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PRIMARY TEACHERS' REASONING AND ARGUMENTATION ABOUT THE TRIANGLE INEQUALITY

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This paper is part of an ongoing study with in-service primary teachers, which has a dual objective of Professional Development and Research. Here we report on a workshop about triangles, focusing on the reasoning and argumentation processes registered in an individual questionnaire and in videotaped team discussions.

With the dual objective of Professional Development (PD) and Research, our Study, called TAMBA, addresses the topics of the Mathematics curriculum for the primary school. The Study was carried out through a series of workshops with in-service teachers of the public schools in a Mexico City working class zone. We have previously conveyed at PME the general design of the Study, some results on the workshop on Fractions, and some previous experiences on the topic of Triangles (Alatorre et al, 2009, 2010, and 2011). This paper reports part of the experience in the TAMBA workshop on Triangles, focusing on the reasoning and argumentation processes rather than on the Geometry aspects, because of space limitations.

FRAMEWORK

The community of mathematics educators concurs in stressing the importance of the mathematical knowledge of teachers; for instance, Southwell & Penglase (2005) sustain that "if teachers are not confident in their mathematical knowledge, they may find it difficult to ensure that their students gain confidence and competence." Therefore, in order to design learning scenarios for teachers, it is also vital to understand how they comprehend and conceptualize the mathematics they teach.

According to Ball, Thames & Phelps (2008), the mathematical knowledge of teachers can be considered as twofold: Common Content Knowledge, CCK, where "common" refers to many other professions or people in general; and Special Content Knowledge, SCK, the mathematical knowledge and skill unique to teaching. CCK and SCK interact with each other; perhaps one of the areas in which this interaction is more evident is in the reasoning and argumentation skills.

We agree with Flores (2007) in the sense that an argumentation is "the set of actions and reasoning that an individual brings into play in order to explain or justify a result or to validate a conjecture raised during a problem solving process". In this construction there are many elements influenced by previous experiences and knowledge. Flores recognizes the following types of argumentation as explanations or justifications of a result: *Authority-based* (arguments based on statements made by some authority –a teacher, a textbook, a principal, etc.), *symbolic* (use of mathematical language and

symbols in a superfluous or naïve way, without really getting to the conclusions meant), *factual* (an account of the actions taken, a repetition of evident facts or a set of algorithmic steps), *empirical* (based on physical facts or drawings as the essence of the argument, not as a visual help for it), and *analytical* (a deductive chain in which each statement follows from the previous one). It is important to add that the latter is not necessarily the only one leading to valid argumentations (for instance, a counter-example can be a valid empirical argument), nor is it always valid (the deductive chain may end in a false or non-pertinent conclusion).

On the other hand, in mathematics an argumentation expresses a reasoning process similar to that of a proof, and although it is not necessarily as rigorous as a proof, it shares with it many of the elements described by de Villiers as cited by Hadas et al.:

> *verification* (concerned with the truth of a statement), *explanation* (providing insight into why it is true), *systematization* (the organization of various results into a deductive system of actions, major concepts and theorems), *discovery* (the discovery or invention of new results), *communication* (the transmission of mathematical knowledge), and *intellectual challenge* (the self-realization/fulfilment derived from constructing a proof). Hadas, Hershkowitz & Shwarz (2000).

These elements are present when in a problem-solving activity students must communicate their ideas and convince others of their points of view. The confrontation of different views implies the creation of a judgement about the pertinence or the inconsistency of an argument, and therefore is also an intellectual commitment. We will use them to analyze teachers' arguments in a Geometry workshop environment.

METHODOLOGY

TAMBA's dual PD/Research objective permeated the modes in which the study was conducted. The workshops were offered to 300-800 teachers (in groups of ca. 20) with topics chosen by them; each took place in a 2-hour session. This allowed us to collect information from a large amount of teachers, but unfortunately gave us no time to further work with them, so, for instance, no interviews were possible. However, similar studies (e.g. Southwell & Penglase, 2005) have encouraged us to present our results.

The PD facet required to have a scenario that would foster cognitive conflict, discussion and re-conceptualization within task-based activities, whereas the Research facet required a means to detect teachers' needs in CCK and SCK. Thus, the sessions were organized in a short individual task (IT) based on a questionnaire, a videotaped team task (TT) as the main activity, and a videotaped group discussion (GD). The PD started with tasks of the IT, developed mainly during the TT and was taken to closure in the GD, while the research needs were covered by the questionnaire and the videotapes; also, in the IT some information about the teachers' characteristics was registered.

Within this common structure, in each workshop both the IT and the TT consisted of several *ad-hoc* designed tasks. In the workshop about triangles the tasks dealt with several geometrical topics; both the IT and the TT started with tasks aimed at the

triangle inequality (*TR.IN*), which are reported in this paper. In the first item of the IT, five sets of lengths in cm were given; the teacher was asked to state whether or not a triangle could be constructed with each, and to briefly explain why. For the first TT task, teams with 3-4 teachers were given several colour Meccano-like plastic strips of different lengths, and clasps (Figure 1). The teachers were asked if it was possible to construct triangles with six sets of strips referred to by their colours. Table 1 reports the lengths in all 11 sets (in the TT the strips' lengths were not explicit, but we report here the amount of units). This part of the TT ended with the question "What conditions must the strips fulfil so that a triangle can be constructed?" In the final GD, both tasks were commented, with the aim of stating the *TR.IN*.

IT =Individual task (cm)	TT = Team task (arbitrary units)
$S_1=\{7, 7, 7\}$ $S_2=\{4, 4, 10\}$ $S_3=\{8, 5, 3\}$ $S_4=\{10, 10, 4\}$ $S_5=\{12, 7, 8\}$	$S_6=\{6, 15, 15\}$ $S_7=\{15, 6, 6\}$ $S_8=\{8, 8, 7\}$ $S_9=\{7, 7, 8\}$ $S_{10}=\{31, 24, 10\}$ $S_{11}=\{5, 6, 15\}$
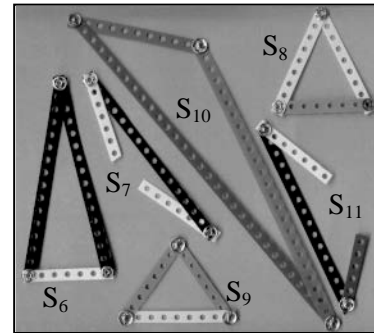
Table 1. Length sets for the different tasks   Figure 1. Strips for the TT

Thus, as a PD setting, the workshop provided three distinct moments. In the IT, the teachers' prior knowledge (CCK/SCK) was at stake; our previous experience had shown that the *TR.IN* is unknown to many teachers (Alatorre et al, 2009). A second moment was provided by the TT, where the team experimentation with the strips fostered the emergence of a cognitive conflict and the analytic reasoning and argumentation skills. Finally, the GD was the scenario in which some systematization and communication skills could be exercised. As a research setting, these three moments can be tracked and analyzed in different ways. In the questionnaire of the IT, the reasons given for the possibility or impossibility of the construction asked for were categorized, and some quantitative methods were applied, whereas the videotapes of the TT and the GD provide information for a qualitative analysis.

RESULTS AND ANALYSIS

The triangles workshop was attended by 353 teachers. We will here report some findings related to each of the three moments described above.

1. Prior knowledge. The responses to the first item of the IT were classified according to two sets of categories: on the one hand the combination of *yes*/*no* answers to the questions about the five sets and on the other hand the kind of justification given to each of the 1460 "*yes*" and the 247 "*no*" answers. In the first case four groups are defined: the correct yes/no/no/yes/yes, a partially correct yes/no/yes/yes/yes, the most frequent error yes/yes/yes/yes/yes, and other answers. For the second classification the reasons given were divided in six groups, regardless of the correctness of the "*yes*" or

"*no*" answer. The relative frequencies for both classifications are shown in Table 2. The fact that only 13% of the answers were yes/no/no/yes/yes corroborates our previous finding in the sense that the *TR.IN* is unknown to the majority of the teachers.

Answers (combinations)		Justifications for the 1707 individual answers		
$S_1 / S_2 / S_3 / S_4 / S_5$		Category	"*yes*"	"*no*"
yes/no/no/yes/yes	13%	A triangle has <u>three sides</u>	12%	0%
yes/no/yes/yes/yes	18%	The triangle's <u>type</u>	62%	0%
		Approaches to the *TR.IN*	5%	54%
yes/yes/yes/yes/yes	46%	Mention of the <u>measures</u>	2%	17%
Other combinations	22%	Other	2%	7%
		No justification	16%	22%

Table 2. Frequencies of the categories for answers to the ɪт and their justifications

We now comment on the categories for the justifications. In the first category are "*yes*" answers that only state that since three measures are given a triangle can be constructed with them; that is, <u>three sides</u> is a sufficient condition for a triangle. In the second category are the answers that contain either the name of the alleged triangle's <u>type</u> (equilateral for S_1, isosceles for S_4, scalene for S_5, but also isosceles for S_2 and scalene for S_3) or the definitions for them (e.g. "two equal sides and one different") or both; in some cases the type was incorrect, such as these two for S_4: "scalene" and "isosceles, they form an angle of 90°". The third category groups not only correct formal expressions of the *TR.IN* (11%), but also correct informal expressions, such as (in *no* for S_2) "4+4 is less than 10" or (in or *yes* for S_4) "the two equal sides are larger than the third" (16%), qualitative comparisons such as (S_2) "one of the measures isn't enough" or "two sides can't reach each other" (62%), justifications showing that the author has a hint about the *TR.IN*, such as (in *no* for S_2) "the third side doesn't fit" (8%), and plain misunderstandings of the *TR.IN*, such as (in *yes* for S_3) "5+3=8" or –misusing the Pythagorean theorem)– "5 and 3 form a right angle and the one with 8 joins the vertexes" (4%). The fourth category groups reasons than only make a vague mention of the <u>measures</u> as a basis for answering *yes* or *no*, such as "because of the measures" or (in *yes* for S_3) "the sides are proportional".

Although much can be said about the different crossings of these two classifications and the correct or incorrect answers to each of the five questions, we will only highlight "the bad news" and "the good news":

- The most striking result is that 41% of all 1707 justifications correspond to the combination yes/yes/yes/yes/yes and one of the categories "three sides" or "type"; among those with the combination yes/yes/yes/yes/yes, these two categories account for 97% of the justifications.
- On the other hand, although only 40% of the teachers said "*no*" to S_2 and 19% to S_3, an approach to the *TR.IN* is the most frequent reason for these answers: 62% for S_2 and 56% for S_3. As many as 85 teachers say "*no*" to S_2 and "*yes*" to S_3; their two main reasons for accepting S_3 are of the category "type" (47%)

and incorrect approaches to the *TR.IN* (19%), mainly recognizing that 5+3=8 but not considering that as a reason to reject S_3.

2. Experimentation. The videotapes registered the work of several of the teams in each group during the TT, but since the teachers organized freely in teams, we have no way of identifying how each of the members of a videotaped team responded to the IT.

For the greatest part of the teachers, the experimentation with the strips produced a cognitive conflict. Many were really surprised that some of the sets could not produce a triangle, regardless of how they tried to assemble them, and said that they had never before realized that some triangles could be impossible. For many this insight created a challenge to understand the conditions necessary for a triangle, and generally speaking the teams had one of the following reactions to this situation. (We describe the reactions and illustrate them with some transcript examples, in which we number consecutively the participant teachers, starting with T1 for each team).

In several such teams one of the teachers started with a tentative formulation (a hypothesis), and another one produced a counterexample, frequently using a different set of strips than the ones proposed. Then a new hypothesis was formulated until no counterexample could be constructed and this hypothesis was accepted as final:

> [E1] T1: But why in this case the triangle cannot be constructed? – T2: Because the measures are different – T3: (shows S_{10}) – T1: It could be that two sides must be larger than the other – T2: That the *sum* of two is larger than any of them.

Other teams had among them one or two teachers who, even with the awareness that there are situations in which a triangle cannot be constructed, denied the possibility of a general condition: they could accept that there are conditions for each type of triangle, but since these conditions are different depending on the type, a general condition was made impossible:

> [E2] T1: At least two sides must be equal, is that a rule? – T2: In this one, all are different – T3: Well, that is a scalene: The rule, to begin with, is that you need three sides – T2: Yes, but we must find the relationship among the sides, because with these… – T3: … nothing can be formed – T2: … I can't form a triangle. So, we need the sum of two to be greater than the other – T3: But I insist, the definition that you are giving rules out the equilateral and the isosceles, in my definition all are considered – T1 (constructs an isosceles) – T2: The sum of these is greater than the other, the condition is fulfilled – T3: In an equilateral? – T2: An equilateral also fulfils the condition – T3: But with those same strips you can't construct an equilateral triangle.

For some teachers the experimentation led to no conflict because they did manage to construct triangles with all the sets. Figure 2 shows one example of a "triangle" constructed with S_7. In other cases the teachers denied the conflict and lost all interest in the task, not looking for explanations or relationships. Some copied in their sheets the answers given by other teachers; some just stopped trying to find a condition, skipped the question and started another of the TT's tasks:

Figure 2

[E3] T1: They need to have straight lines – T2: Here the lines are straight and it's not a triangle – T3: That the measures are proportional – T3: That they are different – (T1 raises, goes to another team, and comes back with an answer, which is accepted).

[E4] T1: They need to have the same size – T2: That could be a condition, but even if they have different sizes you can form one, even if they have different sizes that is not a reason not to form it – T3: That two sides are the same – T2: No, because if we have two equal sides… – (silence) – T2: That the measures of the strips allow for them to join, and that's it.

Some of the teachers had a fairly good idea about the *TR.IN*, so for them the task became a confirmation of their previous knowledge; their challenge was to convince their teammates and to achieve a complete and correct expression of it:

[E5] T1: (shows {11, 7, 24}) We need that sum of these two [11, 7] to be greater than this one – T2: That the sum of two is larger than the other – T3: Prove it – T2: That's what I'm doing – T3: Make these two [11, 7] larger than this [11, 24] – T2: No, it's the sum of these [11, 7] – T1: Oh, the sum of the two together.

3. Systematization / communication. In the third moment, the GD, a common expression for the *TR.IN* was produced, from the contributions of the teams in the group. In this process oftentimes a team that came with an incomplete expression of the *TR.IN* completed the process with the help of the group's conductor (C):

[E6] T1: The condition is that one side must be the same as another, or smaller than another – (C shows a counterexample: S_{11}) – T1: Than *the sum* of the other two sides – (C shows S_3) – T1: That one side must be *smaller* than the *sum* of the other two sides.

Other findings. Although the objective of this paper is not to analyze the use of language, we consider it relevant to mention that during the whole process IT-TT-GD we retrieved a meaningful amount of mathematical terms incorrectly used. Here are some examples:

[E7] "The sides must be *proportional*"… "The condition is that the sum of the *legs* must be larger than the *hypotenuse*"… "(S_2) one *vertex* would be incomplete"… "The *edges* must be larger than 5 cm"… "(S_3) the sum of the *faces* of two barely covers the other"… "One of the sides is larger than the *perimeter* of the other two"… "(S_2) yes, the *base* can measure 10 and the *height* 4".

In other cases the question can be raised about possible misconceptions: "their measures can be joined", "(S_5) they are relatively equal", "(S_5) scalene, because its sides are unequal and its angles are larger than 90°".

Finally, a statistical association was searched between the categories for the answers to the questions of the IT and two other variables: the amount of years teachers have been practicing as such, and the grade they teach (or the highest of both when they attend groups in different shifts). We had found such an association in the TAMBA workshop about Fractions (Alatorre et al, 2011), where the best CCK/SCK levels were attained by the most experienced teachers and also by those who teach in the highest levels of the primary school. However, in this case no statistical association was found

(respectively F=2.43; df=3,286; p=0.07 and F=0.86; df=3,284; p=0.46). Further research should look for a possible association with the teachers' prior training,

DISCUSSION

Teachers can and do build up many of their mathematical concepts and knowledge through their professional practice, as mentioned above for Fractions. The fact that no association was found in this case with the length of service or the grade they teach shows that the <u>TR.IN</u> is not part of the teachers' professional practice. Although some of the teachers may have learnt the *TR.IN* during their high school, they did not see it during their teacher training, and the approach to triangles in the primary school makes it superfluous for the teacher. The usual practice is that triangles are drawn from scratch and not from predetermined measures, so drawing a triangle is always possible, and generally teachers use the prototype of an acute-angled isosceles with a horizontal base. In practice, measures only have two uses: the classification of the triangle's type and the application of formulae for the perimeter and the area. There is a divorce between drawings and lengths, so many teachers, when they require a triangle with measures, assign to a drawing numbers that not necessarily coincide with the actual lengths. This could be at the origin of most of the yes/yes/yes/yes/yes answers to the IT: the sole question whether the triangles could be constructed seemed absurd.

However, we consider that in this case there is at stake something more important than the particular knowledge of the *TR.IN*. The qualitative analysis of the justifications to the IT and of the team processes of the TT suggest that reasoning and argumentation are also not part of many teachers' professional practice, although they are unquestionably part of the CCK and also of the SCK. This lack of habitude of reasoning and argumentation can be seen in many of the behaviours observed in the workshop. Most teachers did not feel the need to justify a *yes* besides pointing to the triangle's type or amount of sides. Many teachers clearly confuse a necessary and a sufficient condition (e.g. "three sides"). For some, the same justification (e.g. "two equal sides") can serve the purpose of explaining a *yes* (for S_4) and a *no* (for S_2). The difficulty with S_3 in the IT may be related with an incomplete learning process about the *TR.IN*, but also with the complexity of dealing with extreme cases. In some cases, the experimentation was denied; apparently some teachers believe that the knowledge of mathematical facts is not obtained through experimentation. Also, many team discussions were aborted because the teachers arrived at a *cul-de-sac* and found no way out of it.

Many of the teams undertook argumentation processes that are far from satisfactory. We found examples of Flores' (2007) argumentations *authority-based* (see e.g. [E3]), *symbolic* ([E7]), *factual* (end of [E4], Fig 2, all the yes/yes/yes/yes/yes), *empirical* (justifications to the IT because of the triangle's type) or incomplete *analytical* ([E3] and [E4]). However, it is also noteworthy that although many of the teachers had previously no idea that a set of three lengths may not lead to a triangle, they tackled the new problem following complete and correct logical processes, discarding successive hypotheses with counterexamples in the discussions and striving to arrive at a general

formulation; that is, many of the processes were valid argumentations, whether *empirical* or *analytical* ([E1]).

In these analytical argumentations, the kinds of reasoning processes described by Hadas et al. (2000) can be found: *verification* ([E2], [E6]), *explanation* ([E4]), *systematization* (search for counterexamples, [E6]), *discovery* (hypothesis, [E1]), *communication* (throughout the workshop, in the justifications of the IT, the discussions in the TT and the final expressions in the GD), and *intellectual challenge* (in the attitude of most teachers towards the task).

As a final assessment, we can affirm that the workshop met its dual objectives. On the one hand, a Professional Development experience was provided to the teachers, which allowed for an awareness of their prior knowledge, a discovery moment involving a cognitive conflict, a reasoning process with the use of particular cases, examples and counterexamples, a peer discussion, and ended with a communication practice. On the second hand, the research facet leads to the knowledge about the need to include, in the professional training of teachers, certain topics and activities that may foster the development of reasoning, argumentation and communication.

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ON STRATEGIES FOR SOLVING INCONCEIVABLE MAGNITUDE ESTIMATION PROBLEMS

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Fermi problems are problems which, due to their difficulty, can be satisfactorily solved by being broken down into smaller pieces that are solved separately. In this article, we present Inconceivable Magnitude Estimation problems as a subgroup of Fermi problems. Based on data collected from a study carried out with 12 to 16-year-old students, we describe the different strategies for solving the problems that were proposed by the students, and discuss the potential of these strategies to successfully solve the problems.

INTRODUCTION

The process of solving problems has received considerable attention in the last few decades within the area of Mathematics Education, but not all of the advances in the research have made it into the classroom. In particular, mathematical modeling is not taught in secondary (12-16 years old) mathematics curriculums in Catalonia (Spain). For this reason, from the teacher's perspective, how modeling could be taught in the classroom emerges as a natural question.

In this article, we suggest Inconceivable Magnitude Estimation Problems (IMEP) as a means for introducing modeling in secondary classrooms. IMEP present the student with a situation in which it is necessary to estimate the value of a considerably large real magnitude, well outside the range of their normal daily experience. These problems can be considered a subgroup of Fermi problems, and allow for different approaches to solving them.

Given that IMEP are problems whose formulation situates them in a specific daily context, distinguishing the main elements from the less relevant ones is a difficult task for students. In this article, we discuss various strategies that students proposed for solving these problems.

THE CONTEXT OF A PROBLEM AND ITS MODELING

According to Van Den Heuvel-Panhuizen (2005), presenting a real context to problems can make them more accessible and suggest strategies to students. Problems that are related to daily life make it possible to begin teaching mathematics within the realm of the concrete and then move on to the more abstract. Chapman (2006) observes that many teachers present real context problems in a closed way which does not allow for discussion of the situations that the problems present. Doerr (2006) explains this by stating that teacher education trains teachers to have this attitude and that ideally,

teachers should be trained to engage with the different kinds of responses that students can present.

According to Winter (1994), the solving of problems with a real context includes the mathematization of a non-mathematic situation, which involves the construction of a mathematical model in accordance with the real situation, calculation of the solution, and transferring the result to the real situation. The most difficult step in this process is to come up with a model that is appropriate for the real situation, as it requires a good understanding of both the situation and the mathematical concepts involved, as well as a great deal of creativity.

In the literature, two principal differences can be found between traditional word problems and modeling activities. Firstly, in modeling, one must connect mathematical concepts and operations with reality, thereby creating meaning for what is being learned, as well as symbolically represent a given situation (Lesh & Zawojewski, 2007). The second difference is related to modeling itself, since students must produce models that are applicable to a given situation and whose solutions can be generalized and interpreted (English, 2006).

FERMI PROBLEMS

Fermi problems are problems which, although difficult to solve, can be solved by being broken down into smaller parts that are solved separately. They are named after the physicist Enrico Fermi (1901-1954), who often gave his classes with such problems. The classic Fermi problem that is most often given as an example is that of estimating the number of piano tuners in Chicago. This is approached by, for example, estimating the total population of the city, the percentage of families that might have a piano, and the time needed to tune a piano.

Ärlebäck (2009) defines Fermi problems as "open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations (p. 331)." Carlson (1997) describes the process of solving a Fermi problem as "the method of obtaining a quick approximation to a seemingly difficult mathematical process by using a series of *educated guesses* and rounded calculations" (p. 308) and asserts that they possess a clear potential to motivate students. Along the same lines, Efthimiou & Llewellyn (2007) characterize Fermi problems as always appearing to be vaguely formulated, giving little information or few relevant facts on how to attack the problem. At the same time, after more careful analysis, they can be broken down into simpler problems which can be used to solve the original problem. These authors argue that this type of problem encourages students to think critically. Others have taken interest in the concrete aspects of solving Fermi problems. Peter-Koop (2004, 2009), for example, gives primary students simple Fermi problems in order to understand the strategies they use to solve them, among other things. In her conclusions, she explains that students solve Fermi problems in many different ways which increase their own mathematical knowledge and that their solution processes are

multicyclic. These conclusions are a starting point that call for more in-depth research. After his observations of students solving Fermi problems, Ärlebäck (2009) concludes that the processes which these activities depict "are richly and dynamically represented when the students get engaged in solving Realistic Fermi problems" (p. 355). In this way, she asserts that this type of problem presents an excellent opportunity to introduce students to mathematical modeling.

INCONCEIVABLE MAGNITUDE ESTIMATION PROBLEMS

Our work focuses on problems based on magnitudes that we can not perceptually estimate without some training, as well as magnitudes which we can imagine, but for which it is difficult to interpret their value. If we think of magnitudes with which we are familiar and to which we have given meaning (the size of a pen, the time that passes during a football match, or the number of people in a classroom), we can metaphorically assert that they are familiar and conceivable. Some examples of magnitudes which are inconceivable in this sense are the quantity of rubble produced by leveling the earth at the construction site of a building, the number of cars that go by a determined point on a motorway in one day, or the number of trees in a forest.

Taking these ideas as a starting point, we define an *inconceivable magnitude* as a physical or abstract magnitude which is beyond our ability to interpret and for which we have not created any meaning. It must be emphasized that, according to this definition, the determination of magnitudes that we consider inconceivable varies from person to person. This determination will be conditioned by their knowledge, abilities or experiences.

Once we attempt to determine the value associated with an inconceivable magnitude, we must by definition work with approximate values. The most natural way of obtaining values for inconceivable magnitudes is to come to an estimation through reasoning. To ask 12 to 16-year-old students to estimate the value of an inconceivable magnitude from their environment is problematic, as it is a type of word problem task which they have not been taught to solve.

Our assumption is that this type of problem should require students to deal with situations that are real for the students, or with which they are familiar. They can be adjusted to different levels, and can help to promote discussion in the mathematics classroom. They can also be used to bring topics that are relevant to the students' personal development into the classroom, thereby improving their knowledge of their environment. At the same time, since exact methods for solving them are not viable, these problems allow students to work on estimation of magnitudes and the assessment of errors in their measurements. Our aim is that, as they solve these problems, the students see the necessity of focusing on the essential components of the situation they are given. In this way, our intention is to introduce the students to mathematical models for solving these problems.

THE STUDY

Pólya (1945) established a problem-solving model with four phases: 1) understanding the problem; 2) making a plan; 3) carrying out the plan; and 4) looking back. The objective of our study is to determine what factors have an effect on comprehension of the problem and the types of solving strategies that students produce when faced with inconceivable magnitude estimation problems (IMEP). In order to focus analysis on the first two phases of Pólya's model, the instructions given to the students explicitly asked them to restrict themselves to explaining how they would solve the problem. We used six estimation of inconceivable magnitude problems which, based on the responses of a reduced group of students in a pilot test, were selected from an initial set of 36.

The problems we used were: A) How many tickets could we sell for a (sold-out) concert in the school schoolyard?; B) How many people are there in a demonstration?; C) How many SMS messages do Catalans send each other in one day?; D) How many drops of water are required to fill a bucket?; E) How many glasses of water are needed to fill a swimming pool?; F) How many one-euro coins fit in a safe with a volume of one cubic meter?

Each problem had instructions which situated it in a real context. For example, the context for problem A was the need to anticipate the number of tickets to sell for the school's year-end party; for problem D, students were told that there was a leak near the computers in the teachers' room. All of the instructions were refined in a pilot test carried out with a small group of university-bound secondary students (students between 16 and 18 years of age who had finished the compulsory phase of secondary education) in order to verify that the students would have no problem understanding the situations presented in the problems.

The problems were given in one-hour class sessions to students in the compulsory phase of secondary education in two schools, one public and one private. They were asked to individually explain the steps they would follow to solve the problem. They were explicitly instructed not to make any calculations and to limit themselves to describing the procedure they considered the best for tackling the problem. The students responded to these questionnaires for 15 to 30 minutes. In this way, we were able to collect responses to several questions from each student. We thereby collected 538 proposals from the 216 students who participated in the study.

We analyzed the students' responses using NVivo 8 software, which permits establishing different categories of analysis and data management, as well as cross-comparison of different types of queries. As Gibbs (2007) suggests, the codification of data into categories establishes a frame of reference for interpreting the data collected, which allows for it to be analyzed from different perspectives.

In particular, our analysis sought to: a) see whether the students' proposals indicated if they were or were not on the right track to solving the problems; b) analyze the different types of strategies they proposed for solving the problems in order to identify

attempts at modeling the situations; and c) determine whether the students' proposals, if carried out, would result in solving the problems.

STRATEGIES AND SOLUTION SUCCESS

Below we present and illustrate the different types of strategies identified in the students' proposals for solving the six problems used in the study, and examine whether what they proposed would result in solving them effectively.

The following is an example of a method proposed by a 15-year-old student to solve problem D, in which students were asked to estimate the number of drops of water that would be needed to fill a bucket:

> "It depends on the dimensions of each drop, on whether it will fall entirely into the bucket, and on the size of the bucket. We'd also need to check that the drops didn't evaporate."

As we can see, this response contains a list of elements that could help in solving the problem, but does not indicate a specific plan or procedure for obtaining the estimate that is asked for. This is an example of a type of proposal that we have classified as proposal lacking strategy.

We also found students whose responses merely proposed an exhaustive count, which is a strategy that can not be considered effective for solving an IMEP. The following is an example of this type of procedure for estimating the number of glasses of water required to fill a swimming pool (problem E):

> "I'd get some glasses and start to fill them with water from the pool. I'd get as many glasses as I needed to take all the water out of the pool. After that, I'd simply count how many full glasses there were to know how many I'd need."

In this case, the student proposes to empty the pool using glasses and then to count them afterwards. The next is an example of the same type of strategy, in which a student suggests counting the number of drops of water contained in a bucket (problem D):

> "In this case, what I'd do is see how long it took for the bucket to fill up (by counting the drops), and then I'd remove all the computers to make sure they were not damaged, and then I'd put another bucket in its place."

We also found other strategies that were more suitable for solving the problems. The following is a response to the problem of estimating the number of people who would fit in the schoolyard for a concert (problem A):

> "First of all, I'd mark out the stage area, then I'd set out a row of chairs, as many as would comfortably fit, to determine the width, and then the next step would be to do the same, but lengthwise, since the others were for the width. Finally, I'd multiply the width by the length, since it's a square or rectangle, and the resulting number would be the number of tickets."

In this approach, the student proposes a rectangular arrangement of the audience as a model, a model which makes successful solution of the problem possible. The student proposes to estimate the number of chairs that would fit in the schoolyard in two

dimensions and then to calculate the product. A different, but equally suitable, approach to solving the same problem is the following:

> "The first thing I'd do would be to calculate the maximum number of people who could fit in the schoolyard. To start, we'd need to know approximately how many people there were per square meter, and then how many square meters the schoolyard was from its length and width. Finally, multiply the people by the number of square meters in the schoolyard. With the resulting number you know how many tickets could be sold."

In this case, the mathematical concept the student proposes to model the situation and thereby obtain an estimate is that of population density. This approach is also valid for obtaining a satisfactory result. Another approach for solving this same problem was the following:

> "I'd get 10 students and calculate the space that each one occupied, and then the average. I'd calculate the total area of the schoolyard and then subtract the space the stage would take up. The resulting space would be the space available for people, which I would divide by the average space occupied by each student. Then, for example, if the result were 108 students, I'd sell 100, because if not, the space would be too tight."

In this case, we can observe that the model used is that of the iteration of a unit. This model is based on establishing a unit of reference which is then applied over the set that is to be estimated, in this case, the average area that a person occupies. As these examples demonstrate, the students' proposals displayed different kinds of strategies for the same problem.

In our analysis, we established different categories to organize the students' proposals for those aspects that were of interest to us: *type of strategy* and *solution success*. We established several categories for the proposed strategies. We found that there were students who did not propose any defined strategy (*lacking strategy*) and others whose proposals employed an exhaustive count (*count*). Yet others relied on seeking information from external sources or who proposed asking someone else (*external source*). On the other hand, there were students who attempted to reduce the problem to a smaller problem within their reach and to use a factor of suitable proportion (*reduced proportion*). Yet others attempted to break the problem down into smaller parts and to solve these separately based on concepts such as population density or points of reference, such as the volume of a glass (*breakdown*). Finally, there was one student who proposed solving the problem by comparing it with a real situation he was familiar with (*real situation and proportion*).

As for the success of the solutions, we established three categories for classifying the proposals according to the degree of success that could be obtained were they to be carried out. Proposals which did not result in a satisfactory estimate, or which did not specify a concrete course of action, or a course of action with erroneous ideas, were classified as *Not solved*. By logic, there should be one other category for proposals which solved the problem, but given the nature of IMEP, we found two clearly differentiated types of proposals which resulted in a valid result.

On the one hand, some proposals relied on exhaustive counts. Taking into account the fact that the magnitudes in question are not within ordinary reach and refer to very large numbers, these proposals would require a long time or excessive resources to be carried out, even if the proposed procedure was valid. For this reason, we classified such proposals as *solved on paper*.

Finally, we established the category *Solved* for proposals that displayed a procedure that could be carried out in practice and which was effective in obtaining a satisfactory solution to the problem. Most of the proposals in this category included some element of modeling to represent the situations in which the problems were developed. Relational analysis produced a table that correlates *strategy type* with the degree of *solution success* proposed by the students.

	Solved	Solved on paper	Not solved	Total
Lacking strategy	0 (0%)	0 (0%)	160	160
Count	0 (0%)	84 (87%)	12 (13%)	96
External source	0 (0%)	14 (70%)	6 (30%)	20
Reduced proportion	13 (42%)	4 (13%)	14 (45%)	31
Breakdown	109 (47%)	52 (23%)	69 (30%)	230
Real situation	1 (100%)	0 (0%)	0 (0%)	1
Total	123	154	261	538

Table 1: Strategy proposed vs solution success

The Table 1 shows that proposals which lacked any kind of strategy and proposals which used an external source or comprehensive count did not result in valid solutions. On the other hand, strategies which reduced the problem to smaller ones in order to carry out a proportion of scale and strategies which broke problems down into smaller parts and solved these parts separately were the strategies that, in at least 40% of cases, successfully solved the problems.

ConclusionS

In this article, we introduce Inconceivable Magnitude Estimation Problems and demonstrate that they can be solved through the use of partial estimates. In this way, following the definition proposed by Ärlebäck (2009), IMEP are a subclass of Fermi problems. We have seen that students produced many different strategies that would result in successful solutions to these problems. We therefore believe that IMEP can be useful for showing students that there is more than one way to solve a problem and that each can result in the same solution. In this way, focus can be shifted to the process of solving a problem, thereby breaking away from the tendency to focus exclusively on the result. By using IMEP, teachers have access to open problems which can be discussed in an open manner due to the existence of different approaches to their solutions (Chapman, 2006).

Furthermore, it is important to note that the strategies which allowed students to solve the problems displayed elements of modeling, which leads us to believe that IMEP could be a useful tool for introducing the processes of modeling into the classroom. More specifically, we believe that group work and project work would allow all students to use the modeling strategies proposed by some of the students in our study. At the same time, exhaustive strategies could be a way to generate discussions that would promote the use of more elaborate strategies that would show students the need to create models which describe the most relevant aspects of a given situation.

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REFLECTING UPON AMBIGUOUS SITUATIONS AS A WAY OF DEVELOPING STUDENTS' MATHEMATICAL CREATIVITY

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The aim of this paper is to show how engaging students in challenging, ambiguous situations through model-eliciting activities can stimulate their mathematical creativity and extend the variety and the quality of their mathematical models. The participants were mathematically talented primary school students who were members of "Kidumatica" math club. We used the "Bigfoot" modeling task to immerse students in an authentic, hands-on mathematical situation. This activity allowed students to use and extend their creative thinking, which was exhibited itself in the diversity of their significant mathematical ideas. Students invented, discovered and created different types of strategies and mathematical conceptual tools.

INTRODUCTION

Learning is the development of both knowledge and skills. We make sense of our world by integrating and analyzing the wealth of information around us. However, rapid growth and development in the 21st century, which touches upon every aspect of our daily lives, requires an educational system that will provide students with authentic learning experiences that reflect this ever-changing, complex and ambiguous environment. Students need to cope with, and assimilate the global changes in technology and information.

The OECD (2008) stated that mathematics "curricula should reflect the reality…[and] should stress innovative applications of mathematics" (p. 18). Relaying on the assumption that education plays an essential role in encouraging and promoting future generations' potential, including cultivating excellence and nurturing diversity (Amit, 2010; Adams 2005), one might question whether mathematics education and educators are relating to the proliferation of technologies and innovations that are globally transforming our lives.

Researchers in mathematics education and developers of model-eliciting activities (MEAs) emphasize the productive aspects of the nature of mathematics and encourage students to develop an explicit mathematical interpretation of ambiguous, authentic and complex situations that might occur in their everyday lives (Chamberlin & Moon, 2005; Lesh & Sriraman, 2005; Della & Cynthia, 2010). From this perspective, the present study explores how engaging students in challenging, ambiguous mathematical situations can stimulate their creativity and extend the diversity of their mathematical ideas.

TOLERANCE TO AMBIGUITY, AND CREATIVITY

The accelerating changes in technology and science in the 21st century are provoking more ambiguity and uncertainty than ever before. Ambiguous situations, which have become an inseparable part of our living environment, were investigated early on in terms of people's predispositions to viewing ambiguous situations as either threatening or desirable (Budner, 1962; Norton, 1975). Budner (1962) defined ambiguous situations as complex, new or contradictory situations and claimed that people who are intolerant of ambiguity have "the tendency to perceive ambiguous situations as sources of threat" (p. 29). Norton (1975) offered eight different causes of ambiguity: (1)multiple meanings, (2) vagueness, (3) incompleteness, or fragmentation,(4) probability, (5) lack of structure, (6) lack of information, (7) uncertainty, inconsistencies and (8) contradictions, and lack of clarity; in each case, individuals' emotional perception of the situation was described as ambiguous tolerance.

Research has revealed a significant role for ambiguity, and the tolerance for it, in creativity, innovation, and problem solving (Guilford, 1973; Kirton, 2004; Adams, 2005; Sternberg, 2006). Guilford (1973), who associated divergent thinking with creativity, argues that "tolerance of ambiguity" is one of the characteristics of a creative individual. Sternberg (2006), in his research on the nature of creativity, claimed that according to investment theory, students can decide when to be creative, and that being "tolerant to ambiguity" is one among 20 decisions which can encourage students' creativity. Kirton's (2004) adaptive-innovative theory, which deals with how people solve problems, differentiates between adaptors, i.e. those who desire to do things better, and innovators, who are more tolerant of ambiguity, are risk-takers and tend to produce more ideas. Adams (2005), in her report on the sources of innovation, offered some recommendations on how the educational system can foster students' innovative and creative skills, arguing that "a rigid environment that adheres too strictly to procedure does not foster creativity. By contrast an humorous, jovial environment where there is comfort with ambiguity and a focus on ideas rather than careers is favourable to innovation (p. 33).

AMBIGUITY IN MATHEMATICAL MODELING ACTIVITIES

Mathematical-modeling activities based on "real-life" problem situations are open-ended, authentic tasks with a high level of complexity, in which students are given the opportunity to construct powerful ideas relating to interdisciplinary data (Lesh & Sriraman, 2005). These activities differ from traditional "word problems" which define static assumptions involving givens and goals (Della & Cynthia, 2010). MEAs require students to make sense of ambiguous situations that can involve uncertainty, lack of information, contradictions or conflicts (Chamberlin & Moon, 2005), with no formula or model provided to complete the MEA (Lesh & Caylor, 2007). The ambiguity of the problem statement and data representation allows diverse interpretations that tolerate more than one single or unified viewpoint or perspective. This suggests that various responses may be appropriate and that there are likely to be

various levels of correctness, depending on students' interpretations, mathematical abilities, general knowledge and skills (Chamberlin & Moon, 2005; Lesh and Doerr, 2003).

AMBIGUITY AND MATHEMATICAL CREATIVITY

Non-routine problems and heuristic tasks that require students to reflect upon complex and ambiguous situations have been suggested by number of researchers as a way of stimulating and promoting students' mathematical creativity (Polya, 1957; Sriraman, 2008; Sriraman & Dahl, 2009). Sriraman (2008), who defines mathematical creativity as the ability to produce novel or original work, claims that "students should be given the opportunity to tackle non-routine problems with complexity and structure—problems which require ….also considerable reflection" (p. 32). Polya (1957), in his book "How to Solve It", advocates a heuristics approach as a way to "study the methods and rules of discovery and invention" (p. 113), but argues that "heuristic argument is likely to be harmful if it is presented ambiguously" (p. 113). Sriraman and Dahl (2009), in a descriptive article explaining the significant role of interdisciplinarity in mathematical education, claimed that "teachers should embrace the idea of 'creative evidence' as contributing to the body of mathematical knowledge, and they should be flexible and open to alternative student approaches to problems" (p. 1248). The emergence of multiple responses according to Guilford's (1973) definition of divergent thinking increases the possibility of arriving at original thoughts.

METHODOLOGY

The following research was aimed at revealing the implications of MEAs on students' creative mathematical thinking. The modeling activity was based on the Bigfoot modeling task (Lesh & Doerr, 2003), which involves four of Norton's (1975) causes of ambiguity: (1) multiple meanings, (2) vagueness, (3) probability, and (4) lack of information. Students were asked to help a scout group discover who fixed their fountain. The only clues the scout group had were "huge" footprints left in the mud. Students had to develop a conceptual mathematical tool that would enable estimating the height of this "giant" man. In addition, they were asked to write a letter mathematically justifying their solution and explaining how to use this tool. Each group of students received a depiction of an authentic large footprint's stride on a piece of cardboard, and measuring tapes and calculators were made available to them. The task was worked on by small groups (3–4 students) for about 50–60 minutes and at the end of that time, each group had to present their models; solutions were shared and discussed by the whole class for about 30–40 minutes.

Participants

Participants in this study included 78 "high-ability" and mathematically gifted students in the 5th through 7th grades who are members of the "Kidumatica" math club. The "Kidumatica" program provides a framework for the cultivation and promotion of

exceptional mathematical abilities in youth from varied socio-economic and ethnic backgrounds (2009).

Data

The data consisted of students' documents written during the MEA, classroom observations, and video-recordings of their model presentations. The written data included students' modeling drafts, their conceptual tools and written presentations. It should be emphasized that the students were asked to write down everything, so that drafts, sketches and the final solutions could be collected. The video-recording included students' oral presentations of their models, researcher interviews and class discussions. Transcripts of these videotapes were used along with students' written data to assist researchers in the analysis.

FINDINGS AND RESULTS

The model-eliciting process requires students to pass through several cycles. Each group went through different cycles of interpretation, development and testing; the students had to construct the data, recognize the important variables and discover the relations between those variables through several phases of development. The first phases were premature and naïve, with some students exhibiting difficulties coping with the complexity and ambiguity of how to use the data to create a meaningful model. However, as the process progressed, they improved their interpretations, and discovered repetitive behavior in the data which led them to mathematize the situation and develop diverse mathematical responses. In their final cycles, the students moved from everyday language to the use of symbols and mathematical formulas which helped them communicate their new ideas.

Students made use of different elements, such as age, gender, different shoe dimensions (width, length, perimeter of shoe), strides and other parts of their body to invent, discover and develop different rules and patterns that would describe and explain the relationships between those elements. The diversity of student responses was also affected by the cognitive and affective abilities they demonstrated during the modeling process. Students' modeling responses were analyzed with respect to the elements selected, and the relationships and operations used to explain and predict or estimate these ambiguous situations. In this paper, the diversity of student responses was identified by the differences in the strategies they demonstrated. The research involved models from 22 groups, which presented at least 12 different models. Some strategies appeared in more than one model but they were used in different contexts, based on different interpretations or with different elements. In the following we focus on six of these models.

The first model was based on the proportional relationship between an individual's height and shoe length: this strategy was generated by most of the students, but their development processes and interpretations differed. During the development process, students used analogies and metaphors to organize their thinking and to reflect on the

situation. The following two examples demonstrate how two different types of analogies led to the same model, as explained during their modeling presentation. In the first group, students drew analogies between getting longer and getting taller. Dan: "We were told that this man is very long [in Hebrew the word for length can be used to indicate a person's height] so we decided to use the length of the shoe to find the length of the man." In the second group, two 6th graders drew an analogy between proportionally growing up and the mathematical notion of proportion. Maor: "We thought that as the man grows, his whole body is growing and also his foot, but the man is the same, he is proportional, his head or his feet cannot grow too large so we wanted to calculate this proportion." Another group used a similar strategy, but instead of shoe length they discovered the correspondence between shoe length and shoe size and used the ratio between height and shoe size.

The second model was based on strides. Here students used the length of their stride and the ratio between it and height. Two students in this group explained how they constructed their model by estimating how many times the average stride is smaller relative to the height. Nitai: "We allowed each of us to walk and for each stride we measured the gap and averaged it." Didi: "Then we measured the height and divided by it and we found that the average stride is three times smaller than the height."

For shoe that is wide *comparing to its length* multiply by X 4
For shoe that is narrow *comparing to its length* multiply by X 5

      than 10 cm   than 10 cm

A/S X (width + height)
      According
      to the width
      of the shoe

The height of the person that repaired the fountain is 2.04 m
  Since the width of the shoe → 13
    the length of the shoe → 38

$$4 \times (13+38) = 2.04$$
    Width length
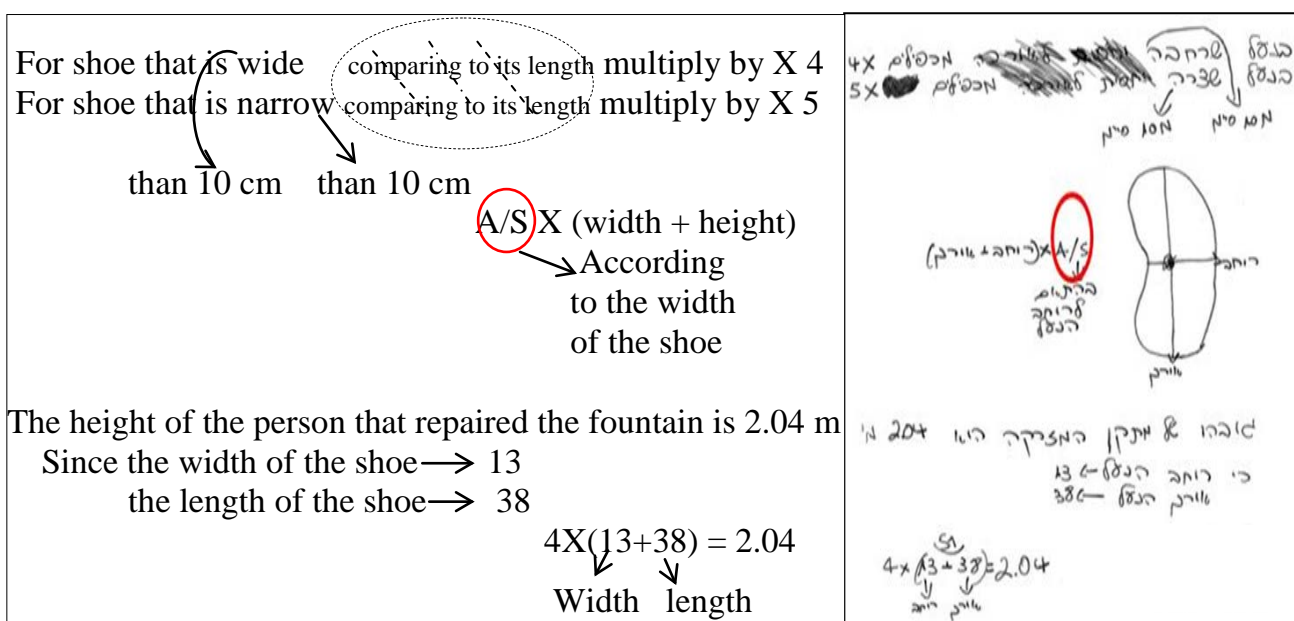
The third and fourth models both involved the ratio between height and the sum of shoe length and width, but the third model involved this rule with an extension: students realized that some children had narrow shoes and some wider. They extended their model by adding a constraint that depended on the width of the shoe. Avia: "Then we noticed that my shoe is relatively wider and Sagi's shoe is narrow compared to its length." Sagi: "So we decided that if the shoe in its narrowest part [pointing to the narrow part in his drawing, which appears in Figure 1] is less than 10 cm we multiply by 5, otherwise we will multiply by 4." During the presentation and class discussion, these students explained how they tried to mathematize the dependence between shoe width and length and its relation to the constraint that could enhance the prediction of

height, but they did not have time to complete it. Researcher: *"I can see that you wrote A/S and erased some words"* (A/S is circled in red, and the erased part is circled with dashed lines in Figure 1). Sagi: "We didn't have time to complete our solution to find the exact ratio between width and length so we compared the shoe's width to 10 cm." Avia: "But we wanted to use the proportion between length and width…so we just wrote A/S."

Formulation for height according to foot length

For child: $a * 7 = H$ For adult: $a * 5\frac{7}{10} = H$

Ratio   Ratio

H - height   a - foot length

For child: I measured myself and checked the proportion between the foot and the height

For adult: I measured Boris [his tutor] and checked the proportion between the foot and the height

How to: measure footprint, estimate age, and compute according to the appropriate equation.

Figure 2: In the Ratios between height and shoe length depending on person's age (fifth-model)

An instrument for estimating man's height
1. We measured the foot length.
2. We checked the average of the times that it fit up to the hip
3. We multiplied the foot by it.
4. We multiplied it by 2 because we found it only up to the hip and it is half of the overall height.

Exercise:

Foot print size: 37.5
In the height up to hip: 3   x
            Up to hip   112.5
            Up to head   2   x
                          225
The result = 225cm
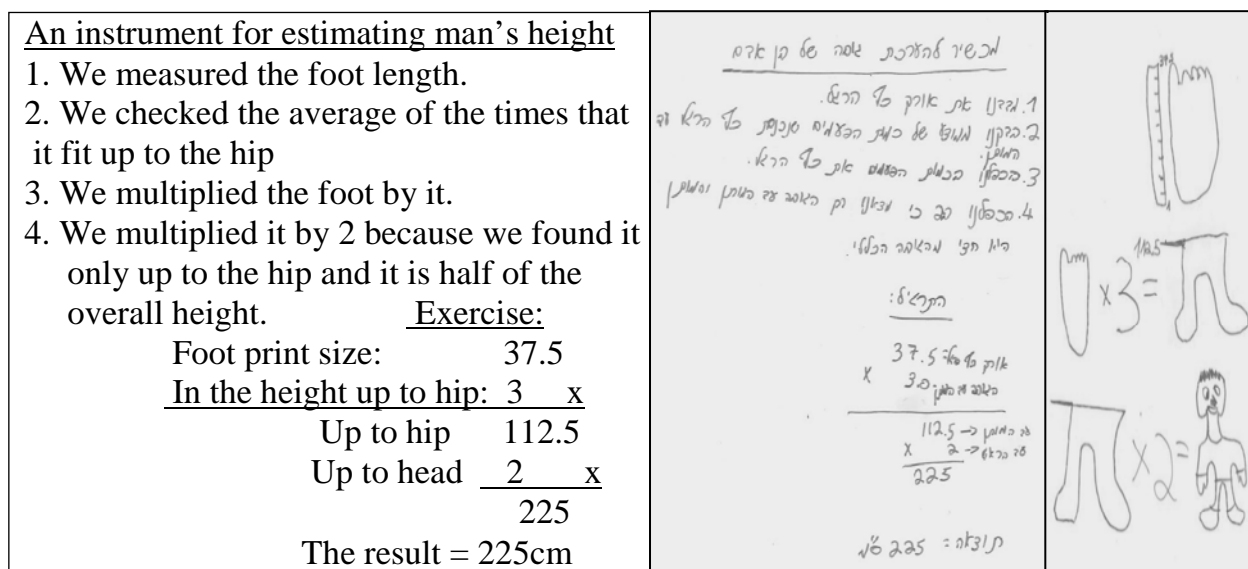
Figure 3: Ratio between shoe length and body length up to the hip (sixth-model)

The fifth model proposed by the students brought together three variables to construct a conceptual tool for estimating a person's height. Different ratios were used between height and shoe length, depending on the person's age. Figure 2 shows their mathematical formulation and an explanation on how to use their conceptual tool.

The sixth model was from 5th graders who used measurements of some parts of their body to estimate the man's height. This model was based on two proportions: the ratio between shoe length and the length of the body up to the hip and that between the latter and total height. Students took measurements of their bodies up to their hips and total height to calculate the ratio of each child in the group, then they calculated the second ratio, averaged it and found that "it is half of the overall height" (see Figure 3). In the second part, they measured their shoe and calculated "the average times that it fit up to the hip."

DISCUSSION

The variety of mathematical responses obtained shows how MEAs can be used to promote the development of more diverse and sophisticated creative conceptual mathematical tools in students. The Bigfoot task was non-routine, complex and structured which, according to Sriraman (2008), is required for the emergence of students' mathematical creativity. In addition, this task inspired students to reflect upon ambiguous situations: students had to construct an estimation to find the height of an unknown person based on uncertainty and lack of information (Lesh and Doerr, 2003; Chamberlin & Moon, 2005). According to Norton (1975), lack of information, probability, vagueness and multiple meanings cause ambiguity, and the students were required to deal with different interpretations. Students asked a broad range of questions, and raised assumptions based on their experience, mathematical skills and general knowledge (Lesh & Sriraman, 2005; Lesh & Caylor, 2007). This led them to discover, invent or develop different mathematical patterns and rules using different pathways and representations, increasing their tendency to produce original ideas (Guilford, 1973; Sriraman, 2008). Finally, the collaborative work, involving model development, model presentation and class discussion at the end of the modeling activity, encouraged students to mathematically communicate their new ideas. The ambiguity and complexity of the task exposed students to different approaches, multiple different pathways and innovative mathematical solutions, described by Sriraman and Dahl (2009) as 'creative evidence'. This creative process not only allows students to apply their mathematical skills and abilities, it also promotes student diversity, comprised of their different perspectives, experiences and backgrounds (Amit, 2010). Reflecting upon ambiguous situations increases the potential for innovation, discovery and creativity.

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COHERENCE AND CONSISTENCY IN SOUTH AFRICAN PRIMARY MATHEMATICS LESSONS

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This paper contributes to the body of research on the pedagogic content knowledge required for primary school teachers to teach mathematics effectively. The particular focus is on teachers from ten schools in South Africa engaged with a longitudinal research and development project: the Wits Maths Connect–Primary project (WMC–P). We report on a video of a lesson that on the surface 'works' in that the teacher provides mediating means (physical, verbal and symbolic) that allow most of the learners to successfully complete the tasks set in the whole-class setting, though not so successfully within individual work. Our analysis reveals, however, that there are mismatches in the coordination of tasks, mediating means and mathematical objects, with each co-varying as tasks unfold, resulting in the mathematical objects not emerging for many learners.

INTRODUCTION

National standardized and international comparative test results continue to paint a bleak picture of performance in mathematics in South Africa. For example, performance on the 2011 Annual National Assessments indicated that the national mean result at Grade 6 (predominantly 11- to 12-year-olds) stood at 30% (Department of Education (DoE) 2002) Previous evidence indicates that the majority of South African learners achieve well below the levels stipulated in the National Curriculum Statement (Department of Education, 2002), and these low levels are entrenched in the national landscape by the end of the Foundation Phase (Gr R - 3, 5- to 8- year-old) (Fleisch, 2008). In this context, we have begun a longitudinal research and development project – the Wits Maths Connect–Primary project (WMC–P) – focused on developing and investigating interventions to improve the teaching and learning of mathematics in ten government primary schools. Project team discussions about lessons observed early in the project noted a lack of clarity both of purpose and coherence in lessons. Supporting connection-making is viewed as central to teaching for conceptual understanding (Askew, 1997; Scott, Mortimer, & Ametller, 2011), so we set out to investigate and understand more rigorously ways in which a lack of clarity or coherence was constituted in lessons.

In the majority of lessons observed (n = 33 of 41 Grade 2 classes), the bulk of time was spent on oral whole class work orchestrated by the teacher with some individual written work within this time, but limited extended individual seatwork. This whole class talk is focused around particular mathematical tasks with the teacher mediating for the learner how to engage with the task. To understand lack of clarity and coherence we focus on how teacher mediation, through talk and artifacts/tools, facilitates (or not) the emergence of mathematical objects for the learners.

THEORETICAL UNDERPINNING

Cole (1996) reminds us that genetic analysis has to operate at different levels: historical, ontogenetic and microgenetic. This paper reports on a microgenetic analysis of mathematical objects as they are co-constructed by teachers and learners. By considering the main activity of lessons, the action learners engage in and the mediating means used by the teachers we reveal aspects of the teachers' understanding of mathematics that allow us to speculate on the ontogenetic origins of their practices. This paper presents one lesson case study as an exemplar.

For our microgentic analysis we build on the Vygotskian ideas of mediation (1978) examining mathematical tasks as they are enacted in terms of the subject (teacher), object (the mathematical object that the lesson is intended to bring into being) and mediating means. We theorise lesson 'objects' as being multiple, in the sense that a teaching episode may have more than one object - an indirect object of learning and a direct object of learning (Marton, Tsui et al. 2004). In the case discussed here we interpret the indirect mathematical object to be understanding missing addends, while the direct object is to complete a number of missing addends calculations. As a mathematical object will only emerge over time, we theorise this as being occasioned through activities, actions and operations (Leontiev, 1978), extending this trifold model to include 'tasks' at a level between 'activity' and 'actions'. We interpret activity as a large coherent 'chunk' of a lesson or lessons activity that appears to be directed at a mathematical object. Within an activity there are a number of tasks learners engage in which are more at the level of the direct objects. Certain actions allow the completion of the tasks, with these being made up of specific operations: we focus our attention on the teacher's mediating means to support these operations, 'chaining' back up through actions and tasks to examine whether the activity is coherent and supporting the emergence of the mathematical object.

A lesson may, as it is enacted, have more than one object in that the object might change as the activity unfolds as a consequence of the teacher's choice and use of mediating means. We thus make the distinction between the intended mathematical objects - what learning a particular activity appeared to have been chosen to bring about - and the enacted mathematical objects - what emerges as the activity unfolds. The intended object may or may not be explicitly articulated by the teacher: when not articulated and in the absence of guidance from the teacher we speculate, given the teachers talk and choice of actions as to what the intended object is.

DATA SOURCES

The case study presented here is drawn from a set of classroom observations collected as baseline data for the 5-year WMC–P project. This data included observations and videos of one mathematics lesson from each of the Grade 2 classes in the ten project schools (n = 41). Preliminary analysis of this data revealed that while the majority of the lesson appeared to run smoothly, closer observation suggested a disconnected sequencing of actions and operations leading to, from the learners perspective,

ambiguity in and obscuring of the indirect learning objects. The lesson focused on in this paper was selected as a 'telling case' exemplar (Mitchell, 1984) of such ambiguity and obscuring. The lesson is typical in having extended instances of whole class talk, which we had observed across the lessons. The tasks the teacher introduced to the class were also quite typical of the range of tasks observed.

The teacher, Pearl (pseudonym), is an experienced the Foundation Phase teacher. Pearl's primary school is located in a township/informal settlement area, with a roll of over 1800 and relatively large classes (37 present in the focal lesson; 3 absent) in temporary classroom buildings. The area has significant inward migration from other parts of the country, and children in the Foundation Phase are placed in classes according to their home language. Pearl's class is taught in Zulu – one of two Zulu classes in the 6 form grade 2. .

The core mathematical task for Pearl's lesson was centred on a resource described as a 'wheel': writing 'addition' as the title for the activity, the resource, stuck up on the board, consisted of three concentric circles – 7 written on the inner circle, and the numbers 0-7 placed in random order around the outermost circle in separate sectors. The task explained by the teacher was to fill in the intermediate circle with the numbers that needed to be added to the outer rim numbers of the 'wheel' to make the number 7. Introducing and mediating the completion of the missing addends for 7 wheel tool up the middle 26 minutes of a 50 minute lesson, and was followed by an individual worksheet activity based on a missing addends for 11 wheel.

In terms of operational number range, the whole class missing addends to 7 task and the individual missing addends to 11 task tend to relate to the curriculum specification given for Grades R and 1: e.g. in the Grade R specification on work within Learning Outcome 1 – 'Numbers, operations and relationships', which includes 'Solves verbally-stated additions and subtraction problems with single-digit numbers and with solutions to at least 10.'

DATA ANALYSIS

Our analysis included five phases: (a) creation of a detailed transcript, (b) identifying the substantive intended activity (indirect mathematical object) (c) identifying the direct mathematical objects of the lesson (d) analysis of the actions and operations directed at completing the tasks and (e) theorising the coordination of the mediating means (operations/actions) and intended objects of learning (activity) and the mathematical object as it played out in comparison with what was intended. In essence, as the analysis developed, phases (b) and (c) occurred concurrently: we present them separately here merely for ease of discussion.

Initial creation of transcript

Following a classroom observation, a bilingual English-Zulu transcribed the video recording, following our instruction to capture all the teacher's talk within the lesson and the objects/ representations referred to within her talk – learner work/ diagrams on

the board/ work with manipulatives. This was then divided into episodes, based on shifts to a different segment of the activity, usually marked by the introduction of a new task. To improve accuracy and detail, the project team viewed the video recordings several times to clarify the interaction between teacher talk and the use of mediating means (e.g., fingers, objects, diagrams, gestures).

Identifying the 'activity'

We focus on activities, rather than lessons, interpreting activity as a sequence of tasks that appears to be directed at the same mathematical object. The sections of the lesson discussed here comprising an activity were framed by main tasks: whole class introduction of 'wheels'; individual students completing a new 'wheel'.

Identifying the mathematical object

The indirect mathematical object may be made explicit by the teacher, fully or partially, or left implicit. In this lesson the teacher began by announcing 'we are learning about addition'. However this only partially announced the mathematical object. The activity - wheels - provided openings for the mathematical object to potentially emerge to be 'missing addends', suggested not only by the construction of the wheel, but by the teacher's repeated articulation of the task being to find what to add to the outer number to make the middle number.

Analysis of the mediating means

The activity in and of itself only provides the broad frame through which the mathematical object might emerge. It is the mediating means used in the enactment (actions and operations) of the tasks making up the activity that support the emergence and establishment of the mathematical object.

CASE STUDY: WHEELS

We present our analysis interwoven with the data from the activity (in italics). (T is teacher, Ch whole class chorus of answers, L1 learner 1 and so forth).

The teacher introduced the lesson with a brief discussion on addition:

 1 T: Today we are going to add. We are adding the numbers. We all know how to add, right?
 2 Ch: Yes.
 3 T: Who can tell me, adding is to do what? When we say we are adding what are we doing to those things?
 4 L1: It's adding two things together.
 5 T: When we add it is to take two things and make it one thing.

The teacher attached a paper with the word 'Addition' (in Zulu) to the board and asked the children to read it, three times, and then to inscribe the addition symbol in the air. She fixed to the board the paper displaying the 'wheel'.

 6 T: We are going to use this wheel today to add.

After counting with the class from zero to seven to check that all the corresponding digits were in the outer ring, the teacher introduced the tasks.

> 7   T: Here I want you to tell me which numbers we can add to each number on the wheel to give us seven.

Utterance (7) is the first indication of 'missing addend' (although not spoken of in those terms) as the indirect object of the activity. This is the articulated object that the teacher spoke of most frequently during the activity, although it differs from her first statement of adding being to 'take two things and make it one thing' (line 5).

> 8   T: I will make an example. Seven. When we add it with zero (pointing to zero on the other ring) will the number change? (8)

Here we see the first shift in the mathematical object. The teacher's style of questioning is consistent with her earlier articulation of the mathematical object in line 5. Rather than asking 'what do we need to add to zero to make seven' (consistent with the mathematical object of missing addend) she asks 'When we add it with zero will the number change?' 'It' is problematic here: does it refer to the seven in the centre of the wheel or to the seven that needs to be added to zero to make seven? In the light of subsequent actions by the teacher (see below) our interpretation is that the teacher uses her knowledge of seven as the answer to the first (implicit) question ('what needs to be added to zero to make seven?') and bases the articulated question on this answer rather than the mathematical object of the activity. (The choice of zero as the first digit to work with is also not helpful as it does not make clear the distinction between 'What needs to be added to zero to make seven' and 'Add seven and zero' as both actions yield the same answer.)

> 9   T: I want you to look for numbers on the inner wheel (pointing to the blank disc) that we will add with the number on the small wheel (pointing to the outer disc) so that the answer is seven. We will start with one (pointing to '1' on the outer wheel, holding up a toothpick)
>
> 10   T: This is one (showing the toothpick). Right?
>
> 11   Ch: Yes.
>
> 12   T: I want you to tell me which number we are going to add one with it to give us seven, which number is that?
>
> 13   L2:   Six.
>
> 14   T: Which number is that? Six. Right?
>
> 15   Ch: Yes.
>
> 16   T: Let's check if she is telling us the truth.
>
> 17   T: (Holds up one toothpick) Let's add six and see if we get seven.

Teacher adds toothpicks to the one held up, the class counting along until she is holding six in that hand. Holds up a seventh in the other hand.

> 18   T: On this side I have six, but on this other side I have one and when we add them (moving the two hands together) it will give us what?
>
> 19   Ch: Seven.

With the task focused on '1' the teacher again articulates the object in the spirit of missing addend (line 12), as is the learner's answer. But the mediation with the

toothpicks is incoherent. Starting by holding up one toothpick and saying 'let's add six and see if we get seven' again is consistent with checking the missing addend. What was actually modelled, however, was six add one, with the original 'one' that was being added to becoming subsumed within the group of six: the single seventh toothpick becoming signified as the one that needed to be added to the six to make seven, effectively 'flipping' the missing addend from being six to being one.

20	T:	What do we need to add to two? (pointing to '2' on the wheel)

Boy (L3) holds up eight fingers.

21	T:	He has jumped to eight. No. We want a number that will give us seven. You must first count two and then tell me which number must I add with that two to give me seven.
		The boy holds out two fingers.
22	T:	I will hold up two fingers for you (puts out two of her fingers) and you can add it with that number.
23	L3:	(Counts his and the teachers fingers) Four.
24	T:	Four?
25	L3:	Yes
26	T:	No. (Ruffles boys hair) Count well. Which numbers can we add with two and give us seven? (Several hands up)
27	L4:	Nine
28	T:	Make seven with your fingers
		Everyone holds up seven fingers (following what the teacher does as five on one hand and two on the other)
29	T:	Now hide two and which number are you left with? Make your seven first and hide two. Which number can we add with this two to make seven?
30	L5:	Eight
31	T:	No, we made seven and hide two and what is left? The number that we will add with two to give us seven?
32	L6:	Five

This marks the beginning of defining subsequent tasks as 'taking away'. The answer 'nine' (27) suggests many children are interpreting the task as 'add two and seven' so the teacher introduces the model of holding up seven fingers and 'hiding' two of them - her articulation now elides between 'taking away' and missing addend (line 29). The children can follow the action of taking away and succeed at showing the five fingers left: thus an action is hit upon that leads to the correct answers being produced. But in doing so a shift in the mathematical object fundamentally is not established as a valid action through the mediation that occurs: while a child might eventually solve 5 + __ = 7 by subtraction rather than counting on, the teacher's mediation assumes that this is obvious. As most of the children now articulate the answer that the teacher wants this becomes the action for the remaining calculations: for the remaining digits the action is to put out seven fingers then 'hide' some of them. So for three:

33	T:	Make three and add the number that will give us seven when added. Let's make seven with our fingers.
		Children hold out seven fingers (again as five and two)

34 T: Hide three fingers, hide three fingers. How many fingers are left now?

Having established that there are four fingers left, the teacher checks this by counting out three toothpicks, counting out four and then counting them all. The teacher continues to articulate the mathematical object as missing addend (utterance 33) and models this when checking, but the mediating actions for arriving are the answer are not consistent with this articulation.

35 T: I want a number that we will add with four and gives us seven. Four apples. Let's first do our seven. (holds up seven fingers, class follows)
36 T: Hide four fingers, right?
37 Ch: Yes.
38 T: After hiding four which number is left?
             This routine continues for five and six, after five the teachers saying:
39 T: Do you see how we add?
40 Ch: Yes.
41 T: Do you see?
42 Ch: Yes.
43 T: Does anyone not see if we add what we must do?
44 Ch: No.
             For six, a boy struggles folding six fingers from his seven: teacher helps.
45 T: One plus the hidden six. How many do we have now?
             Teacher and pupil together count the one finger and six hidden ones.
46 T: Which means we add six with what?
47 L7:   Seven.
48 T: No, we don't add it with seven, we add six with what?'
49 L8:   Seven .

After this was completed, the children were given individual versions of the task to complete with 11 in the inner circle and cubes to use to help them. Many children continued to add the numbers in the outer ring to 11. Of the minority that did appear to attempting missing addends, a small number some worked without the cubes, with only a few succeeding in modelling the 'taking away' method with the cubes. Others struggled to set up a model that worked.

DISCUSSION

It is easy to simply see this case as an example of poor teaching and of a teacher with limited knowledge and skills. We suggest otherwise, and that lack of consistency and coherence in the lesson can be examined in terms of the teacher's own subject knowledge and the style of teaching that she is trying to enact.

The fact that the mathematical object appears not to come into being for many of the learners is not a direct consequence of the teacher's lack of understanding of the object of missing addends. Much of what the teacher articulates indicates that this mathematical object exists for her. The issue lies in her having chosen tasks and mediating means based on her prior knowledge of the mathematical object but the

enactment of these has to be based on bringing the mathematical object into being for the learners, who do not share these prior understandings. There is nothing lacking in the teacher's mathematics here - the shifts in meaning/interpretation that she makes are ones that experienced calculators can do, almost without awareness. What is missing is the 'unpacking' of where her fluency first arises from, which could form the basis of imagining the task from the position of the novice rather than of the expert. A consequence of this lack of bringing the mathematical object into being is learners who can imitate their way to correct answers when funnelled and supported by the teacher, but evidence of many who cannot transfer this competence to even the structurally identical follow up individual task involving missing addends to 11.

In working with the detail of the microgenetic analysis of enacted objects, we gain insights into the poor performance that we highlighted at the start of this paper. If neither objects, nor shifts in objects, are established through coherent mediation in the classroom, it becomes hard for novices to appropriate the operations and actions needed to, at the most basic level, produce correct answers independently – as what they have to draw upon are experiences of disconnected actions that have to simply be taken on trust. Of interest in this paper is that the problem is not reducible to one of poor content knowledge. Instead, a more complex phenomena is seen – where a teacher's prior knowledge of how to solve missing addend problems leads to her 'assuming' the answer in some instances, and assuming the equivalence of the shift to a subtraction-based object, rather than working to establish this shift.

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VALUING MATHEMATICS EDUCATION CONTEXTS

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In this paper, the mathematics learning story of a student named Sandra demonstrates how student engagement changes with the learning contexts, via the identity narratives which are told with reference to different levels of contexts in and outside the mathematics classroom. Data were collected from a survey, interviews, spontaneous conversations, students' blogs and project logbooks. Changes in identity narratives and engagement appeared to be rooted in the relatively stable valuing of achievement, explanation, application and sharing. The extent to which Sandra's valuing was aligned with these facilitates our understanding of the complex interplay amongst context, valuing and agency. That is, sociocultural and personal valuing, and the extent to which these are aligned, promise to regulate and explain the role of learning contexts in student agency, including engagement and hence learning.

INTRODUCTION

Student engagement in (mathematics) learning is an important variable, which determines the extent to which a learner interacts with the subject content in effective ways. However, analysed data in places such as Sweden (see, for examples, Andersson, 2011a, 2011b) have suggested that engagement is not a trait, but rather, a state of a mathematics learner that is regulated by the contexts within which the learner finds him/herself in. That is to say, contemporary research which identifies and labels particular learners as engaged (or not) so that 'something can be done about it' may not yet present the spectrum of experiences which (mathematics) learners go through as their learning contexts change.

Through the story of mathematics learning that developed for a student in Sweden named Sandra, this paper presents a window into the ways in which learners' engagement shifts with changing identity narratives, that in turn are functions of learning contexts. We will explore how these changing variables might be rooted in the cultural values, which are internalised within individuals' experienced contexts. Recognising what the various contexts value is important, we will argue, as it serves two purposes. Firstly, it anchors change in engagement and identity narratives against a relatively stable variable (i.e. values). Secondly, this offers opportunities for teaching practices to be planned for in ways which optimise positive mathematical wellbeing (see Clarkson, Bishop & Seah, 2010) of students.

CONTEXT

In mathematics education research, context tends to be restricted to the immediate context of a particular classroom or studied activity episode (Morgan, 2006). Efforts

have been made to challenge this statement in Sweden (Andersson, 2011a, 2011b). 'Context', however, is complicated to grasp as a single concept. The word is a reference to circumstances, but in our language use it also refers to – and makes possible – discursive spaces. Context hence comprises the network of relationships and available recourses in the social practises in which we act, but at the same time contexts form the ways and spaces where we act.

Contexts can be considered in a number of ways. First, we recognised task contexts as the referents to which a particular task appeal in order to invite students to engage in mathematical activity. Task contexts are expressed in textbooks exercises and through developed pedagogical projects (Wedege, 1999). Research reported by Stocker and Wagner (2007) who introduced tasks influenced by critical education exemplify research addressing the contexts in which exercises and tasks are presented and thus situated. Second, there are situation contexts, understood as the array of "current activities, the other participants, the tools available and other aspects of the immediate environment" (Morgan, 2006, p. 221) in the classroom. A situation context thus also refers to the communicative understanding of contexts. Third, we recognised a wider socio-political context of schooling, referring to contexts outside classrooms that influence what occurs within the mathematics classrooms, operationalized through governmental policies on schools and the national curriculum, ideologies and school policies (Valero, 2004). This school context refers to layers of school organization that shape possibilities for engagement. These include, for example, school structures such as timetables and school leadership, as elaborated by Martin (2000) when addressing the complexity of reasons behind African-American youths' achievement or failure in mathematics education. Fourth, we recognised a societal context as the impact of societal discourses in mathematics classrooms. 'Specialness' when being 'good at mathematics' (Mendick et al, 2009) is an example of discourses within the socio-political societal context that impact on what occurs within the classrooms.

These contexts exist within a socio-cultural setting, and as such they cannot be perceived as being free of the values which underlie cultures (Bishop, 2008). To the extent that contexts influence discourses in the mathematics learning process, it is useful for us to understand contexts also from the perspective of the cultural values that contribute to its occurrence. This is especially meaningful when we find ourselves analysing contexts that might be taking place across different cultures.

VALUES PORTRAYED THROUGH CONTEXTS

Values may be considered to be the window through which an individual views the world around him/her. They are the convictions which an individual has internalised as being important and worthwhile. Values regulate the ways in which the learner utilises his/her cognitive skills and emotional dispositions to learning. Often they contribute to the traits of the individual, who seek to enact these values through the decisions selected, actions taken, and evaluations made. Values in mathematics education are "the deep affective qualities which education fosters through the school subject of

mathematics" (Bishop, 1999, p. 2). They represent "an individual's internalisation, 'cognitisation' and decontextualisation of affective constructs (such as beliefs and attitudes) in her socio-cultural context. Values related to mathematics education are inculcated through the nature of mathematics and through the individual's experience" (Seah, 2005, p. 43), thus becoming the personal convictions which an individual regards as being important (Seah & Kalogeropoulos, 2006) in the process of learning and teaching mathematics.

This focus on values has meant a need to differentiate amongst the many values that are portrayed in the classroom. Bishop (1996) had emphasised three categories of values in the numeracy classroom, namely, mathematical, mathematics educational, and general educational. As he explained:

Mathematical values: values which have developed as the knowledge of Mathematics has developed within 'Westernised' cultures.

General educational values: values associated with the norms of the particular society, and of the particular educational institution.

Mathematics educational values: values embedded in the particular curriculum, textbooks, classroom practices, etc as a result of the other sets of values. (Bishop, 2008, p. 83)

In the light of the literature that had been reviewed, we are interested to explore how identifying the values that underlie narrated identities might provide a means of interpreting these identities in planning effective mathematics lessons.

THE RESEARCH CONTEXT

The data for this paper comes from a one-year research study exploring upper secondary students' learning of mathematics within a social science program in Sweden. Students commonly complete this program because it provides entry into university studies in the social sciences and language faculties. Also, students who do not enjoy mathematics and thus do not want to take the alternative natural science or technical programs often see this social science program as a good option.

Annica, in collaboration with Elin (pseudonym), a mathematics teacher introduced teaching sequences that, enabled students' mathematics learning to be connected to societal topics inspired by different aspects from critical mathematics education (Skovsmose, 2005). How mathematical topics related to societal contexts regarding mathematics as a tool for identifying and analysing contemporary features in society was one important aspect. These aims matched curriculum objectives, which asserted that mathematics education for social science students should "provide general civic competence and constitute an integral part of the chosen study orientation" (Ministry of Education, 2000). A second aspect concerned the epistemological point that an educational practice was considered to involve learning and becoming, rather than a simple transmission of knowledge (Skovsmose, 2005). A third aspect involved how power relations between the actors supported a classroom environment where students could become agentic in a positive way towards their learning and where students had

access to and contributed to the discourse between participants (Andersson & Valero, in press).

