying that "the Greeks elevated mathematics from the status of practical science to a study of abstract entities" in addition to producing axiomatic proofs. We should therefore not only consider students' reasoning, but also their recognition of geometric figures. The learning of inclusion relations is related to both of these processes. In that sense we regard the learning of them as a bridge between empirical and deductive reasoning.

Several additional points from previous studies should be noted. First, students tend to conceive geometric figures as fixed and static images. Fischbein (1993) stated that a geometrical figure is a "figural concept" with both conceptual (ideality, abstractness, generality and perfection) and figural (shape, position, and magnitude) aspects. Specifically he noted that "the figural structure may dominate the dynamics of reasoning" for many students.

Second, students are likely to tacitly add properties such as "unequal adjacent angles" to a parallelogram besides its true properties (p) (Hershkowitz, 1990). Okazaki (2009) found one way to eliminate these tacit properties, not by removing the tacit property ($\neg q$), but rather adding q (e.g., there are cases in which the sizes of adjacent angles of a parallelogram are equal). We described this method as $p \wedge \neg q \rightarrow p \wedge (\neg q \vee q) \rightarrow p$. Although mathematically this is the same as the simple removal of $\neg q$, it is different psychologically. Simple removal implies that the teacher denies the child's tacit properties. In the alternative way, however, the child's concept ($\neg q$) is protected, and for this reason it becomes necessary to newly construct (q).

Third, Ito (1978) suggested difficulties with customary naming. There is a natural resistance to using the same name for what we have classified using different names. Also, we customarily choose the most appropriate name. For instance, if there is a cat, it is better to call it "cat" rather than "animal."

Fourth, de Villiers (1994) suggested difficulties students have with "the meanings of the activity, both linguistic and functional: linguistic in the sense of correctly interpreting the language used for class inclusions, and functional in the sense of understanding why it is more useful than a partition classification." In the linguistic

sense, we have to examine for example whether a students' words "a square is a rectangle" mean "a square is the same as a rectangle" or "a square is a type of rectangle." In the functional sense, we will investigate in what situations they recognize their need to classify geometric figures inclusively.

DESIGN EXPERIMENT

Our design experiment was conducted according to the methodologies of Cobb *et al.* (2003). We conducted the experiment with 30 fifth graders in a classroom of a university-attached elementary school, with the collaboration of a teacher who had 18 years of teaching experience. We found from our questionnaire (Okazaki and Fujita, 2007), conducted prior to the experiments, that the abilities of the students in our experimental group were almost the same as those of students in public schools. However, we observed during our lessons that the students in our experiment were somewhat better in their communication abilities. They had already learned the geometric figures and their properties, and also the area formulas of the figures through activities of isometric changes.

The experiment consisted of three stages: fostering dynamic views (three lessons), exploring inclusion relations between figures (four lessons), and constructing definitions (three lessons).

In the first stage, we aimed at fostering the students' dynamic views using "operative sheets" in which the sizes of angles and sides could be changed freely. The tools are made of transparent sheets on which pictures of a line, parallel lines, a right angle and 60 degree angle are drawn (Fig. 1). The combination of two

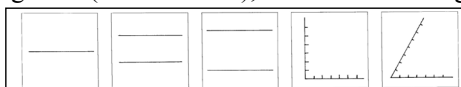


Figure 1. Operative sheets

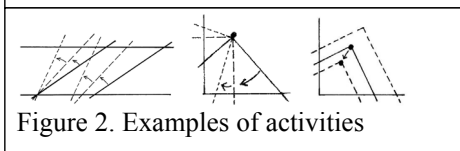


Figure 2. Examples of activities

sheets enables one to make various figures, and if a sheet is moved by a translation or rotation, a group of figures can be constructed. The students explored what figures could be made and how one figure could be transformed into another. Okazaki and Kageyama (2010) analyzed these lessons and identified the types of dynamic views: concrete manipulation, idealization and mental operation, gesture, grasping the movement of points during the whole figure change, recognition of invariants, reverse operation, and simultaneous identification of invariants and variables (slightly modified).

The second stage was conducted in a way designed to show how students give and develop their arguments in the learning of quadrilateral inclusion relations. We found beforehand from our questionnaire that they held a variety of opinions about the relations. For example, some of them accepted the statement that a rhombus is a parallelogram but others did not. Moreover, the parallelogram-rhombus relation and the rhombus-square relation were relatively easy to recognize (58 % and 65 %, respectively), while the parallelogram-rectangle relation and rectangle-square relation

were difficult (38 % and 15 %, respectively). Thus, we decided to organize the lessons as a form of debate in order to stimulate their discussion. They did not have to decide on a definite position initially, but were permitted to change their stance. The teacher's main role was to organize their discussions and debates. Our conjecture was that they would give various types of arguments and develop them by their interactions. The following activities were conducted during four lessons:

First lesson: Reviews, parallelogram-rhombus and parallelogram-rectangle relations

Second lesson: Parallelogram-rectangle (continued) and rhombus-square relations

Third lesson: Rectangle-square relation and comprehensive inquiry of all relations

Fourth lesson: Comprehensive inquiry of all relations

The lessons were recorded with video cameras and through taking field notes. We then made transcripts of the video data. Two types of data analysis were conducted. The first was from a reflective session after each lesson. Here we analyzed what happened in the classroom in terms of the students' activities and interactions. The lesson plans were thus modified by taking into account both the original plan and our analysis of each lesson. Second was a session of retrospective analysis after all the classroom activities had finished. We used the grounded theory (Glaser and Strauss, 1967) to encode and conceptualize the students' arguments by analyzing the transcripts and the video data.

RESULTS

We have labeled the students' arguments alphabetically.

Parallelogram-rhombus relation

In the first lesson, the teacher asked the students if a rhombus can be a parallelogram and let them put their name card on the blackboard under the line that ranged from "it's definitely a parallelogram" (Yes) to "they are completely different" (No). The result was that 23 students indicated Yes, 3 said No, and 6 decided on something in between (Fig. 3).

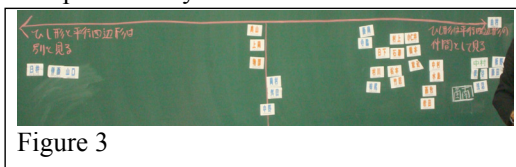


Figure 3

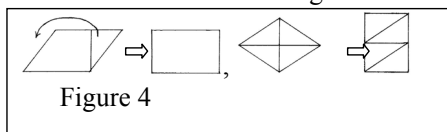


Figure 4

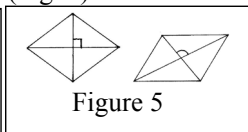


Figure 5

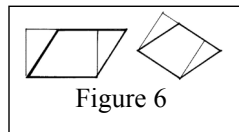


Figure 6

First, Usu stated that we should see the rhombus and parallelogram as different because of the difference in how to transform each into a rectangle, as was seen in the learning of areas (Fig. 4; A: Difference of isometric change). Yama also said No because of the difference between the angles of the diagonals (Fig. 5; B: Difference between symbolic parts).

Next, Naka stated that "they are of the same sort because the opposite angles are equal, the opposite sides are parallel, and the lengths of the opposite sides are the same" (C: Same properties). Moreover, he added an argument against Usu's opinion

“if we are forced to consider this, the methods used to get the areas are also the same” (Fig. 6; D: Similarities of isometric change). At this point in the discussion, three students asked to change the place of their name cards, and interestingly, Naka, one of the three, moved his card a little to the left, in the No direction. We found that they experienced a change in their thoughts on this matter.

After that, Ito (whose view was No) stated, “a rhombus has four equal sides, but only two sides are equal in the parallelogram” (B), and he furthermore generalized: “if there is even one different property, they must be different.” Meanwhile, Shou expressed the Yes view. He explained the properties of the parallelogram “while pointing to the rhombus” (E: Seeing the properties of the general figure in the special figure). We think that this argument implies the general-special relationship, although (C) above only indicates that there are common properties, namely that they are of the same sort.

Parallelogram-rectangle relation

During the last part of the first lesson, the teacher asked if a rectangle was a parallelogram. The students then underwent a great change in their opinions (Fig. 7. Responses of Yes (7), No (15), and those in between (10)).



Figure 7

Yama responded in terms of the dynamic views: “Yes, because the rectangle and parallelogram are made in the same movement” (cf. the left figure in Fig. 2) (F: Two figures made in the same movement). Moreover, he added that a parallelogram can be transformed into a rectangle using isometric change (D). It followed by explaining that a rectangle has the properties of a parallelogram, pointing to the corresponding sides or angles of the rectangle (E).

However, many students held the No view. For example, several students stated “the parallelogram has different sizes of adjacent angles. But a rectangle has all right angles” (B). We found that they strongly held the tacit property “adjacent angles in a parallelogram are different.”

In the second lesson, the arguments (B), (C) and (D) and several further characteristic arguments were brought forth. First, they began their explanations by combining several arguments. Naka stated several common properties and then “both areas can be found by length times width” (G: Plural arguments). Moreover, he added “none of their diagonals intersect at a right angle.” Thus, even a nonessential attribute was used.

Meanwhile, there was an argument used for both stances. The continual change (F) had initially been used to justify that a rectangle is a parallelogram. However, Usu used it to support a No view: “In a parallelogram, the lengths of sides can be changed by a rotation. But we can’t rotate the sides in the case of a rectangle.”

Then the students began to metonymically express a figure transformation as a change of its attribute: “When the sides or angles become the same, a parallelogram becomes a rhombus or rectangle. That means that the adjacent sides or angles become the same” (H: Metonymical thought). However, this thought evoked their tacit

properties, and it worked for those who thought No: “If I am asked whether the adjacent angles can be the same, I think the answer is no,” “a rectangle is made of 90 degrees and a parallelogram is not 90 degrees.” In overcoming this situation, Fuji put forth the idea: “I don’t think that it’s good to be committed to one angle. Instead, as the sums of the adjacent angles are both 180 degrees, we should consider that they belong to the same class” (I: Broad view). However, this was soon refuted by Mizu: “that makes no sense, because the sum of 180 degrees works out for all.”

Rhombus-square relation

The rhombus-square relation was recognized by many students (Yes (19), No (3), and in between (7)). The new argument brought up during the discussions on this relation was: “They are different because the names are different. That is because some scholar gave them names to show they are different things.” (J: Difference by naming) (Ito, 1978). However, the counterstatement was put forth: “The different names are necessary because it’s wrong to call a rhombus a square.” (K: Naming based on general-special relationship)

Rectangle-square relation

In the third lesson, many of the students indicated No in their view of the square and rectangle relation, i.e. “a square is Not a rectangle” (Fig. 8. Yes (2), No (12), and in between (17)). This tendency is common in Japan (Okazaki and Fujita, 2007).

Here, as they had already identified many arguments, most of these reappeared in the students’ discussion. The most frequent argument was that a rectangle has different adjacent sides (B). Meanwhile, the argument of the inclusion view also had an impact: “When we think in terms of a broader view, they are of the same sort because all inner angles are 90 degrees” (I). At this point, four students moved their name card in the Yes direction.

Comparing among relations

Last, the teacher asked students to compare four relations and tried to make them aware of their contradictions, as they tended to see rhombus-parallelogram relations and square-rhombus relations as inclusive, and rectangle-parallelogram and square-rectangle relations as different. In doing so, he introduced Usu’s statement:

“It’s contradictory to consider some figures as the same and the other figures as different.” Furthermore, he proposed to view these relations consistently: “Let’s think of all relations as being either the same or different.” After this, the students could



Figure 8

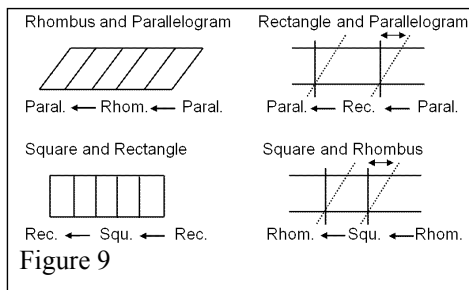


Figure 9

only restate the arguments already put forth from both stances, but could not examine the relations among the geometric relationship pairs.

However, in the fourth lesson, the students began to recognize “the similarities of dynamic changes” between rhombus-parallellogram and square-rectangle, and between rectangle-parallellogram and square-rhombus (L: Similarity between the movements). Taking this opportunity, they noticed that on many occasions they had inconsistently regarded one relation as Yes and the other as No for the same types of relations. We found that it was more natural for them to see how movements were alike than to see how relations were alike. Furthermore, Usu stated that “if we don’t conceive of them in a consistent way, we can’t persuade others” and many students agreed (M: Inquiry of consistency and conviction of others). It seemed that inquiry of consistency and conviction of others developed hand in hand.

DISCUSSION

We have identified the students’ arguments using labels A to M. We also find it useful to classify them into three types.

The first type is arguments based on the similarities or differences between the properties of figures. We here include the similarity or difference in isometric changes (A, D), the difference between symbolic parts (B) and the same properties and a combination of them (C, G). Difference by naming (J) may also be included in this type. Here, we note that nonessential attributes are sometimes included in the students’ arguments.

The second type of argument supposes the general-special relation. It is basically stating the properties of the general figure by pointing to a particular aspect (E). We include in this type the argument of a broad view (I) where students attempted to consider an overall feature rather than regard the tacit properties as minor parts. We also include in this type namings based on general-special relationships (K) (de Villiers, 1994).

The third type of argument is based on the consistency among the relations and the conviction of others (M), where these may emerge as two sides of a coin. These types of arguments constitute layers where the former is the premise for the latter. Since the generalization, the consistency and the conviction of others are also the characteristics of mathematical proof, they suggest a partial transition from an empirical to a deductive proof scheme (Harel and Sowder, 2007), namely from elementary to secondary geometry.

Many arguments are based on dynamic views. The appearance of two figures related by a continuous transformation and the isometric change of areas were the basis for the students’ arguments. Seeing a figure transformation as a change of some attribute sometimes evoked a broader view. Furthermore, the similarity among relations was recognized through an awareness of the similarity of the movements.

We do not believe that the last type of argument is the upper limit for the fifth grade students on the route to deductive geometry. It is a key for them to enhance their recognition of mathematical definitions and concise definitions in particular (de

Villiers, 1994). We have already conducted a third stage of experimentation in which the purpose is for the students to come to know the meanings of the definitions of geometric figures through constructing them by themselves. Thus, our future task will be the analysis and eventual clarification of this overall process. We hope it will result in students enhancing their progression from elementary to secondary geometry.

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FRACTIONS ON A DYNAMIC NUMBER LINE

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This paper provides an example of the potential of a Dynamic Number Line to provoke powerful mathematical thinking in two fifth-grade children about the relative size and position of fractions. The research is part of the evaluation of the Dynamic Number Project, a federally funded research and development project that is currently designing and testing dynamic computer tools for students and teachers. An Integrative Framework drawing from Constructivism, Instrumentation Theory and Semiotic Mediation is used for the analysis of both the tool and the children's mathematical thinking (as they are both shaped by the use of the tool). The analysis indicates that the interactions among teacher, task, tool and students did provoke new mathematical thinking on the part of one student and perhaps both.

INTRODUCTION

We now live in the 21st century, although you might not realize that fact if you were a student sitting in a “typical” mathematics class in most rural or urban school districts in the USA or in other so-called “developed nations” around the world. Outside of school, technology tools and their applications are an integral part of modern life. We use and depend on them for entertainment, information, communication, transportation, commerce, research, comfort, shelter, safety, food production, medical treatment, as well as creative, self-expression. The explosion in web-based resources for finding information, for social networking, for entertainment and for collaborative problem solving in on-line communities has changed the way we live our lives – outside of school. Perhaps one powerful reason for why almost a third of the students entering high schools in the USA “drop out” before completing their high school diploma (Gonzalez, 2010) is that education in many schools is presented in the same way as it was in the 19th and 20th centuries. The educational process in school bears little resemblance to how people learn outside of school. As educators, we need to investigate how children and young adults are making use of the technological environment in which they live and what they are learning from that use. As mathematics educators, we need to understand how we might harness this technological environment to enhance the learning and teaching of mathematics – both in-school and out-of-school.

Purpose of the Dynamic Number Project.

Dynamic Number is a research and development project undertaken by KCP Technologies, the developers of *The Geometer's Sketchpad* software. Its purpose is to extend the functionality of Sketchpad to grades 2–8 mathematics, specifically for the study of integers, fractions, decimals, real numbers, operations, and early algebra.

Through the development of software tools that allow for the direct manipulation and construction of integers, fractions, decimals, and variables, the Dynamic Number project aims to produce powerful, classroom-tested software tools and accompanying curricula to address these pivotal concepts in early and middle-grades mathematics.

Theoretical Framework

What are appropriate theoretical frameworks for investigating the use of such a dynamic tool with children? Drijvers, Kieran and Mariotti (2009) provide an historical overview of theoretical perspectives they consider relevant to integrating technology into mathematics education. They pay particular attention to *Instrumentation Theory* and *Semiotic Mediation*, but make “a plea for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives.” (p. 89)

Instrumentation Theory

Many European researchers (especially in France) have adopted and adapted *Instrumentation Theory* (Verillon & Rabardel, 1995) for their research on the use of technological tools. Central to this theory is the process of *Instrumental Genesis* -- How a tool changes from an artifact to an instrument in the hands of a user, and how both the tool and user are transformed in the process. Kieran and Guzmán (2005) describe this process as follows: “A tool, which starts out merely as an artifact, becomes an instrument for the user only when he or she has been able to appropriate it for himself or herself and has integrated it fully within his or her activity.” (p. 36) They explain further: “in this process of transforming the artifact into an instrument, the learner is not just simply learning tool-techniques that permit him or her to respond to given mathematical tasks. Mathematical concepts codevelop [sic] while the learner is perfecting his techniques with the tool.” (p. 36). Following from the work of Artigue (2002) and Lagrange (2000) they adopt a dialectical interaction triad of *Task*, *Technique*, and *Theory* that serves as their conceptual framework for their study of the role of technology in the development of mathematical thinking. They propose that tool-techniques constitute a bridge between tasks and the emergence of theoretical (i.e. mathematical) knowledge. “It is by looking at the *techniques* that students develop with their technological instruments, in response to certain tasks, that we obtain a window into the evolution of their mathematical thinking.” (Kieran and Guzmán, 2005, p.36)

Semiotic Mediation.

The notion of *semiotic mediation*, according to which cognitive functioning is intimately linked to the use of signs and tools, and affected by it was introduced by Vygotsky (1978). Elaboration of this notion with respect to both mathematics learning and the use of technological tools has proved to be a useful theoretical

framework for many researchers. Drijvers et al (2009) describe a semiotic approach to mediation as follows:

The mediating potential of any artifact resides in the double semiotic link that such an artifact has with both the meanings emerging from its use for accomplishing a task, and the mathematical meanings evoked by that use, as recognized by an expert in mathematics. In this respect, any artifact may be considered both from the individual point of view — for instance, the pupil coping with a task and acting with a tool to accomplish it — and from the social point of view — for instance, the corpus of shared meanings recognizable by the community of experts, mathematicians or mathematics teachers. From a socio-cultural perspective, the tension between these two points of view is the motor of the teaching-learning process centered in the use of an artifact. (pp. 116-117)

Thus any artifact (including those belonging to our new technologies) may offer valuable support to the learning of mathematics according to its *semiotic potential*. These researchers recommend that the determination of the semiotic potential of any learning tool should be an element in the design of any pedagogical plan centered on the use of that tool. They also suggest that:

The construct of instrumental genesis, discussed above, provides a crucial contribution to such analysis. As long as the evolution of personal meanings is related to the accomplishment of a task, it can be analyzed in terms of instrumental genesis, that is, meanings may be related to specific utilization schemes that themselves are related to the specificity of the tasks proposed to students. Thus, an instrumental approach becomes fundamental not only in the identification of semiotic potential but also in the design of appropriate tasks, as well as in the interpretation of pupils' actions and 'speech' acts. (p. 117)

An Integrative Framework

The frameworks discussed above focus mainly on the interaction of the learner and the tool. We need to take into account the role of the teacher (or more experienced other) in the didactical situations made possible by the integration of technology. Olive and Makar (2009) focus on the mathematical knowledge and practices that may result from access to digital technologies. They put forward a new tetrahedral model derived from Steinbring's (2005) didactic triangle (see Figure 1) that integrates aspects of instrumentation theory and the notion of semiotic mediation. This new model illustrates how interactions among the didactical variables: student, teacher, task and technology (that form the vertices of the tetrahedron) create a space within which new mathematical knowledge and practices may emerge. Olive and Makar state "It is not arbitrary that we place the student at the top of this tetrahedron as, from a constructivist point of view, the student is the one who has to construct the new knowledge and develop the new practices, supported by teacher, task and technology." (p. 168)



Figure 1: The Didactical Tetrahedron (from Olive & Makar, 2009, p. 169)

RESEARCH METHODOLOGIES

Appropriate methodology for any research study depends primarily on the research question to be investigated and the theoretical framework within which the research is being conducted. As my primary role in the Dynamic Number Project is to evaluate how students make use of the various tools and what they may learn from that use, I need to use methodologies that focus on *design and use of tools* as well as student *learning*.

Design and use of Technology.

Research on how the specific interface design of a technology impacts its use is critical for understanding the ways in which humans interact with the technology. Investigating how the different uses of technology affect learning and teaching outcomes should also be a concern of the design research. Methodologies that focus on *how* the technology is used (by students and teachers), and not just whether it is used or not, are essential for understanding any affects on outcomes. Instrumentation theory provides an appropriate theoretical framework for investigating how technology is used and shaped by that use.

Learning

From a constructivist perspective, research related to how and what students learn through use of technology should aim to build models of “epistemic students” (Steffe and Olive, 2010) that would be useful for teachers, parents, software designers, designers of learning environments and policy makers. As Steffe and Olive point out, the researcher needs to be engaged with the students in a teaching-learning situation in order to build second-order models of students’ ways and means of operating when engaged in challenging mathematical tasks with the aid of technology. I, therefore, employ a modified *constructivist interview* as the means by which I investigate both how the students make use of the dynamic number tools and what they learn from that use.

RESULTS OF THE FIRST INTERVIEW

The first Dynamic Number tool to be used with students in the project is called *scooting tick marks*. A fifth grade teacher in a small rural elementary school had used the tool to help her class of students make connections between fractions and decimals on the dynamic number line. I followed up on this class lesson by interviewing two pairs of students from her class outside of the classroom for 45 minutes each, two weeks after the lesson. I used a laptop computer enabled for use of the *scooting tick marks* tool. The *scooting tick marks* tool is a dynamic number segment, labelled from 0 to 1 on which a tick mark is located, indicating the position of a fractional number determined by two dynamic *Sketchpad parameters*, arranged in the form of a vertical fractional number. Figure 2 illustrates the view the students saw when being interviewed. When the numerator or denominator of the fraction is changed, the tick mark moves, or “scoots” automatically to the new value.

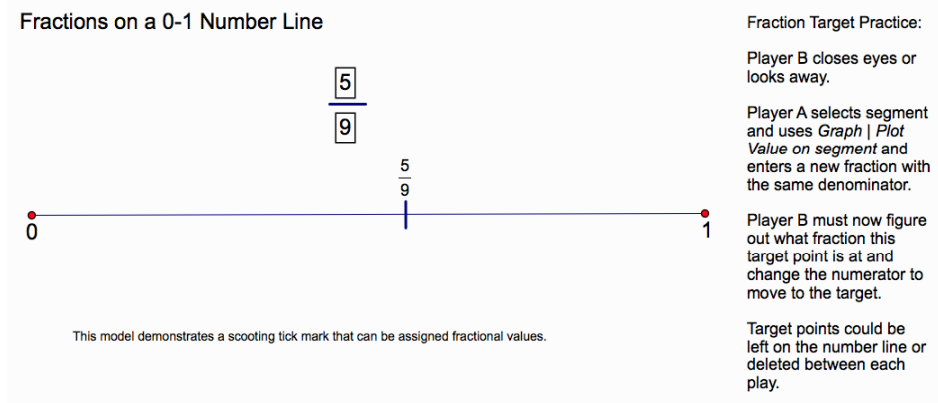


Figure 2: Scooting Tick Marks screen from first interview

I began the interview by asking the students (Eddy and Jason¹) what they saw on the screen. They used appropriate mathematical terms to identify the number line segment and the fraction ($\frac{5}{9}$) as well as the position and label for the tick mark.

The interviews were videotaped and the computer screen and video of the participants captured in real time using *ScreenFlow*. This screen capture application creates a real-time movie of your computer screen and can also insert a secondary video feed into a rectangle in the bottom right-hand corner of the movie. For our interviews, this secondary video feed was provided by a video camera focused on the two students and myself, to capture the interactions among us.

¹ All names are pseudonyms.

The first task in the interview was to play the *Target Practice* game indicated in Figure 2. One player (Player B) closes his eyes while the other player (Player A) uses the Graph menu from the *Sketchpad* menu bar to plot a target point on the unit segment. Player B then had to estimate the fractional position of the target. The understood rule was that the target was at a fractional point with the same denominator as the present position of the tick mark. Both students took turns in being Player A and Player B. Eddy's first attempt was right on target and Jason was very articulate in explaining how he estimated the position of his target.

An example of how the students shaped the tool and how the tool helped shape their understanding of fractional relationships came about 5 minutes from the end of the interview after a couple of near misses and successes. An observer had posed a new target, having changed the denominator to sixths and moved the tick mark to the $\frac{2}{6}$ -position. Figure 3 shows the arrangement of the screen at this point in the interview.

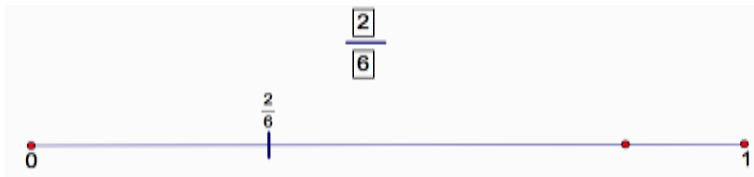


Figure 3: New target for the scooting tick marks game

I explained that we were now using sixths and that the tick mark was at $\frac{2}{6}$. I asked “where do you think the target is?” The two boys stared intently at the screen for 10 seconds, deep in thought, before Jason reiterated the question “Where do I think the target is?” I pointed to the extra point on the segment towards the right end of the unit segment. After another 2 seconds, Jason said “five sixths” followed by Eddy also saying “five sixths”. Before checking their answer by changing the numerator parameter on the screen, I asked them why they thought it was $\frac{5}{6}$. Eddy said he just made his best estimate. Jason explained as in the following transcript:

Transcript from Interview with Jason and Eddy: Where is the Target?

Jason: I can tell you why I thought of it. First, I was thinking $\frac{2}{6}$, $\frac{2}{6}$ in front of zero, then $\frac{2}{6}$. Then I was getting to thinking if you cut that in half that kind of looks right if you... from the (Jason points to the screen).

Interviewer: Show me with the mouse too (so we could see it on the screen capture). So you're saying this is $\frac{2}{6}$ (pointing from the 0 to the $\frac{2}{6}$ tick mark) and if you cut this in half?

Jason: Uh-huh.

Interviewer: And that seems to be about the same as this (pointing from the target to the 1 on the number line segment), and so, why does that tell you it's $\frac{5}{6}$? What's half of $\frac{2}{6}$?

Both students: One sixth.

Interviewer: So why would this be $5/6$? (pointing to the target)

Jason: Because it's one away from the whole (making a sweeping motion with his right hand, indicating the end of the segment).

Interviewer: Which is?

Jason: Five sixths.

Interviewer: Why is it...?

Jason: Six sixths.

Interviewer: Six sixths. Oh, the whole is $6/6$, so one away is $5/6$. Oh! I like that reasoning!

Eddy: Oh, yeah! Yeah! (with a look of surprise and understanding on his face)

Interviewer: (To Eddy) Does that make sense?

Eddy: Yes it does!

Interviewer: So, are you sure now?

Eddy: Yes I am.

In my excitement with the students' reasoning I banged the table and the tick mark jumped to the target, Jason having changed the numerator to 5. We all laughed and congratulated the boys on their correct solutions.

ANALYSIS & CONCLUSIONS

The boys' brief introduction to the scooting tick marks tool was enough for them to provoke an instrumental genesis. The tool became an instrument for them that they could use to support their reasoning. Their reasoning was shaped by the tool (for Jason, especially) in that he was able to use the denominator of the given fraction as a partitioning template (a "connected number" with which to partition the unit segment – Steffe & Olive, 2010) and to estimate a unit fraction as a measurement unit with which he could segment a portion of the number line. I make this conclusion based on his physical actions on the screen. He used the first two digits of his first finger as an estimate for $1/7$ when estimating the target position to be at $3/7$. The scooting tick marks tool was also transformed through these activities in that the unit segment became a partitioned segment (for Jason). The $5/6$ -target episode portrayed in the transcript above provides evidence for how the interaction of "teacher" and the scooting tick marks instrument helped shape Jason's reasoning strategy. The judicious placement of the tick mark at $2/6$ by the observer (teacher at that moment) provoked a multiplicative relation among fraction quantities. Jason estimated that the target's distance from 1 was half of the length from 0 to the $2/6$ -tick mark. He then reasoned that that length must be $1/6$ (half of $2/6$) and that the point that is $1/6$ to the left of 1 must be $5/6$, as 1 is equivalent to $6/6$ and $6/6$ minus $1/6$ is $5/6$. Such reasoning is not that common among 5th graders in American elementary schools (as can be evidenced from the results of the 3-year teaching experiment with third through fifth graders

reported in Steffe & Olive, 2010). In deed, Eddy's surprise at Jason's explanation and eventual understanding of why it made sense, indicates that he had not used such sophisticated reasoning in his own estimation. Olive's and Makar's didactical tetrahedron provides an appropriate framework for this episode (see Figure 1) in that the interactions among Teacher (the observer who set the problem and then the interviewer who asked appropriate questions) the Task, the Tool (scooting tick marks) and the Student (Jason) brought forth new mathematical thinking (for Jason and possibly for Eddy) through the processes of instrumental genesis and semiotic mediation.