Collecting information

In order to understand students' relationships with mathematics from their perspectives, ethnographic methods were used for data collecting (Hammersley & Atkinson, 2007). Annica established a trustful environment through engaging with the students in both formal and informal settings. In this way, she interacted closely with the students, and experienced the contexts and discourses. The research methods deployed included a survey, interviews, spontaneous conversations, a blog and students' project logbooks. Through the survey, students were asked about their prior experiences of mathematics learning and their personal goals in the current course, and hence these narratives referred to different context levels. The interviews also provided reflective data about the different context levels. The blog was a course activity and provided data mainly about task contexts. Students' actions, hence their reflections of their agency (including resistance), also appeared in the blog. The logbooks provided data about the students' learning in relation to task and situation contexts. Annica's research-diary described different school and societal contexts and allowed the students' stories to be related to what went on in school and society at particular times.

Data analysis

The data analysis mainly acknowledged Sfard and Prusak´s (2005) proposal to "equate identities with stories about persons" (p. 14) if the story is reified, endorsed and significant for the identity builder. The students were the significant narrators of these identities and they drew of stories from their parents and their mathematics teacher (Andersson, 2011a). These stories were then located in relation to the different contexts in which they were told at those particular times they were told. Talk about agency in a relational understanding was also connected to the stories. In this way chronological storylines emerged where it became visible how contexts, agency, values and identity narratives were related.

In this paper, we share the story told to us by one of the student participants, Sandra (pseudonym). In particular, four identity narratives in contexts from Sandra's course trajectory will exemplify changes in the students' narrations of themselves and how contexts impacted on the students' engagement through changes in their expressed narratives at particular times. We then filtered students' narratives further to reveal the cultural values which are internalised within Sandra's identity narratives.

SANDRA'S IDENTITY NARRATIVES

Four identity narratives from Sandra 's course trajectory have been chosen as a frame within which to theoretically consider interplays between values, agency and context. They are chosen in that they provide four qualitative different ways of narrating the self, hence supporting the theoretical discussion above.

1. Sandra initially shared with Annica that she had always disliked mathematics because she had 'mathematics anxiety'. This label was Sandra's way of objectifying herself, causing her not wanting to spend more time with mathematics than was absolutely needed. That is the reason why she earlier had not wanted Annica to interview her, which would, as Sandra said, result in more 'mathematics related time'. However, Annica was very welcome to read her blog comments, evaluation sheets and logbook and to talk with her during mathematics lessons.

Sandra told she desperately wanted to pass the mathematics course, as it was required for her future university studies. Foregrounding herself as a university student had shaped her intentions for attending and passing the mathematics courses that is required by society. The socio-political context which appeared to value *achievement*, underlying which might be the societal valuing of *masculinity* (see Hofstede, 1997), had constrained Sandra's achievement of agency; she could not decide to not participate, as her designated identity was to become an university student. Within the situation context, objectifying herself with the label 'having math-anxiety' – and, in so doing, reflecting her lack of mathematical wellbeing – seemed to have impact on her decisions on how to act within the classroom (e.g. spending a minimum of time with mathematics).

2. During a two-week project where the students were given opportunities to decide on task contexts, personal time and work distribution, Sandra talked about herself thus:

> We distributed the time well, I think. […] The group worked well. We were good at different things, and helped each other. I am proud of the work I have done as I felt I could contribute a lot in the beginning when we talked about borrowing money and interest rates. To self decide on time and content made me feel it was related to me. I think mathematics has been a little more fun than usual. […] I feel the project has been meaningful and to look at mathematics from different angles (vända och vrida på matematiken) was positive. But I would have liked more time for explanations from the teacher, as mathematics is difficult for me. (Sandra evaluation sheet, 10-2009)

During this project Sandra achieved agency in relation to task context and situation context. Her personal influence on content, time and work distribution reflected an alignment between what she and the task valued similarly, that is, *application*. This impacted on her decisions to engage in the classroom activities in a different way than she intended at the beginning of the course. In addition she experienced feelings of 'a little fun' and mathematics as 'meaningful'. At this time Sandra took a projective action for learning differently to the initially intended and got rewarded with feelings of 'being proud' of her work. However, even if she was proud of her work and actually passed this sequence with distinction (teacher, results sheet), the last sentence indicated that being objectified with 'mathematics anxiety' still implied her wishing for extra support from her teacher. Here, there is an indication that her valuing of *explanation* (one which is also reported by many students in Seah, 2011) might have accounted for her low mathematical wellbeing.

3. In the middle of the semester the students were expected to work with textbook algebra exercises over two weeks. In contrast to the identity narrative told during the project above, Sandra's two entries on the blog during these textbook work periods emphasised Sandra 's worries and feelings of stress for not passing a coming test:

> I am currently worried about the test. I have received help with things I need help with. Stress. Stress. (Sandra, blog, 07-10-2009).

In class she repeatedly asked the teacher about what would happen if she did not pass the test, and she asked for advice on exercises that was 'extra smart to calculate' when preparing for the test (Annica, field notes). The assessment context which values *assessment* had Sandra feeling worried, and her achievement of agency seemed to be restricted to doing what was required for just passing the test. Sandra's positive experience of the prior project appeared to have vanished, and her mathematical wellbeing suffered consequently. The interplay between her task contexts (restricted to advised exercises on given topic), the situation context within the classroom (to pass a test) and her foreground to become a university student – underlined by a societal valuing of *achievement* – was obvious in her actions. Her 'math anxiety', imagining herself not passing and thus not becoming what she wanted, became problematic and restricted her achievement of agency at this particular time.

4. Later in the semester, there was a larger cross-subject project themed 'Students' Ecological footprints on earth'. At that time Sandra's logbook was rich with comments regarding her and her work-friend's collaborative work. This excerpt exemplifies her reflections on her mathematics learning during the project:

> During the project I have learnt about different diagrams. E.g. I did not know about histograms before the project. I think it has been really interesting with manipulated diagrams and results – now I will be more observant when reading newspapers etc!
>
> What surprised me most though was how important role mathematics plays when talking about environmental issues. With support of mathematics we can get people to react and stop. […] I am so interested in environmental questions and did actually not believe that maths could be important when presenting different standpoints. I have probably learnt more now than if I had only calculated tasks in the book. Now I could get use of the knowledge in the project and that made me motivated and happy! I show my knowledge best through oral presentations because there you can show all the facts and talk instead of just writing a test. To have a purpose with the calculations motivated me a lot. (Sandra, logbook, conclusions).

The project's valuing of *applications* appeared to be aligned with Sandra's values. The oral presentations also afforded her the chance of enacting her valuing of *sharing*. Consequently, Sandra was awarded the best possible grade for this project. Orally she clearly, correctly and convincingly presented her results and answered questions in front of an audience of 50 students, two teachers and one researcher (Annica, fieldnotes) in ways which she believed she could not achieved in a written test setting (Sandra, classroom conversation).

DISCUSSION

Sandra's mathematics learning experience demonstrated the complex interplays amongst learning contexts, the valuing involved, and student agency that resulted. The data suggests that the sociocultural valuing of *achievement* and *applications* affected

Sandra's state of mathematical wellbeing, and thus engagement, in different ways. The former seems to threaten the development of her mathematical wellbeing. It is likely that the contexts did not co-value (with Sandra) *explanation* too. On the other hand, we saw in two of the sequences the enabling effects to wellbeing and engagement when there was alignment between the contexts' valuing and personal valuing of *applications*. The context's valuing of *sharing* also matched Sandra's valuing of the same, further boosting her level of mathematical wellbeing and sense of agency.

Thus, while changes in learning contexts lead to variations in student agency with regards to engagement, Sandra's story demonstrates how the interplay may be accounted for when we are able to reveal what these contexts value and whether these values are aligned (or not) with what Sandra values as learner. The stability of values (Krathwohl, Bloom & Masia, 1964) should thus facilitate an useful means of interpreting the variety of contexts and identity narratives, in so doing fostering mathematical wellbeing, and regulating student agency (including engagement).

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CHINESE AND AUSTRALIAN PRIMARY STUDENTS' MATHEMATICAL TASK TYPES PREFERENCES: UNDERLYING VALUES

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This paper reports on part of a study which investigated the mathematical task type preferences of Grade 5 and 6 students from Victoria, Australia and Chongqing, China. Through the administration of a questionnaire to 1109 Chinese and 689 Australian students, it was found that across the topics of Number and Geometry, tasks situated within a contextualised situation were the most preferred, whilst the other task types were preferred differently between the two topics. Based on the reasons provided by the students, underlying values for the most preferred task types could be suggested. For the topic of Number, each of the three task types appeared to be most preferred for reasons which are encapsulated by the following values: 'challenge', 'multiple solutions', 'real life problems' and 'easiness'.

INTRODUCTION

The teacher and his/her professional practice are important factors in lesson effectiveness (Askew, Hodgen, Hossain, & Bretscher, 2010; Rice, 2003). This professional practice is guided by the teacher's values (Seah, 2005) and beliefs (Barkatsas & Malone, 2005), and is reflected in mathematical tasks, each of which is "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (Stein, Grover, & Henningsen, 1996, p. 460). This paper reports on an exploratory study into the nature of mainland Chinese and Australian Grade 5 and 6 students' interaction with mathematical tasks. The study has been conducted under the umbrella support of 'The Third Wave Project', an international consortium of research teams which is interested in exploring how values might be harnessed to optimise students' learning of mathematics. Specifically, this paper examines the findings to the research questions:

(1) What are the preferences amongst mathematical tasks of Grade 5 and 6 students from Victoria, Australia and Chongqing, China?

(2) What might be the underlying values?

MATHEMATICAL TASKS

In this study, it is assumed that mathematical tasks constitute the gateway to student learning of mathematics. The Task Types in Mathematics Learning [TTML] project (Sullivan, Clarke, Clarke, & O'Shea, 2009) examined teacher use of three types of mathematical tasks: Type 1, in which the tasks are designed to exemplify the mathematics through the use of models, representations, tools or explanations; Type 2,

in which mathematics has been situated within a contextualised practical situation; and Type 3, which are open-ended tasks. By the end of the project, participating teachers were confident and skilled enough to increase the adoption of contextual tasks (task type 2) to a level similar to their use of the other two task types (Clarke & Roche, 2010).

Yet, the ways in which tasks relate to student learning have not always been made explicit (Simon & Tzur, 2004). What sort(s) of mathematical tasks do students prefer, and why? To what extent does preference relate to effective learning? The study within which this paper is contextualised aims to contribute to our understanding in this regard, by identifying the reasons students consider important – and value – in their preference for particular task types.

RESEARCH DESIGN

The study within which this paper is contextualised adopts the sequential mixed methods design (Creswell, 2009). Reflecting our epistemological stance of constructivism, the intention here has been to understand the pedagogical enactment of tasks in primary school mathematics lessons, rather than establishing relationships, determining effects and identifying causes.

This paper reports on the quantitative phase, which aims to map the field relating to the preference for and use of different mathematical task types in Australian and Chinese classrooms. The research method adopted for the phase is the 15-item survey questionnaire that had been constructed earlier for the TTML project and that was translated into Chinese for this study, containing a mix of Likert-type items, ranking exercises, and open-ended questions. In translating the questionnaire to the Chinese language, the contextual information of several items in the TTML version was changed to accommodate the societal realities in mainland China (see Seah, Barkatsas, Sullivan, & Li, 2010). Culturally-different ways of describing phenomena and of teaching during the translation process were accounted for through the process of back-translation (see Seah, Barkatsas, Sullivan, & Li, 2010, for examples).

Data were collected from 1109 Grade 5 and 6 students in Chongqing, a major inland city of about 31 million residents in Southwestern China, and also from 689 Grade 5 and 6 students from Victoria, Australia. This paper reports on the findings relevant to two of the research questions of the wider study, as stated above. The corresponding questions in the questionnaire were items 9 (relating to Number) and 11 (relating to Geometry), whose original English version is shown in Tables 1 and 2 respectively.

In this table there are four maths questions that are pretty much the same type of mathematics content asked in different ways.

We don't want you to work out the answers.

Put a 1 next to the type of question **you like to do most**, 2 next to the one you like next best, and 3 next to the type of question **you like least**:

9ai An adult cinema ticket costs RMB25, and a child ticket costs RMB12. How much would the tickets cost for 2 adults and 4 children to watch a movie?	
9aii 2 adults and 4 children spent RMB120 on movie tickets. How much might an adult ticket and a child ticket cost?	
9aiii 25 X 2 + 12 X 4 =	
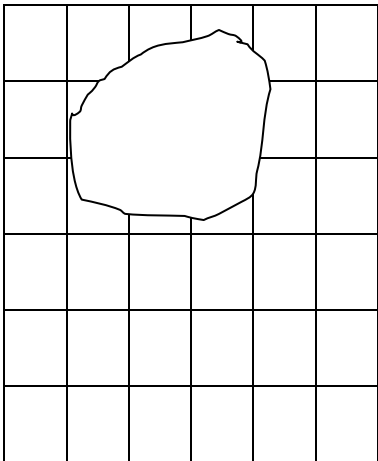
You like to do this type of question (the one you put a 1 against) the most because:

_____

Table 1: Questionnaire item 9.

In this table there are four more maths questions that are pretty much the same type of mathematics content asked in different ways.

We don't want you to work out the answers.

Put a 1 next to the type of question **you like to do most**, 2 next to the one you like next best, and 3 next to the type of question **you like least**:

11ai Find the area of the following figure. 	
11aii If the area of a figure is 10 square units, what might the shape of the figure be?	

11aiii   An athletic track is made up of two straight sections and two semi-circles. The straight section is 100m long. What is the area of the athletic track?

You like to do this type of question (the one you put a 1 against) the most because:

_____

Table 2: Questionnaire item 11.

RESULTS

A Friedman test was used to test for statistically significant differences in the ways students rank ordered the three types of mathematical tasks (items 9ai-iii and 11ai-iii). The results are shown in Tables 3 and 4.

Item	Mean rank (Chinese students)	Mean Rank (Australian students)
9aiii (task type 1)	2.09	1.82
9ai (task type 2)	1.76	2.44
9aii (task type 3)	2.14	1.73

Table 3: Mean ranks for student rank ordering of Items 9ai – iii.

Item	Mean rank (Chinese students)	Mean Rank (Australian students)
11ai (task type 1)	2.26	1.85
11aiii (task type 2)	1.75	1.87
11aii (task type 3)	1.99	2.29

Table 4: Mean ranks for student rank ordering of Items 11ai – iii.

The differences in rankings for both topic types were statistically significant for both the Chinese students: [$\chi^2 (2, 1001) = 97.45$, $p < 0.001$ (question 9)] and [$\chi^2 (2, 1058) =$

153.44, p < 0.001(question 11)] and the Australian students [χ^2 (2, 688) = 207.75, p < 0.001 (question 9)] and [χ^2 (2, 689) = 25.43, p < 0.001(question 11)], respectively.

Thus, it may be said that in the area of Number (Item 9), Grade 5 and 6 students in Chongqing, China, preferred mathematical tasks in the order of types 2 (contextualised tasks), 1 (modelling tasks) and 3 (open-ended tasks), whereas in the area of Geometry (Item 11), the order of preference is task types 2, 3, 1. Their peers in Australia, on the other hand, preferred Number task types in the order of 3, 1, 2, and Geometry task types in the order of 1, 2, 3.

Respondents were also asked to provide a reason for the nomination of a particular question as being the favourite in each of the two sets of questions. The reasons given by the respondents were coded into 7 categories, as shown in Table 5.

1. Challenging (more complex, lots of steps / have to think / I learn something new / improve)

2. Easy to do / understand (instructions clear) / I'm good at this / we do this a lot

3. Real life scenario

4. Involves a model / drawing / grid

5. Multiple solution strategies available, need to devise own strategies

6. Has more than one possible answer

7. Fun / I like this type of operation (e.g. division) or topic (e.g. area)

Table 5: Codes for reasons cited by respondents in ranking each task.

A polychotomous (or polytomous) logit model was used to investigate the significance of these coding categories. This model is a special class of loglinear models and it is used to model the relationship between one or more dependent categorical variables and a number of independent categorical variables.

When the dependent variable has more than two values, the researcher can construct many odds ratios for the same combination of values of the independent variables. The logit procedure (SPSS) considers the last category of each variable as the reference category. In our case, the category 'Fun/I like this type of operation' (coding category 7) is set to zero, and 9ai=3, 9aii=3 and 9aiii=3 are all set to zero respectively in the corresponding logit models. The last two categories from Table 6 had not been considered because there were less than ten responses in each of these categories. Given the space constraints, the results of the polychotomous logit statistical analysis for the Number item (item 9) only are shown in Table 6.

The design for this test is governed by the following models: constant + q9ai + q19ai * q9b, constant + q9aii + q19aii * q9b, constant + q9aiii + q19aiii * q9b. The first number

in each cell is the parameter estimate. The first number within the parenthesis is e^λ, followed by the p value (in the case of statistically significant results). Two cell entries form Table 6 will be discussed in what follows; all other cell entries may be interpreted in the same way.

Reasons cited in ranking the items	Item 9ai=1 (Number *Type 2*)		Item 9aii=1 (Number *Type 3)*		Item9aiii=1 (Number *Type 1)*	
	Chinese	AUS	Chinese	AUS	Chinese	AUS
1:Challenging	-1.85*** (.16)	-.64 (.53)	1.20** (3.32)	1.26*** (3.52)	-1.24*** (.29)	-.38 (.68)
2:Easy to do	-.24*** (.79)	-.02 (.98)	-2.83*** (.06)	-1.13** (.32)	1.79*** (5.99)	.86*** (2.36)
3:Real life scenario	.54 (1.71)	1.22 (3.39)	-.26 (.77)	.61 (1.84)	-1.27 (.28)	-1.90* (.15)
4:Involves a model	-1.32* (.27)	No data entries	-2.25** (.11)	No data entries	1.64* (5.14)	No data entries
5:Multiple solution strategies	-1.66 (.19)	-.90 (.41)	1.11 (3.02)	3.30* (27.11)	-1.88 (.15)	-2.24* (.11)
6:Has more than one possible answer	-3.27 (.04)	-2.51 (.08)	.37 (1.45)	4.30** (73.70)	-.42 (.66)	-3.0 (.05)
7:Fun/I like this type of operation	0 (1)	0 (1)	0(1)	0 (1)	0(1)	0 (1)

Table 6: Parameter estimate (λ, e^λ) summary.

The parameter estimate for *multiple solution strategies* (shaded cell, row 7, column 4), being the favourite for the Chinese students for item 9aii is 1.11. The value of e^λ is $e^{1.11}$

= 3.02. This tells us that based on the model, the Chinese students in the study are three times more likely to have nominated *multiple solution strategies* as the reason for item 9aii being a favourite over nominating the same reason when the same item is the least liked, compared to nominating *fun/I like this operation* as the reason for item 9aii being a favourite over it being nominated when item 9aii is the least liked.

The cell to the right of the shaded cell (row 7, column 5) shows the results for the Australian students on the same questionnaire item. As shown in this cell, the parameter estimate for *multiple solution strategies* being the favourite for item 9aii is 3.30 for the Australian students. The value of e^λ is $e^{3.30} = 27.11$. We can therefore claim that the Australian students in the study are statistically significantly at least twenty-seven times more likely to have nominated *multiple solution strategies* as the reason for item 9aii being a favourite over nominating the same reason when the same item is the least liked, compared to nominating *fun/I like this operation* as the reason for item 9aii being a favourite over it being nominated when item 9aii is the least liked.

CONCLUDING REMARKS

Three types of mathematical tasks were investigated with Grade 5 and 6 students in Chongqing and Victoria in this research study. The data suggest that for both Number and Geometry items, Chinese students preferred most to engage with tasks involving contextualised situations. Their peers in Australia, however, appeared to have different preferences. For the Australian students, open-ended tasks were the most preferred for Number items, whereas modelling tasks were the most preferred for Geometry items.

While a variety of reasons were given for preferring particular task types, a majority of these fell into one of four reason categories for Number items, which we will loosely associate with the valuing of *challenge*, *multiple solutions, real life problems and easiness.* The corresponding reason categories for the Geometry items (the parameter estimates table is not shown here due to space restrictions) are the following: *challenge, multiple solutions, multiple answers and easiness.*

The results demonstrate that different mathematical topics appeal to different students differently and that pedagogical considerations should be mindful of this. As far as Number items are concerned, the students' task preferences seemed to be guided by their experiences in a challenging, real life context, in which they have access to multiple solutions in order to answer the relevant mathematical questions. The possibility that effective mathematics learning is associated with particular features of mathematical thinking and activity, and that the underlying values are manifested through particular task types, would be one objective of investigation in the next phase of this study, involving targeted inquiry with a sample of participants purposively selected from amongst the participants in the study.

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PSYCHOLOGY STUDENTS' ESTIMATION OF ASSOCIATION

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Contingency tables are useful for practitioners in psychology and health sciences, since providing a diagnostic requires an association judgment in a contingency table. In this research we analysed the perception of association in contingency tables and the accuracy in the estimation of its strength in a sample of 414 psychology students in three different Spanish universities. Results show a good perception and estimation of association in both direct and inverse association, misperception of independence and the effect of illusory correlation. Performance is similar in the three universities, and better that reported in a previous study with high-school students.

INTRODUCCIÓN

Contingency tables are common to present statistical information; however little attention is paid to this topic in university education, in assuming that its interpretation is easy. These tables are often presented in diagnosis and psychological evaluation, where psychologists are confronted with different symptoms that may be associated with a disorder or not (Diaz, & Gallego, 2006). Moreover, association judgments are priority learning issues in university statistics courses (Zieffler, 2006).

This study was aimed to evaluate the accuracy in the estimation of association in contingency tables by students entering the Bachelor of Psychology and how different variables affect their association judgments and accuracy. Results will be compared with a previous study by Estepa (1993) with high school students.

PREVIOUS RESEARCH

Research on association was started by Inhelder and Piaget (1955), who conceived association as the last step in the development of probabilistic reasoning, and described the strategies used at different ages when judging association in tasks that were formally equivalent to a 2x2 contingency table (see Table 1). Later psychological studies were developed with adults. Crocker (1981) shown that the accuracy in the estimation of association increases when data are presented simultaneously, frequencies are small, data are presented in a table, and the events co-vary simultaneously along time. Allan and Jenkins (1983) showed the tendency to base the association judgments on the difference between confirmatory cases (cell *a* in Table 1) and contradictory cases (cell d). Erlick and Mills (1967) found that negative association is estimated as close to zero. Three additional factors that influence the judgments of association suggested by Arkes and Harkness work (1983) are: (a) the frequency in cell *a* (which has the greater impact on the estimates), (b) the labelling of rows and columns, and (c) the presence of small frequencies in the cells (which can influence an overestimation).

	A	Not A	Total
B	a	b	a+b
Not B	c	d	c+d
Total	a+c	b+d	

Table 1. A simple contingency table

Other authors studied the influence of previous theories about the context of the problem on the accuracy of the association estimate (Jennings, Amabile, & Ross, 1982; Wright & Murphy, 1984; Alloy & Tabacnick 1984; Meiser & Hewstone, 2006). The estimates are more accurate if people have no theories about the type of association in the data. If the subject's previous theories agree with the type of association reflected by the empirical data, there is a tendency to overestimate the association coefficient. But when the data do not reflect the results expected by these theories, the subjects are often guided by their theories, rather than by data. Chapman (1967, pp. 151) described "illusory correlation" as "the report by observers of a correlation between two classes of events which, in reality, (a) are not correlated, (b) are correlated to a lesser extent than reported, or (c) are correlated in the opposite direction from that which is reported". Kao and Wasserman (1993) also found that most subjects were quite inaccurate in perceiving independence in 2x2 contingency tables, when all the frequencies in the cells are different, while they perform better as frequencies values are closer to each other.

According to Barbancho (1992), an association between variables may be explained by the existence of a unilateral cause-effect relationship (one variable causes the other), but also to interdependence (each variable affects the other), indirect dependence (there is a third variable affecting the other two), concordance (matching in preference by two judges in the same data set) and spurious covariation. In addition to the estimate accuracy, understanding association involve the discrimination of these types of relationships between variables.

Estepa (1993) studied the pre-university students' conception of association in a sample of 213 and analysed their association judgments. He also analysed the accuracy in the estimation of the association coefficient in a subsample of 51 students. The author defined the causal conception according to which the subject only considers association between variables, when it can be explained by the presence of a cause - effect relationship. He also described the unidirectional conception, by which the student does not accept an inverse association, considering the strength of the association, but not its sign and assuming independence where there is an inverse association (see also Batanero, Estepa, Godino, & Green, 1996). In a subsequent study (Batanero, Godino, & Estepa, 1998) the authors found that the unidirectional conception improved with teaching, but not the causal conception. Our research is aimed to assess the students' accuracy in estimating the association coefficient, which was only studied by Estepa in a subsample of students (n=51). We also try to compare

the correctness of the association judgment with the results obtained by Estepa and the influence of some task variables on this judgment. Finally, we focus on Psychology students, while Estepa's research was carried out with high-school students.

METHOD

The sample included 414 students in their first year of Psychology studies from three Spanish universities: Almeria (115 students), Granada (237 students) and Huelva (62 students), all of them taking an introductory statistics course. The questionnaire was given to the student as a part of a practical task that scored in the final marks in the course, in order to assure their interest in completing the task. The samples included all the students enrolled in the course and attending the session; the difference is sample sizes was due to the size of the University: Almeria with 2 groups of students, Huelva with 1 group of students and Granada with 4 groups of students. Though they had not yet studied association in the course they were following, these students had studied statistics and probability in Secondary Education.

Item 1. A researcher is studying the relationship between stress and insomnia. In a sample of 250 people he observed the following results:

	Stress disorders	No stress disorders
Insomnia	90	60
No insomnia	60	40

a. Looking to these data, do you think there is a relationship between stress and insomnia?

b. Please mark on the scale below a point between 0 (minimum strength) and 1 (maximum strength), according the strength of relationship you perceive in these data.

Figure 1. An item example

The questionnaire was adapted from Batanero, Estepa, Godino and Green (1996). The context was changed to a context of psychological diagnose in two items (1 and 2); the frequencies in the table cells were increased in items 2 and 3, since in the original questionnaire the small sizes made invalid the application of the Chi-square statistics (due to the small sample sizes, the association coefficients computed by the authors were, moreover, not statistically significant); the sign and strength of association was the same than in the corresponding item in the above studies. In Figure 1 we present Item 1. The format and questions were identical in the remaining items. The following task variables (Table 2) were considered in the questionnaire:

1. *Sign of association:* We include the three possible cases: direct and inverse association and independence.

2. *Strenght of association,* that was measured by the Pearson's Phi coefficient in 2x2 and Cramer's V coefficient in 2x3 tables. An item with moderate-low association and two items with moderate-high association were included.

3. *Agreement between association in the data and previous theories* suggested by the context. There were two items were the empirical association matched the prior expectations, one where it contradicted the expectations and another with a neutral context suggesting no previous theories.

4. *Type of covariation.* We used three categories of Barbancho's (1992) classification: unilateral causal dependence, interdependence and indirect dependence.

	2x2 table			2 x3 table
	Item 1	Item 2	Item 3	Item 4
Dependence	Independence	Inverse	Direct	Direct
Association coefficient	0	-0,62	0,67	0,37
Agreement with prior theories	No	Yes	There is no theory	Yes
Type of covariation	Inter-dependence	Causal unilateral	Indirect Dependence	Causal unilateral
Context	Insomnia vs stress	Being only vs being problematic	Sedentary life vs allergy	Time of study (3 values) vs passing an exam

Table 2. Task variables in the items

A qualitative analysis of students' responses served to define two different variables. In part (a) of each item, students are asked to provide an association judgment. We considered 3 different categories in their responses: (a) the student consider that the variables in the item are related (judging association); (b) the students considered the variables to be not related (judging independence); and (c) the student was unable to decide (no judgment). The estimation for the association coefficient estimation is deduced measuring the exact position of the point drawn by the student on the numerical scale (0-1) in the second part of the item.

RESULTS AND DISCUSSION

Association judgment

To assess the students' competence to judge the possible association between the variables presented in each item, we present in Table 3 the percentage of students who considered (or not) the existence of a relationship between the variables. In the last columns we add the association coefficient for the data in the item and the relationships between prior theories and data. Most students indicated the existence of association in

all items, in particular when the association was confirmed by the data, but also in item 1 (perfect independence). This result can be explained by illusory correlation (Chapman, 1967) since in this item data contradicts the students' previous theories (that stress is related to insomnia) and is also consistent with Kao and Wasserman's (1993) suggestion that independence is hard to be perceived if the frequencies in the table cells are different. Our students showed a greater effect of previous theories on this item, as well as the causal conception of association, linking the concepts of association and causality that in Batanero et al. (1996), where only 55.4% of students indicated association (the numerical data in this item are the same in both studies, while we changed the context to one more familiar to a Psychology student.

Ítem	Judging Association	Judging Independence	No judgment	Association coefficient	Prior theories vs data
1	323 (78.0)	90 (21.8)	1 (0.2)	0	Do not agree
2	398 (96.1)	14 (3.4)	2 (0.5)	-0,62	Agree
3	386 (93.2)	24 (5.8)	4 (1.0)	0,67	No theories
4	402 (97.1)	4 (1.0)	8 (1.9)	0,37	Agree

Table 3. Frequency (and percent) of students according judgment of association

Our students outperformed in item 2 (inverse association) those in Batanero et al. (2006), where only 47.1% of students considered association. This result could be explained by the change in context and the increased the sample size in our item. Consequently, the unidirectional conceptions of association described by Estepa hardly appear in our research. Results in item 3 were very close in both studies (92.5% in Batanero et al., 1996), where we only increased the frequencies without changing the context or the strength of association. Our results also improved a little in item 4 (95.5% of students considered association in Batanero et al., 1996), where we slightly increased the intensity of the association holding the other variables fixed. Results were very close in the different universities and were not statistically significant in a Chi-squared test of homogeneity (Chi= 0.99; 6 d.f., p=0.9861), which suggest the samples homogeneity in their association judgments.

Estimating the strength of association

In the second part of each item, the students provided a score between 0 and 1 according to the intensity they perceived in the association. This value can be considered an estimate of the coefficient of association (disregarding the sign in 2x2 tables). Table 4 shows the mean score obtained in the whole sample, and each university. The most accurate estimate was given in Item 3, where students had no prior theories: the estimate mean value is very close to the empirical coefficient in all the samples and overall. There is an over-estimation of the coefficient in the other three items, showing the effect of students' prior theories.

In item 1, which corresponds to perfect independence, the global mean value was 0.47, and about this value in each sample (in his subsample Estepa found 0.56). Many students followed their previous theories in this item, as was show in some answers: "You should have some relationship, since in my experience stress due to family or other type of problems may be a cause of insomnia". "In my opinion insomnia and stress are related, since most people who have insomnia suffer from stress", or "Yes, because people with insomnia do not rest well and this causes extra stress that is added to stress due to other external factors". On the other hand, in this item cell a, which corresponds to the simultaneous presence of both stress and insomnia and that, according to Arkes and Harkness (1983) has the greater impact on attention present the maximum absolute frequency.

The estimate for item 2 (inverse dependence), was higher than the empirical value association in all Universities, in particular in Almería. Thus, in our students we did not find a significant presence of the unidirectional conception, while in Estepa's (1993) study the estimation for this item in the subsample was much lower (0.48). Moreover, both the whole sample and in each university most students indicated that there was association in this item (the sign of association was not requested).

	Mean estimate					
Ítem	Almería (n=115)	Granada (n=237)	Huelva (n=62)	Total (n=414)	Association coefficient	Prior theories vs data
1	0.51	0.47	0.44	0.47	0	Do not agree
2	0.78	0.72	0.73	0.73	-0.62	Agree
3	0.75	0.68	0.68	0.70	0.67	No theories
4	0.84	0.81	0.81	0.82	0.37	Agree

Table 4. Results in estimation of the association coefficient

In item 4 (2x3 table, positive association), the difference between the estimate and the coefficient true value was high, showing again illusory correlation (Chapman, 1967). Students overestimated the association, as they were guided by their previous theories that matched the type of association in the data. These theories were fostered by a context so familiar for students (study time, passing or failing an exam), and possibly driven by personal experience. Results were very close all the universities; with a smaller difference with the true value in Granada.

The students from Granada and Huelva estimated an average lower association in all the items than the students from Almeria; however, when performing an Anova comparison of means (two factors: item and university) no statistically significant differences by university or interaction between university and item was obtained. This result suggest that student responses were similar, despite the difference in educational context.

IMPLICATIONS

Results suggest that most psychology students in our study judged association, even in cases where there was none, due to the illusory correlation phenomenon and their previous theories, which affected their accuracy in estimating the association coefficient. Regarding the conceptions described by Batanero, Estepa, Godino and Green (1996), we observed the causal conception, but not the unidirectional conception, since most students perceived the association when this was negative. The estimates of the association improved in our study, as compared with Estepa's (1993) results, in all items except in case of independence were our students gave a higher association coefficient. Results were very close in all participating universities.

According to Schield (2006), an educated person should be able to critically read tables in the press, Internet, media, and professional work. This involve not only the literal reading, but being able to identify trends and variability in the data, including the correct judgment of association. All these reasons and our results suggest the need for further research about teaching association, since the causal conception and the effect of illusory correlation does not seem to improve with traditional instruction (Batanero, Godino, & Estepa, 1998). Our purpose is to continue this work by designing an alternative teaching with activities that confront students with their biases and help them overcome them. This proposal will be tested and students will be assessed in order to compare their intuitive ideas with those acquired as a result of teaching.

Acknowledgements

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ONE COMPUTER-BASED MATHEMATICAL TASK, DIFFERENT ACTIVITIES

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I examine how two sets of in-service mathematics teachers (the students) engage with one GeoGebra-based mathematical task. I compare an a priori analysis of the intended pedagogical purpose of the task with an a posteriori analysis of the actual activities by these students. The analysis shows how one student uses GeoGebra as a tool with which to make sense of the particular mathematical task; in contrast, the other set of students use GeoGebra as a tool with which to explore various aspects of the given functions, without addressing the given task in an adequate way. This suggests that attention may need to be paid to using GeoGebra as a tool for exploration, in the task setup.

RATIONALE

A considered use of technology may assist mathematical understanding (Zbiek & Hollebrands, 2008) and may promote deeper understanding of advanced mathematical concepts (National Council of Teachers of Mathematics, 2000). At the same time most mathematics educators agree that the sort of tasks that students engage with while using these technologies is of fundamental importance (for example, Zbiek & Hollebrands, 2008).

What is meant by a 'task'? Mason and Johnston−Wilder (2006, p. 22) contend that "the purpose of a [mathematical] task is to initiate mathematically fruitful activity that leads to a transformation in what learners are sensitised to notice and competent to carry out". In line with this, I use the term 'computer-based mathematical task' to refer to a mathematical task that exploits the affordances of a computer or similar technology. Computer-based mathematical tasks may provide opportunities for learning which may not be available in the paper and pencil world (for example, the task presented in this paper). At the same time, these opportunities may be diminished if the design or structure of the task is not appropriate. For example, the use of certain computer algebra systems (CAS) often requires knowledge of specialized syntax, and learning to use this syntax may shift the students' attention away from the mathematical focus of the task. Also computer output may differ in form from that of pencil and paper mathematics; this may contribute to students' difficulties with interpretation of the output. And so on.

Elsewhere I have developed a framework which can be used to isolate*, a priori*, the possibilities and limitations of computer-based mathematical tasks (Berger, 2011). In this report I use this framework to compare the extent to which the pedagogical

intentions of a specific given task are realized in the actual implementation of this task by students.

This leads to my research question: In what way(s) may the pedagogic purpose and intended focus of a computer-based mathematical task get transformed when executed by students?

THE FRAMEWORK

I offer a summary of the four components - mathematical focus, cognitive demand, use of CAS' affordances, and technical demand - of the framework. For further details see Berger (2011).

Mathematical focus: A computer-based mathematical task is a task that uses technology to help focus the learner's attention on a specific mathematical concept and/or process. Through this focused activity, the learner is expected to make sense of the particular mathematical notion. Mathematical focus is thus a key feature of the framework.

Cognitive demand: According to Stein, Smith, Henningsen & Silver (2009), the most important characteristic of a mathematical task is its cognitive demand, that is, the "kind and level of thinking required of students to successfully engage with and solve the task" (p. 1). 'Memorisation' tasks involve reproduction of previously learnt formulae or definitions. They are not ambiguous (Stein, et al., 2009). In a CAS context, many mathematical facts are actually reified in the software. So memorization tasks appropriate to the technological environment may involve the verification of a particular fact or formula (for example, $sin^2x + cos^2x = 1$). 'Procedures without connections' tasks are algorithmic; they are focused on the implementation of appropriate algorithms rather than development of conceptual understanding (ibid.). In the CAS environment users can use the computer to execute algorithms and so 'procedures without connections' tasks usually have little cognitive value in and of themselves. Nonetheless the farming out of computations to the computer may free the user to focus on more conceptual aspects of the task. Also the relative ease with which the user may use the CAS to execute procedures may support pencil and paper algorithmic skills. For example, Kieran and Damboise (2007) report on a study in which poorly performing Grade 10 learners used CAS to generate factorisations and expansions of expressions. Being able to examine the patterns of these factorisations and expansions and knowing that they were correct, supported the development of these students' pencil and paper skills. 'Procedures with connections' tasks focus on the use of procedures for the purpose of developing deeper levels of mathematical understandings of specific concepts (Stein et al, 2009). These tasks usually suggest general procedures which illuminate the underlying concepts and they often involve making connections across multiple perspectives (ibid.). 'Doing mathematics' tasks require "complex and non-algorithmic thinking" (ibid.) in which the learner has to determine her own route through the problem. Such tasks require the learner to analyse the task and to consider task constraints; successful execution of the task involves the

learner exploring and using various mathematical concepts, processes or relationships. As with 'procedures with connections' tasks, many opportunities for the design of 'doing mathematics' tasks are opened up when the use of CAS is permitted.

Use of technological affordances: Different technologies offer different affordances for the learning and teaching of mathematics. For example, the use of CAS affords movement between different representations (algebraic, graphical, numerical) of one mathematical object. Seeing different representations of a single mathematical object may illuminate crucial properties of the object. Another affordance is that of dynamic representations. In this regard, the user may define a specific function using one or more parameters. By changing the value of this parameter dynamically the user may be able to see how certain properties of the function change as the parameter changes. As with multiple representations of a single object, this may give insight into invariant or variant properties of families of functions. These are just two of very many possible affordances of CAS. See Berger (2011) for further examples of CAS' affordances.

Technical demands: An important aspect of a computer-based mathematical task is its technical demand. Such a task may appear to be interesting and worthy in terms of its mathematics content but it may require such sophisticated technological skills that it has very little, if any, value in the mathematics classroom. This may be particularly relevant in a heterogeneous country such as South Africa where certain groups of students historically have limited access to, and experience with, computers. The technical demand of a task is classified according to the number of different commands required (single step, several steps or many steps) and the familiarity of the set of commands (standard, non-standard). The familiarity of the commands is a context-dependent category. For example, if users have experience with using the slider in GeoGebra the use of a slider is standard; if they do not have this experience, the use of a slider is non-standard.

Implementation of the task: A further consideration in the design or selection of appropriate tasks is that the learners may approach the tasks in ways not envisaged by the teacher. Stein, Grover & Henningsen (1996) show how the cognitive demands of (non computer-based) tasks in reform-orientated classrooms significantly declined as a result of certain types of assistance by the teacher. In this paper, I show how the use of powerful software may encourage a shifting of mathematical focus away from the intended focus of the task to a completely different focus. I also postulate that prior mathematical and technological knowledge profoundly effects the implementation of the task.

EXPECTATION VERSUS IMPLEMENTATION

Context

The example I present derives from a course on functions given to in-service high school teachers in South Africa. The purpose of the course was to revisit an old topic, functions, from different perspectives. For reasons born out of South African history,

many of the mathematics teachers in South Africa have a degree or diploma in education rather than in mathematics. Most of the teachers in our course came from this group. These teachers' content knowledge is fairly weak and so our aim in this course was to revisit functions, extending and deepening teachers' knowledge of this basic concept. The course was structured as a part-reading, part-activity course. The class met once a week for a three hour session over twelve weeks. Students were expected to study a specific chapter from the prescribed mathematics textbook, Sullivan (2008), prior to their weekly session. During the class, the students discussed the topic they had studied at home for the first hour. They were then presented with tasks around the topic which they did on their own or in pairs. Some tasks required the use of GeoGebra, others did not. Many of the in-service teachers were newly arrived digital immigrants. Data relating to the implementation of specific tasks was collected throughout the course in the form of handed-in students' worksheets as well as audio and screen-recordings.

In this paper, the activities of one single student (Dawn) and one pair of students (Sipho and Lebo) with one specific computer-based mathematical task are examined. Dawn is a very experienced mathematics teacher with a B.Sc degree in pure mathematics. She also has experience with CAS and dynamic geometry software. Sipho and Lebo have degrees in Education, rather than mathematics. Neither Sipho nor Lebo have used a computer in the learning or teaching of mathematics previous to this course.

Task

Is it possible to find a value for *a* such that $ax^2 > x^4$ for all *x*. Explain why or why not.

Pedagogic Expectation: In this task, students were expected to use GeoGebra to plot the graphs $y = x^4$ and $y = ax^2$ on the same set of axes. See Figure 1.

Figure 1: and
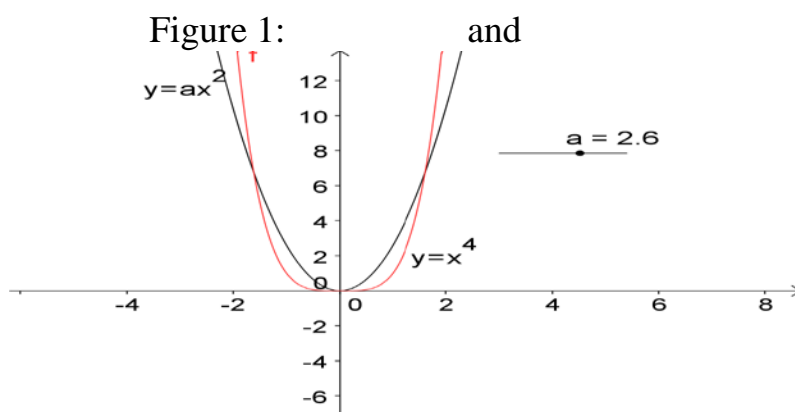
Since *a* is a parameter, students were expected to define *a* in terms of the GeoGebra slider tool and to use this tool to dynamically change the value of *a*. By dynamically altering the value of parameter *a*, it was hoped that students would notice how the graph of $y = ax^2$ changes in relation to the graph of $y = x^4$, for different values of *x*. In particular it was hoped that the students would appreciate that, no matter how large *a*

is, $x^4 > ax^2$ for large x.   With reference to technical demands, students had experience with working with the slider tool in GeoGebra before attempting the task. In particular, they had all engaged with a task in which they had examined the change in properties of the quadratic function $y = a(x + p)^2 + q$, for changing values of parameters a, p, q.

Table 1: Pedagogic expectation of computer-based mathematical task – an a priori analysis

	Category	Explanation of categorization
Mathematical Focus	Relationship between parameter and variable	
Technical demands	Standard; multiple-step	Requires use of several tools such as slider from GeoGebra toolbox. Construction may also require change in window size.
Cognitive demand	Doing mathematics	Task requires non-algorithmic thinking; requires students to distinguish between parameter and a variable. There is no prescribed pathway but dynamic exploration should suggest how change in parameter a effects shape of ax^2.
Use of GeoGebra's affordances	Dynamic representation	GeoGebra allows for the generation of a dynamic graph which changes shape as parameter changes. Slider is a useful tool for systematic exploration available in GeoGebra but not in pencil and paper maths.

Students' activities: Description and Analysis

Dawn's written submission shows that she uses GeoGebra to construct graphs of $y = ax^2$ and x^4 on the same set of axes and that she defines a through a slider. She uses this slider to see the effect of the changing value of a on the relationship between the two functions. Her written argument is:

"Using a slider for $y = ax^2$, I am able to see that $y = ax^2$ is always going to intersect $y = x^4$, and at some point, $x^4 > ax^2$".

She further argues that:

"Power predominates over the coefficient of x^2, ie x^4 means that x^2 gets squared. $\therefore$ Taking x^2 and squaring it will be larger than taking a 'factor' of x^2 only provided $x > \sqrt{a}$ or $x < -\sqrt{a}$ ".

Later she correctly concludes that

"We cannot find an 'a' for which $ax^2 > x^4, \forall x \in R$".

In this submission, Dawn implicitly distinguishes between the effect of a change in parameter value and the effect of a change in variable on the function. She approaches the task systematically and she stays focussed on the required activity. See Table 2

Like Dawn, Lebo and Sipho start off by using GeoGebra to draw $y = x^4$ and $y = ax^2$ on the same set of axes. They define the slider correctly and then use the slider to change the value of *a*. However they do not change the value of *a* in any systematic way. Nor do they use the graphs to consider how a change in value of *a* effects the relationship between $y = ax^2$ and $y = x^4$. Indeed, shortly after generating the graphs, Sipho writes:

$$ax^2 > x^4$$
$$ax^2 - x^4 > 0$$
$$x^4 - ax^2 < 0$$
$$x^2(x^2 - a) < 0$$

He has incorrectly assumed (with no objection from Lebo) that $ax^2 > x^4$. This is followed by a manipulation of symbols without any regard for their status as variable or parameter. A little later, as Sipho changes the shape of ax^2 on the screen through manipulation of the slider, Lebo and Sipho start focusing on the horizontal and/or vertical stretching of the graph of ax^2. Although this is an interesting issue in itself, it does not contribute to their consideration of the relationship between $y = x^4$ and $y = ax^2$. Soon after, they digress further from the intended focus of the task when they start comparing the two graphs for changing values of *a* and fixed value of *x*. Finally they stop examining the graphs and engage in unhelpful algebraic manipulations. Specifically Lebo writes,

" If $a = x^2$, $ax^2 = x^4$
If $a > x^2$, $ax^2 > x^4$
So, Yes it is possible to find the value of *a* such that $ax^2 > x^4$."

These algebraic manipulations are misleading and incorrect: *a* is a parameter and *x* is a variable but in these manipulations Lebo uses these symbols without regard to their status. Indeed ax^2 should be compared to x^4 for specific *a*, and all *x*. Thus, the concluding statement that "it is possible to find the value of *a* such that $ax^2 > x^4$" may be true for specific *a* and specific *x*, but it is not true for all *x*. In fact, for very large *x*, $ax^2 < x^4$, any *a*.

Thus Sipho and Lebo, despite seeing the graphs in GeoGebra and successfully managing the value of *a* through the slider, do not execute the task successfully. They use GeoGebra as a tool for drawing and manipulating graphs rather than as a tool for interpreting the specific task. Partly this is because they do not distinguish adequately between a parameter and a variable (mathematical focus). Also their attention is easily diverted by the ease with which they can manipulate the graphs in any which way. Furthermore they move into a superficial mode of symbol manipulation. See Table 2.

Table 2: An analysis of the implementation of the task

	Dawn	Sipho & Lebo
Mathematical focus	Focus on distinction between parameter and variable. Focus on rate of increase of powers of variables in comparison to multiples of variables.	Do not distinguish between parameters and variables. Do not focus on relationship between ax^2 and x^4 for specific a, all x.
Technical demands	Met	Met
Cognitive demand	Doing Maths	Interpretation of GeoGebra output not adequate. Cognitive demands not met.
Use of GeoGebra's affordances	High: used slider to examine relationship between ax^2 and x^4 for many values of x, specific values of a.	Low: used slider to manipulate graphs in unsystematic way. Did not interpret the changing graphs in terms of changing values of x and specific values of a.

DISCUSSION

In this particular example we see how the intended cognitive demands of the task were not met by Sipho and Lebo. This was despite their technical proficiency with the slider and their prior experience with parameters in the GeoGebra context. Indeed they oscillated between treating a as a parameter (when they use the slider) and a as a variable (in their symbolic manipulations). Furthermore while working with the slider, they shifted their attention away from the question of the task and began to explore how changes in the value of a affected the horizontal or vertical stretching of ax^2. That is, they used GeoGebra as a tool for undirected and unsystematic exploration. In contrast, Dawn was able to exploit the affordances of GeoGebra to see how the graphs of $y = ax^2$ and $y = x^4$ changed in relation to each other for different values of a, and for all values of x. That is, Dawn used GeoGebra as a tool for interpretation of the assigned mathematical phenomenon.

Several reasons for Sipho and Lebo's inappropriate activities suggest themselves. Although Sipho and Lebo were able to manipulate the slider (a technical demand), they were not able to interpret the graphical information on the computer screen as a varied. I suggest that their limited exposure to non-standard mathematical tasks (they both have B.Ed degrees in which the level of mathematics is usually quite basic) and to technological tools for learning mathematics, contributed to their difficulties. This was in stark contrast to Dawn who completed the task systematically and with focus. Secondly Lebo and Sipho may have been seduced by the power of the dynamic representation. That is, with the slider they could easily vary the value of a and see the effect on stretching even though this was not the intended focus of the task. Prior to this course, they had had no contact with graphical software and arguably they were still in

thrall of the potency of dynamic representations. In contrast, Dawn was already using graphical software in her teaching at high school. She was thus not sidetracked by the power of the dynamic representations.

In summary, the analysis shows how one student uses GeoGebra as a tool with which to successfully interpret a particular mathematical phenomenon; the other pair of students use GeoGebra as a tool with which to draw and explore various aspects of the given functions, without addressing the given task in an adequate way. Thus although a computer-based mathematical task may be designed with one pedagogic purpose and with appropriate mathematical and technical demands, different students may engage in the task at different levels and with diverse foci. In particular, the educator may need to suggest ways of using GeoGebra for systematic exploration in the task setup especially when some students are relatively new to the use of technology for the learning of mathematics.

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COMMUNICATING MATHEMATICS OR MATHEMATICAL COMMUNICATION? AN ANALYSIS OF COMPETENCE FRAMEWORKS

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In this study we analyse the communication competence included in two different frameworks of mathematical knowledge. The main purpose is to find out if mathematical communication is primarily described as communication of or about mathematics or if it is (also) described as a special type of communication. The results show that aspects of mathematics are mostly included as the content of communication in the frameworks but the use of different forms of representation is highlighted both in the frameworks and also in prior research as a potential cause for characterising mathematical communication differently than "ordinary" communication.

INTRODUCTION

It is often stated that reading mathematics demands a specific type of reading ability, separate from an "ordinary" reading ability, that needs to be taught at all educational levels (e.g. Burton & Morgan, 2000; Shanahan & Shanahan, 2008). Research has also indicated that it might be the presence of symbols in mathematical texts, and not the mathematics in itself, that primarily creates such a demand of a specific type of reading ability (Österholm, 2006). This discussion about aspects of reading in mathematics can be expanded to aspects of communication, and it is relevant to examine how mathematical communication is described within frameworks that describe (school) mathematics (e.g. NCTM, 2000; Palm, Bergqvist, Eriksson, Hellström, & Häggström, 2004) to determine the relation between mathematical communication and communication in general, as well as between mathematical communication and other aspects of mathematics. The following overarching question is focused on in this paper: Is mathematical communication described simply as communication of mathematics (i.e. ordinary communication but regarding a specific topic) or as a special type of communication?

BACKGROUND

At a general level, two "extreme" examples of different theoretical perspectives can be given regarding relationships between communication and mathematics. Sfard (2008) does not describe communication and cognition as separated, but sees thinking as the individualised form of interpersonal communication and mathematics as a form of discourse. From this perspective, a mathematical communication competence is the same as mathematical knowledge in general, and whether there is something special about mathematical communication is the same as asking if there is something special about mathematics. Another perspective is to see a separation between mathematics and mathematical knowledge on the one hand and the communication of mathematical

content and the ability to communicate on the other hand. From this perspective, a certain use of language "indicates" whether individuals "in fact" conceive of something a certain way (Tall, Thomas, Davis, Gray, & Simpson, 2000, p. 230).

Other researchers focus more specifically on potentially special properties of mathematical communication (or communication in any other content area), when focus tend to be on literacy, and primarily reading. For example, McKenna and Robinson (1990) define the concept of content literacy as consisting of three components; general literacy skills, content-specific literacy skills, and prior knowledge of content. Similarly, Behrman and Street (2005, p. 8) suggest that "the ability to read with understanding would not be constant across disciplines, since learning depends upon domain-based declarative knowledge [prior knowledge of content] and domain-related strategies [content-specific literacy skills], in addition to more generalized strategies [general literacy skills]". These three components frame our discussion and analysis in this paper, and we focus on content-specific literacy skills.

A question addressed in some research studies is whether there are such things as content-specific literacy skills, examined by comparing reading in different domains. Results from such empirical studies tend to highlight similarities between domains. In particular, several studies show strong or moderate correlations between different tests of reading comprehension; between social studies and general reading comprehension (r = 0.79) (Artley, 1943), between reading comprehension in an anatomy course and general reading ability (r = 0.72) (Behrman & Street, 2005), and also between reading comprehension for a mathematical text and a historical text (r = 0.47) (Österholm, 2006). These results are seen as evidence of general literacy skills.

Another type of comparison between domains shows that experts from different domains read texts within their domain in different ways (Shanahan & Shanahan, 2008). However, a limitation in this study is that it is based on the reading of singular texts from each domain, but there is a great variety of texts within a domain (Burton & Morgan, 2000), making it difficult to draw conclusions about domains in general.

Another way to address the issue of content literacy is to think about what could be seen as content-specific literacy skills. At a general level, to be familiar with a certain genre or linguistic register (i.e. that mathematical texts might have a certain style or form, and that they might use words and formulations for purposes different than in other domains) could be seen as part of content-specific literacy skills. However, it is difficult to find a common description of all kinds of mathematical texts, since even when limiting the selection to mathematical research articles, Burton and Morgan (2000) notice a large variety of writing styles.

Empirical studies of students reading comprehension of mathematical texts have highlighted the use of symbols in mathematical texts as the most important potential cause for a need of content-specific literacy skills (Österholm, 2006). The use of different forms of representation is often also noted as a critical property of

mathematics, for example in Sfard's (2008) commognitive theory and in the cognitively oriented framework by Duval (2006).

In summary, we have not found any studies focusing more broadly on characterizing mathematical communication, except at more general theoretical level. When comparing different domains, focus tend to be on aspects of reading, where we have not found any clear empirical evidence for separating reading in different domains in general, instead the variation within a domain seems equally important. For mathematics, some empirical and theoretical evidence exist that different forms of representations can create a potential need for content-specific literacy skills.

PURPOSE

As a way of expanding our knowledge of a potential need for content-specific literacy skills in mathematics, in this paper we examine if and how content-specific literacy skills are described as part of a mathematical communication competence within frameworks of mathematical knowledge. Our research questions are:

1. What aspects of communication are included in frameworks describing mathematical competence?
2. How is mathematics described as the content of communication in frameworks of mathematical competence?
3. How is communication described as having special character due to aspects of mathematics in frameworks of mathematical competence?
4. Is mathematics described mainly as the content of communication or as part of other aspects of communication, in frameworks of mathematical competence?

METHOD

We acknowledge that many different types of analyses could be used to fulfil the described purpose, but in this paper we focus on one type of linguistic analysis, and do not include several different types of analyses, partly due to space limitations. However, we aim to expand our analyses in future publications, since different types of analyses might give different types of information.

Our method for analysing competence frameworks consists of two main steps. In step 1 we read each framework and highlight parts that specify some aspect of communication. In step 2 we analyse the highlighted parts from step 1 regarding how aspects of mathematics are related to the noted aspects of communication, in particular if mathematics is described as the content of communication or related to other aspects of communication. In both these steps, both authors perform the analysis separately and we then compare our results. Before performing the second step, we compare our results from the first step and agree on how to interpret the text and code the data, and we use our common agreement as a basis for the second step.

In this study we analyse two different frameworks of mathematical competence; a framework from NCTM (2000) and a framework created based on an analysis of the Swedish national curriculum (Palm et al., 2004). We shortly refer to these frameworks

as the NCTM framework and the Swedish framework respectively. These frameworks are chosen since they include a communication competence, and we only analyse the parts of the frameworks that explicitly address the communication competence. Aspects of communication could be included also in other frameworks of mathematical competence, which do not include a communication competence, and also in other parts of the analysed frameworks (e.g. when representations are discussed in a separate competence), but we limit our analysis to the communication competence. The main reason for this limitation is that another type of method of analysis could be needed to handle more implicit descriptions of aspects of communication.

The main analytical tool used in this study consists of a description of different aspects of communication. Based on definitions of *communication* we create a description of these aspects. We use definitions from dictionaries; from Merriam-Webster Online and the Swedish National Encyclopaedia (NE) for a standard type of definition and from Wikipedia (in English and Swedish) for a more colloquial type of definition, and also the definition from Sfard (2008) for a more non-standard perspective. We use different types of definitions in order to not exclude potential references to communication in the analysed frameworks. Based on these definitions, the following aspects of communication are identified; agent, technique, quality, content, and unspecified (first column in Table 1, in the results section). Common for all definitions is a focus on some *type of exchange of "information" between agents*. Deliberately, we do not define notions used here, but instead focus on words or phrases that in some way signal or specify some aspect of this "exchange" (third column in Table 1). The components within each aspect (second column in Table 1) are added in order to distinguish between words and phrases that specify a certain aspect of communication differently.

The list of words and phrases is created according to the following procedure: First we include words used in the definitions of communication in the dictionaries and also add words from a brainstorming activity around the different aspects and components. Then we look up the included words in dictionaries and include more words from the given definitions, and repeat this procedure for all new words. The purpose with the list of words and phrases is not only to search for those specific words included in the list, but also to more easily find relevant *types* of words when analysing the frameworks. That is, new words and phrases are also added to the list during the process of analysis.

In the first step of the process of analysis, each framework is read from start to end and all relevant words and phrases are highlighted in the text. The context is taken into account in the process of analysis to decide if a certain word should be highlighted. For example, "understand" could refer to the process of understanding a written text, an aspect of communication, but could also refer to a cognitive state that does not fit our (broad) type of characterisation of communication. All highlighted words are then included in a table as shown by Table 1, which is used for answering research question 1, regarding what aspects of communication are included in the frameworks.

In the next step of analysis, focus is on relationships between aspects of mathematics and aspects of communication, and each framework is read from start to end again. For

each occasion when some word has been highlighted in a framework, it is decided if and how any aspect of mathematics is included in the highlighted aspect of communication, based on the following six types of how an aspect of mathematics is specified:

1. Some form of the word "mathematics" or "mathematical" is used.
2. Some mathematical form of representation is referred to (e.g. through words like table, graph, or symbol).
3. Some mathematical concept or object is referred to (e.g. through words like triangle, number, or function).
4. Some mathematical activity is referred to, by referring to any other type of mathematical competence (e.g. problem solving) or to any procedure or operation that can be linked to a mathematical concept (e.g. derive or multiply).
5. Something mathematical is referred to, other than what is included in types 1-4.
6. Nothing mathematical is referred to.

For each occasion when one of types 1-5 has been noted, it is also noted what aspect of communication the mathematics is related to (i.e. agent, technique, quality, content, or unspecified). All occasions when some aspect of mathematics is specified in relation to some aspect of communication are then used when answering research questions 2-4, regarding how mathematics is included in different aspects of communication.

The following is an example of the process of analysis. In the excerpt below from the NCTM framework, the relevant words and phrases are highlighted:

> Students in the lower grades need help from teachers in order to share mathematical ideas with one another in ways that are clear enough for other students to understand.

Three aspects of communication are here noted; "share" refers to a creative agent, "ideas" refers to content, and "clear enough..." refers to quality regarding the exchange. One occasion is noted where an aspect of mathematics is specified; type 1 (using "mathematical") and related to the aspect of content in the communication.

In this paper, focus is not on quantifying occurrences of different aspects in a detailed manner, but rather on the existence of different aspects and general tendencies. Although the two authors' separate analyses resulted in several discrepancies, the main results and conclusions reported in this study are representative of each of our separate analyses and our common agreement regarding interpretation and analysis of data, which shows good reliability of the procedure in order to produce answers to the specific research questions of the present study.

RESULTS

Table 1 shows the words and phrases found in the NCTM framework. Due to space restrictions, the table for the Swedish framework is not presented, but the result is summarised. In both frameworks of mathematical competence, all aspects of communication are described through the use of corresponding words or phrases. The NCTM framework describes all components (i.e. specifications of aspects) while the Swedish framework does not describe *bodily* as technique or *breadth of information* as quality.

Aspect	Component	Words and phrases
Agent	Sender/ creator	Write; Draw; Speak; Talk; Describe; Explain; Convey; Express (oneself); Articulate; State; Build; Use (e.g. certain technique); Present; Reason; Claim; Justify; Give account of; Describe; Clarify; Formulate; Share; Convince; Act out; Think out loud; Pose a question; Question; Complete; Make public; Work out (in public); Provide; Critique.
	Receiver/ interpreter	Read; Listen; Interpret; Analyze; Evaluate; Examine; Consider; Probe; Explore.
	Both of the above	Converse; Discuss; Dialogue; Respond; Paraphrase; Participate (in conversation).
Technique	Oral	Talk; Speak; Listen; Oral; Discuss; Dialogue; Converse; Think out loud.
	Written	Write; Draw; Read; Symbol; Diagram; Picture; Mathematical expression; Sketch.
	Bodily	Act out; Use object.
	Unspecified	Tools; Ways (of communicating); Informal means; Verbal; Word; Vocabulary; Terminology; Terms; Representation; (Some type of) language; Genre.
Quality	Depth of info	Precise; Coherent; Clear; Thoughtful; Rigorous.
	Breadth of info	Complete; Rich; Elaborate.
	The exchange	Understandable; (Sufficiently) convincing; Audience; Purpose; Communicative power.
	Unspecified	Mathematically; Sophisticated; Well-constructed; Exemplary; Problematic; Informally; Formally; Standards (of dialogue/argument); Carefully.
Content	-	Understanding; Viewpoint; Argument; Idea; Situation; (Result of) thinking; Strategy; Explanation; Solution; Mathematics; Reasoning; Method; Task; Problem; Question; Answer; Evidence; Example; Procedure; Result; Insight; Claim.
Unspecified	-	Communicate; Discourse.

Table 1: Aspects of communication found in the NCTM framework.

Regarding how aspects of mathematics are included in aspects of communication, the analysis of the NCTM framework shows that most often mathematics is part of content (approximately 60 % of all occasions) and otherwise part of technique, except on one

occasion when it is part of quality and one occasion when it is unspecified, using the following words and phrases:

- *Specifying content*: mathematical thinking, strategy, mathematical idea, solution, mathematics, mathematical/procedural task, method, reasoning, mathematical argument, proof, procedure, result, mathematical property, mathematical understanding.
- *Specifying technique*: language of mathematics, (mathematical/algebraic) symbol, diagram, communicate in mathematical ways, mathematical terminology/term, mathematical writing, write mathematically, mathematical language, mathematical style, mathematical expression.
- *Specifying quality*: mathematically rigorous.
- *Unspecified*: communicate mathematically.

The same type of analysis of the Swedish framework shows that most often mathematics is part of content (approximately 70 % of all occasions) and otherwise part of technique, using the following words and phrases (translated from Swedish):

- *Specifying content*: mathematics, information/question with mathematical content, mathematical idea, mathematical line of thought, (mathematical) concept, the concept of pie chart, law, method, reasoning.
- *Specifying technique*: language of mathematics, mathematical language, symbols of mathematics, mathematical terminology, pie chart.

CONCLUSIONS AND DISCUSSION

Communication in general is well represented in the frameworks of mathematical competence through many specifications of different aspects of communication, although all specifications focused on in this study are not included in both frameworks. Besides the general aspects of communication, for both frameworks, aspects of mathematics are mostly included as the content of communication and otherwise as technique, except one occasion when an aspect of quality is specified. Mathematics is often specified through labelling something as "mathematical" in some way (e.g. by referring to the language of mathematics or mathematical ideas/thinking), thereby tending to keep descriptions at a general level, since it is not clear in itself what the notion of "mathematical" refers to.

In prior research no clear evidence of the need for content-specific literacy skills have been found, and similar can be said about the analysis of competence frameworks since aspects of mathematics are mainly included as content of communication and aspects of mathematics are often referred to only by labelling something as "mathematical", and it is not clear if or how this could be seen as creating a need for content-specific literacy skills. This conclusion is valid at least for communication using natural language, but the use of different forms of representation is highlighted both in prior research (empirical and theoretical) and in the frameworks (through certain mathematical techniques) as a potential cause for the need for content-specific literacy skills.

Is there a need to teach a specific kind of communication ability in mathematics? There exist much literature about content literacy that discuss benefits of teaching reading also in content areas (Hall, 2005), but perhaps it is not about learning a special kind of reading ability but an effect of a good way of teaching the content that focuses on processes of interpretation and comprehension (Draper, 2002). This perspective can perhaps also be applied on the NCTM framework, since there is much focus in this framework on effects and benefits of using communication in teaching and learning, and guidance on how to create communication-rich mathematics classrooms.

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DEVELOPING ALGEBRAIC AND DIDACTICAL KNOWLEDGE IN PRE-SERVICE PRIMARY TEACHER EDUCATION

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This study analyzes the contribution of a teaching experiment for the development of prospective primary teachers regarding knowledge of algebra and of algebra teaching as well as their professional identity. The case study of a prospective teachersuggests that an exploratory approach combining content and pedagogy supports this development, especially in the need to propose challenging tasks, to provide opportunity for students' autonomous work and collective discussions and to be attentive to children's representations and strategies in order to promote algebraic thinking.

INTRODUCTION

In addition to consistent mathematical knowledge, prospective teachers need to have an appropriate knowledge of curriculum and didactics. Such knowledge is essential to select tasks and prepare and manage students' work, providing a classroom dynamics that promotes seeking generalizations, sharing strategies, and establishing connections among mathematical ideas. A major challenge in the prospective teachers' future teaching practice is supporting the development of students' algebraic thinking. The goal of this paper is to analyse the contribution of a teaching experiment in an algebra course, in preservice primary and kindergarten teacher education, for the development of prospective teachers' algebraic thinking and knowledge of key aspects for teaching this subject, so that, in the future, they may use them in their teaching practice. In addition, we seek to know the influence of this teaching experiment in the development of the participants' professional identity.

ALGEBRAIC THINKING AND TEACHER EDUCATION

Recent curriculum guidelines (NCTM, 2000) and researchers (Carraher & Schliemann, 2007; Kieran, 2004) point the importance of promoting algebraic thinking from an early age. This does not mean that the topics usually taught in algebra in later school years now arise in primary school (Carraher & Schliemann, 2007), but rather that algebraic ideas are tacked in an informal way. Cai and Knuth (2011) indicate that the development of algebraic thinking requires analyzing relations between quantities, paying attention to structures, studying changes, generalizing, solving problems, modeling, justifying, proving and predicting. Generalization is central to algebraic thinking as well as expressingit symbolically (Kaput, 2008; Mason, Graham, & Johnston-Wilder, 2005). A generalization may be expressed in different ways and, at primary school, students may do this in their own words, based on what they observe

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and learning gradually to express it symbolically (Blanton, 2008). Algebra also involves "syntactically guided actions on reasoning and generalizations expressed in conventional symbol systems" (Kaput, 2008, p. 11).

In preservice teacher education, prospective teachers must have learning experiences aimed at different aspects of algebraic thinking so that, in their teaching practice, they can promote it in their students (Magiera, van den Kieboom, & Moyer, 2011). Ponte and Chapman (2008) address three aspects to consider in preservice teacher education: (i) knowledge of mathematics for teaching, (ii) knowledge of mathematics teaching or didactics, and (iii) professional identity. Both knowledge of mathematics and mathematics teaching are included in the development of identity. Knowledge of mathematics involves knowing how to use mathematics and also understanding its meanings and foundations (Albuquerque et al., 2006). The teacher must know and use procedures and why these procedures work. NCTM (2000) states that "teachers must know and understand deeply the mathematics they are teaching" (p. 17). Prospective teachers need to know also about mathematics teaching, namely about tasks to propose, classroom work, students' learning processes and curriculum guidelines. Ponte and Chapman (2008) suggest that preservice teacher education faces the challenge of combining content and pedagogy, as well as "teaching preservice teachers the same way that they are expected to teach their students" (p. 256). Therefore, they must knowledge algebra and what its teaching involves in primary school to be able to mobilize it later in their practice, creating learning situations to develop their students'algebraic thinking.

Teacher education must also foster the development of prospective teachers' professional identity. This includes the appropriation of the values and standards of the profession, the notion of what is teaching in the envisaged school level, an image of the teacher he/she wants to be, as well as an understanding of his/her own learning and of the role of reflecting on experience (Ponte & Chapman, 2008). Prospective teachers' past experience as school students influences their identity, bringing up these memories in shaping their role as teachers (Brady, 2007). With regard to algebraic thinking in primary school, future teachers will face challenges and demands, most of which they did not experience as students.

METODOLOGY

This research is carried out in the context of a teaching experiment that takes into account current guidelines for preservice teacher education and for kindergarten and basic education (grades 1-6). It aims at two intertwined aspects, the development of participants' algebraic thinking and their learning how to promote the development of students' algebraic thinking. The teaching experiment follows an exploratory approach (most tasks are exploratory and investigative) and the classroom dynamics aims at involving participants in discussing algebraic concepts and analyzingissues on algebra teaching and learning. The experiment involves 7 tasks on topics such as relationships, patterns, sequences, functions, and modeling, their mutual relationship and with other themes. Each task aims at deepen aspects of algebraic knowledge and provides

opportunities to discuss learning situations seeking to develop participants' didactical knowledge. Some situations refer to primary classroom episodes involving students' work, teaching practice or to students' solutions of algebraictasks. Therefore the teaching experiment addresses mathematics and didactic knowledge, providing participants with learning experiences regarding aspects that they will meet in their future practice (Albuquerque et al. 2006; Ponte & Chapman, 2008).

The first author is also the teacher in this experiment. This option establishes a close link to the classroom, allowing the results to inform her practice. We present the case of Diana, a prospective primary school teacher that was a successful mathematics student up to grade 12. Data was collected by two questionnaires with mathematical and didactical tasks, administered before (Qi) and after (Qf) the teaching experiment and three interviews (E1, E2, E3) madebefore, during, and after the teaching experiment. Data is also collected by participant observation, recording field notes (FN) and collecting documents produced by the prospective teacher. Data analysis is descriptive and interpretive, seeking to highlight the contribution of the teaching experiment for the participant's developmentof knowledge of algebra and algebra teaching and professional identity.

DEVELOPMENT OF DIANA'S ALGEBRAIC THINKING

Since the beginning of the study, Diana demonstrates significant algebraic thinking, making generalizations, using algebraic representations and procedures, and relating natural language, algebraic and graphical representations. For example, regarding modeling situations, in the initial questionnaire, she represents in algebraic language a problem with two unknown quantitiesstated in pictures and natural language. She writes a system of two 1st degree equations with two unknowns and solves it by the substitution method. She displays ease in using and manipulating algebraic symbols, showing to know formal procedures to solve systems of equations (Qi).

In another problem involving three unknown quantities (Task 2 of the teaching experiment), Diana identifies relationships between known and unknown quantities and performs basic operations. She writes a system of three 1st degree equations that she uses to find the value of one unknown, showing some difficulty in solving it. After the collective discussion of the solution of the system, she improves her understanding of the procedures (FN). Then, she analyzes solutions of grade 6 students, identifying strategies and representations, and a new problem is proposed (figure 1):

Three friends walk in different routes. We know thatJoão and Tiago together walk 19 km, Tiago and Diogo together walk 24 km, and João and Diogo together 29 km.What distance does each friend walk?

Figure 1: Problem from Task 2 of the teaching experiment

Diana writes the system of three 1st degree equations that she solves correctly by the substitution method. Based on the solution of a student to the former problem (FN), she learns a new strategy that she also uses to solve this problem (figure 2):

Figure 2: Diana's solution to the problem of the friends walking in different routes

Diana adds the three totals for each pair getting the double of the combined walking of the three friends. She considers this an efficient strategy for this context and uses algebraic language to present relationships in a formal way. She continues to do generalizations and to use different representations, notably pictorialand algebraic and torelate different representations. She improves her comprehension of the algebraic language and procedures that she uses, indicating that she now understands "why".

In the final questionnaire, Diana represents the problem proposed in figure 3 by a system of equations and solves it correctly by the substitution method (Qf).

Maria and Raquel went shopping. Maria bought glasses and two equal bags by 64 euros. Raquel spent 101 euros buying similar objectsbut in different quantity, as she brought two glasses and three bags. Find the price of the glasses and the bag. Explain what you did.

Figure 3: Problem from final questionary

In the interview, she thinks in another strategy to solve the problem that may be closer of the strategiesof primary students, without using a system of equations: "Maybe multiply this [picture 2] by 2 would yield 6 more 4 [6 bagsand 4 glasses] and then take this2 from here [picture 1]… Or 3, that is. Exactly" (E3).

Figure 4: Drawing made by Diana

As the interviewer asks her for clarification, she goes on:

> This is what is here [drawspicture 2]. If I multiply this by 2 I get… [draws 4 glasses and 6 bags]. And here, if I multiply by 3 I would get 3 glasses… [draws 3 glassesand 6 bags]. And then, if I go to this one [points towards4 glasses and 6 bags] and took out this [points to3 glasses and 6 bags], this is eliminated [3 glasses] this also [6 bags] and I would get just the glasses. That is, 64 had to multiply by 3, 64×3 [writes in the picture] and here 202 [writes in the picture]… This less that yields the costof the glasses. (E3)

The interviewer asks what the final result is and she indicates:

> $202 - 192$ is 10. Exactly, the glasses cost 10 euros. (E3)

Diana relies on the pictorial representation that she considers to promote students' understanding of the situation. However, this strategy involves the method of

subtraction. She multiplies each equation by the values that she chooses, subtracts the two equations and obtains an equation with one unknown. She shows, once more, a good command of the algebraic language and procedures and her ability to make generalizations and to interpret and use different representations.

The exploratory approach of the teaching experiment promotes Diana's involvement in different learning situations in algebra that contribute towards the development of her knowledge regarding generalization, using and understanding different representations and learning the justification of procedures. In addition to algebraic and graphical representations, she uses also pictorial representations.

KNOWLEDGE OF ALGEBRA TEACHING

Before the teaching experiment, Diana shows to know the main topics of school algebra, functions and equations. However, she indicates that these topics will not be addressed in primary school, at least in the formal way she learned them. She considers that the problems involving unknown quantities may be complex for primary students and therefore the unknown values must be numbers that students can easily find by trial and error. The strategy she suggests does not show the relationships between given and unknown values, verifying that it is necessary to satisfy each condition.

During the teaching experiment, Diana recognizes the possibility of working with situations concerning unknown quantities in primary school although these involve equations and unknowns that are not formally addressed by students and she suggests that this work may takes place supported in pictorial representation and in the establishment of relations based on this representation:

> More through images... I think it's much better, at least for children from grades1-4, because if we put this on paper with no pictures I think it would be much harder for them to understand the exercise. In this way they have something tangible. With images in the exercises it is easier for them to work. (E2)

As a school student, Diana learned in a very different way: "we got to some point and it was just mathematics, mathematics... Everything with computation and we did not ever think of simpler ways" (E2). Thus, she recognizes that some tasks may contribute to the development of students' algebraic thinking and knows how to propose them. She says that, if students just practice exercises, they memorize the procedures without understanding: "if the exercises are similar, just with different numbers, they end up memorizing, they just copy from above just changing the numbers, and often do not understand the exercise" (E2). The teaching experiment led her to solve problems using strategies and representations tailored to the skills and knowledge of her future students, establishing relationships and meeting conditions. She adds that this work in primary school may improve students' understanding, particularly, of equations. She appreciates the practical work of analysis of students' solutions because she considers important to understand what students do and how they think and the discussions about their understanding in different situations.

For Diana, it was important to examine different strategies and representations to identify the work that may be developed with primary students in identifying regularities and establishing generalizations in order to promote their algebraic thinking. Furthermore, the analysis of teaching practice and the dynamics created in the teaching experiment contributed for her recognition of the importance of classroom working modes and the roles of students and teacher, highlighting moments of group work and collective discussion and the teacher's questioning. After the teaching experiment, she relates algebra teaching and learning, in general, to algebraic thinking. She also refers specific aspects related to sequences and functions. She recognizes now that students may solve problems involving unknown quantities based in the exploration of relationships and not just by trial and error as she formerly thought. Concerning the tasks to propose, she indicates: "in the second task they may get some lessons they gained in the first, [tasks may] form a sequence… They may learn in the second task something else, using what they learned in the previous example" (E3). That is, Diana emphasizes the sequences of tasks that gradually increase the cognitive level.

PROFESSIONAL IDENTITY

Diana indicates a clear intention of becoming a teacher for grades 5-6. At the beginning of the teaching experiment, influenced by her former experience as a secondary school student, she views work on algebra as very formal, and does not regard that as appropriate for these grade levels. However, the work on the teaching experiment allows her to verify that working on algebra may be a rather different activity, exploring relationships and patterns aimed at developing students' algebraic thinking. The proposed activities provided her more confidence to work with her future students, especially in grades 1-4, a level that she originally did not intended to teach.

During and after the teaching experiment she identifies important features of the professional knowledge of the teacher of this subject, with which she identifies herself. She considers that the teacher must be able to solve a task in different ways, analyzedifferent answers from students and support them learning from their mistakes:

> [The teacher] has to know how to solve [the task] in a variety of ways, because a child can get there with a different solution and the teacher cannot say that is wrong, because something may be right. And the teacher must know, must understand what the child did... And use what the child knows (…). The child may know something, he/she may be wrong, but not totally wrong, one may use something... (E3)

Diana stresses that teachers must hold a formal knowledge in algebra, knowing the algebraic language and procedures. In her view, the teacher must use this knowledge to "getting the simplest ways to do and to explain" (E2) and to prepare tasksfor her students. The teacher must understand grade 1-6 students' thinking, the strategies that they use and adapt her language to the knowledge and understanding of students. She shows capacity to reflect about her experience and about her development and recognizes the importance of analyzing the students' answers and reasoning:

I think it is very good that we analyzed how children solved the exercise, because, on one hand, we must know to solve the exercises, and, on the other hand, we must understand what kids do. Because sometimes they think in a way that we do not thought of, and it may be correct. And I think it's good that we do not practice just how the exercises may be done, but also understand how they did them. Because, in our future practice we will need, we have to understand. (E2)

The analysis of teaching situations and their relation to experience contributes for her understanding of the work to be done on these gradesand her recognition of some of the challenges that the teacher faces and of the specificity of professional knowledge.

CONCLUSION

This study aims to contribute for understanding how to integrate mathematics and didactical knowledge in prospective teachers' educational programs, in particular, in an algebra course and to identify its contribution for the development of these two aspects as well as in the development of professional identity. Diana intends to become a primary school teacher (grades 1-6) expressing preference for teaching mathematics and science. Being a successful secondary school mathematics student, before the teaching experiment she already makes an effectively use of the algebraic language, solving most tasks with no difficulty but in a formal way, using algebraic procedures, but she does not know what work may be developed in primary school.

With the teaching experiment, Diana recognizes that many algebraic tasks may be addressed in a different way. She strives to find different ways to solve them, and values the solution of a problemusing different strategies and representations, feeling much more prepared to interpret the diversity of students'solutions. The focus on relationships and seeking generalizations provides her a deeper understanding of the procedures she already knew and often used in a mechanized way, showing evolution of her syntactically guided reasoning. Diana considers that primary school students' algebraic thinking may be developed by the exploration of relationships, contributing to a better understanding of formal aspects of algebra later on (Blanton, 2008). In this experiment, she develops an understanding of the knowledge that the teacher need to promote algebra learning. She recognizes that the teacher must have mathematical knowledge to use in his/her teaching practice to prepare suitable tasks for students and to solve correctly different kinds of situations, and also have a deep knowledge of students, their prior knowledge and the way how they learn. She is also aware of the ways she can communicate with students in an effective way. The teaching experiment also influenced the way she regards the work with her future students, highlighting moments of autonomous work and the moments of collective discussion.

Besides changing her view regarding the role of teaching and learning of algebra in primary school, Diana also developed a much better image of the teacher that she wants to be, based on the reflection that she makes about her experience and the development provided by the teaching experiment. As Brady (2007) indicates, initially, her past experiences influence her identity. The memory of how she learned algebra makes she think it will be difficult to address this subject with primary students. This

view changes with the teaching experiment. Contrarily to the focus on calculations that she experienced as student, she now underlines the role of non-routine tasks aimed at the students' understanding and the activities that promote algebraic thinking in primary school. The integration of content and didactic knowledge (Ponte & Chapman, 2008) and the exploratory approach used in the teaching experiment contributed for development ofher knowledge of algebra for teaching, her knowledge of mathematics teaching, and her professional identity. In particular, the emphasis on prospective teachers working on algebraic tasks and analyzing learning situations, combining autonomous work and collective discussions, helped Diana to deepen her mathematical knowledge, understanding the rationale for certain procedures, and to develop her understanding of learning processes and knowledge of teaching practice.

Acknowledgement

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MATHEMATICAL KNOWLEDGE FOR TEACHING USING TECHNOLOGY: A CASE STUDY

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This paper analyses data from a PhD pilot study to explore the nature of mathematical knowledge for teaching using technology, as represented by the central construct of the TPACK framework. The case study of teacher Alice is used as an illustrative example to suggest that the central TPACK construct may be better understood as a transformation and deepening of existing mathematical knowledge rather than as a new category of knowledge representing the integration of technology, pedagogical and mathematical knowledge.

INTRODUCTION

This paper explores the nature of mathematical knowledge for teaching using technology. In particular, it explores whether teachers' mathematical knowledge for teaching using technology should be conceptualised as a new domain of knowledge integrating knowledge of mathematics, pedagogy and technology or rather as a transformation or re-contextualisation of existing *mathematical knowledge* for teaching using technology. In this paper, technology is used to indicate digital technologies, commonly referred to as Information Communication Technologies (ICT). In recent PME conferences, mathematical knowledge for teaching has been a sustained research interest, see for example the RF1 papers on teacher knowledge (Ball et al., 2009) in the 33rd conference and last year's plenary lecture on designing settings for teachers' disciplinary knowledge (Davis, 2010). Although substantial research effort has been focused on conceptualising teacher knowledge (Rowland & Ruthven, 2011; Sullivan & Wood, 2008), it has rarely considered teachers' mathematical knowledge for teaching in the context of technology use. Correspondingly, research on mathematics teachers' knowledge and use of technology is rarely informed by studies of teacher knowledge in mathematics education or in the wider field of education (see for example Hoyles and Lagrange, 2010), thus such research tends not to build towards a systematic analysis of mathematical knowledge for teaching using technology. These omissions are surprising given widespread recognition of the complexities of technology integration experienced by teachers and the corresponding gap between aspirations for technology use in schools and the classroom reality of technology use (Lagrange & Erdogan, 2008), with teacher knowledge often cited as an explanatory factor. Nevertheless, the nature of teachers' mathematical knowledge for teaching using technology remains unresolved.

THEORETICAL BACKGROUND: THE TPACK FRAMEWORK

This study adopts Mishra and Koehler's (2006) Technological Pedagogical Content Knowledge (TPACK) framework due to the juxtaposition of technology knowledge alongside pedagogy and content knowledge, enabling an explicit focus on technology and thus an exploration of the nature of teachers' mathematical knowledge for teaching using technology. The TPACK framework represents Shulman's (1986) pedagogic content knowledge diagrammatically as the intersection of two circles representing general pedagogic knowledge and content knowledge. Extending this representation using a Venn diagram with three overlapping circles, they incorporate technology knowledge as a third domain of teacher knowledge, to indicate the skills or knowledge needed to successfully operate technology. The inclusion of technology knowledge introduces two new dyads, technological pedagogical knowledge (TPK) and technological content knowledge (TCK), representing the intersection of technology knowledge with pedagogic knowledge and content knowledge respectively, and a triad representing the intersection of all three types of knowledge: technological pedagogical content knowledge (TPACK, see Figure 1).
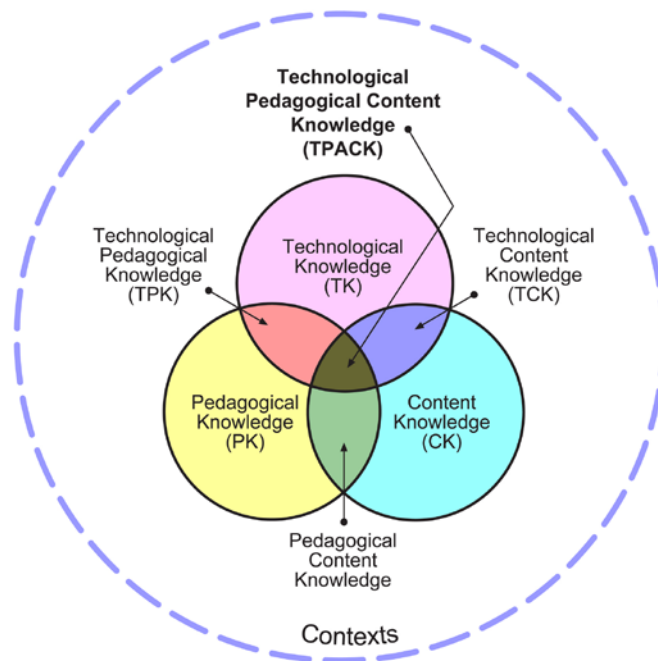
Figure 1. The TPACK framework, source http://tpack.org/

The TPACK framework was developed in the field of educational technology, hence its components require contextualising in the field of mathematics education. Mishra and Koehler (2006) define TCK as knowledge about the manner in which technology and content influence and constrain one another. TCK can be conceptualised as knowledge of how software models mathematical concepts and relations and of how the software design may therefore affect both the substantive and syntactic structures of mathematics. TPK comprises knowledge of the existence, components and capabilities of various technologies for use in teaching and learning settings and

pedagogical considerations for their selection (Mishra & Koehler, 2006). For example, teachers need to be able to reinterpret the function of generic software and hardware, such as word-processing, spreadsheet or presentational software or interactive whiteboard hardware, to suit their own pedagogical purposes. This might include how to manage changes in the *working environment* and *activity format* (Ruthven, 2009), requiring the adaptation of strategies for classroom management and organisation. Finally, Mishra and Koehler (2006) suggest that TPACK is a special form of knowledge, different from that of the technology expert, subject matter specialist or the general pedagogic knowledge shared by teachers across disciplines. In teaching mathematics, TPACK could be exemplified by the knowledge underlying a teacher's selection of spreadsheet software for the capability to manipulate variables and formulae dynamically for the pedagogic purpose of supporting an investigative approach to learning algebra, whilst understanding the limitations of the mathematical representation, such as the discrepancies between spreadsheet and standard algebraic notation, and recognising and developing strategies to deal with the pedagogical implications of these limitations.

The nature of the central TPACK construct remains weakly conceptualised (Graham, 2011). For example, Bowers and Stephens (2011) conclude that the central TPACK construct may represent the empty set in terms of particular knowledge or skills. Instead, they suggest TPACK should be regarded as an orientation or disposition towards viewing technology as a critical tool for identifying mathematical relationships. In contrast, Niess et al (2009) propose TPACK as integrated knowledge, representing the intersection and interconnection of content, pedagogy and technology knowledge. As a result, the nature of teachers' mathematical knowledge for teaching using technology, represented by the central TPACK construct, remains unresolved.

DATA COLLECTION AND CONTEXT

As part of a pilot study for the author's PhD project, three case study teachers were observed teaching a lesson involving technology and subsequently asked to reflect on the lesson in a post-observation interview. Initially presented in Bretscher (2009), the data has been re-analysed using the TPACK framework for the purposes of this paper. Here the case study of Alice is used as an illustrative example to explore the nature of teachers' mathematical knowledge for teaching using technology, as represented by the central TPACK construct. Alice was an experienced mathematics teacher, working at a private girls' school in the UK. She was teaching a group of 14 girls aged 14-15 years, who had just sat their end of school-year exams. Alice noticed that the majority of the group had incorrectly answered a standard question on the nth term of linear sequences and this lesson was intended as a revision lesson of the topic. In Alice's selective school, this group were regarded as low-attaining in mathematics, although according to their predicted grades for the national school-leaving exam (GCSE[1]) they would generally be considered as having average or above-average attainment. The lesson took place in a computer room specially booked for the occasion. Alice used a

PowerPoint presentation on the interactive whiteboard to introduce the topic, demonstrating the differencing method to find the *n*th term of a linear sequence, followed by a pencil-and-paper worksheet. After going through the answers to the worksheet on the interactive whiteboard, the students worked on a spreadsheet exercise where they had to provide the *n*th term for a series of sequences.

ANALYSIS

Demonstrating TPK: generating questions randomly as classroom management

In the interview, Alice demonstrated technological pedagogical knowledge (TPK), articulating how she uses her knowledge of the existence and capabilities of the PowerPoint and spreadsheet software to enhance her pedagogy. She explained how her use of technology enhanced her classroom management, helping her to maintain students' engagement in the tasks she set them and contributing to her smooth handling of the lesson. In particular, the downloaded spreadsheet had an important pedagogic advantage over non-ICT resources such as a textbook or paper worksheet: it incorporated a button that when clicked would re-generate all the questions to be different. For Alice, this was the "cleverness of the spreadsheet…, the thing that I couldn't have written personally" which meant that, during the lesson, she could prevent one student from copying another without a disruptive intervention such as moving her to another seat. Alice used her knowledge of this capability of the spreadsheet to allow her to maintain a less intrusive style of classroom management.

Alice also identified the provision of immediate feedback as a significant feature of the spreadsheet exercise in enhancing her teaching as compared with traditional tools. When the students entered a potential *n*th term for a sequence, the spreadsheet provided immediate feedback: 'well done' for a correct answer and 'try again' for an incorrect one. She explained that the spreadsheet improved pupils' confidence, thereby having a positive impact on their engagement and productivity.

> Once they've done three or four, they know they can get the next few right. It tells them immediately that they have got them right, and then they feel that here's something I can do.

Linked to increasing the students' engagement and productivity, the immediate feedback from the spreadsheet enhanced Alice's capacity for effective classroom management. It freed her from constant requests from students asking for validation, allowing her to target her own skills more efficiently to ensure the smooth running of the lesson.

> …they must all have done more than 18 questions. Now with that group, that's quite a lot of questions for them to have done in a 10 minute time, because of this thing that they tend to stop after one question and wait for reassurance before they carry on to the next.

Significantly, Alice did not indicate how the mathematical knowledge she makes available to her pupils in the lesson is altered by the transformation of pedagogical techniques she is able to enact through her knowledge and use of technology. Indeed,

during the interview, she appears to suggest that the mathematical content of the lesson remains unaltered, identical to a lesson conducted without digital technology, using a traditional whiteboard and textbook exercise on sequences.

> I used presentation software with a little bit of interaction in it, you know, a few claps when they got something right, and I could just as well have done that on the board though it might not have [appeared] so neatly and it wouldn't have looked so neat, but the spreadsheet that they used, that was essentially just like doing a series of questions from a book except that they got immediate feedback.

Alice's focus on enhancing her pedagogy through her use of technology indicates a lack of depth in her consideration of the changes to the mathematical content made available to the students through her teaching using technology. Her apparent demonstration of TPK in fact serves to highlight the shallowness of her TPACK, since her belief that the mathematical content of the lesson remains unaltered suggests a weakness in the transformation of her *mathematical knowledge* for teaching using technology. Thus it is not that Alice has a thorough grasp of TPK but has yet to integrate her knowledge of mathematics with her knowledge of technology and pedagogy to achieve TPACK. Rather it is that the *depth* of her mathematical knowledge is insufficient to appreciate and critique the changes in her teaching of mathematics brought about by her use of technology. For example, Alice's use of the capability of the spreadsheet to randomly generate a set of questions to enhance her classroom management suggests an explicit disregard for the pedagogic advantages and disadvantages of choosing specific examples over others. Indeed, she explained during the interview that what this class needed was "lots and lots of questions that are all identical, so it builds confidence". Rowland et al (2009) suggest that random generation of examples might be reasonable as a means of demonstrating the efficacy and general application of an established method. However in this lesson, Alice's aim was to counter a particular misconception she had noticed in the pupils' recent exam, namely that if a linear sequence has a common difference of a between one term and the next, then it has an nth term of $n + a$. Random generation of examples may be inappropriate here since it may give rise to examples like $3n + 3$ which obscure the role of variables and may unintentionally act to reinforce such a misconception. In addition, Alice did not explain the benefits of the immediate feedback provided by the spreadsheet in terms of the mathematical insight her pupils might gain. Instead, she struggled to find a rationale based on mathematics pedagogy, hoping that the pupils' increased productivity might improve retention, whilst acknowledging that it might not. Thus for Alice, the significance of the spreadsheet's provision of immediate feedback lay solely in enhancing her capacity for classroom management and not in the possibility of altering the mathematical knowledge made available to her pupils through her teaching.

Demonstrating TCK: identifying discrepancies in spreadsheet notation

Alice also demonstrated TCK in her lesson and interview, recognising discrepancies between standard algebraic notation and the algebraic input accepted by the

spreadsheet as a valid *n*th term. Articulating these discrepancies demonstrates Alice's understanding of how the capabilities of the software may alter the presentation of mathematical content, hence her TCK. During the lesson, she raised the pupils' attention to the issue that the spreadsheet would, for example, only accept $3n + 0$ as a valid *n*th term for the three times table, rejecting the standard $3n$ as invalid. In this instance, she suggested they ignore the spreadsheet, remembering that for the exam they would need to write $3n$. In another departure from standard notation, the spreadsheet accepted both $1n + 5$ and $n + 5$ as equally valid answers. Alice did not raise this issue with pupils in the lesson. During the interview, she explained why she had raised one issue but chose to ignore the other.

> Coming back to the [GCSE] exam, I think they would get the mark for 1n+5 and one other mark for n+5, so the fact that the spreadsheet would take either didn't seem to me to be a problem. I thought it was more of a problem […] it wouldn't take 3n, it would only take 3n + 0. That is a problem because obviously, you know, because 3n+0 is not nearly as good an answer as 3n.

Thus she intentionally overlooked this discrepancy between standard algebraic notation and the spreadsheet notation, whilst drawing attention to the issue of the spreadsheet accepting $3n + 0$ but rejecting $3n$. This suggests an explicit ignorance on Alice's part of the pedagogic advantages or disadvantages of her *choice of examples* (Rowland et al., 2009). In addition, by asking students to ignore the spreadsheet, Alice reinforces her position of authority as the source of mathematical knowledge, undermining her argument that the immediate feedback provided by the spreadsheet can act as an alternative source of mathematical knowledge for the students to rely on. There is no point in the students following the spreadsheet's instruction to 'try again' when they appear to get a question incorrect, since it may be the spreadsheet in error. Instead, from the students' point of view, they are better off turning once again to Alice for ultimate validation. Further, by asking students to ignore the spreadsheet and rely instead on her judgement of what is expected in the exam, she misses an opportunity to examine *why* $3n$ may be conceived as an equally valid, if not better notation for the *n*th term of the three times table. She therefore misses the opportunity to build her students' ability to rely on themselves as a source of mathematical knowledge. Thus it seems the depth of Alice's *mathematical knowledge* is insufficient for her to recognise the pedagogic value in discussing explicitly the discrepancies spreadsheet and standard algebraic notation. In particular, Alice's demonstration of TCK serves to highlight the shallowness of her TPACK, again indicating a lack of depth in her consideration of the potential changes to the mathematical content made available to the students through her teaching using technology. Although she recognises using technology may lead to alterations in the presentation of mathematical content, she fails to consider the implications for mathematics pedagogy of such alterations. That she does not see the changes to mathematical content through technology use as impacting on her teaching of mathematics suggests a weakness in the transformation of her *mathematical knowledge* for teaching using ICT. Significantly it is not that Alice has a thorough grasp of TCK but has yet to integrate her pedagogical knowledge with her knowledge

of technology and mathematics to achieve TPACK. Instead, it is that the *depth* of her mathematical knowledge is insufficient to appreciate and develop the changes in the presentation of mathematical content through technology use for pedagogic purposes.

DISCUSSION

A major advantage of the TPACK framework is that by emphasising technology as a knowledge domain alongside pedagogy and content knowledge, the existence of teachers' mathematical knowledge for teaching using technology is highlighted through the central TPACK construct. To an extent Alice exhibited some level of TPACK. She demonstrated sufficient mathematical knowledge to select appropriate technological resources to teach the given mathematical topic with some degree of competence to her students. However, her demonstrations of TPK and TCK both serve to highlight the shallowness of her TPACK, by indicating a lack of depth in her consideration of the potential changes to the mathematical content made available to the students through her teaching using technology. Importantly, in each case it was not that she had a thorough grasp of the dyadic components, TPK and TCK, but had yet to integrate her knowledge of content and pedagogy respectively. Instead, it is that the *depth* of her mathematical knowledge was insufficient to appreciate and develop the changes in the presentation of mathematical content through technology use for pedagogic purposes. Explicit recognition of how changes in the presentation of mathematical content could be transformed for pedagogic purposes would entail a deepening of Alice's existing mathematical knowledge for teaching using technology. The analysis presented above suggests that the central TPACK construct may be better understood, not as a new category of knowledge representing the integration of technology, pedagogy and mathematical knowledge, nor as an orientation towards using technology, but rather as a transformation and deepening of existing *mathematical knowledge* for teaching using technology. A further hypothesis is that the dyadic constructs TPK, TCK and also PCK may not exist as distinct categories of knowledge in the actuality of classroom practice. However, these constructs do provide useful analytical tools for identifying weaknesses in teachers' *mathematical knowledge* for teaching in the context of a particular technological tool.

Notes

1. GCSE stands for General Certificate of Secondary Education.

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A MATHEMATICIAN'S DOUBLE SEMIOTIC LINK OF A DYNAMIC GEOMETRY SOFTWARE

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When a person works on a task using dynamic geometry software (DGS), a double semiotic link is recognizable between this software and both the task and one's mathematical knowledge. In this paper, a mathematician's double semiotic link of a DGS is discussed. The participant imposed the system of Euclid's Elements on DGS. This influenced how he used the software to accomplish the tasks given to him. At the same time, his perception on DGS was also shaped by these tasks. It is the author's hope that this paper could initiate further discussions on the nature of geometry (or geometries) embedded in DGS.

INTRODUCTION

Theoretical framework and research focus

Dynamic geometry software (DGS) is an artifact which carries mathematical meanings. It is a "tool of semiotic mediation" for experiencing the development of mathematical theory (Mariotti, 2000). Bartolini Bussi & Mariotti (2008) points out that a "double semiotic link" between this artifact and both the task and mathematical knowledge is recognized when it is used to accomplish a specific task. They further point out that:

> The main point is that of exploiting the system of relationships among artifact, task and mathematical knowledge. On the one hand, an artifact is related to a specific task … that seeks to provide a suitable solution. On the other hand, the same artifact is related to a specific mathematical knowledge. (Bartolini Bussi & Mariotti, 2008, p.753)

The 'interaction' between these two semiotic links is shaped by and shapes one's perception on the mathematical meanings embedded in DGS. In this paper, a mathematician's double semiotic link of a DGS is discussed. It reveals the complexity of the development of one's utilization and understanding on DGS.

METHODOLOGY

The data reported in this paper was collected as part of the author's Ph.D. study (Chan, 2009). It aims at investigating the participants' working processes of DGS explorative tasks. An ethnographic investigation approach is adopted. Samuel (pseudonym), as one of the participants, is a male university mathematics teacher. He obtained a Ph.D. degree in mathematics. He did not know how to use Sketchpad (a DGS) before he participated in this research study. He met the researcher (the author of this paper) once a month in a year. In each of the meetings, he worked on a geometric explorative task by using Sketchpad. After that, a semi-structured interview was conducted in order to clarify his mathematical thinking during the working process. After he finished 10

sessions of explorative tasks, a 'round-up' semi-structured interview about his perceptions on using DGS to explore geometrical problems was conducted. All the sessions were video-recorded and the interviews were audio-recorded.

SEMIOTIC LINK BETWEEN SKETCHPAD AND EUCLID'S ELEMENTS

Samuel regarded Sketchpad as a computational tool which embeds the system of Euclid's *Elements*. He tried to develop a semiotic link between Sketchpad (the artifact) and the system of Euclid's *Elements* (mathematical knowledge). This is evidenced from the following excerpt of interview transcript[1]:

Samuel: Euclid's *Elements* is a kind of background knowledge. A kind of… I do not want to use the word 'culture'; it is a kind of… I would regard it as a fundamental understanding.

Samuel: It does not necessary to be a specific theorem or a specific construction but a kind of analogy. For instance, the book [Euclid's *Elements*] mentions about A, B, C; then, it may lead to an association with A", B", C" -- this kind of constructional model. This is a sense or an intuition based on Euclid's *Element*.

While trying to impose the system of Euclid's *Elements* on Sketchpad, he realized that it is not so straightforward. He thought that some Sketchpad commands may violate the axioms in Euclid's *Element*. Doing actual measurement is one such example.

Interviewer: You avoided measurement. Am I correct?

Samuel: [I] avoided actual [direct] measurement. If it [Sketchpad] can provide a method to measure the ratio [of the lengths] rather than measuring the actual lengths, I will consider.

Interviewer: But, it is not OK if it gives 1.5 inches, right?

Samuel: It is not acceptable because I do not know whether 1.5 inches is actually 1.500031100009 [inches].

Samuel's avoidance of using measurement tools directly is consistent to how Euclid's *Elements* view 'measurement'. Hartshorne (2000) points out that numbers and magnitudes are regarded as two different things in Euclid's *Elements*. The former are positive integers whereas the latter are geometrical quantities. 'Ratio' in Euclid's *Elements* is neither a number nor a magnitude but a way to define the concept of 'proportion' by comparison of magnitudes. Apart from his avoidance of using measuring tools, Samuel developed his own 'priority list' of Sketchpad commands according to *his* semiotic link between Sketchpad and the system of Euclid's *Elements*. The following excerpt of interview transcript describes the principle of his priority list.

Samuel: My principle is: if possible, we should try best to use compass and ruler.

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[1] All the interviews were conducted in Cantonese (mother tongue of both the researcher and the participant). The transcripts reported in this paper were translated by the author.

Interviewer: Why?

Samuel: It is because you do not know whether some other things would be produced by using these tricky methods [i.e. using tools other than compass and rulers to do construction].

Interviewer: What do you mean by "other things"?

Samuel: That is…during the process of exploration, some circular arguments may be used. I am not sure whether it would occur or not, but this is a sufficient reason to avoid using them.

Interviewer: How does Euclid's *Elements* influence your way to construct figures and do exploration by using Sketchpad?

Samuel: I tend to use some specific Sketchpad commands more frequently… more frequent to use…have a higher priority to use them.

Interviewer: For instance?

Samuel: The main….. Maybe, let me describe the criterion [of selecting Sketchpad commands]. If it is explicitly stated in Euclid's *Elements* as…. it is called….. the postulate or notion, that is what we call axiom [nowadays]… it says that it can be used, then I will use it in higher priority.

Samuel: For instance, [we] can construct a circle, and… [we] can join [two points by] a line. I will use [these commands] repeatedly. Next, those propositions, according to the sequence in Book I… this is not an accurate way of saying... I should say, mainly according to its logical order. For instance, Proposition 1 will be used in higher priority. Proposition 1 says [that we] can construct an equilateral triangle. And then, Propositions 2 and 3, which says how to use compass and ruler to do addition and subtraction. And next, at later [propositions], construct perpendicular lines. Some things like it… in a specific order. And the parallel line which is [stated] at later part [in Euclid's *Elements*], as far as I remember, I seldom construct a parallel line by applying the command directly.

In short, Samuel adapted Sketchpad so that this artifact fitted to *his* semiotic link between this artifact and its embedded mathematical knowledge.

SEMIOTIC LINK BETWEEN SKETCHPAD AND TASK

When a task was given to Samuel, a semiotic link between Sketchpad and this task was established. For some tasks, this semiotic link was inconsistent to his semiotic link between Sketchpad and the system of Euclid's *Elements*. The following two excerpts of exploration episodes illustrate the inconsistency and describe how he overcame it.

Excerpt 1

The following task given to Samuel is based on *Haruki's Cevian Theorem for circles* (Honsberger, 1995, p.144-145) which states that:

Three circles intersect each of the others in two points where A, B, C, D, E and F are the intersection points. We have $\frac{AB}{BC} \times \frac{CD}{DE} \times \frac{EF}{FA} = 1$.

The task

The figure shows three circles intersect each of the others in two points. *A*, *B, C*, *D*, *E* and *F* are the intersection points. *X*, *Y*, *Z* are points that control the sizes of the circles and *P*, *Q*, *R* are the centers of the circles. The aim of this task is to find a relation connecting the lengths of the line segments *AB*, *BC*, *CD*, *DE*, *EF* and *FA*.
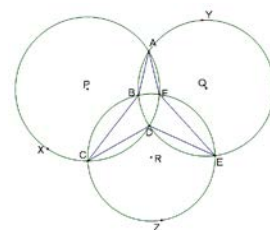
Samuel's solution process

As this task involves the lengths of various line segments, a natural approach of starting up the exploration is to use measuring tools in Sketchpad. However, this is inconsistent to Samuel's semiotic link between DGS and mathematical knowledge. He needed to find a way to resolve this inconsistency.

> Samuel: What is [the meaning of] measuring lengths? It is to draw circles!

After thinking for a while, he constructed three pairs of concentric circles with different colours and then dragged some points for a while. Then, he discovered a property of this configuration:

> Samuel: If AB = AF, it seems that the other four circles, which are indeed two pairs of circles, i.e. the two pairs of circles with center C and center E respectively, have radii in same proportion… it seems to be the same. Is there anything symmetric?
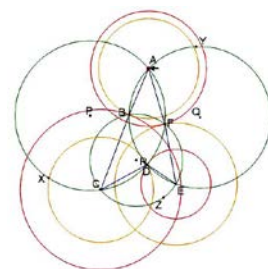
What Samuel meant is indeed a special case of *Haruki's Cevian Theorem*, i.e. BC: CD = FE: DE when AB = AF.

Analysis

In this excerpt, 'measurement' constitutes a semiotic link between DGS and this task. However, as Samuel did not want to use measuring tools in Sketchpad because it did not match with his semiotic link between DGS and the mathematical knowledge, he re-interpreted the meaning of measurement (according to Euclid's *Elements*) as 'drawing circles'. (The first few propositions in Book I of Euclid's *Elements* discuss about 'operations' of magnitudes by using circle-constructions.) He also realized that this task does not really involve measurements but only the ratios of magnitudes. (Ratio, as a concept of proportion, is discussed extensively in Book V of Euclid's *Elements*.) The consistency of the semiotic links between DGS and both the system of Euclid's *Elements* and the task is restored. Samuel used circle-constructions and dragging, instead of applying the measuring tools directly, to explore the ratios of the lengths.

Excerpt 2

The following task given to Samuel is based on the concept of radical axis of two intersecting circles which is their common secant (Posamentier, 2002, Theorem 3.11).

The task

The figure shows two intersecting circles. Let *P* be a point. Lines *PA* and *PB* are tangents to the two circles at points *A* and *B* respectively. Find the locus of *P* such that the distance between *P* and *A* is equal to the distance between *P* and *B*, i.e. *PA=PB*.
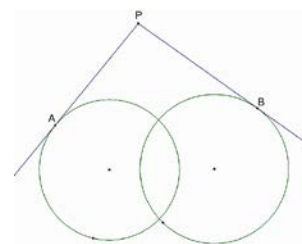
Samuel's solution process

This task involves two mathematical concepts: distance and locus. Similar to Excerpt 1, Samuel used circle-constructions to handle 'measurement' of distances. He constructed two concentric circles both centred at A and have radii PA and PB respectively.

> Samuel: AP and PB are equidistant if and only if these two
>   circles are overlapped.

He dragged *P* so that the two circles coincide exactly. He tried to keep these two circles 'overlap' while dragging point P. He emphasized that the two circles in questions are kept unchanged.
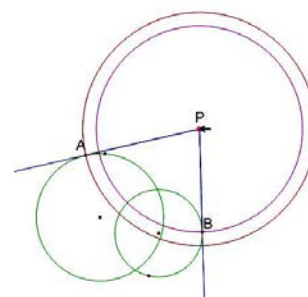
> Samuel: Now, note that I have not 'touched' the two green circles. I only change the
>   position of P. While changing the position of P, the positions of A and B will
>   also be changed. And then the lengths of PA and PB will also be changed.
>   But I also require to keep PA and PB approximately the same lengths. Let me
>   try to use… see whether I can use the *trace*
>   function [command]. Try to use.

He used 'trace' command to keep track of the dragging path which provided him a visual clue to discover the required locus. After dragging for a while, he guessed that the locus is a straight line.
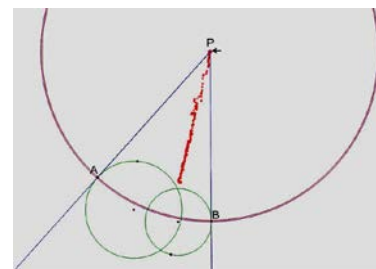
Analysis

Distance and locus constitute a semiotic link between DGS and this task. Samuel's measuring method is consistent to the concept of 'measurement' in Euclid's *Elements. In terms of measurement,* the semiotic link between DGS and the system of Euclid's *Elements* is consistent to the semiotic link between DGS and the task.

> In Euclid's *Elements* there is an undefined concept of equality (what we call congruence)
> for line segments, which could be tested by placing one segment on the other to see
> whether they coincide exactly. In this way the equality or inequality of line segments is
> perceived directly from the geometry without the assistance of real numbers to measure
> their lengths.   (Hartshorne, 2000, p.461-462)

To find the locus, Samuel dragged the points and used 'trace' command. This dragging technique is usually called *dummy locus dragging* (Arzarello, Olivero, Paola, and Robutti, 2002). Dragging plays a special role in Samuel's exploration process. On the one hand, Samuel seems rather comfortable to do dragging. On the other hand, dragging looks inconsistent to the system of Euclid's *Elements*. (See for example, Lopez-Real & Leung, 2006.) It is worth to investigate how Samuel interpreted the meaning of dragging.

INTERPRETATION OF DRAGGING

The following excerpt of interview transcript reveals Samuel's interpretation of dragging in DGS:

Samuel: Although [result in] Sketchpad is just an approximation, it is actually a simulation of R^2. In most cases, questions in R^2 involve properties of continuous [objects]. Of course, each individual diagram is discrete but the underlying construction may depend on some variables. For instance, although I did not do measurement, I realized that some variations of points may control the lengths of some line segments. These are some things that change continuously. So, ultimately, this is the concept of function.

Interviewer: Does the operations in Sketchpad match your way of thinking - function?

Samuel: At the level of 'theatre', it is. As a tool, it is useful. Dragging is a way to express how a function changes. In Sketchpad environment, [the purpose of] dragging is to control variables and to see different outputs. The dragging process simulates a function - an abstract function. Dragging is really something new to me.

Interviewer: How does it [dragging] influence your way of exploration?

Samuel: It is a kind of sensational stimulation. [It is] a tricky method that can help thinking.

Interviewer: It is an interesting idea. Can you say more?

Samuel: It is tricky because it is not conventional. You cannot drag [a geometric object] in paper-and-pencil [environment]. In paper-and-pencil [environment], you can only draw different discrete cases. However, if you have not used any invalid things in your Sketchpad construction, dragging is an acceptable tricky method.

Interviewer: OK, why can it help you to think?

Samuel: It [Dragging] displaces a function.

The above excerpt of interview transcripts describes Samuel's 'mental struggle' on the two semiotic links. On the one hand, dragging is a useful tool for working the tasks. It gives a semiotic link between DGS and the tasks. On the other hand, Samuel also realized that one cannot drag in paper-and-pencil environment. It seems that he thought that dragging makes DGS unable to link up with the system of Euclid's *Elements*. He re-interpreted the meaning of dragging as "a displacement of a function". Dragging

becomes a "tool of semiotic mediation" (Bartolini Bussi and Mariotti, 2008) of the concept of function. His understanding on dragging is consistent to findings in existing literatures (see for example, Falcade, Laborde & Mariotti, 2007). Dragging is a new experience to Samuel. It re-shaped his understanding on the mathematical meanings carried by DGS. Initially, Samuel imposed the system of Euclid's *Elements* on DGS and established a semiotic link between the artifact DGS and mathematical knowledge. The dragging experience (initiated by the semiotic link between DGS and the tasks given to him) 'modified' his semiotic link between DGS and mathematical knowledge. It 'extended' his understanding on the geometry (and more generally, mathematical meanings) embedded in DGS.

DISCUSSION AND CONCLUSION

In this paper, a case study about 'interaction' of a mathematician's double semiotic link in a DGS is discussed. On the one hand, Samuel thought that Sketchpad, as a DGS, is a computational tool for the system of Euclid's *Elements*. He adapted the software (by developing his own 'priority list' of Sketchpad commands) so that this software fitted to his semiotic link between DGS and mathematical knowledge. On the other hand, while working on the explorative tasks, he experienced the powerfulness of dragging and developed a new understanding towards DGS. A new semiotic link between DGS (the artifact) and mathematical knowledge is established. For instance, dragging is regarded as a "sign" (Bartolini Bussi & Mariotti, 2008) of the concept of 'function'. It is worth to note that 'function' as a concept is not included in Euclid's *Elements* but has been emerged at the end of the 17th century. In other words, Samuel's understanding on the mathematical meanings of DGS has been enriched. This is evidenced from the following excerpt of interview transcript conducted after the ten explorative sessions:

Samuel: It [The geometry in Sketchpad] may cover Euclid, I guess. Sketchpad may most likely contain Euclid's *Elements*. Will there be something outside Euclid *Elements*? I am not sure.

Interviewer: What do you mean by "contain"?

Samuel: [Samuel drew a diagram to illustrate his idea.]

Interviewer: Is there any reason for one [circle] is larger than another?

Samuel: Yes, it is true that one is larger and another is smaller.

Interviewer: So, why is DGS so large? At least, it is larger [than Euclid's].

Samuel: You can do anything by using computer programming skills, at least at the level of approximation.
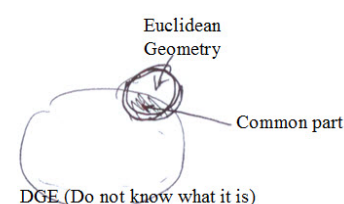
What is geometry (or geometries) of DGS? What is the relationship between Euclid's *Elements* (or more generally, Euclidean geometry) and DGS? Does DGS provide an opportunity to learn *new* geometry? These questions are topics of interest at least since the 1990s (for example, Hölzl, 1996; Lopez-Real and Leung, 2006; Straesser, 2001)

but there are no conclusive answers so far. It is the author's hope that this paper could initiate further discussions on these fundamental questions.

Acknowledgement

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DO OUR FIFTH GRADERS HAVE ENOUGH MATHEMATICS SELF-EFFICACY FOR REACHING BETTER MATHEMATICAL ACHIEVEMENT?

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The main purpose of this study was to examine the effects of fifth-graders' MSE on their mathematical achievement in school, as well as examining the effects of family socio-economic status, and parenting styles on MSE. A students' background sheet and a mathematics self-efficacy instrument were administered to 1244 fifth-graders for gathering data, associated with their mathematical achievement scores in school. Corresponding statistical analyses were applied to the obtained data. The findings showed that fifth-graders' family SES and parenting styles were ascertained as critical elements in the development of their mathematics self-efficacy. It also revealed that their MSE ratings could effectively predict their mathematics achievement. Consequently, suggestions derived from findings and discussions were proposed for further improvement of these fifth-graders' mathematics self-efficacy and the future study.

INTRODUCTION

Self-efficacy (SE) had a great influence on one's task choices, effort, persistence, and achievement (Bandura, 1997). Students who are self-efficacious in learning are likely to put forth more effort, persist longer if they have learning difficulties, be more flexible, and, ultimately, reach a higher level of success. Several studies also found that students' self-efficacy is positively correlated with their academic achievement in various content domains and in different levels of academic settings (Bandura, 1997; Lent, Lopez, & Bieschke, 1991; Multon, et al., 1991; Pajares, 1996; Schunk & Miller, 2002). Evidence also has showed that students' self-efficacy can have a direct influence on their academic achievement and performance (Pajares & Miller, 1994). In fact, students with higher efficacy beliefs performed better and persist longer in the face of learning difficulties or occasional setbacks (Chang, 2010). Similar findings revealed that this superior academic performance came from applying more effective learning strategies (Pintrich & Degroot, 1990). It follows, logically, that students with a strong sense of SE would approach learning difficulties as challenges to be conquered and have a strong commitment to goals they establish, which then results in better academic performance.

Regarded to mathematics, Pajares and Miller (1994) examined the role of mathematics self-efficacy (MSE) in mathematical problem solving for college students, yielding that students' MSE was significantly predictive of their problem solving in mathematics. Similar result was found from the study of Pajares and Kranzler (1995) for high school students', their MSE had a direct effect on their mathematics problem solving skills. Besides, MSE could predict adolescent mathematics achievement (Lent,

et al., 1991; Pajares & Miller, 1994; Pajares & Kranzler, 1995). Betz and Hackett (1989) also showed that students' MSE was significantly correlated to their mathematical achievement. Recently, Kitsantas, Cheema, and Ware (2011) conducted a series of data from Program for International Student Assessment (PISA) and illustrated that 15-year-old students' MSE was a predictor of mathematics achievement in addition to gender, race, relative time spent on mathematics homework, and homework support, which accounted for an additional 20% of the total variation in mathematics achievement. In summary, MSE beliefs had a powerful impact on the level of accomplishment they might ultimately achieve in learning mathematics.

Given the robust literature regarding the effects of MSE on mathematics achievement for adolescents, little knowledge, however, was shown for children. It was showed that self-efficacy began to decline in grade 7 or earlier (Anderman, Maehr, & Midgley, 1999; Urdan & Midegley, 2003), particularly evident in mathematics at the transition to middle school (Jacobs, et al., 2002). For fifth and sixth grades, children are positioned right at the developmental transition period, in which they encounter with significantly psychological, physiological, and social changes. Since new challenges await them in this fast-growing stage (Schunk & Meece, 2006), how to prevent this possible decline becomes more beneficial to their mathematical learning. Especially in Taiwan, no evidence was found in assessing these students' MSE along with their mathematics achievement (Chang & Wu, 2010). Accordingly, the main purpose of this study is to assess the effect of MSE on mathematical achievement of fifth-graders, who are at the beginning stage of this transitional period.

In order to obtain better predictive and possibly explanatory results, items for assessing SE should be context and task specific (Zimmerman, 2000) and designed by using a multidimensional construct (Bandura, 2006; Pajares, 1996). Based on Bandura's (2006) guidelines and his multidimensional scales, the first set of questions, "General Self-Efficacy—Related Mathematics (GSE-M)" subscale, was designed to assess children's general SE that is relative to their mathematical learning, including items of enlisting social resources & parental support, academic achievement, self-regulated learning, and meet others' expectations. Additionally, in Taiwan, children's mathematical learning in the higher-elementary grades begins to be more focused on the knowledge memorized and the skills used, which leads to more test-oriented learning activities. Consequently, the second set of questions, named as "Self-Efficacy for Mathematical Learning (SEML)", was designed more contextually to measure children's realistic learning situations both in and after school, including mathematics cognitive, strategy, and test preparation items. Beside the MSE whole scale, it is also intended to examine the effects of the two subscales on mathematical achievement.

Moreover, two variables were generalized from previous research findings as the contextual factors of the development of students' MSE: family's socio-economic status (SES) and parenting style. In regard to the SES factor, economic hardship and low parental education were positively correlated to difficulties in students' learning (Bradley & Corwyn, 2002; Schunk & Miller, 2002). More importantly, students

usually obtained certain amount of the self-efficacy from their families and home environment (Schunk & Miller, 2002), and thus family's economic status and parental education might influence the development of their children's MSE. Another variable, parenting style, would also had a great influence on students' SE (Schunk & Meece, 2006). Among the four major types of parenting styles identified by Marccoby and Martin (1983), the authoritative-reciprocal parenting style had the best combination of warmth, responsiveness, and control that would foster the development of children's SE (Schunk & Meece, 2006). However, less empirical evidence existed in supporting the effect of parenting styles on children's SE or MSE, especially in Taiwan. Consequently, it is essential to investigate the possible effect of different parenting styles to students' MSE. Consequently, family's SES and parenting styles were included as the background variables for further analyses.

Based on the background and motivation stated above, the two purposes of this study are as follows: (a) to investigate the effects of family factors (SES and parenting style) on MSE; and (b) to assess the effects of MSE on mathematical achievement which were converted to mathematical achievement T score (MA-T). Based on foregoing purposes, this study has three research hypotheses as follows:

- H1: SES has a significant effect on MSE.
- H2: Parenting style has a significant effect on MSE.
- H3: MSE significantly predicts MA-T.

METHOD

A total of 1244 fifth-graders were selected by a stratified random sampling method (by school size) in elementary schools in Taiwan. Based on the purposes of this study, data were collected through a background sheet, MSEI, and their mathematics achievement in school. Students' background sheet mainly delineate students' basic information, family's SES, and parenting styles, which was sent home and filled out by student's parents with a consent form. The family's SES was calculated according to the rules of Lin's (1982) framework, with the equation of "SES = Parents' occupation index $\times 7 +$ Parents' education index $\times 4$". For parenting styles, a dual-dimensional system identified by Maccoby and Martin (1983) was applied with four types of statements for parenting (authoritarian-autocratic, indulgent-permissive, authoritative-reciprocal, and indifferent-uninvolved patterns). Also, mathematical achievement in school was represented in terms of their overall mathematics scores at the fifth-grade level. Mathematics scores, named as mathematical achievement T scores (MA-T), were collected at the end of the school year and then transformed into T scores for further analyses. To measure MSE, Mathematics Self-Efficacy Instrument (MSEI) was developed on the basis of Bandura's (1977, 2006) theory and his guidelines, which consists of 24 items for "General Self-Efficacy—Related Mathematics (GSE-M)" and 23 items for "Self-Efficacy for Mathematical Learning (SEML)", rated on a 100-point scale. MSEI has high internal consistency of .96, .93, and .95 for the total scale,

GSE-M, and SEML subscales respectively (Chang & Wu, 2010). Also, GSE-M and SEML accounted for 27.68% and 20.41% of variance, respectively. Both subscales significantly correlated, $r = .74$, $p < .001$.

RESULTS

The effects of fifth-graders' SES, and parenting styles on MSE

The mean rating of all 1244 fifth-graders on MSE was 69.84, which meant that on average they had nearly 70% confidence in their own mathematics learning abilities. Regarding the effect of SES on MSE, the results showed that there were statistically significant differences in fifth-graders' MSE ratings among the three revised levels of SES, $F (2, 1241) = 6.75$, $p < .01$. The post hoc comparison based on LSD concluded that fifth-graders with the high SES ($M = 72.06$) scored significantly superior in MSE than did those with medium ($M = 69.72$) and low SES ($M = 66.67$). In addition, fifth-graders with the medium SES scored higher MSE than did those with low SES. Accordingly, H1 was supported in this study.

In regard to the effect of parenting style on MSE, the results demonstrated statistically significant differences in fifth-graders' MSE ratings among the four types of parenting styles, $F (3, 1240) = 12.881$, $p < .001$. The post hoc comparison based on LSD yielded that fifth-graders under the discipline of the authoritative-reciprocal parenting pattern (about 71%) tended to possess greater MSE ($M = 71.75$) in learning mathematics than those with other three parenting patterns. Besides, fifth-graders under the discipline of the authoritative-autocratic parenting pattern (about 17%) were likely to possess greater MSE ($M = 71.75$) that those with indifferent-uninvolved parenting pattern ($M = 62.46$). Accordingly, H2 was also supported in this study.

The effects of fifth-graders' MSE on MA-T

To determine whether students' MSE (containing both GSE-M and SEML) could predict their mathematical achievement, multiple regression analyses of GSE-M and SEML regressing on MA-T were conducted. The findings showed that GSE-M and SEML significantly predicted MA-T, $F (2, 1241) = 171.23$, $p < .001$, suggesting that 21.6% of MA-T variance was explained by GSE-M and SEML. The standardized regression coefficients indicated that SEML ($B = .30$, $t = 5.41$, $p < .001$) had greater effects on MA-T than GSE-M ($B = .18$, $t = 3.34$, $p < .01$). In brief, these findings indicated that fifth-graders with the higher MSE would get higher scores on MA-T in school. Therefore, H3 was supported in this study.

DISCUSSION

The influence of fifth-graders' backgrounds on their MSE development

First of all, the result showed that fifth-graders with low SES tended to have lower confidence in their own capability while learning mathematics, which is accordant

with the findings of Bradley and Corwyn (2002). Parents with lower family income and less educated usually have inadequate capital in assisting their children's cognitive development (Schunk & Meece, 2006), which might result in less supplementary resources in learning mathematics. Besides, parents' income levels were positively related to their expectancies of their children's both current and long-term educational attainments (Alexander & Entwisle, 1988). Consequently, "dropout" might be the predictable condition for low SES students (Sherman, 1997). In short, if low parental expectancy does exist, children at home might not obtain sufficient psychological support and behavioral help, which in turn jeopardizes children's MSE.

Secondly, it was revealed that fifth-graders under the authoritative-reciprocal parenting pattern scored higher in MSE than those with other patterns, which was also similar to previous research findings (e.g., Baumrind, 1991; Schunk & Meece, 2006). The authoritative-reciprocal parenting style is the best combination of warmth, responsiveness, and control to support their children's learning in school, which is a well-balanced parenting style. This type of parent is both demanding and responsive. As Baumrind (1991) stated, "They monitor and impart clear standards for their children's conduct. They are assertive, but not intrusive and restrictive. Their disciplinary methods are supportive, rather than punitive. They want their children to be assertive as well as socially responsible, and self-regulated as well as cooperative" (p. 62). Under such parenting style, children are assisted to not only enthusiastically confront learning challenges but also persist longer and solve learning problems effectively. In a word, this parenting style was gainful in promoting children's MSE, which would also bring on a positive impact on their mathematical achievement in school (Schunk & Meece, 2006).

As stated previously, there were a certain number (17%) of parents who used the authoritarian-autocratic parenting pattern. Although their children's MSE ratings were relatively higher (i.e. ranked second in this study), this type of parenting style is essentially problematic. Since the discipline under the authoritarian-autocratic pattern is more arbitrary (Maccoby & Martin, 1983), it often leads to a unidirectional parent-child relationship that prevents parents from understanding real thoughts in their children's minds or needs. Therefore, if their children struggle with mathematical learning problems in school, parents my not be able to provide efficacious support, which would be harmful for the development of MSE. In general, it is recommended that, in Taiwanese elementary schools, we must provide better authoritative-reciprocal models in our future parenting education, endeavoring to assist more parents in adapting their own parenting styles and then enhancing their children's MSE as well. Besides, this exploratory finding reminds us that there needs further investigations on the effect of parenting styles on children's MSE in Taiwan.

Fifth-graders' MSE had a great effect on their mathematical achievement

Averagely, these fifth-graders had nearly 70% confidence in their own mathematics learning abilities. Because "self-efficacy" was considered as one of eight powerful factors for students' learning performance (Bandura, 1977), which was evident in this study that the higher MSE the better mathematical achievement, how to increase or maintain the status of their MSE became more essential to help them be successful in learning mathematics in school both at this transitional period and in the future. As

mentioned above, the higher-elementary school grades in Taiwan begin to be more focused on the acquirement of the subject-matter knowledge, which prepare them for the intensive tests in school and the subsequent entrance examination three years later. Accordingly, two challenges are anticipated in learning mathematic at this transitional period: The content changes from more concrete to more abstract, and more test-oriented instruction and assessment replace hands-on activity and authentic assessment, which leads to more technical practices in mathematics. Also, students are gradually in the face of more competitive learning environment that is originated from the pressure of the entrance examination in the junior high schools. We should advocate enhancing students' self-efficacy in this fast-growing stage, to assist them both to manage and conquer these developmental and academic challenges, and then finally to achieve the intended learning content (Pajares, 2006; Schunk & Meece, 2006). Besides, based on the viewpoint of Urdan and Midgley (2003), a goal structure of elementary classrooms that emphasize one's effort, meaningful learning, and individual mastery would be beneficially enhanced or maintained students' efficacy and competence at this transitional period. As a result, effectively sustaining this positive learning environment would help to prevent possible declines of their MSE (Jacobs, et al., 2002; Wigfield, et al., 1997).

Furthermore, the results of multiple regression revealed that the both subscales (GSE-M and SEML) significantly predicted mathematical achievement with 21.6% variance. This finding of significant effects of MSE on mathematical achievement in school is corresponding to the previous studies (Lent, et al., 1991; Pajares & Miller, 1994; Pajares & Kranzler, 1995). It is also remarkable that SEML (the subscale) had greater effects on students' mathematical achievement in school. Therefore, this finding clearly indicate that the more efficacious on mathematics cognitive, strategy, and test preparation aspects the better mathematical achievement in school. Additionally, it implies that this context and task specific design is tailored to children's actual learning context in mathematics, which also conforms to Bandura's (2006) guideline and has better predictive and possibly explanatory results. Consequently, it is recommended that this instrument and its constructs are effective as a major reference in measuring students' MSE.

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PRACTICE-BASED CONCEPTION OF SECONDARY SCHOOL TEACHERS' MATHEMATICAL PROBLEM-SOLVING KNOWLEDGE FOR TEACHING

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Mathematical problem solving and contextual problems are central to doing and learning mathematics and should also be central to teachers' mathematical knowledge for teaching. This study investigated secondary school mathematics teachers' mathematical problem-solving knowledge for teaching from a practice-based perspective. Data obtained from several sources were analyzed thematically in terms of knowledge of problems, problem solving, problem solver, and instructional practice. Findings highlight a combination of conceptions of these knowledge that determines problem-solving proficiency in teaching and supports students' develop- ment of proficiency in problem solving, with implications for teacher education.

INTRODUCTION

Mathematical problem solving and contextual problems are central to doing and learning mathematics with meaning and deep understanding. Thus, they should be critical factors in understanding and addressing mathematical knowledge for teaching (MKT). This aspect of MKT is considered here as mathematical problem-solving knowledge for teaching (MPSKT). What should this knowledge include? In particular, what should teachers know to teach for problem-solving proficiency? What knowledge should teachers hold to help students to become proficient in problem solving? These questions provide the focus of this paper, which reports on a study aimed at developing a practice-based conception of MPSKT. The study investigated secondary school mathematics teachers' MPSKT *in* teaching in order to understand and conceptualize it in terms of the knowledge they held and used, how they used it, and the part it played in meaningful teaching with contextual/word problems.

THEORETICAL PERSPECTIVE AND RELATED LITERATURE

Significant contributions have been made by Deborah Ball and co-researchers (e.g., Ball, Thames, & Phelps, 2008; Hill & Ball, 2009; Hill et al., 2008; Hill, Schilling, & Ball, 2004; Thames & Ball, 2010) on the nature of MKT. They propose a view of MKT based on how it plays out in practice as a means to develop measures of teacher knowledge. This view includes knowledge of content and students, knowledge of content and teaching, specialized content knowledge, and common content knowledge. In particular, they suggest that general mathematical ability does not fully account for the knowledge and skills needed for effective mathematics teaching. A special type of knowledge is needed by teachers that is specifically mathematical, separate from pedagogy and knowledge of students, and not needed in other professional settings.

This specialized content knowledge uniquely enables "teachers … to do a kind of mathematical work that others do not" (Ball et al., 2008, p. 400). This work requires *decompressed* or *unpacked* mathematical reasoning, in addition to pedagogical thinking, demanding teachers to know *more* and *different* mathematics than what is needed by other adults, i.e., common content knowledge which allows a person to successfully solve mathematical problems in non-classroom contexts, including "being able to do particular calculations, knowing the definition of a concept, or making a simple representation" (Thames & Ball, 2010, p. 223).

While the work of Ball and colleagues offers an important view of MKT, it has not led to consensus regarding the nature of this knowledge. For example, in Rowland and Ruthven (2011a), which deals with "mathematical knowledge in teaching," the work presented by several authors reflects different perspectives about this knowledge of mathematics teachers and different ways of *knowing* within teaching. As the editors noted, the coherence of book "comes less from consensus on the issues and more from a collective understanding and appreciation of the different perspectives and convictions of the contributions as a whole" (Rowland & Ruthven, 2011b, p. 2). Thus, while MKT may have general characteristics as suggested by Ball and colleagues, the specifics involved are dependent on the aspect of mathematics education or mathematics teacher education one is interested in. In this paper the interest is in the specifics regarding MPSKT.

While problem solving (PS) is not explicitly considered in current studies of MKT as discussed above, it is implied as an integral part of mathematics. The intent here is to address it explicitly as MPSKT from a perspective of teachers' PS knowledge in teaching and for PS proficiency. PS could mean different things to teachers depending on their experiences with it as learners. For example, they could correlate it with solving routine word problems or rote exercises, a view that will not support student's development of PS proficiency. The position taken in the study being reported is that teachers need to hold knowledge from a perspective of PS proficiency for teaching.

Mathematical problem-solving proficiency

PS proficiency is being used to represent what is necessary for one to learn and do PS successfully. For example, according to Schoenfeld (1985, 1992), for successful PS, one must be equipped with and competently use appropriate resources, heuristic strategies, metacognitive control, and appropriate beliefs. PS proficiency is also being linked to the components of mathematics proficiency proposed by Kilpatrick, Swafford, and Findell (2001): conceptual understanding; procedural fluency; strategic competence (i.e., ability to formulate, represent, and solve mathematical problems); productive disposition; and adaptive reasoning (i.e., capacity of logical thought, reflection, explanation, and justification). Kilpatrick et al emphasize that these components are interwoven and interdependent in the development of proficiency in mathematics, which, then, should be the same for PS. They also explain,

> Problem solving should be the site in which all of the strands of mathematics proficiency converge. It should provide opportunities for students to weave together the strands of proficiency and for teachers to assess students' performance on all of the strands. (p. 421)

These notions of PS suggest that in order to help students acquire PS proficiency instruction should address all of these characteristics proposed by Schoenfeld and Kilpatrick et al. This thus leads to the consideration of the knowledge teachers should hold to support such instruction.

Teacher knowledge of and for Problem-solving proficiency

Based on the preceding discussion of PS proficiency, teachers should be equipped with those characteristics proposed by Schoenfeld (1985, 1992) and Kilpatrick et al. (2001) and hold knowledge of how and what it means to help students to become better problem solvers. Other research also highlights or implies the importance of holding knowledge of problems, problem solvers, PS pedagogy, the PS process, metacognition, and technology as a PS tool (e.g., Chapman, 2009). While these are key aspects of a teacher's knowledge, it is not the knowledge of itself, but knowing what to do with it – being able to use it, that is important. In particular, how this knowledge is held by the teacher is also important in terms of whether or not it is usable in a meaningful and effective way in supporting PS proficiency in his or her teaching. Thus, a teacher's knowledge of and for PS proficiency must be broader than competence in PS. In this paper, this knowledge is considered in terms of the following four categories:

1. *Knowledge of problems* – teachers should have conceptual understanding of "worthwhile mathematics tasks" (National Council of Teachers of Mathematics, 1991) and problems that will support proficiency in PS.

2. *Knowledge of problem solving* – teachers should have conceptual and procedural knowledge of mathematical PS. This includes understanding the stages problem solvers often pass through in the process of reaching a solution, that is, models of PS such as those of Polya (1957), Schoenfeld (1985; 1992), and Mason, Burton and Stacey (1982).

3. *Knowledge of students as problem solvers* – teachers need to understand students as problem solvers, for example, what constitutes productive beliefs and dispositions toward PS; what one knows, can do, and is disposed to do; and adequate level of difficulty of the problems assigned. They should have knowledge of skills students need to be competent technological problem solvers and how to evaluate students' PS process and progress.

4. *Knowledge of instructional practices* – teachers need to understand instructional practices for PS, including instructional techniques for strategies and metacognition. They must have strategic competence to face the challenges of mathematical PS during instruction. They must perceive the implications of students' different approaches, whether they may be fruitful and, if not, what might make them so. They must decide when and how to intervene – when and how to give help that supports students' success while ensuring that students retain ownership of their solution strategies; what to do

when students are stuck or are pursuing a non-productive approach or spending a lot of time with it; and what to look for. They will sometimes be in the position of not knowing the solution, thus needing to know how to work well without knowing all.

RESEARCH METHOD

This study is part of a larger, four-year, funded research project, involving 26 elementary and secondary teachers, with a focus on mathematics teachers' thinking and teaching of PS using contextual/word problems. The focus here is on the 11 practicing secondary school (grades 9 – 12) teachers. These teachers had 16 to 30 years of teaching experience. They were from different local schools and volunteered for the study. Six of them were considered in their school systems to be exemplary mathematics teachers. They had received teaching awards and/or were involved in co-authoring or reviewing mathematics textbooks and in leading professional development for other mathematics teachers. This combination of teachers turned out to represent a broad range of thinking and teaching approaches in regard to PS.

Main sources of data for the larger study were open-ended interviews, problem-solving tasks, classroom observations, teaching artifacts, and students' work. The interviews explored participants' thinking, knowledge, and experiences with contextual problems (CPs) and PS in three contexts: past experiences as students and teachers, current practice and knowledge, and future practice. This included the relevant prior knowledge, abilities, and expectations they brought to their experiences with PS in their teaching; current knowledge, task features, classroom processes and contextual conditions relating to PS; and planning and intentions for PS in their teaching. Participants were also given relevant, curriculum-based examples of different types of CPs (based on the six types of Charles & Lester, 1982) to solve, critique, and discuss use in their teaching and students engagement in them. This included:

> A road up one side of a hill is 12 km long, and it is 12 km down the other side. Suppose you can cycle up the hill at 6 km/h. How fast would you have to cycle down the other side to average 12 km/h for the entire trip?

Classroom observations and field notes focused on the teachers' actual instructional behaviors during lessons involving PS. Eight to ten lessons (60 to 85 minutes each) were observed and audio taped for each teacher. Post-observation discussions, when necessary, focused on clarifying the teachers' thinking in relation to their actions.

Data analysis involved the researcher and two research assistants working independently to thoroughly review the data and identify attributes of the teachers' thinking and actions that were characteristic of their conceptions of CPs and PS and teaching with CPs. Transcripts were read, initially to gain a general impression of the participants' thinking and then significant statements and behaviors were identified and coded. For the aspect of the study reported here, the coding was based on the four categories of knowledge described earlier in the theoretical perspective, i.e., problems, PS, learners, and instruction/teaching. The coded information was grouped by emerging themes of the teachers' knowledge and validated through an iterative process

of identification and constant comparison. The themes (e.g., CPs as computation, text, and experience) were then analyzed by comparing the significant statements associated with them for points of variation and agreement around which they could be grouped to form general perspectives of knowledge for each of the four categories. The findings reported here consist of these perspectives of knowledge for each category.

FINDINGS

A summary of the findings is presented in terms of the perspectives of knowledge held by the teachers collectively for each of the categories of teacher knowledge proposed in the theoretical perspective; the relationships among these categories of knowledge; and the combination of knowledge that is consistent with proficiency in PS and supported students' PS learning effectively and meaningfully.

Knowledge of problem – collectively, the teachers held six conceptions of CPs: computations, objects, text, problems, tools, and experience. These were categorized in terms of three philosophical perspectives of knowledge (Table 1).

Objectivist perspective	Utilitarian perspective	Humanistic perspective
CPs are: 1. Computations 2. Objects 3. Problems, i.e., algorithmic	CPs are: 1. Text 2. Tools, i.e., - illustrate concept - promote thinking - frame teaching	CPs are: 1. Experience 2. Problems, i.e., - depend on relationship with student - depend on teacher intent - non-algorithmic - meaningful algorithmic

Table 1: Perspectives of knowledge of problems

Brief descriptions are provided for the three less obvious items of Table 1 with quotes from participants. For objects, CPs can be generalized, e.g., by: "concept taught, for example, systems of equations," a pre-determined algorithm, and "type of problem [context], for example, coin, age, distance, number." They "have clear language, no extraneous information, clear about what they want, not ambiguous." For text, CPs are "[a way] to transfer information to somebody else;" "a way to share mathematical experience with another;" For experience, CPs become and provide lived realities for the students. The nature of the CP thus depends on how they are experienced by the student – the particular association, emotions or images they excite.

Knowledge of problem solving – collectively, the teachers held three conceptions of PS: algorithmic, directed non-algorithmic, and open non-algorithmic. The directed situation involves applying predetermined specified strategies while the open involves determining and applying one's own strategies.

Knowledge of instructional approaches – collectively, the teachers held four conceptions of teaching approaches: imposition, abandonment, directed-inquiry, and dialogic-inquiry. Briefly, for imposition, the teacher imposes on students an interpretation and algorithm for CPs; for abandonment, the teacher abandons students to interpret and solve CPs based on algorithms of examples of non-CPs; for directed-inquiry, the teacher directs students' inquiry process and interpretation of CPs; and for dialogic-inquiry, the teacher facilitates students' inquiry process and interpretation of CPs.

Knowledge of students as problem solvers – collectively, the teachers held four conceptions of problem solvers, categorized in terms of: agency, connectedness, separatedness, and inquirer. Briefly, agency deals with students taking more control of their mental activity while connectedness and separatedness deal with the relationship between students' personal experience and problem context.

Table 2 shows relationships among these categories of knowledge. Two teachers were oriented to row 1, one to row 2, three to row 3, and the six exemplary teachers to row 4.

Teaching	Contextual Problems	Problem Solving	Students
Imposition	Objectivist + partial utilitarian	Algorithmic	Separatedness
Abandonment	Objectivist + partial utilitarian	Algorithmic	Separatedness + naïve agency
Directed-inquiry	Partial humanistic + partial utilitarian	Directed non-algorithmic	Directed inquirer + connectedness
Dialogic-inquiry	Humanistic + utilitarian	Open non-algorithmic	Agency + inquirer + connectedness

Table 2: Relationships among categories for knowledge

For the most part, the exemplary teachers demonstrated knowledge of PS consistent with Schoenfeld's (1985, 1992) criteria of appropriate resources, heuristic strategies, metacognitive control, and appropriate beliefs and of PS proficiency based on the components proposed by Kilpatrick et al (2001), i.e., conceptual understanding; procedural fluency; strategic competence; productive disposition; adaptive reasoning. They also demonstrated understanding of "*unpacking* mathematical reasoning" (Ball et al., 2008. While, like the other participating teachers, they started the year with students whose PS experience was predominantly with routine/algorithmic problems, their students demonstrated a much higher level of motivation and success in PS through their actions/work in class than those of the other teachers. Thus their knowledge suggests the type of PS knowledge for teaching that could support students' development of proficiency in PS. These teachers, unlike the others, held knowledge indicated in the last row of Table 1, i.e., their PS knowledge for teaching and use of it included: contextual problems as humanistic and utilitarian situations; PS as open, non-algorithmic processes; students in terms of agency, inquirer, and connectedness;

and teaching as dialogic inquiry. For example, their utilitarian view of problems focused on CPs as a basis of conveying mathematical and social knowledge, of meaningful illustration or application of mathematical concepts, of promoting thinking, and of framing teaching. The humanistic view emphasized the importance of associating CPs with experience and their relationship to students.

These exemplary teachers held knowledge of the other aspects of Table 1, but did not prioritize them in their teaching. For example, they minimized the use of computational-algorithmic CPs. One explained, "They're extra, they're not necessary, they're trivial and they do little most of the time to enhance a topic." Another noted, "They aren't all that important, so if you have to cut corners some place and you don't have a lot of time … they can be dismissed." So, when necessary for them, they used the other aspects of Table 1 strategically, but their focus was always student-centered.

In general, based on classroom observations of all participants, the exemplary teachers were more flexible in their teaching and more successful in motivating students to work with CPs and helping them to learn to solve CPs and develop proficiency in PS. Their teaching was also different from that of the other teachers in terms of integration of CPs throughout their courses as a way of teaching for, about, and through PS and engaging students in developing general PS heuristics and their own solution processes. For example, the Grade 9 teacher started the school term teaching about PS by allowing students to develop a PS model for themselves. Students worked in groups to solve the following problem they did not previously encounter.

> Three water pipes are used to fill a swimming pool. The first pipe alone takes 8 hours to fill the pool, the second pipe alone takes 12 hours to fill the pool, and the third pipe alone takes 24 hours to fill the pool. If all three pipes are opened at the same time, how long will it take to fill the pool?

One student in each group observed the PS process. Each student got a turn at being observer for a different problem. Discussions followed each round of observations.

Another example involved teaching linear systems of equation though PS. Towards the beginning of this unit, this teacher gave her Grade 10 students the following task.

> If you have a weekly part-time job in sales, is it better to have a fixed hourly rate or a fixed weekly salary plus commission?

Students were to consider what information they needed to solve it, use direct or indirect real-life experience to provide realistic information, and determine a way to solve it. This task was new for them but done after PS experience with this teacher.

CONCLUSIONS

MPSKT is complex and includes much more than how to solve mathematical CPs. The study suggests that general PS ability does not fully account for the knowledge and skills needed for effective PS teaching. The tasks of teachers require knowledge beyond that which is needed to reliably solve CPs. This study identifies a practice-based conception of the nature of this knowledge that could support students'

development of proficiency in PS. It provides a framework of key knowledge secondary mathematics teachers could hold in relation to MPSKT which can be used to help teachers to understand the nature of this knowledge. It can also be used to offer opportunities to prospective teachers to help them to be prepared to teach PS meaningfully by exploring these conceptions in terms of their nature and possibilities.

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THE EXPERIENCE OF SECURITY IN MATHEMATICS

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In this paper, we report some findings from an investigation of a topic related to affect and mathematics which is not well-represented in the literature. For some mathematicians, mathematics itself is a source of security in an uncertain world, and we investigated this feeling and experience in the case of 19 adult mathematicians working in universities and schools in Greece. The focus reported here is on ways that a relationship with mathematics offers a sense of permanence and stability on the one hand, and an assurance of novelty and progress on the other. Semi-structured interviews with these participants revealed that they valued mathematical modes of thinking, both within mathematics and in everyday life.

INTRODUCTION

In the introduction to the Research Forum on affect at PME28, Hannula (2008) wrote that emotions "have an important role in human coping and adaptation" (p. 108). The literature concerning emotional responses to mathematics is dominated by investigations into negative responses to instruction and testing, and by constructs such as mathematics anxiety, fear of failure, and mathematics avoidance (Zan et al, 2006). However, there is another side to this coin, with many individuals attesting to a positive response to mathematics, and deriving satisfaction, or pleasure, from it, for various reasons. Bertrand Russell, for example, speaks for those who find a pure, cold beauty in the subject:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show (Russell, 1919, p. 60).

The affective issue under investigation in this particular paper is concerned with an important component of "human coping": for some mathematicians, mathematics contributes to their sense of security, and thereby to their well-being. This is an under-researched topic in the domain of 'positive mathematics-affect'. We explored the experience, for some individuals, of mathematics itself as a 'safe place'. The aim of the research was to explore the concept of security, as it emerges from the relationship of mathematicians to mathematics. To speak of mathematics as offering a haven of some kind may seem strange, if one thinks of mathematics as a body of knowledge. In the next section we shall draw out conceptions of mathematics that could be appealing to certain individuals in terms of security, and describe our conceptualisation of security for the purposes of this study. We then proceed with an account of our findings from interviews with a sample of mathematics professionals.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

In this section, we frame our investigation in terms of the nature of mathematics itself, and of security as a psychological phenomenon.

The nature of mathematics

A wide range of perspectives about the nature of mathematics has evolved since early Greek civilisation up to the present time (Davis and Hersh, 1980, Friend, 2007). According to Plato, material objects are mere shadows of an ideal counterpart, or 'form'. Mathematical objects are paradigmatic exemplars of forms, and they pre-exist, awaiting human discovery. In early modern philosophy, Descartes (1596-1650) continued the Platonic tradition, privileging reason over the sense-experience (Hutchins, 1952, p. 3). The subsequent scientific and industrial revolutions led to the quest for secure foundations for mathematics, and notably to the formalist perspective that mathematics could be reduced to a few axioms and deductive rules as the source of all mathematical knowledge. This vision was undermined by Gödel's proof that any such system complex enough to include arithmetic is necessarily incomplete. A notable response to the collapse of the formalist project is Lakatos' (1976) position, that mathematical knowledge (in keeping with Popper's view of science) is a human and fallible enterprise. This view of mathematics as a human, social construct, negotiable and consensual, is emphasised in social constructivism as a philosophy of mathematics (Ernest, 1998). We pause here to comment that these recent ontologies of mathematics seem to place the mathematician on shifting sand, but nevertheless give them agency in an unfolding mathematical story. On the other hand, Platonism accords well with the mathematician's experience of 'discovery' (Huckstep & Rowland, 2001), is consistent with a stable and dependable mathematical universe, and is frequently considered the default metaphysical position regarding mathematics (Friend, 2007).

Security

Maslow (1970) has proposed that human needs are organised on five priority levels. In this hierarchy, Maslow includes security within a more general 'safety' category, along with stability, structure, order, and freedom from fear. This category is located in the second level of Maslow's hierarchy, preceded only by physiological needs related to survival. In this research, and this paper, we operationalise the concept of security in a two-stage process: (i) by reference to dictionary definitions of security as 'freedom from fear or anxiety' (e.g. www.merriam-webster.com); (ii) a typology of fear due to Riemann (1970), who proposed four types of personal need, organised into two opposing pairs. Each type of need brings with it an associated fear. The first pair opposes the need to be an individual against the need to be part of a group: the corresponding fears are fear of *assimilation* [our translation] and fear of *isolation* and loneliness. The second pair opposes the need for stability with the need for development: the corresponding fears are fear of *change* and fear of confinement and *stagnation*.

As an indication of the relevance and potential application of Riemann's framework to the topic under investigation here (security in mathematics), consider Mendick's (2005)

account of the 'identity work' of two young persons, expressed in terms of their enjoyment of mathematics. Mendick comments (p. 175) that "Phil finds a security in mathematics that enables him to construct himself as intellectually mature and as distant from his working-class, minority-ethnic self". Phil's security can readily be construed in terms of his response to fear of assimilation – through his engagement and success in mathematics, he positions himself as distinct and distinctive, in terms of his distance from his origins and his intellectual capacity.

METHODS

Data Collection. We explored the concept of security with adult mathematicians, since these could be expected to have a well-developed relationship with mathematics, and to be able to articulate it. The participants' professional mathematical roles were in teaching and/or research in Greece. Nine were in university positions: (pseudonyms) Faidra, Paraskevi, Themis, Vasilis, Sofoklis, Periklis, Alvertos, Dimitris, Kleitos. Ten were teaching in secondary schools: Stamatia, Eleftheria, Aris, Sokratis, Avgoustis, Marios, Nestoras, Fanis, Thodoris, Loukas. This was an opportunity sample, determined by existing connections with one university department and several schools. Only four of the 19 participants were female (the first two in each of the lists above), reflecting the population of mathematicians in Greece (Kotarinou, 2004). Most of the participants had substantial professional experience (15 years or more); Themis, Vasilis, Sofoklis, Periklis, Faidra and Eleftheria had been in post between 2 and 8 years.

One semi-structured interview was conducted by the first author with each participant, aiming to probe for unconscious feelings which might be difficult to access directly, but could be hinted at during a conversation (Rubin & Rubin, 1995). The interviewer approached the topic indirectly, by discussing with the participants their relationship with mathematics in a general way. This approach minimised the risk of participant discomfort on being asked to disclose personal information (Robson, 2002). The interviews, which mostly lasted up to 30 minutes, were audio-recorded and transcribed in Greek. The semi-structured interviews were organised around the following four themes: the participant's personal history regarding mathematics; their views about mathematics; the relevance of mathematics in everyday life; and their feelings about mathematics. The interviewer had a repertoire of questions from which she drew in a flexible way.

Data analysis. The scale of the data analysis task was such that it could be handled manually. In a first pass over the interview data, utterances were coded as relevant, or probably relevant, to one of Riemann's four types of fear. Sometimes just one type could be applied to a whole paragraph, at others to only part of a sentence. For example, Stamatia's analysis of mathematical modes of thinking included characteristics referring: to communication, which was connected with fear of isolation; to structured thought, which was connected with fear of change; and to creativity, which was connected with fear of stagnation. In a second pass over the data the initial fear-type codings were reconsidered, and changed in some cases, and some additional utterances

were coded. Several cases of multi-coding arose, even including coding some utterances to opposing fears, as two sides of the same coin. Subsequently, as the data were revisited again and again, a method of constant comparison was applied in a more fine-grained coding, giving rise to broad themes and related sub-themes, associated with each type of fear. The themes and sub-themes related to *fear of change* are listed in Table 1, by way of illustration. Note that whereas themes 1-3 emphasise aspects of mathematics and mathematical activity that have the potential to offer protection against change, themes 4-8 acknowledge *interconnections* with other fears, and *limitations* in safeguarding against change.

Themes	Sub-themes
1. Mode of thought	precision; connectedness; systematisation; orderliness; verifiability; consistency; sense-making; realism; real life
2. Inferences	certainty; reliability; one reality; real life
3. Art	harmony; beauty; balance
4. Assimilation	self-awareness and mode of thought; self-fulfilment and mode of thought; self-confidence and mode of thought
5. Isolation	historical continuity and reliability; precision and one reality; omnipresence and connectedness
6. Stagnation	change and creativity; change and mental activation; change and diversity
7. Limitations to Assimilation	realism
8. Limitations to Change	mode of thought

Table 1: Fear of Change – themes and sub-themes

FINDINGS

In this paper, we restrict our report to those findings from the analysis of the interview data that shed light on the participants' views with regard to the second of Riemann's opposing pairs: *fear of change* and *fear of stagnation*. The analysis is restricted here to those themes (like Mode of thought, Inferences, and Art; Table 1) that relate specifically to the fear-type under examination, rather than those that indicate interconnections and limitations. In the case of *fear of stagnation,* these were Creativity; Problem solving; Diversity.