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EFFECTS OF FORMATIVE ASSESSMENT IN MATHEMATICS EDUCATION

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The purpose of the research in this paper is to evaluate the effectiveness of formative assessment on student achievement. For an experiment, an intervention was designed that involves eight degree students of a randomly chosen middle school located in Istanbul. 27 students, 14 of which are the experimental group and the rest is the control, took parts as the subjects and “pre-test, post-test with control group” was used as the experimental model. Maths and retention tests were conducted in the end to draw the conclusion and statistical analysis of the results of those have shown that formative assessment is positively influential on math achievements as well as memorizing capabilities of students.

INTRODUCTION AND THEROTICAL FRAMEWORK

The subject of assessment has always been complicated and important for teachers. Teachers try to discover what students understand, learn and recall with the help of traditional assessment tools. The tests that teachers administer sometimes offer poor content coverage. According to Kamphaus (1991), it is “nothing short of appalling” (p. 301). Resnick (1987) noted that teacher tests emphasize a narrow range of cognitive skills that are often disconnected from what students will face beyond the classroom.

Writing in 1967, Michael Scriven suggested two roles that evaluation might play. One is that, “it may have a role in the on-going improvement of the curriculum” (p. 41); in another role, the evaluation process may serve to enable administrators to decide whether the entire curriculum, refined by use of the evaluation process in its first role, represents a sufficiently significant advance in the available alternatives to justify the expense of adoption by a school system (pp. 41–42). He then proposed “to use the terms ‘formative’ and ‘summative’ evaluation to qualify evaluation in these roles” (p. 43).

Benjamin Bloom (1969) suggested 2 years later that the same distinction might be applied to the evaluation of student learning—what we today would tend to call assessment. He acknowledged the traditional role that tests played in judging and classifying students, but noted that there was another role for evaluation:

Quite in contrast is the use of “formative evaluation” to provide feedback and correctives at each stage in the teaching-learning process. By formative evaluation we mean evaluation by brief tests used by teachers and students as aids in the learning process. While such tests may be graded and used as part of the judging and classificatory

function of evaluation, we see much more effective use of formative evaluation if it is separated from the grading process and used primarily as an aid to teaching (p. 48).

Explicit in these early uses is that the term formative cannot be a property of an assessment. As Bloom (1969) made clear, the same tests could be used for formative or summative purposes, although he suggested that the formative use will be less effective if the tests are part of the grading process. The crucial feature of formative evaluations, for Scriven (1967) and Bloom (1969), is that the information is used in some way to make changes. Whether it is a curriculum or student achievement that is being evaluated, the evaluation is formative if the information generated is used to make changes to what would have happened in the absence of such information.

As it is stated by Gipps (1994a), the main functionality of formative assessment is that the process of assessment by the teacher is to be assistive in deciding how to present the course material in hand and to yield information about the way students learn concepts and about how to teach them more effectively.

Black and Wiliam (1998a) conducted an extensive research review of 250 journal articles and book chapters winnowed from a much larger pool to determine whether formative assessment raises academic standards in the classroom. They concluded that efforts to strengthen formative assessment produce significant learning gains. Gains are measured by comparing the average improvements in the test scores of the students who were involved in the innovation with the range of scores found for typical groups of students on the same tests. Effect sizes ranged between .4 and .7, with formative assessment apparently helping low-achieving students, including students with learning disabilities, even more than it helped other students (Black and Wiliam, 1998b).

Formative feedback

Feedback given as part of formative assessment helps learners become aware of any gaps that exist between their desired goal and their current knowledge, understanding, or skills and guides them through actions necessary to attain the goal (Ramaprasad, 1983; Sadler, 1989). The most helpful type of feedback on tests and homework provides specific comments about errors and specific suggestions for improvement and encourages students to focus their attention thoughtfully on the task rather than on simply getting the right answer (Bangert-Drowns, Kulick, & Morgan, 1991; Elawar & Corno, 1985). This type of feedback may be particularly helpful to lower achieving students because it emphasizes that students can improve as a result of effort rather than be doomed to low achievement due to some presumed lack of innate ability. Formative assessment helps support the expectation that all children can learn to high levels and counteracts the cycle in which students attribute poor performance to lack of ability and therefore become discouraged and unwilling to invest in further learning (Ames, 1992; Vispoel & Austin, 1995).

The feedback taken by the students is the most important component of the process. Providing feedback, as a tool in this process, has to be employed powerfully for

formative assessment to be positively influential on learning. Moreover, the assessment shows itself in the communication of student and teacher by the existence of this feedback and attains the goal by the help of effective use of it (Türnüklü, 2003).

The main purpose of this study is to research whether formative assessment affects on student success or not. In order to research this effect, two groups of students in the same semester with no significant difference in mathematical success and attitudes are chosen and several applications are held not to evaluate students with just the methods considered classical methods such as tests, true-false questions, fill in the blank questions..., but to help develop assessment methods which can reveal their meta-cognition, improve their mathematical attitude and increase their level of mathematics success.

RESEARCH QUESTIONS

“Pre-test, post-test with control group model” was used as the model for the experiment involving 27 participants. In particular, two questions were raised: 1) does formative assessment affect students’ math success, and 2) does formative assessment affect retention.

METHODOLOGY

Sampling

Convenience sampling method was used for collecting data in a primary school.

During the application, students with parental permission for being in the school after classes, with no transportation problems and with self-motivated to attend these activities were chosen. 14 volunteer students set the experimental group and 13 non-volunteers set the control group.

Research Process

While starting the research, in order to state the level of knowledge about the combination and permutation subject of the experimental and the control groups, a pre-test was made. In order to determine mathematical attitudes, mathematical attitude scale was implemented. In order to examine the differences after the research a post-test was made. To determine whether the students remember the subject or not, a retention test was made eight weeks later. In the beginning, mathematical attitude scale was implemented for both of the groups, but worksheets were only used in the experimental group.

The experiment took five weeks, but with retention period of eight weeks, total process time was about 13 weeks.

Mathematical attitude scale was developed by Erkin & Nazlıççek (2002).

Both in the classes and after the classes, worksheets were given to students. These worksheets included regular test questions about the subject and also essay questions aiming to learn students’ thoughts on the subject. After students completed

worksheets, teacher collected them and gave feedback. They were ruled to re-submit the sheets according to feedback. According to their needs, one to one or group studies were made after classes with the students who showed necessity after the second collection of sheets. Worksheets or tests not given to others were given to these students.

RESULTS AND DISCUSSION

Results

To analyse effect of formative assessment on math success and retention, students' maths achievements test scores were used. Problems in the research were explained in tables along with the data provided by quantitative analyses. Cronbach's Alpha was set to 0.05.

Table 1 shows differences of achievement scores between two groups before and after using formative assessment. Analysis of two groups' achievement in the beginning is necessary because difference with regarding to Alpha may affect results. Considering table 1, pre-test results ($p > \alpha$; $p = 0.281$ and $\alpha = 0.05$) indicate that there is no significance difference between experimental and control groups. However, according to post-test scores ($p < \alpha$; $p = 0.000$), there is an important difference between two groups. In addition, experimental group post-test average is bigger than control groups, so, it is clear that experimental group is successful.

	N	\bar{X}	sd	t	p
Experimental Group (pre-test)	14	41,14	14,109	1,101	0,281
Control Group (pre-test)	13	35,85	10,463		
Experimental Group (post-test)	14	74,43	17,814	4,658	0,000
Control Group (post-test)	13	47,54	11,148		

Table 1: Results of pre-test and post-tests of two groups

Scores in table 2 is about both pre-post test of experimental group. $p < \alpha$; $p = 0.000$, hence, difference between two scores is meaningful. Increase of average in scores could be the result of formative assessment. Correlation coefficient r indicates positive correlation; meaning that student with high point in pre-test tends to have high point in post-test as well.

	N	\bar{X}	sd	t	p	r
Experimental Group (pre-test)	14	41,14	14,109	-8,641	0,000	0,614
Experimental Group (post-test)	14	74,43	17,814			

Table 2: Results of pre-test and post-tests of experimental group

Results in table 3 are similar to the ones in table 2; difference is significant between pre-test and post-test of control group ($p < \alpha$; $p = 0.000$) and correlations are positive. Comparing table 2 with table 3, formative assessment is observed to be non-effective. However, table 1 shows significant difference (post-test) between two groups which suggests that formative assessment could be effective.

	N	\bar{X}	sd	t	p	r
Control Group (pre-test)	13	35,85	10,463	-5,622	0,000	0,761
Control Group (post-test)	13	47,54	11,148			

Table 3: Results of pre-test and post-tests of control group

		N	\bar{X}	sd	t	p	r
Experimental (post-test)	Group	14	74,43	17,814	2,000	0,067	0,863
Experimental (retention test)	Group	14	69,43	17,917			
Control (post-test)	Group	13	47,54	11,148	5,089	0,000	0,799
Control (retention test)	Group	13	38,08	8,995			
Experimental (retention test)	Group	14	69,43	17,917	5,674	0,000	
Control (retention test)	Group	13	38,08	8,995			

Table 4: Retention test results

According to analysis of retention test which can be seen in table 4, in control group, students start to forget the subject. There is no important difference between post-test and retention test scores of experimental group. This means, these scores are similar and students do not forget the subject at an important level. In addition, two groups' retention test scores show meaningful difference. Average of scores also indicates that students in experimental group are more successful than others because they can remember the subject well. Both groups have positive correlations; hence, if one student is successful in post-test, s/he is also successful in retention test.

Discussion

It is observed that, experimental and control group students differ from each other in terms of mathematical success. Data collected before the experiment, were showing no difference between the groups in terms of mathematical success. However, when the pre-test and post-test analysis were made separately for experimental and control groups, significant differences were viewed in the tests made before the subject started and after the subject ended. Therefore, significant difference between the post-test data of experimental and control groups can be considered as the result of formative assessment. The difference between the control group's pre-test and post-test data can be caused by that the students had just learned the subject. Those results correspond to Özçelik (1992:196) show that using feedback and mistake correction help to increase time for active learning and it is possible to try the behaviour mostly which is meant to teach. Change in time for teaching-learning and usage of other sources may increase the pace of students' learning. When learning pace increases, efficiency of learning also increases.

To test the effect of formative assessment on retention, post-test was distributed to students again as retention test to two groups 8 weeks after post-test. Having in mind that using same test as retention test with post-test may affect recalling, according to test scores, while students in experimental group were successful in retention test, others could not achieve it although all of them had same questions with post-test. When retention test results of two groups are compared, it is clear that difference between two groups is meaningful and average of experimental group is higher at important level.

Results mentioned above are mostly similar to other studies addressed in the paper. However, this paper also indicates the relation between formative assessment and retention. In addition, this study would be helpful to motivate teachers to use formative assessment and formative feedback.

IMPLICATIONS FOR TEACHING

Although formative assessment is a time-consuming and endeavour activity for teachers, it can be used in all subjects and all lessons. Feedback is one of the main properties of formative assessment. Teachers can make an effort to give clear feedback especially immediately because feedback is helpful when it is given at the same time with activity. It is important that to apply formative assessment teachers

may need extra time except duration for lessons. Therefore, teachers may use after-class activities to complete curriculum properly. Feedback is also helpful for teacher to reshape the lecture. If students do not understand the subject well, extra time may be spent in class or devoted to specific students who are not clear about the subject, or, if whole class learn the subject well, duration of the subject could be shorten. Feedback also can indicate that whether teacher is successful at teaching the subject or not. S/he can try to develop teaching techniques and study the subject more. It is a kind of self-criticism for teachers.

With the help of formative assessment, teachers can step in to students immediately when they do not understand well, not just before exams. It is easy to see where students have problems with subjects. In addition, due to the positive effects of formative assessment on retention, students could be more successful on exam which are done at the end of the year like SAT or SBS (exam for defining students' level).

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PROSPECTIVE MATHEMATICS TEACHERS' VIEWS ON THE USE OF COMPUTER ALGEBRA SYSTEMS

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This study investigated the views of Turkish and U.S. prospective mathematics teachers on the use, in algebra instruction, of advanced calculators with Computer Algebra Systems (CAS). The possible roles for CAS suggested by Heid and Edwards (2001), along with the black and white box dichotomy were used as a conceptual framework. Participants' responses to an open-ended questionnaire and group interviews revealed their views and beliefs about possible uses of CAS. Results also revealed the similarities and differences between Turkish and U.S. participants' views regarding the effects of using CAS on teaching and learning algebraic manipulation.

INTRODUCTION

Even though the Computer Algebra Systems (CAS) on handheld calculators might create new promise for teaching and learning algebraic manipulation, they have not received the same attention in algebra instruction as the use of the graphing and statistical capabilities of the calculators. Perhaps one possible explanation of slow or no adoption of CAS is rooted in the beliefs that teachers hold regarding the powerful algebraic capabilities of CAS and their effects on secondary school algebra teaching and learning. Using CAS, with a few key presses, students can solve equations, factor and expand very complex algebraic expressions. These new capabilities and potential different uses of CAS bring many debates with them, such as possible positive and negative influences of CAS on the curriculum, learning, and teaching of school algebra.

Research Questions

In 2005, at the Fourth Computer Algebra in Mathematics Education (CAME) Symposium, the Teachers and CAS Theme Group Discussion highlighted the call for obtaining and studying teachers' views of CAS. This issue was echoed at the Fifth CAME Symposium in 2007 by Böhm, "CAS in the classroom is impossible without teachers who like to work with CAS and who hopefully are using CAS not only as calculation tool" (p. 11). In this quote, Böhm emphasizes not only teachers' attitudes but also their technological pedagogical knowledge in using CAS in their classrooms. The current study investigated the views of Turkish and U.S. prospective mathematics teachers on the utilization of advanced calculators with CAS in algebra instruction before and after having a brief experience with CAS.

The research question of this study was: What are Turkish and U.S. prospective teachers' views of CAS in algebra instruction, especially their views of black box and white box uses of CAS?

CONCEPTUAL FRAMEWORK

This study employs two conceptual frameworks —the black and white box dichotomy of CAS in algebra teaching and learning, as well as the possible roles for CAS suggested by Heid and Edwards (2001). The white box and black box dichotomy as different approaches of using CAS in a teaching and learning environment is well established in the mathematics education literature. Buchberger (1990) explains the black box approach as when CAS produces algebraic manipulations without showing their inner workings. In the black box phase, algebraic manipulations involving more than one step can be done all at once using symbolic computation software systems with intermediate steps hidden from the user. On the other hand, in the white box phase, algorithms must be studied thoroughly in the CAS environment and intermediate steps are not hidden (Buchberger, 1990).

The second part of this framework is adapted from Heid and Edwards' (2001) list of four possible roles for CAS in mathematics education: CAS as computational and transformational tool acts as the primary producer of symbolic results in which CAS makes computations in order for the user to focus on the concepts; CAS as an assistance for students to generate many examples in order to search for symbolic patterns; CAS as a generator of results for problems posed in abstract form; and CAS as a pedagogical tool, which creates and generates symbolic procedures to assist students in constructing conceptual understanding (pp.130-132).

Figure 1 displays the framework of this study. CAS as a computational/transformational tool is placed on the intersection of the framework since it could be used in both black and white box approaches. This is the first role defined by Heid and Edwards (2001), which could be used “to liberate students ‘from the technical aspects of paper-and-pencil computing’ while encouraging them ‘to keep sight of the main goal’” (Lagrange, 2005b, p. 118).

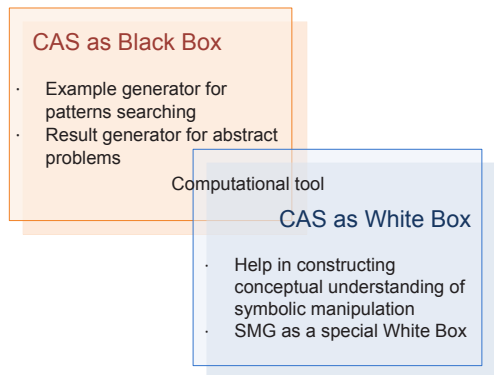


Figure 1. Conceptual framework for CAS Methodology

CAS, especially in the black box approach as an example generator, could be used for pattern searching/discovery (Schneider, 2000; Stacey, Kendal, & Pierce, 2002). So students' focus would be on input and output, but not intermediate steps. CAS as a result generator could be used to check one's hand-calculated responses or to have quick results for complicated algebraic manipulations during a problem solving process where the aim is not to teach the algebraic manipulation. CAS as a white box can support students in constructing conceptual understanding of symbolic manipulation. Here CAS, as a pedagogical tool, behaves like a guide helping students

to focus on the concept behind algebraic manipulation rather than the actual algebraic calculations. In this study, Symbolic Math Guide, which provides extra guidance to users, is used as a special version of the white box approach (in addition to the general white box).

DATA COLLECTION METHODS AND ANALYSIS

The first setting of this study was located in a 5-year teacher education program in Turkey and the other was in a university in the Midwestern United States. Both samples were selected as a result of convenience typical case sampling, which “highlights what is normal or average” (Miles & Huberman, 1994, p. 28). Turkish participants included twenty-seven prospective secondary mathematics teachers. They had no prior experience with advanced graphing calculators in their K-12 education. However, advanced (not graphing) calculators were used in their numerical analysis course for computational purposes. They had a chance to experience some graphing activities with the graphing calculators in their mathematics methods courses. However, they did not have an opportunity to use these calculators in their field studies or student teaching. Therefore their views were mainly based on the experiences they had in the last part of their teacher education program, which included mostly methods and education courses.

The U.S. setting included 22 students working toward mathematics teacher certification. U.S. participants provided a diverse background in their education when compared to Turkish participants whose undergraduate and graduate education was identically the same. U.S. participants also represented a range of backgrounds, experiences, and technological expertise regarding graphing calculators. Most of the U.S. participants categorized themselves as intermediate level users of graphing calculators. Many of them had used graphing calculators when they were in high school or at college when learning mathematics. Even though the selection of these two countries was a result of convenience sampling, comparing the naïve views provided by Turkish prospective teachers to those of their more experienced American counterparts resulted in interesting results, especially when identifying similar views in spite of the different experience levels with technology in general.

First, participants were presented with an activity for solving linear equations in which Buchberger’s black box and white box ideas were introduced. Additionally, Symbolic Math Guide (SMG) which is a special white box was introduced. SMG is an application created by Texas Instruments for their TI-89 and TI-92 calculators only at this time. This application aims to help students with symbolic and algebraic transformations by providing step-by-step transformations. After these introductory activities, participants were asked to respond in writing to discuss the utilization of calculators with CAS capability in algebra instruction, the advantages and disadvantages of black box and white box in a teaching and learning environment, and so on. Then, all participants participated in semi-structured group interviews in which the same questions from the writing portion were asked. These activities lasted approximately for one three-hour blocked class period out of 14 classes.

Due to the nature of qualitative data, the analysis was based on categorizing in order to investigate the emerging themes throughout. Checklist matrices were created in order to analyze participants' views on the use of CAS in algebra instruction. According to Miles and Huberman (1994), "the basic principle is that the matrix includes several components of a single, coherent variable" (p.105). Therefore, several elements of the participants' views on the advantages, disadvantages, and possible utilizations of black box and white box (including SMG) were revealed and compared. After participants' individual writings were analyzed for common codes to create patterns using the CAS, the group interviews were analyzed according to those codes. Additionally, dialogs in the group interviews were examined for rival explanations. Data triangulation and member checks were used to ensure the trustworthiness of this study. Two forms of data collection with different encounter levels provided rich sources of data. Moreover, the synthesis of two conceptual frameworks also strengthened the data analysis. Member check questions during the interviews were used in order to ensure correct understanding of what the participants meant.

RESULTS

Three main themes resulted from the data analysis. Teachers viewed CAS as an interesting and potentially effective tool. However, some teachers preferred to use CAS after students have mastered skills, especially for the black box use. Participants perceived the use of CAS as a white box as more of a pedagogical tool than CAS used as a black box. Participants' responses regarding the effects of white box on students' learning and how to use CAS in teaching algebra differed somewhat and provided a spectrum of views on the use of CAS in algebra teaching and learning.

Learning with CAS

Mainly participants suggested that black box could not be used as a pedagogical tool. Thirty-six percent of the U.S. and 52% of the Turkish participants believed the main weakness of this method is that students would not comprehend what was happening, mainly because they would not be aware of the intermediate steps: "They are not learning the processes...You are not learning what is going on. If you have a problem and you get an answer...Actually they are not learning anything" (U.S. Group Interview). Thirty-three percent of the Turkish participants believed that the black box method would lead to student memorization rather than meaningful learning. Prospective teachers mainly see black box as a "generator of results for problems" as in the CAS conceptual framework. Participants liked that the black box could provide fast accurate calculations when students know the subject and just need the result. Participants emphasized the potential for fast calculation, but they did not mention that the time saved could be used for concept development or interpretation as proposed in the literature (Heid & Edwards, 2001; Lagrange, 2005a, 2005b). Following this use, participants also thought that CAS as black box could be used to check the results of by-hand work. One U.S. and two Turkish prospective teachers implied that when black box is utilized as a "generator of many examples for pattern

searching,” as in the CAS conceptual framework to create an environment for students’ discoveries, it would become a learning tool:

Because the student sees many examples in that way, s/he can directly start discovering by seeing all of the examples at one time. If we had done that on the board one-by-one, it would have taken more time. (Writing-Turkish Participant).

This [black box method] finds the answer quickly, then students can experiment until they figure out how that answer happened (Writing-U.S. Participant).

Some of the U.S. and Turkish participants mentioned that the white box method could become a learning tool with the aim of helping students to comprehend the concepts as in the CAS conceptual framework. Thirty-six percent of the U.S. prospective teachers highlighted the importance of a step-by-step approach which can be accomplished with the white box: “While learning the essential steps in working out an algebra problem, students could easily see which steps would or would not work with the programs white box and SMG at their disposal” (Writing-U.S. Participant). Forty-one percent of the Turkish and 27 % of the U.S. participants believed another benefit of the white box method is the control that students have over this methodology: “The advantage [of white box] is that it does not just give answers and the student has to put forth an effort to obtain answers” (Writing-U.S. Participant). A few participants suggested that students could use white box as a checking tool to strengthen their newly constructed knowledge and reflect on the process of algebraic manipulation. Others mentioned that white box could provide an environment for trial-and-error exploration which might help students to conjecture and reflect on the process: “[The white box method] could be used when learning the material. Students see each step of the operations one by one; therefore it could be beneficial” (Writing-Turkish Participant). A few participants discussed the advantages of this exploratory process, where students could get immediate feedback on their actions from the technology, and sometimes they can experience disequilibrium, which might help their learning: “In this [white box] method, the steps of the operations are controlled by us; therefore we could see where we made a mistake” (Writing-Turkish Participant). This result suggests a revision of the framework of this study. CAS as an example generator for pattern searching/discovery could be placed in the intersection of the framework and could be named as CAS as conjecturing and discovery tool.

A concern voiced by some participants regarding the white box method was that students could become totally lost and would not know what to do next. But it could be the same in a traditional environment: “Students may go in circles, doing unnecessary operations (as they may on paper)” (Writing-U.S. Participant). Another contradictory result was that while some participants liked that the white box does the calculations and allows students to focus on the transformations or algebraic steps, others mentioned this as a disadvantage of the white box method, because “the calculator is doing the basic calculations so students aren’t practicing these” (Writing-U.S. Participant). Those participants, who mentioned the advantage of being

able to focus on the algebraic steps and not be bothered with arithmetical calculations, highlighted the interaction between the uses of CAS as a computational tool and a pedagogical tool in the CAS conceptual framework.

Some participants clearly expressed their preference for SMG over the other two methods. Sixty-three percent of the Turkish and 55% of the U.S. participants discussed that the potential positive influences on students' learning were the list of possible options/steps for algebraic manipulations and the step-by-step approach:

I liked the way it offers choices. Because, if a student thinks of one or two ways of solving the problem, with these choices, the number of methods s/he thinks of would increase, which would broaden the student's horizon (Writing-Turkish Participant)

I think that the white box is a little harder than the SMG, but for kids having a hard time recalling or knowing what to do, it [SMG] gives them a choice. At least they can get started (U.S. Group Interview).

A few warned that this could also become a guessing game. Similar to the general white box, some participants thought that students could experience disequilibrium in the SMG environment when their chosen transformation does not have the intended effect.

Teaching with CAS

When the participants were asked how they would utilize CAS in their classroom, some considered all three roles. Some suggested that they would use SMG first to teach algebraic concepts, and then they would use the white box method for students to make use of and reinforce their newly constructed knowledge. After this was accomplished, the black box method could be used to check work or get immediate results for complex algebraic computations.

I would definitely use the SMG to first expose algebra to students because it is a step-by-step process while giving the students the steps on the screen for them to review. After I would use the white box to get them to solve expressions on their own. Once they totally understand the process I would show them the black box as a short hand process for them to use (Writing-U.S. Participant).

This resembles Edwards's (2001) suggestion of a 'white box first' approach. One U.S. participant offered a different sequence where s/he would start with using the black box for a discovery introduction and then use the SMG for students to "go through step by step until the steps are memorized. Then we would use the white box to see practiced the steps one at a time" (Writing-U.S. Participant). Here s/he followed Drijvers' (2000) view of the use of the black box first to get students' curiosity going in a discovery activity. While the first preservice teacher highlights the importance of guided practice at first, the second teacher prefers to start with a discovery activity to get the thinking going which is followed by guided practice. But in the end, some participants still had some doubts. Twenty-seven percent of the U.S.

participants emphasized that CAS in general could be utilized in the classroom after students have mastered the necessary skills: “They should be used after teaching the material and when the students are proficient at doing equations” (Writing-U.S. Participant). Twenty-two percent of the Turkish participants questioned the use of CAS due the possibility of ineffective consequences or students’ becoming dependent on technology “When used inappropriately, I mean, if students see the procedural steps without making guesses or interpreting, this could weaken their reasoning skills and eliminate the opportunity to develop different strategies” (Turkish Group Interview).

DISCUSSION

Even though U.S. participants had more experience with advanced calculator technology, CAS was a novel instrument for many of them. CAS was certainly a novelty for Turkish prospective teachers, who had very limited experience with the use of calculators in mathematics education in general. However, Turkish and U.S. participants mainly had similar views regarding the use of CAS in algebra instruction; primarily teaching algebraic manipulation. Some suggested the use of CAS after students mastered their skills by paper and pencil. They both again preferred the white box methods to the black box method as a pedagogical tool. There were subtle differences in the viewpoints of Turkish and U.S. participants. U.S. participants focused on more practical issues in their writings and discussions. Some U.S. participants mentioned that SMG providing the steps in words and a better visual environment would help students more. The calculators have been used especially for graphing purposes in the U.S. classrooms at least for two decades. Therefore, U.S. participants tended to have more experiences both as students and student teachers.

In 2005 at the Fourth CAME Symposium, the Teachers and CAS discussion group called for teacher training which would help teachers “to see CAS beyond its potential for generating mathematical answers and develop an image of CAS as a tool for learning” (*CAME 2005 theme group 3: Teachers and CAS: Summary of the theme group discussion*, 2005, p.1). CAME underscores that teachers should be knowledgeable about the potential of CAS, so that they can make informed decisions about using it. Before having the experience with CAS in this study, many of the participants could not imagine how CAS could be used to help students improve their algebraic manipulation skills. Some of them had mentioned that algebraic manipulation skills could only be obtained by paper and pencil, but reconsidered their views after this experience. Some mentioned that their knowledge was extended or their views were confirmed. One of the major possible sources for the knowledge and experience which shapes teachers’ beliefs and practices is their teacher education program. Mathematics teacher educators may help them not only construct and develop their technological, pedagogical, and content knowledge (TPACK) separately but also consider the relationships among those knowledge types; so they can develop and improve their TPACK as a whole (Niess et al., 2009). Having both

practically and theoretically rich experiences where the potential positive or negative effects of CAS on learning, instruction, or curriculum are discussed and reflected on could help teachers to make better decisions regarding the use of in their instruction. In order to be able to do that we would need more research produced on the possible influences of CAS on learning, instruction, and curriculum, as well as students' and teachers beliefs and attitudes with respect to CAS.

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DEVELOPMENT OF PRESERVICE TEACHERS IN LINKING MULTIPLE REPRESENTATIONS VIA TECHNOLOGY

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In this paper, we examine the development of pre-service mathematics teachers' use of multiple representations during teaching in technology-rich environments. The pre-service teachers took part in a preparation program aimed at integration of technology into teaching mathematics. The pre-service teachers' development was scrutinized in terms of their knowledge of and the connections established among the representations. We discuss the educational implications of the study for the design and conduct of successful technology integration programs.

INTRODUCTION

The issue of multiple representations (MRs) are long on the agenda of mathematics education. The MRs became the centre of attention especially after the NCTM standards in 1989. The standards strongly suggested teaching mathematics with different representations so that students would be given different lenses to approach mathematical problems flexibly. Since then, many research attempts were undertaken to investigate the benefit and importance of MRs in teaching and learning mathematics. The researchers repeatedly pointed out that MRs lead to better learning outcomes as they cater for a wider range of students with different ability levels and learning styles (Ainsworth, 1999). Kaput (1989) argues that use of MRs enables one to 'see' complex ideas in a new way and apply them more effectively. Also as different representations stress different aspects of the same concepts, teaching with MRs is found to be useful in making connections (Berthold et al., 2009).

With digital technologies beginning to spread, MRs received a renewed interest. In 2001, for example, the Council's yearbook focused on the roles of representation in school mathematics (Cuoco, 2001). The yearbook attaches considerable importance to the use of digital technologies in establishing links among representations. As NCTM suggests, digital technologies provide visual models or representations that many students are unable to generate through their independent efforts.