Fear of change

First, we will report the participants' views which we associated with fear of change. These views explain how mathematics could make the participants feel that they were protected against, or ready to confront, the unexpected changes of life.

Mode of thought

The participants perceived the mathematical mode of thought as precise, interconnected, systematic, rule-governed, verifiable, non-random, absolute, and sense-making. For example:

> mathematics makes you feel secure because it reveals harmony and orderliness. Every system functions with certain rules; if you violate them, then the system collapses. … It makes sense how one [statement] is linked to another . . . What seems complicated and difficult can be broken down into the links that produce it . . . it is not randomly produced (Fanis)

> In order to prove . . . you need absolutely rational thought, absolute logic (Aris)

> you start from a point arbitrarily, but everything you say afterwards is established (Dimitris)

> mathematics is precise; its results are verifiable . . . you know if you were right or wrong, you have no doubt (Avgoustis).

The interviewees also believed that these attributes of mathematical thought could be transferred to everyday life, and improve it:

> I say to students: maybe you won't use mathematics after [school], but from your mathematical experience you may acquire a mode of thought (Thodoris)

> mathematics reformulates the problem you want to solve, until it becomes comprehensible. The same you do with a real problem. You distinguish and organise [the data] in hierarchies depending on the values you have in your head (Alvertos)

> If you've been taught by mathematics and if you've conquered your passions . . . you can see more clearly, and consequently you are better equipped to confront [a problem] successfully (Sokratis).

Thus mathematics was seen as being ordered itself, and inculcating orderly behaviour.

Inferences

The interviewees asserted that the mathematical mode of thought starts from sound foundations and leads to certain, reliable and permanent conclusions:

> mathematics is logic; there are axioms and a stable basis […] mathematics doesn't change; what has been found remains as it is (Vasilis)

> you may say that proving makes the knowledge secure (Alvertos)

> mathematical thought engenders and answers 'whys' . . . through indubitable arguments (Stamatia)

> [in mathematics] for every problem we can obtain one unique correct solution (Loukas)

The participants transferred this certainty to real life, where mathematical 'sound foundations' was translated into pragmatic 'realistic assumptions':

> mathematics influences our decisions . . .; [it allows us to judge] what our abilities are, so that we make correct choices (Aris)

> Mathematics helps you . . . to put the assumptions in order and to reach the best possible solutions (Fanis)

> You can distinguish between right and wrong . . . in life, contradiction is allowed to some extent; but even though you may not be able to prove something, you'll be able to exclude something [else] (Sofoklis).

The logic inherent in mathematics was especially prized for the 'certainty' guaranteed, from both Platonist or formalist points of view, and the same modes of inference were seen as valuable in everyday affairs.

Art

The orderliness of mathematics suggests harmony and balance, and these in their turn imply beauty. The interviewees judged mathematics as beautiful:

> there are proofs which display harmony . . . I love the logic hidden in mathematics and its beauty . . . There are questions which simply emerge and they are beautiful (Sofoklis)

> after solving a problem, I imagine the solution as a work of art (Dimitris)

> mathematics is something like music: once you hear it, it sticks in your mind (Avgoustis).

This beauty provides equilibrium in the chaos of an uncertain world.

> The world is chaotic; through the symmetry of mathematics I find balance. (Eleftheria)

Fear of stagnation

Here we report the participants' views which we interpreted in relation to the opposing fear, of stagnation. These views explain how mathematics could make the participants feel that they have powers of self-determination, and ability to change the status quo.

Creativity

Some interviewees affirmed that mathematics gives rise to original creations which shape the present and will influence the future. For example:

> in mathematics you are expected . . . to explore existing paths, and potentially to create [new ones] (Alvertos)

> mathematics contributes to contemporary development, it influences the present (Eleftheria)

> differential geometry and vector spaces are a glance into the future (Fanis)

> science fiction is born of the womb of the science of mathematics (Stamatia)

The interviewees also believed that mathematical creations adhere to logic but are not restricted by any physical laws or limitations. Kleitos dissociates himself from a view of a pre-determined mathematical universe:

> other sciences discover, while mathematics creates; there isn't something specific you're looking for … mathematics uses the least possible rules (Kleitos)

Stamatia commented on the transfer of mathematical creativity in real life, in a bold assertion of self-determination:

> mathematics is the science that cultivates independence, boldness, and the love to explore the unknown. You dream an imaginary world, and mathematics allows you to make it real.

Problem Solving

The participants observed that mathematical problems can be tackled using various approaches.

> everyone approaches mathematics differently (Alvertos)

> I like to read about [mathematical issues] which are examined from different perspectives (Nestoras)

Furthermore, the participants observed that problem-solving offers great intellectual independence and stimulation.

> I'm pleased when I watch my students reaching a solution using their brain instead of parroting others' opinions (Stamatia)

> mathematics keeps you vigilant; your mind doesn't get the chance to be idle (Eleftheria)

The participants also commented on problem-solving being an unexpected experience:

> I like it when students see things that I haven't (Themis)

> I see mathematics as an ongoing route and not as something which I've learned and I can rest upon (Periklis)

These contributions present mathematics as offering scope for novelty and originality, taking pleasure in diversity and in the unexpected.

Diversity

As an occupation, mathematics can give rise to a range of emotions.

> mathematics engenders thousands of feelings; from vanity for one's efforts to surprise, hedonism, fury, and stubbornness (Alvertos)

Mathematics was considered to be a tool, both with respect to other sciences and for organising one's thought, and this tool can be used in many different ways:

> it can be a hobby, a profession, a means to get rich, a means of deceit, a means of exploration, and an object of research … [however] applying mathematics is not bloodless; the missiles have been made by mathematicians (Sokratis)

Here, Sokratis disputes G. H. Hardy's (1940) claim that mathematics is benign, harmless and practically useless, and mathematicians detached from practical affairs. Like the other informants here, he attests to the endless diversity and variety of experience and emotion derived from mathematics, which we interpret as another safeguard against stagnation.

CONCLUSION

This investigation into feelings of security in mathematics was underpinned by a conception of security as relief from fear, and by Riemann's (1970) focus on four types of fear, in two opposing pairs. In this paper we reported findings relating to *fear of change*, and *fear of stagnation*. Mathematics was perceived to offer a balance between these opposing anxieties. As many philosophers have suggested in the past (e.g. Hutchins, 1952), the interviewees believed that what distinguished mathematics from the other sciences, natural or human, was its mode of thought. This mode was considered to lead to an exceptional kind of knowledge, indubitable and unchanging, whether discovered or invented. This infallible knowledge was believed to be continually increasing, to the benefit of both mathematics and other disciplines. The former enjoys results unbound by any physical law, the latter findings which could be used to change the world (Guillen, 1995). Mathematics was perceived as a realm of

creation, of beauty and balance, in which everything makes sense. Furthermore, insofar as the mathematical mode of thought can be transferred from mathematical to real-life problems, it was valued as a tool of unique precision, both in handling the unavoidable changes of life (fear of change) and in escaping from undesirable situations (fear of stagnation).

Several lines of further research are suggested by these findings: perhaps the first fruitful avenue would be to investigate the extent to which these findings might be replicated in other cultures, within and beyond Europe.

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AN EXPLORATION OF MATHEMATICS TEACHERS' DISCOURSE IN A TEACHER PROFESSIONAL LEARNING COMMUNITY

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This study investigated the evolution of mathematics teachers' discourse in a teacher professional learning community with an aim to contribute to the discussion on teacher professional development. Four mathematics teachers teaching in diverse socio-economic status (SES) schools participated in the study. Qualitative methods were applied to learn the complexity of teacher discourse in depth. Research findings suggested that the focus of teacher discourse moved toward student mathematical thinking. Teachers teaching in low-SES schools benefited more from the participation in the learning community than their high-SES counterparts.

INTRODUCTION

The engagement of teachers in a professional learning community is suggested to be the most critical and effective way of affecting teacher professional development (National Staff Development Council, 2011). Such a community involves teachers organizing a team for professional development in order to cooperatively improve their instruction. Teachers meet regularly to discuss learning goals, lesson plans, problems they might encounter in teaching, and to reflect upon the lessons they have already taught. Teachers can play an active role in educational reform and discover the main problems relating to school education and teaching.

Although the idea of a teacher professional learning community has been valued for teachers' professional development, the use of learning communities has not yet entered the mainstream of professional development for teachers in Taiwan (Taiwan Ministry of Education, 2008). The Ministry's relevant information and documents are limited in the nation. Moreover, scholars have very few resources in terms of the interaction of mathematics teachers within professional learning communities, such as how a dialogue proceeds.

This study investigated teacher learning from the perspective of teacher discourse. It focused on changes in the content of discourse after mathematics teachers' participation in a professional learning community. The professional learning community focused on improving teachers' discourse-based assessment practice (DAP) from convergent formative assessment to divergent formative assessment (Pryor, & Crossouard, 2008). DAP is a type of formative assessment practice which consists of

questioning and feedback (Tunstall, & Gipps, 1996). In the community, teachers watched video clips of their own teaching to reflect upon their DAP and discussed what they had seen.

Teachers are viewed as learners in a professional learning community. However, literature about how teacher learning occurs, especially the evolution of teacher discourse, is limited. This study is significant because it contributes to the knowledge base regarding the professional learning behavior of mathematics teachers.

LITERATURE REVIEW

Research evidence has indicated that mathematics teachers can better develop their profession if they regularly watch recordings of classroom teaching and discuss what they have seen (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Sherin & van Es, 2009). For example, van Es and Sherin's study (2010) indicated the focus of teachers' professional discourse shifts from teachers' pedagogy to students' mathematical thinking in a video club. Consequently, teachers began to pay more attention to, and exhibited greater understanding of, students. In Borko et al.'s study, when mathematics teachers gained more experience in analyzing video clips, they were able to have more extended discourse, and the four categories, teacher's thinking, students' thinking, pedagogy, and mathematics, appeared more evenly in teacher discourse. In Chung Jing, Shen Shu-Yu, and Huang Mei-Ling's research about the situation of elementary school teachers' professional discourse (2008), it was shown that the content of discourse in the voluntary mathematics teachers' group was more in-depth when compared to the mathematics study group and to the group of classroom teachers. The mathematics study group dealt with the tasks assigned by school authorities by holding ad hoc meetings. In Chen Yen-Ting, Kang Mu-Suen and Leou Shian's research (2010), two junior high school teachers reported that discourse among peer groups had helped them reflect upon and develop mathematical pedagogical knowledge. In short, it is an emerging area in the study of the evolution of teacher professional discourse. Research evidence has shown the complexity of teacher professional discourse. In order to better support teacher professional development, more research is needed to explore the professional learning of mathematics teachers in a learning community.

METHODOLOGY

Context of the Study

Four mathematics teachers teaching in Hua-Lien County and New Taipei City agreed to participate in the study. They all had more than five years of teaching experience in mathematics. Two teachers served in a top-flight urban school in Hua-Lien County. Most students who study in this school come from middle- or high-SES families. Two teachers came from low-SES schools with many minority students. This teacher professional development program for DAP was executed from August 2010 to April 2011. The teachers gathered once every two or four weeks. Each discussion lasted two-and-a-half to three hours.

The first author played dual roles as a facilitator of teacher discussion and the researcher of its effectiveness. We collectively watched teaching video clips and examined the quality of questioning and feedback in teaching episodes. Each teaching episode was viewed as a case. Case discussions have the potential for developing teachers' professional knowledge and for contributing to teachers' movement toward student-centered instruction (Clare & Hollingsworth, 2000). When engaging in case discussions, the teachers can provide their colleagues with not only support but also critiques for the implementation of DAP.

Transcripts of Teacher Professional Development Meetings.

Except for the first three informational sessions, every teacher professional development meeting was videotaped and transcribed verbatim using Transana™ software. The transcripts take up about 33 hours for 12 meetings in all.

Data Analysis

When analyzing the transcripts of the professional development meetings, the transcripts were first broken down into idea units (Jacobs & Morita, 2002), which are fragments of transcripts. In a fragment, there is only one specific idea, which would be discussed by teachers—that is, when teachers' discourse moves into a new topic, it is then counted as another idea unit.

The approach of manipulating idea units was taken from Sherin and van Es (2009). The idea units were broken down into three categories for further analysis: 1. Who initiates idea units? 2. The objects of teachers' discussion, and 3. Discussion topics. "Who initiates idea units?" means that the researcher or teachers initiate the discussion in the idea unit. "The objects of teachers' discussion" refers to whether students, teachers, or other people are the objects of teachers' discussion. The coding scheme of "discussion topics" was developed according to the interaction between the reading of professional discourse literature of mathematics teachers (Chen Yen-Ting, Kang Mu-Suen, & Leou Shian, 2010; Manouchehri, 2002; Sherin & van Es, 2009) and the feedback taken from the data analysis. The discussion topics were categorized as "Teaching techniques," "Mathematical thinking," "Mathematics," "Discourse," "Management," "Atmosphere," "Assessment," "Reflection," and "Others," all of which are in Appendix A. Sherin et al.'s methods (2009) were applied, and only "discussion topics" that were triggered by teachers in the idea units were coded. The result of this coding is presented in a table, displaying the frequency and percentage of each code. Member check was applied to ensure the credibility of data analysis (Schwartz-Shea, 2006). Due to the page limit, we present representative findings.

FINDINGS AND DISCUSSION

Table 1 shows that participating teachers initiated most "idea units" in teacher meetings. This implies that the teachers played an active role in the learning community. This trend (almost 60%) began early on and was maintained until the end of the research project. What is worth noting is that the number of "idea units" had a

tendency toward diminishing over time. For example, teachers opened 84 "idea units" in the meeting on September 12th, but only 56 "idea units" on April 10th. This is because teachers' discourse in the early stage involved many short discussions and several spoken sentences, before it turned toward other topics for discussion. This is why the meeting in the early stage had many "idea units." However, in the late stage of the study, teachers usually focused on one topic and spent time talking about it before entering another topic. Discussion of an "idea unit" usually lasted for some time, which is why at the same meeting time a smaller number of "idea units" appeared in the late stage. This seems to show that teachers' discussions were more focused and deeper in the late stage when compared to the early stage of the study.

The object of teachers' discussion also appears as a different percentage in the early and late stages. Although the focus had been on teachers, the percentage of students in the teachers' discourse had been gradually increasing (9%, 22%, 13%, 15%, and 31%). This suggests that teachers increasingly attended to students as the objects of their attention.

Date	09/12	11/21	01/22	02/27	04/10
Person Who Initiates Discussion					
Researcher	34%	37%	22%	42%	37%
Teacher	66%	63%	78%	58%	63%
Discussion Object					
Teacher	64%	56%	57%	59%	50%
Student	36%	39%	41%	38%	43%
Others	0%	5%	2%	3%	7%
Discussion topic					
Teaching Techniques	39%	28%	19%	15%	23%
Mathematical Thinking	9%	22%	13%	15%	31%
Mathematics	1%	0%	0%	0%	0%
Discourse	2%	1%	6%	3%	3%
Assessment	12%	15%	13%	24%	17%
Reflection	1%	11%	22%	15%	14%
Management	5%	2%	6%	15%	2%
Atmosphere	12%	3%	13%	3%	0%
Others	19%	18%	9%	9%	11%

Table 1: Percentage of "idea units" of teacher professional development meeting

Correspondingly, in the discussion topic, teachers spent less time discussing "teaching techniques," but more time discussing students' "mathematical thinking." The two low-SES teachers discussed more mathematical thinking about students than did their colleagues. Interestingly, the percentage of teachers' discussion about reflection increased until the end of the fall semester and then decreased until the middle of the spring semester. Observations suggest that the teachers increasingly discussed the topic of reflection as they were trying to handle DAP in the first half term of this study. When they were able to handle DAP, the teachers decreased their percentage of time spent in discussion of reflection. It is noteworthy that the two low-SES teachers initiated the most discussion on the reflection topic (80%) which suggests that they benefited more from the participation in the study than their high-SES counterparts. Below is an excerpt which illustrates teachers' reflective behavior among two low-SES teachers in the sessions (Teachers' names are replaced by pseudonyms, and the words in the brackets are those added by researchers for understanding):

> Teacher Lily: Teacher Hua played the video of our previous teaching in the classroom. Then we started talking (discussing about it). Then I feel that the process has allowed us to think about many issues. For example, at the time Teacher Fang said that even I couldn't answer your question (that you proposed to students). You then thought that…

> Teacher Jiang: The feedback from colleagues (in the teacher professional meeting).

> Teacher Lily: When colleagues told me this, I would then start to think that I thought I had expressed myself clearly enough. I might have some language habits which I didn't think would be problematic. But I was very shocked at the time when Teacher Fang and Teacher Wang said even they couldn't answer my question and didn't understand what I was trying to ask. I was kind of agreeing with them. After this I always watched my own teaching videos and considered if I have made any sentences out of my own habits. Students may not have time to think or they didn't understand what my question was about if I didn't ask precise questions. I then started reflecting on this. It was still the most important part because everyone could discuss it together.

> Teacher Jiang: Discuss.

> Teacher Lily: Then giving you some opinions…So I think the discussion of teaching part is very useful for me. … throwing out questions in the discussion process was very good.

(Meeting transcript, 1/22/2011)

Teacher Lily appreciated the way that the researcher led teachers in their discussion by playing teachers' videos. Partners' feedback has allowed her to examine and reflect upon the problems of her questioning. She pointed out that Teacher Fang and Teacher Wang did not know how to reply to her question in the meeting held on September 12, which shocked and inspired her to watch her own teaching videos and to reflect on her own teaching. She came to understand that students did not answer questions because

the teacher did not give sufficient time for them to reply, or, they did not understand what the teacher was asking. She gave insightful comments about the learning experiences she had obtained from the teachers' discussion process.

The development of discussion topics is similar to that of Sherin et al. (2009) but different from that of Chen Yen-Ting et al. (2010). In the latter study, the concept, "knowledge of mathematics course"–which is teachers' understanding of content knowledge of mathematics and of the mathematics curriculum--occupied the highest percentage of time in the teachers' discussion. "The understanding of learners" is the teachers' understanding of students' mathematics learning characteristics and prior knowledge, which is similar to the mathematical thinking topic applied in this study. Its percentage had the tendency to rise, then fall, in the discussion of different periods of two junior high school mathematics teachers.

The difference in the development of teacher discourse between Chen et al. (2010) and this study may be due to the way that researchers conducted case discussions. In their study, the two teaching colleagues observed classroom teaching with each other. After classroom observations, they met to discuss what they observed without watching video clips. The researchers only presented and played the role of an audience. In this study, the researcher played an active role in facilitating teachers' attention in session discussions to students' thinking about mathematics. The role that a facilitator plays has a significant impact on teacher professional learning.

It is noted that teachers participating in video clubs did not demonstrate reflective behaviors in sessions, which is not the case for Borko et al.'s study (2008) and this study. The difference may be explained by whether teachers have the chance to view their own lessons or not. Only a few teachers participating in the video clubs provided their own teaching video for session discussions, while all teachers who participated in the latter's studies shared their teaching videos with their colleagues. Seidel and colleague's experimental study (2005) suggests that teachers' experience of watching their own videos is more stimulating and emotionally arousing than that of watching someone else's videos. This experience of watching one's own videos can better support teacher learning and promote changes in teaching practices. Thus, the facilitators of teacher learning should make efforts to create and manage an environment that makes teachers feel safe to share their own teachings and carefully deal with teachers' feelings when watching their own teaching videos.

CONCLUSIONS

This study explored the development of mathematics teacher discourse in a professional learning community. The research findings suggested that the focus of teacher discourse gradually shifted from teaching techniques to student mathematical thinking. Teachers began to pay attention to student mathematical thinking when they were examining the quality of DAP. This suggests that teaching and learning became a constitutive activity (Crockett, 2007) for the teachers. Teachers teaching in low-SES

schools demonstrated the most reflective behaviors which imply that they benefited more from participation in the learning community than their high-SES counterparts.

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APPENDIX A: CODING SCHEME FOR TEACHER DISCOURSE

Code	Contents
Teaching techniques	It demonstrates teachers' presentation of information in the class, their choices of class tasks, instructional strategies, and instructional decision-making.
Mathematical thinking	Talking about students' mathematical understanding in the class: this includes the comments given to the entire class about mathematical understanding and the discussion of individual student's mathematical thinking. It can be encoded when demonstrating the ability to understand another person's mathematical thinking.
Mathematics	It consists of questions and comments about mathematics concepts that were taught in a class. It does not include the mathematical understanding of students but rather, focuses on the mathematical understanding of teachers.
Discourse	Paying attention to the ways of communicating and discussing ideas between teachers and students. For example, whether or not many students participate in classroom discussions or how students know when they should speak.
Assessment	Focusing on the application of formative assessment. For example: initiation, feedback, and using a peer group as students' learning resources or discussing students' learning performance.
Reflection	Teachers spontaneously challenge their own views about teaching and learning, or reveal their intent to re-organize teaching actions.
Management	Talking about class organization, such as use of time, dealing with any disturbances, and transitions in activities.
Climate	In contrast to classroom management, it refers to the social environment of a classroom. For example: the relationship between teachers and students, students' treatment of one another, or students' level of participation.
Otherwise	The idea units cannot be encoded by the previous seven codes--for example, the discussion of video image and sound quality.

A SIXTH GRADER APPLICATION OF GESTURES AND CONCEPTUAL INTEGRATION TO LEARN GRAPHIC PATTERN GENERALIZATION

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This is a study on the construction and interpretation of mathematical meaning associated with gestures, oral speech, and drawings. We describe a learning episode on adjacent graphical pattern in a sixth grade class. The investigators utilizing the tools of linguistics and the study of gestures to analyze the conceptualization of mathematical concepts related to graphic pattern generalization. Case study method is adopted. The results of this study indicate that, even for elementary topics, the abstract nature of mathematics was made evident through gestures demonstrated during the episode. Instructional implications of this research are included.

1. INTRODUCTION

Researchers in the fields of linguistics, cognitive science, and psychology have recently turned their attention to the phenomenon of spontaneous gestures associated with speech to deal with communication and the construction of mathematical meaning, (McNeill, 2005). In Taiwan, a number of mathematics educators addressed gestures and body movement as either potential sources of information on how we think about mathematics, or as contributors to mathematical thinking and communication itself. Gestures are a crucial tool in the learning of mathematical concepts. Alibali, Kita and Young (2000) claimed that gestures are involved in the conceptual planning of messages, helping students to "package" spatial information into verbalizable units, by exploring alternative ways of encoding and organizing spatial and perceptual information.

The goal of this study was to collect and analysis a corpus of gestures related to the learning of one mathematical topic, pattern generalization. The topic of pattern generalization was selected because the ability to generalize is critical to algebraic thinking and reasoning; however, the point at which the nature of everyday generalizing begins to deviate from the more formal activity of mathematical generalizing is a fundamental issue that remains unresolved (Carpenter, Franke, & Levi, 2003; Radford, 2006).

2. THEORETICAL FRAMEWORK

This study employed the tools of cognitive linguistics and the study of gestures (Fauconnier & Turner, 2002; McNeill, 2005) to form a fundamental theoretical commitment, namely, that human mathematical thinking is integrated at multiple levels: through imagery, bodily motion, and gestures. Cognitive linguistics views language as a dynamic construction, which reflects a series of unconscious mental

mappings created through our experiences based on current understanding. An important mechanism within this framework is the conceptual integration of mental spaces, defined as input spaces projected selectively onto an integrated space, resulting in an emergent structure (Fauconnier & Turner, 2002). Thus, the notion of a pattern is an integration of two input spaces: Knowledge of the common difference value between terms, and our conception of the geometric entity referred to as a figural structure. This integrated entity draws particular elements from each of the input spaces and applies gestures to connect these elements in the creation of a pattern. For example, in the problem given later in Fig. 2, the notion of the "pattern" in the triangular graphic pattern is a blend that draws from two input spaces: common difference (D) and figural structure (FS). The blend draws certain elements and relationships from each of the input spaces, and creates a pattern. The basic elements of this integration are illustrated in Fig. 1.
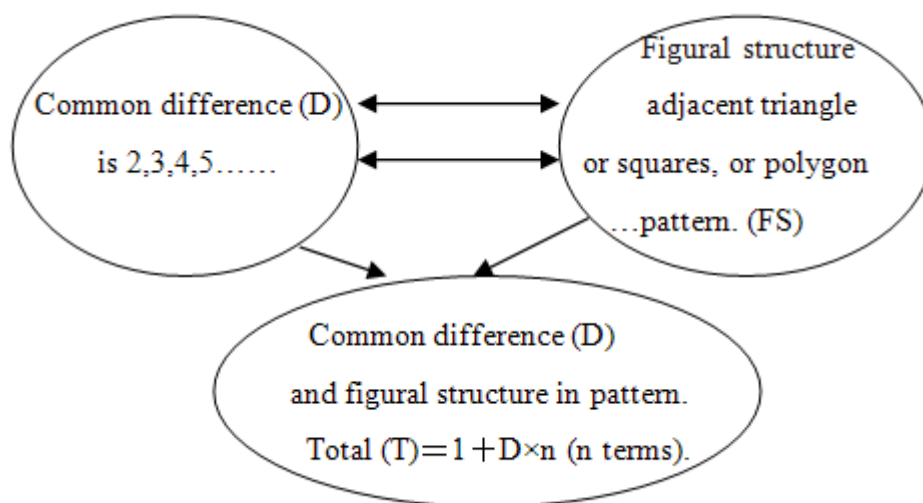
Fig. 1. Conceptual integration associated with figural pattern generalization

Conceptual integration has been applied in the analysis of mathematical ideas ranging from arithmetic to algebra. In this paper, the theory of conceptual integration is used to analyze both speech and gestures, to describe student thinking about graphic pattern generalization. Parrill and Sweetser (2004) defined the meaning of a gesture as, "the relationship between how the hands move in producing a gesture, and whatever mental representation underlies it, as inferred both from the gesture and the accompanying speech" (p. 197). Clearly, the researchers had no direct access to whatever mental representation underlies a gesture, and must therefore use the linguistic device in conjunction with the activity in which the speaker is engaged, as a means to construct a plausible interpretation of gestures. In the current study, gestures are viewed as conceptual integrations. In the analysis of gestures, the inputs of the integration are not abstract conceptual spaces, but rather an awareness of one's immediate physical environment provided by the hands, surrounding objects, and physical space.

McNeill (2005) classified gestures according to four types that are not necessarily exclusive: Deictic, Metaphoric, Iconic, and Beat. All of these four play essential roles in the consideration and communication of mathematics.

3. RELATED RESEARCH

Previous research relating to gestures and mathematics examined a variety of mathematical tasks, including the conservation of volume (Alibali, Kita, & Young, 2000), learning to count (Alibali & diRusso, 1999), and solving simple equations (Goldin-Meadow, 2003). They found that gestures and speech can "package" complementary forms of information to support the speaker's thinking and problem-solving (McNeill, 2005; Radford, 2006). Several studies have shown that learners are able to express their understanding of a new concept through gestures before they are able to express it in speech, and a mismatch or non-redundancy between the information expressed through the gesture versus speech can be an indicator of a readiness to learn the new concept (Alibali et al., 2000; Goldin-Meadow, 2003). In the current research, we examine specific gestures to illustrate the process of conceptualizing mathematical notions related to graphic pattern generalization.

4. METHODOLOGY

4.1 The case

The method we use is case study (Yin, 1994). From a class with thirty-one primary sixth graders in south of Taiwan, one student was selected to be the case because he exhibited a diverse range of mathematics strategies and competency in problem solving. Students had prior experience with repeating patterns. However, none of the students had worked with growing patterns nor received formal instruction in graphic pattern generalization.

4.2 Research context

This study introduced a special unit on concepts related to graphic pattern generalization, which was videotaped using two digital cameras. The cameras were positioned to ensure that all speakers were recorded and all actions could be seen. All artifacts produced by the teachers and children were recorded, including markings on the chalkboard and any graphs or classifications that the children produced.

The students had been introduced to the recognition of figural patterns and various algebraic representations (numerical tables, Cartesian graphs, and symbolic formulas). Students stood at the blackboard and explained how to recognize the pattern and generalize the graphic for the entire class.

4.3 A learning episode

Patterning and generalization provide students with an opportunity to engage in problem-solving situations to acquire the formal mathematical requirements of algebraic generalization. We asked students to establish a general formula for the total number of sticks at any stage for the patterns shown in Fig. 2. Students first read the problem given in a worksheet and used paper and pencil to explore and analyze.

Finally they used geometric sticks to explain their solutions on generalization or the refining of existing models.
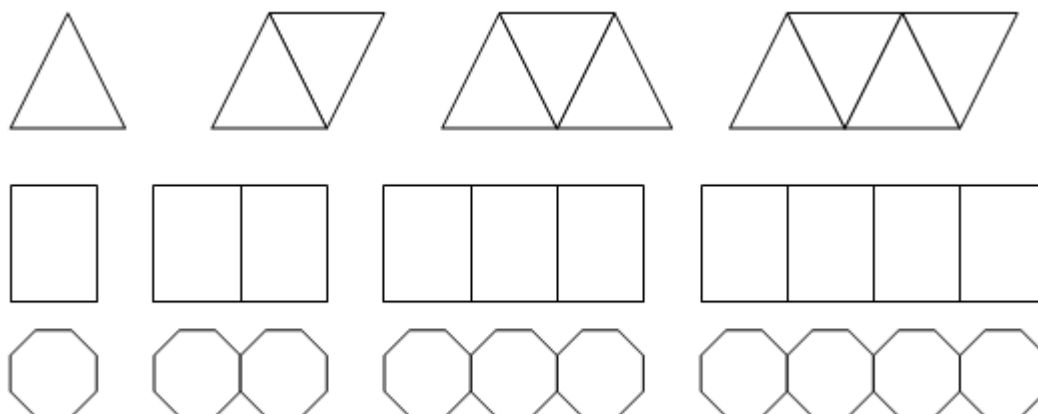

Fig. 2. Adjacent pattern problem (triangles, squares, or polygon)

A. How many sticks are there altogether when there are 10 polygon? Draw and explain.

B. How many sticks are there altogether when there are 15 polygon? Explain.

C. Find a direct formula for the total number of sticks (T) in any pattern number "n". Explain how you obtained your answer.

D. If you obtained your formula numerically, what might it mean if you think about it in terms of the above pattern?

E. If the pattern above is extended over several more cases, and a certain pattern uses 76 sticks in all. Which pattern number is this? Explain how you obtain the answer.

4.4 Data analysis

Gestures were classified using the model created by McNeill:

Deictic: Any extensible body part or held object that is used for pointing.

Metaphoric: Gestures can also be used to present images of abstractions. Some gestures involve a metaphoric use of form—the speaker appears to be holding an object, as if presenting it, yet the meaning is not to present an object but rather to show that she is holding an 'idea' or 'memory' or some other abstract 'object' in her hand.

Iconic: Presenting images of concrete entities and/or actions, in which the form of the gesture and/or its manner of execution embodies picturable aspects of semantic content.

Beat: Flicks of the hand(s) that appear to 'beat' time along with the rhythm of speech. They have meaning and signify the temporal locus in speech of something the speaker feels is important.

Two independents viewed the video and categorized in terms of which type was most salient of four gestures. Discussions were made after the coding until the two agreed on the gesture type.

5. RESULTS

Sin attempted the triangle problem first. Sin first picked up three sticks to make a triangle [*iconic*, Fig.3.1], explaining that the triangle had three sides [1,2,3, *iconic*]. He then added three sticks to the first triangle to make a second triangle, but found that the three sticks could not be combined to make a second triangle [*iconic*, Figure. 3.2]. He said: The side [pointing to the right side of the first triangle] could be used to make a second triangle [*deictic*]. In the same manner, he made a third triangle [*iconic*, Fig. 3.3]. Sin said: The first triangle had 3 sticks and adding 2 sticks could make two triangles, $3 + 2 = 5$, 3 triangles require $3 + 2 + 2 = 7$, 4 triangles would require two more [moving his finger from the right side of the third triangle to the fourth triangle position]. The total number of sticks is $3 + 2 + 2 + 2 = 9$ [*iconic*, Fig. 3.3]. He explained that each additional triangle required 2 additional sticks, so 3 triangles would require 2 groups of 2s, 4 triangles 3 group of 2s, etc. [using a finger to draw the triangle on the blackboard] [*deictic*]. How many sticks would pattern 10 have? He replied that the formula $3 + (10 - 1) \times 2 = 21$ could provide the answer. One pattern used 51 sticks in total. Which pattern is this? Sin pointed to the first triangle and explained that the figure had 3 sticks [*iconic*, Fig. 3.4]; therefore, 3 is first subtracted from 51. Then $48 \div 2 = 24$, $24 + 1 = 25$, resulting in 25 triangles [*metaphoric*].

He said that adding a triangle requires 2 additional sticks, which means that a total of 48 sticks could be combined to form 24 triangles, and after adding the first triangle, there were 25 triangles [pointing to the triangle on the blackboard] [*deictic*].

| 3.1 | 3.2 | 3.3 | 3.4 | 3.5 | 3.6 | 3.7 |

Fig. 3. Sin's performance in graphic pattern generalization

Sin's second attempt was on the square problem. Sin looked for a pattern and said：
For every square you add three more [*iconic*, Fig. 3.5]. He said: That would be 4 plus 3 for two squares [pointing to the second square] [*iconic*, Fig. 3.6]. Plus 3 more for three squares [pointing to the third square]. So that makes 10 sticks. Two 3s would be for three squares. Three 3s would be for four squares, and four 3s for five squares [*deictic*, Fig. 3.7]. For n squares, it would just be n minus one 3s. Which pattern would use 61 sticks? Sin pointed to the first square and explained that the figure had 4 sticks; therefore, first subtract 4 from 61 and then $57 \div 3 = 19$, $19 + 1 = 20$, would result in 20 squares [*deictic*].

Finally, Sin compared a triangle with the square on the blackboard and said that increasing the number of triangles would require 2 sticks but adding squares would require 3 sticks [***deictic***, Fig. 3.6]. Sin felt that adding pentagons would require 4 sticks, because the figures would require the number of sides minus 1[***iconic***, Fig. 3.7]. How many sticks are needed to form an octagon? Sin said that the formula $8+(10-1)\times7=71$ could provide the answer. One pattern used 106 sticks in total [***deictic***]; Which pattern number is this? Sin explained that the octagon graph had 8 sticks, requiring 8 to be subtracted from 106 and then $98\div7=14$, $14+1=15$ [***metaphoric***].

6. DISCUSSION AND RECOMMENDATION

The goal of the study was to analyze a corpus of gestures related to one mathematical topic, the figural pattern problem.

Several researchers have pointed out that the initial stage of generalization involves focusing on or drawing attention to a possible invariant property or relationship (Lobato, Ellis, & Mun~oz, 2003), grasping a commonality (Radford, 2006), and becoming aware of one's own actions in relation to the phenomenon undergoing generalization (Mason, Graham, & Johnston-Wilder, 2005). McNeill's categorization of gestures assisted us to understand how children used gestures to generalize a figural pattern.

In this study Sin used three types of gestures by McNeill (2005) to point out that the structure of the graph and the common difference value changed, and utilized these gestures to present the difference value. By integrating the generalization of each term, the common difference value and the entire structural relations, he was able to explain the significance and the concept of the algebraic expression. Two types of cognitive behaviors demonstrated by these students are worth noticing. One type, "***deictic***", referred to pointing to concrete objects often related to tangible materials utilized (e.g geometric sticks) in instruction about graphic pattern generalization. In the other type, which was given the label "***iconic–metaphoric***," participants' gestures re-enacted the physical process of drawing out a mathematical procedure, or referred to visual locations and elements of mathematical structure. This latter kind of gesture highlights the importance of the abstract form in these students' thinking about mathematics, and the way that structure can form a "chain" of meanings in this domain. Finally, Sin did not used ***beat*** to solve this figural pattern problem.

Our findings indicate that the abstract nature of mathematics was made evident through the high proportion of gestures appearing in the episode. Thus, teachers could encourage the gestures and expressions used by students to promote the learning of concepts related to generalization.

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MATHEMATICAL DISCUSSIONS IN PRESERVICE ELEMENTARY COURSES

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Towson University   Boston University   Boston University

Enacting tasks at high levels of cognitive demand helps preservice teachers make sense of mathematical ideas and serves as a model for instruction. We contrast two small group discussions within a preservice elementary geometry classroom to illustrate characteristics of productive small group discussions. We observed that the group whose members felt comfortable periodically shifting their roles was able to maintain the task's high level of cognitive demand during task implementation. We conjecture that instructors of preservice teachers should foster small group discussions in which participants have opportunities to contribute via a variety of roles and focus on conceptual understanding.

PREPARATION OF TEACHERS IN MATHEMATICAL KNOWLEDGE

Improving the mathematics preparation of elementary teachers is a necessary step toward improving student learning of mathematics (National Council on Teacher Quality, 2008). Additionally, teachers who possess mathematical knowledge for teaching are more likely to implement challenging mathematical tasks in their classrooms (Charalambous, 2010). By 'challenging mathematical tasks,' we refer to tasks that require thinking at high levels of cognitive demand (Stein, Grover, & Henningsen, 1996). Such tasks prompt students to make mathematical generalizations, explain their reasoning, and focus on making sense of important mathematical ideas (Stein, Smith, Henningsen, & Silver, 2000).

These findings suggest that in order for preservice teachers to learn the mathematics required for teaching, they should be exposed to working with high cognitive demand tasks in their teacher training programs. The rationale for engaging preservice teachers in high cognitive demand tasks is that those who have experience providing mathematical explanations and justifications and reflecting on the mathematical connections inherent in the tasks might better engage their future students in similar types of activities (Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003). However, solving high cognitive demand tasks is often a new activity for preservice teachers.

How can teacher educators, then, engage preservice teachers in these types of mathematical activities? While there is a broad literature base identifying the benefits of engaging elementary students in productive classroom discourse (e.g., Choppin, 2007), there are few examples in the research of preservice teachers engaging in productive classroom discourse and how such discourse can help to maintain high levels of cognitive demand of tasks.

The purpose of this article is to present an analysis of preservice elementary teachers' small group discussions involving a mathematical task that was written at a high level of cognitive demand. The context of this analysis is a mathematics content course for preservice elementary and special education teachers taught by one of the authors of this paper. In our analysis, we extend the current research to illustrate how small group interactions can play a key role in helping preservice teachers make sense of important mathematical ideas. We conclude our analysis by discussing ways in which instructors of such courses can help preservice teachers maintain the cognitive demands of challenging tasks in small group settings.

Method

The study was conducted in the spring semester of 2010, during a semester-long preservice elementary geometry course at a large private university in the United States. Two classroom sections of this course were videotaped and audio taped during eighty-minute class sessions over a two-week period.

Transcripts of all videotaped class sessions were analyzed using two sets of rubrics. We used the IQA-AR rubrics for the potential of the task and for the implementation of the task (Boston & Smith, 2009) in order to assess cognitive demand levels before and during implementation, respectively, as well as to identify any possible changes in levels. In order to assess the nature of student-to-student talk within small group discussions, we used the *Levels of Math Talk* framework (Hufferd-Ackles et al., 2004).

Analysis

This report examines two small group interactions within a preservice elementary teacher mathematics course and their ability to maintain the cognitive demands of a task during task enactment. Group X was able to maintain the cognitive demands of the task, while group Y lowered the cognitive demands of the task. The task came in the form of a final question at the end of the two-day lesson on surface area, and involved comparing and contrasting two methods for finding the surface area of a rectangular prism.

One method for calculating the surface area of prism is known as the *lateral surface area method*, and follows in its most generic form:

SA= (Area of lateral surfaces rectangle) + 2 × (Area of a base)

This formula is based on the fact that the surface of a prism consists of lateral faces that can be composed into a rectangle and two congruent bases. Participants further refined the formula above once they discovered that the area of the lateral surfaces rectangle of a prism is the product of the prism's base perimeter and height. Their revised formula for the total surface area of a prism became as follows:

SA= (Perimeter of base × Height) + 2 × (Area of a base)

The second method for calculating the total surface area of a rectangular prism requires calculating the area of each individual face of a prism and summing the areas. The

choice to focus on the *lateral surface area method* was twofold: most of the preservice teachers had not previously encountered the lateral surface area method; and, unlike the traditional method described above, this alternative method generalizes extremely well to all prisms and cylinders.

The question presented to the preservice teachers is as follows:

> Another formula for the surface area of a rectangular prism is given below: *SA=2lw+2wh+2lh*. Explain how this formula determines the surface area of a rectangular prism. Compare and contrast this formula to the lateral surface area method for surface area.

Each group of preservice teachers was given approximately fifteen minutes to complete this question.

Cognitive Demand Analysis for Potential of Task

Using the IQA-AR rubric for the potential of the task (Boston & Smith, 2009), the cognitive demand of the final question as it is written is at a level 4. The question was presented to participants as a way to assess and extend their developing understanding of the lateral surface area method. It requires participants to make connections between two different methods for computing the total surface area of a rectangular prism. Participants must be able to see past the symbolic form of each method to notice similarities and differences. This type of analysis requires complex, non-algorithmic thinking and prompts preservice teachers to make their reasoning explicit to others.

Introduction to Case Studies

Recent research also suggests that small-group instruction can provide preservice teachers with opportunities to examine and respond to each others' misconceptions and interpretations, which in turn may inform their future teaching (Van Zoest, Stockero, & Edson, 2010). Based upon Stein et al's (1996) suggestion that further research is needed to provide details on factors which lower the cognitive demands of tasks during the implementation phase, we not only coded for the cognitive demands used during the solution process but also we examined the roles that preservice teachers adopted during parts of the conversation. We call these roles *participant-explainer*, *participant-questioner*, or *non-participant*. A participant-explainer is someone whose primary role is to explain a mathematical concept or procedure to the other members of the group. A participant-questioner is someone whose primary role is to ask pertinent, mathematically relevant questions of the other members of the group. A non-participant is someone who, for reasons not explored in this study, does not contribute to the group discussion in a mathematically relevant way.

Case Study of Group X: Interchanging Roles in a Small Group Discussion

The preservice teachers in group X, Sam, Morgan, Alice, and Lisa, were at first unsure that the formula given in the final question worked. They decided to verify whether the two methods yielded the same result for a square prism with dimensions 8, 8, and 6,

which was provided in a prior task. The excerpt below follows a discussion involving the computation of the surface area of the 8×8×6 prism in two ways. Morgan refocuses the group's conversation to answer the first part of the task regarding why the first given formula works.

[Beginning of Recorded Material]

171 Morgan: So how does this relate to the lateral surface area? Basically [the formula] takes the area of every individual face [makes faces with hands] and so…

172 Sam: They're parallel.

173 Morgan: Right, so it takes the parallel faces.

174 Sam: So can you go, [makes faces with hands – see prisms with opposite bases highlighted] you have this one, this one, and this one.

[End of Recorded Material]
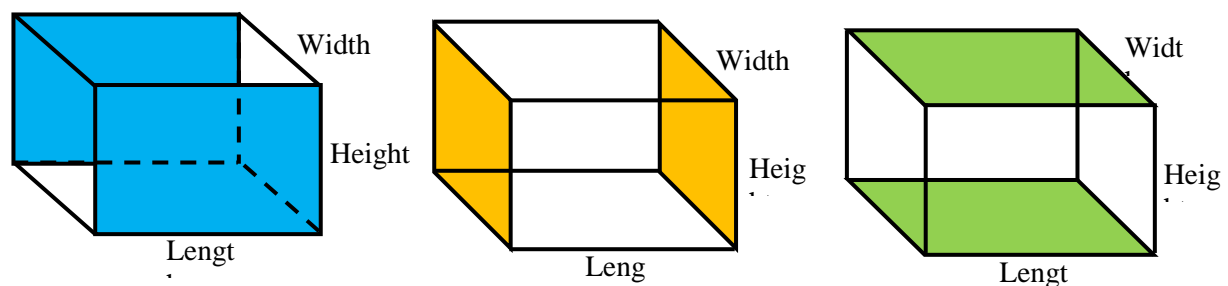
Figure 1: Rectangular prism with 3 pairs of parallel faces highlighted (created using Geometer's Sketchpad software from Key Curriculum Press).

175 Lisa: Wait so, it ends up taking the surface area of each face?
176 Morgan: And, with the lateral surface [method], what it does is it takes the area of the bases [makes bases with hands] but then all around it [makes circular motion], so you don't have to do [makes dimensions with hands] this, and this, and this. It's two calculations instead of three.

177 Lisa: Yeah.

178 Sam: But then really, we have to figure out what this length is. So if we only know the dimensions…

179 Lisa: Yeah. Like here we're given 32, but if we were just given 8,8,8,8…

180 Sam: Then we'd still have to calculate the…

181 Lisa: We'd still have to add up the… I mean… adding up the 8s is [easy]…

182 Alice: It'd still be easier to do the two calculations instead of the three. Like if you had 8,8,8,8, it'd be easier to just add those up, find 32 and do it that way. You know?

[End of Recorded Material]

Using the IQA-AR rubric for implementation of the task (Boston & Smith, 2009), this discussion rates as a level 4 in cognitive demand because preservice teachers solved a

genuine problem for which their reasoning is evident in their work on the task. The preservice teachers developed an explanation for the derivation of the new formula for lateral surface area and made connections between two strategies to find lateral surface area.

Using the Levels of Math-Talk rubric (Hufferd-Ackles et al., 2009), this conversation is rated at a level 3 in *questioning* because during this discussion, preservice teachers initiate conversations among themselves, without any instructor prompting. This conversation is rated at a level 3 in *explaining mathematical thinking* because preservice teachers offer their ideas, and spontaneously compare and contrast the number of calculations that need to take place using each of the two strategies of finding total surface area. This conversation is rated at a level 3 in *source of mathematical ideas* since the preservice teachers often finish each others' sentences and freely interject to repeat, explain, or build upon their classmates' thoughts. This conversation is rated at a level 3 in *responsibility for learning* since the preservice teachers listen to understand each others' ideas, and clarify others' work for themselves.

Case Study of Group Y: Stagnant Roles in another Small Group Discussion

Preservice teachers in group Y (Jean, Sarah, Paige, and Tiffany) begin discussing Question 16 together by trying to make sense of the formula provided in the question ($SA = 2lw + 2wh + 2lh$):

[Beginning of Recorded Material]

    196 Jean: This finds the area of each side [holds hands in front of her]…

    197 Sarah: Yeah

    198 Jean: And then multiply by 2 because there is two of each side.

    199 Sarah: Two of each side.

    200 Paige: Why?

    201 Sarah: Because if you draw a rectangular prism like so per se [draws prisms on worksheet]… So this is length and this is width [points to length & width] on this side. This is height [labels height of prism] and this is width [labels width of prism]. Height and width you have it twice because you have it here and here [points to two rectangles]. Length and width, you have it here and here [points to two rectangles]. And width (sic… should be "length" but the PST repeated width) and height you have it here and here [points to two rectangles].

    202 Paige: I think I got it.

    203 Sarah: It kind of sort of got the point across.

    204 Paige: Well because there are three things…

    205 Jean: Do you get it, Tiffany?

    206 Tiffany: No.

207 Jean: Here, I'll make one, ready? [Makes rectangular prism out of paper] Now I have a rectangular prism. So now you have a length and then a width and a height. So the 2 length times width is you're finding like the top and the bottom. Right? And you multiply it by 2 because there's 2 and then you do that for each one. So you find like this [points to bottom and top of prism], this [points to parallel sides of prism], and then the ends [points to the two bases of the prism].

[End of Recorded Material]

Using the IQA-AR Mathematics Rubric for Implementation of the Task (Boston & Smith, 2009), the cognitive demand of the task was decreased to a level 2 since the preservice teachers only explained how the formula determines the total surface area, a procedure which was specifically called for by the task (since the formula was given to them). The preservice teachers did not compare this method to the lateral surfaces area method for finding the total surface area of a prism. There was little ambiguity of how the given formula worked, since preservice teachers had prior knowledge of adding up faces to determine surface area. However, by not making any connections between the formula and the lateral surfaces area method, the preservice teachers missed an opportunity to develop a deeper understanding of lateral surface area –comparing and contrasting different methods forces students to think about why each method works and to make judgments about the reasonableness of each method. As a result, these preservice teachers avoided solving the most cognitively challenging part of the task; it is unclear whether the latter part of the question was skipped because the preservice teachers inadvertently forgot to read it, needed more time in order to answer that portion of the question, or whether they read the full question but thought that their answers were sufficient.

Using the Levels of Math Talk rubric (Hufferd-Ackles et al., 2004), this discussion's math talk is rated between levels 1 and 2. In the *questioning* component of math talk, the discussion is rated at a level 1 because Tiffany fails to initiate questioning when she did not understand Jean's and Sarah's ideas. Tiffany only speaks after Jean asks her if she understands, at which point Tiffany admits in line 206 that she did not understand the explanation. In the *explaining mathematical thinking* component of math talk, this discussion is rated between levels 1 and 2. Tiffany and Paige do not attempt to explain the mathematics behind the task at any point during the conversation, while Jean and Sarah fully explain and justify their thinking. In the *source of mathematical ideas* component of math talk, this discussion is rated at a level 1 because group discussion focuses exclusively on Jean's and Sarah's ideas; Tiffany and Paige do not contribute their own ideas to the discussion. In the *responsibility for learning* component of math talk, this conversation is rated at a level 1 because Tiffany does not take responsibility for her own learning, as she could have done by questioning Sarah's work; instead, she remained silent until Jean asked her directly if she understood (line 205). The explanations made by Sarah and Jean are brief, but the other preservice teachers in the

group did not ask for additional clarifications or further probing of how the formula and lateral surfaces area method are similar.

Discussion

In this article, we have provided detailed examples of small group interactions among preservice elementary teachers as they solved a cognitively challenging mathematical task. A focus on developing understanding versus finding answers may help to explain why group X exhibited a shifting roles dynamic while group Y did not. It is possible that if group X had a shared goal of understanding the mathematics of the task, then each member may have felt the need to make sense of the solution for themselves by asking questions or explaining their own or others' reasoning. On the other hand, if group Y focused their efforts on finding a solution without making sense of the underlying mathematics, then as long as one member got an answer, the rest of the group might not have felt the need to contribute any further. This can help to explain why some group Y members did not contribute to small group discussion.

Prior research supports our notion that discourse is a complex, messy process (Franke, Kazemi, & Battey, 2007). We observe that shifting roles during small group interactions between preservice teachers can help groups maintain high levels of cognitive demand and can also help explain productive talk that appears unstructured. However, more research is necessary before we can claim that shifting roles are necessary traits of productive small group interactions. Would a small group whose members all immediately take on the role of participant-explainer not be as productive simply because member roles do not shift? Are there other variables at play that helped group X maintain high cognitive demand levels? Further research should examine the potential interplay between the levels of math talk and tasks' cognitive demands to see if it is possible to generalize that small groups' success in solving cognitively challenging problems are related to small group interactions at high levels of math talk.

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EXPLORING HIGH-SCHOOL MATHEMATICS TEACHERS' SPECIALIZED CONTENT KNOWLEDGE: TWO CASE STUDIES

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This article draws upon an ongoing research in Taiwan which explores senior high school teachers' Mathematical Knowledge for Teaching (MKT). We use both quantitative and qualitative approaches to investigate two high-school mathematics teachers' specialized content knowledge (SCK) and its relationship to other domains of MKT, and revise the coding rubrics, developed by learning mathematics for teaching (LMT) project, to adapt to Taiwanese high-school classroom teaching practice. Results indicate that the revised coding system of classroom observation reveals different elements of mathematics teachers' SCK related to their knowledge of content and curriculum (KCC).

INTRODUCTION

Widespread agreement exists that what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn (Fennema & Franke, 1992). Recently, the introduction of MKT seems to have progress on answering this question (Hill, Ball & Schilling, 2008). The SCK, as a sub-domain of MKT, is the mathematical knowledge and skill unique to teaching, and is the greatest predictor which contributes to students' achievement (Hill, Blunk, Charalambous, Lewis, Phelps & Sleep, 2008). On the other hand, high-school teaching in Taiwan which is supposed to be in exact accordance with the national curricular standard might differ from the United States. Moreover, high-school teachers in Taiwan must carry out those duties to help students pass the college entrance examination that is held by the official assessment system and the curricular organization every year. Hence, high-school classroom teaching is highly oriented by the national curricular standard. This study, at first, aimed to explore the Taiwanese high-school mathematics teachers' SCK. Furthermore, we also examined the possible relationships of the participant teachers' SCK and their KCC.

THEORETICAL FOUNDATION

Teaching refers to the person who owns the specific knowledge, skills, attitudes and content, and who imparts intentionally to those without the specific content. To achieve this goal, Shulman (1986) suggested that teacher knowledge consisted of subject matter content knowledge (SMK), pedagogical content knowledge (PCK), and curricular knowledge. It's particularly interesting that PCK represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of

learners, and presented for instruction (Shulman, 1987, p. 8). PCK seems only to help teachers in the beginning stage of classroom teaching. However, while these teachers encounter all difficulties in their teaching, they might transform their mathematical knowledge into pedagogically useful forms (Ball & Bass, 2000). When discussing the idea of uncertainties in teaching, Ball and Bass thought it important to seek to complement the examination of curriculum and of what experienced teachers know with a mathematical analysis of core teaching activities, and to seek to identify the underlying resources entailed by these teacher activities.

In 2008, Ball, Thames, and Phelps identified MKT based on analyses of the mathematical problems that arise in teaching, and suggested six categories of MKT as following: common content knowledge, horizon content knowledge, SCK, knowledge of content and students (KCS), knowledge of content and teaching (KCT), and KCC. Particularly, they thought SCK is specialized to the work of teaching and only teachers need to know it; it is the mathematics knowledge and skill unique to teaching and requires knowledge beyond what teachers taught to students; and, it might help teachers overcome difficulties in their teaching practices. However, the problem of boundaries of these six categories confused Ball and her colleagues.

Davis and Simmt (2006) suggested four intertwining and fluent aspects of mathematics-for-teaching, including mathematical objects, curriculum structures, classroom collectivity, and subjective understanding. And the distinction between some purely mathematical knowledge and mathematical knowledge used in teaching is not appropriate (Huillet, 2009). Furthermore, Petrou and Goulding (2011) thought that PCK, SMK, the curriculum and its associated materials and the assessment system should be interplayed in the context. Figure 1 (Petrou & Goulding, p. 21) shows the relationships among different categories of teacher mathematical knowledge.
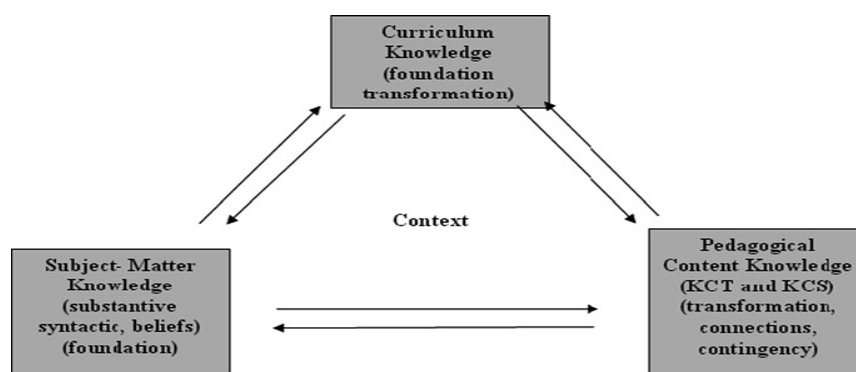
Figure1. Synthesis of models on teacher mathematical knowledge

This article attempted to explore the Taiwanese experienced high-school mathematics teachers' SCK, for Cannon (2008) found that based on researching their teaching practices under the framework of MKT, the training teachers lacked SCK. We also wanted to explore the relationship between teachers' SCK and KCC for two reasons. One is that there are limited studies focusing on exploring the relationship of teachers' SMK and KCC; the other is that SCK, as one form of knowing mathematics, is

excluded from those who do not teach mathematics (Ball, Thames, Bass, Sleep, Lewis & Phelps, 2009).

METHODOLOGY

The study, funded by the National Science Council in Taiwan, was framed in a qualitative research perspective that focused on interpreting people's thinking and actions based on actual settings, and provided the quantitative data in terms of building the system of classroom observation. The case study was chosen to be a way of investigating an empirical topic by following a set of pre-specified procedures (Yin, 1994). Three participant high-school mathematics teachers, each of whom have been teaching for more than 10 years in a public high-school in Taiwan, were purposefully selected for studying their MKT. The research period was lasted for two semesters, during which the researchers entered the participants' classrooms to observe and videotape their teaching. Our research group consisted of an experienced teacher educator, a retired consultant high-school mathematics teacher, and four graduate students. The observed and videotaped units were chosen by the consultant teacher, who had taught for more than 35 years in both public and private high-schools. In the first semester, we observed two teaching units (planes in space and lines in space); in the second semester, we observed another two teaching units (repetition combination and mathematical expectation). Due to limited space, only two participants' (Yan and Li) SCK will be reported here.

Data sources include the participant's own lecture notes, textbooks and handouts used by his mathematics department, as well as videotapes and interviews. All members of the research group worked collaboratively to discuss teaching contents, the way of the presentation, and the participant's possible teaching consideration. In this study, we conducted face-to-face, semi-structure, in-depth interviews with each participant before or after his classes, focusing on eliciting the teacher's mathematical knowledge and understanding that were presented in and related to his teaching practice.

Systematic classroom observation is used to provide evidence about what happens in classroom through a process of non-participant observation (McIntyre, 1980). This study attempts to revise the coding rubrics that were developed by LMT. The coding rubrics provide an instrument to measure the mathematics quality of instruction (MQI), and explore and name new elements of MKT (LMT, 2010). Given the purpose of this study, we focused on three sections: instructional formats and content, knowledge of mathematical terrain of enacted lesson, and use of mathematics with students. And we revised some terms of the original MQI instrument in terms of the participants' teaching features. The revised observational system is given in Table 1.

Categories	Subcategories	
Instructional Formats and Content	Format for segment	1. Whole group 2. Individual work
	Content	1. Geometry 2. Algebra 3. Probability and Statistics 4. Analysis 5. Discrete mathematics

		1. Review, warm up or homework 2. Introducing major task or concept
	Lesson type	3. Teacher's illustration 4. Student work time
		5. Synthesis, or closure
Knowledge of Mathematical Terrain	1. Conventional notation 2. Technical language	
	3. General language for expressing mathematical ideas	
	4. Selection of numbers, case and contexts for mathematical ideas	
	5. Selection of correct manipulations, and other visual and concrete models to represent mathematical ideas	
	6. Multiple models	
	7. Makes links among any combination of symbols, concrete pictures, diagrams	
	8. Mathematical descriptions 9. Mathematical explanations 10. Mathematical justifications	
	11. Computational error or mathematical oversights 12. Multiple perspectives	
	13. Comparison 14. Conceptual connection	
Use of Mathematics with Students	1. Uses student's errors 2. Elicits student's description or explanation	
	3. Interprets unusual/tentative/promising student productions	
	4. Answers student's problem 5. Launch of tasks/problems	

Table1: The revised system of classroom observation in Taiwan

In order to achieve the credibility of coding system and to reduce coders' biases, we examine the reliability of the system. LMT (2010) suggested that the coders must possess high levels of mathematical knowledge, and knowledge of mathematics for teaching, to code accurately. This study used Cohen's proposed *k*-coefficient for checking inter-observer agreement on the three selected videotaped lessons. *K*-coefficient should probably exceed 0.75 for acceptable observer consistency (Frick & Semmel, 1978). We called on another graduate student, who had been in charge of studying another teacher case, to examine the reliability. About the codes of Lesson type, we arranged them chronologically to get teachers' teaching modes.

RESULTS AND DISCUSSION

Table 2 shows the coding results of lesson type and use of mathematics with students. Yan and Li's teaching was basically following lecture-illustration-exercise mode, but their teaching was still different in some aspects. Li spent more time reviewing previous learned concepts and student homework, and helping students synthesize and unite all features of a concept. Yan, then, often immediately introduced major mathematical ideas and the related mathematics problems, but he provided students more time to think about what they had just learned and to solve the problems given in the class. On the other hand, although there were very few interactions with students, Yan and Li still attempted to encourage students to describe and explain the related mathematics ideas that they learned in the class. Li provided few problems for his students, and the solutions were always given immediately. Therefore, Li neither had chances to use student's errors to distinguish students' understanding, nor had opportunities to interpret students' unusual, tentative or promising productions to reinforce the learning of concept.

	Lesson type					Use of mathematics with students				
	RWH	IMT	TI	SWT	SC	USE	EDE	ISP	ASP	LTP
Yan	3%	19%	49%	29%	1%	6%	20%	5%	10%	39%
Li	15%	13%	57%	8%	7%	0%	16%	0%	7%	2%

1. RWH: Review, warm up or homework, IMT: Introducing major task or concept, TI: Teacher's illustration, SWT: Student work time, SC: Synthesis, or closure.

2. USE: Uses student's errors, EDE: Elicits student's description or explanation, ISP: Interprets unusual/tentative/promising student productions, ASP: Answers student's problem, LTP: Launch of tasks/problems.

Table 2: The proportion of lesson type and use of mathematics with students

Table 3 exhibits the proportion of presentation of knowledge of mathematical terrain. Their teaching would vary according to the characteristics of different teaching units. For equation of plane and line in space, Li used visual and concrete manipulative, multiple models and examples to present mathematical ideas and meanings more often. Moreover, he used the previous learned concepts to compare and connect to each other. However, Yan often illustrated and explained the links among symbols, concrete pictures and diagrams. In repetition combination, Li chose suitable manipulation, as well as other visual and concrete models to represent the formula of repetition combination. Yan, then, provided the comparison between the repetition permutation and repetition combination, and elicited the relevant characteristics of repetition combination. Finally, Yan provided students examples to think of the relationship between mathematical expectation and the weighted average. But Li led his students to learn mathematical expectation at the angle of the random variable, and suggested that the operational method of mathematical expectation should be the weighted average method.

		CN	TE	GL	SN	SC	MM	ML	MD	ME	MJ	MO	MP	CP	CE
EPL	Yan	2%	6%	1%	16%	18%	11%	37%	64%	19%	5%	3%	13%	7%	7%
	Li	2%	7%	11%	48%	62%	27%	21%	62%	51%	6%	19%	31%	33%	12%
RC	Yan	4%	11%	0%	14%	0%	7%	18%	82%	43%	0%	11%	18%	25%	11%
	Li	3%	3%	0%	38%	21%	13%	5%	41%	62%	0%	13%	28%	21%	8%
EX	Yan	6%	6%	0%	6%	0%	0%	18%	76%	41%	0%	6%	18%	29%	12%
	Li	0%	14%	0%	33%	5%	0%	0%	62%	52%	0%	29%	33%	24%	5%

1. EPL: Equation of plane and line in space, RC: Repetition combination, EX: Expectation.

2. CN: Conventional notation, TE: Technical language, GL: General language for expressing mathematical ideas, SN: Selection of numbers, case and contexts for mathematical ideas, SC: Selection of correct manipulations, and other visual and concrete models to represent mathematical ideas, MM: Multiple models, ML: Makes links among any combination of symbols, concrete pictures, diagrams, MD: Mathematical descriptions, ME: Mathematical explanations, MJ: Mathematical justifications, MO: Computational error or mathematical oversights, MP: Multiple perspectives, CP: Comparison, CE: Conceptual connection.

Table 3: The proportion of presentation of knowledge of mathematical terrain

We discussed two selected teaching tasks in more details as follows. Firstly, after introducing the major concept of the mathematical expectation, Yan brought forth a question in the textbook as follows: "There are six same-sized coins, including four 5-dollar and two 10-dollar coins in the bag. After taking a coin out and putting it back, you can take a coin again. Please find the mathematical expectation of amount of the two coins." During the process of solving this problem, Yan asked student whether $\frac{4 \times 4}{6 \times 6}$ could be presented by $\frac{4}{6} \times \frac{4}{6}$. This question aroused our interest because the two sides of equal sign revealed different mathematical meanings. Figure 2 snapshots a critical part of the video clip.

Figure 2: The problem of mathematical expectation (video, Yan, 20100603)

However, the explanation of equal sign must be based on conditional probability and independent events, and Yan said in the interview:

> Some teachers would agree, but some teachers would disagree…Although the conditional probability does not belong to the second-grade curriculum of high school, students may be confronted with some problems that could be solved by the classical or conditional probability, and I thought that some problems solved by the conditional probability were natural…So I taught this unit before the mathematical expectation (interview, Yan, 20100810).

Secondly, Li directly used the multiplication law shown above, when he taught a similar problem as follows: "There are five red balls and two black balls in the box. After taking a ball out, you are not required to put it back. Please find the mathematical expectation of the times of taking balls out before you take the red one." Figure 3 shows a critical part of the video clip.

Figure 3: The problem of taking a ball (video, Li, 20100602)

In the interview, Li said:

> I did not think too much, and in fact, every student can accept this multiplication law.But, the idea of conditional probability is the concept of reducing or changing the original sample space. According to the perspectives of textbooks, I thought the order of these

teaching units were inappropriate. I did not follow the concept of changing the original sample space to teach the idea of the multiplication law…for I considered teaching the idea of reducing the sample space was enough. Surely, the multiplication law must be based on the conditional probability and be extended to the independent events and to the Bayes' theorem. Actually, I had taught this idea in the previous permutation lessons, and I thought that the objective of this problem could be introduced to the independent events…Certainly, the standard writing was $\dfrac{P_1^2 \times P_1^5}{P_2^7}$, but, for my students, $\dfrac{2}{7} \times \dfrac{5}{6}$ was clearer and more efficient (interview, 20100623).

Although both Yan and Li realized the distinction between the classical and conditional probability, the ways they taught were very different. Yan took the accuracy of the mathematical explanation and the practicality of problem-solving into consideration so that he modified the sequence of the curricular structure. Ball et al. (2008) pointed out that KCT includes the arrangement of the sequence of the curriculum, and teachers with SCK can choose and develop workable definition (Charalambous, 2008). Therefore, there exists the multi-dimensional fluidity among SCK, KCT, and KCC. However, Li chose workable definition that did not make students confused. Particularly, his workable definition hid his understanding of his students under his teaching practice and was connected with the previous, present and even future concepts included in the curriculum. So Li's SCK reveals the multi-dimensional fluidity between KCS and KCC.

IMPLICATION

Although the system of classroom observation reveals two high-school teachers' different teaching modes, their SCK also reflects the latent parts between the interactions of KCC and teachers' understanding of the content. It seems possible that some degree of overlapping exists between sub-domains of MKT in different countries (Delaney et al. 2008), and this exploratory study indicates the fluidity of Taiwanese high-school mathematics teachers' knowledge. And, in particular, SCK, the form of knowing mathematics, is excluded from those who do not teach mathematics (Ball et al., 2009). Thus, the findings of this study also point out new directions for further research such as estimating to what aspects can SCK be extended, investigating how SCK might improve teacher's teaching and student's learning in mathematics classroom, as well as how to develop high-school mathematics teachers' SCK.

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THE EFFECT OF DIFFERENT PATTERN FORMATS ON SECONDARY TWO STUDENTS' ABILITY TO GENERALISE

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This paper reports on the test performance of 105 Singapore secondary school students in pattern generalising tasks to determine whether the format of pattern display hinders students' pattern recognition and ability to generalise. Data were collected through administering a written test comprising four figural generalising tasks involving both linear and quadratic patterns, presented in two different formats. The students, assigned to work on tasks in only one of the formats, had to establish the functional rule underpinning each pattern. The findings revealed that the students could generate the functional rule regardless of the given pattern format. Further, there was no gender difference in student performance for each task.

BACKGROUND AND THEORETICAL FRAMEWORK

Pattern generalising tasks typically involve getting students to examine specific cases to search for a pattern, extend the pattern to predict other cases, and articulate the functional relationship underpinning the pattern using mathematical symbols. These tasks can be classified as (1) numerical, which lists the pattern as a sequence of numbers, or (2) figural, which presents the pattern as a sequence of figures. The functional relationship follows either a linear or non-linear rule.

Several past studies have drawn attention to students' difficulties in dealing with such generalising tasks. These difficulties are often traced to student-related factors such as inexperience in working with generalising tasks (Stacey, 1989; Warren, 2005), ignorance of appropriate generalising strategies (Moss & Beatty, 2006), lack of spatial visualisation techniques (Becker & Rivera, 2006; Warren, 2005), lack of an understanding of the variable concept (Becker & Rivera, 2006), and inexperience in using the highly specific mathematical language of algebra to express generality (Hoyles, Noss, Geraniou, & Mavrikis, 2009). But a few recent studies seemed to throw up suggestions that certain features of the generalising tasks could have added to students' difficulties. We shall first describe what we mean by task features and then follow with a discussion of some of these studies.

We take task features to mean defining characteristics that make up the problem situation in a mathematical task. For pattern generalising tasks, the features can include whether (1) the given pattern is presented as a sequence of numbers or diagrams, or simply as a single diagram; (2) the functional rule describes a linear or non-linear relationship; and (3) the diagrams are depicted two-dimensionally or three-dimensionally (Chua, 2009). Task features such as these three can co-exist in a

generalising task. The first feature, which we called as *the format of pattern display*, is the main focus of this paper.

In a study by Hoyles and Küchemann (2001) where one of the tasks was to find the number of grey tiles needed to surround a row of 60 white tiles when given just a single diagram and a description of how it was constructed, the success rate for high attaining students was rather low. Taking into account the students' abilities, this familiar tile-pattern task should not be so difficult. Thus could the students' difficulty have been a result of being given just a single diagram in the task? In another study by Becker and Rivera (2006), students were found to benefit from the given two-dimensional and sequential diagrams which helped to direct their attention to the basic core of the figural pattern that remains invariant and the part that is growing, enabling them to establish the functional rule for predicting any term. On the contrary, students in Warren's (2000) study were unable to even spatially visualise a sequence of two-dimensional diagrams in a classic matchstick problem involving a row of squares. Could adding more diagrams to the sequence have helped Warren's students to better visualise the pattern? If the diagrams are not presented sequentially, will Becker's and Rivera's students still be able to generalise the pattern?
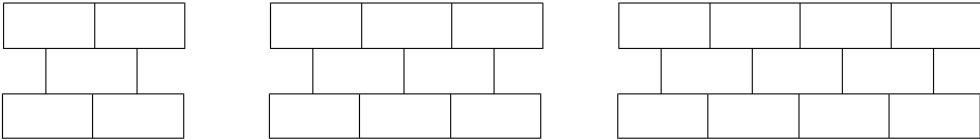
So far, no study seems to have attempted to examine the effect of different formats of pattern display on students' pattern recognition and ability to generalise. Thus our present study sought to fill in what appears to be a gap in this worthy research theme. This paper aims to add to the body of work on pattern generalisation by exploring these questions: Is there any effect of the format of pattern display on students' rule construction? Is there any difference in students' rule construction between the format of pattern display and gender?

METHODS

Our present study used a between-subjects experimental design to examine whether different formats of pattern display had any effect on students' rule construction. Four linear and four quadratic figural generalising tasks were developed for this investigation. All the eight tasks were deliberately made less structured without any part questions that gradually led students to detect and construct the general rule. This was to allow the students a greater scope for exploring the pattern structure so that we could then see how they recognised and perceived the pattern without any scaffolding. Each task existed in two different formats, with its pattern depicted as (1) a sequence of three successive diagrams, and (2) a single diagram or a sequence of two or three non-successive diagrams. For each format, the eight tasks were divided into two sets of four tasks, administered on two separate days. The task distribution was done in such a way that produced parallel sets of tasks, differing only in pattern format. We report here on the two linear and two quadratic generalising tasks in the first set. Figure 1 below shows the two different formats of a linear task from this set.

John used identical bricks to make several designs of different sizes on a long wall.

The diagrams below show three designs he made.

Size 1   Size 2   Size 3

As the size number became larger, more bricks were used.
John wanted to find the number of bricks he had to use to make any size.
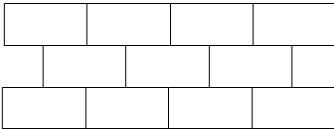He used a rule to find this number.

(a) Successive format

John used identical bricks to make several designs of different sizes on a long wall.

Each design is made up of three rows of bricks.
The top and bottom rows are identical, containing the same number of bricks.
The middle row is shorter and has one fewer brick than each of the other two rows.

The diagram below shows how a Size 3 design that John made looks like.

Size 3

As the size number became larger, more bricks were used.
John wanted to find the number of bricks he had to use to make any size.
He used a rule to find this number.

(b) Non-successive format

Figure 1. Bricks

The *Bricks* pattern was represented by three consecutive diagrams in the successive format (Figure 1a) and by a single diagram in the non-successive format (Figure 1b). The latter format included a description of how the pattern grew, which was deemed as essential information for students when given only a single diagram. For the other three generalising tasks, the successive format of the pattern was similar but the non-successive format now included one with two diagrams and two with three diagrams. All the tasks required students to work out a general rule for the pattern in terms of the size number individually, as well as to justify how they obtained the rule. Figure 2 below offers an overview of the three patterns in their respective formats.

The four generalising tasks, to be completed in 45 minutes, were administered to 105 Secondary Two students (aged 14 years) from a secondary school in Singapore. The students, 55 girls and 50 boys from three intact classes in the Express course selected by the school, belong to the top 60% of the entire Secondary Two cohort of students in Singapore. The students were separated into two groups, Group 1 (n = 55, 30 girls, 25 boys) and Group 2 (n = 50, 25 girls, 25 boys), using their results in a 50-mark baseline

mathematics test administered a few weeks earlier. The mean baseline test scores for Group 1 (G1), Group 2 (G2) and the total sample were 43.11, 42.60 and 42.87 respectively, which were roughly similar. G1 worked on generalising tasks with successive diagrams whereas G2 was given tasks with non-successive diagrams.
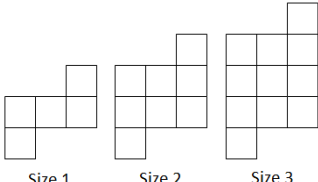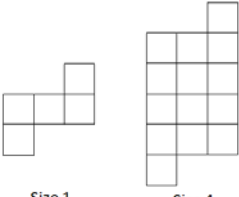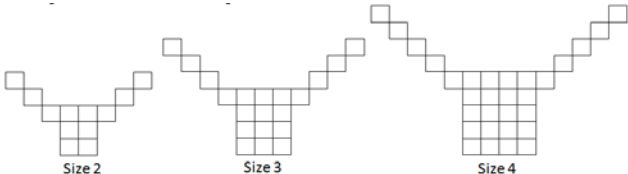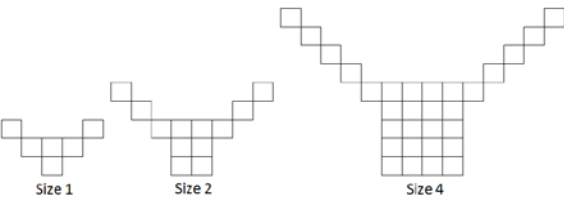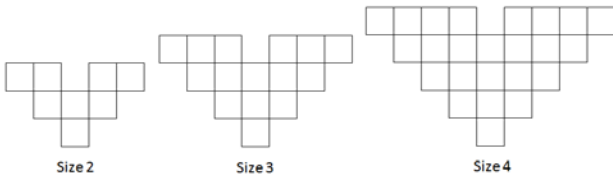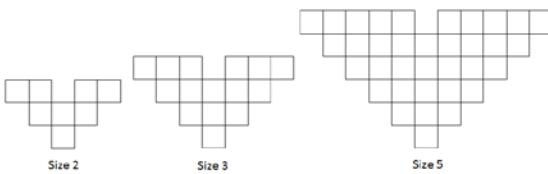
	Successive format	Non-successive format
Linear Birthday Party Decorations		
Quadratic Oh Deer		
Quadratic Tulips		

Figure 2. Three other generalising tasks

Having learnt the topic of number patterns in the Singapore mathematics curriculum before participating in this study, these students should be able to continue for a few more terms any pattern presented as a sequence of numbers or figures, make a near and far generalisation and establish the general rule in the form of an algebraic expression for predicting any term. Further, they should also be far more familiar in dealing with linear patterns than with non-linear ones, which are less commonly featured in their mathematics textbook.

Student responses for individual generalising tasks were scored using an analytic rubric with a six-point (0, 1, 2, 3, 4 or 5) marking scheme for rule construction and for generalising strategy used. Using responses to the *Bricks* task, a score of 5 points for rule construction was given to a correct general rule (e.g., $5+3(n-1)$), 4 points to an incorrect rule due to minor slips in algebraic manipulation (e.g., $5+3(n-1) = 5+3n-1=3n+4$), 3 points to using a functional relationship to show the structure of terms without deriving the general rule (e.g., $5+3(10-1)$ for Size 10), 2 points to a correct recursive rule (e.g., add 3 to get the next term), 1 point to an incorrect recursive rule (e.g., $n+3$), and 0 point to an incorrect or blank response. For generalising strategy used, 5 points were given to showing clear evidence of using numerical or

visual cues from the pattern to derive the correct general rule, 4 and 3 points to working out the structure of non-immediate and immediate terms respectively, 2 points to looking at the differences between terms, 1 point to looking at the nature of the terms (e.g., some terms are even, some are odd), and 0 point to a blank response or incorrect strategy. Figure 3 shows a G2 student's response to the *Oh Deer* task that was awarded 3 points for rule construction and 4 points for generalising strategy used, totalling up to 7 points.
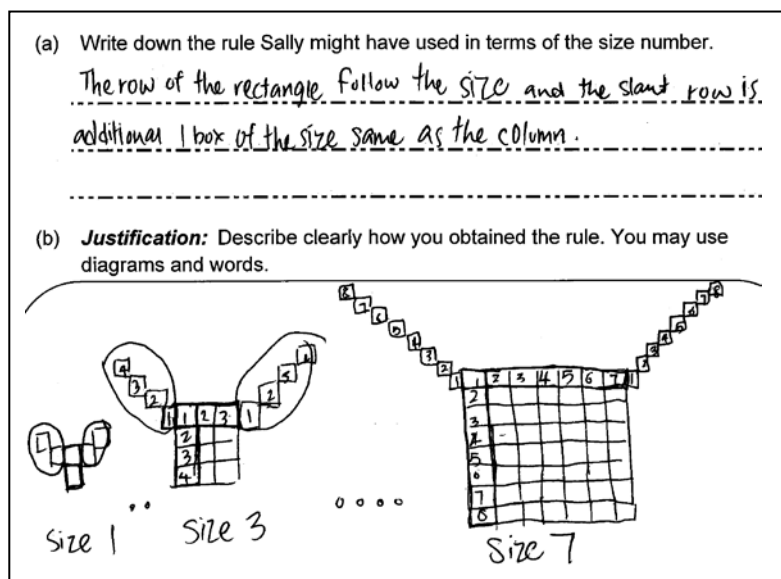
Figure 3. A 7-point student response to Oh Deer

For each pattern format, the mean score and standard deviation by gender for each task were worked out and then used to measure students' performance in that task. Independent *t*-tests were conducted for each task to test for any significant differences in students' performance between G1 and G2.

RESULTS

This section presents the findings to the following two questions that guided this study.

> *1. Is there any effect of the format of pattern display on students' rule construction?*

Table 1 shows the mean scores and standard deviations of the four generalising tasks, as well as the t-statistics for the differences in mean scores between G1 and G2 students. The mean scores of G1 students spanned a wider range, from 5.89 in *Tulips* to 7.93 in *Birthday (BD) Party Decorations*, than those of G2 students, from 6.18 in *Tulips* to 7.20 in both *Bricks* and *BD Party Decorations*. Students in G1 and in G2 produced fairly similar mean scores for the two linear tasks. As for the two quadratic tasks, the mean scores of G2 students were also consistent, but those of G1 students differed by more than 1 point.

G1 students obtained higher mean scores than G2 students in *Bricks*, *BD Party Decorations* and *Oh Deer*, but vice versa in *Tulips*. The greatest difference in mean

scores existed in *BD Party Decorations* (0.73), favouring the G1 students, while *Tulips* had the least difference (0.29), favouring the G2 students instead. However, t-test showed that the differences between G1 and G2 students in all four tasks were not statistically significant at the 5% level.

		Bricks		BD Party Decor		Oh Deer		Tulips	
		mean	sd	mean	sd	mean	sd	mean	sd
Girls	G1 (n = 30)	7.20	2.987	7.63	3.327	6.53	3.646	5.67	3.661
	G2 (n = 25)	6.84	3.636	6.52	3.842	5.24	4.186	5.96	3.422
Boys	G1 (n = 25)	8.28	2.638	8.28	2.638	7.36	3.695	6.16	3.508
	G2 (n = 25)	7.56	3.959	7.88	3.308	7.32	3.727	6.40	3.253
Total	G1 (n = 55)	7.69	2.860	7.93	3.024	6.91	3.658	5.89	3.568
	G2 (n = 50)	7.20	3.780	7.20	3.614	6.28	4.061	6.18	3.312
Difference between G1 and G2 students	t	p-value	t	p-value	t	p-value	t	p-value
	.755	.452	1.122	.265	.835	.406	− .429	.669

Table 1. Results of students' performance in each generalising task

2. *Is there any difference in students' rule construction between the format of pattern display and gender?*

	Bricks		BD Party Decor		Oh Deer		Tulips	
	t	p-value	t	p-value	t	p-value	t	p- value
Diff between girls and boys in G1	− 1.407	.165	− .787	.435	− .832	.409	− .507	.614
Diff between girls and boys in G2	− .670	.506	− 1.341	.186	− 1.855	.070	− .466	.643
Diff between girls in G1 and G2	.403	.688	1.152	.255	1.225	.226	− .305	.762
Diff between boys in G1 and G2	.757	.453	.473	.639	.038	.970	− .251	.803

Table 2. Student's Performance Between Pattern Format and Gender

Table 2 above shows the t-statistics for the differences in mean scores between gender in each group and between groups for each gender.

From Table 1 above, G1 boys and G2 boys outperformed G1 girls and G2 girls respectively in that the boys obtained higher mean scores than the girls in every task. However, Table 2 shows that the differences in mean scores between girls and boys in G1 and in G2 were not statistically significant for every task.

When the mean scores of each task were compared between the two groups of girls, it was found that the mean scores of G1 girls were higher than the mean scores of G2 girls in *Bricks*, *BD Party Decorations* and *Oh Deer*, but lower in *Tulips*. As can be seen

from Table 2, t-tests showed that the differences in mean scores for each of the four tasks were not statistically significant. Similar findings were also observed amongst the boys.

DISCUSSIONS AND CONCLUSION

This paper explored whether the different formats of pattern display in pattern generalising tasks played a role in students' pattern recognition and ability to make generalisations. The results suggest two main preliminary conclusions. First, students could generate the functional rule underpinning a pattern even if the pattern deviated from the typical and familiar format of three successive diagrams to involve non-successive diagrams. For instance, a sizeable number of G2 students (nearly 70%) did not seem to flounder when asked to construct the linear rule for *Bricks*, when given only a single diagram, and for *BD Party Decorations*, when shown two diagrams. It was also encouraging to note that the other two non-successive quadratic generalising tasks did not seem to disconcert more than half the G2 students who derived the functional rule successfully.

Students' ability to establish the rule seemed to be assisted very much by their awareness of the structure inherent in the pattern. To become aware of the structure, some students needed to draw additional diagrams themselves before they could see the structural relationship from the geometrical arrangement of tiles or cards. For some other students, drawing such diagrams was not necessary at all. By treating the given diagrams generically, they were able to abstract the structural relationship from them. For instance, some G2 students construed Size 1 of *BD Party Decorations* as a row of three cards plus two more, Size 4 as four rows of three cards plus two more, and hence, Size n as n rows of three cards plus two more, or $3n+2$ when expressed in symbols. This finding lends support to the view of Mason, Stephens and Watson (2009) that teaching students to identify structure in the learning of mathematics is crucial. This is because being able to recognise structure is an extremely useful skill for students to have in that their attention will no longer be drawn to focus on the usual counting of tiles or cards but on abstracting relationships between sets of objects, then followed by articulating a rule that captures this relationship.

Second, although the mean scores of boys for all four generalising tasks in both groups were higher than that of girls, there were no significant gender differences within each group. This suggests that both girls and boys in this sample performed equally well on pattern generalisation, a topic that had gained some notoriety for its difficulty.

Finally, our present study had just recently completed at the time of preparing this paper. We will need to analyse all the data collected from other schools to see if the findings presented in this paper still remain consistent. Thereafter, we might then have more conclusive evidence to decide whether or not the format of pattern display is really a hindrance to students' pattern recognition and ability to generalise.

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STUDYING, SELF-REPORTING, AND RESTUDYING BASIC CONCEPTS OF ELEMENTARY NUMBER THEORY

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The objective of this case study is to look in depth into personal factors affecting metacognitive monitoring and control in self-regulated study and restudy of basic concepts of elementary number theory. We incorporate a theoretical framework of embodied cognition and learning with a wide spectrum of observational methods ranging from audiovisual, keyboard and screen capture, eye-tracking, and self-report data, to psychophysiological data including electrocardiography (EKG) and respiration rate data. Our aim is to generate "learner profiles" that provide deeper insights into personal factors implicated in motivation, metacognition, and beliefs, pertaining to self-regulated learning and mathematics anxiety, which can be used to better inform assessment and tailor instructional design in mathematics education.

OBJECTIVE AND PROPOSE

The broader objective of this program of research is to look in depth into personal factors affecting metacognitive monitoring and control in self-regulated study and restudy of basic concepts of elementary number theory that include the division theorem, divisibility, divisibility rules, factors, divisors, multiples, and prime decomposition (Campbell, 2002; Campbell, Cimen, & Handscomb, 2009). We begin doing so in this research report with tight observational control (Campbell, 2010) of a single case study into personal factors affecting study and restudy of this material, interjected with self-reports of judgments of learning (JOLs) (Nelson, Dunlosky, Graf, & Narens, 1994). Our ultimate objective and purpose is to generate "learner profiles" which can be used to better inform assessment and tailor instructional design in prospective and preservice mathematics teacher education.

THEORETICAL FRAMEWORK

We take the position that all subjective experience is manifest in some way in brain and body behavior, which justifies taking a more rigorous approach to behavioral control stems from a theoretical framework that views cognition and learning as embodied (Campbell, 2003a; Varela, Thompson, & Rosch, 1991). Accordingly, recording and integrating embodied, i.e., psychophysiological, behavioral responses should shed light on cognition and learning that could otherwise remain hidden using more limited traditional techniques such as field notes, self-reports via interviews, talk-aloud protocols, psychometrics, and audiovisual recordings of overt behavior (Campbell, 2003b; Campbell with the ENL Group, 2007).

METHODOLOGY

Our instrument for investigating metacognitive monitoring and control of study-restudy content is presented to our participant, a prospective mathematics teacher, in six pages delivered using gStudy (Perry & Winne, 2006). This subject matter content for study-restudy was specifically designed to involve three levels of learning: computation (C); understanding (U); and reasoning (R). Our participant was allowed to study this material at her leisure. The study material was then presented to our participant once again in a manner that highlighted different parts thereof, enabling her to provide judgments of learning (JOLs), i.e., to indicate whether she had learned that content very well, well, or not well (Figure 1). Once she completed the JOLs, she was given an opportunity to restudy the material in preparation for a test on that study material.
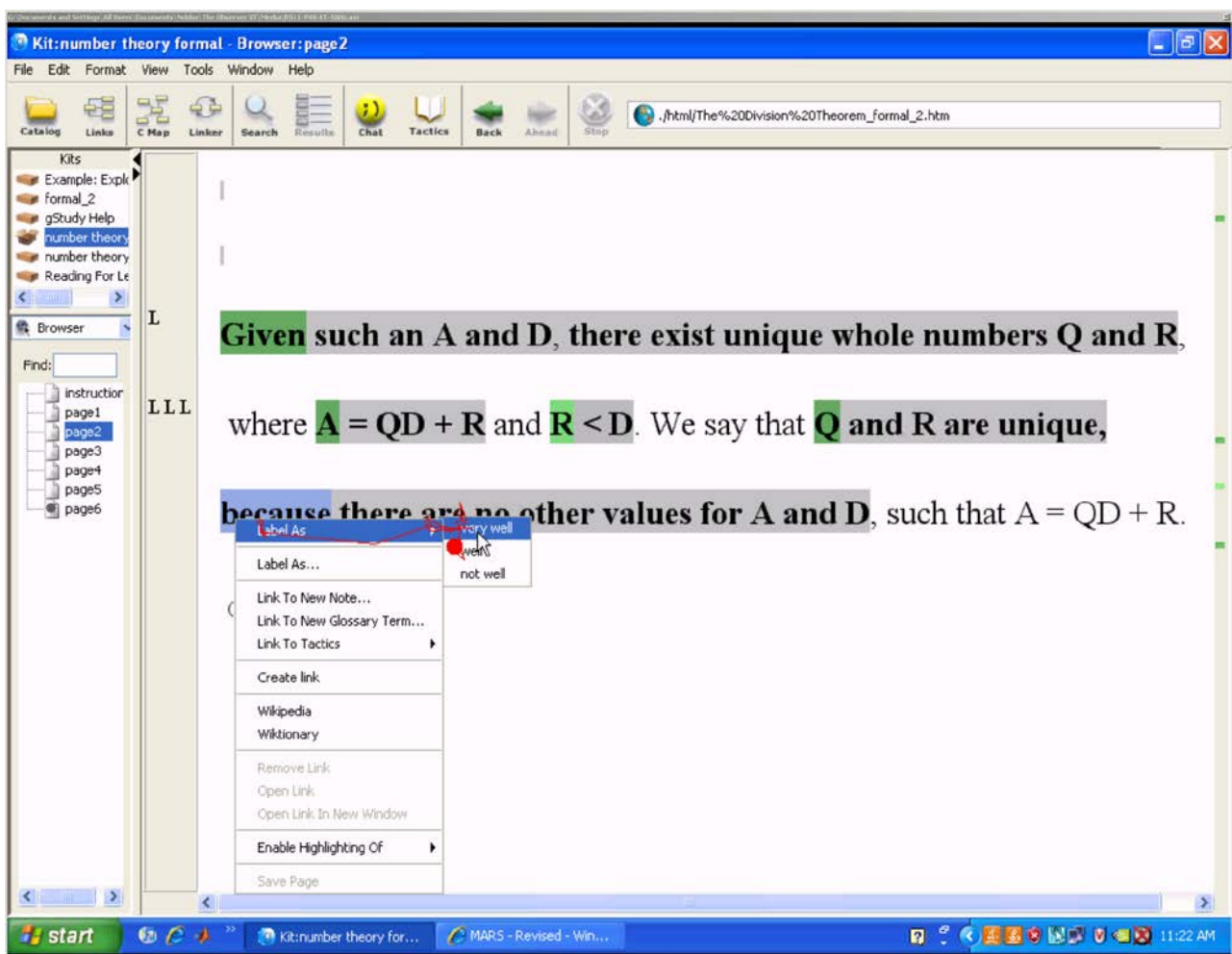
Figure 1: Screen capture of Page 2 of study material with participant indicating JOL

All methods of observation and measurement have intrinsic limitations. Thus, it is not possible that every subjective nuance of learning and lived experience can be objectively observable, measurable, and identifiable in brain and body behavior. Hence, we will likely meet with greater success using psychophysiological means of observational control to matters involving lived experiences that are more intensely

embodied, such as anxiety. Indeed, the fact that anxiety is such a deeply embodied phenomenon, to the extent of being physically disabling, in itself warrants inclusion of psychophysiological methods into our repertoire of observational methods.

We expect to detect evidence of anxiety with increases in heart rate and respiration (e.g., Kelly, 1980; Dew, Galassi, & Galassi, 1984). Accordingly, we incorporate a wide spectrum of observational methods enabling us not only to record overt behavior, using audiovisual techniques, and self-report data, using psychometric questionnaires and JOLs, but also covert behavior related to psychophysiological responses of various organs, including brain, heart, lung, and skin, along with muscle response and eye movement. We further augment our observational control by presenting our stimuli via computer and using screen and keyboard capture (Campbell, 2010).

Our model for interpreting our data on metacognitive monitoring and control is an adaptation of Elliot (1999) and Elliot and McGregor's (2001) motivational distinctions between mastery-performance and approach-avoidance, fused with Nelson, et al's (1994) notion of self-reported judgments of learning (JOLs) resulting from metacognitive monitoring (Table 1).

	Mastery / intrinsic motivation	**Performance / extrinsic motivation**
Approach / taking time	JOL: not well understood	JOL: very well understood
Avoidance / not taking time	JOL: very well understood	JOL: not well understood

Table 1: Metacognitive monitoring and control model for interpreting motivation in restudy

Mastery is learning something for its own sake. Performance is focused on outcome. Here, we interpret *mastery-approach* to represent taking time in restudy to learn something self-judged to be not well understood, whereas *mastery-avoidance* represents not taking time for restudy of content self-judged to be well understood. We interpret, *performance-approach* to involve taking time to better consolidate content self-judged as well understood, whereas *performance-avoidance* represents not bothering to take additional time restudying content already self-judged as poorly understood. In sum, mastery and performance represent intrinsic and extrinsic motivation, respectively, whereas approach and avoidance represent taking or not taking time in restudy.

DATA SOURCES AND EVIDENCE

Behavioral data

Our participant was "wired up" to monitor fluctuations in heart and respiration rates. We presented the gStudy stimulus to our participant using a Tobii 1750 eye-tracking monitor, which detects reflections of infrared light pulses on a participants' retina to trace what is being looked at from moment to moment. An ultra sensitive microphone allowed for high-quality recordings of think-aloud narratives. Infrared video cameras record important aspects of the participant behavior, such as facial expressions and

body movements, from three vantage points. Steps were taken to maximize the accuracy of eye tracking data of the study-restudy material such as increasing font size and spacing of the study material (Figure 1). Data streams were integrated, time synchronized and analyzed using Noldus's Observer XT (Figure 2). We relied on cross-calibrating audiovisual, eye-tracking, and other data to ensure we were selecting behavioral data for analysis at the appropriate times (Campbell & the ENL Group, 2007).
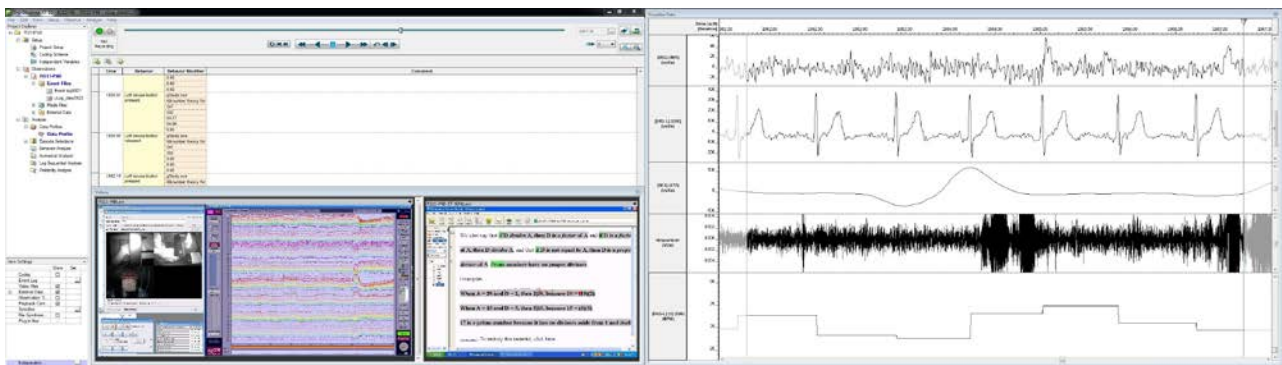
Figure 2: The integrated and synchronized data set using Noldus's Observer XT

Self-report data

The participant was given informed consent. She filled out a demographic questionnaire. Pre- and post-questionnaires were used prior to and after engaging our participant in the study-restudy activity. Pre-questionnaires, we do not go into detail here, included the Motivated Strategies for Learning Questionnaire (MSLQ) (Duncan & McKeachie, 2005), the Epistemic Belief Inventory (EBI) (Schraw, Bendixen & Dunkle, 2002), the Metacognitive Awareness Inventory (MAI) (Schraw & Dennison, 1994), the Math Anxiety Rating Scales (MARS) (Hopko, 2003).

A Number Theory Pre-Questionnaire (NTPreQ) designed to gain insight into how comfortable the participant was with their abilities regarding calculation, reading, recall, comprehension, and reasoning. After completing the pre-questionnaires, the participant engaged upon the study component of the experiment. Following completion of this initial study period, the participant labeled their judgments of learning (JOLs) pertaining to how well she learned computational, conceptual, and inferential aspects of the study material. The specific aspects of the study material to be self-judged were highlighted (see Figure 1). After labeling the JOLs, the participant was given a 10-question true/false test on the study material and was asked to rate her confidence in her answers on a scale of 0-10 for each question. After a short rest, the participant engaged in restudy of the material, and then rewrote the same test. Finally, the participant filled out a metacognitive Number Theory Post-Questionnaire (NTPostQ) pertaining to her experiences in the experiment.

Participant

Our participant was a 22 year-old female undergraduate student and prospective teacher, of Vietnamese descent and a major in molecular biology, with no previous exposure to the basic concepts of elementary number theory presented in the study/restudy material. Her health was self-reported as good (no anxiety disorders or symptoms, no physical problems). After the observation she reported being "a little worried that it was going to be hardcore math theory that was being tested on the exam part" before the observation.

RESULTS

The participant's average heart rate for the study period was 75.1 beats per minute (bpm), and reduced to 69.0 bpm for the self-report period, and reduced further to 67.0 bpm for the restudy period of the same subject content material. Her respiration rates were 20.3, 18.0 and 17.8 breaths per minute for the study, self-report and restudy periods, respectively, while her respective eye blink rate over those three time periods were 37.5, 16.0, and 34.3 blinks per minute. These values are summarized in Table 2.

	Time Spent (seconds)	Heart Rate (beats per minute)	Respiration rate (breaths per minute)	Eye Blink Rate (blinks per minute)
Study	608	75.1	20.3	37.5
Self-Report	278	69.0	18.0	16.0
Restudy	98	67.0	17.8	34.3

Table 2: Time and physiological data summary for study, self-report, and restudy periods

As heart rate is a strong indicator for the level of stress and anxiety (Kelly, 1980; Dew, Galassi & Galassi, 1984), the results clearly indicate that the participant was less anxious, i.e., more relaxed, for the restudy period, in comparison with the study period.

During the self-report period, the participant was re-shown the six pages of study material with items highlighted and she was asked to report her judgment of learning (JOL) regarding them (35 in total). She was asked to choose among three options per case for her self-reporting: not well, well and very well (Figure 1). We substituted scores of -1 for not well, 0 for well, and +1 for very well. We then tallied this scoring to give us a total JOL confidence indicator of +11.

All the JOLs labeled by the participant as "not well" learned involved calculations, and our data indicate she did not spend much time on these tasks. Hence, in accord with Table 2, the participant can be classified as having a performance-avoidance orientation in this regard. The participant reported she learned most of the understanding tasks very well, while reporting most reasoning tasks she had learned, well or very well.

Question	Question Type	Test 1 Results	Test 1 Confidence	Test 2 Results	Test 2 Confidence	NTPreQ
1	Calculation	Incorrect	7	Correct	8	3
2	Calculation	Correct	10	Correct	10	3
3	Understanding	Correct	10	Incorrect	10	4
4	Understanding	Correct	10	Correct	10	4
5	Reasoning	Incorrect	9	Correct	10	3
6	Reasoning	Incorrect	10	Incorrect	10	4

Table 3: Number theory test and NTPreQ results

Table 3 summarizes our results from the test that was administered after the study period and the results from the same test, which was administered once again after the restudy period. These results align well with results of her self-assessment from the NTPreQ, in which she reported her level of comfort on a scale of 1 (not comfortable at all) to 5 (completely comfortable), with calculation tasks as 3, while reporting her level of comfort with understanding involving recall and comprehension as 4, and with aspects of reasoning as 3.5. Test results substantiate these reports, reinforcing that she is less confident with her answers with calculation tasks compared to understanding and reasoning tasks. Although she reports a somewhat higher confidence for reasoning tasks, she is less successful on this type of task compared to understanding, which she self-reported in the NTPreQ prior to the study/restudy periods as being most comfortable with. Again, her JOLs indicate she spent less time on the pages that involved calculation.

Another interesting result concern answers provided for NTPostQ, which was presented to her after restudying the material and having taken the test for the second time. She stated that the learning task was not interesting for her (ranked 0 out of 7) and it was not challenging for her (ranked 2 out of 7). She also indicated that she restudied the items she found most difficult to understand. These answers indicate she is a mastery-oriented learner when it comes to subject content involving understanding and reasoning.

DISCUSSION AND CONCLUSIONS

Based on our theoretical framework of embodied cognition and learning, we have felt compelled to augment traditional audiovisual, psychometric, and other self-report data sets with psychophysiological observations (Campbell, 2010). Doing so provides an additional dimension for empirical grounding and cross validation of our results. Our most salient results to this point, for the purpose of this report, focus on the internal and external consistency of the self-report data (i.e., the NTPreQ, JOLs, and NTPostQ) and the psychophysiological data (i.e., the average heart and respiration rates).

With regard to the self-report data, interpreted with our metacognitive monitoring and control model (Table 1), we see that our participant has a performance-avoidance

orientation to calculation, whereas she has more of a mastery-approach orientation to understanding and reasoning. Subsequent analysis of our psychometric data will likely help us to further refine and expand upon this incipient "learner profile."

With regard to our psychophysiological data, it is evident that our participant became more relaxed through the metacognitive process of providing JOLs, and further to some extent through restudy. Changes in heart rate are mirrored in changes in respiration rate, in that both are component parts of a deeply connected cardiovascular system. There is, however, a substantive difference in eye-blink rate with regard to metacognitive and cognitive activities. This may be accounted in part by greater attentiveness to mouse pointing and clicking in providing JOLs. However, further analysis of that difference is warranted. Moreover, we also acquired electroencephalographic (EEG) data we are currently analyzing that may shed further insight into our participant's cognitive states.

The conjunction of our self-report and psychophysiological results analysed thus far suggests that reporting JOLs, as a means of metacognitive reflection, could also serve a pedagogical purpose (as *formative self-assessment*) beyond just being a research tool, in helping reduce anxiety and helping improve learner motivational awareness regarding restudy. Our analysis of this case study is on-going. We have acquired similar data sets from other individuals, and are in the process of expanding this study accordingly.

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INTERNATIONAL COMPARISONS OF MATHEMATICS CLASSROOMS AND CURRICULA: THE VALIDITY-COMPARABILITY COMPROMISE

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The pursuit of commensurability in international comparative research by imposing general classificatory frameworks can misrepresent valued performances, school knowledge and classroom practice as these are actually conceived by each community and sacrifice validity in the interest of comparability. The "validity-comparability compromise" is proposed as a theoretical concern with significant implications for international cross-cultural research. We draw on current international research to illustrate a variety of aspects of the issue and its consequences for the manner in which international research is conducted and its results interpreted. The effects extend to data generation and analysis and constitute essential contingencies on the interpretation and application of international comparative research.

INTRODUCTION

This paper identifies key considerations affecting the conduct and utility of international comparative research. Central to the design of such research studies are the dual imperatives of validity and comparability. Unfortunately, as will be illustrated, these imperatives are inevitably in tension. This paper identifies, illustrates and discusses these tensions, utilising very specific examples from current international comparative research. We argue that any value that might be derived from international comparisons of curricula or classroom practice is critically contingent on how the research design addresses the dual priorities of validity and comparability. We further argue that since these priorities act against each other, researchers undertaking international comparative research must find a satisfactory balance between these competing obligations.

Perhaps only the drive to categorise is more fundamental than our inclination to compare (cf. Lakoff, 1987). Indeed, the two activities are intrinsically entwined. In this paper, commensurability is interpreted as the right to compare (cf. Stengers, 2011). And it is our central assertion that this right to compare cannot be assumed, but is contingent on our capacity to legitimise both the act of comparison and the categories through which this act is performed. The need for such legitimisation has been raised for international comparisons of student achievement, but less frequently and less carefully for the cross-cultural comparison of curricula and classrooms.

Critical in the legitimisation of these acts of comparison are the validity of the categories we employ and of the act of comparison itself. Much of our focus in this

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paper is on cultural validity, which we interpret (with Säljö, 1991) as a key determinant of practice in the international settings we aspire to compare. Research designs, especially data generation and categorisation processes, can misrepresent or conceal cultural idiosyncrasies in the interest of facilitating comparison.

This paper considers this validity-comparability compromise in relation to both curriculum and classroom practice research. Curricular comparisons raise issues related to the structure of school knowledge and the aspirational character of valued performances. Comparisons of classroom practice foreground the performative realisation of school knowledge and introduce the teacher as curricular agent (among other roles), modelling, orchestrating, facilitating and promoting performances aligned with the educational traditions of the enfolding culture. Any cross-cultural comparative analysis faces the challenge of honouring the separate cultural contexts, while employing an analytical frame that affords reasonable comparison.

The paper utilises seven "dilemmas" to reveal some of the contingencies under which international comparative research might be undertaken. The issues raised by each dilemma are not mutually exclusive sets. Specific empirical examples from current international research provide the vehicle by which the entailments of each dilemma can be explored to identify areas of cross-cultural research requiring critical examination. Relevant theory is invoked as required by each emergent contingency.

COMPARABILITY AND VALIDITY IN CROSS-CULTURAL STUDIES

In an international comparative study, any evaluative aspect is reflective of the cultural authorship of the study.

> Culture is thus what allows us to perceive the world as meaningful and coherent and at the same time it operates as a constraint on our understandings and activities. (Säljö, 1991, p. 180).

In seeking to make comparison between the practices of classrooms situated in different cultures, the most obvious comparator constructs become problematic.

Dilemma 1: Cultural-specificity of cross-cultural codes
Use of culturally-specific categories for cross-cultural coding (eg participation, mathematics).

In the Chinese adaptation of the research design for the Middle School Mathematics and Institutional Setting of Teaching (MIST) project, the decision was made not to use the Instructional Quality Assessment (IQA) (Silver & Stein, 1996), but instead to develop a local instrument for the evaluation of mathematics classroom instruction. The reason for the rejection of the IQA instrument for use in Chinese school settings reflected the embeddedness, within the instrument, of particular values characteristic of the cultural setting and educational philosophy of the authoring culture (USA). For example, for the measurement of students' participation in classroom instruction, new criteria are needed that accommodate the larger class size and norms of social interaction of the Chinese mathematics classroom. Figure 1 shows the criteria for

evaluating the level of student participation in teacher-facilitated discussion in mathematics classes.

A. Participation

Was there widespread participation in teacher-facilitated discussion?

4	Over 50% of the students participated consistently throughout the discussion.
3	25 to 50% of the students participated consistently in the discussion OR over 50% of the students participated minimally.
2	25 to 50% of the students participated minimally in the discussion (that is, they contributed only once.)
1	Less than 25% of the students participated in the discussion.
N/A	Reason:

Figure 1. Participation criteria from the *Instructional Quality Assessment* (IQA) instrument (Silver & Stein, 2003).

In countries such as China and Korea, teachers in both primary and secondary schools make extensive use of elicited student choral response as a key instructional strategy (Clarke, 2010). In the lessons analysed from one Shanghai classroom, a large number of choral responses (~ 80) were used in each lesson. In the analysis of a classroom in Tokyo, there were a similar number of individual student public statements, but no evidence of choral response. Applying the IQA participation criteria (Figure 1), the regularity and frequency of the use of choral responses would characterise this classroom as participatory at a level comparable with the classroom in Tokyo. Yet the students in the Tokyo classroom participate primarily through individual contributions rather than choral response and the type of teacher-facilitated discussion and the nature of student participation in that discussion in the two classrooms are sufficiently different to make their comparability with respect to participation highly questionable.

Dilemma 2: Inclusive vs Distinctive
Use of inclusive categories to maximise applicability across cultures, thereby sacrificing distinctive (and potentially explanatory) detail (eg. mathematical thinking).

In a recent study undertaken by the authors, we compared the ways in which mathematics curricula are framed in Australia, China, Finland and Israel. We sought to identify the similarities and differences in the organisation of mathematics curricula in the four countries in terms of their aims, content areas and performance expectations. In particular, we investigated the ways in which "mathematical thinking" was framed through curricular statements.

The key documents analysed in this study were: the Victorian Essential Learning Standards (VELS), the Chinese Mathematics Curriculum Standards (CMCS), the Finnish National Core Curriculum (FNCC) and the Mathematics Curriculum (Israel)

(MCI). The four curricula are structurally quite different and prioritise different performance types. The excerpts below capture some of these qualitative differences.

See mathematical connections and be able to apply mathematical concepts, skills and processes in posing and solving mathematical problems (VELS).

[Translation] Obtain important mathematics knowledge that is essential for functioning in society and further development (including mathematical facts and experience in participating in mathematics activities) and basic mathematical thinking skills as well as essential skills of application (CMCS).

The task of instruction in mathematics is to offer opportunities for the development of mathematical thinking, and for the learning of mathematical concepts and the most widely used problem-solving methods (FNCC).

[Translation] Mathematics is not only a collection of calculated algorithmic operations that serve an applied purpose but also a subject with its own structure that includes unique thinking and investigation methods. The goal of the curriculum is to generate a change in the way that students view the subject (MCI).

Any attempt to characterise the relative emphasis given to particular types of valued performance at different grade levels can only be undertaken if a common classificatory framework can be imposed on all curricula. But such a general framework must not be allowed to mask the significant emphasis given to Geometry in grades 7 to 9 in China, or to "Communicating" in grades 3 to 5 in Finland, or the idiosyncratic prioritizing in grades 7 to 9 in Israel of "the evolution of phenomena from the perspective of mathematics." The danger is that the commensurability demands of such comparisons conceal major conceptual differences in the curricular expression of categories of school knowledge. The act of reconstructing culturally-specific categories to enable cross cultural comparisons runs the risk of distorting the knowledge categories we seek to compare. In cross-cultural research the imposition of an "external" classification scheme for the purposes of achieving comparability can sacrifice validity by concealing cultural characteristics and by creating artificial distinctions. Comparability is achieved through processes of typification and omission, and each has the potential to misrepresent the setting.

> **Dilemma 3: Evaluative Criteria**
> Use of culturally-specific *criteria* for cross-cultural *evaluation* of instructional quality (eg. Student spoken mathematics).

Where research is specifically constructed to be evaluative, the question arises as to the legitimate application of criteria developed in one culture to the practices of another culture. The use of evaluative criteria posits an ideal of effective practice that should be substantiated by reference to research. Problems arise when the research on which a criterion is based is itself culturally-specific.

For example, despite the emphatic advocacy in Western educational literature, classrooms in China and Korea have historically not made use of student-student spoken mathematics as a pedagogical tool. In research undertaken by Clarke, Xu and

Wan (2010), classrooms were identified in which student spoken mathematics was purposefully promoted in public but not in private interactions (eg Shanghai classroom 1), in both public and private interactions (eg Melbourne 1) and in neither public nor private interactions (eg Seoul 1). Each of these classrooms models a distinctive pedagogy with respect to student spoken mathematics.

If the occurrence of student-spoken mathematics is identified with quality instruction, then the instructional practice of the classroom in Seoul would be judged to be deficient. The classrooms in Shanghai and Melbourne differed significantly in the extent to which private student-student interactions were encouraged, but the teachers in both classrooms prioritized student facility with spoken mathematics. In the Shanghai classroom, promotion of this capability was developed solely through public discourse, whereas in the Melbourne classroom, private student-student mathematical speech was an essential pedagogical tool. Interestingly, in post-lesson interviews, the students from Melbourne and Shanghai showed comparable fluency in their use of the language of mathematics, while students from the classrooms in Seoul showed little evidence of such a capacity. Evaluative judgments of instructional quality made in the context of international comparative research must justify the model of accomplished practice implicit in the criteria employed and provide evidence of the cross-cultural legitimacy of these criteria.

Dilemma 4: Form vs Function

Confusion between form and function, where an activity coded on the basis of common form is employed in differently situated classrooms to serve quite different functions (eg kikan-shido or between-desks-instruction).

Kikan-shido (a Japanese term meaning "between-desks-instruction") has a *form* that is immediately recognisable in most countries around the world. In kikan-shido the teacher walks around the classroom, while the students work independently, in pairs or in small groups. Although kikan-shido is immediately recognisable to most educators by its form, it is employed in classrooms around the world to realise very different functions. A teacher undertaking kikan-shido in Australia, will do so with very different purposes in mind from those pursued by a teacher in Hong Kong, or, for example, a teacher in Japan. In reporting the frequency of occurrence of an activity such as kikan-shido for the purposes of comparative analysis, the researcher conflates activities that are similar in form but which may be employed in differently-situated classrooms for quite distinct functions. Such conflation can create an impression of similarity although differences in practice are actually quite profound (for more detail, see Clarke, Emanuelsson, Jablonka & Mok, 2006).

Dilemma 5: Linguistic Preclusion

Misrepresentation resulting from cultural or linguistic preclusion (eg Japanese classrooms as underplaying intellectual ownership).

The analysis of social interaction in one culture using expectations encrypted in classificatory schemes that reflect the linguistic norms of another culture can

misrepresent the practices being studied. This can occur because characteristics of social interaction privileged in the researcher's analytical frame may not be expressible within the linguistic conventions of the observed setting. For example, the Japanese value implicit communication that requires speaker and listener to supply the context without explicit utterances and cues. This tendency is typically found in leaving sentences unfinished. As a consequence, in Japanese discourse, agency or action are often hidden and left ambiguous. In English, when introducing a definition, the teacher might employ a do-verb: "We define". In a Japanese mathematics classroom, the teacher often introduces a definition in the intransitive sense (*Sou Natte Iru* = "as it is" or "something manifests itself") as if it is beyond one's concern. Such differences in the location of agency, embedded in language use, pose challenges for interpretive analysis and categorisation of classroom dialogue.

Dilemma 6: Omission

Misrepresentation by omission, where the authoring culture of the researcher lacks an appropriate term or construct for the activity being observed (eg. Pudian).

The Sapir-Whorf hypothesis suggests that our lived experience is mediated significantly by our capacity to name and categorise our world.

> We see and hear . . . very largely as we do because the language habits of our community predispose certain choices of interpretation (Sapir, 1949).

Marton and Tsui (2004) suggest that "the categories . . . not only express the social structure but also create the need for people to conform to the behavior associated with these categories" (p. 28). Our interactions with classroom settings, whether as learner, teacher or researcher, are mediated by our capacity to name what we see and experience. Speakers of one language have access to terms, and therefore perceptive possibilities, that may not be available to speakers of another language. For example, in the Chinese pedagogy "Qifa Shi" (Cao, Clarke, & Xu, 2010), the activity "Pudian" is a key element. Pudian can take various forms: Connection, Transition, Contextualising, but its function is to help students develop a conceptual, associative bridge between their existing knowledge and the new content. There is no simple equivalent to Pudian in English, although teacher education programs delivered in most English-speaking countries would certainly encourage the sort of connections that Pudian is intended to facilitate. Many such pedagogical terms have been collected in a variety of languages (Clarke, 2010), describing classroom activities central to the pedagogy of one community but unnamed and frequently absent from the pedagogies of other communities. It follows that an unnamed activity will be absent from any catalogue of desirable teacher actions and consequently denied specific promotion in any program of mathematics teacher education. It is also likely that such activities will go unrecognised in reports of cross-cultural international research, where the authoring culture of the research report lacks the particular term.

> **Dilemma 7: Disconnection**
> Misrepresentation through disconnection, where activities that derive their local meaning from their connectedness are separated for independent study (eg. teaching and learning (cf obuchenie), public and private speech).

 Whether we look to the Japanese "gakushu-shido", the Dutch "leren" or the Russian "obuchenie", we find that some communities have acknowledged the interdependence of instruction and learning by encompassing both activities within the one process and, most significantly, within the one word. In English, we dichotomise classroom practice into Teaching or Learning. One demonstration of the consequences of the inappropriate disconnection of actions that should be seen as fundamentally connected is evident in the comparison of two published translations involving Vygotsky's use of the term "obuchenie" (discussed in Clarke, 2001).

> From this point of view, *instruction* cannot be identified as development, but properly organized *instruction* will result in the child's intellectual development, will bring into being an entire series of such developmental processes, which were not at all possible without *instruction* (Vygotsky, as quoted in Hedegaard, 1990, p. 350).

> From this point of view, *learning* is not development; however, properly organized *learning* results in mental development and sets in motion a variety of developmental processes that would be impossible apart from *learning* (Vygotsky, 1978, p. 90).

The analogous disconnection of public and private speech in classrooms, and of speaking and listening (Clarke, 2006) has the same effect of misrepresenting activities that may be fundamentally interrelated (not just conceptually but also functionally connected) in their enactment in particular classroom settings.

CONCLUSIONS

The pursuit of commensurability in international comparative research by imposing general classificatory frameworks can misrepresent valued performances, school knowledge and classroom practice as these are actually conceived by each community and sacrifice validity in the interest of comparability. In this paper, the "validity-comparability compromise" has been proposed as a theoretical concern that has significant implications for international comparative research. The identified dilemmas offer different perspectives and illustrate some of the consequences of ignoring this central concern. Partnerships with those being compared can minimise misrepresentation, but the necessity of the compromise is inescapable. The interpretation and application of international comparative research will be critically contingent on researchers' capacity to address those "dilemmas" pertinent to their particular design. We hope this paper fuels a wider engagement in the critical interrogation of international comparison as a socio-material knowledge practice.

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SUCCESS AND STRATEGIES IN 10 YEAR OLD STUDENTS' MENTAL THREE-DIGIT ADDITION

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In this study, 4ᵗʰ grade students' achievement and strategy use on three-digit addition tasks are presented. 78 students (40 boys, 38 girls, mean age 10 year 4 months) participated in the study. Students solved 8 tasks of various difficulties aiming to evoke the use of typical strategies revealed by previous research: stepwise, split, compensation, simplifying strategies, and indirect addition. The results show that students used the split strategy for the majority of tasks independently of how effectively that strategy could be used. There was no sign of using compensation, simplifying and indirect addition strategies. The results points to the potentials addition strategy trainings may have in developing students three-digit addition skills.

INTRODUCTION

The title of this paper paraphrases the title of Selter's (2001) work on success, methods and strategies of German elementary school children solving three-digit addition and subtraction. Success refers to students' achievement in terms of correct solution to mathematical, namely, addition problems. Methods of solution can take either written or oral computation forms. In the current study, only oral computation procedures are investigated. The term strategy remains implicit in the majority of recent articles on children's and adults' computation. However, a rather general definition given by Richard Mayer (2010, p. 164.) may serve well the purposes of the current study: "Strategies are general methods for planning and monitoring how to accomplish some task." In the case of mental arithmetical computations, strategies are therefore conscious planning and monitoring processes that can be used for solving a variety of different tasks.

The importance of research on elementary school children's success and strategies on mental computation can be supported not only by the widely recognized importance of mathematical skills (see e.g., Smith, 1999), but also by the challenges raised by research on adaptive strategy use. These two aspects are intertwined, and – from an educational point of view – there may be a bidirectional link between them. Developing expertise in mental computation may lead to a broad repertoire of calculation strategies, and at the same time enrichment of students' strategies may lead to better results both in correctness and the time needed for the solution. There is a growing body of evidence pointing to the importance of adaptive strategy use in mathematics (De Corte, Mason, Depaepe & Verschaffel, 2011).

Strategies in three-digit addition

Selter (2001) stated that there had been barely any research on addition and subtraction with three-digit numbers, except for a study by Fuson et al. (1997). In the past decade, some new findings have been reported, and besides investigating the achievement on and the strategies used for three-digit addition and subtraction, results of educational intervention programs have contributed to extend our knowledge of the topic. As for the categorization of strategies used for addition with three-digit numbers, there are different category systems using different labels for slightly different (or identical) strategies. The most recent one is provided by Heinze, Marschick and Lipowsky (2009) and is "denoted as an idealized because [it is] based on a mathematical systemization" (p. 592). There are five strategies listed by them:

- *stepwise strategy*: when the second addend is added in three steps. For example: 123+456=((123+400)+50)+6. This is called the "begin-with-one-number" method by Fuson et al. (1997).
- *split strategy*: adding first the hundreds, then the tens, and finally the ones. For example: 123+456=(100+400)+(20+40)+(3+6). This is called the "decompose hundreds-tens-and-ones" method by Fuson et al. (1997), and "htu (hundreds, tens, units)" strategy by Selter (2001).
- *compensation strategy*: one of the addend is rounded off to the nearest hundreds number. For example: 527+398=527+400-2. This is very similar to the *simplifying strategy* when both addends are changed by moving some from one of them to the other, e.g., 527+398=525+400. This latter strategy is called the "change-both-numbers" method by Fuson et al. (1997), and is labelled auxiliary or simplifying by Selter (2001).
- the strategy called *indirect addition* refers to a subtraction strategy when mental computation is executed like it was an addition task. For example: 701-698 is the number to be added to 698 in order to get 701.

All of the examples above were borrowed from Heinze, Marschick and Lipowsky's (2009) study. The tasks administered to students in the current investigation represent these four main bullet list categories. It means that although each three-digit addition task can be solved by any of the first three methods, and all three-digit subtraction tasks can be solved by means of "indirect addition", there are tasks that are especially suitable for effective use of the above-mentioned strategies.

Aims of the current study

The current paper presents results of a larger research project aiming at enriching students' mental computation strategy use. The research presented here can be considered as the pre-test phase of an intervention program. Due to the sample size and sample heterogeneity (in terms of SES-background and type of residence) the following research topics can yield generalizable data and results.

(1) Students' achievement on three-digit addition problems by means of mental computation, and in terms of correctness and the time needed for the solution.

(2) Students' errors during the mental computation process. These errors are often 'rational errors' (Ben Zeev, 1996), and may refer to a misused or inefficient strategy.

(3) Students' self-report of their strategy use.

(4) Inter-relations among achievement, errors and strategy use.

METHODS

Sample

The students involved were recruited from two different schools: one school is situated in a county seat town and the other in a village of Hungary. Both schools have two 4th grade classes, and the students come from rather diverse socio-economic background families. The sample comprised 78 students (40 boys and 38 girls). Their mean age was 123.92 months (10 years and 3.92 months).

Test and procedure

Eight tasks were developed for this investigation. There were six three-digit addition tasks and two three-digit subtraction tasks. The first task was considered as a warm-up one. Students had to compute the following operations:

(1) $342 + 235 = 577$ (2) $143 + 426 = 569$ (3) $702 + 105 = 807$

(4) $284 + 202 = 486$ (5) $527 + 398 = 925$ (6) $498 + 256 = 754$

(7) $701 - 694 = 7$ (8) $646 - 583 = 63$

The first four tasks could be effectively solved either by the stepwise or the split strategies. The 5th and 6th ones were planned to evoke the compensation or simplifying strategies, while the last ones gave the opportunity for using the indirect addition strategy.

All tasks were printed on a separate A4 sheet of paper, and were handed over to the students. At the moment of handover, timing was started. Students saw the operation to be computed in a form like e.g., "$342 + 235 =$", and they were not allowed to write down anything to the paper.

The interviewers noted all erroneous answers (if any) to their answer sheet, and at the moment of hearing the right answer, they stopped the watch, and wrote down the time, then proceeded to the next task. The maximum time allowance for a task was 60 seconds.

After having completed all the eight task, they turned on the dictaphone, and asked the students to tell how each task was solved. The students could saw again the tasks while talking about their solution strategy. The key encouragement question in case of silence was: "What partial results did you have?"

Students were tested individually in a quiet, separated room of the school. Data collection was managed by three university students who were previously trained and

then paid for their contribution. Data collection took place in the form of an interview, the protocol of which had been previously rehearsed during the training session with the interviewers.

RESULTS

Achievement in three-digit addition

The rate of correct solutions within the 60 second time limit is shown in Table 1, along with the average time needed for the correct solution. Please note that the first task can be considered a warming-up one.

Task	Rate of correct solutions (%)	Mean time (SD in parentheses)
342 + 235 = 577	94.9	13.35 (10.36)
143 + 426 = 569	97.4	10.95 (9.57)
702 + 105 = 807	98.7	5.53 (5.65)
284 + 202 = 486	100.0	8.39 (8.90)
527 + 398 = 925	70.5	24.14 (17.90)
498 + 256 = 754	69.2	22.02 (15.02)
701 – 694 = 7	52.6	24.37 (16.82)
646 – 583 = 63	50.0	28.28 (14.75)

Table 1: The rate of correct solutions yielded within 60 seconds, and mean response time (SD in parentheses) N = 78

The results suggest that the first four tasks were solved by almost everyone within a rather short time. However, the fifth and sixth tasks that would have been easily solved by the so-called compensation or simplifying strategies required much longer solution time, and about one third of the students failed to solve them. The two subtraction tasks proved to be even more difficult.

"Rational errors"

Students' erroneous answers were noted down. In some cases, there were several erroneous answers provided; in Table 2 only each student's first non-correct solution is considered (if there were any). Please note that only the incorrect answers given by at least 3 students (3.8%) are shown. Table 2 includes incorrect answers of those who later (within 60 seconds) gave the correct answer as well.

Task	The most frequent incorrect answers (relative frequency in parentheses)
$342 + 235 = 577$	5707 (5.1%); 5777 (3.8%); 587 (3.8%)
$143 + 426 = 569$	579 (3.8%); 590 (3.8%)
$702 + 105 = 807$	
$284 + 202 = 486$	
$527 + 398 = 925$	915 (6.4%); 625 (5.1%)
$498 + 256 = 754$	654 (10.3%)
$701 − 694 = 7$	193 (15.4%); 5 (7.7%); 16 (5.1%); 13 (3.8%); 93 (3.8%)
$646 − 583 = 63$	43 (9.0%); 143 (9.0%); 163 (6.4%); 57 (3.8%); 67 (3.8%); 137 (3.8%)

Table 2: The most frequent incorrect answers.

Students' self-report of mental computation strategies

Having completed all eight tasks, students reported of their strategy use task by task. In the simplest cases, the split (or decompose hundreds-tens-and ones) strategy was the most commonly used. The majority of them continued to use this strategy for the fifth and sixths tasks (albeit the compensation or simplifying strategies would have easily worked). For example in the case of Rozália (code number #106), the following self-report was received:

> Rozália: 498 plus 256. I added in a way that 400 plus 200 is 600. 9 plus 5 is 14. This is 900… 600 and twelve. And 8 plus 2, no plus 6 is…

Finally she gave 625 as an answer which is not correct. Her self-report clearly indicates the insistence on using the split strategy. However, with these addends, the split strategy requires rather heavy memory load and fair computational skills. Another student (code number #125) tried to use the stepwise strategy in this task:

> Boglárka: 498 plus 200 makes 698, plus 50 [pause], is 748, plus 6 [pause], is 713…

Neither Rozália nor Boglárka gave the correct answer in the first phase of the investigation. Rozália gave the same incorrect answer, Boglárka had 915 as her first erroneous answer. A third student (code number #126) had the correct answer before without any incorrect solution attempts, and he described his strategy in the following way:

> Bendegúz: 498 plus 256. 498 plus 200 is 698; plus 50 is 748, plus 6 is 754.…

In this case, the stepwise strategy was correctly used. A final example is given for the sixth task showing a "pseudo" mental calculation strategy. This student (code number #221) solved all the previous tasks; too, in a way as they were written computational tasks.

Tamás:	This was done in the same way, that is 8 plus 6 is 14. The remainder is 1, this is added to 9 to get 15…
Interviewer:	You mean 9 + 1 + 5 = 15.
Tamás:	Yes, and then again the remainder is 1, and then it will be 7.

This student solved the tasks in a way that he mentally put the addends one under another, and followed the algorithm learnt for written computations.

There was no sign of the compensation or simplifying strategy use in the case of the fifth and sixth tasks. Similarly, the last two tasks may have evoked the indirect addition method, but students (please note that half of them failed to give the correct answer within 60 seconds) used the split or stepwise strategies.

DISCUSSION

The results can be discussed along three lines. *Students' achievement* (success) on different types of three-digit addition tasks show that in the case of simpler tasks where there are less then ten tens, and less then ten ones in the addends, the solution is straightforward. In the tasks where the compensation or simplifying strategies might have given an easy solution, about one third of the students failed to give the correct answer. In the subtraction tasks, only half of them succeeded.

An *analysis of incorrect answers* shows that in some cases computational errors made while otherwise using an appropriate strategy led to incorrect answers.

In several cases, typical rational errors described in the literature can be observed. For example, $701 - 694 = 193$ indicate that those students who had this solution, subtracted always the smaller digit from the bigger one: $7 - 6 = 1$ for the hundreds, $9 - 0 = 9$ for the tens, and $4 - 1 = 3$ for the ones. This obviously erroneous strategy might reflect an early over-automatization of a wrong written subtraction algorithm.

Students' self-reports of their strategy use may point to two relevant phenomena. First, they are well aware of what they are doing when adding two numbers, at least in terms of the mathematical description of the process. They use the terms hundreds, tens, ones, remainder etc. Second, there are a rather limited variety of strategies used, at least the lack of the compensation and simplifying strategies, and the absence of indirect addition have been revealed. The narrow range of strategies used can be in part due to the perseverance effect known from the literature (Schillemans, Luwel, Bulté, Depaepe & Verschaffel, 2009).

According to Peters, De Smedt, Torbeyns, Ghesquière and Verschaffel (2010), adults tend to use the indirect addition for subtraction problems in rather reasonable cases, when the subtrahend was larger than the difference. Consequently, the indirect addition method can be labeled as a relatively late developmental stage in computational strategy use for subtractions.

Nevertheless, a kind of re-orchestration of the written computation algorithm for mental computation has been demonstrated. Therefore, this strategy might be considered as a real archetypical mental strategy.

IMPLICATIONS

There is an agreement in the literature on the need for greater flexibility in computations (Beishuizen & Anghileri, 1998). How it can be achieved raises several questions. One debate is about how teachers can become capable of fostering students' addition strategies. In Carpenter, Franke, Jacobs, Fennema and Empson's (1998) study, teachers themselves took part in a 3-year training program before the experiment. There are successful intervention studies with less demanding prerequisite resources, like that of Hiebert and Wearne's (1996) experiment. The second big issue is whether (and how) explicit addition strategies are taught. In Hiebert and Wearne's experiment "students were encouraged to develop their own procedures and to explain them to their peers" (p. 258). The debate on whether addition strategies should actively be taught to students or they can be left for spontaneous development is analyzed by Murphy (2004).

Our suggestion is – and this is in line with the results of the current investigation – that students should be actively taught to use a wide repertoire of addition strategies. Adaptive strategy use, i.e. when strategy choice is made according to task, individual and context variables, requires a range of possibly available strategies. While learning this strategy repertoire, students can constructively develop new strategies they have been never taught. Keeping in mind the educational goals of developing mathematical skills, fostering students' active and conscious strategy use in mental computation may well support the development of adaptive expertise.

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ADVANCED COLLEGE-LEVEL STUDENTS' CATEGORIZATION AND USE OF MATHEMATICAL DEFINITIONS

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This qualitative study of five undergraduate mathematics majors found that some students, (even students at an advanced level of undergraduate mathematical study) have a mathematician's perspective neither on the concept of mathematical definition nor on the structure of mathematics as a whole. Participants in this study were likely to reason from incomplete concept images rather than from concept definitions and were likely to perceive that definitions (like theorems) need to be verified. The results of this study have implications for college-level mathematics instruction.

INTRODUCTION AND FRAMEWORK

There is a large body of literature that documents students' misconceptions and difficulties with mathematical proofs. Some of these difficulties have to do with their perceptions of the nature and logical structure of proof, some with students' inadequate problem-solving skills, and some having to do with mathematical communication and concept understanding (Moore, 1994). He identified seven difficulties that students typically have with proofs of which Knapp (2006) points out that all but one are related in some way to students' facility with definitions.

The Nature of Mathematical Definitions

Definitions play a central role in mathematics. Mathematicians and students of mathematics use definitions routinely but seldom think about the nature of mathematical definition (Wilson, 1990). The process of defining in mathematics is the process of giving names to mathematical objects. In natural language, most definitions are *extracted*; that is they describe how a word is used and what is meant by it. In mathematics, definitions are *stipulated*; that is created on the advice of experts. "Extracted definitions report usage, while stipulative definitions create usage, indeed create concepts, by decree" (Edwards & Ward, 2004, p. 412).

Definitions are arbitrary (Winicki-Landman & Leikin, 2000). There may be many ways to define an object and ultimately one must be selected as *the* definition. A square can be defined to be a regular quadrilateral, or it can be defined to be a polygon whose diagonals are equal and perpendicular, or it can be defined in some other way. Once a definition is selected, then all other equivalent, biconditional statements become theorems that need to be proved. A definition is neither true nor false; is merely accepted or rejected (Wilson, 1990).

Student Understanding of the Concept of Mathematical Definition

One does not learn mathematics as quickly and easily as it can be presented. Tall and Vinner (1981) state that the human brain is neither that efficient nor logical in its operations and suggested that how students think about mathematical concepts may be quite different than how the concepts are formally defined. Students sometimes argue from their concept images rather than the concept definitions and in so doing they are not using definitions the way a mathematician would (Edwards and Ward, 2004). But further, students' concept images of *mathematical definition* might be faulty or incomplete. Students who perceive that mathematical definitions are no different than other mathematical statements that require justification are not categorizing definitions the way a mathematician would (Edwards and Ward, 2004).

Students do not use definitions the way mathematicians do.
One way students misuse definitions stems from their perception that mathematical ideas are extracted rather than stipulated. In a study of 14 undergraduate mathematics majors, Edwards and Ward (2004) report that even when students can correctly state definitions, sometimes they abandon the definitions and argue from their personal concept images. When for one participant, a concept definition conflicted with her concept image she seemed to think that the definition had not been extracted correctly and argued (incorrectly) from her concept image instead.

Another way student misuse definitions stems from a poor intuitive understanding of the concept in question. Knapp (2006) provides a framework for understanding how students' use definitions in proofs. Her participants were 10 undergraduate students in a first course in real analysis. She parses knowing a definition into *ventriloquating* (reciting without fully understanding) and *appropriating* (being able to use a definition). "Appropriating a definition requires students' personally meaningful understanding to match the culturally meaningful understanding" (Knapp, 2006, p. 18). Students who can state a concept definition but who revert to a faulty or incomplete concept image when making arguments are not appropriating that definition.

Students do not categorize mathematical definitions the way mathematicians do.
In a study of 251 college mathematics majors, Vinner (1977) reported that students frequently mis-categorize mathematical definitions as theorems and axioms, or even as non-mathematical statements such as facts and laws. This could be because teachers try to justify definitions (e.g., $x^{-a} = 1/x^a$) and in so doing, they might give students the impression that they are proving these definitions. Further, he states that some familiar definitions are introduced to students in middle school before the definitional structure of mathematics is made clear to students. These early impressions of certain specific definitions, of mathematical definition in general, or of the overall structure of mathematics may have lasting effects.

Edwards and Ward (2004) found that students of advanced undergraduate mathematics (abstract algebra) had difficulties "arising from the students' understanding of the very nature of mathematical definitions" (p. 412) not merely the

content of the definitions or the loaded nature of certain terms with their non-mathematical usage. Similar to Vinner's (1977) study, Edwards and Ward (2004) reported that one of their participants believed that once a theorem "is proven, it becomes a definition." Some participants in their study viewed mathematical definitions as extracted rather than stipulated. One participant believed that mathematicians are not free to create definitions, but "have to make the definitions from what something actually is" (p. 415) and another believed that when a definition is created, that it must pass peer review to be sure that it is error free.

METHODS

The purpose of our study was to understand how students of advanced undergraduate-level mathematics perceived the concept of *mathematical definition* and how they use definitions to verify simple conjectures. We know from Edwards and Ward (2004) and from Vinner (1977) that college students do not categorize or use definitions the way mathematicians do but we wanted to probe more deeply into how this group fit definitions into the structure of mathematics.

The data from this study comes from semi-structured, task-based interviews (Goldin, 2000) with five undergraduate mathematics majors. Our participants were juniors or seniors currently enrolled in a course in real analysis. Each was interviewed once for approximately 45 minutes. The interviews were audio-recorded and transcribed. The transcripts were coded separately by each of the two authors. At the end of this coding phase, the two authors met with each other and compared and refined their codes, and grouped the codes into broader categories.

There were several factors that informed our choice of tasks in our interview protocol. Each task was intended to elicit a discussion regarding some aspect of definition and its place in mathematics. Participants were asked to select one of among several definitions and to discuss what made it preferable to the others, to use two competing definitions for *even number* to determine the parities of certain integers, and to discuss whether definitions needed to be justified or proved. These tasks elicited the discussions that became the data for this study.

RESULTS

The two findings of this study largely confirm previous studies in the area. Our findings are (1) participants did not make clear distinctions between definitions and theorems; and (2) participants were also likely to argue from their concept images rather than from the concept definitions. While both Vinner (1977) and Edwards and Ward (2004) report that students sometimes perceive that definitions need some kind of justification to be accepted, our evidence suggests a different possible source for students' failure to categorize definitions the way mathematicians do.

Students' Categorization of Definitions

We found that our participants did not separate definitions (whose meanings are stipulated) from other mathematical statements whose validity must be verified with a proof. We presented our participants with the definition $x^{-a} = 1/x^a$ and so that there would be no doubt, they were told that this was the definition for x^{-a}. They were then

asked, "Does this definition need to be proved?" All five participants indicated that this definition (and others too) needed to be proved.

> Alan: I definitely could prove this because I had to do a proof just this semester on why 0 is less than 1. I didn't think that needed to be proved but apparently… I'd have to stare at for a while before [I got it] in my head of how to start to work it out but I just like defining it better personally [because] if somebody already proved it then there's a definition from that [and] then you don't need to… If it's already been proved… somebody else has already done the work… So it's already been proved. I think a definition is fine.

Alan has two ideas here. First, that in upper division college mathematics, students are frequently asked to prove things that are obvious and that a possible way around these difficult proofs is to define things. That his ideas are not well formed is apparent when he discusses defining as merely relying on a theorem that someone else proved. Other participants share his ideas about his experiences in upper division mathematics classes. Fredrick for example discussed the necessity of a proof based upon the mathematical level of the audience. "It depends on who you're saying this to. If you're talking to high schoolers, then I would say 'no.' But if it was like college or something and you're doing abstract algebra or something… I guess it's necessary." Fredrick and Alan both perceived that they had been asked to prove intuitively obvious facts that do not need justification outside of upper division mathematics classrooms, and at least some definitions might fall into this category.

Other participants discussed the reasoning behind the definitions and why they have been defined in a particular way. Colleen, for example discusses the justification of the definition of x^{-a}. "There's a reason why $x^{-a} = 1/x^a$. So I guess because there's a reason, that it probably would be a good idea to be proved. [But] I get lost when you try and prove it to me because some brilliant, crazy mathematician proved it… If you just tell me $x^{-a} = 1/x^a$, I'm good with that. Somebody already did all the legwork." Similar to Alan, Colleen saw a definition as an end run around a difficult proof and believed that some mathematician had to prove it sometime in the past.

When asked about the definition $x^{-a} = 1/x^a$, Dori said, "I would say you should prove it. That's not obvious to people who are just seeing it [for the first time], so yeah." She went on to discuss the necessity of proving that all squares are regular quadrilaterals. "I feel like at some point we [proved] that… So, I say, 'prove everything.'" At first, she said that definitions that are not immediately obvious to someone seeing them for the first time should be justified, but then followed by saying that all mathematical statements needed justification and that at some point she proved the definition "A square is a regular quadrilateral."

Wendy believed that definitions have to be justified in order to be incorporated into the structure of mathematics.

> Wendy: They are [proved]. It's not should they [be proved?]. Definitions have to come from somewhere. We learned there's different forms like lemmas and

> stuff like that… I don't remember the order; I used to. I know the lemma is the least right thing and the theorem is like the top or something… [I don't] remember the order of them but I know there's different degrees, I think.

Unlike Alan and Colleen, Wendy did not seem to resent being asked to prove things she thought were obvious. To her, proving definitions was just part of the work of a mathematician. In her view, once a definition has been proved, it becomes something similar to (but possibly not the same thing as) a theorem.

All of the participants believed definitions needed to be proved. Colleen wanted a justification for why a definition was the way it was. Alan, Dori and Fredrick said that all mathematical statements required proof. And Wendy described how a proved definition might fit into the structure of mathematics. To varying degrees they tended to confound definitions with other mathematical statements requiring justification and most indicated they had seen proofs for definitions in some of their college-level mathematics courses.

Students' Use of Definitions

We found that our participants were more likely to argue from their concept images than from the concept definitions. After reading, and comparing and contrasting the two definitions for *even number* given below, four out of five of our participants indicated that zero was neither even nor odd. Only two recognized that there was a discrepancy between their concept image and the definition, of which only one determined (very tentatively) that zero is even.

> 1. *A number is called even provided it represents a number of objects that can be placed into two groups of equal size.*
>
> 2. *A number is called even provided it is an integer multiple of 2.*

Notice that under both definitions, zero is an even number. Zero objects can be placed into two piles of zero items each (perhaps a bit of a stretch for some), and zero is 2•0 which is an integer multiple of 2. The following excerpts all attempt to answer the question, "Is zero even, odd, or neither?"

Wendy's concept image of *even number* was that even numbers represent a collection of objects in which all objects can "pair up" simultaneously. She could talk about more formal definitions of *even* and explicitly mentioned both $n = 2a$ and $2|n$ from her mathematics classes but she kept coming back to her concept image of objects pairing up. In answering our question, she said, "It's neither because you're not starting with anything. It's not paired out or anything. Is that right?" Although Wendy understood the definitions on the paper, and even proposed alternative formulations, when it came to deciding if zero was even or odd, she reasoned *entirely* from her concept image rather than the concept definition to make that determination.

Colleen and Dori both recognized that their concept image that zero was not even was at odds with at least one of the definitions provided. Colleen said, "I don't remember. I think it's neither, personally. But wait! It can be divisible by 2. I don't know, I don't

remember from [class] what we decided. Isn't there still a big argument about whether it is [even or not]? But technically, if you go by the definitions, it would be even. I don't know. I think it's neither. It's neither even nor odd." Although Colleen recognized that the definition demanded that zero be even, her concept image was strong enough for her to discard the definition. Dori, on the other hand tentatively discarded her concept image that zero was not even in favour of the definition. She said, "Neither. It definitely cannot be odd, but I'm torn between the neither and even. I would say 'neither' because you can't put it into two groups. [Under Definition #2], I would say 'even' but I'm sure, I'm positive that somebody could dispute me with 'neither' for the same reason I said [about Definition #1]. But I would say 'even.'" Similar to Colleen, Dori said that if she restricted herself only to the definitions (specifically Definition #2) then zero would be even but she still wasn't 100% convinced. Eventually, she decided it might be even although she was certain someone would have a problem with it. Still, she was uncomfortable with the notion that zero could be even so she offered her own addendum "zero doesn't count" to the definition of even to make it fit with her concept image that only positive and negative numbers could be even.

DISCUSSION AND CONCLUSION

We were interested in junior and senior mathematics majors' ideas about the concept of mathematical definition and found that at least some participants were still unclear as to the structure of mathematics as a whole despite the advanced level of their studies. They did not separate definitions from other kinds of statements that required justification and adhered very strongly to faulty or incomplete concept images. For example a concept image common to all of our participants excluded zero from the set of even numbers. Our participants were all familiar with the definition of even number and some suggested alternate definitions. But most of our participants seemed to be merely ventriloquating (Knapp, 2006) rather than appropriating the definition. Similar to Edwards and Ward (2004), one of our participants preferred to argue solely from her concept image rather than the concept definition, but two found that their concept image of even number differed from the concept definition and indicated that if they restricted themselves to only the concept definition, then zero would have to be even but neither were comfortable stating this claim with certainty. In this last case, it seems likely that these two participants perceived that the definitions had not been extracted properly.

Beyond corroborating the findings of previous studies, this study provides some evidence that students even at the advanced undergraduate levels are still developing an understanding not only of the concept of *mathematical definition*, but also of the mathematical system as a whole and their concept image of this entire system may not be fully formed. For example, all of our participants believed that they could prove a definition. Two possible reasons for this is given by Vinner (1977); first, students have seen their teachers motivate definitions before and so perceive that the definitions were proved, and second, certain familiar definitions are introduced to students before

the structure of mathematics is made clear to them. We find something quite different; some of our participants perceived that in their advanced mathematics courses, they had frequently been engaged in proving completely obvious facts (e.g., $0 < 1$). In such courses, their notions of what did and did not require a proof were challenged to the point where they perceived that absolutely nothing, not even definitions could be taken for granted.

It may be only natural for students at this level to perceive that all basic information such as intuitively obvious theorems, definitions, and possibly even axioms must be verified because up to this point in their mathematical educations, they have been *learning* mathematics, but not really *doing* mathematics. It may be that they perceive that they have been asked to prove things solely for the purpose of demonstrating to their professors that they can reproduce some such verifications and do not see themselves as active participants within the mathematical system. Perhaps for some, the distinction between *definition* and *result* becomes clear only when one attempts to create one or the other. If so, it seems likely that engaging students in creating mathematics might help them better understand the mathematical system, and make the distinctions between definitions and theorems more apparent to them.

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STUDENTS' PROPORTIONAL REASONING IN MATHEMATICS AND SCIENCE

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Proportional reasoning is increasingly being recognised as fundamental for successful operation in many topics within both the mathematics and science curriculum. However, research has consistently highlighted students' difficulties with proportion and proportion-related tasks and applications, suggesting the difficulty for many students in these core school subjects. As a first step in a major research project to support the design of integrated curriculum across these two disciplines, this paper reports on students' results on a proportional reasoning pretest of mathematics and science items. Administered to approximately 700 students across grades 4 to 9, results anticipated increased gradual progression in results, but surprising similarities in performance on particular items for student groups at each year level.

PROPORTIONAL REASONING IN MATHEMATICS AND SCIENCE

Many topics within the school mathematics and science curriculum require knowledge and understanding of ratio and proportion and being able to reason proportionally. In mathematics, for example, problem solving and calculation activities in domains involving scale, probability, percent, rate, trigonometry, equivalence, measurement, algebra, the geometry of plane shapes, are assisted through ratio and proportion knowledge. In science, calculations for density, molarity, speed and acceleration, force, require competence in ratio and proportion. Proportional reasoning, according to Lamon (2006) is fundamental to both mathematics and science.

Proportional reasoning means being able to understand the multiplicative relationship inherent in situations of comparison (Behr, et al., 1992). The study of ratio is the foundation upon which situations of comparison can be formalised, as a ratio, in its barest form describes a situation in comparative terms. For example, if a container of juice is made up of 2 cups of concentrated juice and 5 cups of water, then a container triple the size of the original container will require triple the amounts of concentrate and water (that is, 6 cups of concentrated juice and 15 cups of water) to ensure the same taste is attained. Proportional thinking and reasoning is knowing the multiplicative relationship between the base ratio and the proportional situation to which it is applied. Further, proportional reasoning is also dependent upon sound foundations of associated topics, particularly multiplication and division (Vergnaud, 1983), fractions (English & Halford, 1995) and fractional concepts of order and equivalence (Behr, et al. 1992). Although understanding of ratio and proportion is intertwined with many mathematical topics, the essence of proportional reasoning is the understanding of the multiplicative structure of proportional situations (Behr, et al., 1992).

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In the middle years of schooling, ratio and proportion are typically studied in mathematics classes. In fact, ratio and proportion have been described as the cornerstone of middle years mathematics curriculum (Lesh, Post & Behr, 1988). However, research has consistently highlighted students' difficulties with proportion and proportion-related tasks and applications (e.g, Behr, Harel, Post & Lesh, 1992; Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller, 1998; Lo & Watanabe, 1997), which means that many students will struggle with topics within both the middle years mathematics and science curriculum due to their lack of understanding of ratio and proportion. Understanding ratio and proportion is more than merely being able to perform appropriate calculations and being able to apply rules and formulae, and manipulating numbers and symbols in proportion equations. It is well-accepted that students' computational performances are not a true indicator of the degree to which they understand the concepts underlying the calculations.

THE STUDY

The research reported in this paper is part of a larger study entitled the MC SAM project, the acronym for 'Making Connections in Science and Mathematics'. The project aims to take a "conscious, systematic and explicit…. structured and goal-oriented" learning by design approach (Kalantzis & Cope, 2004, p. 39) to support the careful design of an integrated curriculum to promote students' connected knowledge development across these two disciplines. In this project, researchers and teachers are collaboratively developing, implementing and documenting innovative, relevant and connected learning in mathematics and science, and hence redefining classroom culture as well as redefining curriculum. This paper presents results of a proportional reasoning pretest, the results of which highlight great variance of proportional reasoning in students across Years 4 to 9, and simultaneously underscores the importance of a more systematic and structured approach to promoting proportional reasoning across mathematics and science.

The pretest was to designed to provide a snapshot of a large group of students' proportional reasoning on tasks that relate to mathematics and science curriculum in the middle years of schooling. This aspect of the research was concerned with the development of an instrument that would provide a 'broad brush' measure of students' proportional reasoning and their thinking strategies, and that would have some degree of diagnostic power. This challenge was undertaken with full awareness of both the pervasiveness and the elusiveness of proportional reasoning throughout the curriculum and that its development is dependent upon many other knowledge foundations in mathematics and science.

Instrument design

A large corpus of existing research has provided analysis of strategies applied by students to various proportional reasoning tasks (e.g., Misailidou & Williams, 2003; Hart, 1981). Such research has highlighted issues associated with the impact of 'awkward' numbers (that is, common fractions and decimals as opposed to whole

numbers), the common application of an incorrect additive strategy, and the blind application of rules and formulae to proportion problems.

To identify more specific links across both mathematics and science, we consulted the Atlas of Scientific Literacy (American Association for the Advancement of Science (AAAS), 2001). The AAAS has identified two key components of proportional reasoning: Ratios and Proportion (parts and wholes, descriptions and comparisons and computation) and Describing Change (related changes, kinds of change, and invariance). Using this as a frame, we devised the test to incorporate items on direct proportion (whole number and fractional ratios), rate, and inverse proportion as well as items relating to fractions, probability, speed and density. Guided by the words of Lamon (2006) who suggested that students must be provided with many different contexts, 'to analyse quantitative relationships in context, and to represent those relationships in symbols, tables, and graphs' (p. 4), the items included contexts of shopping, cooking, mixing cordial, painting fences, graphing stories, saving money, school excursions, dual measurement scales. For each item on the test, students were required to provide the answer and explain the thinking they applied to solve the problem.

The pretest consisted of 16 items, split into two sections of 8 items each. Bearing in mind that the test would be administered to 4th Grade students, we wanted to avoid test fatigue and provided students with 30 minutes to complete each section of the test on two different days. Most students completed each section of the test within 15 minutes. Table 1 provides the title of each test item and a brief description of its focus.

A1	*Butterflies*. 5 drops of nectar for 2 butterflies; *x* drops of nectar for 12 butterflies? Missing value – simple numbers.
A2	*Chance Encounters*. Which of 4 bags of counters (B/W) has best chance of selecting black: 4B 4W; 1B 1W; 2B 1W; 4B 3W. Probability
A3	*Shopping Trip*. $6 remaining after spending $^1/_3$ of money. How much at the start? Part/part/whole – complex ratio.
A4	*Three Cups*. Full cup, ½ cup, $^1/_3$ cup water; 3, 2, 1 lumps sugar respectively. Which is sweetest? Intuitive proportion, small numbers.
A5	*Sticky Mess*. Recipe: 4 cups of sugar, 10 cups of flour; 6 sugar for *x* flour? Missing value – complex ratio.
A6	*Fence Painting*. 6 people take 3 days; how many in 2 days? Inverse.
A7	*End of Term*. Comparison of preferences for an end of term activity in two classes of students (different totals). Absolute vs relative thinking.
A8	*Number Line*. Reading dual-scale representation using two measurement scales. Scaling.
B1	*Speedy Geoff*. Distance covered when speed halved. Speed.

B2	*Balancing*. Identifying impact of weights on Balance Scale.
B3	*Washing Days*. Powder A: 1kg, 20 loads, $4; Powder B: 1.5 kg, 30 loads, $6.50. Which is better buy?
B4	*Funky Music*. Mum pays $5 for every $2 saved to buy item for $210. How much did each person paying? Part/part/whole
B5	*Cycling Home*. Matching graph to speed of bicycle.
B6	*Sinking and Floating*. Density of object in liquid.
B7	*Juicy Drink*. Mixing cordial; two-step ratio problem.
B8	*Tree Growth*. Non-proportional situation, trees grow at same rate.

Table 1: Proportional Reasoning Pretest Item Overview.