Research studies, implicitly or explicitly, regard teachers as an important factor in reaching at the expected benefits from the use of MRs in technology-rich environments (e.g., Alagic & Palenz, 2006). Therefore what teachers know about and how they use MRs as well as how they draw on technology to interrelate MRs become important issues while determining the effect of teaching mathematics with MRs through technology. Despite its importance, there appear few studies on how (pre-service) teachers employ MRs in technology-rich environments (Juersvich et al., 2009; Alagic & Palenz, 2006). These studies, however, do not give much details as to

how pre-service mathematics teachers (PSMTs) use MRs and how their competence to effectively make use of MRs in technology-rich environments can be developed. In this connection, we will share, here in this paper, some of the results of our research project which focused on technology integration process designed for PSMTs. We will report our findings regarding PSMTs' development with regard to use of MRs in technologically rich environments and interconnecting them for a deeper understanding. Before doing this, however, we first describe the project and background of the study in the next section.

BACKGROUND OF THE STUDY AND METHODOLOGY

This paper stems from a research project for which a course was designed for PSMTs to develop technological pedagogical content knowledge (TPCK) (Mishra & Koehler, 2006). TPCK framework has been recently used to investigate the characteristics of knowledge required by teachers for successful technology integration. It was originated from the notion of "Pedagogical Content Knowledge (PCK)" offered by Shulman (1986). As the importance and potential of technology in teaching and learning is realized, Pierson (2001) has included the technology component into the idea of PCK and considered TPCK as a blend of three categories of knowledge: content, pedagogy and technology. We use TPCK framework with its various components to develop contents for two courses (Methods for Teaching Mathematics II and Technology-Aided Mathematics Teaching) as part of a project for PSMTs in Turkey. The aims of these courses were, broadly speaking, to get PSMTs equipped with the skills of teaching mathematics with the aid of technology at secondary level.

This study focuses on a particular component of TPCK, namely knowledge of using MRs of a particular topic with technology. We brought the content aspect into play and aimed to exemplify how this component can be applied with a particular mathematical concept for which we used the concept of derivative. Two workshops were designed: the first aims to develop pedagogical content knowledge (PCK), and the second to develop technological content knowledge (TCK), and (TPCK).

During the PCK workshop, PSMTs were asked to share their existing knowledge of MRs and provided with the knowledge of algebraic, numerical and graphical representations of derivative, the relationships among them, and how to take them into account in teaching. Limitations and affordances of each representation were also discussed. TPCK workshop firstly focused on TCK. Technological content introduced to pre-service teachers is a Turkish version of Graphic Calculus software. The software and an activity book in Turkish (Akkoç, 2006) were given to each PSMTs. Graphic Calculus software provides graphical and numerical representations of derivative at a point which are dynamically linked. During the TPCK workshops, MRs of derivative at a point was introduced. A discussion among PSMT's about the software focused on the following questions:

- What kinds of representations of derivative at a point are available and what kinds of opportunities are there to make links between representations?

- What kinds of opportunities are there to relate different aspects of derivative (derivative as rate of change, derivative as slope and derivative as a limit) using the software?

Forty PSMTs in Istanbul, Turkey, took part in the courses. Completion of the course along with some others would award them a certificate for teaching mathematics in high schools for students aged between 15 and 19. All forty PSMTs enrolled this course which was designed on the basis of TPCK framework as explained hitherto.

During the research, several data collection tools were employed, including: lesson plans, detailed teaching notes and questionnaires. Participants were given open-ended questionnaires before and after the courses to find out their initial understandings of several issues including MRs. The participants were also asked to prepare three lesson plans: one before the PCK workshops, one after the PCK workshops and the last was after the TCPK workshops. All three lesson plans were on the same topic: introduction of derivative. To prepare the plans, the PSMTs were allowed to make use of any textbook they wish and also required to examine the curriculum scripts.

DATA ANALYSIS AND FINDINGS

The concept of derivative at a point can be represented in algebraic, graphical and tabular (numerical) forms. Derivative at a point has three different aspects: (i) the slope of the tangent line to a curve at a particular point, (ii) the limit of the difference quotient and (iii) the instantaneous rate of change (Bingölbali, 2008). All these three are closely interrelated and an understanding of any one of these supports the comprehension of the others. As Ainsworth's (1999) study suggests, constructing a deeper understanding involves interconnecting the MRs of derivative with reference to different aspect of the concept. Hence in our analyses of PSMTs lesson plans, we focused on the MRs of derivative in tandem with its different aspects.

Before the TPCK program started, we wished to find out participant PSMTs' initial ideas on MRs in general. To this end, we collected data via questionnaire where we asked the participants to explain their understanding of MRs. The questionnaire was applied twice: before PCK and after TPCK workshops. Participants' responses involved graphical, tabular and algebraic examples in explaining the meaning of MRs. Before the workshops started there were only 4 (10%) responses within these three categories. The number of PSMTs with regard to these three representations increased dramatically to 36 (90%) after the completion of workshops. Interesting was to see that eight PSMTs considered different symbols used for mathematical expressions as representations of the concepts. After the workshops, however, this misconception disappeared. Before the program, 15 (37.5%) PSMTs left this question unanswered but this figure dropped to none after the workshops. Generally speaking, PSMTs' grasp of the MRs has improved over the course with clear terms.

We now turn our attention to the PSMTs' lesson plans along which participants prepared rather detailed teaching notes. Three sets of lesson plans with teaching notes were analyzed: (1) before the workshops started, (2) after PCK workshops and (3)

following TPCK workshops. In analyzing the plans, we aimed to determine whether PSMTs made connections among three MRs (graphical, numerical and algebraic) of derivative and whether they employed technology for that purpose (see Table-1).

	First plans	Second plans	Third plans
Categories	N (%)	N (%)	N (%)
MRs of derivative (Graphical (G), Numerical (N) and Algebraic (A)) are not linked.	34 (85.0)	9 (22.5)	1 (2.5)
Only one pair of representations are linked (any one of G-N, G-A, or N-A)	3 (7.5)	5 (12.5)	5 (12.5)
Only two pairs of representations are linked (any two of G-N, G-A, and N-A)	1 (2.5)	8 (20.0)	1 (2.5)
Three pairs of representations are linked (pairs of G-N, G-A, and N-A are all present)	0	2 (5.0)	2 (5.0)
All three representations are interconnected to one another (G-N, G-A, & N-A and G-N-A are present)	0	12 (30.0)	27 (67.5)
No response	2 (5.0)	4 (10.0)	4 (10.0)
Total (N=40)	40	40	40

Table-1. Frequency analysis of the links among different representations

It is worth noting that 85% of initial lesson plans did not link MRs of derivative. However, this figure drops to 22.5% after PCK workshops and to a remarkable 2.5% following the TPCK workshops. There is also an outstanding increase in the number of plans which links all three MRs of derivative. There was not a single plan that linked all three MRs of derivative at the beginning; yet this figure increases to 30% and to 67.5% after, respectively, PCK and TPCK workshops. Please note that all the links among the MRs of derivative in the third lesson plans (after TPCK workshops)

were established through technology (Graphical Calculus software).

Before presenting our analysis of PSMTs in relating three aspects of derivative using MRs, let us provide an example from one particular PSMT's third lesson plan in detail. Figure 1 presents a screen-shot using Graphic Calculus

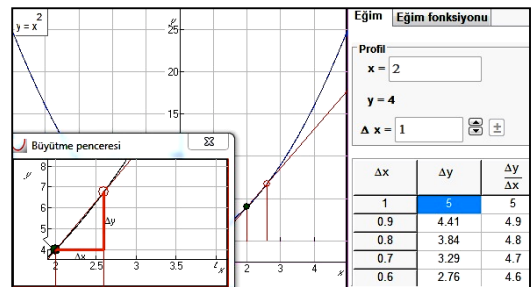


Figure 1. Screen-shot of Graphic Calculus

software taken from the PSMT's lesson plan. As can be seen from the figure, the software provides numerical and graphical representations of derivative at a point and dynamic links between them. As the software calculates the average rates of change, it draws the secant lines spontaneously. The PSMT used this activity in her third lesson plan and explained what she aimed with it in the open-ended questionnaire she filled. She mentioned that the numerical values of rates of change could be represented graphically by the secant lines through pairs of points. She also mentioned that using the zoom-in tool a right triangle could be constructed on which the rate of change represents the slope of the secant line. She later explained that values for rates of change in the table approach to a number and this promotes an intuitive understanding of limit. She added that secant lines approach to the tangent line and rates of change approach to instantaneous rate of change. This way she related three different aspects of derivative using MRs.

We analyzed forty PSMTs' lesson plans with detailed teaching notes in a similar way (just as described above) to see if they related different aspects through MRs of derivative (see Table-2).

	First plans	Second plans	Third plans
Categories	N (%)	N (%)	N (%)
None of the aspects of the derivative is explicitly addressed.	28 (70.0)	6 (15.0)	0
Only one aspect addressed: any one of Derivative-Rate of Change (D-RoC), Derivative-Limit (D-L) or Derivative-Slope (D-S) relationship focused.	8 (20.0)	6 (15.0)	2 (5.0)
Two aspects are addressed: any two of D-RoC, D-L, D-S emphasized.	1 (2.5)	6 (15.0)	2 (5.0)
All three aspects are addressed but not interconnected	0	2 (5.0)	2 (5.0)
All three aspects are both addressed and interconnected	0	19 (47.5)	30 (75.0)
Unanswered	3 (7.5)	1 (2.5)	4 (10.0)
Total (N=40)	40	40	40

Table-2. Frequency analysis of the aspects of derivative addressed in the plans

As seen in Table-2, 70% of the PSMTs did not address any aspect of derivative in their first lesson plans. However, this figure decreased to 15% and to none following the completion of respectively PCK and TPCK workshops. A notable difference is an

increase in the number of those who both addressed and interconnected different aspects via MRs of derivative. This figure, increased from zero to 47.5% after PCK and to an outstanding 75% after TPCK workshops. No less important is the fact that all the PSMTs employed technology for their lesson plans and devised ways to employ it while interrelating different aspects through MRs of derivative.

DISCUSSION

PSMTs participating in our research has shown great improvements not only in their knowledge of MRs but also in establishing connections among MRs and doing this with the help of technology. Participants' responses before the workshops provide evidence that they knew little about the issue of MRs and in fact many could not explain the meaning of MRs or even some considered mathematical symbols as representations. However, following the TPCK workshops, their responses convincingly suggest that they developed insights into the issue of MRs. The way that MRs were employed in the first lesson plans suggested to us that PSMTs could not appreciate the importance of MRs in constructing deeper understanding of derivative as they were not able to use representations in a connected manner, at least, in the context of teaching derivative. Having completed the TPCK workshops, PSMTs were not only able to explain the meaning of MRs but also to use MRs by interconnecting them with regard to teaching derivative. They also showed great developments in making use of technology while establishing interconnections among the MRs and relating MRs to different aspects of derivative.

The question of interest here is: why is the professed development on the part of PSMTs regarding the use of MRs in technology rich environments important? Our answer is "for at least two reasons". First, as Moreno & Mayer (1999) point out, establishment of the links among the MRs of mathematical concepts present exceptional potentials for learners to achieve deeper understandings and this is cited as one of the most important reasons to justify the use of MRs in mathematics instruction. However, several studies (such as that of Juersvich et al., 2009) suggest that the links among the MRs are not often established by teachers during the instruction. Hence this heavy burden is left to the shoulders of the learners. In fact, our analysis of the first lesson plans support this argument that 85% of our participants did not make connections among MRs in their lesson plans. Unless these connections are established, the "expected" deeper understanding is not readily available due to the fact that learners often focus on one type of representations or fail to connect them (Berthold et al., 2009). Hence if the links among the MRs is not explicitly considered during the instruction, then use of MRs does not necessarily result in a better understanding. Considering that 67.5% of the lesson plans prepared after TPCK workshops attended to the connections among the MRs of derivative, PSMTs in our project seemed to have grasped the significance of making the links among the MRs an explicit focus of their instruction and also were able to make use of the potential that technology offers.

Secondly, the observed development of our participants with regard to the use of MRs through technology underlines certain features for the design of successful technology integration programs: the content, method of delivery of the content and hands-on activities (see also Hew & Brush, 2006). The content of PCK and TPCK workshops included the issue of MRs. During the workshops, particular attention was paid to the examples of MRs in different topics of mathematics (limit and functions etc.), functions of MRs, affordances and limitations of particular representations, significance of the connections among MRs, representational power of the specific technology employed during the workshops (i.e. Graphic Calculus), and the strengths and weaknesses of this particular technology regarding MRs of derivative.

In delivering the content, we employed TPCK framework which shaped our method to deliver the content during the workshops. We initially focused on the definition, the value of employing MRs for teaching and provided particular examples of MRs from different topics (Pedagogical Knowledge). Later, we attended to the ways for the utilization of MRs, their functions, the necessity and benefit of interconnecting representations in the context of derivative (Pedagogical Content Knowledge). Then technology dimension was brought into attention for a discussion of affordances and limitations that the particular software, Graphic Calculus, has for the MRs of derivative (Technological Content Knowledge). Afterwards, we brought into play the particular topic under consideration (derivative) and encouraged our participants come up with ways as to how to combine MRs of derivative with the help of technology, how to explicate the aspects of derivative via MRs, how technology can make explicit the links among the MRs and aspects of derivative (Technological Pedagogical Content Knowledge). Our participants' development suggest that this way of delivering the content was, at least in our case, effective.

While delivering the content, an extra care was taken to ensure that our participants find opportunities to get involved into hands-on activities. We aimed to provide PSMTs with a "space" where they can explore their own ideas, in their own ways with the technological tools available to them. Therefore, the PSMTs were guided by thought provoking questions but they were given responsibilities to discover in group-work sessions. We believe that PSMTs should be given chances to discover alternatives by themselves. Only then can they gain insights and develop competencies for an effective integration process of technology. Considering all these we argue that technology integration programs could be successful when they have at least these three as features in the design and conduct of the courses.

We end our discussion with a final implication of our findings. Having witnessed the PSMTs development and their detailed lesson plans, we are convinced that an effective teaching depends very much on the use of MRs regardless of the teaching medium. Hence we think that MRs and interconnections among them should be one of the foci of technology integration programs. (Pre-service) teachers should be given a chance to realize the important roles that MRs play for a robust understanding and the potential that technology offers interconnecting different representations.

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TEACHER EDUCATION MODELS FOR PROMOTING MATHEMATICAL THINKING

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This report compares and contrasts two models for teacher education for pre-service and in-service teachers. Both models utilize a unique video collection on student reasoning in mathematics, which recently has been made accessible through a web-based repository, Video Mosaic. The repository provides access not only to videos but also to related metadata (e.g., transcripts, written work of students) and innovative tools for using the materials in a collaborative online environment. We introduce Video Mosaic, present both models, and highlight their distinguishing features. We refer to preliminary findings from design research in studying implementation of the models and discuss their significance in light of reform efforts that emphasise promoting mathematical thinking in both teachers and their students.

INTRODUCTION

An extensive program of longitudinal and cross-sectional research has been ongoing for nearly 25 years at Rutgers University to investigate how students build mathematical ideas and forms of reasoning when invited to work on cognitively challenging tasks under conditions that support and encourage student engagement. The research has produced a unique video collection with over 4,500 hours of raw video, which has been the data source of analyses resulting in numerous publications and over 30 unpublished doctoral dissertations reporting on students learning and reasoning in mathematics. These studies detailing the development of student mathematical thinking initially generated, and subsequently have been based on, an evolving model for video data analysis involving seven, non-linear interlacing steps (Davis, Maher & Martino, 1992; Powell, Francisco & Maher, 2003). Resulting from this analytical method, the studies yielded transcripts of video data and identified critical events in the development of mathematical thinking and reasoning.

With a grant from the U.S. National Science Foundation¹ (NSF), we are making the fruits of this research program accessible over the World Wide Web through development of the Video Mosaic Collaborative² (VMC). The grant also sponsors use the videos and related resources to conduct design research and empirical studies in teacher education. Grounded in previous teacher professional development work

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² The Video Mosaic website is located at: <http://videomosaic.org/>

(Maher, Davis & Alston, 1992; Maher et al., 2000), we have adapted models that utilize VMC resources in helping teachers to attend students' mathematical reasoning. One model is for the education of pre-service teachers, and the second is a professional development model for in-service teachers. To put them into context, we first outline the theoretical perspectives that underlie our research program, and then describe the video collection and what the VMC offers to the worldwide community of researchers, teacher educators, and practitioners.

THEORETICAL PERSPECTIVES

Our research program has been guided by constructivist views on the learning and teaching of mathematics (Davis & Maher, 1990; Maher & Davis, 1990; Davis, 1990). When mathematics is viewed as a sense-making activity, learners cycle through a process of creating representations, retrieving or constructing relevant knowledge, mapping representations to knowledge, and using them as a basis for action toward problem solving. Teachers, or researchers functioning in their stead, assume the role of facilitator in the learning environment, with the goal of getting students to see mathematics as a subject where you think creatively and understand what you are doing. They empower students to discover ways to solve problems and learn the big ideas, rather than focus on meaningless rules to be followed mindlessly. Teacher-researchers also encourage students to create personally meaningful representations and foster their abilities to communicate explanations in justification of strategies used and solutions found. Within this view, pedagogy is based on creating an appropriate assimilation paradigm as an experience where students build representations and map to knowledge for constructing new ideas through discovery learning. As Davis (1990, p. 102) asserts, "Your mental representations must give you the power to see new possibilities and new constraints in new situations."

Our research team created conditions in the learning environments where studies were conducted that enabled discovery learning by students. The conditions included provision of problem-solving tasks that are accessible yet challenging, sufficient time to explore problems, opportunities to revisit problems after some time had passed, and establishing norms for both social and mathematical behaviours (Maher, 2005). Norms included working with a partner or in a small group, sharing ideas within and among groups, criticising ideas but not the people offering them, and offering arguments in support of solutions to problems. Facilitation and questioning by researchers prompted students to offer convincing arguments, which evolved into the expectation that it should be a *mathematically convincing* argument. Importantly, the researcher was never the sole arbitrator of what was convincing, as students also had to convince one another that solutions were valid. Thus, the learning environments supported the development of students' reasoning and justification. Videotaping with multiple cameras captured the developmental process and enabled it to be studied carefully. Findings from such studies inform the research community (e.g., Maher &

Martino, 1996); and videos of students reasoning and justifying in their problem solving serve as an important tool in mathematics teacher education (Maher, 2008).

THE VIDEO COLLECTION AND REPOSITORY

Our video collection features students engaged in mathematical problem solving across multiple content strands in both classroom and informal settings. The videos were collected during a two-decade period through research supported with three previous NSF grants. The research includes (1) a yearlong study in a fourth-grade classroom where students make sense of fraction as number and build conceptual understanding for operations with fractions; (2) a three-year study of informal mathematics learning with urban, middle-school students in an after-school program; and (3) a seminal longitudinal study that followed a cohort of students from elementary through high school and beyond to adulthood. All these studies created the conditions described above to extend student mathematical thinking. Notably, this was accomplished by engaging students in problem solving *before* formal introduction of the underlying mathematical content and algorithmic procedures in the regular curricula at their schools. Capturing these activities on video and collecting written work of students enabled researchers to study in fine detail how certain mathematical ideas were built and how forms of reasoning naturally emerged in the various learning environments. Well-documented examples of students' mathematical reasoning now populate a searchable database called the Video Mosaic.

The Video Mosaic has been constructed by layering it over the Fedora architecture of a digital repository that preserves the scholarly work produced at Rutgers University (Agnew, Mills & Maher, 2010). The development team is committed to using open source software and ensuring interoperability, which has included advancing the field of digital library sciences with development of new metadata for making the resources findable. Customized portals with a variety of search options make it easy for users to identify video and related resources (e.g., transcripts, students' written work, dissertations that analysed the data). For instance, a teacher educator might search by content strand for resources to use in an elementary math methods course; a researcher might search by name of a particular student for videos to do a case study of how a learner builds mathematical ideas over time; and a teacher might search for math problems that align with NCTM standards for a particular grade range.

Our team also has developed innovative tools for collaborative work using the Video Mosaic, such as the VMC Analytic (Agnew, Mills & Maher, 2010). In simple terms, the Analytic tool enables users to link together video clips or selected segments of video clips as related events, where icons (i.e., first frame of the video) can be organised much like slides in a PowerPoint presentation to build a video narrative for a particular purpose. Each video event can be annotated with text providing salient information to further define the narrative. There are endless possible stories that might be told, but let us consider here creation of a narrative that is used by a teacher educator. She wants to illustrate to prospective or practising fourth-grade teachers

how children reason about the problem, which is bigger $\frac{1}{2}$ or $\frac{1}{3}$ and by how much, when Cuisenaire rods are available as a manipulative for building models. There are multiple pathways for finding such video, such as searching on Cuisenaire rods in menu choices for term manipulatives or comparing fractions as a choice for math problems. Descriptive metadata aid the user in determining whether or not results of a search have yielded desirable resources. Searches can be done within the Analytic tool itself, as well as in the main portals to the VMC. Once desired video has been found, the teacher educator plays a video clip and can make new start and stop points within that clip to highlight certain actions for subsequent work with teachers. Annotations can record questions to pose about video segments for stimulating discussion with teachers. The full video clip could be placed as the last event in the narrative with an annotation prompting teachers to write a short reflective essay about implications for classroom implementation of same comparing fractions problem based on having studied the video. This is but one example of how the Video Mosaic supports technology-enhanced lessons for teachers, and hints at what is possible for instructional inventions such as we have been conducting in our current research.

TWO MODELS FOR TEACHER EDUCATION

We premise our work with teachers on the notion that they must know more than how to solve math problems; they must come to understand well the reasoning that justifies valid solutions to those problems. They also must move beyond their own way of approaching and solving a problem to recognize that more than one solution strategy and form of reasoning may be valid when working on a particular task, as all students do not think alike and one of our goals is development of adaptive expertise (Bransford, Derry, Berliner, & Hammerness, 2006). We therefore begin by engaging teachers as learners in problem-solving tasks and require that they provide convincing arguments for solutions they find. Their own problem solving is an important prerequisite for subsequent study of videos. Our work thus far has focused on the mathematical strands of combinatorics and fractions. The model for proceeding varies depending on whether they are pre-service or in-service teachers.

Model for Pre-Service Teachers

Our intervention model for pre-service teachers involves working with them in the context of a one-semester university course within a teacher education program leading to certification to teach at elementary or secondary level. The course occurs before they have had a teaching internship, so participants typically do not have any experience in the role of classroom teacher. With elementary-level prospective teachers, interventions focus on either one or both content strands and tend to last through approximately one half to two thirds of the semester. The need to cover other content, such as place value, tends to limit how much time can be devoted to counting and fractions in what typically is a single elementary math methods course in their teacher education program. Interventions at the secondary level focus on the combinatorics strand and vary in duration depending on how much time the instructor

chooses to spend working with the tasks and videos, as there is opportunity to extend mathematical thinking to advanced levels (Ahluwalia, 2011).

An essential component of problem solving is the construction of representations that model solutions, and manipulative tools sometimes are made available for use. In the fractions strand, our tasks involve modelling with Cuisenaire rods, and the teachers are given a little time to explore the rods and their properties. Instructors establish the condition that the rods have permanent colour names but that their number names can vary. Examples of initial tasks are: *If we call the orange rod one, what number name would be given to the yellow rod? And what number name would we give to the red rod? Now if we call the blue rod one, what number name would be given to the light green rod?* Then, once the teachers have grasped this way of working with the rods, tasks move on to finding fractional relationships and comparing fractions. From these activities, conceptual understanding of operations with fractions naturally emerges (Steencken & Maher, 2003), and certain tasks tend to elicit particular forms of reasoning (Maher, Mueller & Yankelwitz, 2009).

For the fraction strand tasks, there are videos of students engaged with those same tasks, which the pre-service teachers study with guidance from their instructor. Instructors typically show the first few video clips during class to draw attention to the way the teacher-researcher engages students with tasks, the kinds of models students build with the rods, the way they express their ideas, and the forms of reasoning conveyed in their explanations. The practice of using class time to view and discuss videos may continue; or instructors may choose to assign videos as homework with pre-service teachers preparing written observations or engaging one another in online discussion about what they notice.

Interventions in the combinatorics strand take a similar form of pre-service teachers working on problem-solving tasks and studying videos of students engaged with the same tasks. Student reasoning in this strand has been researched extensively (Maher, Powell & Uptegrove, 2010). A new book that also is available in online version provides readings that complement problem solving and video study in the effort to help teachers better understand the developmental process of reasoning. Among the great challenges for teachers is making sense of student representations (Maher & Davis, 1990).

Model for In-Service Teachers

Our intervention model for working with practicing teachers shares many similarities with the model for prospective teachers, yet has some notable differences. One distinction is flexibility in format for implementation. Interventions take the form of a series of professional development workshops or a university course bearing graduate credit. The in-service teachers also begin by working in pairs on the same kinds of tasks in one of the strands. After studying videos of students investigating those tasks, however, the teachers prepare to bring the tasks into their classroom for their own students to explore. Preparation includes discussion with workshop/course

instructor about how teachers should facilitate administration of the task with their students, such as considerations for pairing students and what to say to a student who is stuck without telling him or her how to proceed. The statements of tasks always include a prompt for a convincing argument to be provided by students, which gets recorded on their written work. Teachers then collect the student work and bring it with them to the next session with their instructor. Together they analyse examples of student work to look at how they reason about the problem. Analysing the work of one's own students thus is a distinguishing feature of the model for in-service teachers. More details of this model, including descriptions of specific tasks and videos for the combinatorics strand suitable for middle-school level, appears elsewhere (Maher, Landis & Palius, 2010).

PRELIMINARY FINDINGS AND DISCUSSION

Among the data we collect in studying the interventions are participant responses to a beliefs inventory, from which we analyse change from before and after the intervention to measure its impact. In studies reported elsewhere, we present analysis of belief change among pre-service elementary teachers at multiple sites (Maher, Palius & Mueller, 2010) and analysis of belief change for in-service middle-school teachers at one site (Maher, Landis & Palius, 2010). Each of these studies reports statistically significant change on a subset of belief inventory items that are aligned with the nature of intervention conducted. For in-service teachers, including regular and special education teachers, there were changes in beliefs in how mathematics is learned and how teachers can influence children's learning. Also, there was no significant difference in the belief changes for regular and special education teachers (Maher, Landis & Palius, 2010). However, for pre-service teachers, while there were significant changes in beliefs about how children learn, there was no evidence of change in beliefs about how teachers can influence that learning (Maher, Palius & Mueller, 2010).

The contrast in results between pre-service and in-service teacher study participants is not surprising; and the difference is echoed in the reflection essays that more recent groups of study participants submitted following interventions in Fall 2010 courses. While comments about insights to student learning are ubiquitous, only in-service teachers reflect on how specific practices have evolved as a result of what they saw on the videos. A poignant example comes from a middle-school teacher whose intervention consisted of three cycles of (i) problem solving, (ii) studying video, (iii) implementing task in the classroom, and (iv) analysing student work. She writes

Interestingly, I always knew from my own research and education to employ effective questioning as a technique to improve inquiry in a student. I always thought I did a very nice job of this in my classroom. However, after watching the video clips on the Rutgers website that show teacher questioning and allowing the student to work through their strategies by giving them time to communicate without leading them shed new light on how I will continue to work toward achieving this important skill. I find myself keeping this at the forefront of my mind each day. When I first started questioning in this way and

requiring students to convince me of their solutions, my students were very resistant. I realized this phenomenon may have occurred because I was giving too much away in my original technique. As a result, I know I have students who are deeper thinkers and less reliant on me for the answer.

In closing, we note growing awareness of the need to reform programs that prepare new teachers to emphasise more classroom training. A major U.S. newspaper reported recommendations by an expert panel convened by the National Council for the Accreditation of Teachers with representatives from higher education, K-12 school districts, teachers unions and state boards of education. Recommendations “urge teacher-training programs to operate more like medical schools, which rely heavily on clinical experience” (Banchero, 2010). This is not a mandate for imminent change, as the panel is not empowered to institute its recommendations. In the meantime, the Video Mosaic offers accessible resources for teacher education that give evidence of the robust mathematical thinking that students are capable of doing, along with innovative tools for designing or replicating interventions that can positively affect classroom teaching practices.

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WHAT DO PROSPECTIVE TEACHERS ANALYZE WHEN THEY WATCH A MATHEMATICS LESSON?

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This study examines changes in prospective teachers' analytic foci as they learned how to teach elementary mathematics through a specific case-based pedagogy designed to help them pay more attention to the mathematics-specific features of a lesson. The comparison between early and late comments on cases showed that the participant teachers' analytic foci moved from general to substantive features of a mathematics lesson. This result was also confirmed by the teachers' self-assessment of their learning. Given this, issues and suggestions to promote teacher expertise by mathematics-specific analysis ability are discussed.

INTRODUCTION

Mathematics teachers need to have professional appreciation toward and ability of mathematics-specific analysis in classroom events. Without this appreciation and ability, it is no wonder teachers have such difficulties understanding teaching practice. A teacher may think that her mathematics teaching approaches have been significantly changed by simply adopting new materials and new policies, although such changes are neither substantive nor promising from a teacher educator's perspective (Cohen, 1990). Thus, what teachers pay attention to, when they analyze others' teaching practice as well as their own practice, is the most fundamental element to improve their teaching expertise.

A specific case-based pedagogy with videotaped mathematics lessons and their analytic narratives has been developed and implemented in Korea to provide pre-service teachers (PSTs) with knowledge and skills to analyze and reflect on a lesson by mathematics-specific ways beyond superficial or general features. The term case-based pedagogy is used to underline a series of pedagogical flow by which teachers first analyze others' teaching practice and then design, implement, and reflect on their own instruction both individually and collectively.

This paper examines what the PSTs learned through the case-based pedagogy as evident mostly in the changes between the early and the late comments on others' teaching practices. The analysis is also confirmed by the PSTs' self-assessment. As such, this paper can be a catalyst for mathematics teacher educators to design and implement teacher education programs in order to promote teacher expertise.

THEORETICAL BACKGROUND

Shulman (1986) points out the need of teacher knowledge that is specific to the contents and students to be taught. Building on this claim, Borko et al (2000) emphasizes ‘mathematics-specific pedagogy’ in which two aspects, mathematical tasks and discourse, are highlighted as central to recent reform-oriented mathematics teaching. In fact, NCTM (2007) proposes that mathematical tasks and discourses should be constantly analyzed in relation to what students actually learn. Reports on the effects of professional development show that programs focused on specific mathematics content and students’ learning are helpful in comparison to those on general teaching strategies (Desimone et al. 2002). Therefore, this mathematics-specific analysis ability should be the essential factor to improve teacher expertise.

However, it is not easy for PSTs to develop such professional analysis ability. PSTs are able to focus their attention on classroom environment and management by observing videos at most twice but have difficulty in noticing more key features of mathematics teaching such as mathematical content and communication (Star & Strickland 2008). This difficulty is even more challenging for elementary PSTs because they are educated to teach almost all subjects, which may prevent them with understanding the substantive characteristics of a *mathematics* lesson.

It is hard to imagine that teachers develop this discipline-specific analytic focus without any specific interventions throughout their long-term teaching career, if they are not educated during teacher preparation programs. This is the main reason why PSTs need to have opportunities to develop their specific skills of observation, interpretation, and analysis. Teacher education programs, however, are often criticized due to the sporadic and superficial curriculum, uninspired instruction, or little connection between theory and practice (Borko et al. 2000).

Against this trend, teacher educators have emphasized systemic or fundamental renovations in teacher education programs (Ball & Forzani, 2009). Given that, case use has drawn many teacher educators’ attention because it helps PSTs to be engaged in thoughtful inquiry and analysis into classroom practices while maintaining the complexities and demands of everyday teaching (Markovits & Smith, 2008). More recently, video cases have been extensively used because of their specific characteristics of reflecting on the complex nature of teaching and making noteworthy events salient. They were used to stimulate productive discussions of mathematics and teaching strategies (Borko et al. 2008), to increase teachers’ ability to notice classroom events, specifically students’ mathematical thinking (van Es & Sherin 2010), or to develop teachers’ orientations, analysis, planning, and enactment abilities (Santagata & Guarino 2010).

In line with this recent trend, this paper illustrates the extent to which PSTs notice the main substantive features of mathematics lessons while learning how to teach elementary mathematics through case-based pedagogy. It should be noted that the

argument is not to neglect general classroom events that are common across multiple subject matters (e.g., classroom management). Such features are worthy of notice specifically for PSTs. However, more substantive elements of mathematics lessons (e.g., mathematical tasks and content-specific teaching strategies) should be paid purposeful attention to when using cases in teacher preparation programs. It becomes critical to develop a detailed framework with supporting materials by which PSTs develop their mathematics-specific analytic ability, beyond just reading narratives or watching videos. In this way, case-based pedagogy can embrace both the practical and the theoretical aspects of a mathematics lesson.

METHOD

The videotaped mathematics lessons for this study were collected from the mathematics teaching demonstrations for PSTs at attached elementary schools, public lessons recognized as good instruction from teaching contests, and PSTs' classroom teaching during their practicum period. In addition, some lessons were purposefully planned and implemented to address key ideas of mathematical teaching and learning which might be difficult to observe in ordinary classrooms. A preliminary analysis of the 30 collected lessons was conducted with three criteria: (a) productivity of the lesson to raise important issues of mathematics instruction, (b) specificity of the lesson to understand what happens in the classroom, and (c) the representativeness of the lesson to cover big mathematical ideas taught across grade levels. Fifteen lessons were selected using this process.

A comprehensive written case was then developed for each videotaped lesson with two purposes. One is intended to help PSTs contextualize the lesson so that the written case included 'overview of the case', 'detailed description of the lesson', and 'supplementary materials'. The other is to foster PSTs' learning by watching, analyzing, and reflecting on the specific lesson so that the case included 'theoretical background', 'focused analysis', and 'additional analysis'. Each narrative case is comprehensive and lengthy for these purposes.

The participants were 16 elementary PSTs enrolled in the university course named as study and practice of elementary mathematics education. They were at the first semester of the final fourth year in their teacher preparation program. The course consisted of two phases. One was to discuss the developed cases (i.e., videotaped lessons and written narratives) and the other was to discuss their own mathematics lessons implemented during the practicum period. This paper focuses on the first phase.

The PSTs were asked to read a part of one or two written cases, specifically from 'overview of the case' to 'detailed description of the lesson', per week before each class. In the class they were asked to write down whatever stood out while watching the videotaped lesson together. The lesson was then extensively discussed on the basis of the PSTs' comments. A total of 11 cases were used for the semester. The PSTs' comments on each case were collected in order to keep track of their learning

of what to analyze a mathematics lesson and to explore changes in the process. In addition, a survey was conducted to understand the PSTs' overall perception of elementary mathematics instruction and to probe how they would perceive their learning from case discussions.

The PSTs' comments on the videotaped lessons were analyzed to explore changes in their ability to analyze a mathematics lesson. Initial coding categories were guided by previous research on the criteria of analyzing mathematics lesson (NCTM, 2007). However, it became clear from early analysis that more specified categories were needed to capture the nuanced and significant differences in the comments. For instance, one described the teacher's general teaching strategy which might be the same across other subject matters, whereas another commented on the teacher's specific instructional strategy related to the mathematics topic. These led to differentiate mathematics-specific features from general ones. This differentiation indeed coincided with the research purpose to explore in what ways the PSTs would develop mathematics-specific analytic lenses. More specified coding categories (see Table 1) evolved to account for subtle but significant differences in the comments. For this paper, the PSTs' comments on Case 2 (early) and 11 (late) were compared and contrasted. The content area and the key issue for discussion were the same in both Cases. Two research assistants independently coded all the comments about Case 2 and 11. Inter-rater reliability of both cases was more than 80%.

Analysis of the survey was done in a bottom-up approach. Any phrases in which the PSTs described as their foci in observing a mathematics lesson and as their perception on their learning through case discussions were identified, respectively. These phrases were then combined on the basis of their similarities into topics. Frequency of these topics was finally counted.

RESULTS

Early perception on what to analyze in mathematics teaching

Analysis of the survey reveals what the PSTs considered the most important in analyzing mathematics instruction. Most PSTs perceived learning goals, teacher behavior, and students' participation as the most significant topics. First, nine PSTs pointed out the achievement of learning goals. They thought that the achievement of learning goals is significant because teaching should be a purposeful act and because mathematics is hierarchical so that incomplete attainment of an instructional objective in one lesson may hinder the subsequent lessons.

Second, seven PSTs claimed a primary focus on teacher behavior such as raising questions and responding to students. Specifically, they were cautious about the possibility that a teacher's inaccurate explanation or words may cause misconceptions on the part of students.

Third, four PSTs claimed students' participation and interest as the analytic focus of mathematics teaching. They considered students' interest in the lesson both as a

precursor of good instruction and as the motive to pursue further study in mathematics.

PSTs' analytic focus in early and late comments on cases

The PSTs' analytic focus from the early to late comments shifted to attend more to the mathematics-specific features of a lesson (see Table 1). They decreased from 37% to 13% in the percentage of comments they wrote about the general features of the lesson. In the early comments, the teachers focused a lot on the classroom atmosphere and general teaching strategies, but in the late comments, these foci decreased substantively.

Feature	Topic	Early Comments	Late Comments
General	Instructional Materials	11 (6%)	2 (1%)
	Classroom Atmosphere	31 (17%)	3 (2%)
	Physical Environment	4 (2%)	0 (0%)
	General Teaching Strategies	22 (12%)	11 (6%)
	Motivation	1 (1%)	6 (4%)
	Subtotal	69 (37%)	22 (13%)
Mathematics	Mathematical Tasks (Content)	23 (12%)	28 (16%)
	Mathematical Tasks (Student)	10 (5%)	11 (6%)
	Teaching Strategies (Content)	27 (15%)	39 (23%)
	Teaching Strategies (Student)	33 (18%)	24 (14%)
	Mathematical Use of Instructional Materials	11 (6%)	10 (6%)
	Mathematical Communication (Teacher)	5 (3%)	12 (7%)
	Mathematical Communication (Student)	1 (1%)	7 (4%)
	Students' Mathematical Thinking	6(3%)	17 (10%)
	Subtotal	116(63%)	148 (87%)
Total		185 (100%)	170 (100%)

Table 1: PSTs' analytic focus in early and late comments on cases

The PSTs increased from 63% to 87% in the percentage of comments they made about the mathematics-specific features of a lesson: in mathematical tasks (from 17% to 22%), in teaching strategies (from 33% to 37%), in mathematical communication (from 4% to 11%), and in students' mathematical thinking (from 3% to 10%).

On the one hand, the most frequent topic both in the early and in the late comments is related to teaching strategies. A subtle difference was that in the early comments teachers focused more on teaching strategies related to students' mathematical

abilities and interest, whereas in the late comments they commented more on teaching strategies related to mathematical contents being studied. On the other hand, a noticeable shift from the early to the late comments happened with regard to mathematical communication and students' mathematical thinking. Specifically, the PSTs made comments to the degree by which mathematical communication was emphasized and fostered in terms of the case teacher's adequate questioning and timely feedback tailored to students' understanding. They also came to be more sensitive to students' mathematical thinking. For instance, the PSTs made comments on individual students' specific solution methods and their concomitant mathematical reasoning. Despite the relatively small percentage, the PSTs' salient notice of students' mathematical thinking seems worthy to be mentioned.

This promising shift from the early to the late comments may reflect dramatic change in a few PSTs rather than consistent change in most PSTs. The analysis of individual PST's analytic focus in the early and in the late comments reveals that this is not the case. Whereas in the early comments only 4 PSTs made mathematics-specific comments more than or equal to 80% out of their total comments, in the late comments as many as 14 PSTs did.

PSTs' self-assessment of their own learning

Analysis of the survey reveals three areas the PSTs claimed to have learned by case discussions. First, 12 PSTs claimed that they learned how to analyze a lesson in a way to reflect on the specificity of mathematics lesson. They often compared what they previously looked at with what they currently focused on. The following is a representative description:

I used to analyze a lesson in terms of general features such as students' participation, teachers' attitude, and teaching strategies to induce students' interest. But now I analyze in a way to reflect on the characteristics of a mathematics lesson. For instance, are mathematical tasks appropriate to achieve the instructional goals? How about the level of such tasks? Was teacher's mathematical knowledge accurate? How was the communication between the teacher and students, or between students? I came to know how to look at a mathematics lesson more meaningfully.

Second, 10 PSTs also claimed to have developed professional insight into quality mathematics lessons. Some teachers expressed their excitement to see classroom contexts in which productive mathematical communication, which was very often emphasized in the literature, was really happening. Others differentiated effective mathematics lessons from seemingly good, but indeed unsuccessful ones in terms of students' understanding. They were able to recognize that effective mathematics lessons are related not to splendid instructional materials or students' fun activities but to the degree to which key mathematical content are meaningfully explored with students' thinking. The PSTs claimed that this vision came from a vivid discussion of multiple cases in class including unsuccessful and thought-provoking ones.

Third, half of the PSTs described the importance of thorough design of a mathematics lesson by recognizing from the case discussions that good lesson design is the prerequisite to good implementation. With regard to lesson design, some PSTs emphasized the importance of teachers' accurate knowledge of mathematical content as well as systematic analysis of curriculum materials. Others focused on teachers' comprehensive knowledge of students including their misconceptions related to the topics to be taught.

DISCUSSION

Development of mathematics-specific analytic ability becomes critical, specifically for elementary PSTs. The PSTs in this study demonstrated substantive increases in mathematics-specific analysis abilities while taking a university course in which a specific case-based pedagogy was designed and implemented. This result makes us pay attention to the characteristics of such pedagogy. The PSTs in a group context had rich opportunities to analyze various cases on the basis of detailed information of the classroom events and key features of each lesson with strong theoretical basis. While interpreting the same event from multiple perspectives, the PSTs learned to see beyond the superficial features of classroom practice. This experience of examining critically one's own initial analytic focus and elaborating it toward mathematically specific features seems fundamental in using cases in teacher preparation programs.

Another aspect to be discussed is a teacher's professional vision on what is effective mathematics instruction (Cohen 1990; Feiman-Nemser 2001; NCTM 2007). This vision will guide PSTs to pay more attention to the critical aspects of classroom events in designing, implementing, and reflecting on a lesson. In this respect, it should be emphasized that case-based pedagogy should provide PSTs with opportunities to expose their initial perception on desirable mathematics teaching, to explore alternative approaches, and to develop a more elaborated vision. The PSTs in this study claimed to have developed this vision through case discussions. This makes us consider how to carefully choose cases. Note that the cases in this study varied in terms of their characteristics. Whereas some cases were based on exemplary effective instruction that might be difficult to observe in other contexts, others served as a catalyst by which PSTs could examine the weaknesses of such lessons and seek for alternatives. Better understanding of what constitutes an effective lesson may lead PSTs to focus on essential mathematical content and students' mathematical thinking with strong commitment.

The last aspect to be noteworthy is the PSTs' self-assessment on their learning. There has been a tendency of advocating case use mainly from teacher educators' perspective. Listening to how participant teachers perceive their learning from new learning contexts may provide not only the objectivity of research but also significant information for mathematics teacher educators who are interested in using cases.

Because of some practical and contextual factors, the results reported in this paper have limitations to provide a full generalization or a precise pre-and post-

comparison. Nevertheless, the compelling evidence of the PSTs' improvement in terms of their keen analytic focus gives mathematics teacher educators much impetus to design and implementation of case-based pedagogy. Given that there has been little systematic information on new types of teacher preparation programs and their impact on teacher expertise, this paper is expected to be a catalyst for mathematics teacher educators to pursue such an endeavor.

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USING GOOGLE SKETCHUP TO RESEARCH CHILDREN'S EXPERIENCE OF DIMENSION

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Following on from a phenomenographic study of how dimension is experienced, the study reported here explored how the software Google SketchUp and the use of its dimensional tools can facilitate further experiences. Clinical interviews based around carefully designed tasks were conducted. This paper reports evidence from one pair of students on how the dimensional tools prompted the construction of ideas about dimension that could be regarded as situated abstractions of the powerful mathematical notions of vector and vector space.

EXPERIENCING DIMENSION

Dimension is a powerful mathematical construct that is rarely taught or researched explicitly and not normally construed as something that is 'experienced'. We start nevertheless from the assumption that even very young children might have experiences of dimension that could be seen as the situated root of sophisticated mathematical concepts. This assumption was in fact confirmed by our earlier work, when we had mapped out through a phenomenographic study (Marton, 1981) the different ways that children experience dimension. In this preceding research, dimensional experience was characterised as a type of measurement (Dimension as Action), as a domain incorporating shapes (Dimension as State), as an object having materialistic attributes (Dimension as Material), as a property for distinguishing between 2D and 3D shapes (Dimension as Cross-dimensional) and as a property for creating a hierarchy of relationships among shapes (Dimension as Hierarchy).

In the study reported here, we sought to build on that prior work by observing experiences not previously identified. We report how we exploited the software, Google SketchUp, as a window (Noss and Hoyles, 1996) on experience of dimension.

DESIGNING A WINDOW ON DIMENSIONAL EXPERIENCE

We aimed to design a window that embedded notions of dimension (in the sense that an expert might recognise dimension within the window) in such a way that experiences of dimension might be stimulated. Pratt and Noss (2010) have referred to this process as 'designing for abstraction'. We intended to observe how children might concretise dimension through the use of these embedded notions (Wilensky, 1991).

The window consisted of particular tools in Google SketchUp alongside tasks, carefully designed to foster possible utilities of dimension through purposeful activity

(Ainley, Pratt and Hansen, 2006). For example, the tasks would often involve modelling a real world phenomenon (Noss and Hoyles, 2006; Simpson, Hoyles and Noss, 2005) and resolving curious illusions raised by modelling in 3D on a 2D screen. In order to trigger dimensional experience in particular, the students constructed objects within and across dimensional spaces. In this way, we expected that the window would perturb students' experiences and the tools would provide them with a means to begin to resolve the conflicts raised by those perturbations.

A set of principles was formed in order to choose the software that could act as a window in the way set out above. To begin with, the mathematical ideas embedded in the tools had to be visible to the child. The tools had to have an *expressive power* (Abelson and DiSessa, 1986) and be transparent by making "visible its operations and how they are integrated with the embedded context" (Orhun, 1995). What is more, they had to have restrictions, which act as obstacles to the children's construction and could lead them into a debugging process that might lead to the construction of utilities for dimension (Edwards and Benedickt, 1995; Gargarian, 1996; Osta, 1998; Papert, 1980). We expected that these utilities would be expressed as situated abstractions (Noss and Hoyles, 1996), heuristics for making general sense of the environment, couched in the language of the specific tools within the environment.

METHODOLOGY

It was decided on this basis to use Google SketchUp 7 (<http://sketchup.google.com/>), an accessible general-purpose 3D content creation tool. Experiences of dimension were generated from twelve 10-year old students from a primary school in London. The students worked in pairs (six pairs in total). The clinical interview, as described by Hunting (1997), based on the completion of tasks using Google SketchUp, was the main method in this study. Each interview lasted around 120-180 minutes. This paper focuses on how the use of the software, together with the use of its 'special' tools, facilitated the students' experiences of dimension.

The tasks

We report below activity in two tasks. In the main task (A), students had to use SketchUp to build a neighbourhood. The subsidiary tasks were: (i) B1, where the students were asked to complete many pre-constructed incomplete frames (see Figure 1); (ii) B2, where students were asked to construct a square in different ways; and (iii) B3, where student were asked to construct a cube in different ways.

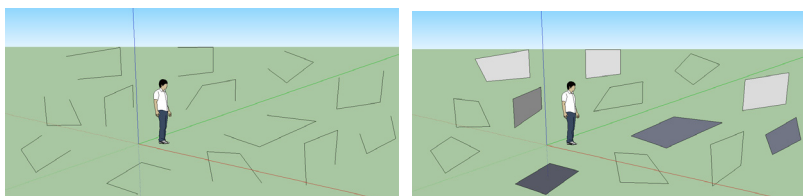


Figure 1: Task B1, before and after completion of the frames by the students

The dimensional tools

In the SketchUp environment there are three axes by default: the red, the blue and the green axes. The axes are perpendicular and their extensions in the negative direction are marked with dotted lines. We identified a set of tools within SketchUp, which, because of their special affordances, became known by us as dimensional tools.

A *dimensional tool* is a tool whose use is likely to raise opportunities for the mathematical experience of dimension. For example, below, we describe two such tools, the *Line tool* and the *Coloured surface tool*. In each description, we give an overview of how the tool is used, including its potentials and constraints, and their embedded mathematics, in the sense of its potential to raise opportunities for mathematical experience of dimension. Exactly whether or how that potential was in practice realised is a focus for the analysis. Given the sophisticated nature of the mathematics, we are not of course suggesting that such young students learned the mathematical notions identified. Rather, we are arguing that the analysis reveals various ways in which the students' experiences might be interpreted as situated accounts of mathematical experience as might be recognised by someone enculturated in those sophisticated mathematical notions. We refer here to the *utility* of the tools, arguing that the use of both tools described here was important for the completion of all the tasks in order to create lines and surfaces for constructing shapes.

The Line tool: While drawing with the Line tool, the colour of the line being drawn changes and mirrors that of any parallel axis or is black if not parallel to any default axis. We believe that the Line tool supports the construction of situated abstractions for direction vectors by allowing the student to create vectors and by drawing attention to their relationship with axes. In particular, through the focus on the default axes, attention is drawn to the idea of a base vector. For example, to create a 2D shape, a student would need to use 2 different direction vectors whereas a 3D shape would require three direction vectors, possibly one parallel to each of the three different axes.

The Coloured Surface tool: In order to create a surface, the student might create a closed loop of lines, the edges of the surface. If the edges are co-planar, the surface is coloured. If the closed loop is twisted, the surface will remain uncoloured. We believe that the Coloured Surface tool supports the construction of situated

abstractions for spanning, that is to say the linear combination of independent vectors. The potential for experiencing spanning lies in the feel gained by observing how the edges of the rectangle combine and result in a coloured surface whenever the edges are multiples of coloured base vectors.

AN INTERPRETATIVE ACCOUNT OF STUDENTS' EXPERIENCE

In order to analyse how the dimensional tools afforded experience of dimension, we traced activity with the dimension tools within the tasks described above. Here we focus on the interpretation of the data on one pair of students, to illustrate the activity across all pairs.

At the beginning of Task A, the students distinguished between the “flat” shapes that belonged to the “ground” (the green-red axes plane) from those which were “coming out” or “standing up” (Figure 2).

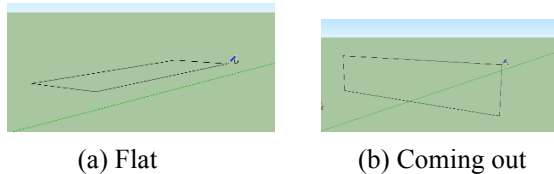


Figure 2: ‘Flat’ vs. ‘Coming out’ shapes

First, the students created a ‘flat’ rectangle and a ‘flat’ circle. To do this, they used the Circle and the Rectangle tools. For the Circle tool, the student simply indicates the centre of the circle and then drags out the radius. For the Rectangle tool, the student indicates one vertex, drags one side, makes a second vertex and then drags a second side, which is taken to be perpendicular to the first. When a third vertex is indicated the rectangle is automatically drawn and will always be coloured. The students were then asked to use the Line tool to construct their next shape. The Line tool acts differently by allowing the construction of lines in any possible plane in space. The students used the Line tool to create a pentagon, which was in a ‘coming out’ position (see Figure 2b) and coloured. The students argued the shape “*is bending*” and “*it is kind of 3D but it is flat*”, similar to their expressions of ‘coming out’ and ‘standing up’ before. However, after moving around the shape (SketchUp includes an Orbit tool to provide different perspectives), they realised that the shape was not in the position they thought: “*It is a flying sign!*”. They thought of it as being “*on the ground*” but after orbiting they realised that it was drawn “*in the air*”. They decided to keep it and to draw a post under it, which, after several trials, they succeeded in doing.

The creation of the pentagon and the post introduced students to the degrees of freedom of moving in 3D space. Before that point, they thought of SketchUp environment as any other 2D software for drawing, similar to their familiar pencil and paper procedure. The students connected the bottom edge of the pentagon to the

red axis (Figure 3, Shape a). To facilitate that, the students used their newly discovered freedom to switch to a side elevation so that the side of the pentagon and the red axis looked perpendicular to each other and it was easier to connect them.

The students began to construct another shape and the researcher (the first named author) drew their attention to the change of colour in the line being drawn. They argued that this was *“because it is in different directions”* and that the colour depends on the direction the line is following, *“like when it goes north, south, east, west, when it goes to these directions it has a colour, when it is going diagonal it is black”*.

Although the students noticed the change of the line colour, they did not use this knowledge as the polygon that they created was not coloured (Figure 3, Shape b) (the shape was not coplanar: see Coloured Surface tool above).

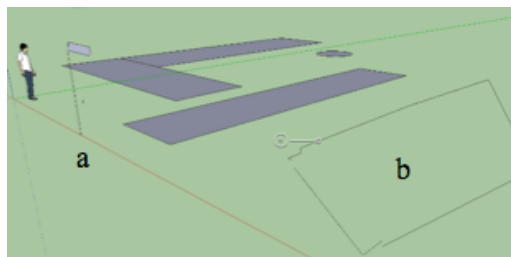


Figure 3: Polygon created by lines

The students explained that it depended on the tool; the Rectangle and Circle tools created coloured shapes but the Line tool did not (In fact, the Rectangle and Circle tools always create coplanar shapes but the Line tool may not). To probe further, the researcher introduced Task B1 (Figure 1). After using the Orbit tool, the students' argued that the coloured shapes were *“just great”* and *“proper”* while the non-coloured ones were *“twisting and turning”*.

After Task B1, the students continued working on their neighbourhood. Along the way they created non-coloured shapes and referred to them as ‘twisted’, ones that *“change form with Orbit”*. They also pointed out that the colour of the line being drawn by the Line tool *“has to do with these lines (the axes) when it is going to the direction of the lines”*. The students gradually became more fluent in identifying which lines were on the same plane and which were not, by reference to their colour while being drawn.

The polygons created by the students were transformed into 3D shapes using a SketchUp tool that ‘pulls’ the plane shape into a solid prism. At the end of the Task A, we asked the students about the differences between the 2D and the 3D versions of their neighbourhood. The students talked about the lines they drew as ‘shaking’ (going in unexpected directions).

During Task B2, the students used the Line tool to create a coloured rectangle using green and blue lines. They stated that their rectangle was coloured because they used

coloured lines to create it. They also drew two examples of rectangles, one with coloured lines and one with black lines in order to demonstrate their conjecture. They argued that the colours of the lines corresponded to the cardinal directions i.e. “*red means south*” and that, when the lines go in the direction of the axes, they take on that colour.

Thus, the students understood a relationship between the colour of the lines and whether the shape was coloured. The researcher wished to explore further whether they understood that for a shape to be coloured it has to belong to only one plane, an idea clearly connected to the twisted nature of the shape. Therefore, the researcher drew a rectangle using all three different colours of the lines. The rectangle looked perfect, created by coloured lines but it was not coloured. The students used Orbit to turn around and see that indeed the rectangle that looked perfect was not even a rectangle, as the three (perpendicular) lines were not connected. The apparent connection was illusory. Their first reaction was that it was not symmetrical adding that in order for a rectangle to be coloured it had to be created by using only two colours of lines and not three. They reiterated that a green line (a'), a red line (b') and a blue line (c') would go in the directions of the green (a), the red (b) and the blue (c) axes respectively and created lines, one of each colour, to illustrate the relationship (Figure 4):

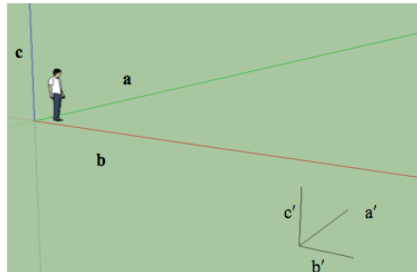


Figure 4: Red, Green and Blue lines

Subsequently, in Task B3, attention was drawn to how many coloured lines were used, and the students responded that they used all three colours whereas in the rectangle task they had used only two colours.

DISCUSSION

We provided an environment, which was intended to act as a stimulus for purposeful activity within which situated abstractions for dimension would be constructed. Using this environment as a window on that activity, we observed evidence of how the tasks and the dimensional tools enabled the articulation of ideas about direction and freedom of movement in 2D and 3D. In particular we observed the students expressing various situated abstractions:

- (i) Polygons can be ‘flat’ (in a 2D space) or ‘coming out’ (in a 3D space)

- (ii) Polygons that look flat in 3D can be disconnected.
- (iii) Polygons that look flat in 3D can be twisted.
- (iv) The axes provide useful reference points for judging the direction of lines drawn in 3D.
- (v) Rectangles are created from the use of two differently coloured lines as drawn by the Line tool.
- (vi) Cubes are created from the use of three differently coloured lines.

These situated abstractions pointed to experiences of dimension that in some cases went beyond those identified in the previous phenomenographic study. We regard (i), (ii) and (iii) as experiences of the quality of 3D space to provide more capacity and more potential for freedom of movement. In (i), shapes in 3D are able to move beyond their trapped state in 2D. In (ii), the additional direction of movement allows the possibility of illusory effects. In (iii), the additional freedom throws up the possibility of 2D shapes that are twisted. A key idea about dimension seems to be that it in some sense depicts the level of capacity of the space and we see echoes of this idea in the first three situated abstractions.

We regard (iv), (v) and (vi) as related to manner in which we ‘measure’ the dimension of a vector space in terms of the number of linearly independent vectors that span that space. In (iv), the role of axes as a key to the colours of the lines being drawn by the Line tool, gives the axes a special position. They afforded an emphasis on the ‘three’ in 3D and gave a reference point for the students when exploring the direction of lines. In (v), rectangles were seen as being ‘spanned’ by two lines. When three lines were used there was redundancy, which could lead to disconnected shapes. The tool would inevitably create a coloured rectangle when exactly two different coloured lines were used. In (vi), a similar relationship was found between the number of colours needed when creating a cube. A key idea about dimension seems to be that it indicates what is needed to create that space through linear combinations of independent vectors. The coloured lines can be seen by us as base vectors and indeed the number of base vectors needed is the dimension of the vector space.

Of course the situated abstractions are truly humble expressions of these sophisticated mathematical ideas and we do not know whether, as we hope, the situated abstractions might provide suitable resources for the development of more sophisticated ideas of vector and vector space in the future.

This study considered the significance of the setting for acting as a window on students’ experiences of dimension, focussing on the situated abstractions formed by the children within the situation designed. Google SketchUp acted as an expressive window into students’ experiences to dimension and has thrown light upon dimensional experiences relating to vectors and vector spaces that they were not present during the other situations designed. We conclude that dimension, not normally explicitly taught, can be experienced in situated ways through an

environment like SketchUp together with carefully designed tasks. SketchUp contained key dimensional tools, which help us to understand how SketchUp provided the window we needed on activity that illustrated how dimension can be experienced as a capacity for containment and as a measure of freedom of movement.

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MATHEMATICALLY GIFTED STUDENTS' ANALOGY IN STATISTICS

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Statistical thinking handles concrete data. However, many of the concepts used in statistics are abstract in nature. In statistics, analogy can be used and should be used to grasp and to represent abstract concepts. Statistics is regarded as different discipline from mathematics due to various aspects of statistics, especially the crucial role of context. This study investigated mathematically gifted students' analogy in statistics. The gifted were asked to construct similar problems to base problem that is a statistical problem. On the basis of analysis on students' new problems, researchers could classify them into four types of analogy in statistics: Success in both structural and contextual analogy, success in structural analogy, structural generalization, and failure in both structural and contextual analogy.