RESULTS

Approximately 700 students across Grades 4-9 completed the test. Students' results on this assessment are presented in Figure 1.
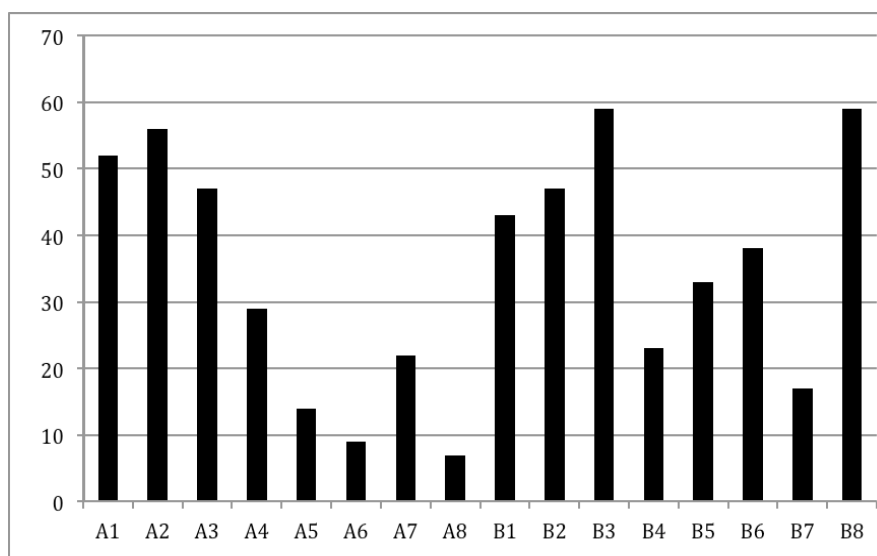
Figure 1: **Percentage correct for each test item**

Students' responses for each test item were coded. Coding occurred at two levels, and hence a two-digit code was assigned to each response. The first digit in the code identified whether the item was correct (code 1), incorrect (code 2), or not attempted (code 0). The second digit in the code identified the thinking strategy utilised by the student in solving the problem, as gleaned from the explanation of how he/she solved each problem. In particular, a solution strategy that showed application of elegant ratio thinking (that is, direct use of multiplication and division strategies) was assigned a code of 1, with a solution strategy that showed application of a repeated addition strategy (use of tables of values) assigned a code of 2. These two codes were considered indicative of appropriate proportional reasoning. A code of 3 was given to thinking that suggested (incorrect) additive thinking had been applied, and a code of 4

was given to thinking that suggested that the student's strategy would never lead him/her to the correct solution. A code of 0 was given when the student left this section blank. Scores of 11 or 12 thus indicated a correct solution and application of proportional reasoning. A score of 23 indicated an incorrect solution with inappropriate additive thinking. Table 2 shows the percentage of responses for each particular code.

	Response Code										
Item	**11**	**12**	**13**	**14**	**10**	**21**	**22**	**23**	**24**	**20**	**00**
A1	31	19	0	0	2	4	11	2	28	1	2
A2	16	2	36	2	0	2	1	32	8	1	0
A3	31	9	4	3	2	2	4	5	32	7	3
A4	12	0	8	6	3	0	0	27	38	4	2
A5	9	4	0	1	0	1	0	66	7	8	4
A6	1	0	4	2	2	1	0	33	42	11	4
A7	7	0	6	7	2	1	0	57	15	3	2
A8	2	1	2	2	0	2	0	13	53	13	12
B1	35	5	0	1	2	0	0	4	40	10	3
B2	10	10	20	5	2	0	1	6	40	3	3
B3	22	3	20	10	4	1	0	25	9	3	3
B4	11	9	0	1	2	1	10	1	32	12	21
B5	11	16	2	2	2	0	15	27	15	6	4
B6	1	6	27	1	3	0	8	33	9	3	9
B7	15	1	0	0	1	7	0	15	34	11	16
B8	0	0	56	0	3	3	1	14	13	3	7

Table 2: Percentage of strategy use for correct and incorrect responses.

DISCUSSION

Lamon (2006) described proportional reasoning as a web of interrelated "concepts, operations, contexts, representations, and ways of thinking" (p. 9) to highlight the complexity of proportional reasoning and hence advocating a rich, recursive curriculum across rational number domains for promoting proportional reasoning. Central core ideas for proportional reasoning, as identified by Lamon, include rational number interpretation, measurement, quantities and covariation, relative thinking, unitizing, sharing and comparing, reasoning up and down. And all these are "recurrent, recursive and of increasing complexity across mathematical and scientific domains" (p. 9). Inherent in these words is a call for change of focus to mathematics instruction in ratio and proportion topics, and a new look at the traditional separatist demarcation of mathematics and science curricular.

The pretest designed in the MC SAM project is only a tentative first step for emphasising the centrality of proportional reasoning across mathematics and science topics. In this test, the items were designed to capture students' proportional reasoning in its broadest sense. Some items were very typical ratio tasks (items A1, A5, B7), but some were specifically linked more directly to science. Item B1 was a simple speed situation: *Geoff runs 100 metres in 12 seconds. If he runs the same distance at half the speed, how long will it take him?* This item was correctly answered by less than 50% of the students, but was comparatively well-answered by the fourth graders (just above 30%), and was one of the best-answered items on the test for these students (see figure 2). This suggests that intuitively, fourth graders can understand simple speed situations. Interestingly, the ninth graders' mean score for this item was only approximately 63%, and was not the best-answered item for this cohort. Item B2 was a classic balance beam problem, frequently cited in science research as a science reasoning task (see for example, Shayer & Adey, 1981). The mean score for this item was 47%, and was also well-answered by the fourth graders (35%). Item B5 required students to select (from 6) the appropriate graph for the following situation: *Anne was cycling home from school. She rode for a short time at a steady speed then stopped for a rest. When se started again, she rode twice as fast to get home quickly.* This item was devised to link to the AAAS's 'Describing Change' component of scientific reasoning, but clearly graph interpretation is a key component of rational number understanding (Lamon, 2006). The mean score for this item was 31%, with the fourth graders responding relatively well at 15%, which is higher than for many other items. This suggests that fourth graders can interpret situations graphically, and has implications for instruction at much earlier junctions than typically occurs in primary school. Compared to the seventh to ninth graders, the fourth and fifth graders's results were impressive. However, not as impressive as for item B6, which was also a specific science item relating to density. This item had several parts, providing students with a data table that displayed the mass and volume of a collection of cubes and information about one cube in the collection that is known to sink. The students had to determine which other cubes would sink. The fourth graders scored higher than the ninth graders

(38% and 22% respectively), and the fifth graders scored highest of all cohorts (48%). The reasons for these curious differences can only be speculated, but the impact of instruction upon students' intuitive knowledge of density warrants scrutiny in relation to performance.

Item B3 and B8 were the best answered of all items on this test (both 59%). B8 was a non-proportional situation (two trees of different height grow at the same rate; find the height of the second tree after a period of time given the height of the first tree). Students' capacity to distinguish proportional from non-proportional situations is a key for proportional reasoning that indicates reasoning capacity as compared to blind application of formulae (Lamon, 2006). B3 was a 'better buy' situation (briefly described in table 1), and results of this task may not be as exciting as they appear, as this item was essentially a two-choice item (A or B). This is where the second level of coding gives further insight into students' reasoning. From Table 2, it can be seen that students who selected the correct washing powder equally used multiplicative and additive reasoning (22% responses coded 11 and 20% coded 13). Ten percent of students selected the correct answer (code 10) without stating how they achieved this answer. Approximately 25% of students selected the wrong powder (code 23) and used additive thinking in their response. Hence, for this particular item, students may have selected the correct powder but used inappropriate faulty additive reasoning.

The coding of responses and the use of additive and multiplicative thinking is most starkly revealed in items A1 and A5 (see table 1 for an overview of these items). Approximately 50% of students used appropriate multiplicative thinking for item A1, but for A5, 66% of students used inappropriate additive thinking on a standard ratio task that involved a fractional ratio. Item A1 was one of the better-answered of all items by the ninth graders, where the mean score for this cohort was approximately 73%. But for item A5 involving a fractional ratio, performance overall is merely 15% overall, and 20% for ninth graders. Students clearly recognised the multiplicative relationship of the butterflies to drops of nectar in item A1, but alarmingly abandoned this thinking and used an additive strategy for item A5 in the recipe question. The power and stability of additive thinking is clearly an issue for successful operation in domains that require proportional reasoning. Although this finding is not new, the overwhelming incorrect use of additive thinking for this item further highlights the instability of relational thinking of students in the middle years of schooling.

Conclusion

The results reported in this paper are the first steps towards taking a more structured approach to a connected curriculum across the domains of mathematics and science. The proportional reasoning test devised for this project makes no claims of comprehensively assessing students' proportional reasoning for mathematics and science. However, its purpose was more fundamentally to raise awareness of the pervasiveness of proportional reasoning across the domains of mathematics and

science and to assist teachers to target instruction more specifically to promote students' proportional reasoning.

Acknowledgement

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JUSTIFICATIONS AND EXPLANATIONS IN ISRAELI 7TH GRADE MATH TEXTBOOKS

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This study compares six 7th grade Israeli mathematics textbooks, examining the opportunities provided by the textbooks to justify and explain mathematical work, in two central topics: equation solving and triangle properties. Using two different units of analysis, initial findings reveal that all six textbooks included considerably larger percentages of geometric tasks that required students to justify or explain their solutions than such algebraic tasks. Moreover, considerable differences were found among the six textbooks in the percentages of tasks that required students to justify and explain in both topics, more so in the algebraic topic. Analysis of the nature of the student tasks – whether the tasks include a given mathematical claim for the students to justify or not – also revealed substantial differences among the textbooks.

INTRODUCTION

The Israeli school curriculum is developed and regulated by the Ministry of Education. In 2009 the Ministry of Education launched a new national junior-high school mathematics curriculum that comprises three strands: numeric, algebraic and geometric. The new curriculum stresses problem solving, thinking, and reasoning for all students, emphasizing the development of students' ability to explain, justify and prove, in both domains of algebra and geometry (Ministry of Education, 2009). In response to the introduction of the new national curriculum, several teams, from the academia and from the private sector, began to develop parallel experimental curriculum programs that include textbooks, teacher guides, and other teaching and learning resources. Our study compares a sample of these new 7th grade textbooks, examining the opportunities provided for students to explain, justify and prove.

BACKGROUND

Comparative studies of mathematics textbooks conducted in recent years examine a variety of aspects and issues. Some centre on the issue of justification and explanation (e.g., Hanna & de Bruyn, 1999; Stacey & Vincent, 2009; Stylianides, 2008), which is central to work both in the discipline and in school mathematics. These studies reveal differences among textbooks intended for the same grade level even in countries with a national or a provincial curriculum (e.g., Hanna & de Bruyn, 1999), and show quantitative and qualitative differences among the justifications and explanations provided or expected from the student, in different topics within the same textbook. For example, Hanna and de Bruyn (1999), who examined Canadian grade 12 textbooks, showed that about one-half of the tasks in geometry were proof-related whereas less than 5% of the tasks in algebra.

The research methods used in these studies vary. Some researchers focused in their analysis only on aspects related to the notion of proof (e.g., Hanna & de Bruyn, 1999) whereas others examined the nature of explanations and justifications presented or required – not restricting themselves to proof-related aspects only (e.g., Stacey & Vincent, 2009). Some researchers analyzed only the explanatory text presented in textbooks (e.g., Stacey & Vincent, 2009), whereas others analyzed only the tasks intended for student work (e.g., Stylianides, 2008), or both textbook components (e.g., Hanna & de Bruyn, 1999). When analyzing textbook tasks, researches used different units of analysis. Some used the numbering system of the textbooks and referred to a single numbered problem or exercise as one task (e.g., Hanna & de Bryun, 1999), whereas others defined a task as "an activity, exercise or a set of exercises in a textbook that has been written with the intent of focusing a student's attention on a particular idea" (Jones & Tarr, 2007, p. 13).

Building on these studies our larger comparative research study examines: (1) the justifications to mathematical statements offered in 7th grade textbooks (direct instruction), and (2) the opportunities provided for students to justify and explain their own mathematical work (individual/small-group work). In this paper we report initial findings from the second part of our research, which compares the opportunities provided for students to justify and explain their own mathematical work in two central topics in the 7th grade curriculum: equation solving in algebra and triangle properties in geometry.

METHODOLOGY

Nine parallel new textbook series were developed in Israel after the introduction of the new national curriculum. They can be classified into three groups, according to how they are commonly perceived in the public eye: (1) four textbooks are associated with commercial publishers, (2) three textbooks are associated with the academia or with a non-profit organization dedicated to the advancement of the education system in Israel, and (3) two textbooks were written by research mathematicians. From these nine textbook series we selected six 7th grade textbooks for analysis (textbooks A, B, C, D, E and F). The selected textbooks represent the wide-range of Israeli textbook developers and publishers: two textbooks were published by commercial publishers (A and B), three by academic publishers/non-profit organization (C, D and E), and one textbook was written by a research mathematician (F).

Two topics were selected for analysis: equation solving from the algebra strand and triangle properties (area and angle sum) from the geometry strand. These topics were selected because they are central in the 7th grade curriculum, and have a significant procedural characteristic. Table 1 shows, for each topic and each textbook, the number of pages selected for analysis out of the total number of textbook pages, and the number of lessons suggested for teaching the content of these pages (based on the authors' recommendations).

Textbook (# pages)		Solving equations		Triangle properties	
		# pages	# lessons	# pages	# lessons
A	(643)	44	12	49	16
B	(623)	53	11	31	12
C	(746)	58	12	60	9
D	(591)	31	9	45	13
E	(456)	31	13	27	9
F	(430)	24	-	21	-

Table 1: Sample selection from each textbook

The first stage of data analysis was, for each topic and each textbook, to count the number of tasks suggested for individual or small-group work, either in class or at home (based on the authors' recommendations). The first count (Count 1) was based on the numbering system of the textbooks themselves. However, we found that this way of counting caused some distortion when comparing the textbooks, because different textbooks used different numbering systems for problems and exercises. For example, solving equations exercises were numbered separately for each equation in textbook A, whereas several such exercises were often grouped together and were numbered only once in the other textbooks.

In order to overcome such a distortion, we employed a second count of the number of tasks (Count 2), based on the definition of tasks proposed by Jones and Tarr (2007). According to this definition, a set of exercises that are built on each other are considered as a single task, even if they were numbered separately in the textbook. Likewise, a sequence of successive exercises dealing with the same mathematical idea, or practicing the same skill, is also considered as a single task, regardless of the numbering set out in the textbook. For example, the exercises in Figure 1 are counted as two tasks when employing Count 1, but as one task using Count 2, because they both deal with the same mathematical idea and practice the same skill of solving simple equations by considerations.

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1. Find the solutions of the following equations:

   a. $4x = 12$ b. $6 + x = 12$ c. $x - 4 = 12$ d. $6 - x = 12$

2. Find the solutions of the following equations:

   a. $4x = -12$ b. $6 + x = -12$ c. $x - 4 = -12$ d. $6 - x = -12$

Figure 1: Exercises counted as two tasks (Count 1) and as one task (Count 2).

Table 2 presents, for each textbook and each topic, the number of tasks suggested for individual or small-group work, according to Count 1 and Count 2.

	Solving equations		Triangle properties	
Textbook	# tasks (Count 1)	# tasks (Count 2)	# tasks (Count 1)	# tasks (Count 2)
A	335	83	92	85
B	85	45	61	56
C	116	81	118	96
D	55	45	83	80
E	72	61	64	64
F	72	54	64	61

Table 2: Number of tasks according to Count 1 and Count 2

As shown in Table 2, the discrepancy between the number of Count 1 and Count 2 tasks is considerably larger for tasks in algebra than in geometry.

In the second stage of data analysis we coded each task, using both counts, either as requiring students to justify or explain their mathematical work (J-tasks) or as not (NJ-tasks). J-tasks comprised all tasks in which such a requirement was explicit, expressed by phrases such as "Justify", "Explain why", "Describe your considerations", "Explain your solution", etc. In addition, we classified tasks as requiring a justification or an explanation even if such requirements were not explicitly stated, in cases where it was clear that explanation is expected. For example, clearly the solution of the task "Is there a triangle whose three altitudes are outside it?" could not be a yes/no answer only without any explanation.

Finally, we analyzed all the J-tasks, examining whether students are asked to justify a given claim (GC/J-tasks), or whether no claim is stated and students are asked to solve a problem, and then to justify their solution (NC/J-tasks). GC/J-tasks commonly contained phrases, such as "Show that…", "Prove that…", "Explain why…". NC/J-tasks typically started with questions, such as "Solve and explain", "Is it true or false?", "Could it be that…?" Figures 2 and 3 exemplify GC/J-tasks and NC/J-tasks (respectively).

Show that the following equations are equivalent:

    a) $3x + 2 = 17$ b) $4x - 6 = 14$

Figure 2: Example of GC/J-tasks

A right triangle was obtained by multiplying the side lengths of a given right triangle by 2. How much larger is the perimeter of the new triangle? How much larger is its area? Explain.

Figure 3: Example of NC/J-tasks

RESULTS

Substantial differences were found among the six textbooks in the percentages of algebraic J-tasks (i.e., tasks that required students to justify or explain). The differences were found using each of Count 1 and Count 2 (see Table 3).

| Textbook | Count 1 | | | Count 2 | | |
| | Algebraic tasks | Algebraic J-tasks | | Algebraic tasks | Algebraic J-tasks | |
	n	n	%	n	n	%
A	335	22	7	83	15	18
B	85	1	1	45	1	2
C	116	22	19	81	18	22
D	55	4	7	45	4	9
E	72	15	21	61	14	23
F	72	5	7	54	5	9

Table 3: Numbers and percentages of J-tasks in the algebraic topic

As shown in Table 3, the two counts produced similar percentages for each textbook, except for textbook A, for which the percentage of algebraic J-tasks out of the total number of algebraic tasks was much larger using Count 2 (7% according to Count 1 and 18% according to Count 2). Analysis shows that less than 10% of the algebraic tasks of three textbooks (B, D and F) required students to justify or explain – textbook B being the extreme case – whereas about 20% of the algebraic tasks in each of textbooks C and E were J-tasks.

In the geometric topic – triangle's area and angle sum – the differences among the six textbooks were less prominent than in the algebraic topic (see Table 4). As shown in Table 4, the two counts produced similar percentages for each textbook. More than 20% of the geometric tasks of all textbooks required students to justify or explain, according to each of the two counts – more than the corresponding percentages of algebraic tasks – textbook A stands out with the largest percentage of such tasks (almost one-half of the tasks).

	Count 1			Count 2		
Textbook	Geometric tasks	Geometric J-tasks		Geometric tasks	Geometric J-tasks	
	n	n	%	n	n	%
A	92	41	45	85	39	45
B	61	13	21	56	13	23
C	118	38	32	96	35	36
D	83	24	29	80	24	30
E	64	21	33	64	21	33
F	64	21	33	61	21	34

Table 4: Numbers and percentages of J-tasks in the geometric topic

Analysis of the nature of the student tasks – whether the tasks include a given mathematical claim for the students to justify (GC/J-tasks), or not (NC/J-tasks) – revealed substantial differences among the textbooks. Table 5 presents the frequency of NC/J-tasks for each topic in each textbook, according to Count 2 (similar outcomes were obtained using Count 1).

As shown in Table 5, more than 90% of the J-tasks in three textbooks (A, C and D) were NC/J-tasks, both in algebra and in geometry, i.e., tasks that do not state a mathematical claim that should be justified. Textbook F stands out with the smallest percentages of such tasks – for both topics about 20% of the J-tasks of textbook F were NC/J-tasks. Apart from textbook B (which is an uninteresting case because it had only one algebraic J-task), prominent discrepancies between the percentages of algebraic and geometric NC/J-tasks occurred in the case of one textbook only (E).

	Solving equations		Triangle properties	
	J-tasks	NC/J-tasks	J-tasks	NC/J-tasks
Textbook	n	%	n	%
A	15	100	39	92
B	1	0	13	85
C	18	94	35	97
D	4	100	24	92
E	14	71	21	86
F	5	20	21	24

Table 5: NC/J-tasks frequencies

DISCUSSION

Analysis revealed substantial differences among the six textbooks in the percentages of algebraic tasks that required students to justify and explain (J-tasks). There were also differences among the textbooks with regards to such geometric tasks, but they were less prominent. Analysis of the nature of the student tasks (i.e., whether the tasks included a given mathematical claim for the students to justify or not) also showed considerable differences among the textbooks – one textbook adopting a completely different approach than the other five – but not between the two topics.

Our findings reveal that all six textbooks included considerably larger percentages of geometric J-tasks than such algebraic tasks. This finding, which is consistent with findings of other studies (e.g., Hanna & de Bruyn, 1999), might be related to several factors. For example, for many years geometry has been viewed as the most appropriate domain for teaching students proof, and for developing students' ability to reason logically. To achieve that, traditional geometric tasks required students to use proof to justify their work. This has not been the case with algebra, which historically has been a domain "concerned with generalized computational processes" (Sfard, 1995, p. 17). Also, algebra, which is known to be difficult for many students (e.g., Sfard, 1995), is formally introduced in the Israeli curriculum in the 7th grade. Consequently, textbooks' authors may have assumed that algebraic tasks that require justifications or explanations might be too difficult for 7th grade students, who for the first time need to deal with algebraic representations and language. The smaller gap found between the percentages of algebraic and geometric J-tasks in some of the textbooks may reflect a different approach that is in line with two important goals for the new national curriculum (Ministry of Education, 2009): (a) understanding the essence of algebra as a mathematical branch that deals with generalization processes, raising hypotheses and justifying them, and (b) developing argumentative discourse: ways to explain or prove algebraic properties and rules.

Another interesting finding was the high percentages of geometric J-tasks that did not include a given claim for students to justify, in all textbooks but the one written by a mathematician. Most geometric J-tasks in the other five textbooks were not in the traditional form of "Prove that…". Instead, students were expected to propose hypotheses and justify or refute them. This approach is in line with last decade calls for changing the traditional way of teaching geometry, introducing investigation and problem posing into geometry classes (e.g., Yerushalmy & Chazan, 1987).

Another important aspect this research illuminates is the interconnections between research methods and findings. We saw that the use of different units of analysis sometimes produced different findings. We feel that our use of two units of analysis strengthened our findings, and addressed well the methodological problem caused by different textbooks' structures.

Our study portrays the opportunities provided by six new textbooks for 7th grade Israeli students to justify and explain their mathematical work. The findings may

reflect also the approaches of the textbooks' authors to teaching and learning mathematics, but not necessarily in a simple way. For example, low percentages of J-tasks in specific textbooks may reflect an authors' view that justifications and explanations are not important at this learning stage. But such low percentages may also reflect a different view: that justifications and explanations are important, but it is the teacher's and not the textbook's role to encourage students to justify and explain their answers.

We also need to be cautious when attempting to make simple links between textbooks and classroom instruction. Textbooks are, in a way, the potentially enacted curriculum. Yet, accumulating research suggests that different teachers enact the same curriculum materials differently (e.g., Even & Kvatinsky, 2010), and variations in curriculum enactment were found even in cases where the teacher used the same textbook in different classrooms (e.g., Eisenmann & Even, 2011).

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PRE-SERVICE TEACHERS' VIEWS ON USING MULTIPLE REPRESENTATIONS IN MATHEMATICS CLASSROOMS – AN INTER-CULTURAL STUDY

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Dealing with representations and changing between them plays a key role for both mathematics as a discipline and for building up mathematical knowledge in the classroom. Hence, professional knowledge and views of teachers related to using multiple representations can be considered as a prerequisite for creating conceptually rich learning opportunities. However, specific empirical research is scarce – in particular there is a lack of studies taking into account that culture might influence such views. Consequently, this study focuses on views about using multiple representations held by more than 100 British and more than 200 German pre-service teachers. The results indicate that culture might influence the views of the pre-service teachers, but also that there are common needs for further professional development.

INTRODUCTION

Since mathematical concepts can only be accessed through representations, teachers should be aware of their crucial role for the construction processes of the learners' mathematical knowledge. In particular, they should have developed a profile of views on reasons for using multiple representations. Perceptions of such reasons can have a significant impact on the teachers' abilities to design rich learning opportunities. For instance, acknowledging that only the combination of different representations affords rich insights into mathematical concepts may better support teachers in designing mathematical activities than seeing the main purpose of multiple representations in keeping pupils' interest. Despite the obvious importance of such views for the mathematics classroom, specific empirical research is scarce.

Hence, this study focuses on such views on using multiple representations. We use a trans-national design with British and German pre-service teachers to explore whether the views are strongly culture-bound. In line with a multi-layer model of professional knowledge, these views are examined on different levels of globality to find out how general views on using multiple representations translate into views about the use of representations in a content domain and in specific tasks. The results suggest cultural differences, but also that there are common needs for professional development, since pre-service teachers of both subsamples appear not to fully understand the crucial role of multiple representations for mathematical thinking and learning.

In the following first section, we briefly introduce into the theoretical background of this study, the second and third sections present research questions and the research design. Results are reported in the fourth section and discussed in the fifth section.

THREORTICAL BACKGROUND

In mathematics and consequently also in mathematics classrooms representations play a special role. Since mathematical objects are never directly accessible, experts as well as learners have no choice other than using representations when dealing with them (Duval, 2006). We take the notion "representation" to mean something which stands for something else – in this case for an "invisible" mathematical object (cf. Goldin & Shteingold, 2001). Since usually a single representation can only make visible some properties of the corresponding object, multiple representations which can complement each other are needed for getting hold of it (Gagatsis & Shiakalli, 2004). Hence, representations are not only tools for mathematical thinking and communication, but also essential accesses to mathematical objects. This characteristic of the discipline entails many possible problems for learners. In particular conversions from one mode of representation to another often pose a crucial obstacle to comprehension and at the same time the ability to recognize a mathematical object behind its different representations and to use them flexibly is key for successful mathematical thinking and problem solving (i.e. Lesh, Post, & Behr, 1987; Gagatsis, & Shiakalli, 2004; Panaoura et al., 2009).

Consequently fostering the pupils' competencies in dealing with multiple representations should be a central goal in the mathematics classroom (cf. i.e. KMK, 2003; NCTM, 2000). In particular for the content domain of fractions – which is the focus of the domain specific parts of this study – there is broad consensus on the significance of multiple representations for the pupils' learning (i.e. Ball, 1993; Padberg, 2002).

Against this background the question arises as to what professional knowledge and views teachers have with respect to this (special) role of multiple representations in mathematics and for teaching mathematics. For exploring such views, this study uses a multi-layer model of professional knowledge (Kuntze, 2012), that combines the spectrum between knowledge and beliefs (e.g. Pajares, 1992), the domains by Shulman (1986; cf. also Ball, Thames & Phelps, 2008) with levels of globality, i.e. a distinction between general and specific views resp. knowledge (cf. Törner, 2002; Kuntze, 2012).

RESEARCH INTEREST

According to the need for research pointed out in the previous sections the study presented here aims to provide evidence for the following research questions:

What views do British and German pre-service teachers have on the role of multiple representations for learning mathematics? In particular: Which reasons for using multiple representations in mathematics classrooms are most important to them?

Do inter-cultural comparisons reveal any differences regarding such views?

Are views on different levels of globality interrelated? In particular: Are global views on reasons for using multiple representations interconnected with domain-specific

views on dealing with representations when teaching fractions and with views on specific problems that do or do not support using multiple representations?

SAMPLE AND METHODS

For answering these research questions, a questionnaire was designed in German and was then translated into English. This translation was examined carefully by two native speakers of English, one of whom is also fluent in German and has taught mathematics both in the UK and in Germany.

The questionnaire was administered to 139 British (99 female, 22 male, 18 without data) and 219 German (183 female, 26 male, 10 without data) pre-service teachers before the beginning of a course at their university. The British participants had a mean age of 27.9 years (SD = 6.9), while the German participants were on the average 20.7 years old (SD = 2.5), but (with only a few exceptions in both samples) all the participants were at the beginning of their first year of teacher education at university. Corresponding to the research questions for this study three parts of the questionnaire were included in the evaluations, each of them assessing views on using multiple representations on a different level of globality. There was one part about reasons for using multiple representations in mathematical classrooms in general, then there was a part focusing on specific views related to the use of multiple representations while teaching fractions and furthermore the participants were asked to evaluate the learning potential of a specific fraction problem in which multiple representations were not used appropriately. All these questionnaire sections used scales consisting of several multiple-choice items each. The pictorial representations in the problem given to the participants and shown in figure 1 are not really helpful for solving the problem, since they can't illustrate the operation needed to carry out the calculation. Thus, solving this problem is just a matter of carrying out the calculation on a symbolic-numerical representational level and ignoring the given pictorial representations.
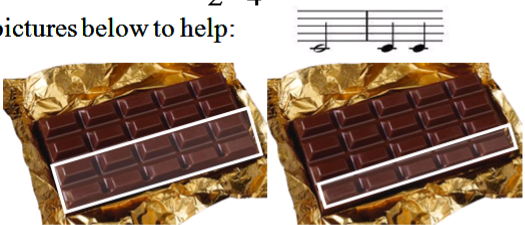

Do you know what $\frac{1}{2} \cdot \frac{1}{4}$ is? You can use the pictures below to help:	The role that representations play in this problem is such that pupils' mathematical skills will improve.
	Pupils can learn a lot from this problem.
	The way in which representations are used in this problem aids pupils' understanding.
	It is important that pupils are given such problems to build up their knowledge of basic principles.

Figure 1: Specific fraction problem

At the beginning of the questionnaire there were explanations of the notions "representation" and "pictorial representation" in a mathematical context given in order to ensure that all participants have a similar understanding of these key terms for the study. The data was analysed using quantitative methods. In order to be culture-fair, the analyses were done firstly separately for both of the subsamples in order to check for culture-specific patterns.

RESULTS

We start with the results concerning the most global views investigated – the rating of the importance of reasons for using multiple representations in mathematics classrooms. In line with the design of the questionnaire, separate factor analyses for both subsamples yielded six 3-items-scales with high reliability values each of which reflects a specific reasoning. Three of these scales express reasons for using multiple representations which are not really specific to mathematics. Sample items for these scales were, respectively (identifiers of the scales in brackets):

"They make it easier to keep pupils' attention and interest." (motivation & interest)

"Pupils can use pictorial representations as mnemonics." (supporting remembering)

"Different learning types and input channels can be addressed." (learning types and input channels)

The other three scales correspond to reasons for dealing with multiple representations in mathematics classrooms taking into account the key role which representations play for mathematical thinking. Here are sample items for these scales:

"Enhancing the ability to change from one representation to another is essential for the development of mathematical understanding." (necessity for understanding)

"Many mathematical problems can only be solved by changing from one representation to another." (supporting problem solving)

"Only the combination of different representations can make a mathematical concept accessible." (making mathematical concepts accessible)

Figure 2 shows the means and standard errors of these scales for both subsamples. The value 1 stands for "not important" and the value 5 corresponds to "extremely important". First, it's noticeable that both subsamples rated the more general reasons that do
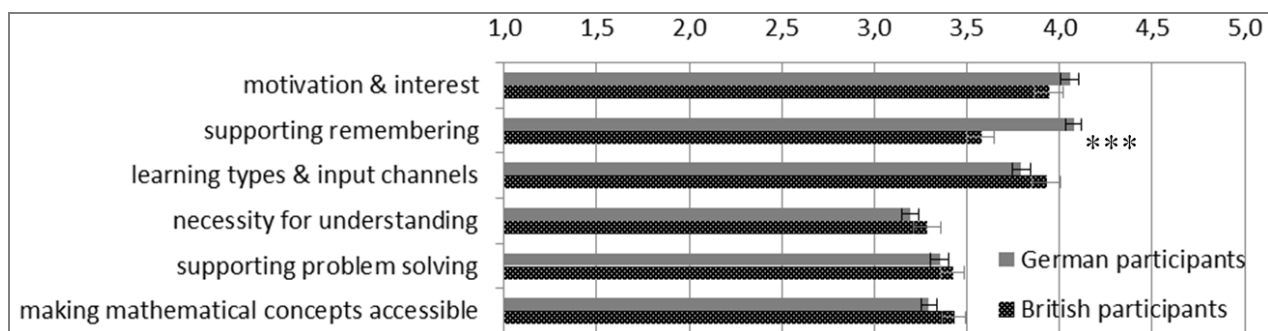

Figure 2: Views on the importance of reasons for using multiple representations

not require the awareness of the special role of multiple representations in mathematics as more important than the other reasons. Furthermore there are no significant differences between the ratings of the British and the German pre-service teachers, except for the second scale: The German pre-service teachers attributed a higher significance to the contribution of multiple representations to remembering mathematical facts than did their British counterparts (T=6.016, df =206.8, p<.001, d=0.731).

Looking at the results for the questionnaire section about views on the role of multiple representations for teaching fractions reveals more differences between the views which were expressed in the two subsamples. Here our factor analyses have yielded eight different scales, again in line with the theory-based design of the questionnaire. Five of them revolve around the question of whether one should use multiple representations for teaching fractions or not; however they focus on different aspects. Sample items for these scales are (identifiers of the scales in brackets):

"To understand fractions properly, it is necessary to use many different representations in class." (multiple representations for understanding)

"In order to give pupils the opportunity to choose their preferred type of representation, which they most easily understand, they should be provided with many different representations." (multiple representations for individual preferences)

"It's best to use only one kind of pictorial representation for fractions in lessons, so that you can always come back to this as a 'standard' representation." (one standard repres.)

"Several different pictorial representations for fractions could confuse pupils, especially the weaker ones." (fear of confusion by multiple representations)

"If pupils pay too much attention to pictorial representations, their ability to confidently do calculations with fractions is impeded." (multiple representations impede learning rules)

The remaining three scales express views concerning the question of whether one should use pictorial representations for fractions consistently until the end of the teaching unit or rather just for the introduction and then foster abstraction. Samples for items of these three scales are:

"For optimum learning of fraction calculations it is important to use pictorial representations consistently until the end of the teaching unit." (ongoing use of pictorial rep.)

"After the introductory stage of the teaching unit, the teacher should move away from pictorial representations in order to strengthen the pupils' calculation skills." (pictorial representations just for introduction)

"In order to gradually encourage the pupils to think of fractions in an abstract way, teachers should move away from pictorial representations in the course of the teaching unit." (fostering abstraction)

These eight scales consist of three items each (with one exception where we had to exclude one item for reliability reasons) and they are highly reliable for both subsamples. Comparing the means of the two subsamples shown in Figure 3, one discovers an interesting pattern: The British pre-service teachers compared to the German pre-service teachers were more in favour of using multiple representations for teaching fractions and they saw fostering abstraction as less important. Cohen's d shows that the difference concerning the scale "multiple representations for understanding" can be neglected, whereas the other significant differences correspond to weak or medium effects ($.34 < d < .69$). However, despite these differences, similar views can be identified: The participants of both subsamples rather opposed the consistent use of pictorial representations until the end of the teaching unit on fractions.
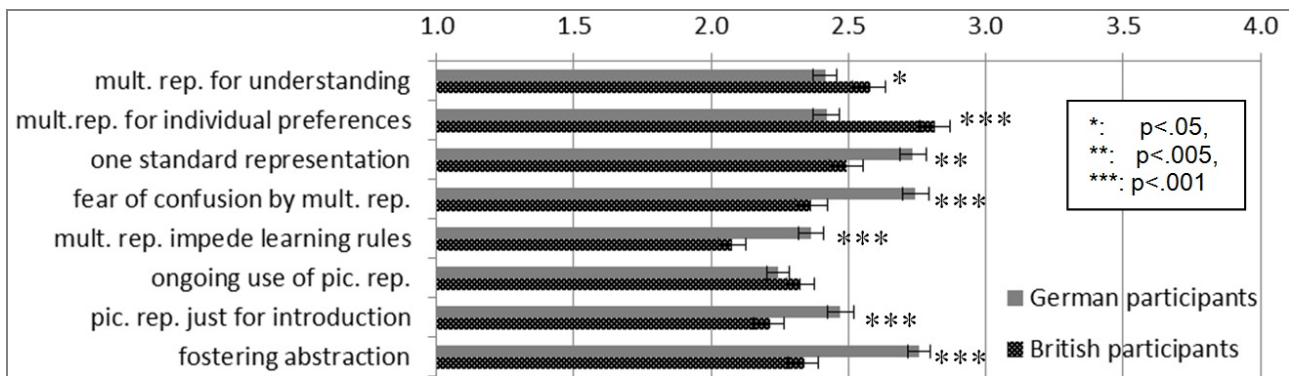
Figure 3: Views on the role of multiple representations for teaching fractions

The evaluation of the learning potential of the specific fraction problem discussed above was based on the four items shown in Figure 1. Factor analyses showed that for both sub-samples all these items formed a single highly reliable scale ($\alpha=.84/ \alpha=.84$).
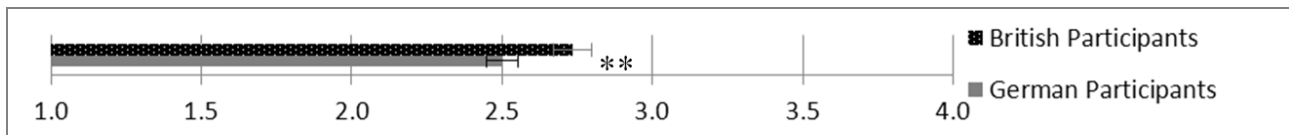

Figure 4: Evaluation of the learning potential of the given problem

As can be seen in Figure 4 the learning potential of the problem was rated as medium by the German and slightly positive by the British pre-service teachers.

The third research question focused on relationships between views of the pre-service teachers. We would like to recall that these views were located on different levels of globality, i.e. ranging from relatively general views to relatively content-specific views. For meeting space limitations, we focus on selected findings.
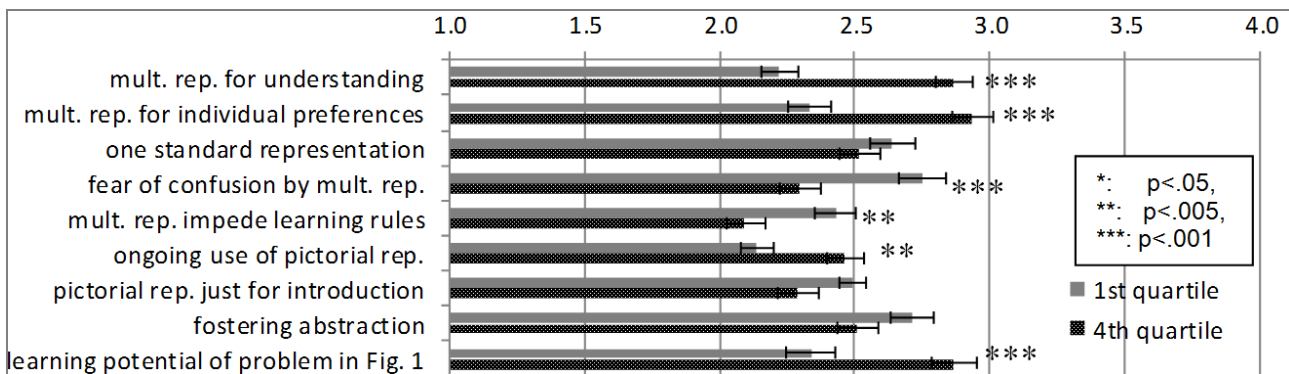

Figure 5: Views on content-specific levels of globality for 'extreme' groups

Seen against the theoretical background of our study, a key idea is that mathematical concepts often need the approach through multiple representations for building up satisfactory understanding. The corresponding view is reflected in the last scale in Figure 2. If we split up the sub-samples into quartiles according to this view, we can compare a group of pre-service teachers who acknowledge this aspect (the upper quartile) with the 25% of teachers who see this aspect as relatively unimportant. For these 'extreme' groups, Figure 5 presents their more content-domain-specific views,

related to the domain of fractions and to the perceived learning potential of the specific problem in Figure 1. The data in Figure 5 indicates significant differences.

DISCUSSION AND CONCLUSIONS

The results suggest that pre-service teachers' views on using multiple representations in the mathematics classroom are framed by culture, but also that there is a common lack of awareness regarding reasons for emphasizing multiple representations and their interrelations, which are intrinsic to mathematics as a discipline and essential for the pupils' understanding of mathematical concepts.

We start with the discussion of the differences between the two subsamples, which our evaluations have yielded. With respect to reasons for using multiple representations in mathematical classrooms in general the only significant difference we identified is the greater emphasis of the German pre-service teachers on remembering facts. Concerning more specific views related to the use of multiple representations connected to teaching fractions however, the comparison of the subsamples appears to reveal more cultural differences: For example, the British pre-service teachers attached significantly greater importance to multiple representations for teaching fractions than their German counterparts – at least when reasons not specific to mathematics were in the focus. Considering the scales showing the strongest effects suggests that this discrepancy could come from the German pre-service teachers rather fearing confusing their pupils, whereas for the British participants taking into account individual preferences was predominant. This may also explain why the British pre-service teachers have evaluated the learning potential of the problem, in which multiple representations were used inappropriately, more positively. The task suggests that every pupil can choose a pictorial representation according to his/her preferences in order to find the solution. The fact that these representations don't match appears to be often undetected by pre-service teachers of both countries. This demonstrates that the pure conviction of "Using multiple representations is good" is not enough for designing rich learning opportunities, but instead a deeper understanding of the role which representations play in mathematics is needed for being able to analyse how multiple representations should be used in mathematical classrooms. The impression that this is a common need for professional development is reinforced by the results in Figures 2 and 3: The pre-service teachers of both countries attached comparatively less importance to the reasons for dealing with multiple representations related to mathematical thinking or the development of conceptual understanding and they rather opposed the ongoing use of pictorial representations when teaching fractions.

Referring to the third research question, comparing the pre-service teachers' views on using multiple representations in mathematical classrooms on different levels of globality yields interrelations, which are consistent with our theoretical assumptions and with the findings related to needs for professional development. The results shown in Figure 5 indicate that the components of professional knowledge examined in this study belong to a bundle of convictions and knowledge that merits further attention and deepening research. Such research should also include a focus on specific content

knowledge, as knowledge appears to play a role especially for views on the more situated levels of globality. For research questions in this domain, ongoing analyses of data from other questionnaire sections of this survey could give insight.

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INFRASTRUCTURES WITHIN THE STUDENT TEACHING PRACTICUM THAT NURTURE ELEMENTS OF LESSON STUDY

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A single case of student teaching program in a Fuzoku school in Japan was highlighted in the hope of elevating certain ruminations on how the essential features of Lesson Study are legitimized and nurtured in pre-service mathematics teacher education. Findings of this study attempt to elucidate how, via mechanisms or infrastructures within student teaching practicum, significant development of habits of mind could be possibly forged among prospective mathematics teachers in order to maximize their student teaching experience and to facilitate them into participating in Lesson Study as they step into the profession.

INTRODUCTION

In the Japanese educational system, national universities that offer courses in Education have attached institutions that serve as laboratory schools for student teachers, among some of their functions. These schools are called *Fuzoku* schools, which exemplify a well-defined function in the pre-service (and in-service) teacher education in Japan. Whereas Lesson Study (from now on, LS) is widely performed in in-service mathematics teacher education in Japan, the student teaching program (STP) in *Fuzoku* schools integrate vital elements of LS as a part of the practicum.

This inquiry is a part of a larger phenomenological study that seeks to understand the underlying principles behind the accession of LS in pre-service mathematics teacher education. It is believed that undertaking an investigation on how LS is being integrated and nurtured in STP in Japan will contribute to the ongoing reflections on how pre-service mathematics teacher education could be made relevant towards a smooth transition of perspective teachers into becoming in-service mathematics teachers. In my earlier discussion connected to this study, I suggested several habits of mind that need to be cultivated amongst pre-service mathematics teachers in order for them to be able to participate in LS endeavours as they commence their practice in the profession: (1) making sense of powerful resources for classroom instruction; (2) utilizing the school and classroom contexts as venues of inquiry; (3) engaging in critical reflections; and (4) forging the spirit of collaboration (Elipane, 2011).

In the light of necessitating, nurturing, and reinforcing certain skills or habits of mind that would facilitate among prospective teachers into being able to optimally make sense of their learning experiences in the STP, further reflections on the integration of LS in pre-service mathematics teacher education context would lead to reflections on whether these were done in an intentional or systematic manner. This paper, therefore,

yields implications on the forcible mechanisms, or infrastructures within the STP, that nurture such habits of mind.

THEORETICAL BACKGROUND

The simple premise behind conducting LS is that in order to improve teaching, the most effective place to do so would be in the context of a classroom lesson (Stigler & Hiebert, 1999). Furthermore, Lewis, Perry and Hurd (2009) proposed a theoretical model of LS, which identifies three *pathways* wherein LS could improve instruction, and these are changes in (1) teachers' knowledge and beliefs, (2) professional community, and (3) teaching-learning resources (p. 285). Also, with an emphasis on the live Research Lessons (RL) in Lesson Study, Murata (2011) identifies five key characteristics of the practice that must be preserved if teachers or stakeholders are contemplating this kind of endeavor: (1) centered around teacher's interests; (2) student focused; (3) has a RL; (4) a reflective process; and (5) collaborative (p.10).

Extending rationalizations on LS to the teacher education context would necessitate looking further into several perspectives in framing studies that surround pre-service mathematics teacher education. Ponte (2011) succinctly articulates the distinctiveness between the practices inherent in the in-service and the pre-service contexts by saying that " the primary practice of practicing teachers is professional practice, and the primary practice of the prospective teachers is their learning practice" (p. 415). Here, the notion of recontextualizing yields certain relevance. Ensor (2001) suggests that this notion highlights the "transformation of discourses as they are disembedded from one social context and inserted into others" (Ensor, 2001, p. 297).

Moreover, the Notion of Emergent Perspectives (NEP), as described by Cobb (2000), posits that learning is a cognitive process of individual construction, but concurrently a sociological process of participation in a group. Hence, in the context of mathematics teacher education, NEP situates the cognitive development of mathematics teachers, prospective and practitioners alike, within various educational learning environments, (mathematics) classrooms, or communities of practice. Thus, NEP marries the cognitive/psychological construct, in which learning to teach mathematics involves active individual construction, and the sociological stance of mathematics teacher education, a process that involves adaptation and assimilation of levels of social dimension embedded in certain communities of practice.

METHODOLOGY

Prior to undertaking this particular investigation, I have already been amply acquainted to the social context of the practice by observing a number of LS in several schools in Japan for about four years. In this report, a pre-service teacher was observed daily in his activities as a student teacher (ST) over the span of the STP, which lasted for four weeks. As my objective is to make an inquiry regarding pre-service teacher education in mathematics as a specialized area, the subject for this investigation was purposively selected to be a prospective middle school mathematics teacher. The other

criterion for the selection of the subject was the willingness to be a part of this investigation. The placement school was a *Fuzoku* Middle School of a national university named Saitama University, which is located in Saitama (prefecture bordering the north of the Tokyo Area). It is composed of three year levels, each with four sections and an average class size of 40.

The subject for this inquiry was assigned to the cooperating teacher (CT) who handles all the four classes of first year; and he was to handle only one section. On the other hand, another ST was also assigned to the same CT; she was supposed to teach another class. The CT and the two STs together formed a group, wherein, with the guidance of the CT, they observed, commented, discussed, and reflected on each other's lessons. This small community of practice simulated a small LS group.

The STs engaged in a series of classroom observations, lesson preparations, actual classroom instructions, and *hanseikai* (reflection meetings held after every lesson done by the STs). A RL was also undertaken towards the end of the practicum. All the 7 STs in mathematics observed each other's lessons, together with all the 3 CTs, and 2 mathematics teacher educators from the university. They all participated in the *hanseikai* after all the lessons were delivered.

The observations allowed me to become acquainted with the contextual environment of the STP, and to be able to generate conjectures regarding the underlying principles that are engendered in the program. Moreover, the interviews, informal conversations, and the analysis of the activities that the ST underwent and several artifacts (the ST's observation notes, daily journal, etc.) were utilized as rich sources of data that facilitated the crystallization of findings. The interviews with the ST and the CT were audio-taped, along with the reflection meetings and the RL undertaken by the ST. Though it was not possible for me to videotape all the activities and the RL, some photographs were taken when they were allowed. Van Mannen's (1990) phenomenological method was employed in analysing the data. Significant statements, utterances, and actions were highlighted to provide an understanding of how the participants experienced LS in the *Fuzoku* school. From these, clusters of meanings were formulated into the emergent themes that pertain to mechanisms or infrastructures that forge optimization of the learning experiences during the STP.

FINDINGS

The discussions engendered in this particular investigation revolve heavily on the daily activities that were made accessible to the STs during the STP. The way that the STs were immersed into the structure of the program sheds light on the collaborative nature embodied in LS endeavours. Moreover, engaging in preparations, planning, and enactment of daily lessons showed consistent associations with the structure and educational values being reinforced in LS. The enactment of ordinary lessons, which were observed by the other ST and the CT, then passed through lesson debriefs. Therefore, the ordinary lessons could be seen as congruent to RLs being done during the normal LS activities in in-service context, especially that these lessons were able to

facilitate the creation of shared classroom norms and understandings, which provided a powerful learning opportunities for the STs. Indeed, as the elements of LS are embedded in the STP, it appeared that it's already in the tacit core of beliefs and practices of the ST. He said, "I didn't treat the RL as something special. But I believe that I always have to think of the students' learning, be it a RL, or just a simple day."

Four mechanisms that facilitated towards learning to teach mathematics were extracted from the analysis of data: (1) sensitization to images of reform; (2) forged reifications of learning experiences; (3) student feedback and communications; and (4) immersion in communities of practice.

Sensitization to Images of Reform

It was apparent that the ST was able to develop most of the practices that the CT was usually doing in the classroom. Hence, the propensity of the apprenticeship model (Lortie, 1975) of learning to be a teacher in STPs surfaced. In any case, it must be noted that the teachers of *Fuzoku* schools themselves are actively and continuously participating in research activities (e.g., LS) that espouse reform-oriented views on subject matter teaching and learning, which, in a way or another, addresses the danger of socializing STs in continually adhering to traditional ways of teaching. This is because of one of the intrinsic roles of *Fuzoku* schools in Japan in providing leadership in teaching innovations. In this particular case study, the STs also had an opportunity to observe their CTs deliver their RLs. In reaction to this experience, the ST wrote in his journal (25 May 2010):

> I was able to observe how classes are done by pros. I was enthusiastic about the content of the lesson more than ever, but I am anxious about how I could elicit ideas from my own students [when I do my own lesson].

Moreover, amalgamated with his realizations of the complexity of the school and classroom environment, the ST's engagement in critical reflections prevented him from the trappings of conceptualizing teaching mathematics as a technicized endeavour wherein the STs merely imitate the techniques of the CT. Indeed, engagements in critical reflections are a vital habit of mind that allows STs to optimize their learning during the STP (Elipane, 2011).

In his first time to observe a mathematics lesson for first year in the *Fuzoku* Junior High, the ST wrote in his journal (11 May 2010)

> I was surprised with the high level of mathematics instruction the first time I observed the mathematics lesson for first year. Not only were [the students] well equipped to respond to difficult tasks, but [they] also had good attitude towards the class and they showed aptitude in note-taking and recitation… I think that since most of the students also came from *Fuzoku* elementary school, they are being trained well on their knowledge and thinking. This kind of rearing emanates from the spirit and environment embodied in *Fuzoku* schools. I think that the lessons are student-centred. So, when I do my own lesson, I will carefully think of the students' independence and initiatives.

Apparently, also embedded in these ruminations are widely held beliefs about the quality of *Fuzoku* schools in Japan. This perceived reality in *Fuzoku* schools presented potential leeway for the ST into being able to explore more challenging tasks for his actual lessons, and employ a range of strategies in the classroom. Also, his observations have ushered him into further inquiry regarding how he could improve his class, with implications on continuous and recursive learning.

Forged reifications of learning experiences

Reifications of learning experiences have proven to be a very effective way to bring about change to the ST. Not only can one clearly map out the progress that has been taking place by looking at the reified works, it could also be a potential way to facilitate deeper learning. Wenger (1998) describes reification as "the process of giving form to our experiences by producing objects that congeal this experience into thingness" (p.58). Liljedahl (2007), on the other hand, elaborates this notion through the concept of movement of beliefs/knowledge from the tacit to the explicit, and perceives it as a way to stimulate teachers' learning experiences. Various ways or modes of reifications were able to elucidate the ST's transformative learnings. For example, the student teaching journal he completed daily during the practicum illuminates his emotional engagements, which were also considered crucial towards the fruition of the whole learning experience engendered and nurtured during the STP (Elipane, 2011). Third week into the STP, the ST has explicitly articulated his desire towards change (Journal entry, Week 3):

> I felt that the third week went extremely faster than how I felt during the second week. I thought that Monday has just come, but I realized it was already Thursday afternoon. With this kind of pacing, I wonder if change could happen based on the points of reflections discussed last week. I don't really think I changed.

In addition, one of the responses of the CT on the journal entries of the ST read (Journal entry, Week 2): "Writing the journal is for nobody else but you. Though facts/realities are important, don't forget the emotions felt [during the practicum]."

The lesson plans, on the other hand, were able to exemplify the ST's improvement in mathematical content for teaching. Central to lesson planning is the privileging of tasks. The tasks can be suitably considered as a reification of how a teacher has engaged in *kyouzaikenkyuu*, a very important component of LS wherein teachers get the opportunity to reinforce and rectify their mathematical understandings, pedagogical content knowledge, and epistemological assimilations on student thinking. In using the metaphor of icebergs, Doig, Groves and Fujii (2011), relate that it should not only be on its visible tip – the tasks as they are privileged and utilized in the classrooms – that practitioners must be able to focus their attention as they engage in LS endeavours. They go on by suggesting that what lies beneath the iceberg – the rigorous process of *kyouzaikenkyuu* – is very important in planning lessons.

Student feedback and communications

For the whole span of the STP, the STs were encouraged to take advantage of all the possible opportunities for them to communicate with the students and, ultimately, to gather information on their state of learning. Several mechanisms for this purpose were embedded in the system. For one, the STs were not only put in-charge for the subject matter they were supposed to teach, but also for the homeroom periods of the classes assigned to them. They were also impelled to participate during the non- or extra-curricular activities of the students, such as taking their meals with the class during lunch time, joining them during the cleaning time, or participating in club activities where some of the students of their class were enlisted. It turned out that these opportunities to connect with the students outside academic curricular activities have been very important sources of information for the ST to know his students deeper. In his journal, he wrote (12 May 2011):

> I have been gradually coming to understand the atmosphere/character of the class assigned to me. Mainly by having lunch together with the students, and while observing them the classroom, I observed the manner/character of each one of them. Since I have already remembered most of their names and faces, I want to focus more on those who are good in math, and also those who aren't.

Of course, mechanisms within the academic curricular activities that facilitate assessments of students were also existent, aside from the conventional ones. For example, one socio-mathematical norm present in this particular case was the use of students' self-assessment cards. These cards were filled-out by the students after every lesson. It was apparent that these cards were helpful for the ST in being able to have more informed epistemological understanding on how the students are cognitively and affectively responding to his instruction. They served as a tool for the ST to guide him in designing and planning lessons, and they even served as a source of motivation. In his journal entry (14 May 2010), he wrote:

> I replied to the comments written on the self-assessment cards. Receiving lessons from an amateur teacher like me, I was happy to see comments like "I understood", "I got it", "It was interesting", etc. I want to improve further and do a good lesson.

In addition, the mathematics lesson diary, which is a compilation of notes on lessons in mathematics, has been another important source of feedback for the ST. Every student in the class take turns in doing the task of writing the notes, and the ST gave his comments after every completed note for a lesson. It must be noted that the content of this diary does not only consist of the topic or task for the day, the ideas and opinions of the student-in-charge and his/her classmates, solutions, and summary of the lesson, but also some reflections of the student in-charge about the lesson.

Immersion in Communities of Practice

Enlisting the ST in an environment that nurtured his systematic development afforded him certain engagements in collaborative experiences that tackled their shared understandings and reflections on their lived experiences during the STP. Working in

a community of practice presented certain transformative values in the STs' development towards becoming mathematics teachers, as they learn from the diversity of each other's perspectives. In order to be able to do so, the STs were impelled into expositions of their thoughts and making their learning public.

Moreover, by letting the STs formally articulate their comments and thoughts on lessons they have actually seen via the class observation report sheets when they did their RLs, it could be ascertained how competencies in being able to consciously and explicitly generate and convey their thoughts are being deliberately and systematically honed in the STP. In this way, STs were not only encouraged to engage in deep reflections about their own lessons, but also on the lessons of their peers. This could directly draw implications towards being able to initiate themselves into the notion of collegiality as they step in the actual profession of teaching. Below are some comments from other STs on RL (which topic was about addition and subtraction of positive and negative numbers) of the subject of this study, as written on the class observation report sheets:

> The introductory problem seemed interesting. The anticipated reactions [of the students] were well thought of. However, the rules for the task did not have clear rule/standards.

> It is not good to acknowledge only those answers/ideas that [you were] expecting.

> The concept of subtracting negative numbers becomes addition was thoroughly conveyed. It was clearly understood that adding -1 and subtracting +1 is just same.

It could be seen from these comments that STs were able to comfortably convey positive and negative points regarding the lesson. This could be due to the fact that the STs have been situated in an affable and collaborative infrastructure within the practicum wherein they work and support each other towards fulfilling unified goals from the STP. They have been together every school day for virtually a month seeking help and affirmation on tasks related to their student teaching.

DISCUSSION

As exemplified by this particular case of student teaching in a *Fuzoku* School, forcible infrastructures within the STP that integrates the elements of LS could be substantiated could facilitate pre-service mathematics teacher education. Hence, even for just a short (albeit intensive) period of time, prospective teachers were able to optimally make sense of the learnings and experiences systematically provided for them. Each of the lessons planned, enacted, and jointly reflected upon served as informal simulations of RLs, as their executions were consistent with the essential features of LS. Having the opportunity to iteratively engage in this activity certainly effected transformative learning to the ST.

Moreover, it can be said that the strong linkage between the mathematics teacher educators of the university and the teachers at the *Fuzoku* Middle School has been greatly beneficial in nurturing a shared common teaching culture between the institutions. Thus, the process of recontextualization and enculturation were well

coordinated, fortifying the socialization of STs from the context of the university into the teaching culture of the school. It was made apparent from this investigation that the STP held in this particular *Fuzoku* school was able to provide an authentic context wherein STs are afforded facility into developing rich understandings of reform-oriented mathematics classroom instruction. The STP was able to situate the pre-service mathematics teacher education into the crux of the actual professional practice that diverges from the traditional training model of teacher improvement that is usually emphatic on a technicized view of learning and may sometimes be external from the crucial aspects of the actual practice.

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'ROBOTS CAN'T BE AT TWO PLACES AT THE SAME TIME': MATERIAL AGENCY IN MATHEMATICS CLASS

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This article aims to discuss the role and impact of robots on mathematics learning. Analysing students' participation in mathematics classes, when using robots, we discuss their agency and its role on fostering participation and consequently on the learning of mathematics[i].

INTRODUCTION

Learning mathematics has been traditionally seen as a cognitive and individual activity and mathematics itself as the subject for the mind. The conception of learning we talk about in this article is considerably different. We consider learning mathematics as an aspect of participation in social practices (Lave e Wenger, 2001) in which people get engaged in solving problems and making sense, using mathematical representations, concepts and methods (Boaler e Greeno, 2000).

This idea of learning goes beyond the idea that social practices make rich contexts for learning mathematics - it defends that being part of social practices is what learning mathematics is. School mathematics classes which allow students to engage in practices of negotiation and interpretation, using physical and discursive tools and resources, provide learning scenarios in which students participate by adapting to the constraints and agreements of it (Greeno & MMAP, 1998).

Through the project DROIDE II – we have created learning scenarios[ii] in which robots are physical artefacts with which students think during school mathematics practices, aiming to understand how students produce meanings and develop their learning of topics and mathematical concepts when robots are mediators' artefacts.

In this article we will analyse students participation in mathematics classes, in the sphere of the scenarios created, discussing the role of material agency in the learning of mathematics.

LEARNING AS PARTICIPATION IN SOCIAL PRACTICES

Learning is a process that takes place in a participatory structure (Lave e Wenger, 1991). This means, amongst other things, that learning is mediated by the different perspectives existing between co-participants.

The focus of Wenger`s (1998) theory in his book Communities of Practice –Learning, Meaning and Identity - is in 'learning as social participant'. Participating does not only refer to events of local engagement in a certain kind of activities or with a certain type of people, but to a wider process of being an active participant on the practices of social communities. That participation makes us not only what we are but also who we

are and the way we look and interpreter what we do. It also shapes the communities in which we participate; in fact, our ability or lack of it to shape our communities of practice is an important aspect of our participating experience.

Participation in a social practice implies constant negotiation. To negotiate a shared enterprise implies responsibility among the parts involved. This relations include what matter and what doesn`t, what is important and what isn`t, what to do and not to, what parts need attention and what to ignore, what to say and not to, what should be justified and what to presume justified, what to show and what to conceal, to understand when actions and artefacts are good enough and when they still need improvement or refining.

Learning as participation in mathematics classes

Analysing students' participation in mathematics classes becomes important when we want to understand and discuss learning as an emergent phenomenon from participation in social practices.

Learning mathematics is a process of people becoming more capable of participating and a social practice which encompasses the relations between people and knowing. Boaler and Greeno (2000) consider knowing and understanding mathematics as aspects resulting from participation on social practices, in particular, in those which the individuals engage themselves on making sense and solving problems using mathematics representations, concepts and methods as resources. Throughout this process many moments of negotiation take place. These moments of negotiation that occur in mathematics classes shape the practice of school mathematics, affecting participants and their way of participating.

Recently Greeno (2011) has been elaborating about students' interactions and stated that we can analyse the way students position themselves in the interaction by two ways: the systemic positioning considering the other students and the teacher; and the semantic positioning considering the concepts and the mathematics methods. The systematic positioning implies the levels of expectations of others to who is expected to be the first to contribute, to question other`s proposals and to whom to explain about the methods and processes involved in the tasks. The semantic positioning implies what Pickering (1995) called conceptual agency, in which the individual makes choices and emits judgements based on the meanings and adaptation of methods and interpretations.

To think about the systemic position we can analyse two aspects: the negotiated structure of participation and the way students understand the proposed task.

To explain the negotiated structure of participation we will focus on question such as: (i) how is one idea appropriated by the collective? (ii) Who is expected to assume or criticise the mathematics ideas? (iii) How do students become encouraged to speak with each other, meaning, what rules of argumentation are at stack in that practice? To explain the way students understand the proposed task we will focus on: on the sense-

making requirement, the task`s structure and on the requirements to fulfil the task with success (Gresalfi, Martin, Hand, Greeno, 2009). These aspects were the focus of our analyses because analysing these aspects of the interaction allows us to explain students' participation in school mathematics practices and to make quite visible the positioning students assume concerning agency and accountability.

On Agency

"An individual's agency refers to the way in which he or she acts, or refrains from acting, and the way in which her or his action contributes to the joint action of the group in which he or she is participating"(Gresaldi et al, 2009, p. 53).

Pickering (1995) made a difference between human agency and material agency. Humans are active and intentional beings. Human agency has an intentional and social structure. Physical artefacts are essential for the modern world. People manoeuvre in a field of material agency "capture, seduce, download, recruit, enrol, or materialize that agency, taming and domesticating it, putting it at our service, often in the accomplishment of tasks"(p.6). Human agency is itself emergently reconfigured in its engagement with material agency.

"There is no way that human and material agency can be disentangled. Or else, while agency and intentionality may not be properties of things, they are not properties of humans either: they are the properties of material engagement, that is, of the grey zone where brain, body and culture conflate"(Malafouris, 2008, p. 22).

Pickering (1995, in Gresalfi et al., 2009) made a distinction, when he developed the terms conceptual and disciplinary agency, in his sociohistorical analysis of a case of research in mathematics. Mathematicians "exercise conceptual agency when they engage in decision making, exploration, and strategizing" (p. 53).When they decide to use an established method, agency is turned over to the discipline.

According to Pickering (1995) what happens generally, in physics and mathematics, is 'a dance of agency' that combines the conceptual agency with the disciplinary agency or conceptual agency with the material agency. Pickering did not consider material agency significant in mathematics (Wagner, 2004). Although Pickering rejects the possibility of material agency in mathematics, Wagner (2007) considers that question should be pursued. In this article we`ll discuss how is this dance between the material agency and conceptual agency or disciplinary agency in mathematics classes.

METHODOLOGY

The nature of the research related in this article is qualitative due it aims to develop an understanding of human systems, such as a technology-using teacher and his or her students and classroom (Savenye and Robinson, 2004).

To use the Situated Learning Theories as theoretical foundation, when doing research, implies some methodological assumptions such as assuming that investigating is to participate in a wide range of practices in which the investigation occurs (Matos e Santos, 2008). That was the positioning assumed by the researcher involved in the data

collection. To be part of the research was also to learn. So, the observant participating was a central strategy and acquired the status of data collection methodology.

The data collection was made in two months, between February and April of the school years 2010-2011. We chose to work with two classes of the 7th grade students (ages between 13 and 15 years old) studying functions. There was an initial session where students, had their first contact with the robots. It took place in the Droide Laboratory of University of Madeira. A video cam was used, focused on a group. Four 90 minute classes were recorded (also with a video cam focusing in a group).

The analysis was made based on the video transcriptions and on the notes taken by the researcher and teachers involved in their project notebook. The units of analysis include person, activity and the contexts where activity takes place (Matos, 2010). We tried to find patterns of interaction, among students and students and teachers, using questions above to think about data. Bellow we present a short part of the analyses that we have been doing.

DISCUSSION –THE CASE OF 'HE'

The initial session, students went to University of Madeira, to the DROIDE Laboratory, to assemble and program robots. Students were made accountable to the assembly of a robot that could be programmed and that could function as they were eager to see the robot (a car) moving. They had to convince themselves and other groups and teachers that they were capable of doing it, by simply doing it, even because despite the great ambience, cooperation and companionship there was a certain competition going on to see who would be the first group to finish the project and who would do it better. There wasn`t any type of explicit negotiation as they worked thru building the robot or programming it. Each element of the group has taken on a task and the others simply assumed another task or function.

On the following day, back to school, they worked on a worksheet 'notion of a function' which aim was that students working with robots, oriented by the questions of the worksheet, understand, learn and define the concept of function. It was been made over two 90 minute classes. The worksheet had a closed and very scholar structure. The innovation was the inclusion of the robots to think about the mathematical concepts involved.

Each group of students received a worksheet and, even before robots were distributed, the teacher asked them to read attentively the issues on the proposal.

The task is to think about two robot trips given two graphics. The first question was about students analysing both graphics and to make a description of the robot trip having the starting point as a reference. The second question was about the robot`s programming in order realize the trips, if possible. The graphics presented were the following:
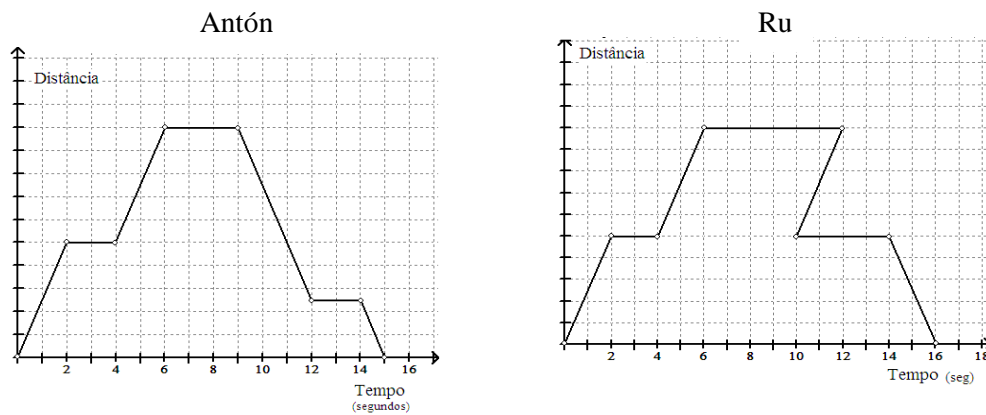
Figura 1 – Graphics presented on the worksheet

The school mathematics practice analysed can be characterized by the resolution of mathematical questions on group, in which students had to discuss every task, to describe the processes that leaded them to results and finally they had to present results and conclusions to the rest of the class. The wider group discussion was mediated by the teacher.

'He' was one of the 10 boys that had failed the preceding year and these school year had had a marginal participation on mathematics classes. Since the moment he began to work with the robots, 'He'`s posture, in mathematics classes, changed. 'He' was the one handling the robot in the group, programming it and checking the programming results.

'He'`s group read the graphic (on the left side) concerning António`s trip with few hesitations. After analysing António`s graphics and programming the robot to make that trip, experimenting it on the floor and verifying it's well done, they came back to the desktop and asked teacher's help. 'He' asked:

He: In the second graphic we don`t really have to do anything, right?

Teacher: Why do you say that? What do you mean "don`t have to do anything"?

He: We already analyzed Rui`s graphic and we can`t program it.

Teacher: And why can`t you?

He: We can`t because there`s no command that allows us to make the robot go back in time.

Teacher: But where in the graphic do you see that the robot has to go back in time?

He: Right here teacher (Rui pointed the graphic to the 12s moment), at 12 second the robot was at a distance 10, but also at a distance of five, because the robot went back and time does not go back. It can`t be at two places at the same time. We can`t program it because it isn`t possible.

'He' was very much convinced that this programming wasn`t possible. Even so, he couldn`t convince his colleagues, that were not, at the time, able to see his point. After discussing his point of view with the teacher he left his colleagues to proceed with the task of programming the second trip even knowing it wasn`t possible, as he pulled

aside and started writing. After some time, the teacher went back to the group and asked if they already had reached to a conclusion. One of students in the group replied:

> Pe: Yes we did. We can`t program it. We only made it until here (pointing on the Robotics Invention System programming interface, to the path until the 12 second).

The inclusion of the robots motivated 'He' and made him to commit to the resolution of the working proposal. His point of view has convinced the teacher but not his colleagues. Probably due to the way other students saw 'He' in terms of mathematical knowledge. He was a student with marginal participation and, maybe because of that his mathematical explanation wasn`t accepted by the group. It was not supposed that 'He' was accountable to the solution of mathematical question due his trajectory in mathematics classes along all the school year until robots arrive.

'He'`s questioning to the teacher was very useful in order to include the teacher himself in the responsibility system, there is, if teacher approved his answer, the other students of the group will be convinced, once he wasn`t being able to convince his colleagues. But it was probably a way for 'He' to show what he was capable of (accountable for). After solving every other question of the worksheet, that included writing the condition needed to allow for a correspondence to be a function, students had to comment on the following sentence "The correspondence presented by António is a function. Rui`s correspondence isn`t a function"

'He' again asked the teacher a question, for what he already seemed to have the answer, showing once again what they had been able to achieve making himself accountable to the idea.

> He.: Teacher, can we say that Rui`s graphic isn`t a function because there is one single time corresponding to two distances?

> Teacher: And that's what can't happened for a trip to be possible?

> He.: Yes it is. For a trip to be possible, it can`t be at the same time at two different places. Rui`s robot at 10s is at the distance of 5 and 10.

'He' was the 'motor' of this group for the 'good' resolution of mathematical question proposed, displaying his conceptual agency, that was emergently reconfigured in its engagement with material agency[iii]. Using the robots by which he showed great interest since the first session, seemed to be de leverage to operate the change on 'He'. He was able to explain why the correspondence is not a function in terms of the robots functioning '[the robot] can`t be at two places at the same time'. The robot, associated to the notion of function, was part of the shared repertoire of this class seeing that they always used those sentence every time they have to justify that a correspondence is a function and after they 'translated it' to the situation they had to solve.

FINAL CONSIDERATIONS

During these classes there were times where mathematics contents assumed some invisibility in order to provide the robots with some more visibility and there were times where the robots or even its programming allowed mathematics contents a wider visibility. The duality among visibility and invisibility of the artefacts (physical and conceptual) shaped the participation and by that, students mathematics learning.

Introducing robots in the school mathematics learning scenarios displayed a dynamic link between the work with these artefacts and the way that students think about the notion of function. The agency emerges from the action of using robots to think mathematically. We need to recognize that material agency is irreducible to human agency. Nevertheless, we need to stress that the trajectory of emergence of material agency is bound up with that of human agency (Pickering, 1995). In this case, it displayed conceptual agency in a student that usually uses to do almost nothing in mathematics class.

Students thinking about the notion of function with robots displayed a dynamic coupling between dealing with the mathematical concepts and dealing with the robot that looks like a dance of agency. We have to underline that the dance is between equal patterns. This equality does mean that one of the two dancers is not at times leading the dance. What is does imply is that we can not separate both thinks. We can not separe what has been learned form the action of dealing with robots. Trying to separate it is like 'trying to construct a pot keeping your hands clean from the mud'. Agency is relational and emergent product of material engagement (Malafouris, 2008). Robots were determining on the kind of participation that students had, on there material engagement and material agency is strongly coupled with conceptual agency displayed on students.

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[ii] The concept of learning scenarios adopted in this article was stories of what might be. Unlike projections, scenarios do not necessarily portray what we expect the future to actually look like. Instead scenarios aim to stimulate creative ways of thinking that help people break out of established ways of looking at situations and planning their actions (Wollenberg, Edmunds & Buck, 2000)

[iii] Thru several classes working with robots 'He' was been accountable to and accountable for by the colleagues and by the teacher. There is no space for these analyses on the pages of these article.

THE CONCEPT OF FINITE LIMIT OF A FUNCTION AT ONE POINT AS EXPLAINED BY STUDENTS OF NON-COMPULSORY SECONDARY EDUCATION

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Abstract

We review various educational studies of the mathematical concept of limit of a function at a point that indicate how colloquial uses of the terms "to approach," "to tend toward," "to reach," "to exceed" and "limit" influence students' conceptions of these terms. We then present the results of an exploratory study of this question performed with Spanish students in non-compulsory secondary education and analyze the responses they provide to justify the truth or falsity of statements related to the different characteristics of the concept of finite limit of a function at a point when they use these terms. Finally, we organize their answers according to the kinds of arguments made. Using the response profiles detected, we discuss the influence of everyday usage on the students' arguments.

PROBLEM

The language used to describe the properties of the concept of limit includes terms that have diverse colloquial uses in everyday life. Monaghan (1991), Tall (1980), Tall and Vinner (1981), and Cornu (1991), among others, analyze students' use of the terms "to approach," "to converge at," "to tend toward," "to reach," "to exceed," and "limit" in a mathematical context and the conflicts that arise from using these terms due to the diversity of their meanings. Their analysis suggests a focus for investigating how students' mathematical use of these terms is conditioned by their everyday meanings.

We propose to review and update these studies. Our report characterizes the students' conceptions, as demonstrated in their explanations of certain statements on the concept of the finite limit of a function at a point, when they use the key terms identified in the literature.

ANTECEDENTS

Tall (1980) finds that most students conceive of the limit as a dynamic process and not as a numerical quantity, due to the no less problematic characteristics of the intuitive definition of limit. Tall and Vinner (1981) confirm that the expression "f(x) tends toward L, when x tends toward x_0" is a source of conflict in the formal definition of limit. Among studies that focus on specific terminology, Monaghan (1991) analyzes in greater depth the influence of language on students' ideas about the terms "to tend toward," "to approach," "to converge at" and "limit" when used to refer to different graphs of functions and the examples the students use to explain these terms. Monaghan concludes that many students do not distinguish between "tend toward"

and "approach" in a mathematical context. Cornu (1991) studies the influence that the terms "to reach" and "to exceed" associated with the term "limit" in a colloquial context have on students' conceptions when they classify the value of the limit as not exceedable and not reachable in a dynamic or static environment. This influence is also reported in subsequent studies published in the Proceedings of the PME. Juter and Grevholm (2004) identify subjects who argue that the limit of a function at infinity is not reached because x cannot reach infinity. In analyzing the evolution of different conceptualizations of the limit of a series, Roh (2007) also finds that the unreachability of the limit is an obstacle.

METHOD

Instrument

We worked with a survey of six open-response questions presented in two different questionnaires, A and B. The items are adapted from Lauten, Graham and Ferrini-Mundy (1994); the statements are given in the Appendix.

The questions are open answer. They require assigning, and justifying, the values of true (T) or false (F) to a statement about a property related to the concept of the limit of a function at a point.

Sample

The sample was composed of 36 Spanish students in the first year of non-compulsory secondary education, 16-17 years of age, who were taking Mathematics for the Science and Technology track. The students were chosen deliberately based on their availability.

The survey was administered in the middle of the academic year 2010/2011; the subjects had received prior instruction on the concept of limit. Of the total number of subjects, 18 answered questionnaire A and the other 18 answered questionnaire B. The survey was administered during a regular session of the math class.

RESULTS

Arguments on object-process interrelation in the concept of limit

Item 1.A requires justifying the truth or falsity of the statement: "A limit describes how a function moves as x moves towards a certain point." This item revises a question posed by Tall (1980).

The key idea of the item is that of the movement of the function (dynamic process). Table 1 summarized the three profiles provided by the students' arguments.