INTRODUCTION AND BACKGROUND

It seems reasonable to argue that because statistical thinking handles concrete data, students should have little difficulty with statistical thinking (delMas, 2004, p.86). However, many of the concepts used in statistics are abstract in nature, and according to previous researches, many students have considerable difficulty with statistical thinking. delMas (2004) argues that just as in mathematics, in statistics analogy can be used and should be used to grasp and to represent abstract concepts. Statisticians see statistics as a discipline separated from mathematics (Ben-Zvi & Garfield, 2004, p.5). The first difference between the two disciplines is that statistics is highly dependent on real-world context (Cobb & Moore, 1997, p.801). In statistics, context plays a crucial role, that is, it is difficult to put meaning on the numbers in data sets without considering the context. In statistics, numbers can have different meanings according to the context where they come from.

Analogy is a kind of reasoning to perceive and operate on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar (Richland, Holyoak, and Stigler, 2004, pp. 37-38). Researchers who notice the important roles of analogical reasoning argue that analogy is a source of mathematical discovery and mathematical problem solving (Polya, 1962) and is necessary for development of mathematical thinking (Dreyfus & Eisenberg, 1996, p.260). Gentner (1989) distinguishes between surface similarity and structural similarity in analogy. She argues that surface similarity is based on shared object

attributes, whereas structural similarity is based on the relational structure. However, she does not focus on analogy in statistics.

In mathematics education, researchers have given considerable attention to analogical reasoning in problem solving (English, 1998, p.126). When a new problem is proposed, students have to recognize the similarities between already known problem and the new problem (English, 2004, p.5). However, there are new movements that focus on use of analogy in problem posing rather than problem solving (Do, 2007). Do (2007) suggests that mathematically talented students should be able to propose problems by using analogical reasoning or generalization. He presents models and examples for design of mathematical activities that focus on conjecture and problem posing by analogical reasoning.

Stoyanova (2003) emphasizes problem posing beyond problem solving and proposes three problem-posing situations as free, semi-structured, and structured problem-posing situations. In a free problem-posing situation, students are asked to construct problems without base problems. In a semi-structured problem posing situation, students are given an open situation and are invited to explore the structure of it or to complete it. In a structured problem-posing situation, well-structured problem is given and students are asked to construct new problems related to the given problem. Kilpatrick (1987) and Polya (1957) suggested various strategies that can be used in the structured problem-posing situation. Kilpatrick (1987) used the term "problem formulation" and suggested analogy as one of the strategies for problem formulation. Polya (1957) mentioned problem posing as one of the problem solving strategies and suggested the use of analogy in the process.

Greenes (1981) describes the characteristics of mathematically gifted. They tend to construct problems spontaneously and find solutions of the problems that they develop. They also show good problem construction abilities in a given situation. Klavir and Gorodetsky (2009) used construction tasks of analogical problem to reveal the creativity as common excellence of both gifted students and expert students. The researchers presented several base problems to students and asked to them to solve target problems and construct new similar problems to the base problem. As a result, the gifted and the experts showed a high relative creativity.

It is worthy to investigate analogical reasoning ability of mathematically gifted in statistics which has different features from mathematics. In this study, the gifted were asked to construct similar problems to base problem that is a statistical problem. By analysing students' new problems, we found types of analogical reasoning of mathematically gifted students in statistics, and on the basis of this findings we discussed analogical reasoning which can occur in statistics or should have appeared.

METHOD

The participants of this study are twenty-seven 8th graders (14-year-old). They are all receiving instruction at a university-attached institute for a mathematically talented student education in Seoul, Korea. For this study, students took a 3-hour class. In the

first unit of the class, students dealt with several statistical data from various real life contexts. They were introduced two types of calculation method for average by using the data sets; one is arithmetic mean algorithm and the other is weighted mean algorithm. Students had to decide more proper method between the two methods and to justify the reason why they choose it for each context. At the end of the class, they constructed similar problems to the given problem. Figure 1 presents the problem presented to students.

There are five math classes in school A. Five math teachers teach each class. The total number of students in five classes is 50 and each class has 20, 10, 10, 5, and 5 students. One parent asked to students taking a math class how many students were in their classes. Twenty students presented 20, twenty students gave 10, and ten students offered 5. On the basis of information from the students, the parent calculates how many students are in each class as follows:

$$\frac{20 \times 20 + 10 \times 10 + 10 \times 10 + 5 \times 5 + 5 \times 5}{20 + 10 + 10 + 5 + 5} = 13$$

Explain the strengths of the method that the parent takes to calculate the average of students in one class.

Figure 1: Base problem

To make a new problem similar to the problem in the Figure 1, students have to recognize the key elements of the base problem that are presented in Figure 2. The first three elements in the Figure 2 are necessary for structural analogy, and the last element is essential for contextual analogy. In this problem, what is matter is not the number of the students in classes but the number of students whom teachers take care of. In mathematics, analogical reasoning is regarded as successful when it maintains the relation as well as structure between the elements in base problem and target problem. In statistics, on the other hand, it can be regarded as successful when it maintains the meaning of context as well as structure between the elements in base problem and target problem.

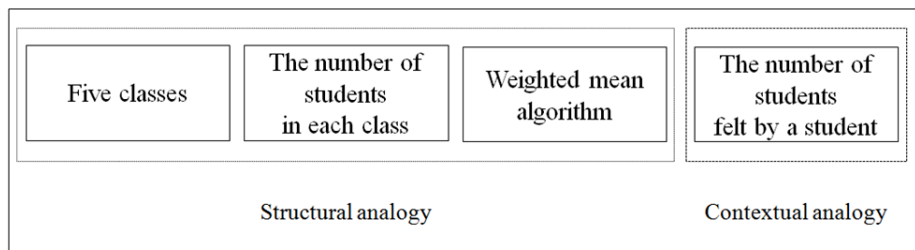


Figure 2: The main elements of base problem

RESULTS AND DISCUSSIONS

Successful analogy in statistics: Success in both structural analogy and contextual analogy

Figure 3 shows a well-made problem which analogize the context and the structure among elements out of the base problem. In this case, the student recognized the context of the base problem (the number of students felt by a student) as well as the other structural elements (five classes, the number of students in each class, and weighted mean algorithm) and transferred it to the similar context, the number of users felt by a user in a swimming pool, with structural elements together. His explanation "Many people said that there were much more people when they got to the pool." shows that he had understood the context in the base problem very well. Figure 4 presents successful analogy in statistics. Unlike areas in mathematics including geometry and algebra, in statistics contextual analogy as well as structural analogy is very important.

People investigated the number of users of a swimming pool for one week. The numbers of users of the swimming pool from Sunday to Saturday were 20, 25, 23, 44, 40, 52 and 40, respectively. The swimming pool manager said that the number of users of the pool per one day is about 35. However, many people said that there were much more people when they got to the pool, so they claimed that the average number should be calculated as follows:

$$\frac{20^2 + 25^2 + 23^2 + 44^2 + 40^2 + 52^2 + 40^2}{20 + 25 + 23 + 44 + 40 + 52 + 40}$$

If any person wants to buy a weekly ticket of the pool, which is better number he or she have to consider?

Figure 3: An example of successful analogy in statistics

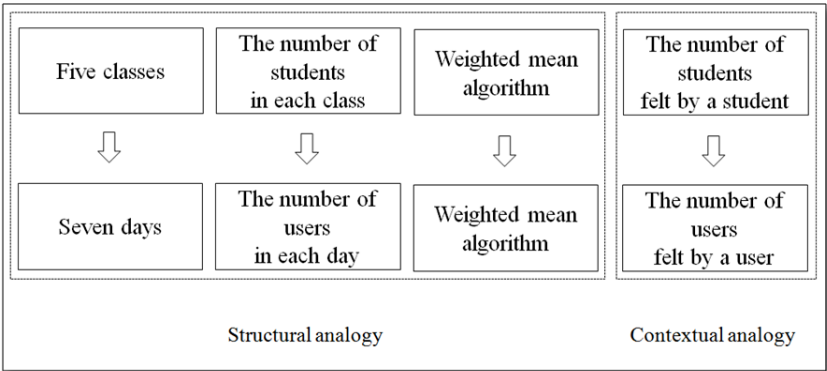


Figure 4: Framework for analogy in statistics

Unsuccessful analogy in statistics 1: Success in structural analogy only

Figure 5 shows an example that succeeded in structural analogy and failed in contextual analogy. The student tried contextual analogy by calculating the number of concert watchers per one broadcaster. However, the student's problem has no similar context to the base problem and is not natural. In the student's problem, the number by weighted mean algorithm has different contextual meaning from the base problem. The number of concert watchers per one broadcaster is not related to each watcher. People do not have any complaint or displeasure even though each broadcaster has too many watchers. The student did not succeed in contextual analogy. In other words, the student failed in analogy in statistics.

There are three broadcasters, KBS, SBS, and MBC, broadcasting the live concert of one famous Korean singer. The total number of the concert watchers through the three broadcasters is 120. People watch the concert through one broadcaster at home. The numbers of audiences who watch the concert on KBS, SBS and MBC is 50, 50, and 20, respectively. In a survey, the whole 120 concert audiences were asked to the number of people who watch the concert on the same broadcast. One hundred people said that the number is 50, twenty people said 20. A researcher calculated the average as follows:

$$\frac{50 \times 50 + 50 \times 50 + 20 \times 20}{50 + 50 + 20}$$

What is the difference between the value of him/her and arithmetic mean $\frac{50+50+20}{3}$?

Figure 5: An example of success in structural analogy only

For analogical reasoning to be successful in statistics, it is very important to transfer the meaning of the numbers within the context because numbers dealt with in statistics are dependent on context. The example in Figure 5 reveals that the student has a poor understanding of the meanings of the data and the context within the base problem. Therefore, we can conclude that problem posing by analogy in statistics gives opportunities to evaluate how well students understand the meaning of the data and the context within problems.

Unsuccessful analogy in statistics 2: Structural generalization

Figure 6 is an example that tries to generalize the use of arithmetic mean and weighted mean. The student succeeded in structural analogy only by generalizing the structure of the base problem, but the weighted mean cannot have any meaning because the context is improper for the weighted mean. This kind of generalization is powerful in geometry and algebra but not in statistics if context is not considered. Because context is the most important in statistics, generalization not considering context is not powerful and does not have any meaning.

There are bags a_1, a_2, \dots, a_n , including x_1, x_2, \dots, x_n numbers of marbles, respectively. Suppose that we know the numbers x_1, x_2, \dots, x_n . Then arithmetic mean of the numbers of marbles is $\frac{x_1 + x_2 + \dots + x_n}{n}$. Explain the reason why the following mean is better than arithmetic mean:

$$\frac{x_1^2 + x_2^2 + \dots + x_n^2}{x_1 + x_2 + \dots + x_n}$$

Figure 6: An example of improper generalization in statistics

Figure 6 shows that the student generalized algorithms for two kinds of averages by corresponding classrooms and the numbers of students to bags and the numbers of marbles. The student could recognize the main elements of the base problem and generalize them. This response supports previous findings (Greenes, 1981; Sriraman, 2003) that mathematically gifted understand structure of problem and attempt to generalize it. In the process of generalization, however, the student omitted context that is important in statistics, so that the meaning of two means in the problem disappeared. He overlooked the fact that numbers handled in statistics depend on context. This implies that mathematically gifted students can fail to recognize the influence of context in statistical reasoning and thinking even though they have high ability in grasping the structures of problems.

Unsuccessful analogy in statistics 3: Failure in both structural and contextual analogy

Figure 7 shows an example of failure in both structural and contextual analogy. The student tried to correspond the classrooms and the number of students to the arrow scoring point system and the number of arrows, respectively. He calculated two kinds of averages of points per one arrow by corresponding the mean value calculated by the coach to arithmetic mean and the mean value calculated by the player to weighted mean. However, the weighted mean algorithm in the student's problem is the same to the algorithm of arithmetic mean. It looks like algorithm of a weighted mean in this situation, but it is another expression finding arithmetic mean in essential. The student made a problem which cannot be applied weighted mean algorithm by focusing on superficial similarities. We can conclude that the student do not have good understanding of arithmetic mean as well as weighted mean.

An archer in Tae-reung Training Center is shooting arrows with a bow. He shoot 10 points in 10 times, 9 points in 9 times and 8 points in 8 times. The coach claimed that the average of the point was $\frac{10+9+8}{3} = 9$, but the archer calculated it as $\frac{10 \times 10 + 9 \times 9 + 8 \times 8}{10+9+8} = 9.4$. What is the good point of presenting the value the archer calculated as "average"?

Figure 7: An example of failure in both structural and contextual analogy

CONCLUSION

In this study, mathematically gifted students were asked to construct similar problems to the given statistical problem related to arithmetic mean and weighted mean. According to analysis on the problems students constructed, their responses are categorized into four types: Success in both structural and contextual analogy, success in structural analogy, structural generalization attempt that they experienced in algebra and geometry, and failure in both structural and contextual analogy.

The students' responses suggested several educational implications. First, this activity gave opportunities for students to experience that analogical reasoning in statistics needs contextual analogy as well as structural analogy. Second, tasks constructing similar problems in statistics can evaluate how well students understand the content that statistical ideas are applied. For example, students who understand the meaning of weighted mean could success in both structural and contextual analogy, but the students who do not have good understanding of the idea constructed problems focusing on structural analogy only, so they constructed problems with inappropriate context.

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REACHING DECISIONS VIA INTERNAL DIALOGUE: ITS ROLE IN A LECTURER PROFESSIONAL DEVELOPMENT MODEL

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In this paper we consider the professional development of university mathematics lecturers. We describe two exemplars from a two-year research process to engage mathematics educators and research mathematicians in a constructive dialogue about teaching. In this study lectures were video recorded and then discussed in a supportive community of practice. Using Schoenfeld's Resources, Orientations and Goals (ROG) theoretical framework we analyse two lecture segments and describe how the two lecturers' ROGs caused them to make decisions that moved them toward different outcomes. The value of explicit ROG-based discussion of small-scale lecture moments to a professional development model is considered.

INTRODUCTION

Delivering mathematics content to large numbers of students via the medium of lectures presents a number of pedagogical difficulties that are seldom explicitly addressed (Speer, Smith, & Horvath, 2010). This paper presents a small part of a two-year project examining a professional development model, with two key questions: whether an effective lecturing professional development strategy can be built around a community of practice; and whether Schoenfeld's Resources, Orientations and Goals (ROG) theoretical framework (see details below) can be adapted to analyse university mathematics lecturing. While the effectiveness of various approaches to teacher professional development has been extensively examined, comparatively little is known at collegiate level (Speer et al., 2010). Addressing this need, this project structured community interactions to prioritise three practices identified as effective. Firstly we focused on "small, but meaningful, aspects of practice" (Speer, 2008, p. 219), "at the very level of detail when development and change appear to occur—the moment-to-moment decisions and practices of teachers" (Speer, 2008, p. 263). Such fine-detailed examination has been successfully used in microteaching in teacher education. Secondly, all discussions within the group followed the protocol that these should develop from concerns identified by the lecturer themselves. This aligns well with Robinson (1989), who argues that professional development that recognises the teacher as a professional and works from the 'bottom-up' is empowering, providing teachers with opportunities to make meaningful choices, and with Paterson and Barton (2009), who showed that teachers evidenced positive changes when working on self-identified areas of concern in their practice. Finally, the development of a community of practice was actively fostered (Buckley & du Toit, 2010).

THEORETICAL FRAMEWORK

Schoenfeld (2008, 2010) has developed a theory of teaching-in-context, with a goal of answering how and why teachers make the in-the-moment choices they do while they are engaged in the act of teaching. The current framework is based on Resources, Orientations and Goals (ROG) that teachers bring to their practice. Thus, what a teacher decides to do while engaged in teaching is a function of the teacher's goals, orientations, including dispositions, beliefs, values, tastes and preferences, (which serve to prioritize goals), and resources (particularly knowledge) (Schoenfeld, 2008). Hence, when a teacher enters a classroom they use their orientations to adjust to the situation. Goals are established based on the orientation, and relevant knowledge is activated. Decisions consistent with goals are made, consciously or unconsciously, about the directions to pursue and the resources to use (Schoenfeld, 2010). Classroom decisions, including those made in-the-moment, are crucial, since "The quality of people's decision making...affects how successfully people attain the goals they set for themselves." (Schoenfeld, 2010, p. 36), and analysing these decisions should be part of a professional development programme. However, an individual's ROG may contain competing goals inspired by differing orientations. This latter situation is the subject of this paper, as we seek to analyse how the conflict of competing goals arising from an internal dialogue between mathematician and teacher are resolved.

METHOD

In this study four research mathematicians and four mathematics educators, all from Auckland University, formed a community to re-examine lecturing practice in the light of educational theories. This paper discusses lectures of two research mathematicians that were video-recorded; each lecturer then chose a small section of less than five minutes that the whole group watched and discussed together in a supportive manner. This focussed discussion was audio-recorded and later transcribed. We found that the discussion often started with the video content but then moved on to examine other relevant, related issues of learning, practice and mathematics. The video and audio transcription data was supplemented by a lecturer ROG written before the lecture and interviews. The data was analysed by focussing on the relationship between decision points and the ROGs.

DESCRIPTION OF TWO SITUATIONS

The two situations we describe involved experienced male lecturers, who we call Sandy and Simon, presenting an applied mathematics lecture to first year students and a postgraduate lecture in number theory, respectively. The primary purpose of Sandy's lecture was to consider solutions, for various values of the parameter q , of a difference equation that reduced to the form $x_m = qx_{m-1}(1 - x_{m-1})$. He revealed a number of orientations relevant to this lecture segment. Firstly, as a teacher he values demonstrating results, even without precise prior knowledge of what may occur:

O1: "It is good to demonstrate things to the students rather than just tell them."

O2: "I'm pretty happy to experiment in this course and things will occasionally go wrong when you do that."

He also has a pedagogical belief that, since he is part of a teaching team:

O3: It is important to stick to the course book and cover all the material.

With regard to the use of Matlab, his orientations were:

O4: It has value for exploration "I just wanted to explore, show them the graphs on the screen using Matlab and change the Matlab a little bit to get a closer look at the graphs"

These were some of the orientations that led to the establishment of a number of goals, some of which Sandy explicitly wrote down, and some we infer, including:

G1: To show students that interesting, unexpected things happen to the solution as the parameter changes.

G2: Students understanding, from the demonstration, that the solution of the difference equation is a periodic function.

G3: To show how easy it is to discover these things by using Matlab.

G4: To have students appreciate the value of Matlab as a mathematician's research tool. "...the use of computers to um.. explore mathematics that's something that I see as a mathematician and that I'd like to impart that..."

G5: To keep students interested. "...we wanted to keep them interested and so this was an extra lecture showing some more advanced features of the logistic equation that's usually taught at graduate level."

G6: To explain clearly the mathematical basis of the construction and solutions of the difference equation.

G7: To stick to the course book and cover all the assigned material.

To achieve these goals he called on resources, including his mathematical knowledge (R1) and the computer program Matlab (R2), which was used to plot solutions of the equation for various values of the parameter. The solutions were then displayed using an overhead projector (R3). At one point Sandy made the decision to move to the projected graph and show the students the periodicity of the function.

Why did he decide to do this? In line with his orientations O1, 2 and 4 it was part of his attainment of the goals G2, 3 and 6. O1 was crucial here, the belief that demonstrating and not just telling leads to understanding better, as he said "In fact the solutions are periodic and it was a bit hard to look and see that straight off that the solution's periodic, so that's why I wanted to do the counting." However, it was during this process of counting the function local minima that a crucial decision point arose. The lecture transcription follows.

1. What's happening here it looks even more complicated, 3.6...[3 to 4 secs] yeh so you can see that if you look at it closely...[walks to screen]

- 2 Suppose you start by looking at this value here [pointing at the graph on the projection] then there's going to be 1, 2, 3, 4, 5, 6, 7, you can count to 8 I think maybe.. do I ever get back to where I started, maybe not [*realises there's a problem*]
- 3 9, 10, 11, 12, 13, 14 um.. how many values? So it looks like there's a period of um.. let's see 1, 2, 3, 4, 5, 6, 7, 8.. [*starts to count again*] it looks like there's a period of 14. Whether that's the case or not I'm not sure.
- 4 We might not have got to the limiting value yet. But it looks like we've settled down to a period of 14. By a period of 14 I mean that it takes 14 um.. we need n to change by 14 to get back to where you started from... So it seems to be settling down to some complicated periodic um.. solution.

We see here that the count of the period arrives at 14, but Sandy knows that the true value is 16, as he later explained "...because the period doubles each time, so it goes from 2 to 4 to 8 to 16, so.. and so on, so there's a theory that actually says the period has to double." This was unexpected, "I guess the thing that I was probably concerned about was um.. observing something that I didn't expect and not being about to explain it immediately". Hence, in-the-moment, he has to decide whether to address what is, for him, a mathematical discrepancy. How did he make the decision?

Analysing the decision—Sandy

Arriving at the decision involved an internal dialogue between the lecturer as a mathematician [M] and as a teacher [T]. This dialogue had as its aim the resolution of conflict between the competing pedagogical goals, G1, 2, 3 and 5, and the mathematician's goals of G4 and G6. We see this from Sandy's comments about this.

Yeah, in fact my decision was based on the fact that I'd already spoken far longer than I'd planned to [G7 T] on the existing equation and it was time to actually go and do some problems [G7 T] which was supposed to be the rest of the lecture so I got onto that [G7 T].

Actually I would have liked to have pursued it a bit [G6 M] but we had already spent more than the allotted amount of time on this demonstration [G7 T] and I had shown them periodicity for shorter periods already [G2 T] so I think they had grasped the concept quite well, so the fact that I didn't actually get a period of 16 bugged me a bit [G6, mathematician] but not enough to ruin the rest of the lecture [G7 T].

I certainly made a decision not to continue with an unexpected outcome on a graph in the first part of the lecture. Part of me wanted to address this at the time [G4, 6 M] but I had already gone over time with this part of the lecture and had achieved the goals I desired [G1, 2, 3 T].

We see that, in *this* situation, with *these* students, the teacher wins out over the mathematician. The reason seems to be that the predominant goal was G2, to demonstrate that 'that the solution of the difference equation is a periodic function'. This had been accomplished, and meeting his pedagogical goals released him from the need to explain the mathematical anomaly, as he said "Actually I didn't get any

reaction from the students...I never did tell them it was really 16.” He confirmed that, as a teacher, he was happy with the outcome of the lecture, including this decision:

I’m pretty happy with the way the lecture went. Students seemed interested [G5 T], in our exploration of the logistic equation [O1, 2, 4; G1, 3 T] and participated in the exploration. In response to questions, we changed the Matlab code to zoom in on graphs of solutions, which allowed us to clearly see the periodicity [G2 T].

The actual lecture itself went pretty well I was pleased with the demonstration [O1, 2, 4; G1, 2, 3 T] um.. I think it was clear enough. The students were able to see what was going on [G2 T].

Simon’s lecture introduced the students to continued fractions. He too revealed a number of orientations relevant to this lecture but we only present here the ones related to the decision we examine in detail.

- O1: To emphasise to students that the right theoretical tools and proof techniques can tame a mathematical problem.
- O2: Some proofs are more interesting and important than others. “The real reason I think it’s a cool proof is the fact that you prove a more general result. It’s one of these things that happens a lot in mathematics, you don’t see it much at the junior level.”
- O3: Mathematics needs to be correct. “Oh, this is not really right, I don’t like it not to be right.”
- O4: Mathematical notation needs to be consistent and accurate. “Right so the symbols h_j over k_j will from now on will mean precisely one of these things for the specific numbers I am interested in.”
- O5: Students who are talented at mathematics, such as this class, can cope when a lecturer dwells on the finer points in mathematics. “There is also a confidence that the students can cope with – if I go off on my own little journey the students will have the tools to deal with that ... Whereas in another class you would be worried that if I’d lost them after 15 minutes then that’s it.”
- O6: Some (but not all) all students at this level are ready to be inducted into mathematics. “Last year’s class I didn’t feel like they were being inducted into mathematics so it wasn’t necessary to dwell on this particular issue of the more general result.”

In the ROG he writes before the lecture he states the following goals for the course:

- G1: To increase the students’ mathematical maturity by helping them understand the theory and do proofs.
- G2: To provide good general preparation for post-graduate study in number theory.
- G3: To give exposure to different proof techniques.
- G4: The most important theoretical part, for this first lecture, is to state and prove correctness of the recurrence formulae for computing the convergents.

Some of Simon’s goals emerge during discussion.

G5: To engage with the mathematics for its own sake, ‘it’s fun.’

G6: To ensure that the mathematics he does is ‘right’; it’s part of his role as a lecturer

G7: To use notation that is consistent.

G8: To induct (some) students into thinking and behaving like mathematicians.

In order to satisfy his goals he draws on a number of resources, including his knowledge of mathematics in general and number theory in particular, (R1) and his assessment of the students' mathematical ability and interest (R2). The decision we will discuss here is one he made when he suddenly realised that he was going to encounter a 'notational conundrum' while proving the correctness of the recurrence formulae for computing the convergents. The lecture transcription follows.

1. So, by the inductive hypothesis, [*starts to write*] I know what this is. [*gestures in swirl over previous line*] It is some $\frac{h_i}{k_i}$. [*looks at board, as if he is thinking*]
2. I'm going to call it... Did I give it a name? [*Looks at paper*] I didn't give it a name. It's just some $\frac{h_i}{k_i}$. But whatever that $\frac{h_i}{k_i}$ is it apparently satisfies the recurrence formula [*points to paper looks at class*] [*Stands back, looks at the board, pauses*]
3. Yeah I mean this is *an* $\frac{h_i}{k_i}$ but it's not *the* $\frac{h_i}{k_i}$ that I am really thinking of [*gestures back to previous expression*] This is a very subtle point.
4. Let's define h_i over... Let's define $\frac{h_i}{k_j}$ to be these things up to a_j where I have worked these out already all the way up to i [*writes down and puts in rectangular box above previous 2 expressions*] Right so the symbols $\frac{h_i}{k_i}$ will from now on mean precisely one of these things for the specific numbers I am interested in. This thing I have written down here [*gestures*] is *not* the $\frac{h_i}{k_i}$ in that notation because this end term is wrong.

Analysing the decision—Simon

When asked why he made a decision to 'labour the point' and disentangle a problem of which the students were not (yet) aware, Simon said it was the mathematician within going "Oh this is not really right" [G6 M] and added "at that point the whole world disappeared and it's just me and the mathematics." [G5 M]. "I suddenly realised that it is sort of not quite fitting how I was using those symbols previously. [G7 M] I was thinking ahead to where the proof was going and suddenly it becomes clear to me that there is a problem ahead but it's not clear to anyone else yet." [R1] He likens himself to the driver of a group of interested, but unworried, tourists (the students) who to his surprise notices a 'Danger Ahead' sign on the roadside while the others are all still happily chatting at the back of the bus. He could 'put his foot down and hope' and tell the students "it's in the notes ... it would be a good exercise for you to do carefully." But he chose to sort it out, to labour the point, and take the bus down the bumpy road. [G5 M and G2, 10 T].

When describing why he chose this section to re-view he spoke about a mismatch between what he did and his written ROG. Discussion uncovered higher order goals that drove the decision—his unwritten orientation and goals as a mathematician [G5, 6, 7 M] and his desire to induct talented students into mathematics [G2, 8 T]. He is not alone in needing to resolve the “tensions experienced by the lecturer in satisfying student needs and mathematical values.” (Joworski et al., 2009 p. 249). It was his estimation of students’ ability [O5] that allowed him to take the ‘detour’. His original assumption that because he saw them as good students [O5] he would say ‘Just do it’ proved incorrect in-the-moment. As he says ‘last year there was not the same ability so I went through the proof much more lightly with them. It wasn’t necessary to dwell on this particular issue of the more general result.’ The need to ‘get it right’ [G6, 7 M] and to induct them [G8 T] won the day.

DISCUSSION

Can we say that this professional development model works? Is the analysis in depth of small parts of a lecture chosen by the lecturer, against an explicit framework of ROGs within a community of practice, of pedagogical value? The feedback from all involved suggests that the answer is yes. A number of important aspects contribute to this. One is that having a ‘mixed’ group of mathematics educators and mathematicians enables cross-fertilisation of ideas. Sandy commented, “I gained a mathematics education perspective ... which clarified in my own mind what I do when I teach”, and “...you come in with your theory from time to time explaining some of the things that we all do and that’s very useful.” One valuable aspect of the process was the opportunity to see others’ teaching. As Sandy said “And also seeing other people teaching, that’s wonderful.” Towards the end of the year it was agreed that it would be a good thing if the practice of watching others became ‘business as usual.’ While we agree, we contend that the subsequent, focussed discussion is extremely important.

Secondly, the community was deliberately set up to be supportive. While Simon’s comment, “It’s pretty revealing watching yourself being videoed isn’t it?” shows that lecturers were sensitive about exposing their practice, Sandy maintained “It was reassuring that nobody thought that [he looked silly]...that was good.” and that the group was “very supportive, very supportive.” He valued feedback and discussion from a practical, teaching perspective: “So it’s good to get that feedback from other people and in some cases people identify things that I do that I wasn’t even aware of...I have my usual techniques for teaching but it’s good to get some opinions on these.” The development of a community of practice is evidenced in the fact that when we observed that “you [Simon] looked how [Sandy] looked when he was worried about the thing”, we all knew ‘the thing’ referred to his 14/16 dilemma and subsequent decision. As we work the repertoire of shared decision moments is growing and enables new ones to feed off them. The process of thinking about, and then writing, one’s ROG was another feature commented on, with Sandy saying how it improved his lecture. In Simon’s case the ‘dissonance’ he perceived between his

stated ROG and his decision, led to a discussion of his higher order goals. This led him to say that he would like to think more about the audience and that “anything that encourages me personally, to put more thought into who the audience really are, what actually they know, that’s extremely useful...and it’s astonishing to think that that is not automatically done”. The ROG structure also provides a framework for discussing the ‘objects of practice’ the lecturers chose. Engaging mathematicians (even more than teachers) in a conversation about pedagogy—particularly their own—is enabled by linguistic and theoretical support that is grounded in their practice. We suggest that a fine-grained analysis of small-scale lecturer-chosen lecture segments, against the ROG framework, activates an awareness of the basis on which we make teaching decisions, prompting examination of these decisions leading to development of practice.