Profiles	Profile descriptions
Profile I (Process)	Responds true. The limit object describes the dynamic process that gives rise to it.

Profile II (Object/Process)	Responds true. Stresses the limit as the result of the dynamic process that, implicitly, is also part of the concept; expresses a dual conception of object and process.
Profile III (Object)	Responds false. Differentiates the limit object from the dynamic process that gives rise to it.

Table 1: Profile descriptions related to Item 1.A

Sample answer for Profile I: "True: Yes, it does, because the limit provides all the possible points that a function can take."

Sample answer for Profile III: "False. A limit is that toward which the function f(x) tends."

Arguments on the exceedable character of the value of the limit

Item 2.A requires justifying the truth or falsity of the statement: "A limit is a number or point past which the function cannot go." This question was analyzed by Monaghan (1991) and Cornu (1991).

The key idea of this item is not to exceed (a limit property). Table 2 summarized the two profiles provided by the students arguments.

Profiles	Profile descriptions
Profile I (Not exceedable)	The value of the limit is recognized as not exceedable in any case. Most of the arguments stress the value of the limit as unreachable **(Subprofile I.1) (Not reachable)**. This property is due to the infinite nature of the numerical process and the exclusion of the image of the point.
Profile II (Exceedable)	The value of the limit is exceedable in certain cases; in fact, examples are given in which the limit is exceedable; in some cases it is considered reachable **(Subprofile II.1) (Reachable).**

Table 2: Profile descriptions related to Item 2.A

Sample answer for Subprofile I.1: "True: Because a limit is a point that a function approaches infinitely but never reaches"

Arguments on the relation between the finite character of the practical process vs. the potentially infinite character of the formal iterative process

Item 3.A proposes justifying the truth or falsity of the statement: "A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached." Key ideas for this item are, following Tall (1980): *to try values* and *to reach*. We find four profiles, or arguments, as shown in Table 3.

Profiles	Profile descriptions
Profile I (Potential infinity)	Responds true. The process is potentially infinite, without further consideration. There are references to both-sides conditions of the process **(Subprofile I.1) (bilaterality)**.
Profile II (Practical finiteness)	Responds false. Stresses the finiteness of the process in practice vs. the potentially infinite formal process.
Profile III (Bilateralness)	Responds false. Does not assume the existence of the limit and requires the both-sides condition not included in the wording of the item.
Profile IV (Not reachable)	Responds false. The limit is not reachable, possibly due to the potentially infinite character of the process.

Table 3: Profile descriptions related to Item 3.A

Arguments on the reachable character of the limit

Item 1.B asks students to argue the truth or falsity of the statement: "A limit is a number or point the function gets close to but never reaches." The item is related to the studies done by Tall (1980), Tall and Vinner (1980), Monaghan (1991), and Cornu (1991). Key terms in this item are *to get close* and *to reach*. Three profiles of arguments emerge, as summarized in Table 4:

Profiles	Profile descriptions
Profile I (Not reachable)	Responds true. Affirms that the limit is unreachable in any case, due to the infinite nature of the process of approaching in some cases.
Profile II (Reachable)	Responds false. The limit can be reachable in some cases. A transition profile.
Profile III (Always reachable)	Responds false. Affirms that the value of the limit is always reachable. Only two cases affirm that the value of the limit is reachable but never exceedable **(Subprofile III.1)(Not exceedable).**

Table 4: Profile descriptions related to Item 1.B

Sample answer for Profile II: "False: It is not necessary to reach the limit exactly but only to approach it enough to know what the limit is"

Sample answer for Subprofile III.1: "False: Function attains the limit but does not go past it."

Arguments on the interrelation of precision and the process of approximating the limit

Item 2.B asks for arguments for the truth or falsehood of the statement: "A limit is an approximation that can be made as accurate as you wish." The main idea of this item is to consider a limit as a process of approximation that can be as accurate as desired. The arguments collected demonstrate the profiles presented in Table 5.

Profiles	Profile descriptions
Profile I (Exactitude)	Responds false. Considers the limit as an exact value, not a process of approximation giving increasingly accurate values. Limit is an object, not a process.
Profile II (Arbitrary precision)	Responds true. Shows the infinite nature of the process followed to calculate the value of the limit. Contrary what we would logically expect, some responses hold that it is not the precision that matters but the values that approximate the limit.
Profile III (Restricted precision)	Responds true. Precision is restricted, given the finite nature of the method of calculating in practice. Only one subject fits **Subprofile III.1 (Conditional precision)**, which establishes that the precision depends on the function and the values of x.

Table 5: Profile descriptions related to Item 2.B

Sample answer for Profile II: "False: Because accuracy is not important—finding the numbers that get closer to the limit is"

Arguments on the arbitrariness of the process of approximation

Item 3.B asks for arguments for the truth or falsity of the statement: "A limit is a number that the y-values of a function can be made arbitrarily close to by restricting the x-values." The key terms in this item are to be made arbitrarily close to and x-value restrictions. The answers show two profiles of response, presented in Table 6.

Profiles	Profile descriptions
Profile I (Non-arbitrariness)	Responds false. Differentiates between approximate and arbitrary approach. The values of f(x) do not approach the limit arbitrarily. Rather, depending on the values of x chosen, their respective images approach or recede from the limit.
Profile II (Arbitrariness)	Responds true. Some subjects consider the monotony of convergence at the limit.

Table 6: Profile descriptions related to Item 3.B

Sample answer for Profile I: "The values of the function become close to the limit in an approximate but not an arbitrary way."

DISCUSSION AND CONCLUSIONS

The subjects themselves provide the criterion for discriminating between profiles based on the arguments used. When we find several different arguments simultaneously, we consider the true/false dichotomy as a secondary criterion for discriminating between the profiles.

Conceptions about the concept of limit

The profiles of response for Item 1.A show subjects who conceive the limit exclusively as a process. There are also subjects who establish the interdependence of the process of obtaining the limit of a function at a point and the object that they call limit. That is, the object determines and describes the process followed to obtain the limit, even when using a dynamic interpretation, conclusions that agree with those of Tall (1980). Finally, we identify subjects who do conceive the limit not as a process but as an object, point or number; the concept of limit establishes not a "how" but a "where."

Second, we can infer from Items 3.A and 2.B that most of the subjects recognize that the practical method for calculating a limit is finite, although some believe that assigning values close to it can be done as many times as one wishes, using the intuitive notion of potential infinity. However, others stress that precision is restricted, or that it is enough to intuit the limit. It is possible that the use of the expression "as one wishes" in Item 2.B introduced some subjectivity and hindered students' understanding of the arbitrary precision of the approximation of a limit.

Third, we do not find indications that the term "restriction" in the wording of Item 3.B led subjects to an intuitive use of the continuum (intervals or graphic representation), since they persist in discrete reasoning. These questions develop those studied by Monaghan (1991).

Exceedable or reachable character of the value of the limit

It is significant that these properties are considered primarily in general and not in particular. The unexceedable character of the limit is attributed to a great extent to its unreachability. However, unreachability is not due to unexceedability; even if the limit is reachable, it is not exceedable for some subjects. It is significant that only 2 of the 18 subjects are aware of the particular character of these and that only one admits different possibilities. Misuse of the expression "f(x) tends toward L" and of strictly monotonous functions may contribute to these obstacles, detected previously by Cornu (1991). Furthermore, we suggest that these properties are considered locally and not globally.

The notion of arbitrariness

The students interpreted the term "arbitrariness" in different ways. One case differentiates between "approaching approximately" and "approaching arbitrarily." Another case interprets this as: "The approximation to the limit is arbitrary if any value of the variable x is verified," an understanding based on its everyday meaning.

Balance

No previous study has analyzed the response profiles presented here. The richness and variety of the responses obtained for the items proposed is organized according to the profiles, which demonstrate the complexity of the notions implied in the concept of the finite limit of a function at a point. It remains to establish connections between the profiles described for the different questions.

Acknowledgements

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APPENDIX: Items included in the questionnaire

Questionnaire A.

(1.A.) A limit describes how a function moves as x moves towards a certain point.

(2.A.) A limit is a number or point past which the function cannot go.

(3.A.) A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

Questionnaire B.

(1.B.) A limit is a number or point the function gets close to but never reaches

(2.B.) A limit is an approximation that can be made as accurate as you wish.

(3.B.) A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values.

MATHEMATICS EDUCATION IN NATURAL LEARNING SITUATIONS: EVALUATION OF A PROFESSIONAL DEVELOPMENT PROGRAM FOR EARLY CHILDHOOD EDUCATORS

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Early mathematical competencies are powerful predictors for further mathematical learning. To provide children with sound basic mathematical competencies present early childhood educators with considerable challenges. To support them in this important task, a professional development program was designed. This was part of a larger project, aiming at improving the quality in early childhood education and in school. The effectiveness of the professional development program was evaluated. Statistical analyses indicate that children whose educators were able to support children in their individual development due to a sound knowledge of early mathematical development, and of ways of engaging young children in mathematics, improved their mathematical abilities compared to children belonging to a control group.

INTRODUCTION

Children bring heterogeneous prerequisites when they start school and even when they enter kindergarten (Anders, Grosse, Roßbach, Ebert, & Weinert, accepted). This is a serious issue as several empirical studies indicate that early numerical competencies and early structure sense are powerful predictors of later mathematical achievement (Dornheim, 2008; Krajewski & Schneider, 2009; Lüken, 2010). The importance to provide all children with basic mathematical competencies is recognized and early childhood mathematics education has become a widely discussed topic in the last years.

Though many materials and learning programs are offered for early childhood mathematics education, it is a difficult task for kindergarten educators to support children in their individual mathematical development processes. They have to decide how they should create and organize substantial mathematical learning situations, and they have to recognize important steps in the development of their children to be able to care for their further development. As learned from several empirical studies concerning teacher competencies (Ball, Thames, Bass, Sleep, Lewis & Phelbs, 2009; Lindmeier & Ufer 2010; Shulman, 1986), teaching requires a wide range of competencies, including, for example, content knowledge, pedagogical content knowledge and action-related competencies. One can assume that similar requirements apply to early childhood educators. Regarding that in Germany – in contrast to many other countries - in pre-service education of early childhood educators, mathematics education has not played an important role over time, this fact

becomes increasingly important. Therefore professional development programs are necessary to guarantee a substantial mathematics education for the young children.

For this empirical study a professional development program was designed and carried out to support early childhood educators in their responsible task. This was part of the academic support in a larger project aiming at improving the quality in kindergarten and in school. A specific intention of the project was to support educators and teachers to foster children in their individual learning processes. The fundament for the professional development program was a concept for early childhood mathematics education, which focuses on natural learning situations and on fundamental mathematical ideas. Mathematical learning in this sense is non-formal and refers mainly to play- and everyday-activities as detailed below. This concept is supposed to provide all children with basic mathematical competencies. Another key aspect of the professional development program was to provide knowledge about the development of mathematical competencies in early childhood to enable the educators to focus on the individual mathematical learning process of each child and to care for an appropriate support. The effectiveness of the professional development program was evaluated through an empirical study, which assessed and compared the mathematical achievement of two groups of children - children, whose educators took part in the professional development program and children in a control group.

THEORETICAL BACKGROUND

Early Mathematics Education in Natural Learning Situations

There are several reasons to assume that early mathematics education in natural learning situations, e.g. during play and everyday activities, provides a solid base for further mathematical learning: Children learn mathematics in meaningful contexts, they enhance their communicative competencies in dialogue with other children and adults and their conceptual and procedural knowledge in several mathematical content areas (Greenes, 1999), and learning in this way corresponds to a constructivist perspective on learning (Reusser, 2001). Natural learning situations are not only situations, which happen more or less by chance. The crucial point is "the progressive development of what is already experienced into a fuller and richer and also more organized form, a form that gradually approximates that in which subject-matter is presented to the skilled, mature person" (Dewey, 1938, p. 48). The educators have to moderate and accompany the learning processes by initiating these learning situations or by using the potential for mathematical learning that everyday activities offer. This means, they have to recognize and to make use of the opportunities for mathematical learning in play and everyday activities. A central demand is that all these planned and initiated activities should be mathematically correct and they should be based on fundamental mathematical ideas to guarantee coherence and consistency in mathematical learning.

Competencies Required of Early Childhood Educators

To implement early mathematics education in the way described above, and to ensure that children with different levels of knowledge and skills can profit, requires wide-ranging knowledge and competencies of educators. They need content knowledge in particular to guarantee coherent mathematical learning, to see the relations between mathematics in the early years and later on, and to judge solutions or statements, children produce (Ball et al., 2009). Their pedagogical content knowledge is necessary to identify individual learning difficulties and to ask deeper questions to understand children's thinking better. It helps to assess the individual competencies over a broader range and to decide whether additional support is needed or not. The knowledge of learning difficulties, misconceptions and prerequisites is also called "diagnostic knowledge" (Weinert, Schrader & Helmke, 1990).

Knowledge as described above is not sufficient for successful action in concrete situations, but it is part of another important competence early childhood educators need: the theoretical construct "action competence" (Weinert, 2001, p. 51). This means not to act in a prescribed manner, but to act adequately to the situation, to the individual person and to the subject on the background of a sound knowledge of content and pedagogical content. For example, educators plan early mathematics education, but in addition spontaneously seize the opportunities for mathematical learning in everyday and play situations. They have to detect mathematically relevant aspects in interactions between children, in play situations or in everyday routines and they have to use these situations by asking relevant questions or by encouraging reflection (van Oers, 2009). In case of individual learning difficulties action competence is indispensable for the selection of adequate and necessary steps for further learning. Successful educators are sensitive to the individual development of their children, they have ideas to foster them and they act in an adequate way (Weinert, Schrader & Helmke 1990).

EVALUATION STUDY OF A PROFESSIONAL DEVELOPMENT PROGRAM FOR EARLY CHILDHOOD EDUCATORS

Thinking of early mathematics education means not only to design materials for early mathematics education. It means especially to have in mind that with increasing demands on educators their professional development is becoming more and more important. Therefore, a professional development program was worked out to support educators in providing opportunities to learn mathematics in natural learning situations and in fostering children appropriate to their individual stage of learning.

Professional Development Program

In three of four modules, the educators worked on content and pedagogical content knowledge in the domains 'number, counting, quantity', 'space and shape' and 'measurement and data'. The fourth module focused on observation, documentation of children's learning progresses and possibilities of intervening, if the observation of

children shows specific problems. The module 'number, counting, quantity' took a whole day, the three others only a half-day. The three content-based modules were structured as follows. The mathematical content was divided into smaller sections, e.g. for the domain 'number, counting, quantity' in counting sequence, counting process, comparing, quantification and structures, pattern and changing. In each section the educators got information about associated mathematical competencies and their development from early childhood up to the first school years. These information sections included thought experiments and self-reflection-tasks like trying to count on, or calculating by, the letters of the alphabet. If possible the information part was illustrated with short video-sequences to reflect the development of children and to train observation. This more theoretic content was enriched by everyday activities and play situations in the context of natural learning situations. Many activities were carried out directly by the educators. Their own learning experiences were reflected in a short discussion afterwards and supplemented by their own ideas. The fourth module aimed at a more conscious approach to traces of mathematical competencies in children's actions. It provided information on observation and diagnostics in general, especially the reasons why this is indispensable in early mathematics education. The main part of this module was a training to observe learning processes, to draw conclusions for further mathematical learning and to reflect on which tasks or situations could help to foster individual learning. Video-sequences were used for this training. To support the educators in these challenging activities, the observation-tool "Lerndokumentation" (Steinweg, 2006) was introduced. It is a chart, where central mathematical competencies are described in natural learning situations in early childhood education.

Research Question, Sample and Methodology

The main research question was if this professional development program for educators has positive effects on the mathematical learning of children. Hence, children's performance in a mathematics test was measured and compared with the performance of a control group. There were three points of measurement over a three-year period in annual test intervals: a pretest, a test during the intervention time and a posttest. The treatment group was a proportionally stratified sample (age, gender, migration-background; pretest: N=21, age 3-4 years; posttest: N=19, age 5-6 years) out of the target population, which was defined by all children in the day-care centers, participating in the project. The control group was stratified in the same way as the treatment group. The day-care centers in both groups are located in comparable districts regarding some social data and the percentage of foreign nationals (Gasteiger, 2010). The intervention (professional development program) was carried out eight months after the pretest and repeated after one year. The posttest was carried out four months after the second intervention. To measure the development of mathematical learning a test instrument was designed (Gasteiger, 2010) with items in the domains 'number and calculation' (19 items, Cronbach's α=.89, .86, .76), 'measurement' (5 items, α=.55, .40, .35), and 'shape and space' (6 items, α=.62, .55, .44) – 30 items all in

all (α=.91, .89, .78). The test includes an interview supported with some material, e.g. counting objects, number- and quantity-cards, and a paper-pencil test. For the data analysis, the whole test was videotaped. The reliabilities of the subscales 'measurement' and 'shape and space' are low. Therefore, for further data analysis, only the whole scale and the subscale 'number and calculation' will be used.

RESULTS

To analyse the results of the evaluation study we used the data of the group of children who took part in the mathematics test three times (N=19 in each group), with one exception: at the second point of measurement one child in the treatment group was not available for the test, but it took part at the pre- and posttest.

Pretest scores for treatment group (M=28%, SD=17%) and control group (M=40%, SD=22%) differ not significantly but considerably (t(36)=1.89, p>0.05), even though the samples were stratified in parallel ways. From the first to the second point of measurement both groups' mathematical competencies are developing in parallel. At the third point of measurement, the results of both groups approximate to each other (treatment group: M=80%, SD=13%; control group: M=84%, SD=13%, t(36)=.89, p>0.05).

ANOVAS with repeated measures are used to examine the development of mathematical competencies and differences between the treatment, and the control group. The focus is on the interaction effect between group and point of measurement. Considering the whole scale of test items, the main effect of mathematical development over the period of two years is highly significant as expected (p<.001), because children improve their mathematical competencies over time – independent from any intervention. The interaction effect is not significant (F(1.730)=1.687, p=.20). If only the subscale 'number and calculation' is considered, in addition to the main effect of mathematical development in general, the interaction effect is significant as well (F(1.924)=4.468, p=.02, η^2=.11). Children in the treatment group differ significantly in their performance in the content domain 'number and calculation' from children in a control group. The comparison of means shows that they can improve their competencies between the second and the third point of measurement:

M (SD)	treatment group	control group
1st point of measurement (pretest)	21% (16%)	39% (23%)
	t(36)=2.68, p<0.05, d=0.87	
2nd point of measurement	55% (22%)	71% (21%)
	t(35)=2.29, p<0.05, d=0.75	
3rd point of measurement (posttest)	83% (14%)	86% (16%)
	t(36)=0.46, p>0.05, d=0.15	

Table 1: Comparison of means in the subscale 'number and counting'.

Examining the subscales, it could be detected that the differences in the results of the two groups at the beginning of the evaluation study and also at the second point of measurement are caused by the differences in the subscale 'number and counting'. At first and second point of measurement they are even significant (see table 1), while the performance of treatment and control group in the two other subscales nearly do not differ (Gasteiger, 2010).

DISCUSSION

In this study, the intervention addressed the professional development of the *educators* but the mathematical competencies of *children* were the decisive factor to appraise whether the intervention was successful or not. This fact may explain why, from the first to the second point of measurement, both groups' mathematical competencies are developing in parallel (see table 1). For the educators taking part in the professional development program it can be assumed that they get ideas for early mathematics education in natural learning situations, information about mathematical development in early childhood and experiences in the observation of mathematical competencies. The professional development program gave no explicit instruction how the educators should act when they are back at work. This means changes in the daily work might happen when the educators have in mind what they learned about mathematical learning and development and, when they know how to act through their – hopefully – improved content and pedagogical content knowledge. There is a long way to children's improvement of mathematical competencies: the educators need to reflect on their daily work due to their new experiences, they need to try to realize early mathematics education in natural learning situations, to detect individual difficulties and competencies of the children, and to support their development. Not until then can the professional development have an effect on children's performance. So it is remarkable that despite the long way these effects on children's mathematical development can be detected. Obviously, the effects of professional development on children's mathematical achievement do not emerge immediately, but there is reason to believe that this process has a sustainable impact on the daily work of the educators and may lead to an ongoing enhancement of children's mathematical development.

Another interesting point in this evaluation study is that the intervention only had effects in the domain 'number and calculation'. There are some ideas to explain this result. One reason may be that the test instrument was not balanced in the content domains. There were considerably more items in the domain 'number and calculation' than in the other two content domains and the subscales to 'shape and space' and to 'measurement' were not as reliable as the subscale 'number and calculation'. Maybe with a longer test instrument, effects could be detected in other content domains as well. Also it may be assumed that early childhood educators rather think of numbers, counting and calculating than of spatial thinking or measuring time, when they engage with early mathematics education (Lee & Ginsburg, 2007). Moreover, the domain 'number, counting, quantity' took more time in the professional development program

than the other domains. Possibly it was or it is easier for educators to think about natural learning situations in this domain than in the others. Discussions with educators during the professional development program confirm this statement.

CONCLUSION

There are many ways to think about early mathematics education. Today, early childhood educators can use frameworks for an orientation, they have a choice between several materials or training programs and they have access to diagnostic tools. Demanding that early childhood education should meet the requirements on sustainable learning and all activities in early mathematics should be based on individual prerequisites and learning progress of children, it is indispensable to support the early childhood educators (Baroody, 2004). They have to act competent – sometimes spontaneously – and to plan and initiate mathematical learning in a meaningful way, having in mind why some contents or skills are relevant for further mathematical learning, and others are not. Using materials, frameworks, and diagnostic tools without having in mind which mathematical ideas are relevant for children and how they can learn them adequate and matched to their individual learning progress promises not to be successful (Siraj-Blatchford, Sylva, Muttock, Gilden & Bell, 2002).

The results of the study presented here show that professional development can have effects on children's mathematical learning though it is a long way from the development of educators' competencies to children's mathematical achievement. The approach to think about early mathematics education by professionalization of educators is demanding. Short-term effects could not be expected, but there are good reasons to believe that in the long run this approach can lead to a profound change in the thinking of early childhood educators and in their acting concerning early mathematics education.

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THE RELATIONSHIP BETWEEN SELF-CONCEPT AND EPISTEMOLOGICAL BELIEFS IN MATHEMATICS AS A FUNCTION OF GENDER AND GRADE

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Research has shown that self-concept and epistemological beliefs in mathematics have influence on school performance. Also, gender differences and developmental changes in self-concept and epistemological beliefs have been documented. However, the relationship of self-concept and epistemological beliefs in mathematics as a function of gender and grade has not been the object of much research. This study with N = 145 students reveals (1) that the self-concept is related with epistemological beliefs in mathematics, (2) that differences in self-concept and epistemological beliefs in mathematics exist between girls and boys but not between different grades, and (3) that gender differences in the mathematical self-concept increase when taking into account the influence of epistemological beliefs and performance in mathematics.

INTRODUCTION

It is widely acknowledged that school performance is the result of the interplay of multiple factors. Hence, not only cognitive skills but also personality characteristics have an impact on school performance. In this context, the self-concept, that is, the individual's perception of one's own abilities in an academic discipline, plays an important role for school performance (e.g., Marsh et al., 2005). Similarly, epistemological beliefs, that is, one's own beliefs about the nature of knowledge and knowledge acquisition, have been shown to have influence on school performance (e.g., Hofer & Pintrich, 2002). In addition, prior research has provided evidence for gender differences and developmental changes in both self-concept and epistemological beliefs. In this context, however, the question arises as to how the self-concept is specifically related to epistemological beliefs. Therefore, we conducted a study in which we examined the relationship of self-concept and epistemological beliefs in mathematics as a function of gender and grade.

THEORETICAL BACKGROUND

Self-concept in mathematics

The mathematical self-concept refers to ideas of one's own skills and abilities in mathematics (Marsh et al., 2005). It has been shown to be closely related to mathematical performance (e.g., Fredricks & Eccles, 2002). At the same time, however, prior research suggests that boys have a higher self-concept in mathematics than girls (e.g., Skaalvik & Skaalvik, 2004). This seems to be true even though their level of performance might actually be the same.

In addition, prior research has shown that there are developmental changes in a student's self-concept. For example, Fredricks and Eccles (2004) observed declines in a student's mathematical ability perceptions from grade 1 to grade 12. In addition, they found that boys had a higher mathematical self-concept than girls but this gender-related difference decreased over time.

Epistemological beliefs in mathematics

Epistemological beliefs refer to one's own subjective theories and beliefs about the nature of knowledge and knowledge acquisition in academic disciplines such as mathematics. It has been shown that more sophisticated epistemological beliefs are more closely related to deep-processing learning strategies. Such learning strategies are more beneficial to learning than superficial learning strategies that are more typical of naïve epistemological beliefs. The relationship of epistemological beliefs and learning strategies is assumed to be responsible for differences in school performance (e.g., Köller, Baumert, & Neubrand, 2000).

According to Schoenfeld (1992), epistemological beliefs in mathematics can be divided in two epistemological views: a *static system* view and a *dynamic process* view. The static system view emphasizes learning and the application of definitions, facts and routines whereas the dynamic process view is related to discovering and arguing in mathematics. Grigutsch (1996) found in his study that the dynamic process view was related to a higher self-concept and a higher performance in mathematics. In contrast, the static system view was linked with a lower self-concept and a lower performance in mathematics. In addition, girls more often emphasized the process character of mathematics and aspects of application whereas boys more often had a view where a schematic conception of mathematics was predominant.

In general, epistemological models assume that epistemological beliefs develop over time. In other words, there is a developmental transition from more naïve epistemological beliefs to more sophisticated epistemological beliefs (e.g., Hofer & Pintrich, 2002).

RESEARCH QUESTIONS

Prior research provides empirical evidence for gender-related differences in the mathematical self-concept of girls and boys. In addition, previous studies suggest that the mathematical self-concept declines over time in the majority of students. Likewise, it has been shown that gender is related to differences in the level of epistemological beliefs in mathematics. In contrast to the self-concept, epistemological beliefs are assumed to develop over time in students. In this study, we extended prior research by examining the relationship of self-concept and epistemological beliefs in mathematics as a function of gender and grade. More specifically, we addressed the following research questions:

- What level of epistemological beliefs do students of grade 9 and students of grade 10 possess?
- Are more sophisticated epistemological beliefs in mathematics positively related and are more naïve epistemological beliefs negatively related with the self-concept and the performance in mathematics?

- Do boys possess – in accordance with gender-related differences in the mathematical self-concept – more sophisticated epistemological beliefs in mathematics and less naïve epistemological beliefs in mathematics than girls?
- Do students of grade 10 possess – in accordance with stage-related differences in the epistemological beliefs – more sophisticated epistemological beliefs and less naïve epistemological beliefs in mathematics than students of grade 9?

METHODOLOGY

The cross-sectional study is based on a total of $N = 145$ high school students in grades 9 and 10 of a German gymnasium (i.e., a school with the highest track in the German school system) with 79 girls and 66 boys. The questionnaire that was given to the students consisted of (1) 23 items assessing the self-concept (e.g., "I get good marks in mathematics."), and (2) 25 items assessing epistemological beliefs. The items were taken from large-scale assessments such as *PISA* and *TIMSS*, from the *Potsdam Motivation Inventory PMI-M* (Rheinberg & Wendland, 2003), and from the *Self Description Questionnaire II* (Marsh et al., 2005). The epistemological beliefs in mathematics addressed the following six aspects: (1) rigid schemes (e.g., "Exercises in mathematics always have only one right solution."), (2) schematic conception of mathematics (e.g., "Mathematics is a collection of calculation methods and calculation rules that specify exactly how to solve a problem."), (3) realistic conception of mathematics (e.g., "In the meantime, all mathematical problems are solved."), (4) relativistic conception of mathematics (e.g., "Mathematics is a game with numbers, symbols and formulas."), (5) processes (e.g., "In mathematics, you can find a lot of things by yourself.") and (6) relevance/application (e.g., "Mathematics is needed for lots of problems in our daily lives."). The first three scales (i.e., rigid schemes, schematic conception, realistic conception) assess more naïve epistemological beliefs in mathematics. The last three scales (i.e., relativistic conception, processes, relevance/application) assess more sophisticated epistemological beliefs in mathematics. For all items, a 4-point rating scale ranging from 1 = *disagree* to 4 = *agree* was used. The internal consistency of all epistemological beliefs scales as indicated by Cronbach's alpha ranged from .60 to .75 with the exception of the scale assessing rigid schemes with an internal consistency of only $\alpha = .40$. The internal consistency of the self-concept scale was excellent, $\alpha = .95$. In addition to the questionnaire, the students were asked to provide their grade in mathematics. In Germany, grades range from 1 to 6 with lower numeric values indicating a higher performance (e.g., 1 = *very good*, 6 = *unsatisfactory*).

RESULTS

Table 1 displays the correlations among the scales assessing the different aspects of epistemological beliefs in mathematics. They reveal that the more sophisticated epistemological beliefs relativistic conception, processes, and relevance/application

are nearly always significantly and negatively associated with the more naïve epistemological beliefs rigid schemes, schematic conception, and realistic conception. Hence, students who had more sophisticated epistemological beliefs with regard to relativistic conception, processes and relevance/application had, at the same time, less naïve epistemological beliefs with regard to rigid schemes, schematic conception, and realistic conception.

Scales	1	2	3	4	5	6
Rigid schemes (1)	-					
Schematic conception (2)	.09	-				
Realistic conception (3)	.32*	.23*	-			
Relativistic conception (4)	-.18	-.19*	-.12	-		
Processes (5)	-.28*	-.33*	-.10	.32*	-	
Relevance/Application (6)	-.28*	-.34*	-.19*	.35*	.43*	-

Table 1: Correlations among the scales of epistemological beliefs (*$p < .05$).

Epistemological beliefs, self-concept, and performance in mathematics

The results concerning the epistemological beliefs in mathematics showed that, on average, students scored relatively high on the scales that addressed more sophisticated epistemological beliefs with the exception of the scale that assessed a relativistic conception ($M = 1.89$, $SD = 0.62$): relevance/application ($M = 2.99$, $SD = 0.49$) and processes ($M = 2.76$, $SD = 0.75$). Conversely, students, on average, scored relatively low on the scales that addressed more naïve epistemological beliefs with the exception of the scale that assessed a schematic conception ($M = 2.83$, $SD = 0.42$): realistic conception ($M = 2.06$, $SD = 0.69$) and rigid schemes ($M = 1.56$, $SD = .60$). In addition, the students' self-concept in mathematics was, on average, moderately high ($M = 2.56$, $SD = 0.64$) and their performance in mathematics was, on average, relatively low ($M = 3.95$, $SD = 1.00$).

Table 2 displays the correlations of the epistemological beliefs scales with the self-concept and the performance in mathematics as well as the partial correlations between the epistemological beliefs scales and the self-concept in mathematics. The correlations show that the more sophisticated epistemological beliefs relativistic conception, processes and relevance/application were significantly and positively associated with the self-concept and the performance in mathematics. Conversely, the more naïve epistemological beliefs rigid schemes and schematic conception were significantly and negatively associated with the self-concept and the performance in mathematics. Realistic conception was the only scale, however, that was not significantly correlated with the self-concept or with the performance in mathematics. The partial correlations displayed in Table 2 show that nearly all correlations of the more sophisticated and more naïve epistemological beliefs with the self-concept in

mathematics remained stable even when we controlled for the influence of the performance in mathematics.

	Self-concept	Performance	Self-concept controlled for performance
Rigid schemes	-.20*	-.19*	-.10
Schematic conception	-.34*	-.23*	-.25*
Realistic conception	.07	-.10	.18*
Relativistic conception	.44*	.26*	.37*
Processes	.46*	.35*	.33*
Relevance/Application	.52*	.40*	.39*

Table 2: Correlations of the epistemological beliefs scales with the self-concept and the performance in mathematics and partial correlations of the epistemological beliefs scales with the self-concept in mathematics (*p < .05).

Differences in epistemological beliefs, self-concept, and performance in mathematics as a function of gender and grade

In a first step, we examined differences in the epistemological beliefs as a function of gender and grade. To do so, we performed a multivariate analysis of variance in which we included gender and grade as independent variables and the six aspects of epistemological beliefs as dependent variables. The analysis yielded a significant main effect for gender, $F(6, 136) = 3.88, p < .05, \eta^2 = .15$ (large effect). This main effect was mainly produced by significant differences in rigid schemes $F(1, 141) = 3.91, p = .05, \eta^2 = .03$ (small effect), relativistic conception $F(1, 141) = 5.09, p < .05, \eta^2 = .04$ (small effect), and relevance/application $F(1, 141) = 9.36, p < .05, \eta^2 = .06$ (medium effect) between girls and boys. As displayed in Table 3, rigid schemes as an aspect of more naïve epistemological beliefs were significantly more pronounced in boys than in girls. Interestingly, relativistic conception and relevance/application as aspects of more sophisticated epistemological beliefs were also significantly more pronounced in boys than in girls. The statistical analysis failed to reveal a significant main effect for grade, $F(6, 136) = 1.01$, *ns*, and a significant interaction effect between gender and grade, $F(6, 136) = 1.97$, *ns*.

In a second step, we investigated differences in the mathematical self-concept as a function of gender and grade. To do so, we performed a univariate analysis of variance in which we included gender and grade as independent variables and mathematical self-concept as dependent variable. The analysis yielded a significant main effect for gender, $F(1, 141) = 11.31, p < .05, \eta^2 = .07$ (medium effect). More specifically, the boys' self-concept in mathematics was significantly higher than the girls' self-concept in mathematics (see Table 3). However, there was no significant main effect for grade,

$F(1, 141) = 0.13$, *ns*, and no significant interaction effect between gender and grade, $F(1, 141) = 0.20$, *ns* (see Table 3 and left-handed diagram in Figure 1).

	Girls	Boys
Rigid schemes	Grade 9: 1.47 (0.58) Grade 10: 1.46 (0.54)	Grade 9: 1.55 (0.55) Grade 10: 1.79 (0.68)
Schematic conception	Grade 9: 2.85 (0.35) Grade 10: 2.83 (0.48)	Grade 9: 2.86 (0.46) Grade 10: 2.77 (0.43)
Realistic conception	Grade 9: 2.13 (0.66) Grade 10: 1.86 (0.76)	Grade 9: 2.03 (0.74) Grade 10: 2.14 (0.61)
Relativistic conception	Grade 9: 1.75 (0.50) Grade 10: 1.86 (0.71)	Grade 9: 2.24 (0.65) Grade 10: 1.82 (0.56)
Processes	Grade 9: 2.75 (0.69) Grade 10: 2.68 (0.78)	Grade 9: 2.87 (0.75) Grade 10: 2.71 (0.82)
Relevance/Application	Grade 9: 2.82 (0.43) Grade 10: 2.96 (0.51)	Grade 9: 3.20 (0.39) Grade 10: 3.07 (0.55)
Self-concept	Grade 9: 2.44 (0.57) Grade 10: 2.35 (0.78)	Grade 9: 2.75 (0.51) Grade 10: 2.76 (0.64)
Performance	Grade 9: 3.78 (1.00) Grade 10: 4.13 (1.05)	Grade 9: 4.02 (0.96) Grade 10: 3.99 (1.03)

Table 3: Means and standards deviations for the scales of epistemological beliefs, the self-concept and the performance in mathematics as a function of gender and grade.

In a third step, we studied differences in the mathematical performance as a function of gender and grade. To do so, we performed a univariate analysis of variance in which we included gender and grade as independent variables and mathematical performance as dependent variable. The analysis failed to yield a significant main effect for gender, $F(1, 141) = 0.07$, *ns*, a significant main effect for grade, $F(1, 141) = 0.83$, *ns*, and a significant interaction effect between gender and grade, $F(1, 141) = 1.17$, *ns* (see Table 3).

In a fourth step, we studied differences in the mathematical self-concept as a function of gender and grade and controlled for the influence of epistemological beliefs and performance in mathematics. To do so, we performed a univariate analysis of variance in which we included gender and grade as independent variables, mathematical self-concept as dependent variable, and the six aspects of epistemological beliefs together with mathematical performance as control variables. The analysis again revealed a significant main effect for gender, $F(1, 134) = 10.82$, $p < .05$, $\eta^2 = .08$ (medium effect), and no significant main effect for grade, $F(1, 134) = 0.67$, *ns*. In this analysis, however, there was also a significant interaction effect between gender and grade, $F(1, 134) = 5.84$, $p < .05$, $\eta^2 = .04$ (small effect). As displayed in the

right-handed diagram in Figure 1, the gender differences in the mathematical self-concept were lower in grade 9 and larger in grade 10.
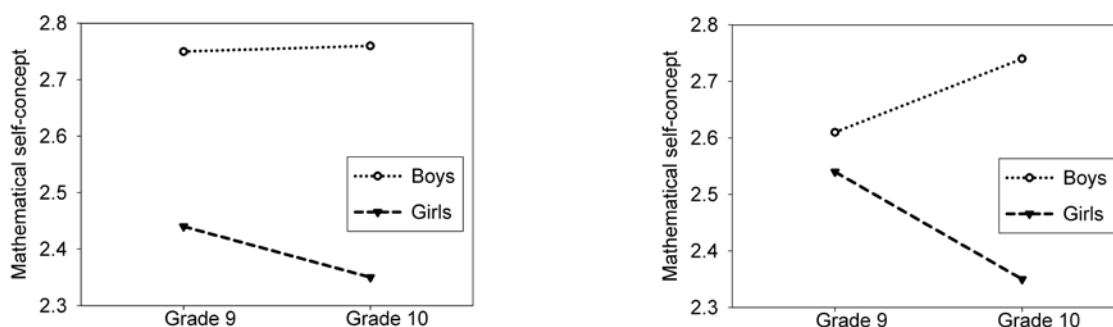

Figure 1: Mathematical self-concept as a function of gender and grade (left) and while controlling for epistemological beliefs and performance in mathematics (right).

DISCUSSION

In this study, we examined epistemological beliefs, self-concept, and performance in mathematics of grade 9 and grade 10 students. First, the results demonstrated the construct validity of the scales assessing epistemological beliefs in mathematics. This was because the more sophisticated epistemological beliefs were negatively correlated with the more naïve epistemological beliefs. Second, the students in this study tended to have more sophisticated and less naïve epistemological beliefs in mathematics. This pattern of results was, however, not obtained for the scales that assessed schematic or relativistic conceptions of mathematics. Third, we showed that more sophisticated epistemological beliefs were positively linked and more naïve epistemological beliefs were negatively linked with self-concept and performance in mathematics. When we controlled for the influence of performance in mathematics, the correlations between epistemological beliefs and self-concept in mathematics were somewhat lower but remained significant. This was, however, not true for the scale that assessed rigid schemes as an aspect of more naïve epistemological beliefs. Fourth, we found differences in epistemological beliefs and self-concept in mathematics between girls and boys. Boys had a more pronounced view where rigid schemes were predominant while, at the same time, they also had, in contrast to the study by Grigutsch (1996), more sophisticated epistemological beliefs in mathematics than girls. In addition, boys reported about a higher self-concept in mathematics than girls. However, there was no gender difference in the mathematical performance. Fifth, we failed to observe any difference in epistemological beliefs, self-concept and performance in mathematics between grade 9 and grade 10 students. Sixth, when we took into account the influence of epistemological beliefs and performance in mathematics there was an interaction effect between gender and grade on the mathematical self-concept. More specifically, the difference in the self-concept was relatively small in grade 9 whereas the difference in the self-concept was relatively large in grade 10. This finding is in contrast to the study by Fredricks and Eccles (2002).

Taken together, the gender differences in self-concept and epistemological beliefs in mathematics that were observed in this study are consistent with findings of previous research. In contrast to previous research, however, there were no differences in self-concept and epistemological beliefs in mathematics between grade 9 students and grade 10 students. An explanation for this finding is that the difference in the grades (i.e., grade 9 and grade 10) was too small to produce an effect on self-concept and epistemological beliefs in mathematics. Alternatively, it might be assumed that, due to the cross-sectional design of this study, it was not possible for us to examine, in contrast to previous research, individual changes in self-concept and epistemological beliefs longitudinally. In this study, we also provided empirical evidence for a relationship of self-concept and epistemological beliefs in mathematics. The nature of this relationship remained nearly the same even when we controlled for the influence of mathematical performance. Finally, we found that gender differences in the mathematical self-concept in fact increased from grade 9 to grade 10 when we took into account the influence of epistemological beliefs and performance in mathematics.

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'EDUCATION FOR GLOBAL CITIZENSHIP AND SUSTAINABILITY': A CHALLENGE FOR SECONDARY MATHEMATICS STUDENT TEACHERS?

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The study reports on the extent to which pre service Secondary Mathematics teachers integrate the sustainable development and global citizenship dimension of the curriculum into their teaching and aims to determine what possible barriers they may conceive to embedding these issues into their mathematics lessons. Data was collected from 25 pre service secondary mathematics teachers through questionnaires and interviews. Predominant in the findings was the teachers' reluctance to integrate ESGC into their lessons, as there was a lack of exposure to these issues in a mathematical context.

INTRODUCTION

Policies to formalise Education for Sustainability and Global Citizenship (ESGC), in the United Kingdom, have grown in emphasis over the years. In 1998 the Government Sustainable Development Education Panel (SSDEP) was formed in the UK to consider ways in which schools could promote education for sustainable development (ESD) and in 2000 the UK National Curriculum established ESD as a statutory requirement in geography, science, design and technology and citizenship. Underpinning this was the statement 'the curriculum should reflect values in our society that promote personal development, equality of opportunity, economic wellbeing, a healthy and just democracy, and a sustainable future' (National Curriculum, 2000).

There are two main ways, distinguished by Bonnett (2002), in which sustainable development could be integrated in education. The first, Bonnett (2002) terms 'environmentalism' which he defines as a measurable approach where schools actively practice sustainable development. Examples of this would include raising funds to build outdoor classrooms, installing solar photovoltaic panels and creating the role of School Travel Advisors. This approach by schools is commendable, but it is also questionable as to the extent to which it raises pupils' awareness of the issues relating to global citizenship and sustainable development. Schools may well involve children in these projects but to what extent does it empower pupils to have a critical awareness of these issues?

Bonnet (2002) also considers the 'school development approach'. This approach encourages pupils to be critical thinkers who are able to reflectively engage in issues of sustainable development as part of their education. According to Bonnet (2002) this approach assumes that,

schools best further sustainable development by encouraging ongoing exploration and engagement with environmental issues in which the promptings of their own rationality are followed. Here the essence is to develop pupils' own critical ability and interpretation of issues in the context of first hand practical situations that they confront (P10).

The approaches identified by Bonnet (2002) manage ESGC in two distinct ways. The 'environmentalism' approach addresses the issue through school initiatives whereas in the 'school development approach' pupils are encouraged to critically think around the issue. However, evidence suggests that, certainly in the area of mathematics education, there is scarcely any indication of the 'school developmental approach'. Apple (2000, p. 243) states that, as a product of neoconservatist policies in education:

It is unfortunate but true that there is not a long tradition within the mainstream of mathematics education of both critically and rigorously examining the connections between mathematics as an area of study and the larger relations of unequal economic, political, and cultural power.

Findings in a study by M Robbins, LJ Francis, E Elliott (2003) further support this assertion. Their study invited 187 student teachers to fill in a short questionnaire in order to elicit their attitude towards education for global citizen citizenship (EGC). Of notable importance was the conclusion that there were significant differences in attitude toward EGC between student teachers pursuing different major fields of study, with Geography as the subject where student teachers had the most positive attitude towards EGC and mathematics where they had the least positive attitude when compared with eleven other subjects.

THE STUDY

The study surveyed pre-service secondary mathematics teachers to determine what impact ESGC had on them and how they embedded these issues into their teaching. The study aimed to identify the barriers preventing student teachers adopting these issues in their teaching. The pre-service teachers in the study (referred to as student teachers in the UK) were on the Postgraduate Certificate in Education (PGCE) Course. The course runs over an academic year and consists of 10 weeks of University based teaching and placements in two contrasting schools with each placement running for eleven weeks. For the most part students are placed in Inner London Comprehensive Schools in which the demography of the pupils is very diverse. One of the University based components of the course is an 'Equality, Inclusion and Citizenship' (EIC) unit which is designed to enable students to critically explore some of the key educational issues concerning equality, inclusion and citizenship and how these impact on their professional role as teachers. ESGC is one of the areas covered in the EIC sessions.

Following the end of their first school placement the cohort of 25 students were asked to complete a questionnaire in order to ascertain their notions of ESGC. The questionnaire was divided into three sections. The first section asked a set of closed 'Yes' and 'No' questions to elicit whether students had prior expectations of the themes covered in the EIC unit to be covered on a PGCE course. In the second section students were asked if they agreed that a particular issue covered in an EIC session had influenced their teaching. Students' responses were recorded on a 5 point Likert scale

ranging from Strongly Agree to Strongly Disagree. The final section used both closed and open ended questions to determine whether the students thought it was important to bring the ESGC into teaching mathematics. There were also two open questions at the end of the questionnaire in which the students were asked to think of ways in which to embed ESGC into their mathematics lesson. As the questionnaires were given to the students after a session at the University, there was a 100% return. Students often fill in feedback forms after taught sessions so filling in a questionnaire at the end of their session was not unusual for them.

The second part of the study was carried out once students had returned from their second school practice. This involved one to one interviews, lasting approximately 15 minutes each, with a random sample of 10 students. These interviews were undertaken in order to establish whether they had observed lessons where issues of global citizenship and sustainability had been addressed, how this had impacted on their teaching and any reasons which prevented them from embedding issues of global citizenship and sustainability in their teaching.

RESULTS
Responses to questionnaire following students' first placement
Table 1 presents a summary of students' responses to their expectations of topics which were going to be covered on the course through the EIC programme.

	Yes %	No %
Inclusive Education	95	5
Gender	79	21
Multilingualism	79	21
Race	79	21
Sustainable Development	63	36
Sexual Orientation	52	48
Social Class	52	48
Global Citizenship	47	53

Table 1: Student's responses to a question asking if they expected these topics to be covered in their Secondary mathematics PGCE Course.

Results show that only 47% of the students had prior expectations of 'Global Citizenship' being covered in their PGCE course. Results for 'Social Class' and 'Sexual Orientation' were similar, with 52% of the students having prior expectations of these topics being covered in the course. However, 63% of the students had expectations of 'Sustainable Development' to be covered. In contrast 79% expected Race, Multilingualism or Gender to be covered and 95% expected inclusive education to be covered. This is an area of concern, that almost half the cohort did not think that issues of social class, sexual orientation and global citizenship would feature in a Secondary Mathematics PGCE course.

Table 2 shows that very few student teachers disagreed with issues of Education for Global Citizenship and Education for Sustainability (EGCSD) being brought into mathematics lessons.

	Strongly agree/Agree	Uncertain	Strongly disagree/Disagree
Sustainable Development	47%	32%	21%
Global Citizenship	42%	32%	26%

Table 2: Student's responses to the statement 'It is important to bring in the teaching of sustainable development / global citizenship into the teaching of mathematics?'

Table 3 shows an even distribution of responses regarding 'Sustainable Development' but not 'Global Citizenship' where more than half the students felt that it should be taught as a stand alone topic in a different subject area.

	Strongly agree/Agree	Uncertain	Strongly disagree/Disagree
Sustainable Development	37%	32%	32%
Global Citizenship	53%	26%	21%

Table 3: Student's responses to the statement 'ESGC needs to be taught as a stand alone topic in a different subject area'.

In order to get an insight into the reasons for their answers students were also asked to explain their responses. There were a number of positive statements from the students emphasising the importance of bringing in issues of sustainability into the teaching of mathematics but, similarly, there were some comments which were opposed to these ideas. These included,

Student A: Another political topic that should stay out of the maths classroom. We have other priorities.

Student B: Some of my students cannot write or read or do 8 x 2 in their head. I have other priorities.

Students C: I could use them in examples but do not feel it is important, maybe more important in PSHE (Personal, Social and Health Education)

Taken at face value these comments appear to reveal a negative attitude towards incorporating issues of ESGC into mathematics lessons. However, closer analysis of the language suggests otherwise. Phrases such as 'We have other priorities' imply that some of the respondents were not averse to issues of sustainable development and global citizenship being taught in lessons but reluctant to integrate them into their own teaching as it could interfere with the 'priority' of teaching mathematics.

Positive statements included :

Student F: Teaching pupils about life outside of the curriculum will make them rounded and more mature individuals, more equipped for the real world. Within Maths this should be brought in naturally as much as possible. In small doses.

Student G: It is interesting, relevant and easy to incorporate into a lesson of math.

Student H: Because it gives a practical side of mathematics that can relate to the outside world and it is important to know.

This enthusiasm was reassuring in contrast to the negative or indifferent comments of some of the responses. However, later interviews revealed that, even with this positive approach students, for the most part, found it difficult to bring these issues into their teaching.

Questions 5 and 6 on the questionnaire were qualitative questions asking students ways in which ESGC could be embedded into their mathematics teaching. Only six students (32%) filled these sections in. Their answers suggested lessons looking at habit cycles, using data from newspapers, interpreting climate change graphs and using examples in teaching to reflect the diversity of the pupils. However, the poor return on these two questions suggest the problem for the students, even those interested in embedding issues of ESGC as part of their mathematics lesson, was that many were not sure how to integrate these into a lesson.

Interviews following students' second school placement.
On returning from their second school placement students were interviewed and asked about any observations they had made in mathematics lessons relating to the integration of global citizenship and sustainability and how this had influenced their own practice.

The students were asked:

1. To give examples of how they had seen ESGC embedded into mathematics lessons or had done so themselves.

2. How they felt about pupils being made aware of these issues in mathematics lessons.

3. How their department encouraged embedding issues of global citizenship / sustainability into lessons.

4. How their school encouraged embedding issues of ESGC into lessons.

In all but two cases, interviewees said they had not observed issues of ESGC in lessons. However, the student teachers noted that the only time these issues were raised in mathematics was when a whole school initiative had taken place. These were special days, in most cases once a year, which, for the most part, were seen as token gestures or tick box exercises by the student teachers. In one case (Student J) a department had taken the initiative to have specific mathematics projects for different year groups. An example of this was a Year 7 project, Me & My World, in which pupils would be given

data, such as Gross Domestic Product and life expectancy, about different countries which they would have to analyse. However, according to Student J, this was the only time in the year that these issues would be raised in a mathematics lesson.

In another case (Student S) there had been sporadic reference to issues of global citizenship and sustainability and even then only when there had been a whole school push on particular issues. Even so, like the other students, Student S felt that these days were token gestures. However, Student S had used ideas of ESGC in his mathematics lessons. He explained that he had set himself a challenge following the EIC lectures and, at the beginning of his lessons, had introduced short activities based on climate change. Further, he described himself as a teacher who, at the start of the course, felt that mathematics should be taught to get the pupils through exams but, having delivered lessons where mathematics was applied to real situations, had now gone full circle and decided that the lessons the pupils most enjoyed were where they could apply the mathematics to something they care about. In three cases students had recognised that ESGC was not necessarily an idea which had to be taught explicitly but that teachers could embed the ideas implicitly in their teaching. Examples included, taking the 'cultural diversity' of pupils into account when promoting group work and enabling pupils from different global backgrounds to exchange experiences and work with each other. However, in most cases students were still unsure of how to embed ESGC into their lesson and believed it to be time consuming.

> Student P: I think it's important, we are all citizens of the world, but I am still unclear as to how to incorporate it into my maths lesson.

Student P's comment reflected the feelings of the other students in the interview, all students interviewed acknowledged the importance of including these issues into their lessons but felt restricted by the constraints of the curriculum.

CONCLUSION

Although it is not possible to draw any firm conclusion, having worked with such a small sample of student teachers, the findings of this short study have identified two main areas of concern. These centre around an uncertainty of how to embed ESGC into mathematics teaching and also dealing with the constraints of the mathematics curriculum, including the content of secondary school mathematics text books (Beg & Ghosh, 2011).

The mathematics curriculum certainly has the opportunity to provide a platform in which ESGC can be quite effectively integrated. An obvious area would be data handling, although other strands of mathematics also have similar potential. Therefore, it seems ironic that when teaching a subject which plays a key role as a tool for social analysis, maths teachers should come across barriers to embedding social factors into the delivery of their lesson.

It is important that student mathematics teachers develop an awareness of how to integrate ESGC into mathematics education and, in doing so, practice a curriculum in which pupils become critical thinkers and not number crunchers. However there are

challenges ahead, with a political climate in which there is rapid change in education and moves towards a narrowing of the curriculum. Indeed, the current UK Government's curriculum review has placed a huge question mark over the future of teaching 'Citizenship', originally introduced in 2002 as a statutory subject in order to develop children's critical awareness. Therefore the prospect of developing the global dimension in the school curriculum looks likely to become an even greater challenge for both teachers and student teachers. The responsibility, therefore, lies with Initial Teacher Training institutions to drive the integration of ESGC into mathematics teaching by providing models of good practice to their students. However, we need to be aware of the challenges for pre-service mathematics teachers when planning and delivering mathematics lessons centred around social issues. Jacobsen and Mistele's (2010) study provides examples of issues and challenges in this area. They give examples where, in their attempts to connect mathematics lessons with social issues, pre-service teachers trivialise the social issue or, by focusing on the social issue, they use the mathematics without any mathematical instruction.

This is further supported by Garii and Rule's (2009) analysis of student teachers integration of social issues into mathematics and science. They concluded that student teachers needed additional support and guidance from the faculty in order to allow them more opportunities to expand their knowledge and confidence to present interesting and relevant lessons that meet both academic and societal needs.

There is clearly scope for pre service mathematics teachers to integrate ESGC into their lessons. I have mentioned the challenges faced by teachers in achieving this and it is the role of initial teacher training institutions to guide teachers to teach in reformed based ways. This has to be approached cautiously as the delivery of the lessons has its own challenges. Finally, once this is achieved, a further challenge will be to sustain these changes throughout the teachers' career.

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TEACHING FOR CREATIVITY: THE INTERPLAY BETWEEN MATHEMATICAL MODELING AND MATHEMATICAL CREATIVITY

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The aim of this paper is to show how engaging students in "real-life" mathematical situations can stimulate their mathematical creative thinking. We analyzed the mathematical modeling of two girls, aged 10 and 13 years, as they worked on an authentic task involving the selection of a track team. The girls displayed several modeling cycles that revealed their thinking processes, as well as cognitive and affective features that may serve as the foundation for a methodology that uses model-eliciting activities to promote the mathematical creative process.

INTRODUCTION

For the past few years, there has been an increasing demand for new ways of structuring mathematics. The OECD (2008) stated that mathematics "curricula should reflect the reality…[and] should stress innovative applications of mathematics" (p.18). Finding or developing diverse dimensions of mathematical education is not enough; one has to consider the rapid progress in science and technology which has characterized the 21st century and its effects. This accelerating progress has become a part of almost every aspect of our changing world, requiring the development of certain abilities and skills among students; among these, adaptability, the ability to solve non-routine "real-life" problems, creativity and systems skills have become crucial factors (Hilton, 2008; Jerald, 2009). Therefore introducing new methods of learning and teaching mathematics should reflect this rapid progress, enabling our students to successfully integrate into the 21st century. Mathematical model-eliciting activities (MEAs) provide the student with opportunities to deal with non-routine "real-life" challenges. These authentic challenges encourage them to ask questions, and to be sensitive to the complexity of mathematically structured situations, as part of developing, creating and inventing significant mathematical ideas. However, the development of students' mathematical creative thinking through MEAs has only been addressed in a few studies to date (Chamberlin & Moon, 2005). The study reported herein examines the role of affective and cognitive elements (Goldin, Epstein, Schorr& Warner, 2011) in facilitating the development of students' mathematical creativity through MEAs.

MATHEMATICAL MODELING

Mathematical models are conceptual systems consisting of elements, relations, operations, and rules governing interactions; these are expressed with external

notation systems which are used to construct, describe, explain or predict the behaviors of other systems. Model-development processes usually involve a series of recursive cycles consisting of developing, testing, and revising phases in which a variety of different ways of thinking are repeatedly expressed, tested, and revised or rejected (Lesh & Doerr, 2003; Lesh & Thomas, 2010). Mathematical-modeling activities are based on "real-life" problem situations in which students are given the opportunity to construct powerful ideas relating to interdisciplinary data (Lesh & Sriraman, 2005). These activities are open-ended in nature. The ambiguity of the problem statement and data representation suggests that various responses may be appropriate and that there are likely various levels of correctness, depending on students' interpretations, mathematical abilities, general knowledge and skills (Chamberlin & Moon, 2005). MEAs are designed according to six principles: reality, construction, self-evaluation, sharability, model documentation and effective prototype. These principles emphasize the importance of stimulating students' competence to extend their own personal knowledge and apply their real-life sense-making abilities to the creation of original mathematical models (Lesh, Amit, & Schorr, 1997; Lesh, R, Hoover, M, Hole, B, Kelly, A, & Post, T, 2000).

MATHEMATICAL CREATIVITY

Many researchers see the potential for mathematical creativity as a dynamic ability that can be developed in students. They associate students' creative ability with cognitive problem-solving abilities and suggest several ways of stimulating and assessing it (Haylock, 1997; Sriraman, 2008; Amit, 2010). Haylock (1997) suggests that breaking mental set or as he described it "overcoming fixation" is a crucial factor in creativity; in his study he demonstrated creative responses which allowed students to overcome fixations and solve complex problems. Sriraman (2008) defines mathematical creativity as the ability to produce novel or original work. He claims that "in order for mathematical creativity to manifest itself in the classroom, students should be given the opportunity to tackle non-routine problems with complexity and structure problems which require not only motivation and persistence but also considerable reflection" (p.32). According to Kruteskii (1976), mathematical creativity appears as flexible mathematical thinking which is "switching from one mental operation to another qualitatively different one" (p.282), which depends on openness to free thinking and exploration of diverse approaches to a problem. Polya (1957) provides heuristics to tackle mathematical problems in his book "How to Solve It" and defines some cognitive characteristics of the ingenious solver that might lead him/her to the discovery of an original solution. He claims that analogous objects agree in certain relations of their respective parts, and explain that "all sort of analogy may play a role in discovery of the solution…" (p.38). Leikin (2009) suggests observing and evaluating mathematical creativity through the lens of multiple solution tasks (MSTs) and states that "solving mathematical problems in multiple ways is closely related to personal mathematical creativity" (p.133). Some researchers have examined the connection between mathematical problem posing and creativity (Yuan

& Sriraman, 2010, Silver, 1997). Yuan and Sriraman (2010) compared problem posing and creative abilities of mathematically gifted Chinese and American students. Silver (1997) demonstrate an approach of fostering mathematical creativity through problem posing and problem solving in terms of fluency, flexibility and novelty.

METHODOLOGY

Research Design

The study reported herein was based on two tasks: a warm-up activity and the MEA. The warm-up activity was aimed at preparing the girls for the modeling task and took about an hour and a half. Each girl received a newspaper article about Usain St. Leo Bolt, the Jamaican sprinter and Olympic gold medalist. The article contained Bolt's records and a qualitative description of his the run in which he set the new world record in the 100 meter dash. After reading the article, each girl had to answer questions about it, constituting the basis for a discussion held between the researcher and the two girls. During that discussion, questions were raised regarding the definitions of the article's concepts (speed, rate, etc.) and their implications. The main purpose of this activity was to stimulate the girls' interest and motivation, and to familiarize them with the context of the modeling task, including factual knowledge, and cognitive and technical skills, so that their solution would stem from their own experience (Lesh, et al., 2000). The modeling task (Fig.1) was designed according to the afore-listed six principles (Chamberlin & Moon, 2005) and based on English and Watters' modeling activity "The Olympic Team" (English & Watters, 2005). It was a non-routine, real-life challenge, which allowed formulation of several (mathematically justified) solutions, depending on each girl's mathematical abilities, general knowledge and skills (Sriraman, 2008). The modeling task was based on a situation that could exist in the girls' daily lives and required a "real" solution (Chamberlin & Moon, 2005). The modeling eliciting activity (MEA) consisted of three sessions: (1) model development: each girl worked by herself (75 minutes), (2) presentation and discussion: each girl presented her solution (30 minutes) and (3) interviews with each girl.

Participants

The participants in this case study were two girls, 10-year-old Rotem and 13-year-old Shir. The girls had high achievements in mathematics and were participating in a special enrichment class for excellent students at their school. In addition, the two girls were enthusiastic about sports and took part in running races at school.

Data sources

The study was based on recorded interviews with each of the girls, their written material collected at the end of both tasks, the researcher's notes taken during the task solving and recordings of conversations during the activities and of the final discussions at the end of each task. It should be emphasized that the girls were asked to write down everything, so that drafts, sketches and final solutions could be collected.

During the interviews and the conversations, the researcher did not accept simple or standard answers. Each answer was discussed with the girls in order to understand their way of thinking. Attention was paid to their body language and the vocabulary they used, in order to understand their experience, and its meaning and importance from each girl's perspective.

Assigning team members for 100 x 4 boys' and girls' relay race

Tables 1 and 2 contain the records of 4 boys and 4 girls who won silver or gold medals in 60 meter or 100 meter runs that took place in autumn, winter, spring and summer of 2010. A relay race for 6th graders is going to take place 2 weeks from now. For the first time, boys and girls will compete together in a mixed race between all of the schools in the city. Due to the short notice, the head of the sports committee at your school needs your help: he has to decide which two boys and two girls to assign to the relay team based on their accomplishments in the 2010 races. Your task is to construct a guide that will help the head of the sports committee choose the best team members for the 100 x 4 relay race.

Sample from Tables 1 and 2:

Ali (name)	Autumn	Winter	Spring	Summer	
60 m	9.5 (gold-medal)	9.7	9.5	9.3 (gold-medal)	Table 1
200 m	38	37.5 (silver-medal)	39 (silver- medal)	38	Table 2

Figure 1. Relay Race Modeling Task
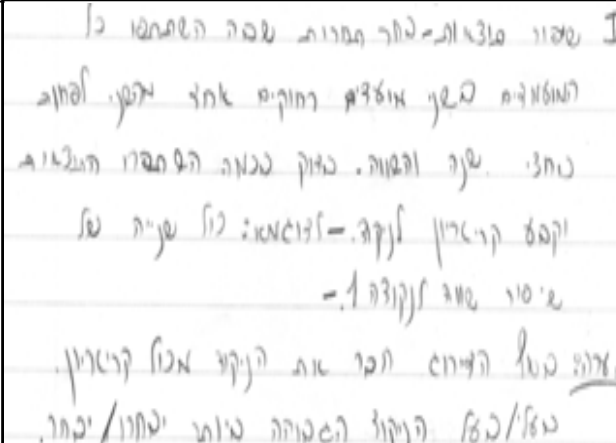
FINDINGS AND RESULTS

Analysis of the findings revealed two types of characteristics involved in the mathematical-modeling process: (1) cognitive and (2) affective. These features influenced the progress of the creative process and the creativity of both girls' conceptual tools. The girls' mathematical models contained unusual criteria for ranking scores to grade all runners.

The mathematical-modeling process

During the MEA, the girls went through several "modeling cycles"; in each cycle, the girls creatively developed mathematics that were new to them. Shir's model-eliciting process consisted of four cycles. In the first cycle, "criterion selection", Shir chose three criteria, based on her notion of fairness. She said "I need criteria to decide, who the best is. I need at least three criteria no less. I have to provide an equal opportunity

for all runners." After quantifying the data she found that three criteria were not enough, as two runners received identical values. She moved to the second cycle, "record improvement", where she added a fourth criterion to resolve the problematic situation: she decided to quantify each runner's improvement over the course of the year. In the third cycle, "scoring system", she realized that quantifying the fourth criterion and comparing runners' results was inconvenient; she therefore ranked the results for every criterion and set up a scoring system: "I have to weigh all of the data to know who the best is. Scoring is much more convenient than comparing each and every one" In the fourth cycle, "generalization", Shir tried generalizing her solution. For each criterion she added a mathematical formulation along with a written explanation that clarified scoring calculation and ranking weight and could be adapted to, and transformed for other, similar situations (e.g. establishing other sports teams). As an example, Fig. 2 presents part of Shir's letter to the head of the committee explaining how to use the improvement criterion.

| Record improvement: Select a competition that all nominees participated in twice, at least half a year apart, and compare. Check how the result improved and set score criterion. For example: 1 second of improvement equals 1 point. Note: At the end of the grading, sum all scores for each criterion and select those with the highest scores. | |

Cognitive characteristics

Flexibility - In the first modeling cycle, Rotem chose three criteria to distinguish between all runners. In the second cycle, she realized that one of these—tallying medals for each runner—was not a sufficient good criterion because two runners got the same score. She found a different way to solve the problem: "…to run in summer is much more difficult, so winning a medal in the summertime is worth more." She used seasons as weighting variables to formulate a weighted sum for her medal criterion.

Combination - In the second cycle, Shir added one more criterion by bringing together the runners' complete record and the season in which they competed in an original mathematical combination. Fig. 2 shows part of Shir's letter to the head of the sports committee, generally explaining how to apply this criterion.

Analogy - During the warm-up activity, Rotem compared Bolt's records; she drew an analogy between getting tired and slowing down: "a 400 meter run is much more tiring than a 100 meter run, so you run much slower because you are getting tired and it takes more time." She continued her solution idea and said "400 is four times 100, but 400

meters he [Bolt] runs in 45 seconds and 100 meters he runs in 10 seconds (approximately) so 400 took him 45 instead of 40." Rotem discovered a new mathematical formulation for the concept 'Speed' (the number of seconds taken to pass a fixed distance) which suited her intuitive 'everyday' thinking.

Affective characteristics

Motivation and interest - During the activities, the girls showed intense involvement, which was reflected in the level of interest, curiosity and meaning they found in the modeling task. Rotem explained: "in all mathematical exercises you only need to calculate and solve, but when 'real life' is involved it is much easier and fun to think, because you don't think about the mathematics you think about life."

Self-efficacy and persistence - The girls' understanding and recognition of the necessity of the task affected their persistence and will to continue, even though it was sometimes difficult and complex. Rotem:"It wasn't easy but I hardly thought about it, I knew and understood that I can find the solution and help him [the head of the sports committee] find the best runner."

Metacognition and self-reflection - Throughout the course of the task activities and each of its phases, the girls were aware of their own thinking in a way that affected and regulated their activities. Rotem, at the end of her first cycle said: "I didn't think well enough about my criteria…to know who the best runner is…I have to think about some more criteria and how to formulate them." During the interview, Shir described her work: "I didn't know how to apply to the task; I had to think in a different way, to think more real thinking, there was no single right solution and it made me think about other solutions, which is the best one, and not to think in a rigid way."

CONCLUSIONS

In the presented case study we examined how teaching for creativity through mathematical-modeling activities encourages the development of students' creative mathematical thinking. The findings clearly show some cognitive and affective characteristics that could establish the foundations for creative process development methodology using MEAs. The participants were two girls aged 10 and 13 years. The modeling task was based on meaningful situations that could occur in their 'real lives' in order to stimulate their motivation and engagement. The results exhibit the essential role of the affective (Goldin, et al., 2011) and cognitive aspects in the development of creative performance during the mathematical-modeling process. The practical implications of the current case study suggest that engaging students with non-routine mathematical problems (Sriraman, 2008) through MEAs can encourage them to develop, create or invent significant mathematical artifacts or tools (Lesh & Thomas, 2010).

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USELESS BRACKETS IN ARITHMETIC EXPRESSIONS WITH MIXED OPERATIONS

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There can be different intentions with brackets in mathematical expressions. It has previously been suggested that mathematically useless brackets can be educationally useful when learning the order of operations in expressions with mixed operations. This paper reports how students (12-13 years) deal with the implicit mental conflict between brackets as a necessary part of the order of operations and brackets to emphasize precedence. The students taking part in this quasi-experimental study were instructed on the order of operations, but were also indirectly exposed to different use of brackets. It is concluded that emphasizing brackets impede the transfer from a left-to-right computation strategy to the use of precedence rules.

INTRODUCTION

Brackets and rules of the order of operations are essential parts of algebra and distinguish the algebraic language from the spoken everyday language (Freudenthal, 1973, p 305). A comprehension of brackets used in the precedence rules are therefore of fundamental importance, not only for numerical calculations, but also in order to create an algebraic structure sense. Precedence errors are one of the most common arithmetic errors among secondary school students, particularly in expressions on the form $a \pm b \cdot c$ with mixed operations (Blando, Kelly, Schneider & Sleeman, 1989). Students do not primarily focus on the operations in arithmetic expressions, and therefore have problems in learning the proper order of operations. Instead they tend to focus on the numbers, detaching them from the operations (Lincheski & Livneh, 1999). Linchevski & Livneh (1999) therefore suggested inserting brackets around the product in $a \pm b \cdot c$ to support the structure sense.

These mathematically useless brackets would then emphasize the precedence of multiplication over addition/subtraction. Previous studies have shown that pupils' structure sense on algebraic expressions can be enhanced with emphasizing brackets (Hoch & Dreyfus, 2004), and more recently Marchini & Papadopoulos (2011) have shown that emphasizing brackets can contribute to an enhanced rate of success in simple arithmetic expressions too.

However, brackets are not easy entities to handle (e.g. Kieran, 1979; Hewitt, 2005; Okazaki, 2006). Moreover, the role of the bracket signs is different for emphasizing brackets compared to the "ordinary" role of brackets (where they typically are vital parts in the order of operations). Previous research on this area lacks a description of how pupils are handling the implicit but double role of the bracket-signs. The aim of

this study is to explore the effect of using emphasizing brackets to assist students' learning the precedence of multiplication.

DESCRIPTION OF THE STUDY

By pre-defined instructions we have studied students' perception and application in connection with expressions with brackets. The data was collected in a quasi-experimental form from an experiment group and a control group. The students were exposed to a pre-test, to instructions, and to a post-test, see Table 1.

Pre-test Item no	expression	Post-test Item no	Expression
F1	$(3+5) \cdot 2$	E1	$(7-2) \cdot 3$
F2	$2 \cdot 7 + 3$	E2	$2 \cdot 5 + 4$
F3	$3 + 5 \cdot 2$	E3	$8 - 3 \cdot 2$
F4	$4 \cdot (5-3)$	E4	$4 \cdot (5-3)$
F5	$8 - 3 \cdot 2$	E5	$2 \cdot 7 + 3$
F6	$3 + 4 \cdot 2$	E6	$3 + 4 \cdot 2$
F7	$(7-1) \cdot 2$	E7	$(7-1) \cdot 2$
F8	$2 \cdot 5 + 4$	E8	$2 + 5 \cdot 3$
F9	$(3+7) \cdot 2$	E9	$(3+7) \cdot 2$
F10	$2 + 3 \cdot 2$	E10	$2 + 3 \cdot 2$
F11	$4 + 7 \cdot 3$	E11	$4 + 7 \cdot 3$
F12	$2 \cdot (3+6)$	E12	$2 \cdot (3+6)$
F13	$4 \cdot (4-2)$	E13	$4 \cdot (4-2)$
F14	$3 + 5 \cdot 2$	E14	$3 + 5 \cdot 2$
F15	$4 \cdot 4 - 3$	E15	$4 \cdot 4 - 3$
F16	$8 - 3 \cdot 2$	E16	$8 - 3 \cdot 2$

Table 1: The different expressions to be computed in the
pre-test (left) and the post-test (right) respectively

Each test included seven tasks (expressions) on the form $a \pm b \cdot c$ (Items No F3, F5, F6, F10, F11, F14, F16, E3, E6, E8, E10, E11, E14, E16) where a preceding or left-to-right strategy could be identified by respective answers. Each test also included three expressions on the form $a \cdot b \pm c$ (Items F2, F8, F15, E2, E5, E15) and the other three expressions on the form $a \cdot (b \pm c)$ (Items F4, F12, F13, E4, E12, E13). All numbers were single digit and the expected answers kept the values to be positive (> 0) and low (≤ 33).

In total 169 students aged 12 to 13 in four Swedish secondary schools were included in this study. The instructions were made for two types, one experiment type which were instructed with emphasizing brackets and one control type which were without emphasizing brackets, see Figure 1. Both types of instructions included calculation of a set of eight different examples. In the experiment type both emphasizing brackets and brackets as part of precedence were used, whereas in the control type only brackets as part of precedence were used. In the experiment type of instruction the examples with emphasizing brackets were articulated as multiplication has higher priority (the direct translation from Swedish to English of the precedence rule would be "the priority rule") and therefore we put brackets around the multiplication to show that this should be calculated first. Care was taken not to mention the word emphasizing or to point out the different use of brackets. The instructions were pre-defined in all eight groups. In two of the groups one of the researchers was carrying out the instructions. In two other groups a class teacher carried out the instructions. The remaining four groups were given instructions by one of two different video-recorded clips.

(a)

$5 + 3 \cdot 2$

"Five plus three times two."

→

$5 + \underbrace{(3 \cdot 2)}_{= 6}$

"Since multiplication have higher priority than addition we put brackets there to show that this should be calculated first. Therefore three times two should be calculated first."

→

$5 + \underbrace{(3 \cdot 2)}_{= 6} = 5 + 6 = 11$

"Three times two is six. Thus it says five plus six, and five plus six is eleven."

(b)

$5 + 3 \cdot 2$

"Five plus three times two."

→

$5 + \underbrace{3 \cdot 2}_{= 6}$

"Since multiplication have higher priority than addition we should calculate three times two first."

→

$5 + \underbrace{3 \cdot 2}_{= 6} = 5 + 6 = 11$

"Three times two is six. Thus it says five plus six, and five plus six is eleven."

Figure 1: Excerpt from the instructions of the (a) experiment type and (b) control type. The difference is that in the experiment type emphasizing brackets are used.

Half of the groups were exposed to the experiment type instructions and the other half to the control type of instructions. In the experiment groups there were in total 83 students, and in the control group there were in total 86 students. The entire experiment was done in about 30 minutes in each group. The experiment groups and the control groups were chosen by convenience, but with the intention that they should be as close to equivalent as possible. However, as we discovered during the experiment, the control groups performed slightly better in that respect that these students' answers on the pre-test were more based on a precedence rule than in the experiment groups. The mathematics tasks in the two tests were designed such that the

strategy of computation should be more or less evident from the answers. The analysis of the protocols was then performed by inspecting the different variations in answers that were given.

RESULTS AND DISCUSSION

The instructions were given in three different ways, by the researcher, by the teacher and by video-clips, as described above. By inspecting the data it appears as if the video-clips gave the most reliable effect. But the difference between the different methods is too small to be significant.

Transfer from left-to-right to precedence rule

The main aim of the study was to look at students response (answers) to expressions on the form $a \pm b \cdot c$. The collected data showed that in the pre-test a left-to-right strategy was dominant in these answers (see Figure 2). The second most common strategy was using the precedence rules. We note that unfortunately the groups were not completely equivalent in this respect, since significantly more (122 compared to 64) answers in the control group can be associated to a proper use of the precedence rules.
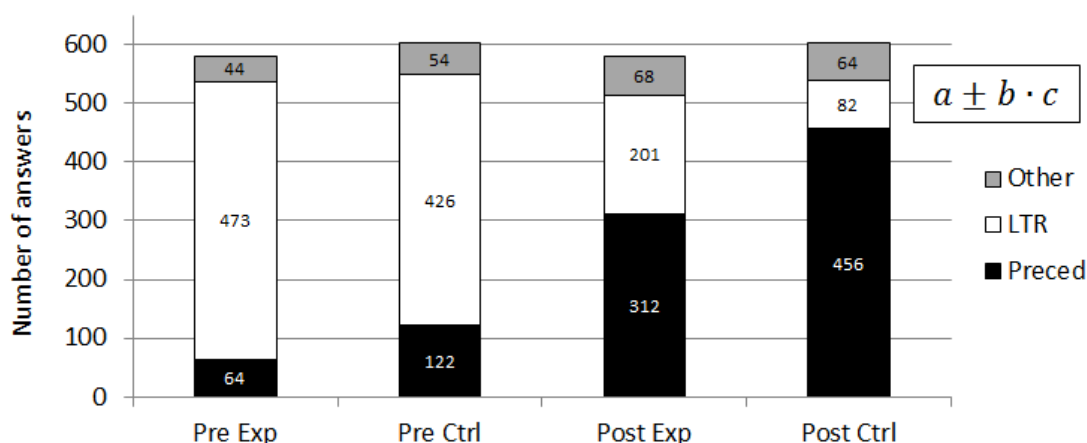
Figure 2: Number of answers on expressions of the kind $a \pm b \cdot c$ in the pre-test and post-test for the experimental group (Exp) and the control group (Ctrl). The answers are sorted as calculated by precedence (Preced) and a left-to-right-principle (LTR). The last category (Others) includes all types of miscalculations, blanks or other possible strategies unaccounted for.