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THEORETICAL DRAGGING: A NON-LINGUISTIC WARRANT LEADING TO ‘DYNAMIC’ PROPOSITIONS

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This study sought to examine how the competence of student's instrumental decoding affects the development of their ability to construct dynamic meanings in a dynamic geometry environment. I shall be presenting types of instrumental obstacles stemming from the students' tools selection or from the Euclidean definition of segment. This sense may lead to the understanding of how the tools the students use play a fundamental role as a non linguistic warrant. Seen in this light, Toulmin's model will be used to represent the construction of a dynamic proposition through a structural analysis of the students' argumentation using the software's DGS tools.

INTRODUCTION AND THEORETICAL FRAME

The present paper attempts to bridge the world of digital technology and the world Euclid bequeathed us in his "Elements". The crux of the paper are the ‘dynamic’ meanings evoked in a DGS environment in tandem with the geometric meanings discussed in their textbooks. These meanings are a response during *instrumental genesis* (e.g., Rabardel, 1995) through the tool use of the software and the development of argumentation as a discursive process, supported by the visualization provided by the dynamic diagram. Dynamic geometry software, such as The Geometer's Sketchpad (Jackiw, 2001) was used in this study. Üstün and Ubuz (2004) consider that “the Geometer's Sketchpad is an important vehicle of technological chance in geometry classroom. [...] The shapes are first created and then they are explored, manipulated and transformed to ideal concept”. Students execute on screen constructions using software's tools and primitive geometrical objects in an effort to decode their mental representations into software actions. This sense of how the student's competence at instrumental decoding affects the development of their ability in constructing meanings, may lead to an understanding of how the tools the students use, play a fundamental role as a non linguistic warrant. Since Plato, argumentation has been considered a core means of building knowledge. In Toulmin's (1958/1993) model “an argument comprises the claim (i.e. the statement of the speaker), data (i.e. data justifying the claim) and the warrant (i.e. the inference rule, which allows data to be connected to the claim)” (Pedemonte, 2007, p. 27). Pedemonte (2007, p.28) has presented Toulmin's basic structure of an argument constructing a figure with the three basic elements mentioned above (fig. 5). In Pedemonte's (2007) opinion “Abduction is an inference which allows the construction of a claim starting from an observed fact. *Induction* is an inference which allows the construction of a claim generalising from some particular cases. *Generalisation* plays an important role in inductive argumentation. Deduction is an

inference allowing the construction of a claim starting from some data and a rule. In Toulmin's model a step appears as a deductive step: data and warrants lead to the claim." (p. 29). In the following sections I shall introduce the meanings of theoretical and experimental dragging, 'dynamic' point and 'dynamic' segment. I shall then examine what the DGS *instrumental decoding* is, and how the structure of an argument is shaped when students use a tool--for example the dragging tool.

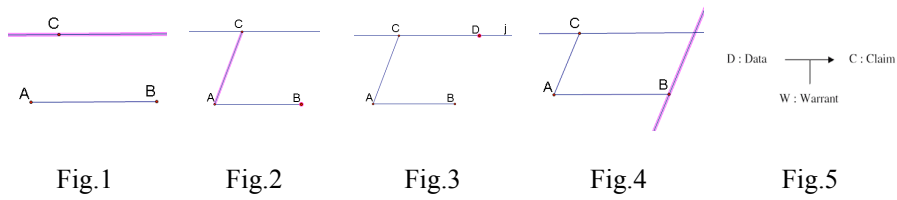
Theoretical and experimental dragging

Diagrams are a kind of external representation (students' geometrical constructions, for example). Gonzalez and Herbst (2009) have defined the *dynamic diagram* as "a diagram made with DGS and that has the potential to be changed in some way by dragging one or more of its parts" (p.154). Students using dragging are led "to understand how a geometric construction can be defined by a system of dependencies" (Jackiw and Finzer, 1993). Researchers (e.g., Parzysz, 1988; Hollebrands, 2007) distinguished between two types of geometrical objects: *figures* and *drawings*. Parzysz (1988) defined a *drawing* as a representation of a geometrical object and a *figure* as the "text defining it [the geometrical object]" (p.80). Dragging preserves the properties of geometrical objects constructed in the DGS environment. According to Mariotti (2000, p.36) "the dragging test, externally oriented at first, is aimed at testing perceptually the correctness of the drawing; as soon as it becomes part of interpersonal activities [...] it changes its function and becomes a sign referring to a meaning, the meaning of the theoretical correctness of the figure." Hollebrands (2007) also supported that the students in her study "used reactive or proactive strategies when dragging, either in response to or in anticipation of the effects on dragging" (cited in Gonzalez and Herbst, 2009, p.158-159). Building on Mariotti's considerations and Hollebrands distinction about dragging strategies, I consider there to be two main diacrisis in dragging utilizations with regard to students actions: (a) the *theoretical dragging* in which the student aims to transform a drawing into a figure on screen, meaning s/he intentionally transforms a drawing to acquire additional properties and (b) the *experimental dragging* in which the student investigates whether the figure (or drawing) has certain properties or whether the modification of the drawing in the picture plane through dragging leads to the construction of another figure (or drawing).

What is DGS instrumental decoding?

The construction of a figure on screen in a DGS environment is a result of a complex process on the student's part. The student has first to transform the verbal or written formulation ("construct a parallelogram" for example) into a mental image, which is to say an internal representation recalling a *prototype image* (e.g., HersHKovitz, 1990) that s/he has shaped from a textbook or other authority, before transforming it into an external representation, namely an on-screen construction. This process requires the student to decode their actions using software primitives, functions etc. In order to accomplish a construction in the software the student must acquire the competence for *instrumental decoding* meaning the competence to transform his/her mental

images to actions in the software. Competence in the DGS environment depends on the competence of the cognitive analysis which students bring to bear when decoding the utilization of software tools, based on Duval's (1995) semiotic analysis of students' *apprehension of a geometric figure*. Duval has distinguished three kinds of operations, one of which is the *place way*, meaning an operation which changes a figure's orientation. During the development of a construction, I believe that the student has to develop three kinds of *apprehension* when selecting software objects which accord with the types of cognitive apprehension outlined by Duval (1995, pp.145-147) namely *perceptual*, *sequential*, *discursive*, and *operative apprehension*. In concrete terms, the competence of *instrumental decoding* in the software's constructions depends on: a) the *sequential apprehension* of the tools selection (i.e. s/he has to select point C and segment AB and then the command (fig. 1) meaning that s/he has to follow a predetermined order); b) the *verbal apprehension* of the tools selection which means the student has to verbalize this process, (i.e. s/he says "I am going to select point C and the segment AB") and c) a *place way type* of elements operation on the figure (i.e. when s/he transforms the orientation of the elements to apply the command selecting point B and the opposite side AC, for example in fig.4) due to his/her *perceptual apprehension* (fig. 2, 4). Then s/he has constructed the *operative apprehension* of the figure's elements for the construction, meaning the competence to operate the construction. The figures below (fig. 1, 2, 4) illustrate the *linking visual active representations* (Patsiomitou, 2010) of the steps in the students' construction of the parallelogram.



The 'dynamic' point

Secondary students in Greece study the axiomatic foundation of Euclidean geometry from the first classes of high school. In terms of Euclid's definitions "*a straight line* is a line which lies evenly with the points on itself" (def. 4). Between the fundamental Euclid's definitions they have heard, is the definition of the segment. The definition of a segment AB given in most Greek geometry textbooks is as follows: "segment AB is the shape which consists of the endpoints A, B and the set of points which belong to the portion of the line between these two endpoints". The DGS environments have been designed to take the Euclidean definitions and propositions into account. I use the term '*dynamic*' point to refer to a point made in a DGS. A '*dynamic*' point is a fundamental element in a dynamic construction. '*Dynamic*' segment is a segment made in a DGS. According to the Geometer's

Sketchpad reference manual (2001) “points are the fundamental building blocks of classical geometry, and geometric figures such as lines and circles are defined in terms of points” (p.11). Hollebrands, Straeser and Laborde (2008, p.165) described the distinction between the three different kinds of points in a DGS environment: (a) a free point “can be directly dragged anywhere in the plane (degree of freedom 2)”, (b) a point on an object “can be dragged only on this object (degree of freedom 1)” and (c) a constructed point “cannot be grasped and dragged (degree of freedom 0) but moves only if an element of which it is dependent is dragged”. Hegedus (2005) also reports about points in a DGS environment as “hot-spots” something “not been an artifact of the environment but an axiomatic part of the system that allows “true” mathematical figures to be built” (p. 2). According to Hegedus (ibid.) “dragging a hot spot illustrates how the Euclidean construction [...] has been correctly implemented in this sketch” so “the hot spot is a critical part of the construction” (p. 2).

RESEARCH METHODOLOGY

The didactic experiment was conducted in a class at a public high school in Athens during the second term of the academic year. The students of the experimental group followed a re-conceptualized *learning path* for the teaching and learning of parallelograms in geometry, using Geometer's Sketchpad environment, reported at the previous PME conference (Patsiomitou and Emvalotis, 2010). The learning path has been “inspired by and utilises the van Hiele theory of geometric thought levels, currently acclaimed as one of the best frameworks for studying teaching and learning processes in geometry” (Atebe, 2008, p.3). Firstly I examined the student's level of geometric thought using the test developed by Usiskin (1982) which is in accordance to the van Hiele model (Fuys et al., 1988). The original van Hiele level classification is the following: Recognition (Level 1), Analysis (Level 2), Informal deduction (Level 3), Formal deduction (Level 4) and Rigor (Level 5). Olkun, Sinoplu and Deryakulu (2005) argued that “the Geometer's Sketchpad is a suitable dynamic environment in which students can explore geometry according to their van Hiele levels” (p.3). In the first phase of the research process the students had to build parallelograms with an emphasis on the “*construction*” menu. My intention was to introduce the Sketchpad tools and commands ‘step by step’, “in parallel with the corresponding theory” (Mariotti, 2000, p.41), because from my previous experience the students too often make mechanical use of the software and, this in return renders them unable to understand the logic behind the command options. I have recorded in detail how the students came to understand the use of the tools and correlated this ability with the partial construction of the meanings. An analysis of these findings led me to conclude that the student's use of the tools in the software encourages them to construct other ‘dynamic’ meanings in addition to the geometric meanings discussed in their textbooks. The sections that follow will present snapshots from the research process which led me to formulate two main research questions: (1) Do students face *obstacles* in a DGS environment due to instrumental decoding? (2) Do students build ‘*dynamic*’ *propositions* in the DGS environment?

RESEARCH PROCESS

Obstacles due to instrumental decoding: I distinguished a few types of *obstacles* due to student lack of competence in instrumental decoding (i.e. this is to say an *instrumental obstacle*). I am going to describe two of them including snapshots of the research process. **A.** The students tried to construct a parallelogram using the Geometer's Sketchpad. Most students at van Hiele level 1 were unable to understand the *sequential apprehension* of the tools selection, because they were unable to understand the logic of the sequence of actions or unable to link this logic with the theory of geometry. For example M14 (van Hiele level 1 at the pre-test) faced an instrumental obstacle which depended on her *sequential apprehension* of the objects to be used for the construction. She tried to construct a parallel line by selecting the line alone and then the menu command, which is to say she followed an irrational sequence of actions. At this point, she faced an *instrumental obstacle* and commended in an informal way on the non-activation of the software's command (saying "[the command] is not illuminated again"). Subsequently, her interaction with the software, led to a cognitive conflict which helped her to apprehend the sequence of actions. Students of van Hiele level 2 developed the three kinds of *apprehension* along with the other members of the group: *verbal apprehension* emerged as a result of the previous action in the software, namely as a result of the interaction with the tools. For example, as a result of the previous action M2 (van Hiele level 2 at the pre-test) states: "this will be a line parallel to segment AB". **B:** The utilization of Euclidean definition 4 mentioned above presented level-2 students with *instrumental obstacles* in the DG environment. Thus: the group prompted student M8 (van Hiele level 2 at the pre-test) to select the segment in order to construct a perpendicular line. Among the definitions he knew was the definition of a segment mentioned above. He therefore followed the definition of the textbook, decoding the verbal expression by selecting the segment and its endpoints. This action results in the command not being activated on screen, so he was unable to continue the process. This is to say a *cognitive conflict* occurred between what the students knew from the Euclidean geometry definitions they had learned and what they encountered in the DGS environment. Exactly the same thing happened to student M2 when she tried to select a segment to construct its midpoint. This action led the students to apply new rules inductively and to understand empirically something that we could define by answering the question "what is a 'dynamic' segment?". The '*dynamic*' segment is a portion of a straight line which does not consist of points. *Dynamic points* can be placed independently on the dynamic segment and move free with one degree of freedom on the path to which they belong. This means that a point placed on a segment has its *two degrees of freedom* transforming into *one degree of freedom*. In a second example, student M3 tried to select a point on the straight perpendicular line intersecting with the segment AB in order to construct the sides of an isosceles triangle. Trying to decode the verbal formulation "select a point on the straight line" in the DGS environment they were unable to do it on the dynamic line (or the dynamic segment) they had constructed. Student M3 thus faced a *cognitive conflict*

which led him to understand that he had to select an independent point and put it on the line. This is exactly the time in which student set a new rule something we could define: the selection of a segment in a DGS environment occurs with the selection of its internal alone, which represents the set of points in the Euclidean definition.

The construction of ‘dynamic’ meanings

Students M9, M10, M14 (van Hiele level 1) tried to construct a parallelogram (fig. 1, 2, 3, 4). They constructed a segment AB and a point C, then a parallel line from the point C (fig.1). They were unable to understand how to continue the process by constructing a parallel line from point B, which is to say they lacked competence in the *place way operation* of the tools. M14 constructed a point D on line j (fig. 3) and began with repeated *experimental dragging*. Visually she understood that point D had to preserve its congruency with the opposite segment AB if the figure were to remain a parallelogram. So she dragged point D again, stating “the dot (i.e point D) must be almost here in order to become the congruent of the segment”. She then used *theoretical dragging* to turn her *drawing* into a *figure* of a parallelogram. By virtue of *instrumental genesis* the student has constructed an *instrument* which includes the *utilization scheme* of dragging and the meaning of the congruent opposite sides of a parallelogram. Seen in this light, the software’s primitives are non linguistic visual data and the tool is the *warrant* for the construction of a *dynamic proposition* which is empowered in a dynamic geometry environment.

DISCUSSION

The diagram below (fig.6) is an adaptation of Toulmin’s model with tool use. Points C, D are the *data* D1, D2 for the actions that follow. *The experimental dragging* tool operates as non-linguistic *warrant* in Toulmin’s model for the students’ understanding of both the stability of point C and of the modification of point D and, hence, of segment CD. The construction of the *claims* C1, C2 begins with “an observed fact” (Pedemonte, 2007, p.29). Through *instrumental genesis*, the tool affects the students’ understanding that opposite sides of a concrete parallelogram should be congruent, making this the abductive part of the process. *Theoretical dragging* affects the construction of an intrinsic *inductive rule* (i.e. The dynamic segment CD can be modified when it is dragged from point D so that it becomes congruent or not with another segment) which leads to a *generalization of the rule* for any dynamic segment. The transformation of the position of the point through dragging leads to the transformation of the segment, which leads in turn to the comprehension of the *dynamic proposition* S1 which is the *deductive claim* (i.e. If a dynamic segment is dragged from its endpoint with one degree of freedom then it will not preserve the visual constraints of congruency with another segment).

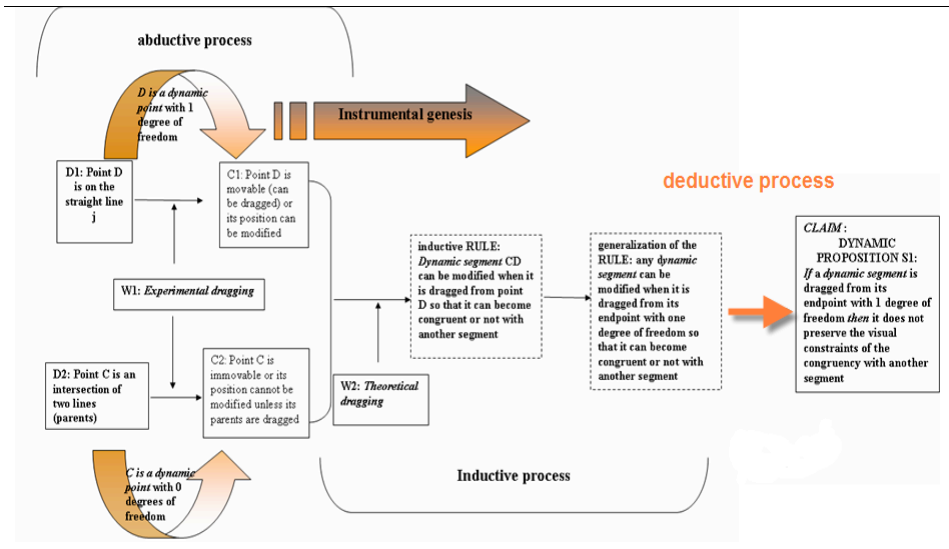


Fig.6

Dynamic proposition S1 leads students to perceptually understand that congruent *dynamic segments* are a result of the degree of freedom of the endpoints of a segment. I should like to conclude by claiming that, for this reason the *instrumental knowledge* of a dynamic geometry environment is “knowledge with [its own conceptual] rules” – to paraphrase Skemp’s instrumental understanding of “rules without reasons” (Skemp, 1978, p.9). These rules reinforce students’ development of geometrical thinking by reinforcing their relational and logical understanding (Skemp, 1978).

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IN-SERVICE AND PRE-SERVICE TEACHERS' STRATEGIES OF TASK ADAPTATION

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In school settings it is expected from teachers to select, formulate and adjust tasks in order to attain specific goals. We investigate in-service and pre-service teachers' strategies for task adaptation. The strategies were explored from the point of view of the targeted mathematical or pedagogical focus. Several types of strategies were identified based on their potential goal. The study shows that in-service and pre-service teachers have different preferences concerning the types of strategies to apply - a situation that is linked, most probably, to practices in teacher education.

INTRODUCTION

Research shows that learning in school settings depends on teacher. Several researchers (Raymond, 1997; Schoenfeld, 2008) suggested that teachers' knowledge, beliefs and attitudes are key elements that define classroom actions and interactions. On the other hand, students' learning depends on the tasks given to them and their implementation determines their cognitive level (Doyle, 1988). As key elements in orchestrating classroom interactions and discussion, teachers decide on what aspects of a task to focus, how to organize the class, what questions to ask to support the mathematical inquiry of students with different knowledge levels (NCTM, 2000). In order for this to happen, the teacher needs to have good tasks at hand. In turn, the design of a good task is an iterative, time-taking process (Liljedahl et al., 2007) that goes through a series of phases: predictive analysis, trial, reflective analysis, and adjustment.

Although it is expected from a teacher to be aware of all the aspects involved in the preparation of an effective task, often they cannot deal with such complexity. The identification of the essential mathematical content asks from the teacher to have a deep understanding of the mathematics they teach (Ma, 1999). The teacher also needs to have techniques to modify the problems at hand and be habituated to analyze the resulting problem from mathematical and pedagogical point of view, beside its overall correctness. Teachers need to understand the potential impact of any change they bring to a task – from mathematical and pedagogical point of view – and build on that understanding in order to elaborate sequences of tasks that will deepen the students' understanding. In the same time, they need to adapt to the flow of classroom dynamics, which makes teachers' activities even more complex (Simon, 1997).

The very first step of this complex process is task preparation. Our focus in this paper is on a particular type of task preparation: reformulation of a problem under given constraint. We analyze and classify the strategies used by three categories of participants: pre-service bachelor, pre-service master level, and in-service teachers in the reformulation of a problem. We argue that there is a link between the category of participants and the strategies employed, as well as their personal arguments for the choices they made.

METODOLOGY

Participants: There were three categories of participants at these experiments: 12 in-service teachers, participants at a summer camp; 16 pre-service teachers who were students - at bachelor level – of a didactics of mathematics course, and 6 master students. Bachelor students had no direct contact (meaning, institutional) with school-children and did not take, yet, a didactics course. Master students had already covered a didactics module (including practice with school-children). They were taught how to put mathematical problems in a realistic context, because the new curriculum requests it. The in-service teachers that participated in the experiment had at least 5 years of teaching in school, at both secondary and college level. However, their training did not include the creation and use of context in mathematics teaching.

Tools: The participants were presented with a mathematical problem and two solutions of it. After the presentation, they were requested to formulate, starting from the displayed problem, questions that would be: easier, harder, more beautiful, and more useful. None of these characteristics were explicitly defined and the reason for it was that we were also interested in their own formulation of criteria. All participants had a few days to complete this task. They had to handle explications concerning the applied criteria and the solutions for the proposed problems.

Design of the study: Initially, the experiment was run with two groups of participants: students at bachelor level and teachers. The starting problem was the following:

$$\text{Solve the system: } \begin{cases} x + y + z = 9 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \end{cases} \text{ in the set of positive real number.}$$

From the analysis of their answers, we formulated some preliminary conclusions. In order to verify those, we reran the experiment with a category that is „between” pre and in-service teachers: the master students. Since we also wanted to see the influence of the starting problem on the participants’ behavior, master students saw just a part of the problem, namely, the part that leads to the solution. They received the following formulation:

If x , y and z are three positive numbers which satisfy the relation $x+y+z=9$, prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 1$.

For both cases, we gave two solutions.

The first one uses the fact that for any a positive real, we have $a + \frac{1}{a} \geq 2$. A second solution uses the Cauchy-Buniakowsky-Schwartz inequality: for the given problem, the inequality applies for $\sqrt{x}, \sqrt{y}, \sqrt{z}$ and $\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{y}}, \frac{1}{\sqrt{z}}$ and leads to the conclusion $x = y = z$ for the first problem and the requested inequality in the second problem, respectively.

ANALYSIS OF THE PROCESS

As specified above, the reformulation task required from each participant to give four new problems related to the initial one. Although not all the participants gave four problems, some proposed more than one for a certain criteria. In our analysis we included only those problems that were not direct textual reformulations of the, initially, algebraically specified one. No doubt, textual reformulations offer a rich context for studying problem solving, but at this point we were interested in the participants' strategies to *mathematically* modify the problems. For example, consider the following text:

Be x , y and z the length of the sides of a triangle. The perimeter of the triangle is 9 and the sum of the inverse of each side is 1. Prove that the triangle is equilateral.

When solving this “new” problem, one has to transform it using an algebraic language, and arrive to the “old” problem. Therefore, we consider that this is a textual reformulation, but not a mathematical modification.

As a first step of our analysis, we clustered the strategies based on the degree to which they seem to focus on the pure mathematical content of the problem or on ways to help students to get to that content. In this very first classification we relied on Liljedahl's et al. (2007) *usage and goals* model of mathematical tasks. We focused on the part that refers to tasks having the goal to promote mathematical understanding. Liljedahl et al. (2007) mentions two ways to reach this: to use mathematics (noted by them as mM) and to use pedagogy (noted as pM).

Once the first classification made, it became obvious that there are important types of strategies inside of both categories. After several steps of refining we obtained the strategies presented in Table 1.

<i>Mathematics for mathematics (mM)</i>	<i>Pedagogy for mathematics (pM)</i>
Maintaining math content of the initial problem	Guide of the solution with specific hints
Focus on a specific element needed in the solution	Building a context for the task
	Manipulating conditions in the problem
	Adapting to a particular school setting

Table 1: Types of strategies used by the participants in the experiment

These strategies, connected to different aspects of the problem solving process, are explained below in more detail and illustrated by examples.

Maintaining mathematical content of the initial problem

The existence of a unique solution for the starting system (in which the number of variables is bigger than the number of equations) is assured by the fact that we can get to an extreme case of an inequality. Consequently, we considered that the mathematical content of the starting problems was maintained if the solution of the newly posed problem needs the use of this extreme case. The goal of such a strategy is to show the use of a similar solving process in another context, sometimes easier, sometimes more difficult. Example of such strategy:

Solve the following system in \mathbf{R}_+ :

$$x_1 + x_2 + \dots + x_n = n^2$$

$$1/x_1 + 1/x_2 + \dots + 1/x_n = 1$$

Solve the following system in \mathbf{R}_+ :

$$x_1 + x_2 = 4$$

$$1/x_1 + 1/x_2 = 1$$

Focus on a specific element needed in the solution

Often, the specific element needed in the solution is studied by its own before being used in the solving process of another problem. In order to build a stand-alone problem, one needs to identify this element. On its turn, it requires deep understanding of the role of each element (conditions, form of equations, etc.) in the problem and within the solution. In this case, later on, the student will identify the particularities of a situation where the fact can be applied. Even more, by presenting a part of the solution as an “introductory” problem, one draws student’s attention to the conditions under which the conclusion remains valid. In some cases, the participants in this study isolated a partial result, needed in the solution for the initial problem: we consider this fact as an independent strategy. Example:

Prove that for any $a, b \in \mathbf{R}_+^$ we have: $a/b + b/a \geq 2$*

Guide of the solution with specific hints

It is common in textbooks to give hints or impose constraints regarding the solution method for a problem. Teachers use hints in order to help the students starting the problem solving process or to make them aware of a particular aspect. In this paper we refer to a special kind of hint such as the use of identities or inequalities. The difference with the previous aspect is that here the indications are contained in the problem rather than being separated. Frequently, student’s attention remains concentrated on the problem and not on the hint. One of the possible consequences of this fact is that the conditions under which the *hint* is valid are not clarified. In the present paper, we considered this aspect present if there were clearly included indications on how to proceed with the problem. Example:

Use the Cauchy-Buniakowski-Schwartz inequality $\left(\sum_{i=1}^3 a_i^2\right)\left(\sum_{i=1}^3 b_i^2\right) \geq \left(\sum_{i=1}^3 a_i b_i\right)^2$ to prove the following: if $x+y+z=9$ then $1/x+1/y+1/z \geq 1$, with x, y and z positive real numbers.

Building a context for the task

Learning in context supposes that students process new information in such a way that it makes sense to them in their own frames of reference. In this paper, we consider that a *context* could be constructed by the inclusion of a problem into a “family”. Such an inclusion has the purpose to facilitate not only the understanding of the solution of that problem, but also connections and transfers. Students can apply the same reasoning to several problems presented together since they share elements in their formulation and strategy of solving (Singer & Voica, 2003). It was considered that a strategy shows a context building if there are several assumptions that work together, in the sense defined above. Example of a master student who formulated a series of questions:

Is the relation from the initial problem true if one of the numbers is negative? In the affirmative case, find the biggest set for which it is true.

Find the maximal set for which the initial problem is true.

Is there a minorant (biggest one) for the expression $1/x + 1/y + 1/z$ when $x + y + z = 9$ and x , y and z are positive real numbers? Is there a majorant (a smallest one)?

These questions are related to the initial problem and highlight the elements used in the solution. For example, the first one emphasizes the fact that all the unknowns are positive numbers.

Manipulating conditions in the problem

Another aspect of interest was the focus on the conditions that appear in the problem. These give a domain of validity to the problem and its ‘solution; therefore, by manipulating it, students’ attention is drawn on the link between conditions and applicable solution method. This was considered present whenever the conditions of the newly posed problem were modified in comparison with the initial one. Example:

Solve in the set of natural numbers the system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 9 \\ 1/x_1 + 1/x_2 + 1/x_3 &= 1 \end{aligned}$$

Adapting to a particular school setting

The last type of strategy to be considered separately was the one that take into account school-settings. From practical point of view, a teacher prepares the tasks for his/her group of students and these tasks have to correspond to the knowledge level and the openness of the students, between many other aspects. In most cases, this means also to stick to the curriculum material. In our analysis a strategy was considered to belong to this type if the comment from the participant (on his own strategies) suggested the consideration of some aspect of a school-setting.

Solve the following system in the set of positive real numbers:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 1/x_1 + 1/x_2 + 1/x_3 &= 9 \end{aligned}$$

The participant’s comment (an in-service teacher) was: *this problem is easier (then the initial) because the student can observe more easily the replacement of 1 than the multiplication of the two relations.*

In table 2 we give an overall summary of the strategies used by the participants. Here are included all the strategies identified in the participants’, independently of the particular request they answered (easier problem, harder, nicer or more useful).

Strategy /Sample	Bachelor students	Master students	In-service teachers
Maintaining math content of the initial problem (mM)	12	8	21
Focus on a specific element needed in the solution (mM)	1	5	1
Guide the solution with specific hints (pM)	-	3	-
Building a context for the task (pM)	-	4	-
Manipulating conditions in the problem (pM)	3	1	4
Adapting to a particular school setting (pM)	11	2	14
Wrong formulations	10	2	4

Table 2: Distribution of strategy types between the participants

During the analysis, we saw that not all problems were well-formulated or solvable. Therefore, it is important that teachers clearly understand the affordances of a problem (and since this requires a correctly formulated situation) therefore it was of interest to include the number of wrongly posed problems by each category of participants. This information is shown in the last row.

DISCUSSION AND CONCLUSION

From the values in table 2 we can observe that the most frequent strategy employed was the first one: *preserving the mathematical content*. There are at least two things to highlight in relation to this. First, even if the solutions were given at the beginning, it was the personal task of each participant to try to make sense of them. Second, flexibility in designing new tasks is influenced by the level of understanding of the solutions (Singer et al., 2011). With two exceptions, all the participants proposed a two variable two equations system - of the very same form as the initial one - as an easier problem. However, we considered that only those who chose some particular constant values kept the mathematical content. Our hypothesis is that they understood the relation between the constants values from the initial equation and its solving process, therefore their understanding allowed them to design a problem of exactly the same type. This hypothesis is further sustained by the fact that all the participants who used this strategy to make an easier problem also used it to get a difficult one.

A second aspect to comment on is the fact that the few participants that used this strategy to propose a more difficult problem do not use it for an easier one. Interestingly, this was the case only for pre-service, bachelor level students. For the

easier problem they all used the last strategy, adapting to school settings, which in their case means an adaptation to the written curriculum and to an abstract school situation. Their arguments for choosing such strategy mostly consisted of references to the availability of the solution tools: „there is no need for artifice”, „one can solve the problem in a natural way, with known methods”, etc. However, this argument is not a correct one, since the easier version of the very same initial system can be solved by „known methods”. Since none of the other participants had such a choice (the adaptation to a pedagogical strategy for the easier problem and the first strategy for the more difficult one) we explain this by the lack of concrete class experience of these participants, on the one hand, and, possibly, by the fact that they have not yet seen examples of such usage during a didactics course.

Surprisingly, the second strategy was not used by teachers as we expected it would happen. No doubt, teachers are used to have situations in which as a first step they solve a problem related to a particular result and, then, they use the result in subsequent problems. The same can be said about the third and fourth strategies. The specification of solution methods, giving hints and having problems with multiple points, multiple questions (that can be considered a particular case of a context) are common in textbooks and in the classroom practice. Therefore, we explain their non-use by the lack of experience of building such situations even when participants might have quite an experience in using them.

In-service and pre-service (bachelor level) teachers seem to prefer the strategies that modify the domain of variables and the one for adapting to school settings. Those who modified the domain of variables did so for the easier and more difficult problem asking to solve the very same system over the set of natural numbers or the set of real numbers. The arguments for choosing such strategy were mostly of the type: „one can enumerate all possibilities” or „reducing specifications usually add to the difficulty”. As said before, manipulating the conditions can give an insight on the relation between the available solving methods and conditions specified in the problem. As expected, most teachers reformulated the task by referring to his/her particular class. This meant also that most of them completely changed the task, so there was no similarity at all in the mathematical content with the starting problem. In this sense our result is similar to the finding reported by other researchers (Doyle, 1988; Kim & Stein, 2006), namely that teachers often provide rote procedures and skills-based practice problems in order to ensure the student’s success. Meanwhile, for the in-service teachers, this might be a direct consequence of their teaching experience. This let us with the question of how to stimulate teachers to take up the challenge to adapt tasks to their class. Research shows that it is not enough to have experience to formulate correct problems (Voica & Voica, 2011) and our results suggest that many teachers simply prefer to propose what they are sure it will work in their class.

As a more general observation, master pre-service students showed more variety in their strategies corresponding to richer pedagogical and mathematical content

considerations than other participants did. Our explanation is that they had been instructed in a different way, mainly due to changes in teacher instruction programs. Bachelor level students still lack the experience of facing concrete school class situations and their instruction is still more directed to the development of mathematical knowledge rather than the pedagogical one. In-service teachers on their part have been through another kind of instruction, much more theoretical and more in the tradition of a „show and tell” presentation.

Alltogether these results led us to the conclusion that there is a need to inform in-service teachers about new practices and give them opportunity to learn about how to develop good material and how to include such material in their class. In the same time, in-service and pre-service teachers need to be put in situations that permit to explore their own understanding of the purposes of tasks used in the classroom and that of their own teaching methods.

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HOW SPECIAL EDUCATION TEACHERS THINK ABOUT THEIR STUDENTS' MATHEMATICAL ABILITIES

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Research has shown that high expectations of teachers about their students' academic development have a positive influence on how these students actually develop. Therefore, when working on better learning results it is necessary to know how teachers think about their students' abilities. The present study was meant to investigate what perceptions primary school teachers in special education have of their students' potential in mathematics and what possibilities they see to reveal this potential. Data were collected through an online questionnaire. Surprisingly, the responses showed that, although they work with weak learners, most teachers were quite positive about the mathematical potential of their students. Moreover, teachers could underpin their view with observations from school practice.

THEORETICAL BACKGROUND

Importance of high teacher expectations

Over the last decades, it has become more and more clear that expectations of teachers about what their students will accomplish in school are a determining factor in how the students actually develop. This awareness about the crucial role of teachers' expectations on student achievement has its origin in research by Rosenthal and Jacobson (1968), who were the first to describe the so-called 'Pygmalion effect'. In short, this self-fulfilling mechanism means that the teachers' ideas about their students' potential in a particular subject determine the way the teachers treat their students. As a result, teachers create a learning environment that contributes to their students' academic development (see e.g., O'Connell, Dusek, & Wheeler, 1974). Consequently, positive teacher expectations are considered a key for achieving better learning outcomes for students (Beswick, 2008; Jepma & Meijnen, 2004). To achieve that teachers have positive expectations about their students, programs for professional development put much effort into opening the eyes of the teachers for the talents of students (see e.g., De Lange et al., 2008).

Focus on recognizing talents of students

In the Netherlands, both the Onderwijsraad [Advisory Council for Education] (2007) and the Inspectie van het Onderwijs [Schools Inspectorate] (2007) emphasize the importance of recognizing students' talent and not leaving it unused. According to the Inspectie van het Onderwijs (2007) the latter is particularly relevant to special education (SE). A similar plea for recognizing the qualities of students with learning difficulties can be found in the *Individuals with Disabilities Education Act*

Amendments (2004), which also emphasizes that recognizing and utilizing students' talents, is an important focus of educational policy.

Although there are not many studies that focused on identifying unused talents of SE students, there is sufficient evidence that SE students are more able than it is generally assumed. An effective approach to reveal the talents of SE students is presenting them with mathematical content that is beyond the regular primary school curriculum in SE. For example, various researchers have shown that students who are weak in mathematics can be successful in problem solving when hints are available (Bottge, Rueda, Serlin, Hung, & Kwon, 2007). Good results have been also found with topics that are generally not dealt with in SE, like ratio (Van den Heuvel-Panhuizen, 1996) and combinatorics problems (Peltenburg & Van den Heuvel-Panhuizen, in preparation).

Furthermore, dynamic assessment approaches that open the zone of proximal development have proven to be successful in disclosing the talents of SE students. Whereas the standardized way of assessing requires an approach in which students are not allowed to receive any help or use auxiliary resources, dynamic assessment creates an environment that concentrates on gaining insight into the students' potential by investigating whether the student is able to solve a problem with some help (Campioni, 1989). For example, a study on the topic of subtraction problems that require 'crossing the ten' revealed that an ICT-based dynamic assessment which allows students to make use of optional auxiliary tools gives them better opportunities to show their mathematical potential compared to a standardized test format (Peltenburg, Van den Heuvel-Panhuizen, & Robitzsch, 2010).

The question is whether the previously described policy of not leaving students' talents unused and the underlying belief that unused talent exists in SE students are also endorsed by teachers.

Teacher expectations of weak students

Research has shown that teachers in general have lower expectations of students who have learning difficulties than of those who have not (see e.g. Zohar, Degani, & Vaaknin, 2001). These lower expectations are also reflected in their teaching. For example, Beswick (2008) found that teachers opt for a less varied and challenging curriculum for students they consider weak in mathematics than for students of whom they have higher expectations.

How teachers think about the academic possibilities of their students is often linked to what they see as the causes of their students' success or failure in school (Cooper & Burger, 1980). Teachers attribute students' weak performance to various factors. Apart from *student factors*, such as cognitive limitations, *environmental factors*, such as lack of parental involvement, they also mention *teacher or teaching factors*, like poor teaching or an inadequate curriculum, as a cause for learning problems (Soodak & Podell, 1994). In other words, these results suggest that teachers are aware of their own role in their students' learning results.

Since teachers seem to realize that factors outside the abilities of students could determine students' weak performance, again the question can be asked whether at the same time teachers acknowledge that unused talents exist in students. Although it could be argued that teachers are pre-eminently in a position to recognize students' talents, one may wonder whether this really happens. Teachers in SE are usually overwhelmed by merely achieving the targets of the existing program. Therefore, they will not often proceed to try out new topics or new didactical approaches which have shown evidence of unused talents in a research setting. Nor will there be much opportunity to develop and try alternative testing methods alongside the often required standardized tests.

RESEARCH QUESTIONS

Based on the above, we formulated the following research questions:

1. What ideas do SE teachers have on the mathematical abilities of SE students? Do SE teachers agree that there is an unused potential in SE?
2. Have SE teachers experienced the necessity to adapt their expectations about their students' development in mathematics? If yes, for what reason?
3. What ideas do teachers have on revealing mathematical potential of SE students?

RESEARCH METHOD

Data collection and group of respondents

To answer our research questions, we sent an email containing a link to an online questionnaire to 298 Dutch schools for SE. This number covers 95% of the total number of schools for primary SE in the Netherlands. A total of 84 completed questionnaires was returned. A non-response study was carried out to be sure that participation was not biased by a particular view on the topic investigated. An a-select sample of 20 schools was approached by telephone and was asked for their reason not responding to the questionnaire. All schools stated lack of time as the reason for non-response.

Because most of the email addresses were general addresses, not linked to a person some of the persons who answered were not teachers. The group of respondents worked all in primary SE and consisted of 52 teachers and 32 other staff members, such as internal coaches, school leaders and other persons working in school. Because teachers are more often in direct contact with the students our focus was on teachers. Therefore, we decided to only include them in our analysis. Most teachers were quite experienced; on average they worked in SE for 16 years. The age of the students that the teachers worked with varied from an average of 9 years to 11,5 years.

Questionnaire

In the online questionnaire that was developed for this study first a number of general questions was asked about the respondent's responsibility in school and his or her

work experience. Then three open questions followed about the respondent's views and ideas about the mathematical potential of SE students. In the first of the open questions the respondents had to react to a particular statement about the idea that there is unused potential in SE. The second question surveyed the respondent's experiences that may have led to raising expectations of SE students' potential. This question was illustrated with an example from a teaching practice in which emerged that SE students could solve a combinatorics problem. The third question asked the respondents in what way they think the mathematical potential of SE students can be revealed.

Data analysis

The respondents' answers were entered into the software program Atlas ti. To carry out the coding a classification scheme (see Figure 1) was developed in which eventually all answers of the respondents fit. The scheme was developed in three rounds. The final coding was done by the first author. A subset of answers was also coded by the second author. The agreement was about 95%.

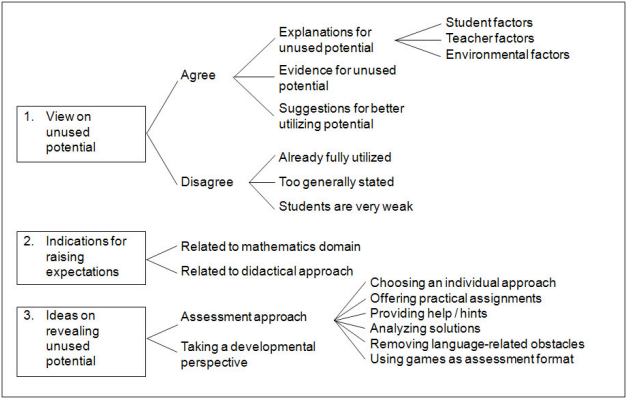


Figure 1: Classification scheme for the teachers' answers

RESULTS

Ideas about the existence of unused potential in primary SE

Agreement on the idea that unused mathematical potential exist in SE

In total, 31 teachers (60%) agreed that there is unused mathematical potential in SE. They clarified their view by giving an explanation for this underutilization (referring to a *student*, a *teaching* or an *environmental factor*), by offering some form of proof, or by giving a suggestion for better utilizing the SE students' potential.

Explanations for unused potential (n=22)

Out of the 22 explanations that were given for the existence of unused potential in SE, only four teachers mentioned explanations that relate to student factors. These teachers indicated that behavioral, concentration and memory problems in students

are obstacles in developing their potential. More frequent were explanations relating to characteristics of the teachers themselves. Seven teachers mentioned these. The one that was mentioned most often was that teachers have low expectations of students. One of the teachers explained as follows: “I think that many SE teachers concentrate on making the child feel safe and comfortable in school, and this is often combined with not making high demands.” Another teacher-related explanation was lingering too long over a particular topic. The most common explanations, mentioned by 11 teachers, were those referring to environmental factors. Eight of them brought up organizational problems. The teachers pointed out the differences in level between students in class. Another explanation, given by two teachers, had to do with the quality of the mathematics textbooks. It was said that they do not connect with the knowledge and the interests of SE students. Finally, one teacher mentioned the influence of standardized testing. According to this teacher, mathematics teaching in SE is holding too much to standardized testing, and this holds back students.

Evidence for the existence of unused potential (n=7)

Seven teachers offered some kind of proof for the existence of unused potential. Three of them described their experiences that students were able to do more when they were given suitable instruction; for instance through giving instruction individually or in small groups, or through offering more practical activities. The other four teachers stated that students in their schools work on activities that belong to the program of grades 5 or 6, which are clearly beyond what is generally seen as what is achievable in SE.

Suggestions to better utilize potential (n=2)

The two teachers who suggested ways to better utilize the potential of SE students raised the matter of changing current mathematics education. Teaching should focus more on the individual student and should be made more challenging, for example, by offering them practical assignments.

Disagreement on the idea that unused mathematical potential exist in SE (n=21)

The 21 teachers (40%) who did not think that there is unused potential in SE, named three reasons for their view. One argument was that the potential is already being fully utilized. This group of ten teachers made it clear that their school already is doing all that can be done to “fully achieve the students’ potential.” They mentioned for instance putting more effort to challenge students, using better testing procedures, and going faster through the curriculum. Nine teachers rejected the statement because they felt that it was too generally stated. According to them unused potential may exist, but certainly not for all students. Two teachers disagreed with the statement because according to them SE students are very weak in mathematics.

Experiences that necessitated teachers to raise their expectations

In total, 32 teachers (62%) indicated they had to adjust positively their expectations of the potential of SE students as a result of particular experiences. Of the remaining teachers, 11 stated that they did not have such experiences, and the other nine left the

question unanswered. All teachers who changed their mind about the potential of SE students, illustrated their answer with an example, either related to a specific mathematics domain or to a specific didactical approach.

Experiences related to a mathematics domain (n=19)

A surprising finding was that seven teachers gave an example related to the domain of combinatorics. These teachers might have been inspired by the example given in the questionnaire. On the one hand, mentioning this example may be a form of socially desirable behavior. On the other hand, this topic is so far removed from the SE mathematics curriculum that it is unlikely the teachers were led by this. Also, unexpectedly, positive experiences were mentioned two times with respect to fractions as well as to the use of graphs. More common mathematics domains were also brought up. Six examples were given on the domain of measurement and geometry, including working with shapes and symmetry. One teacher provided an example of calculating with money, in which a student determined in a very systematic way what products could be bought for so many euros. Finally, there was a teacher who mentioned results relating to calculation up to 100. To his surprise, a student who was working in the number domain up to 20, could calculate problems up to 100 with the aid of the empty number line.

Experiences related to a didactical approach (n=13)

Four teachers changed their mind on the basis of their positive experiences with a specific approach to assessment, i.e. giving children more opportunities by letting them use scrap paper or by allowing them to take more time to do a test. Two teachers had expectation-raising experiences with offering their students practical assignments, such as returning money to a customer. According to these teachers these practical activities challenge the students and connect to their interests, and as a result they are quicker to pick up the topic than when they are working from a book. Two teachers also made discoveries about the children's capabilities by offering them games and another two teachers by letting students work together. Furthermore, two teachers implemented a step-by-step approach in their teaching and one teacher used concrete materials.

Ideas on revealing the mathematical potential of SE students

The third question asked the teachers to give their ideas about how to reveal the mathematical potential of SE students. In total, 23 teachers (44%) either gave no answer or explicitly stated that they did not have any suggestions. The other 29 teachers (56%) did have ideas about revealing potential. Six teachers mentioned formulating a 'developmental perspective' which means that teachers set their expectations for each of their students. However, the majority of the teachers gave ideas related to specific assessment approaches.

Assessment ideas (n=23)

Teachers who suggested specific assessment approaches, showed a diversity of ideas. Seven teachers mentioned individual testing with a diagnostic character, indicated as 'learning interviews', 'diagnostic interviews' and 'diagnostic teaching'. Four teachers emphasized the use of practical assignments. They expected that SE students would have better opportunities to show their abilities if assessment includes more so-called 'do-assignments', like paying in a shop or planning a journey by train. Four other teachers came with the idea of including aids in tests such as giving them hints, allowing them to use multiplication tables and calculators. Within this category the use of so-called 'learnability tests' was mentioned as well. This means that the student is given a worked example. Three teachers suggested not to look just at students' test scores, but also to analyze their solutions. These teachers argued for a qualitative analysis of student work, allowing a better insight into students' strong points. Three other teachers saw opportunities to establish the mathematical potential of SE students by avoiding word problems in assessment. These teachers made it clear that they see the low technical reading level of SE students as an obstacle for successfully solving wordy problems. The solution offered by these teachers was to offer short, clearly formulated problems, and reading aloud the problems. Finally, two teachers suggested assessing students by using games, since, according to them, formal assessment does not always give good insight into students' abilities.

CONCLUSIONS AND DISCUSSION

The present study has revealed three striking outcomes:

- Many SE teachers believe that there is unused mathematical potential in SE students.
- SE teachers can give empirical support from their own teaching practice for the idea that SE students are more able than it is generally assumed.
- SE teachers have clear ideas about how to reveal the potential of SE students which largely reflect a preference for new assessment approaches.

Because of the nonrandom sampling procedure and small sample of teachers, the findings cannot be generalized to the entire population of SE teachers in the Netherlands. Other limitations are the restricted number of questions and the fixed way of questioning that did not allow an in-depth interview of the teachers' thinking about the mathematical abilities of SE students. Nevertheless, the teachers were very rich in their answers and gave us a good picture of their ideas, including helpful assessment approaches. The most important message is that SE teachers are rather positive about SE students' potential. Our findings demonstrate that in SE the climate for raising mathematical performances is advantageous.

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INQUIRING MATHEMATICS TEACHING AT THE UNIVERSITY LEVEL

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The paper investigates mathematics teaching at the university level in a lecture form. The context was a first year calculus course in a programme of study leading to a mathematics degree. The study is based on the collaboration of three researchers of mathematics education one, the lecturer of the course, who was also a research mathematician. The focus is on the lecturer's teaching decisions, actions and reflections and on the way that these were linked to his research and teaching experiences. Through a teaching episode on the use of counterexamples for refuting invalid claims, certain characteristics of university mathematics teaching and its development emerged.

INTRODUCTION

Mathematics Education Researchers' predominant concerns at the University level are students' cognitive processes and their difficulties with advanced mathematical concepts (Artigue, Batanero & Kent, 2007). However, a small number of studies focus on mathematics teaching and its development. Most of these studies deal with specific teaching interventions such as project work (Niss, 2001; Vithal, Christiansen & Skovsmose, 1995) or with a small number of students mainly at the tutorial level (Jaworski, 2003; Nardi, 2008). However, teaching a large number of students in a lecture format which is the usual instructional activity at the university level has very little been investigated (some examples will be discussed below in the theoretical background section).

This paper is based on a wider study of university mathematics teaching and its development in a collaborative frame among lecturers and researchers. Here, the focus will be on one lecturer's teaching practice in a Calculus course and on how this is related to his different sources of knowledge coming from university teaching, mathematical research and mathematics education research experiences.

THE THEORETICAL BACKGROUND

Speer, Smith and Horvath (2010) through a systematic literature review reported that most studies related to college instruction focused on the impact of instructional activities on students' learning while very few on the actual teaching practice. Through an analysis of five research examples related to teaching practice, they clarified what they considered as teaching practice and its particular meaning at college level. In particular, they made a distinction between instructional activities and teaching practice. According to this distinction the lecture, the context in our

study, is an instructional activity while teaching practice concerns teachers' thinking, judgments and decision-making in planning, teaching and reflecting on the lesson. Weber (2004) also investigated teaching practice at the university level through a case study of a lecturer teaching a mathematical analysis course. He distinguished three teaching styles related to the teaching of proof: the logico- structural, procedural and semantic. The styles that the lecturer adopted in his teaching seemed to be influenced by his beliefs about mathematics as a research mathematician and about students and teaching as an experienced mathematics lecturer. Jaworski, Treffert-Thomas and Bartsch (2009) studied university mathematics teaching in a linear algebra course identifying two modes of a lecturer's talk: the expository and the didactic. The first concerns the lecturer's thinking about mathematics and the second about teaching. The above distinction is related to the different sources of experiences that a lecturer utilizes in his planning, teaching and reflecting on the lesson. Similarly, the study of Speer and Wagner (2009) indicated the different aspects of knowledge needed by a university lecturer in order to provide analytic scaffolding during classroom discussions in an inquiry oriented curriculum on differential equations. In the above case studies teaching at the university level was framed by the lecturers' experiences as research mathematicians and university teachers. In our study, the lecturer brings experiences from his research both in mathematics and mathematics education and from his teaching as a university lecturer. He also participates in a community of inquiry with two other researchers. In this community teaching becomes the focus of discussion through critical reflection and the goal of this community is developmental (Jaworski, 2006; Potari et al, 2010). Overall, the lecturer engages in a multitude of simultaneous practices in the sense described by Skott (2010) related to the teaching and learning of mathematics when he designs, acts and reflects on his teaching.

In this paper, we focus on the lecturer's teaching on refuting mathematical claims by using counterexamples. Counterexamples are essential mathematical tools in the generation of mathematics and is an area that is getting an increasing interest in research in mathematics education (Giannakoulis et al., 2010; Potari et al., 2009; Weber, 2009; Zodic & Zaslavsky, 2008). In particular, the underlying reasoning in the process of refuting invalid claims and the generation of counterexamples are important issues in proof research. The lecturer had also been involved in research in this area.

METHODOLOGY

The context of the study

The study was based on a semester calculus course taught to first year students in the mathematics department of a central Greek University. The lecturer of the course, the third author of this paper, was a mathematician and a mathematics education researcher with more than twenty years teaching experience at the university. The other two researchers were a mathematics education researcher and a high school

mathematics teacher who was also a graduate student in mathematics education. The content of the course included limits of sequences and functions, continuous functions and the related theorems, derivative and its applications (local maximum and minimum, the Rolle's and Mean Value Theorem, the study of function's monotony and the L' Hospital rule). The emphasis was on the relevant concepts and on the proof of the theorems while less attention was given to the computations. The course was compulsory and taught in parallel in three classes of approximately 100 students. Less than half of the students are successful every year, so a large number of students retake the course several times throughout their studies.

Data collection and analysis

The first two researchers, observed the 26 two- hour lectures of the course during a semester. The lectures were audio-recorded and transcribed while field notes were also kept. A meeting of the three researchers followed each lecture where the group identified central issues related to the lecturer's teaching actions and discussed about his decisions and the rationale behind them. The discussions in the meetings were also audio-recorded and transcribed. The analysis of the data was ongoing and retrospective. In the ongoing analysis a number of issues emerged concerning mathematics teaching. These issues were traced through our data from the teaching and patterns of teaching strategies were identified. Then, from the lecturer's reflections during the meetings the sources on which the lecturer based his specific teaching decisions were also identified.

RESULTS

In the lessons studied, a number of teaching strategies were identified aiming to the construction of mathematical meaning. These included the use of generic examples as a basis for key ideas for proof construction; connections between visual and formal representation for supporting conceptual understanding; abductive reasoning as a way to understand and develop the deductive proof; examples to illustrate critical characteristics of concepts and; counterexamples for refuting invalid claims and investigating the conditions needed for a theorem to be valid.

In this paper we analyse a teaching episode related to the use of counterexamples and the discussion during the group meeting after that class. The episode refers to the use of counterexamples in the case of proving that the inverse of a theorem is not valid. The teaching was about the convergence of sequences. After the proof of the theorem "if a sequence (x_n) converges to a real number x then the sequence $(|x_n|)$ converges to $|x|$ ", the following discussion took place in the class:

- 1 Lect: What about the inverse? If $(|x_n|)$ converges to x does this tell us
- 2 something for (x_n) ?
- 3 S1: No.
- 4 Lect: Why?
- 5 S2: We will do the reverse process. That is $\|x_n|-|x|| < \epsilon$ for every $n \geq n_0$.

- 6 We know that $\|x_n - x\| < \|x_n - x\|$, so we cannot conclude.
- 7 Lect: Does this prove us that the inverse statement does not hold? How can we
- 8 be sure that $\|x_n - x\| < \varepsilon$ does not hold?
- 9 [no response]
- 10 Lect: For example, let's take $-1 < a < 1$. Then $\min \{|a|, a^2\} < 1$ but
- 11 $\min \{|a|, a^2\} < \max \{|a|, a^2\} < 1$. In this case, if the smallest is less than
- 12 1, then the biggest is less than 1. What do we need to see?
- 13 S2: Let's take two cases [meaning $\|x_n - x\| < \varepsilon$ or $\|x_n - x\| \geq \varepsilon$]. Will one case lead to
- something that does not hold?
- 14 Lect: What do we need to see?
- 15 S2: The relation of $\|x_n - x\|$ with the ε .
- 16 Lect: Do you think that it does not hold? Is there any other way?
- 17 S2: Let's suppose that it does not hold [he tries to use proof by contradiction].
- 18 Lect: No, you want to prove that something does not hold and not that it never
- 19 holds. How can we do this?
- 20 S3: By giving a counterexample.
- 21 Lect: Can you give me one?
- 22 S2: The sequence $((-1)^n)$.
- 23 Lect: That's right. The $\|x_n\|$ tends to 1 but the $((-1)^n)$ does not converge. In
- 24 general, if we want to prove that something does not hold we find a
- 25 counterexample.
- 26 Lect: We said that if (x_n) tends to x then $(\|x_n\|)$ tends to $\|x\|$. The inverse does not
- 27 hold. Is there any value of x where the inverse theorem holds?"

By analyzing the above extract we see that:

- Lines 1-6: The lecturer encouraged the class to investigate the validity of the inverse theorem. A student claimed that it is not valid (line 3). The lecturer asked for justification. Another student tried to prove this claim theoretically.
- Lines 7- 12 The lecturer posed two questions to the class based on the student's (S2) reasoning and he attempted to create the need for a counterexample (lines 7, 8). The students did not respond. The lecturer gave an example to show that in some algebraic inequalities the reasoning of S2 is false. He probed the students to think about how they could refute an invalid claim.
- Lines 13-20 The student (S2) insisted on using theoretical arguments. The lecturer encouraged him to reconsider his approach (lines 14, 16). The student

suggested proving the conjecture by using proof by contradiction. The lecturer made explicit the logical necessities for proving by contradiction and compared them to the specific mathematical problems. Finally, another student suggested the use of a counterexample.

Lines 21-27 The lecturer asked for a counterexample. The student (S2) gave one. The lecturer accepted and verified it. Then he stressed the use of a counterexample as a general process of refuting an invalid claim. He extended the discussion on the conditions under which the inverse theorem holds.

After a short discussion with the students the lecturer drew a straight line and represented the interval $[-x, x]$ and its centre 0. He explained informally that if the sequence $(|x_n|)$ converges to $|x|$ then “if $x \neq 0$ infinite terms of the sequence could approach x and infinite $-x$, so the sequence does not converge. The only case that we do not have this is when $x=0$ ”. Then he carried on formulating the theorem “Let a sequence (x_n) . Then $\lim x_n = 0 \leftrightarrow \lim |x_n| = 0$ ” and he asked the students to prove it.

During the group meeting after the lesson, aspects of teaching and learning related to the refutation of an invalid mathematical claim were discussed. Some of these aspects were: students’ difficulties to understand the role of counterexamples as a way of refuting a mathematical claim and their persistence of using general theoretical arguments despite the lecturer’s efforts; the need for emphasizing in the teaching the extended role of a counterexample in the identification of the conditions under which the claim can be valid; the transparency of an example or counterexample provided by the lecturer. The above issues were also linked to relevant research in which the lecturer and one of the other two researchers had participated (Gianakoulis et al, 2010; Potari et al, 2009). Based on this research that indicated the underlying reasoning for refuting an invalid claim the lecturer reflected on his teaching actions:

I think that I should have valued student’s theoretical reasoning (on lines 5, 6). It was not useless. Although it is not a proof, it could lead us to the conjecture that the claim is not correct. I should have emphasized it in the lesson.

Another issue discussed was whether the students had understood the counterexample given by the lecturer on lines 10-12. The group discussed about the difficulties that may be caused to first year university students by the symbols. During the discussion, the lecturer offered an alternative representation of this example which could be more accessible to the students. While analyzing the above extract, discussing about the role of counterexamples in mathematics and in mathematics teaching, the lecturer brought experiences from his practice as a research mathematician as well as a researcher in mathematics education:

In mathematics the generation of counterexamples is a common practice. You often make claims and get a sense about their validity through specific examples. When you suppose that a claim does not hold, you try to generate a counterexample and then you examine the critical characteristics of the counterexample that make the claim invalid. This could lead to the formulation of a new valid claim. However, in university teaching this process

of thinking is not usually made explicit to the students. In the past, sometimes I followed this process in my teaching without being aware of students' difficulties to understand it. Now, I consider more what the students' think and I am more conscious about what I need to emphasize. I think that my involvement in mathematics education research has contributed to the above shifts in my teaching.

From the above teaching example and the group discussion we could identify a number of teaching strategies that also seemed to be rather typical of the lecturer's teaching. For example, he poses an open question to the class; asks for justification; values students' responses and formulates questions based on these; uses visual representations to allow students get an intuitive sense of theorems and proofs; emphasizes the important mathematical ideas and processes such as in the specific example the role of the counterexamples in the process of refutation and in the generation of a new theorem. His ultimate goal was students' understanding and he was trying to facilitate it in different ways. However, he often faced tensions caused by the large number of students related to managing students' ideas and covering the content. The following are some of his reflections: "I felt that I was discussing with one student and I am not sure what the other students understood", "I go slowly and I am not sure that in this way I will cover the content. I wonder if I am too analytic". He also compared what he was doing in the past with his current approach and he commented that "I think that nowadays I try to encourage greater students' participation in class although this is not always easy".

CONCLUDING REMARKS

In this particular context, where the instructional activity is a lecture, we identified different teaching practices that the lecturer adopted to challenge the students to develop high level mathematical reasoning and thinking and a view of mathematics as a dynamic field. At the same time, he was sensitive to the students' needs by asking them questions and giving them some time to think, enriching their concept images and emphasizing the semantic aspects of proofs. We could claim that there were a number of teaching incidents in the lectures where sensitivity and challenge seemed to be in balance. Jaworski (2003) identified this balance in some cases of tutorial teaching. In a lecture course it seems that can also be achieved with possibly a different meaning. In terms of the resources that the lecturer used to make teaching decisions, we see that he brings experiences from: his practice as a research mathematician as he emphasizes the nature of mathematics and its development; his involvement in mathematics education research; his participation in the group where the focus was to inquire teaching at the university level; and his actual teaching practice. The above experiences seemed to be blended and reflected in his teaching decisions and actions. However, is it realistic to expect a university teacher to have all these experiences? We are aware that the case we present in the paper is a particular one. Nevertheless, it allowed us to see certain interactions between different sources of experiences and teaching. As a research team we inquired university mathematics teaching attempting to develop our knowledge about what

characterizes it. In this process we used self- reflection (the lecturer), critical questioning (the other two researchers), and coordinated different interpretations of specific teaching actions and decisions developing a deeper understanding.

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COORDINATION OF APROXIMATIONS IN SECONDARY SCHOOL STUDENTS' UNDERSTANDING OF LIMIT CONCEPT

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The aim of this research is to characterize the coordination of the processes of approximation related to the understanding of the limit of a function. We analyze the answers of 64 post-secondary school students to 7 problems considering the dynamic and metric conception of limit of a function. Results indicate that the metric understanding of the limit in terms of inequality supports that the student is capable of coordinating the approximations in the domain and in the range when lateral approximations coincide. However, the student is not capable of this coordination when lateral approximations do not coincide. This indicates that the metric understanding of the limit begins with the previous construction of the dynamic conception in case of coincidence of the lateral approximations in the range.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Previous research about post-secondary students' understanding of the concept of limit of a real function (Cornu, 1991; Cotrill, Dubinsky, Nichols, Schwingendrf, Thomas, & Vidakovic, 1996; Hardy, 2009; Moru, 2009; Oehrtman, 2009; Przenioslo, 2004; Roh, 2008, 2010) has shown the influence of the dynamic conception on the metric conception. The dynamic conception of limit concept can be characterized by the following idea: "if x approaches a , its images $[f(x)]$ approach L " and it could be expressed by $\lim_{x \rightarrow a} f(x)$ where f is a function, a a real number and L the limit of the function. On the other hand, the metric conception of limit is given by

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x (0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

Although the first students' approximation to the limit concept is the "idea of approximation" (Cornu, 1991, p.153; Oehrtman, 2009) Cottrill et al. (1996) suggest that the dynamic conception can be relatively complicated for the students, as they must construct "a process in the domain and a process in the range, and using the function for coordinating them" (p.187). These authors suggest what makes the concept of limit inaccessible to many students is the requirement of coordinating two processes of approximation with the quantification derived from the metric conception. In this sense, they indicate that the difficulty of the students in constructing the formal definition of the limit, and especially the development of a metric conception of the limit of a function can be the result of an insufficient development of the dynamic conception. Nevertheless, Williams (1991, 2001)

suggests that the dynamic conception of the limit concept can make difficult the progress towards the development of a metric conception.

In this controversy, the issues about the influence of the different representation modes for supporting or limiting the meaning construction of limit concept emerge. Some previous investigations indicate that the idea of limit presented in a graphical situation is stronger than the presented in a numerical form (Monaghan, 2001). However, the understanding of the limit concept in one of the representations not necessarily implicates its understanding in another representation (Elia, Gagatsis, Panaura, Zachariades, & Zoulinak, 2007). This previous research point out the relevant role played by the coordination of approximation in the domain and in the range in order to generate significant relations between the dynamic and metric conception of the limit concept.

In this context, we formulate the following research questions:

- How does the coordination of the approximations in the range and in the domain support the emergence of relation between dynamic and metric conception of the limit concept?
- How do the different representations influence on the development of this coordination?

METHOD

Participants and instrument

A test consists of 7 problems with 19 items was solved by 64 post-secondary school students (16-18 years old). The different items considered the mathematical elements from limit concept (Table 1) and the numerical (N), graphical (G) and algebraic (A) representations. Table 2 shows the mathematical elements considered from limit concept and the type of representation in each item.

Mathematical elements considered	
E0	The value of the function f in $x = a$, $f(a)$ (f a function and a a real number)
E1	Idea of <i>approximation</i> : x approaches a and $f(x)$ approaches L
E2	Dynamic coordination: when x approaches a , $f(x)$ approaches L
E3	Formalization like a manifestation of being conscious of the existence of the limit L of the function $f(x)$ in the point a , writing as $\lim f(x) = L$
E4	Metric coordination: it is possible to find out a x sufficiently near a , such that $f(x)$ is near L much as you wishes

Table 1. Mathematical elements considered in limit concept.

	P1	P2	P3				P4				P5		P6				P7		
	1	2a	2b	2c	2d	3a	3b	3c	3d	4a	4ax	4af(x)	4b	5	6a	6b	6c	6d	7
E0		X	X							X									
E1	X					X	X				X	X		X	X	X			
E2				X				X									X		
E3					X				X				X					X	
E4																			X
R	N	G	G	G	G	N	N	N	N	A	N	A	A	N	N	N	N	N	N

Table 2. Mathematical elements considered in the test.

Ei (i=0,1,2,3,4) mathematical elements from limit concept considered

Pi (i=1,2,3,4,5,6,7) problems in the test

R: The type of representation (N, G, A)

For example, problem 4 consists of 4 items (Figure 1): 4a – the calculation of the value of $f(x)$ in several points-, 4ax- idea of approximation to a certain number in the domain from the values of x -, 4af(x)- idea of approximation to a certain number in the range from the values of $f(x)$ -, 4b- formalization as a manifestation of being conscious of the existence of the limit . The aim was to analyze if students calculate the value of the function in an algebraic way (E0); construct a process of approximation to a certain number ($x = 2$) in the domain from the values of x for both the right and the left side (E1), construct a process of approximation to a particular number ($f(x) = 0,25$) in the range from the values of $f(x)$ for the right and for the left side (E1), translate the coordination of the both approaches to a formal language, and finally, analyze how students associate the coordination of both processes of approximation (E2) with the existence of the limit of the function (E3).

Analysis

Post-secondary school students' answers to each item were analyzed by three investigators. This process generated criteria in order to realize a dichotomic codification, (1) correct answer and (0) incorrect answer. Afterwards, a statistical implicative analysis was carried out (Gras, Suzuki, Guillet, & Spagnolo, 2008; Trigueros & Escandon, 2008) using the software CHIC (Classification Hiérarchique Implicative et Cohésitive). From this analysis, implicative diagrams were derived and they involve relations between students' answers and the items of the test.

Problem 4

If $f(x) = \frac{x-2}{x^2-4}$ complete:

a) $\xrightarrow{x \text{ tends to } \dots}$

x	1,9	1,99	1,999	1,9999
f(x)				

$\xrightarrow{f(x) \text{ tends to } \dots}$

$\xleftarrow{x \text{ tends to } \dots}$

2,0001	2,001	2,01	2,1

$\xleftarrow{f(x) \text{ tends to } \dots}$

b) $\lim_{x \rightarrow \dots} f(x) = \dots$

Figure 1. Example of a problem of the test: Problem 4.

Figure 2 shows a student answer to problem 4. The answer was codified as (1,1,0,0) since the answers to items 4a and 4ax are correct, but this student solved item 4af(x) incorrectly and did not reply to item 4b.

Problema 4

Si $f(x) = \frac{x-2}{x^2-4}$ complete:

a) $\xrightarrow{x \text{ tiende a } \dots}$ Item 4ax $\xleftarrow{x \text{ tiende a } \dots}$

x	1,9	1,99	1,999	1,9999
f(x)	0'2564	0'2562	0'2562	0'2562

2,0001	2,001	2,01	2,1
0'249937	0'249937	0'24993	0'2499

$\xrightarrow{f(x) \text{ tiende a } \dots}$ Item 4af(x) $\xleftarrow{f(x) \text{ tiende a } \dots}$

b) $\lim_{x \rightarrow \dots} f(x) = \dots$ Item 4b

Figure 2. Example of a student answer.

RESULTS

Access to the concept and formalization

Figures 3 and 4 show the implicative diagrams to 99% and 90% of statistical significance respectively.

The implicative diagram to 99% of significance (Figure 3) generated three independent implicative structures. In two of them, mathematical elements in numerical representation (N) are related, and in the third one, mathematical elements

in algebraic representation (A) are related. The left structure shows, in a numerical context (N), the implicative relation between the approximation to a number in the range when the lateral approximations are equal (E1N3b) and the approximation to a number in the domain (E1N3a). On the other hand, in an algebraic representation (A) the implication between the approximation process to a number in the range (E1A4af) and the calculation of the value of the function in a point (E0A4a) appears. Finally, the last implicative structure shows the implicative relations between the approximation to a number in the range when the lateral approximations are different, the coordination between the approximation in the domain with the approximation in the range, and manifestations of the existence of the limit in the numerical representations.

The relations described in this level of significance indicate that the numerical representation determines the start for the development of the process of coordination in the domain and in the range even when the lateral approximations are different.

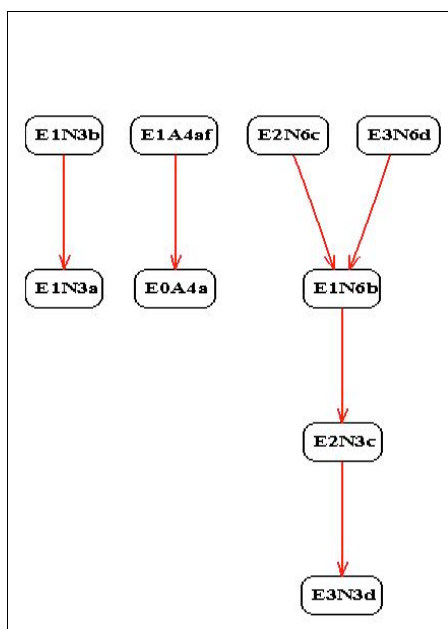


Figure 3. Implicative diagram (99%).

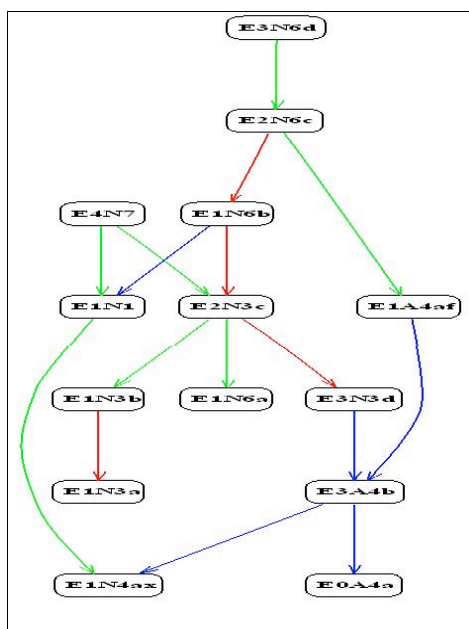


Figure 4. Implicative diagram (90%).

On the other hand, the implicative diagram to 90 % of statistical significance (Figure 4) only generated an implicative structure. This diagram shows in numerical representation, the relation among the coordination of the approximation in the domain with the approximation in the range when the lateral approximations do not coincide (E2N6c), the approximation in the range when the lateral approximations do not coincide (E1N6b), and the approximation to a number in the range when the

lateral approximations coincide in algebraic mode (E1A4af) as well as the relation between being conscious of the existence of the limit by the formalization (E3A4b) and the approximation to a number in the domain (E1N4ax) in numerical way.

This implicative structure indicates that the post-secondary students start coordinating the approximation processes in the domain and in the range when the lateral approximations are not match in the numerical representation. This coordination is added to the coordination in the algebraic way when the lateral approximations coincide with the relation to the conscience of existence or not of limit revealed by the formalization.

Both levels of statistical significance show two ideas. Firstly, the numerical representation of representation supports the process of coordination of the approximations in the domain and in the range so much if the lateral approximations coincide or not. Secondly, the algebraic representation supports to become conscious of this coordination revealed across its formalization.

The coordination of the approximation processes in the domain and in the range and the metric comprehension of limit

Implicative relations between the metric comprehension of the limit in terms of inequalities and the coordination between the approximation in the domain and in the range only appears in the implicative diagram at 90% of significance. The metric comprehension in terms of inequalities in the numerical representation (E4N7) presents two implications: (i) the coordination of approximations in the domain and in the range (when lateral approximations coincide) (E2N3c), and (ii) the approximation to a number (E1N1) (Figure 4).

The implicative relations in the numerical representation, $E4N7 \rightarrow E2N3c$ and $E4N7 \rightarrow E1N1$, indicate that the metric comprehension of the limit in terms of inequalities involves to be capable of coordinating the approximation in the domain with the approximation in the range when the lateral approximations coincide, but it does not imply to be capable necessarily of realizing this coordination when the lateral approximations do not coincide.

Discussion

The aim of this research is to provide information about how students construct the relation between dynamic and metric conception of the limit concept from the coordination of the approximation processes in the range and in the domain, and about which is the role played by representation modes in establishing the relations between these conceptions.

An important fact revealed by our results is that the metric comprehension of the limit in terms of inequalities involves being capable of coordinating the approximations in the domain and the range when the lateral approximations coincide, but does not imply necessarily that the student is capable of establishing this coordination when the lateral approximations do not coincide. This indicates the

cognitive difference between the coincidence or not of lateral approximations in order to coordinate the approximation in the domain with the approximation in the range. That is to say, this fact suggests that the metric comprehension of the limit seems to start with the previous construction of the dynamic conception but in the case of the coincidence of lateral approximations in the range. The results also show the important role developed by the numerical representation in the coordination of approximation processes and as a previous step to the coordination in the algebraic representation and in coming to be conscious of the existence of limit or not.

Finally, the absence of the variables in graphical representation can be explained because the items in the graphical way of representation have been linked to the idea "graph of asymptote" (Kidron, 2010) that according to Cornu (1991) it is an epistemological obstacle in the historical development of the limit concept. In this sense, the results obtained must be interpreted considering this fact. A final issue for further research is to study the role played by the graphical representation in the development of the coordination of the processes of approximation in the domain and in the range to construct the relation between the dynamic and metric conception of the concept of limit of a function in a point.

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MAKING CONNECTIONS BETWEEN THE SAMPLE SPACE AND THE PROBABILITY OF AN EVENT

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In this paper, I explore students' emerging thinking about "sample space" and "probability of an event" through an observation of 9- to 10-year-olds, challenged to reason about conditional probability in both "with" and "without" replacement situations. This study used a cognitive framework that captures the nature of students' probabilistic reasoning across four levels (subjective, transitional, informal quantitative, numerical) in relation to conditional probability tasks. The results show shifts in the structural complexity of students' thinking across the first three levels. However, none of the students reached level 4.

INTRODUCTION AND THEORETICAL FRAMEWORK

A recent curriculum reform in school mathematics (e.g., Australian Curriculum and Reporting Authority [ACARA], 2010) advocates the broadening of probability and statistics in the school curriculum. Conditional probability is introduced in the curriculum at year 9. It is typically introduced by random experiments involving sampling "with" and "without" replacement situations. Australian middle school students are expected to be able: (1) to list all outcomes for two-step chance experiments, both "with" and "without" replacement using tree diagrams or arrays, (2) assign probabilities to outcomes and determine probabilities for events. Although "the notion of conditional probability is a basic tool of probability theory, and it is unfortunate that its great simplicity is somewhat obscured by a singularly clumsy terminology" (Feller, 1973, pp. 114). This is obvious in the various research studies that have reported over the years the middle school students' informal thinking in conditional probability (e.g., Piaget & Inhelder, 1951/1975; Fischbein & Gazit, 1984; Jones, Langrall, Thornton & Mogill, 1997, Watson & Moritz, 2002; Tarr & Lannin, 2005). The seminal research of Fischbein and Gazit (1984) attempted to capture students' (grades 5-7) probabilistic thinking in conditional probability when engaged in tasks required to determine conditional probabilities in "with" and "without" replacement situations. According to Fischbein and Gazit (1984) in a "without" replacement situation, students: (a) struggled to realise that the sample space had changed and (b) their probability judgments of an event were impaired by comparing the number of favourable outcomes for the event before and after the first trial rather than by considering it in relation to the total number of outcomes (pp. 8-9). Students, who engaged in conditional probability judgments, applied the phrase "50-50 chance" to probability situations in which two or more outcomes in the space were not equally likely to occur (Tarr, 2002). The use of the phrase "50-50 chance" when

dealing “with” conditional probabilities in without-replacement tasks prevented students from recognizing that the probabilities of all events was changed by the conditioning event.

In the same vein, Jones, Langrall, Thornton and Mogill (1997) formulated a cognitive framework that captures the manifold nature of middle school students’ (9-13 year olds) probabilistic reasoning and understanding of conditional probability and statistical independence. In fact, they suggested four levels of thinking with respect to the conditional probability. Level 1 is associated with subjective thinking. Children at this level, tend to rely on subjective judgments ignoring completely any numerical information in formulating any conditional probability statements. Additionally, this Level 1 children might struggle to list all the outcomes in a “with” and “without” replacement tasks and their judgements are made with regard to the changing probability of any event in a “without” replacement task.

Children exhibiting level 2 thinking (transitional) begin to gain an awareness of the changing conditional probabilities of some events in “without” replacement tasks but their judgements are still limited to the occurrence of preceding events.

Students at Level 3 (Informal Quantitative) are able to list the complete set of outcomes in a “with” or “without” replacement tasks. These students appreciate the role that quantities play in the sample when making conditional probability judgments. In particular, students exhibiting level 3 thinking recognise that the probabilities of all events changed in a “without” replacement task and begun to use ratios or relative frequencies to determine conditional probabilities.

Students at Level 4 (Numerical) can assign numerical values, ratios rather than fractions, to the changing conditional probabilities when interpreting probabilities in “without replacement” situations. Even when students progress towards Numerical Level, difficulties remain at high school and University (Tarr & Lannin, 2005).

APPROACH OF THE STUDY

According to Diaz and de la Fuente (2007) students’ difficulties might be overcome if the concept of conditional probability is taught in conjunction with material on intuitive strategies and inferential errors so students are confronted with their misconceptions. Instead of having students confronted with their misconceptions about conditional probability at the University level, in this research study my aim is to elaborate the conceptual struggle that needs to take place for young students to engage in conditional probability tasks. In doing so, I acknowledge a constructivist stance in which I search for naïve understandings (in contrast to the misconceptions research as described by Smith, diSessa & Rochelle, 1993) that might effectively be used as resources for the construction of new intuitions that are less apt to subjective judgments (Langrall & Mooney, 2005). I begin by clarifying my perspective on what I consider as the conceptual roots of conditional probability judgments.

Tarr & Lannin (2005) pointed out that understanding the role of “sample space” and “probability of an event” help students to monitor the composition of the sample space, make probability comparisons, and determine that the probability of all events change in non-replacement situations (p. 231). ” Understanding the role of the “sample space” and the “probability of an event” are key factors in making conditional probability judgements.

By focusing on the conceptual roots of conditional probability reasoning, I can imagine young children engaging in conditional probability tasks of making multiple comparisons – including comparisons to the total number of objects. For example, questions can focus students’ attention on the changing number of elements comprising the sample space. This typically serves as a basis for assigning numerical probabilities and help students make conditional judgements.

Understanding the role of sample space in making conditional probability judgments is typically a characteristic of students’ at Level 3 (Informal Quantitative) and Level 4 (Numerical) of the framework of Jones, Langrall, Thornton and Mogill (1997). The focus in this paper will be on young students’ transitional thinking and informal quantitative judgements about the sample space and conditional probabilities. In particular, the focus is on students around the age of 9 to 10 years old, well before any formal teaching of conditional probability. It is therefore likely that intuitions will not yet have been formalized through schooling. While I observe students’ reasoning, I am likely to see (i) naïve conceptions at the transitional level (Level 2) and informal quantitative level (Level 3) of conditional probability thinking, or (ii) meanings that appear to be rooted in a situational way to the abstracted theory of conditional probability (Level 4: numerical).

This research study falls into the category of design experiments (Cobb et al., 2003) based on a hypothetical learning trajectories developed from an instrumentalist philosophical standpoint (Baroody et al., 2004). Design experiments aims to sensitize us towards the complex learning ecology of the domain being investigated through an iterative design process featuring cycles of invention and revision” (Cobb et al., 2003, p.10). Each design stage incorporates a set of conjectures about both students’ mathematical abstraction and the design of tasks. These conjectures are in effect tested while using the tasks. A retrospective analysis follows in which the researcher determines the actual outcomes and then generates and tests an alternative conjecture. As the project unfolds, the design-researcher’s efforts increasingly converge into an integrated coherent framework. This emergent framework, in turn, brings together principles of design for mathematical learning tools and contextualizing activities.

This research study was the first cycle of the current iterative design. It consisted of two tasks:

(1) A container contains 16 stickers: 8 blue coloured stickers and 8 purple coloured stickers. A sticker is drawn from the container and placed back inside the container.

Is the chance of drawing the same colour of sticker from the container the same or different as it was before? Explain. Repeat the same activity several times.

(2) A container contains 16 stickers: 8 blue coloured stickers and 8 purple coloured stickers. A sticker is drawn and stuck on a separate blank sheet of paper. Is the chance of drawing the same colour of sticker from the container the same or different as it was before? Explain. Repeat the same activity until there are no remaining stickers in the container.

Twenty Students¹ in Grades 3, (ranging in age from nine to ten years old) of an elementary school in the state of New South Wales in Australia, formed the population for this study. All the students spent one session to answer the questions of the two experiments. Eight students volunteered to spend another session with me to answer the questions of the two experiments. Each session lasted approximately 40-45 minutes and was conducted with a pair of students, so I (R) spent 4 sessions (in total) to work with each pair of students.

The data included audio recordings of students' voices and field notes. The audio recordings were simply transcribed. The transcript was turned into a plain account of the sessions. The data were subsequently analysed using progressive focusing (Robson, 1993).

RESULTS

For the rest of this paper we will focus on the work of two students Sophie (S), Chris (C), who their reasoning was representative of the data collected for this research study. To illustrate the students' work, I join Sophie and Chris as they began to form predictions of drawing a blue or purple coloured sticker in a "with replacement" situation.

- 1 C: Because look, Blue and purple, and purple. And blue, purple, purple, purple, blue, blue, blue, blue, blue, blue ... Like umm, I've got the answer in somewhere here.
- 2 R: Where?
- 3 C: It's a group. See, you've got a group there and a group there.
- 4 R: ah you mean you've got a group of purple and a group of blue stickers.

Students seemed to place too much faith in the occurrence of previous outcomes (lines 3, and 4). They did assume that the draws of the stickers followed a pattern as it was evidenced in the above slice of data. This is of course at the core of the difficulty with 'randomness', because although the stickers were chosen randomly from the container, the random process itself produced a pattern. Chris drew another sticker.

- 5 C: Yes. Now, It's my turn ... a purple sticker.

¹ Notwithstanding, none of the students had never received formal instruction in probability.

- 6 S: Now we would have purple all the time. A group of purple.
- 7 C: Yeah, we will draw a group of purple and then a group of blue....
- 8 S: Come on, go for purple again.
- 9 C: What?
- 10 S: You are making, ahh haa. You are in a big trouble. I want to know what are you doing?
- 11 C: Now, I got blue.
- 12 S: She always do it.
- 13 R: What is she doing?
- 14 S: She always made bad draws. Let me draw. I can make a good draw. (Sophie drew a blue sticker).
- 15 C: Blue
- 16 R: So now will we have a group of blue?
- 17 S: Um, blueeee?...We drew B, P, P, B, P, P, P, B, B, B, B, B, P, P, B, B. We drew 9 blue out of the 16 draws and 7 purple. So, I go for blue.

Students appeared to be confined to the events that have previously occurred, because they expected a group of purples to be drawn, followed by a group of blues (lines 6 and 7). It would appear that Sophie assumed that variation in the draws was caused directly by Chris' performance (lines 12-14). Sophie was possibly entertaining the idea that her draw may somehow give continuation to the observed regularity (line 14). To her surprise, she drew a blue sticker. Sophie reflected on the sequence of previous outcomes to form a prediction. I challenged Sophie and Chris to consider the composition of the sample.

- 18 R: Remember that we have eight blue and eight purple stickers in the container.
- 19 S: What equals 8? We drew 10 blue and 7 purple. But the blue do not equal 8.
- 20 R: Yes, but in the container we have 8 blue stickers and 8 purple stickers.
- 21 S: Yeah, The blue should be equal 8 and not 10.
- 22 C: Yeah, it's all confusing ... ehm I think that ... what we drew is different.
- 23 R: You drew stickers from the container. Every time you drew one sticker.
- 24 S: ah I know. I can draw 1 blue out of the 8 blue, I put it back and ...then there are still 8 blue and 8 purple in the container.
- 25 C: If we go for a purple this time again it will be one out of the eight purples.
- 26 R: But 8 and 8 is equal to 16.
- 27 S: Ehm... it will be one out of the 16 purple and blue as well.

Sophie seemed to confuse the number of events that previously occurred with the composition of the sample space, since she was expecting to have eight blue stickers drawn from the container (line 19, 21). Sophie recognised that the sequence of the drawn stickers does not reflect precisely the actual composition of the sample space (line 22), and she articulated a situated version of a numerical probability (line 27).

Later on, Sophie and Chris engaged with a “without” replacement situation.

- 28 R: We have drawn 2 purple stickers and 0 blue stickers. Remember, in total there were 8 purple and 8 blue stickers. So how many purples do we still have in the container?
- 29 S: Uhhhh ... 5, oh no 6
- 30 R: So we have 8 blue in the container and 6 purple. I have more blue stickers than purple stickers. Which one has more chance to be drawn?
- 31 C: ... Blue is still got 8 and ...
- 32 R: What did you get?
- 33 C: Purple. There are only 5 purple. So we have more blue here (in the container).
- 34 S: There are only 5 purple and there are 8 blue out of ...
- 35 R: So, Chris what do you think Sophie is going to draw now?
- 36 S: Blue. There are 5 purple ...
- 37 R: Is the chance of drawing another purple sticker from the container the same as it was before?
- 38 S: No it is less chance. There are less purple stickers than there are blue.
- 39 C: Ehm... there are only 5 purple in the container, but ... ehm the ... the chance of drawing a blue is the same.
- 40 R: Why the same?
- 41 C: Blue still got 8 here (in the container)

I drew the girls’ attention on the different composition of the sample space when the stickers were not placed back inside the container (line 28, 30). Sophie argued that the chance of drawing a purple sticker has decreased because there are fewer purple stickers in the container (line 38) compared to the blue stickers. Chris added that that the chance of drawing a blue sticker has remained unchanged (line 31, 39) because there is still the same number of blue stickers in the container (line 41).

DISCUSSION

Sophie and Chris who were working in a “with” a replacement situation, appeared to exhibit Level 1 thinking. They harboured a pervasive belief that the individual could control the draws of the stickers from the container (lines 10-14). Such a statement is incompatible with the concept of independence, given that the probability of drawing any sticker is equal to 0.5. The girls also tended to place too much faith in the occurrence of the previous outcomes, when formulating predictions in a “with” replacement situation. In particular, they assumed that the composition of the sample space was supposed to mirror exactly the events that had previously occurred, so they expected to have eight purple and eight blue stickers drawn from the container. Their attention is focused on the comparisons of the number of events occurred with the composition of the sample space. Later on the number of events was compared to the total number of outcomes (line 22) showing that the student’ thinking moved to the informal quantitative level (Level 3), since Sophie could assign a situated version of

numerical probabilities. There were no obvious connections made with the conditioning events although students recognised that the occurrence of one event did not have an impact on the probability of another event in a “with” replacement situation (line 22).

When students engaged with a “without” replacement situation their thinking reverted to Level 2 (transitional). The students did not realise that the occurrence of one event had indeed an influence on the occurrence of the preceding event although they seemed aware of the changing number of elements comprising the sample space after a certain element was selected. Students did not build any links between the total number of elements in the sample space given the conditioning and the probability of an event.

The present study provides new insights in students’ reasoning about the sample space and the probability of an event in “with” and “without” replacement situations. In this study, students’ thinking was identified and a movement of students’ probabilistic thinking across the four levels was reported.

When I argue that young students’ did not build substantial connections between the sample space and the probability of an event, I do not present this as a misconception to be eradicated. Rather, I see all these findings as identifying students’ starting points, informing how I should be designing future research activities which will offer to students experiences that may enable them to construct more sophisticated meanings for their conditional probabilistic thinking out of these relatively naïve conceptions.

Future research will investigate more systematically students’ thinking while building connections between the sample space and the probability of an event and offer students opportunities to make multiple comparisons between the total number of objects contained in a sample before and after an object is drawn given a conditioning event.

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