After the instructions, the amount of answers that can be associated to a left-to-right strategy decreased in both groups in favour of the precedence rule. I the experiment group the increase of precedence answers was larger (in percentage) than in the control group. However, in the experiment group the number of precedence-answers was initially low. The effect of the specific instruction on this data is therefore complicated. Therefore a more detailed analysis of the transfer between the different strategies when computing expressions on this form was conducted. Figure 3 shows the

percentage of the students that changed from not applying a precedence rule to calculating with precedence on an expression on the form $a \pm b \cdot c$. From the data in Figure 3 we note that the lowest amount of changes was obtained for the expression $8 - 3 \cdot 2$.
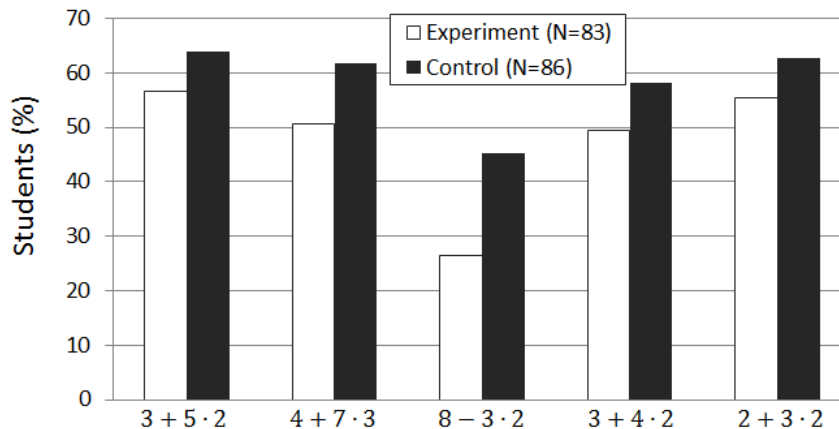
Figure 3: Percentage of students in the experiment group and the control group who have changed their computation procedure from non-precedence to the precedence rule.

With a χ^2-test of independence we obtained a statistically significant (= 99.96%) difference between the two groups of instructions on the total number of changes on expressions on the $a \pm b \cdot c$ form. However, if we look at consistency, meaning that *the same student* should be consistent in *all* responses to the expressions on this form ($a \pm b \cdot c$), we find only a few in each group (16 in the experiment group and 25 in the control group) that actually miscalculate the expressions in the pre-test and changes to a correct precedence-based calculation of these expressions in the post-test. A χ^2-test of independence on this smaller set therefore yield a lower statistical significance (= 80%). Anyhow it appears safe to claim that the experiment instruction – with both emphasizing brackets and brackets as part of the precedence rules – results in lower transfer from a left-to-right to the use of the precedence rules with statistical significance.

Bracket ignoring

The data also reveal other effects of the instructions. It has already been demonstrated by e.g. Blando *et al* (1989) that students can ignore brackets. Our data shown in Figure 4 demonstrate that bracket ignoring is decreased from pre-test to post-test. Most of the students had not previously been exposed to brackets in mathematical expressions. The high amount of precedence strategy must therefore be attributed to an a priori structure sense concerning brackets. Though the differences between the groups are small, it appears as if the effect of the instructions is larger in the experiment group than in the control group (39→4 compared to 34→11 in Fig 4). In this aspect the instructions with both emphasizing brackets and brackets as part of the precedence rule appear to be more efficient to suppress the bracket ignoring effect.

We also find one example of a student in the control group that after the instruction consequently ignores brackets in every expression, hence the student finds, e.g., $(7-2)\cdot3 \to 1$ (on Item E1) and $(3+7)\cdot2 \to 17$ (on Item E7). However, this student changed from consistently using a left-to-right strategy (but with correct handling of brackets) to consistently using a precedence rule on $a\pm b\cdot c$ expressions but with equalling $a\pm b\cdot c$ and $(a\pm b)\cdot c$. This could then be considered as an *incomplete* learning of the precedence rules.
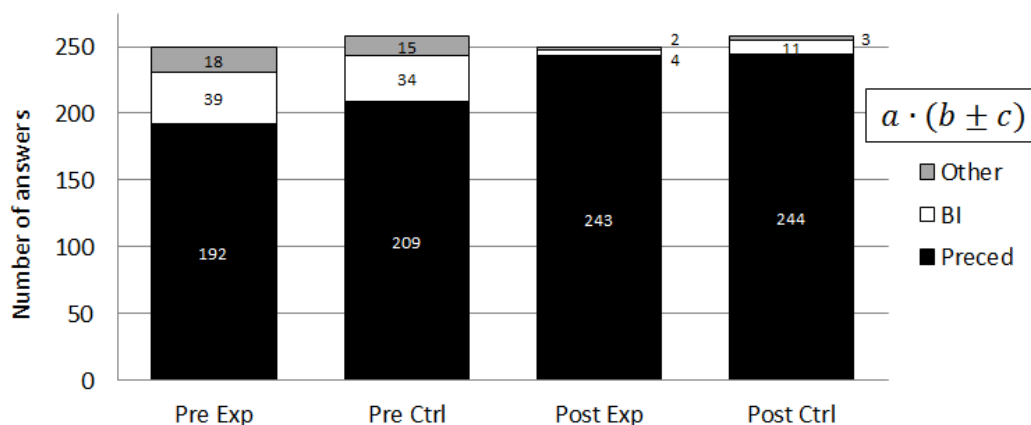
Figure 4: Number of answers on the three expressions of the kind $a\cdot(b\pm c)$ in the pre-test and post-test for the experimental group and the control group, respectively, sorted as calculated by precedence (Preced), a bracket ignoring strategy (BI) or by other strategies (Other).

Reversed strategy

Figure 5 shows the number of responses to expressions of the form $a\cdot b\pm c$. From the structure of the expressions it is impossible to separate the preceding strategy from a left-to-right strategy. However, we note that the second most common response can be associated to a calculation according to $a\cdot b\pm c \to a\cdot(b\pm c)$, i.e. with initially performing the rightmost operation and after that the left operation. This strategy is reversed in the sense that it is either reversed in spatial order, i.e., the calculation is performed from right to left, or it is performed with a reversed precedence rule, addition/subtraction before multiplication. One could suspect that answers in this category can be deduced to a "calculate easiest part first" strategy.

The expressions on this form ($a\cdot b\pm c$) can be compared to the expression $8-3\cdot2$ which according to Figure 3 had low success rate. We found the third most common answer to the latter expression to be -2. The reason for this could be a reversed strategy, evaluating the expression from right to left. However, as the reversed strategy appears to decrease under the influence of the instruction, in the case of $8-3\cdot2 \to -2$ the frequency is significantly increased in the post-test compared to the pre-test, as shown in Figure 6. It is obviously the comparison between the two terms in the expression

$8 - 3 \cdot 2$ that has been "damaged" by the instructions. At present it is unclear why this effect basically does not exist in the pre-test data but only occur after instructions.
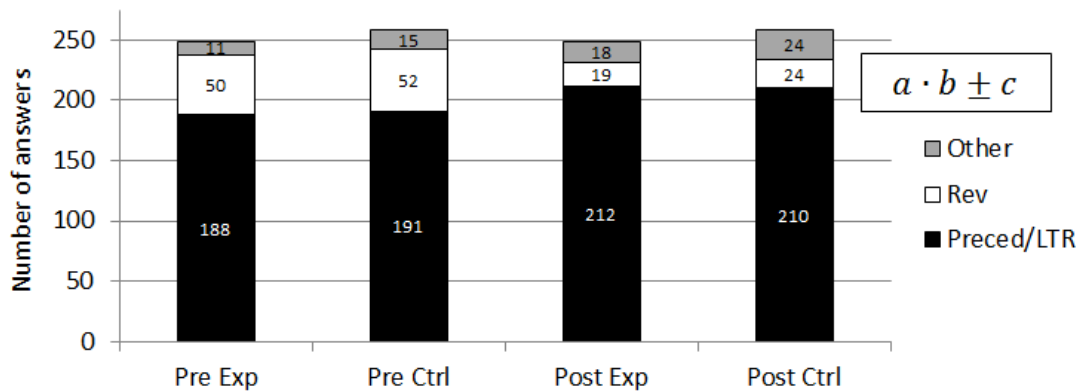

Figure 5: Number of answers on the three expressions of the kind $a \cdot b \pm c$ in the pre-test and post-test for the experimental group and the control group, respectively, sorted as calculated by precedence or left-to-right (Preced/LTR), by a reversed strategy (Rev) or by other strategies (Other).

Our data suggest that the implicit conflict between brackets as part of the precedence rule and brackets to emphasize precedence could be an obstacle when learning the order of operations. In our instructions the difference between the different roles of the brackets were not articulated. Thus, the outcome could possibly be different if the variations of different roles of the brackets would be stressed more clearly.
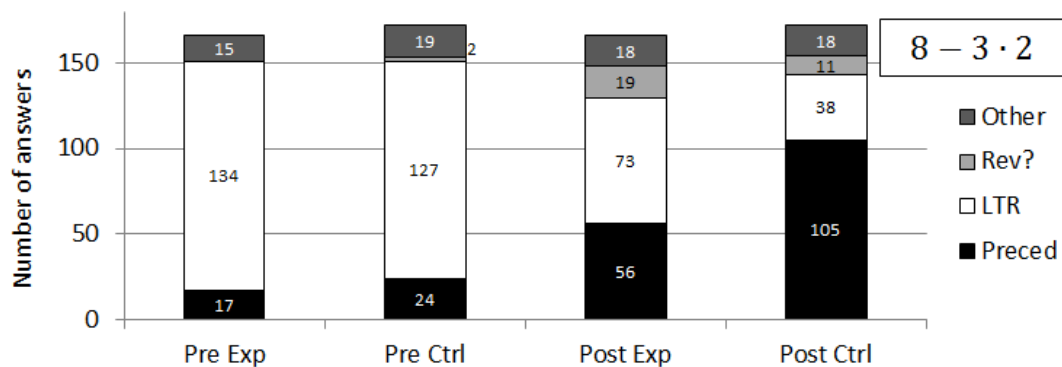
Figure 6: Number of answers to $8 - 3 \cdot 2$. The answers are sorted as calculated by precedence (Preced), by a left-to-right strategy (LTR), by a possibly reversed strategy (Rev?) or by other strategies (Other).

CONCLUSION

Lincheski & Livneh (1999) suggested inserting mathematically useless brackets to emphasize the structure in expressions on the form $a \pm b \cdot c$. Such emphasizing brackets have indeed been shown to help students perceive structure in algebraic expressions (Hoch & Dreyfus, 2004) and arithmetic equations (Marchini &

Papadopoulos, 2011). However, brackets inserted in arithmetic expressions with mixed operations are more often used with the purpose of indicating precedence of an addition/subtraction over a multiplication as in $a \cdot (b + c)$. This dual intention with bracket symbols could possibly be an obstacle when learning the order of operations. We have therefore designed a set of instructions in order to study the effect of emphasizing brackets in an environment of brackets as part of the precedence rule. In total 169 students at the age of 12-13 years were exposed to the quasi-experimental study.

Can mathematically useless brackets be useful tools for teaching? Marchini and Papadopoulos (2011) asked. From our data we conclude that it appears not to support the learning of the order of operations. With statistical significance we claim that instructions using both emphasizing brackets around multiplications and brackets indicating precedence other than the normal order of operations decreased the transfer from a left-to-right strategy to the application of a precedence rule. Possibly the instructions with emphasizing brackets could give a positive effect on the structure sense related to brackets as the bracket ignoring effect decreased.

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STUDENTS CREATING WAYS TO REPRESENT PROPORTIONAL SITUATIONS: IN RELATION TO CONCEPTUALIZATION OF RATE

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This paper examines seventh grade students' processes of creating mathematical expressions for problems involving proportional relationship. Based on the framework of the conceptualization of intensive quantity, I analyze the challenges they experienced during their problem-solving attempts. Interviews with 14 students showed that the students who had a less developed rate conceptualization found it difficult to recognize the constancy of ratio in the presented situations. Moreover, the students who had a more mature rate conceptualization were also faced with challenges, although different, in expressing proportional relationships. The results imply a close link between creating mathematical expressions and conceptualization of rate but there are still other factors that may serve for the process of creation.

INTRODUCTION

Studies on student's progression from arithmetic to algebra have accumulated important findings with respect to algebraic aspects of arithmetic and the methods to successfully prepare children for the formal study of algebra (e.g., Carraher & Schliemann, 2007). Innovative curriculum and teaching methods have been proposed in various countries. In Japan, one of the four content areas in the elementary school curriculum is *quantitative relationships,* which includes functional relationships and ideas related to mathematical expressions as two pillars of elementary school algebra (Watanabe, 2011). Proportional relationship, a core content topic bridging arithmetic and algebra, is taught spirally in grades 5–7 (MEXT, 2008). However, students' achievement is not adequate: especially, the results of assessments by MEXT repeatedly show their difficulty in understanding algebraic equations that express proportional relationship and other functional relationships.

The focus of this study is the learning trajectories regarding various representations for proportional relationship from elementary to lower secondary schools from the perspective of proportional reasoning development. This study considers mathematics learning as inextricably linked to semiotic activity in which students endeavor to create meanings and signs by reflecting on their interrelationships; a constitutive element of this also involves adjusting their creation of signs and meanings (van Oers, 2000). Therefore, it is essential to understand students' processes of acquiring different symbolic means in the classroom in conjunction with the impact they have on their proportional reasoning (Hino, 2011).

In this paper, I report on the attempts of seventh grade students to create mathematical expressions in order to represent proportional situations prior to formal instruction on

an algebraic equation ($y = ax$). In Japan, students are introduced to proportional relationship through the use of tables in the fifth grade: If two quantities □ and ◯ change in such a way that ◯ increases by a factor of 2, 3, … as □ increases by a factor of 2, 3, …, we say that ◯ is proportional to □. In the sixth grade, after a brief introduction to letters, the relationship between □ and ◯ is formulated, such as in the equation $y \div x = $ *fixed number*. In this grade, a graph of a proportional relationship is introduced by plotting several points and observing their arrangement as a straight line that goes through the point where both quantities are 0. In the seventh grade, with the introduction to function, the symbolic representation ($y = ax$) is introduced and proportional relationship is defined in the form of $y = ax$. The extension of domains from positive to negative and the introduction to linear graphs in the Cartesian plane are also new topics. On the basis of these circumstances, I examine the grade 7 students' semiotic activities in relation to their conceptualization of rate.

THEORETICAL BACKGROUND

In the area of proportional reasoning, researchers have proposed various perspectives and frameworks to characterize its development (e.g., Harel & Confrey, 1994; Davis, 2003). In this study, I refer to the framework developed by Kaput and West (1994) and other researchers who discuss the conceptualization of intensive quantity. Their framework was chosen for this study because the focus is on students' developing meaning in regard to the constancy of ratio, which should ultimately be reified as constancy of proportion (*a* in $y = ax$). Moreover, studies of Japanese children in grades 4–6 show a crucial relationship between their conceptualization of intensive quantity and learning of multiplication and division (e.g., Nakamura, 2011).

Kaput and West distinguish two fundamentally different conceptualizations of intensive quantity. One is *rate-ratio,* in which a general description of an entity, situation, or event is conceptualized. The other is *particular ratio,* in which a particular instance of the rate-ratio is described. According to Kaput and West, rate-ratio conceptualization is built on repeated experiences of particular ratio, and it involves complex deployment of three major ideas: "numerical equivalence across particular intensive quantities," "semantic equivalence across situation descriptions," and "homogeneity of the intensive quantity's referent in the situation being described" (p. 241). They emphasize that appreciating equivalence of one given pair of units with another and knowing that it applies across all instances of the situation being modeled are at the heart of the rate-ratio conception.

By using the framework, they offer fine-grained analyses of informal proportional reasoning strategies, i.e., *coordinated build-up/build-down*, *abbreviated build-up/ build-down using multiplication and division*, and *unit factor strategies*. They point out that the construction of a correspondence between the two entities in a situation is the key step in the construction of rate-ratio between the measures of these entities. From this point of view, in the first two strategies, it is important whether the solver coordinates pairs of segments or groups by adjusting unit-size either in bottom-up way

or by "all-in-one" division and multiplication. Moreover, among the three, the unit factor strategy most requires a rate-ratio conception, since "the unit factor *is* a rate" (p. 258, emphasis original).

It should be noted here that these studies have targeted elementary school children. Furthermore, the researchers focused on missing value problems or comparison problems. In this study, I use the framework with problems based on expressing proportional relationship through equations, which constitutes a major activity involving proportional relationship in lower secondary schools (MEXT, 2008).

METHOD OF STUDY

This paper is a part of larger study that aims to understand seventh grade students' learning of numbers, mathematical expressions, and functions and, based on this understanding, to propose teaching interventions. Two public schools in Tochigi prefecture, Japan, participated in the study. In the first year of the project, I implemented a questionnaire at the beginning of the school year with all seventh grade students to assess their proportional reasoning. It contained four tasks involving proportional relationship: three missing value and one comparison. Numerical features of ratios used in these tasks were "not reduced ratio" and "divisible to get unit factor" (e.g., 6 : 390 = 10 : x). Based on the results, I chose 14 students with different performance levels to be interviewed for 4 or 5 times during the school year. Each interview session lasts about 30 minutes. Along with data from their written documents, all interview sessions are audio and video recorded and transcribed. In this paper, I use data from the first and second rounds of interviews conducted prior to the formal teaching of proportional relationship, in which the algebraic equation ($y = ax$) and the coordinate graph were introduced.

All interview tasks concern two quantities that vary proportionally with having the same numerical features as in the questionnaire. In the first round of interviews, students were provided with a situation in which a person walks at a constant speed. They were then asked to express the relationship between time and distance by different representations (e.g., picture, graph or mathematical expression). In the second round, given three situations, they were asked to judge whether each situation is proportional or not, together with questions on expressing the relationships between quantities. Through the questions, I intended to elicit students' ideas and methods for creating signs and meanings and for using or modifying them. Two samples of the tasks and questions are presented below:

Walking task:

Situation: A person is walking eastward at a constant speed of 300 meters in 5 minutes. At ☐ minutes after the person passed point O, he is at △ meters to the east from point O.

A question: "Will you create a mathematical expression that shows the relationship between time and distance after the person passed point O?"

Potato chip task:

Situation: On the back of the potato chip bag, you can find the indication of calories contained in the bag. In one potato chip bag, it says 300 kcal per 50g.

A question: "Will you write a mathematical expression that shows the relationship between the weight of potato chips eaten and the calories being consumed?" (Formula version of the question: "For the people who are concerned about nutrition, will you make a formula that finds the calories they ingest once they know the weight of their consumption of potato chips?")

To obtain information on their semiotic activity, both the product and process of the student's reasoning for each question were summarized and analyzed. Two major focuses of the analysis were the (key) words and the visual mediators (icons, algebraic ideograms, et al.) that the students verbalized or drew for variables and the constancy of ratio (c.f., Capsi & Sfard, 2010). They are thought to contain valuable information on the interplay between sign production and proportional reasoning. The students produced various symbolic forms. Remarkably, each student demonstrated unique aspects on both focal points, but some similar observations were also noted among the students. To note these similarities, I compared and contrasted the responses of all students to generate (and check) categories. The repeated patterns of reasoning and use of signs were also sought for each student across the two interviews to attain further information on their reasoning processes.

SOME RESULTS

Performance of 14 Students in the Questionnaire

In order to catch students' conceptualization of rate, I paid attention to two features in their written solutions. One is the number of tasks that were solved by producing and using equivalent ratios. Here, based on the analysis of informal proportional reasoning strategies described earlier, build-up processes with some unit-size adjustment or unit factor strategies were counted. Another is the content of description in their solutions. I searched whether the solutions included a description of the meaning of equivalent ratios, such as "35 means the weight per 1 meter."

	Performance Feature	
Name of the Student	Number of tasks solved by producing and using equivalent ratios	Whether the solutions included a description of the meaning of equivalent ratios
A1	1	No
B1, B2, B3	2	No
C1, C2, C3, C4	3	No
D1, D2, D3	3	Yes
E1, E2, E3	4	Yes

Table 1: Performance of 14 participants

Table 1 presents an overview of the performance of the 14 participants. They were divided into five groups according to the consistency with which they produced and

stated numerically or semantically equivalent intensive quantities in solving the tasks. The students varied in their conceptualization of rate. A1 failed to produce equivalent ratios by adjusting the unit-size given in the task except for one missing-value task. She and other seven students did not state the meaning of equivalent ratios in any of their solutions. On the other hand, E1, E2 and E3 solved all the four tasks by coordinating pairs of segments or groups by adjusting unit-size. They also stated the meaning of equivalent ratios they developed.

Students Ways of Creating Mathematical Expressions

In the interviews, E1 and E2 were consistent in their creation of mathematical expressions. They could produce standard equations such as $\square \times 60 = \triangle$ or $A \times 4 = B$. However, the other students created a variety of mathematical expressions. In this section, I illustrate some of their ways of creating mathematical expressions by using their responses to the sample questions above.

Stating a Particular Ratio. In the walking task, I first asked A1 to draw a picture of the situation. Her picture (Figure 1) showed the information on rate (300 m in 5 min.) as the whole line from point O to the final destination. She did not state "5 min." in her picture. During her communication with me about the location of the person at the 5- and 10-minute points after passing point O, she changed the information of rate.

A1: (She pointed at the middle of the segment) I think he is about half [in 5 min.]… (Being asked to point to the location in 10 min.,) He would be here (She pointed at the right end of the segment). Probably in 10 min. From here (She pointed at point O), in 5 min. and 10 min. it goes like, 5 and 10.
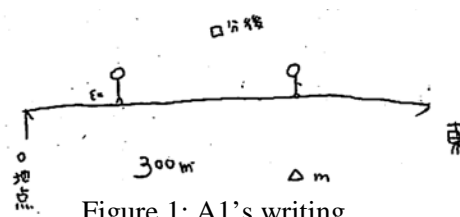
Figure 1: A1's writing

When asked to represent the situation with a mathematical expression, she produced one numerical statement "$150 \div 5 = 30$" but could not explain its meaning. These observations indicate that A1 interpreted the information of rate either just as a number or as a multiplicative relationship between two numbers. C1 and C2 were also observed to state particular ratio relationship of two numbers in their mathematical expressions.

Focus on Co-Variation. Many of the students interpreted the information of rate in the task as referring to more than a particular ratio. However, how they attended to the information and incorporate it into their mathematical expressions varied. B1, C2, C4 and D1 attended to co-variation of the two quantities instead of correspondence between x-y. The expressions they created were often idiosyncratic. B1 created "$\square : \triangle$" in the walking problem.

Figure 2: B1's writing (1)

B1: Well, this is, I mean.. this is a little bit different, I don't mean $\square$ to $\triangle$ (meaning a ratio of $\square : \triangle$), so I will change it to 1 to 2 (meaning a ratio of 1:2), and so (He wrote the expression in Figure 2) …, this is what I mean.

For the formula version of the potato chip task, B1 apologized for only being able to write the formula in a tabular form (Figure 3). In creating the table, he systematically co-varied the two quantities, first with larger numbers and then with smaller numbers. For both

Figure 3: B1's writing (2)

tasks, B1 basically used build-up/build-down strategy without forming a unit factor from the original information of rate.
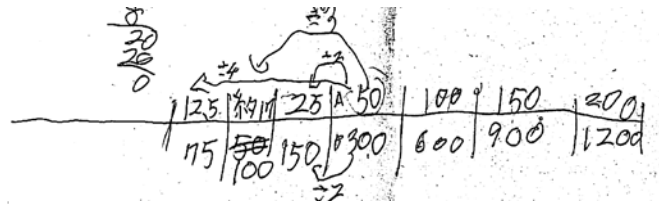
C2 repeatedly had hard time to make mathematical expressions in the interviews. In the formula version of the potato chip task, she developed an expression (Figure 4) that was made sense to her for the first time. In creating her expression, she first posed a question of the weight of potato chip in one bag and then decided herself that the weight for one bag was 100 g. Then her thinking proceeded as follows:

C2: If one bag is 100 g, then probably, A is the weight of the potato chip and B is the weight eaten (She is checking the task). It (meaning formula) should tell the calorie… Oh, I got it!... Well, the quantity eaten? It would be A multiply… well, if this is the case, if I know the grams eaten, then if I multiply by 2 or by 3… probably I think it will derive the calorie. ….

($A \times 2$=quantity eaten)
($B \times 2$=calorie for the quantity eaten)
Figure 4: C2's writing

The protocol shows her idea of creating an expression, namely, to form a unit and to build-up multiple instances based on the unit. As a result, the expression was two mathematical sentences for the weight and calorie. The expression was idiosyncratic in the sense that A and B seem to mean two specific quantities, 50 g and 300 kcal, respectively, and the "2"s mean a variation of numbers. Like B1, C2 did not form the unit factor (6 kcal per 1 g), but she also showed her struggle of spontaneously constructing a unit of the calories per one bag during the process of creating the expression.

Attending to Individual Solutions. E3 was the student who could use the unit factor strategy and adjust unit-sizes in performing the build-up strategy to solve missing value tasks. However, in creating mathematical expression, he was observed paying attention to the individual solutions rather than the common feature across solutions. In the walking task, he wrote $\square \rightarrow \triangle$ and added 何分 (what minutes) and 何m (what meters) (Figure 5). He said that
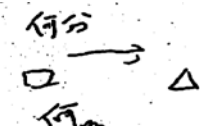

Figure 5: E3's writing

he could not decide on one expression because it would be different when it was used to find the minutes or to find the meters. For the purpose of reducing his difficulty, I asked him to choose one of the two. Then he replied:

E3: All right, then, well let's see, if I know meters, then, it says 300 meters in 5 minutes, so I divide 300 by 5 and get 60 meters. And since it is 60 meters, if the answer is 900 (He wrote "900 m"), the

time spent to walk [900 meters] should be 15, it should be 15 minutes (He wrote "15分"). Well, then it means that I don't need to change it (He means 300 m in 5 min.) to 60 meters in 1 minute. I can do it by using 5 minutes (meaning 300 m in 5 min.). … Well, this is difficult.… (He further wrote "12分".). Let's see, I know in 10 minutes it is 600 meters, and if I change it to 60 meters in 1 minute, then, since it is 2 [more] minutes, I will add 120 meters to this one (meaning 600 meters), then I get 720 for the time.

At first, he seemed to be creating an expression for finding the distance when the time is given. However, he was still confused which of the two expressions he was creating. The protocol shows that E3 was engaged in finding answers for the case of 900 meters and 12 minutes by adjusting the unit-sizes. However, it did not contribute greatly to the development of his expression. He finally produced $\triangle \div \square = \bigcirc$ but could not state the meaning of $\bigcirc$ beyond "the distance from point O."

DISCUSSION

The students in this study created a variety of mathematical expressions in the interviews. Although they were not received formal instruction on $y = ax$, they approached the question of expressing proportional relationship by using various symbolic means they had acquired in elementary schools. Nevertheless, their ways of expressing the information of rate in the situation varied more than expected, which revealed the students' perspectives on what should be attended to, or what needs to be symbolized in what manner to represent proportional situations.

Here, the students' conceptualization of rate was found to play an important role in representing proportional situations through mathematical expression. As illustrated in the previous section, the participants with a less developed conceptualization of rate easily changed the given rate information, which suggest that they did not attach significant meaning to the information, or make an expression that stated the particular ratio relationship of two numbers. Even when the participants produced multiple instances from the rate information given, some of them tended to focus on the aspect of co-variation between the two quantities (e.g., Carraher & Schliemann, 2007; Nunokawa, 2010). They were engaged in build-up/build-down processes without adjusting the unit-sizes to get the unit factors. Here again, their conceptualization of rate was functioning.

It was also found that those with a more mature rate conceptualization were still faced with challenges to express the relationship. They ranged from organizing individual solutions based on the common features as shown earlier, to making distinctions between general and specific for the objects of symbolization and assigning appropriate signs. Thus, the results imply that to be able to represent proportional situations by symbolic representations there are also other factors that we need to consider, which would include semiotic aspects of the concept of rates and variations. Kaput & West (1994) noted the fourth idea that bears on a full conceptualization of rate-rate, namely, that of variation and underlying idea of variable. It is likely that not only the idea of variable, we also need to pay attention to students' understanding of

and skills for the symbolization of variable to represent proportional relationship. Here, the interview data suggest that students' conception of mathematical expression would be an important basis because their struggles show their conception of mathematical expression only as a recipe for finding answers.

At present, I am following these students in the development of their reasoning process and their early use of algebraic equations and other representations learned in mathematics lessons. One of the interests is to get information on the influence of learning of these means on their conceptualization of rate. Gaining such information will be useful in designing the resulting teaching interventions.

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PRE-SERVICE TEACHERS' SPECIALIZED CONTENT KNOWLEDGE ON MULTIPLICATION OF FRACTIONS

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This paper reports pre-service teachers' Specialized Content Knowledge (SCK) on multiplication of fractions. The responses of ninety-two first year Bachelor of Education (Primary) pre-service teachers, enrolled in a regional university in New South Wales, Australia, in a multiplication of fractions mastery test item were analyzed using an analytical tool. This tool, designed by the authors, consisted of four components, of which three are elements of SCK suggested by Lin, Chin and Chiu (2011): correctness of answer, justification, explanation, and representation. Preliminary findings suggest that mathematics pedagogy subjects in pre-service teacher education programmes should expose pre-service teachers to the various explanations and representations of concepts in mathematics.

INTRODUCTION

Pre-service teachers' mathematics knowledge and pedagogical content knowledge have been an international concern since the last two decades, resulting in a growing number of research in this area (e.g., Ball, 1990; Isiksal and Cakiroglu, 2011; Fennema & Franke; 1992, Goulding, Rowland & Barber, 2002; Hill, Ball & Schilling, 2008, Ball et. al., 2009). The introduction of Mathematical Knowledge for Teaching (MKT) − the mathematical knowledge that teacher need to carry out their work as teachers of mathematics − helped clarify the various types of teacher knowledge involved in a teacher's work repertoire. Studies on MKT have indicated that teachers' Specialized Content Knowledge (SCK) is a possible predictor of students' achievement of mathematics (Hill, Ball & Schilling, 2008). This paper reports preliminary findings on the SCK of pre-service teachers from a regional university in New South Wales, Australia.

MATHEMATICS TEACHERS' SPECIALIZED CONTENT KNOWWLEDGE

In Shulman's seminal work, he (1986) suggested that teacher knowledge consisted of subject knowledge, pedagogical content knowledge and curricular knowledge. Ball, Thames and Phelps (2008) extended Shulman's definition of teacher knowledge by coming up with a working definition of MKT. Specialized Content Knowledge (SCK), a component of MKT, is knowledge that involves both the knowledge and skill unique to teaching mathematics, that is, SCK involves conceptual understanding of mathematics concepts and knowledge of students' errors in mathematics (Ball, Thames, & Phelps, 2008). It should be noted that SCK is a form of knowing mathematics needed by teachers (to explain mathematical concepts and ideas to students) and not needed by those who do not teach (Ball, Thames & Phelps, 2008;

Ball et. al., 2009). In a recent study, Lin, Chin and Chiu (2011) suggested that there are three elements of SCK. They are: Justification (how to explain and justify one's mathematical ideas by rigorous arguments based on mathematical definitions and theorems), Explanation (how to provide mathematical explanations for common rule and procedures), and Representation (how to choose, make and use mathematical representations). Hill, Ball & Schilling (2008) indicated that teachers' SCK is a possible predictor of students' achievement. They also noticed that there has been a lack of measures to assess how teacher knowledge is related to student achievement.

Before we could address this concern, we need a measure to assess teacher's SCK first. In this study, an analytical tool was developed to capture pre-service teachers' SCK. Preliminary analysis of data in this study was conducted using this analytical tool.

DESIGN AND METHODS

Background of the study

The study involved ninety-two first year Bachelor of Education (Primary) pre-service teachers at a regional university in New South Wales, Australia. One of the objectives of the first year Bachelor of Education (Primary) mathematics pedagogy subject was designed to enhance pre-service teachers' SCK. As part of the assessment criteria, all the pre-service teachers had to sit for a ten-item paper-and-pencil mathematics mastery test. This mastery test required pre-service teachers to demonstrate their conceptual knowledge of the mathematics topics stipulated in the New South Wales K-6 mathematics syllabus. Each item in the mastery test was four marks. Difficulty was found in deciding how many marks (i.e., 1, 2 or 3) should be awarded for incomplete or partly correct answers. The scores of this mastery test could also not reflect exactly the pre-service teachers' SCK, that is, which mathematics concepts the pre-service teachers understood or did not understand. As a consequence, the lecturers and tutors could neither give feedback nor provide much help to the pre-service teachers on their performance in the mastery test.

The test item

This paper focuses on one test item in the mastery test which involved multiplication of fractions, that is, $\frac{1}{3} \times \frac{3}{4}$:

$\frac{3}{4}$ of a pizza is left after a party. $\frac{1}{3}$ of the left-overs are given to Sarah to take home. What fraction of the pizza does Sarah take home?

The analytical tool

To address the above stated issue, an analytical tool was designed. This tool was used to analyse the pre-service teachers' written responses to this test item. This analytical tool (See Figure 1) consisted of four components − Correctness of answer,

Justification, Explanation, and Representation. Among the four components, three were elements of SCK, as suggested in a recent study by Lin, Chin and Chiu (2011).

In Figure 1, the number in the parentheses represents the marks awarded for displaying the corresponding response or understanding. Each pre-service teacher's solution to the item was first checked for whether the answer to the item was correct or incorrect [Correctness of answer]. If the answer was incorrect, zero mark was awarded. If the answer was correct, 1 mark was awarded. Next, the solution was checked for completeness of the justification given to the correct answer given. Solutions that had 'correct and complete justifications', and 'correct answer and incomplete justification', were further checked for explanation. That is, whether it was a procedural explanation, a conceptual explanation, or both procedural and conceptual explanation were given. An explanation that was procedural, in this case, is explanation based on how to execute the multiplication of fractions algorithm. A conceptual explanation involves knowing why the algorithm works. A conceptual explanation was then further analysed whether it was a mathematically-based explanation or a practically-based explanation (see Levenson, Tsamir & Tirosh, 2010). A mathematically-based explanation is "based on mathematically definitions or previously learned mathematical properties, and often use mathematical reasoning" (Levenson, Tsamir & Tirosh, 2010, p. 346). A practically-based explanation is one which uses "daily contexts and/or manipulatives to "give meaning" to mathematical expressions" (Levenson, Tsamir & Tirosh, 2010, p. 345).

Both authors of this paper coded and scored each of the ninety-two pre-service teachers' written responses. The coding and scoring by each author was checked by the other author. Any discrepancy in coding was discussed and a final coding and score were agreed by both authors.
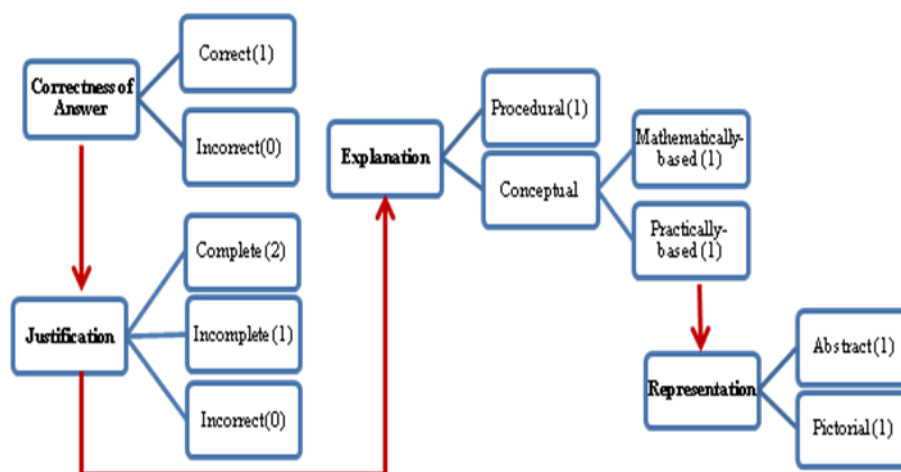

Figure 1: The analytical tool

The research questions

This study addresses the following research questions:

1. Were the pre-service teachers successful in solving the test item?
2. Were the pre-service teachers able to justify their answer to the test item?
3. Which type of explanation did the pre-service teachers use to explain their answer?
4. Did the pre-service teachers make use of representations to justify and explain their answer to the test item? If so, which type of representations did they use?

RESULTS

1. Were the pre-service teachers successful in solving the test item?
Majority of the pre-service teachers (76 out of 92, or 82.6%) gave the correct answer to the test item. Slightly less than a fifth of the pre-service teachers (16 out of 92 or 17.4%) were unsuccessful in getting the correct answer to the test item. Analysis of the overall score for the test item showed that majority (about 38%) of the pre-service teachers scored 3 marks out of a possible maximum of 8 marks. This means that majority of the pre-service teachers in the study were able to provide a correct answer to the test item, but unable to provide a complete justification of their answer.

2. Were the pre-service teachers able to justify their answer to the test item?
Table 1 shows the frequency and percentage of the pre-service teachers' (who gave correct answer to the test item) level of justification of their answer to the test item. Recall that majority of the pre-service teachers was able to provide the correct answer to the test item. However, majority of these pre-service teachers were not able to justify their answer successfully. Only 46.1% of the 76 pre-service teachers (or 38.0% of 92 pre-service teachers) who gave the correct answer to the test item was able to provide a complete justification to their answer.

Justification	Frequency	Percent
Incorrect	11	14.5
Incomplete	30	39.5
Complete	35	46.1
Total	76	100.1

Table 1: Level of justification of the pre-service teachers.

3. Which type of explanation did the pre-service teachers use to explain their answer?

Table 2 shows the type of explanation given by the 76 pre-service teachers who provided the correct answer to the test item. Majority of these 76 pre-service teachers provided an explanation by showing the execution of the multiplication of fractions algorithm, that is, a procedural explanation. Although the test item had a daily context (pizza left over after a party), only about a third of these pre-service teachers (35 pre-service teachers) made use of this daily context to explain their answer. None of the pre-service teachers gave a mathematically-based explanation. None of the pre-service teachers provided all three types of explanations – procedural, practically-based (conceptual), and mathematically-based (conceptual).

Type of explanation	Frequency	Percent
Procedural	51	67.1
Practically-based	11	14.5
Procedural & Practically-based	14	18.4

Table 2: Type of explanation given by pre-service teachers.

4. *Did the pre-service teachers make use of representations to justify and explain their answer to the test item? If so, which type of representations did they use?*

As mentioned above, 35 of the 76 pre-service teachers who gave the correct answer to the test item provided a practically-based explanation in their solutions. All of the 35 pre-service teachers provided pictorial representations of their explanations.

DISCUSSION

This study, unlike other studies on teachers' mathematics knowledge, used an analytical tool to indicate the level of SCK of an individual based on each three components of SCK (Justification, Explanation and Representation). Such information will be helpful for lecturers and tutors to provide feedback to the pre-service teachers and also to design mathematics pedagogy courses in pre-service teacher education programmes.

Using the analytical tool, the preliminary findings are: (a) majority of the pre-service teachers were not able provide a complete justification of their answer, (b) majority of the pre-service teachers provided a procedural explanation, (c) only about a third of these pre-service teachers used a practically-based explanation, even though the test item itself involves a daily context of pizza left over after a party, (d) none of the pre-service teachers provided mathematically-based explanation, (e) none of the pre-service teachers provided all three types of explanations – procedural, practically-based (conceptual) and mathematically-based (conceptual), and (f) only

about 38.0% of the pre-service teachers used a pictorial representation in their explanation.

The above findings echoed the findings from Tobias and Itters (2007) study which found the SCK of pre-service teachers in regional Australia to be lacking. This study found the pre-service teachers' SCK in the topic of multiplication of fractions to be lacking. The findings of this study were also consistent with those of previous studies (e.g., Isiksal and Cakiroglu, 2011) − that many pre-service teachers' understanding of mathematics concepts is characterized by rote knowledge of the algorithm rather than by the concepts underlying procedures. Although the role of procedural learning should not be ignored, if a teacher's knowledge of mathematics concepts is limited to only procedures, we cannot expect his or her classroom to be one of "developing knowledge, skills and understanding through inquiry" (Board of Studies NSW, p. 7), an objective stipulated in the current New South Wales mathematics K-6 Syllabus. Generating representations for a mathematical concept is a common teaching task in mathematics classrooms. Using a pictorial representation to explain the multiplication of fractions concept was found to be missing in majority of the pre-service teachers in the study. Connecting mathematics concepts with representations from daily contexts or the "real world" may help students make more sense when learning mathematics concepts. Further, it is important to note that "[w]ithout a solid knowledge of what to represent, no matter how rich one's knowledge of students' lives and no matter how much one is motivated to connect mathematics with students' lives, one cannot still produce a conceptually correct representation" (Ma, 1999, p. 82). Many studies have indicated that the real mathematical thinking that goes on in the classrooms is dependent on how the teachers' mathematics content knowledge and pedagogical knowledge (Fennema & Franke, 1992; Ma, 1999; Ball, Thames, & Phelps, 2008). Hence the findings in this study suggest that mathematics pedagogy subjects in pre-service teacher education programmes could help improve pre-service teacher's SCK by not only exposing them to the types of explanations of mathematics concepts (procedural and conceptual), it is also pertinent that they be made aware of the various representations of mathematics concepts (pictorial and abstract).

Ma's (1999) seminal work suggested that mathematics teachers should possess the following teaching skills: (a) connectedness (making connections between mathematics concepts and procedures), (b) multiple perspectives (providing explanations of different facets of a mathematical idea and various approaches to a solution), (c) basic ideas (revisiting and reinforcing basic mathematics ideas and concepts), and (d) longitudinal coherence (having a fundamental understanding of the entire elementary mathematics curriculum). The Chinese teachers in Ma's study regarded the meaning of multiplication of fractions as "a "knot" that ties a cluster of concepts (e.g., fraction concept, meaning of multiplication of whole numbers) that support the understanding of the meaning of division of fractions (p. 82). Hence, in a similar sense, teachers' SCK is also like a "knot" that ties the various kinds of teaching

skills and knowledge that teachers need in order to carry out their work as teachers well.

The findings in this study cannot be generalized to the whole of New South Wales, Australia. Also, only the written answers of the pre-service teachers were analyzed. Another study could employ an interview method to further understand pre-service teachers' SCK on the topic of multiplication of fractions. Yet another study could further analyze the types of pictorial representations for multiplication of fractions that the pre-service teachers gave in the mastery test.

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ELABORATING COORDINATION MECHANISM FOR TEACHER GROWTH IN PROFESSION

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This paper presents examination of a new mechanism, namely coordination, to teacher growth in profession. Here coordination is generally defined as the ability to construct novelty for teaching by transforming and coordinating sources of information observed and experienced from different learning environment. Two types of coordination were identified based on analysis of an experienced mathematics teacher when participating in design-based professional development programs. The potential of coordination as a mechanism to outline teacher growth is also detailed.

INTRODUCTION

Considerable studies have investigated mechanisms that can trigger teacher growth in profession. The reflection and enactment are treated as the central mechanisms to the growth. Dewey (1933) indicated reflective thinking as the key to overcome challenges but not routine thinking. Schön (1983) elaborated reflection from a different perspective by distinguished reflection-in-action from reflection-on-action. The former means the feedback and the pragmatic knowledge obtained during the action and the latter refers to the process of making sense of an action after it has occurred to extends ones' knowledge base. Jaworski (1993) indicated reflection and action that can not be separate because self-reflection is a spiral and cyclic process involving planning, action, observation, reflection. But it seems that research in reflection and enactment do not carefully articulate what and how sources of information obtained from different learning environment facilitate the growth in profession.

In the paper, we propose a new growth mechanism, coordination, from a socially situated learning perspective. Here coordination is defined as the ability to construct novelty for teaching by transforming and coordinating sources of information observed and experienced from different learning environment (e.g., professional development or teaching). The way to define coordination is aligned with conceptual coordination in psychology which emphasizes "mechanism as construction of novelties" (Piaget, 1970). Coordination mechanism for teacher growth also focuses on what and how new knowledge for teaching can be erected through the interaction and participation in different learning environment. Thus, this paper aims to investigate the nature of coordination and how coordination can trigger teachers' growth in profession.

THEORETICAL LITERATURE

From socially situated learning perspective, learning is viewed as a form of participation in the social world where the participation refers to "a process of taking

part and also to the relations with others that reflect this process" (Wenger, 1998, p. 55). Learning in line with participation is conceived as a social phenomenon embedded in the experienced, lived-in world involving the transaction between person and social environment where one's experiences are situated. The environment with participants shapes the communities of practices that have a history, norms, tools, and traditions of practices (Vygotsky, 1978). Through legitimate peripheral participation in ongoing social practice (Lave & Wenger, 1991), participants gradually become a member of a community in which they entail the ability to communicate and to act according to its particular norms (Cobb, Yackel, & Wood, 1992). In this sense, knowledge is co-produced in social settings as distributed among people and environments.

Considering the interactions between/among different tiers of participants are complex, Lesh and Kelly (2000), using mathematical model as an example, suggested research design should involve different tiers of participants, which cooperate in an interactive nature linking research and practice to solve problems encountered in classrooms. Specifically, as the transfer of knowledge from educators, teachers, to students in professional development is not a linear and one-way process in which the solutions to problems encountered in teaching and learning can be directly obtained. Rather, the transferring process is complex, cyclic, and sophisticated, involving interplays with multiple tiers of participants (Lesh, Hamilton, & Kaput, 2007). Classroom teachers need much effort to back and forth transform and coordinate the materials and experiences learned from professional development into practical strategies that can be used to communicate the knowledge with students in classrooms.

METHODOLOGY

We elaborated coordination mechanism by selecting an experienced mathematics teacher, Zhang, who made professional growth through the participation of two consecutive professional programs. Each of the programs lasted a semester long. The selected teacher Zhang had more than 15 years of teaching experiences and was a consultant for district school mathematics teachers. The programs were organized with the aims to enhance students' active thinking by means of using an approach of designing tasks rooted in design research paradigm (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). During the professional development programs, teachers were required to initiate tasks, present their designs in professional development for feedback from educators and peer teachers, test the designs with students for revisions. The process of creating tasks, enacting the tasks in classrooms, and collecting students' responses for revisions offer teachers opportunities to enrich their pedagogical power and the growth in profession. The professional development programs were led by an experienced educator who mastered in both research in mathematics education and teaching practices in classrooms. Because of the expertise in both research and teaching, educator was able to elaborate the research and theories

associated with student cognition and provide directions to the refinement of task sequences accordingly.

Data collection included survey and interviews to understand Zhang's perspectives on active thinking and his expectation before attending the professional development program. In addition, situations to administer interviews occur when it is not clear how Zhang initiated the design of task sequences, enacted the design in classroom lessons and made refinement accordingly. Other data sources were Zhang's design products, the video corpus of professional development sessions, video recordings of his classroom teaching for enacting the designed tasks, and the field notes taken by researchers when observing both professional development sessions and the classroom teaching. Qualitative methods (Merriam, 1998) were applied to analyze and triangulate the data sources in order to clarify the storyline for Zhang's professional growth and to establish the reliability of the analyses.

FINDINGS

Based on the analysis, we identify two types of coordination during Zhang's task design and implementation of task in classroom.

Coordination as making connection between others and personal ideas

The coordination occurred when Zhang created a task involving a mathematical formula. Originally, Zhang thought that mathematical formulae were usually quickly reviewed in classroom lessons. He has not considered mathematical formulae as the materials to design tasks that can enhance students to think mathematics actively. But his observation of the task created by peer teacher Shou involving a mathematical formula ($(a+b)^2 = a^2 + 2ab + b^2$) changed his mind.

Zhang: I was touched by Shou's task design…I thought it is easy to teach a mathematical formula. However, I saw how Shou made revisions several times according to the suggestions from the educator and other peer teachers as well as his experiences of implementing the task in classroom lessons. The process of refining the task does impress me and I feel touched…I feel that I still have much room for the growth.

In addition to the novelty of creating an active-thinking task involving mathematics formula, Zhang was also inspired by the process of revising the task with educator and peer teachers. The observation motivated Zhang to make connection of mathematical formula with his personal teaching strategy of paper folding activity that creates opportunities for students to identify the existence of square root numbers.
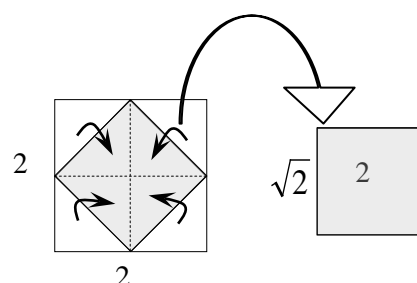

Figure 1: Paper folding activity

As shown in Figure 1, the activity involves the paper folding of a square with area 4 into a small square with area 2. The goal of the activity

is to present the existence of $\sqrt{2}$ which can be denoted as the side length of the small square with area 2. Zhang linked the paper folding activity with the observation in professional development and thought that the activity can be the materials to designing another task involving the formula for multiplication of square roots $\sqrt{a}\sqrt{b}=\sqrt{ab}$. As the design deals with $\sqrt{a}+\sqrt{a}=2\sqrt{a}=\sqrt{4}\sqrt{a}=\sqrt{4a}$, a special case for the formula, he anticipated that the task design can amend students' misconception $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$, a misconception that has been reported in literature (e.g., Hart, 1981). The task created by Zhang generally includes six parts. The first part aimed to scaffold students in defining $\sqrt{}$ by paper folding activity elaborated previously. Zhang expected students to recognize the existence of $\sqrt{2}$ and the notation $\sqrt{2}$ that can be represented as the side length for the square with area 2. The second part of the task further requires students to derive $\sqrt{8}$ based on the constructed square with area 2. Figure 2 shows four squares with area 2 can be constructed into a big square which area is 8. Then, the side length for the square with area 8 can be denoted as $\sqrt{8}$. Zhang expected that the visual diagram representations of the side lengths for the square with area 2 and that with area 8 allow students to recognize the relationships $\sqrt{2}+\sqrt{2}=\sqrt{8}$. The obtained relation $\sqrt{2}+\sqrt{2}=\sqrt{8}$ further offered students opportunity to reason $\sqrt{2}+\sqrt{2}=2\times\sqrt{2}=\sqrt{4}\sqrt{2}=\sqrt{8}$, a supportive example for the formula $\sqrt{a}\sqrt{b}=\sqrt{ab}$.

The third part of the task involved the activity that had students to experience $\sqrt{18}$ and $\sqrt{32}$ by observing big squares made by 9 small squares and 16 small squares with area 2 respectively. Zhang anticipated that the observation based on the side lengths and areas in the big squares can lead students to construct the relationship $\sqrt{2}+\sqrt{2}+\sqrt{2}=3\sqrt{2}=\sqrt{9}\sqrt{2}=\sqrt{18}$ and $\sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{2}=4\sqrt{2}=\sqrt{16}\sqrt{2}=\sqrt{32}$, respectively.
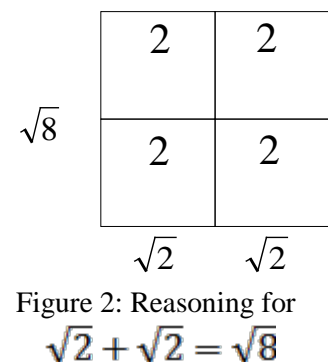

Figure 2: Reasoning for $\sqrt{2}+\sqrt{2}=\sqrt{8}$

Through the observation of the three examples, the fourth part of the task provided students opportunity to conjecture the pattern $\sqrt{a}\sqrt{2}=\sqrt{2a}$ where a is a square number. The five part of the task had to do with the generalization of the conjectured result $\sqrt{a}\sqrt{2}=\sqrt{2a}$ into $\sqrt{a}\sqrt{b}=\sqrt{ab}$ where a and b denote any positive numbers. Finally, Zhang used a variety of calculation items that allow students to become familiarized with the formula and mathematics concepts relevant to square roots (e.g., simplification of additions of square root numbers).

For coordination, what Zhang learned from the designing experiences was that his original instructional strategy can be used to create tasks involving mathematical formula that he has not known previously.

Coordination as integrating sources of information into the creation of novelty

Zhang enacted the designed task involving the formula in a regular class and the after-school class for three low-attaining students who also studied in Zhang's regular class. Zhang's instruction in regular class was more oriented to a teacher-leading instruction as he lectured the lessons and the students were listeners. Thus, his students in the regular class did not have many opportunities to express their thinking and learning difficulties. Consequently, Zhang anticipated that the designed task can scaffold students in successfully conjecturing the formula and doing the calculation items correctly.

But Zhang found that the original design of task could not amend students' misconceptions as he expected in after-school class for the three low-attaining students. The reason that he could perceive students' learning difficulty was due to his change of teaching style by frequently asking students to express their thinking in the lessons instead of directly telling them the mathematics. In this regard, students had more opportunities to actively engage in conjecturing the formula and express their mathematics thinking. The change of teaching in turn offered Zhang chances to notice the pedagogical problems and opportunity to challenge the problems.

In the teaching for the three low-attaining students in the after-school class, Zhang re-led the paper folding activity and expected the activity that can allow students to recognize relation $\sqrt{2} + \sqrt{2} = \sqrt{8}$ by observing side length between big square and small squares in the diagram representations. At the instructional moment, the students showed the agreement by replying that $\sqrt{2} + \sqrt{2} = \sqrt{8}$. However, when the students were asked to work on calculating item $\sqrt{2} + \sqrt{2} = \_\_\_$, they changed the answer to $\sqrt{4}$. Students' inconsistent responses created the pedagogical problem for Zhang. Zhang thought that the task arrangement could help students understand the $\sqrt{}$ concept and then recognize that $\sqrt{2} + \sqrt{2} = \sqrt{2+2} = \sqrt{4}$ was not correct. However, Zhang did not realize the fact that mathematical contradiction does not necessitate provoking students' cognitive conflict. Students did not feel conflict for the inconsistent answers between the observation derived from diagram representations and calculations with written symbols. In this regard, their misconception related to $\sqrt{2} + \sqrt{2}$ could not be amended by the observation of the relations $\sqrt{2} + \sqrt{2} = \sqrt{8}$ in diagram representation. Students still thought $\sqrt{2} + \sqrt{2} = \sqrt{4}$ when doing the calculation tasks and did not see the problems.

Consequently, Zhang decided to challenge his pedagogical problem. He tried to interview the students to understand the reasons for the inconsistent responses between diagram representations and the arithmetic calculations. The interview revealed that students have not viewed $\sqrt{}$ as an operator or a number yet, and thought that "$\sqrt{}$ *is a monster with a hat*". In this regard, similar to the error pattern (e.g., 2a+5=7a) reported by Hart (1981), when working on the arithmetic calculations,

students intended to ignore the notation $\sqrt{}$ and focus on the familiar numbers. For example, $\sqrt{2} + \sqrt{2}$ can be calculated as adding 2 by 2 first without the consideration of the square root notation and then put the square root notation back when obtaining the number 4. In other words, students have not developed the concept as $\sqrt{2}$ is a number.

Having recognized the underlying reasons for students' inconsistent answers, Zhang came up with a resolution according to his prior teaching experiences and the understanding of curriculum materials. He proposed the strategy by checking the square root table as $\sqrt{2} = 1.414$ to help students understand $\sqrt{2}$ is a number. When $\sqrt{2}$ becomes a number, students can further infer $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ on the basis of the calculations $1.414 + 1.414 = 2 \times 1.414$. This time, this proposed strategy did help students overcome their difficulty in inferring $\sqrt{2} + \sqrt{2} = \sqrt{8}$. Consequently, Zhang revised the task again by including an activity of calculating $\sqrt{2}$ by calculators before heading to the paper folding activity.

It is worthy of noting that the existence of researchers in the classroom teaching for the three low-attaining students may inform Zhang the importance of active thinking so that reinforced Zhang to not directly tell students the answers but had to come up with a plan to challenge the pedagogical problem. In this regard, Zhang had to generate a solution plan by coordinating active thinking, his understanding of curriculum materials, prior teaching experiences and students' learning difficulties.

DISCUSSION

The analysis of Zhang's task creation and the implementation of the task in classroom teaching reveals two types of coordination in terms of novelty of knowledge that can be constructed. The first type of coordination can be generally described as the connection between personal ideas and the ideas from others into the creation of the task. The way Zhang created the task was based on coordinating the idea from Shou as mathematical formula can be the materials to enhance student active thinking and his prior strategy by using operating activity to motivate students to learn mathematics. The coordination allowed Zhang to recognize his prior instruction strategy can be extended to another mathematics topic that he has not considered previously. For the second type of coordination, it occurred when Zhang challenged his pedagogical problem. In order to solve the pedagogical problem in classroom, Zhang had to clarify the reasons for why students' misconceptions could not be amended by the original plan. Then, Zhang coordinated active thinking, his prior teaching experiences and the understanding of curriculum materials to solve the problem. The novelty constructed through the coordination is the knowledge as the concept of $\sqrt{2}$ as an operator and a number that should be established before working on the formula for multiplication of square root. Particular attention should be given to the process of identifying students' misconceptions that could not be amended by the original instructional plan and conducting the follow-up interviews to probe the underlying reasons. This process

allowed Zhang to generate an efficient and effective plan to challenge the pedagogical problem.

As coordination mechanism clarifies what sources of information can be transformed and coordinated into the construction of novelty, it possesses the potential to describe different types of learning trajectories. Taking Zhang as an example, his growth in profession by coordination largely occurred in his teaching practices as he noticed students' learning difficulties in his teaching and coordinated a solution plan to solve the pedagogical problem. The learning from professional development functions as a guideline reminding him the importance of active thinking. In this regard, the way Zhang made growth in profession can be described as a pragmatic-oriented trajectory. But alternative coordination trajectory to the professional growth can be the heavy scaffold provided by professional development as the educator offers concrete suggestions that allow teachers to back and forth transform and coordinate the experiences between professional development and teaching environment to enrich their pedagogical power.

While coordination emphasizes the transformation and integration of information sources experienced from different learning environment, the mechanism can play a central role in helping teachers bridge the gap between research/literature and practice. Specifically, recognizing that the knowledge transferring process is complex, cyclic and sophisticated, involving the interplays with multiple tiers of participants (Lesh et al., 2007), coordination mechanism creates a way to investigate the bridging process as how teachers transform and coordinate their learning in professional development to solve the pedagogical challenges encountered in their classroom.

Additionally, coordination also offers the opportunities to understand how teachers perceive, evaluate, and then select the sources of information that they want to transform and coordinate for self-growth. The process of perceiving, evaluating and then selecting the sources of information for coordination are subjective and involves personal learning preferences. While a number of participating teachers presented their task designs in professional development, the selection of Shou's designing experiences into the task creation is very likely made based on the evaluation in terms of what materials that are suitable for his teaching. In this regard, he is not passive receiver but an active interpreter and constructor as his intension determines what he wants to learn from professional development environment.

As coordination itself is also a reflective action (Piaget, 1970), using coordination as a mechanism to examine teachers' professional growth is in line with the research inquiry in reflection and enactment. However, as coordination focuses on how different sources of information can be transformed and coordinated to produce new knowledge for teaching, the point that seems not to be well taken in reflection and enactment research, it offers potential to outline teachers' professional growth in an alternative and detailed way.

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INVESTIGATING ENGINEERING STUDENTS' MATHEMATICAL MODELING COMPETENCY FROM A MODELING PERSPECTIVE

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This study investigates university students' modeling process in one modeling activity. The data is collected by students' individual and group written responses to the mathematical modeling activity, video-taped group discussions and classroom observation by the researcher. The data showed that university students have difficulty in transition between different modes of mathematical representations and the classifications of variables/parameters as known or unknown, implicit or explicit, independent or dependent variables.

INTRODUCTION

Over the past ten years, it has become increasingly important to apply mathematics to other subjects, including engineering, nanotechnology, economics, and biology. Many educators and researchers in mathematics education believe that this should be reflected in the classroom via mathematical modeling activities. Students should be availed with tools in addition to school mathematics, and allowed to glimpse real-world mathematics outside the classroom. The use of models and modeling in enhancing the instruction and learning of mathematics is an indispensable means of cultivating students' mathematical literacy, which they need in the new era of technology (Burkhardt, 2006; English, 1999; Lesh & Doerr, 2003). Several factors, such as entrance examinations and existing teaching materials, fail to create a favorable environment for mathematical modeling in the current mathematics education situation of Taiwan. However, university students have less academic pressure, and calculus is a fundamental course in college-level mathematics and engineering education. Students need to understand the concepts of calculus and be able to apply them. For engineering students, calculus is not only a specialized subject, but also knowledge that they will need in their future workplaces. Thus, integrating modeling activities into calculus courses is a proper approach to implement mathematical modeling instruction.

This study aims to design mathematical modeling activities, based on models and modeling perspectives and embedded into calculus courses, to develop students' mathematical modeling competency. Teaching experiments in this study used the island approach proposed by Blum and Niss (1991) to integrate model-based teaching activities into formal activities for teaching calculus, and is used to avoid resistance from students who are used to traditional teaching. The ultimate purpose of the teaching experiments is to foster students' modeling competency through a modeling process. By implementing such teaching experiments, we investigate the

mathematical modeling process and competency of first year engineering students, which can be used as a reference for designing activities for teaching mathematical modeling to college students.

Theoretical Framework

Mathematical modeling instruction aims to support students in learning mathematics. Through modeling, mathematics can be used to describe, understand, and predict real-world situations. Hence, mathematical modeling can help students gain external mathematical experience, and create the connections between mathematical concepts involved in modeling activities. Mathematical modeling involves multiple processes such as mathematization, interpretation, communication, and even application (Lesh & Doerr, 2003; Maaß, 2006). Unlike traditional problem solving, which focuses only on the representation of mathematical problems and solutions, mathematical modeling focuses on converting and interpreting contextual information, identifying potential problems, establishing models, and reinterpreting the premise, hypothesis, and possible errors of mathematical solutions. These processes are normally described in the form of stages. By following these processes, students can constantly refine and develop their mathematical models in a circular manner. Moreover, students need to be able to engage in mental activities when moving from one stage to another during the modeling process.

Competency indicates that individuals are able to make relevant decisions and implement proper actions in a real-world situation. These decisions and actions are essential for individuals in successfully handling real situations. As Blum and Leiß (2006) indicated, if teaching and learning are emphasized simultaneously, an individual-oriented perspective on problem solving is necessary to better understand what students do when solving modeling problems, and to provide a better foundation for the diagnosis and involvement of educators. This study adopts the modeling cycle proposed by Galbraith and Stillman (2006) as a research framework to investigate the mathematical modeling process and mathematical competency of first year engineering students.

Description of this Study

This study adopts an interpretative orientation based on anti-positivism (Cohen , Manion, & Morrison, 2000, p. 22), regarding case studies as a research strategy for closely examining the modeling process of students. The mathematical modeling problem of this study is as follows:

> A company is carrying out a cost-cutting exercise and requires your help with an investigation into how it can reduce its transport costs. The company employs a number of drivers who cover a substantial amount of mileage every day. There has recently been a large increase in their fuel costs and drivers can achieve a higher rate of miles per gallon from their vehicles by driving at a lower speed. This, however, increases journey times and the cost of the driver's time.

The data reported in this study was gathered from three calculus classes. The entire process of each class, which lasted for 70 min, was recorded and videotaped. The

subjects in this study consisted of 54 first year engineering students of a university. These students were divided into ten groups. Each group included five to six students. The researcher observed and videotaped three groups, including Groups A, D, and F. After each class, the researcher had a retrospective interview with these three groups to understand their real intentions. The researcher showed a certain group of students a video regarding their situation in class and asked them to explain their behavior in detail by asking questions, such as "How did you propose your ideas at this point?". This artical reported the modeling process of the six students in Group D. The grades of these students in calculus were approximately average. Sam, as an instructor in the teaching experiments, had 15 years experience teaching calculus in a university. He was willing to participate in this study because of his interest in fostering students' mathematical thinking through modeling.

Major Activities of Each Class

In the first class, Sam first posed a math problem, and students had class discussions and worked on their own. After some students posed their questions, others could express their opinions. Sam guided or instructed students to implement reflection by asking students several questions such as "Why?", "Then?", and "How will you do it?". Then students had group discussions and tried to understand, structure, and simplify a real-world situation and convert it into a problem statement. During this process, students gradually discovered and verified several keywords in the problem statement, and were encouraged to convert a real-world problem into a problem statement based on keywords. In the second class, students needed to simply or structure a problem situation based on a problem statement, and to further generate real models. Students first held group discussions for 30 min. They were encouraged to reexamine a problem statement, and asked to create variables, parameters, constants, and symbolic representations based on keywords. In the second stage, Groups A, D, H, and G were invited to share their reports, and held class discussions. During the discussions, Sam occasionally asked students questions such as "Why did you think in this way?", and "Do you think this is the best way?". These questions provided students with material for discussion and enabled students to perceive the importance of examining arguments. Sam also took this opportunity to explain the similarities and differences between variables, parameters, and constants. In the third class, students first used known mathematical knowledge to solve mathematical models on their own, and then held group discussions to interpret mathematical solutions as real results.

RESULTS

Figure 1 shows the mathematical modeling competencies in each transition (cognitive activity) that were identified in this implementation of the task. Each element has two parts where key (generic) categories in the transitions between phases of the modeling cycle are indicated (in regular type), and illustrated (in capitals) with reference to the task. Evidence for selected examples of these activities is presented in the analysis of transitions that follows.

Real-world situations → Real-world models. In the first stage, students verified certain keywords in a problem statement, including the driving speed, costs in terms of diesel fuel and driver salary, and transportation costs. There was a mathematical observation during the inquiry process. Inquiry and mathematical observation allowed students to surpass their preconceived opinions of real-world situations, especially when students had talks with others in group discussions. For example, "The truck travels at a constant speed, ignoring traffic lights and jams" was the important concept that was simplified in group discussions.

1.*Real- world situations → Real- world models* (understand, simplify, and interpret context)

1.1 Clarify the context of problems【DRIVE THE OPTIMAL DRIVING SPEED FOR A TRUCK TO MINIMIZE TRANSPORTATION COSTS UNDER CONSIDERATIONS OF THE COST OF DIESEL FUEL AND THE DIRVER'S SALARY】

1.2 Simplify hypotheses【A TRUCK TRAVELS AT A CONSTANT SPEED, IGNORING TRAFFIC LIGHTS AND JAMS】

2.*Real - world models → Mathematical models* (CONSTRUCT HYPOTHESIS AND MATHEMATIZATION)

2.1 Verify variables and parameters【THE DISTANCE DRIVEN BY A TRUCK IS A PARAMETER】

2.2 Use graphical representation 【THE RELATIONSHIP BETWEEN THE SPEED OF A TRUCK AND THE NUMBER OF km/L OF DIESEL FUEL THE TRUCK CAN GET】

2.3 Use situational elements of graphical representation 【USE THE SYMBOL "g" TO REPRESENT NUMBER OF km/L OF DIESEL FUEL THE TRUCK CAN GET 】

2.4 Construct relevant hypotheses【THE RELATIONSHIP BETWEEN THE SPEED OF A TRUCK AND THE NUMBER OF km/L OF DIESEL FUEL THE TRUCK CAN GET】

2.5 Use mathematical knowledge appropriately 【WRITE OUT THE FUNCTION OF TRANSPORTATION COSTS】

3. *Mathematical models → Mathematical solutions* (OPERATE MATHEMATICALLY)

3.1 Representational change 【CONVERT THE RELATIONSHIP BETWEEN THE SPEED OF A TRUCK AND THE NUMBER OF km/L OF DIESEL FUEL THE TRUCK CAN GET INTO AN ALGEBRAIC EXPRESSION】

3.2 Analyse【VERIFY DIFFERENTIAL VARIABLES】

3.3 Apply the concept of derivatives【USE FIRST-ORDER DERIVATIVES TO SEEK EXTREMA】

3.4 Understand the meaning of parameters 【MATHEMATICAL SOLUTIONS INCLUDE PARAMETERS】

4.*Mathematical solutions → Real-world meaning of solutions* (INTERPRET MATHEMATICAL RESULTS)

4.1 Verify mathematical results based on real-world situations【THE RATIONALITY OF $f/w = 0.5$】

4.2 Integrate arguments to verify interpretational results【THE RANGE OF f/w VALUE】

Figure 1. Framework showing transitions and mathematical modeling competencies in the implementation of transportation costs activity

Then, students verified the variables and limitations in the situation to investigate the key factors influencing transportation costs. For instance, John suggested that the cost of diesel is inversely proportional to transportation time. However, Mary had a

different opinion, in that speed is not necessarily inversely proportional to the number of km/L of diesel a truck can get. After group discussions, relevant factors were listed as below: the factors related to driving distance, the factors related to the truck. The preceding has demonstrated that all these students could successfully generate a problem statement. Under considerations of the cost of diesel and of the driver, the students needed to derive the optimal driving speed for the truck to minimize transportation costs.

Real-world models → Mathematical models. In the second stage, students engaged in the work of mathematization. Students encountered the most difficulty and spent most of their time on this stage. The difficulty lied in creating mathematical properties corresponding to situational conditions and hypotheses. It seemed rather important to provide these factors; for instance, "What are parameters?" and "What are variables?". This is also a very important process in mathematical modeling activities.

All the students in Group D had the same hypotheses on the price per L of diesel, the hourly rate of a driver, and the number of km a truck travels. However, their hypotheses on the speed of the truck and the number of km/L of diesel a truck can get were slightly different. Someone hypothesized that a truck can run 40 km at a speed of 20 km/h, and that for every 20 km/h increase in speed, the number of km/L of diesel a truck can get would be reduced by 10 km. Someone hypothesized that a truck can drive for 1 h on 1 L of diesel at a speed of 50 km/h, and that an increase in speed of 5 km/h would reduce the driving time by 0.2 h. Obviously, there was a significant difference between student performance and the meaning and purpose of mathematical modeling. Other groups of students had similar problems; thus, Sam re-explained the meaning of mathematical models, and encouraged students to hypothesize more parameters and variables.

Group D first discussed whether several keywords in the problem were parameters or variables. Gray argued that since the number of km/L of diesel a truck can get may vary with the speed of the truck, that this was a variable. Tom suggested that because the company does not often adjust salaries, the hourly rate for a truck driver can be considered as a parameter instead of a variable. Regarding the cost of diesel, in John's opinion, even though the price of diesel may be adjusted every week, the adjustments were relatively modest. Sometimes the price rises by two cents and falls by two cents afterward, meaning that the price does not change. Therefore, the price of diesel can be hypothesized as a parameter. After a 20 min discussion, Group D reached the consensus that the heart of the question is the speed; any values directly related to the speed are variables, and the rest are parameters. Therefore, all the students finished assigning symbols to represent all the parameters and variables.

The second stage of mathematization was to construct hypotheses. A less controversial hypothesis was that a truck travels at a constant speed, ignoring traffic lights and jams. A more controversial hypothesis was the relationship between the speed of the truck and number of km/L of diesel that the truck can get, which was the

most important hypothesis in this modeling activity. When working on their own, most students in Group D hypothesized that the speed of a truck is inversely proportional to the number km/L of diesel a truck can get. Such a hypothesis differed from real-world situations. Only Gray and Roberto put forward different opinions. Both of them suggested that the faster the speed, the more fuel that can be saved, based on their experiences with riding a scooter.

Hence, the students in Group D decided to investigate the data on their own regarding the speed of a truck and number of km/L of diesel a truck can get, which could serve as a basis for the subsequent discussion. These students tried to identify the relationship between the speed of a truck and number of km/L of diesel a truck can get by gathering relevant data from the Internet and dealer websites, and by calling car dealerships. The relationship between the speed of a truck and number of km/L of diesel it can get was demonstrated by the students not to be a simple linear relationship. For instance, John divided the speed of a truck into two different ranges, from 0 to 50 and from 50 to 80 km/h, and drew two line segments connecting two specific points, (0,0) and (50,8), and (50,8) and (80,10), serving as the hypothesized graphical representations. After discussion, Group D hypothesized that the maximum speed was 100 km/h; beginning with 0 km/h, the number of km/L of diesel a truck can get steadily increases with increases in speed; at a speed of 60 km/h, a truck can travel 12 km on 1 L of diesel. They also hypothesized that after a truck goes over 60 km/h, the number of km/L of diesel it can get steadily decreases; at a speed of 100 km/h, a truck can travel 8 km on 1 L of diesel. Moreover, they represented their hypotheses with graphics.

After finishing its hypothesized graphical representations, Group D began with converting key factors into mathematical representations, including the traveling time ($d\,/\,s$), cost of a driver (wd/s), and cost of diesel (fd/g), and created the function of transportation costs: $C(x) = wd/s + fd/g$. At this point, Group D finished its mathematical models, including the algebraic expression of the cost function and a graphical representation of the number of km/L of diesel a truck can get.

Mathematical models → Mathematical solutions. In this stage, known mathematical knowledge was used to solve mathematical models. Because the form of cost functions differed from the functions with which students dealt previously, the first problem students encountered was that they did not know which variable should be differentiated. The next difficulty was how to use "g" to differentiate "s". Students demonstrated the relationship between s and g using graphical representations; thus, they had difficulty implementing differential operations. After a 5 min discussion, they were still unable to find a solution. Therefore, Sam suggested that the students convert graphics into functional forms by using the combination of an equation of two straight lines. After about 5 min, Group D represented the relationship between s and g with an algebraic expression:

$$g = \begin{cases} s/5, & 0 \le s \le 60 \\ -0.1s + 18, & 60 \le s \le 100 \end{cases}$$

Finally, they used the concept of first-order derivatives to determine a mathematical solution. However, students wondered why the solution for the optimal speed contained parameters, because they thought that the form of an answer should be a number, based on their previous learning experiences. Sam discovered that most students had questions about the form of the mathematical solution. Hence, in class discussions, he explained the purpose and meaning of mathematical modeling and the meaning of a mathematical solution that included parameters.

Mathematical solutions → Real-world meaning of solutions. In this stage, students interpreted mathematical solutions as real results. In the last stage, Group D discovered that the optimal speed was correlated with the value of f/w. They also understood that they needed to propose possible results based on the value of f/w. Tom suggested that when $f/w = 0.5$, the optimal speed and lowest transportation costs can be obtained. However, he did not consider the hourly rate of the driver to be unreasonable, which was only approximately 60 NTD when $f/w = 0.5$. There were three students suggested that f/w should be 0.4 when the price per liter of diesel was 29 or 30 NTD. There were two students listed several solutions such as $f/w = 0.2$, $s = 74.6$; when $f/w = 0.3$, $s = 65.9$; and when $f/w = 0.4$,; when $f/w = 0.5$, $s = 55.6$. They did not provide a single solution; instead, provided clients with the suggestion that the value of f/w be between 0.3 and 0.4.

CONCLUSIONS

This study emphasizes student reflection on mathematical understanding, mathematization and analysis of mathematical systems, and the interpretation of results to a given real-world problem, providing students with an opportunity to learn mathematics in a different way. This study replaced extrema problems in calculus courses with the mathematical modeling activities of reducing transportation costs. Through mathematical modeling instruction, students can gradually develop their mathematical modeling competency by working on their own and through discussion with their peers. The analysis results of research data show that a fundamental and important problem encountered by students is their failure to recognize variables and parameters; and whether these values are known or unknown, obscure or clear, or independent or related. Without this fundamental knowledge, students may have difficulty engaging in mathematical modeling activities, especially during the process of mathematization. Therefore, the insufficient ability of students to categorize variables and parameters should not be ignored. Educators should help students in establishing useful relationships required by mathematical problems. Another obvious problem is representational change. Representational tools and systems such as tables, graphics, and drawings are important parts of the mathematical modeling process. This research result accords with the argument emphasized by Lesh and Doerr (2003): that representational fluency plays an important role in the model-documentation principle of mathematical modeling.

The study results verify the cognitive activities in which students engaged and the competency required during the mathematical modeling process of solving the problem of reducing transportation costs. Researchers and teachers can use the transitional framework proposed in this study to verify whether students have the specific abilities necessary to successfully finish particular mathematical modeling problems. Teachers who want to implement mathematical modeling instruction can also use the transitional framework to ensure that students are able to develop their mathematical modeling competency, even though not every modeling activity includes every element. From the perspective of student learning, the transitional framework can be used to predict difficulties that students may encounter.

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AN EXPLORATION OF COMPUTER-BASED CURRICULA FOR TEACHING CHILDREN VOLUME MEASUREMENT CONCEPTS

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This study examines the effectiveness of two sets of computer-based curricula for teaching volume measurement concepts to fifth-grade children. Findings show that curricula that involve a set of core concepts of volume measurement and physical manipulations by means of guided question-and-answer instruction facilitate children's acquisition of volume measurement concepts. Moreover, children are likely to show gains in test scores in explaining mathematical thinking for volume measurement if they have been exposed to an enriched curriculum that integrates more geometric concepts with volume measurement.

INTRODUCTION

Volume, which involves the spatial organisation of quantities, is the measure of a three-dimensional (3-D) space. Understanding the set of core concepts for volume measurement is a general goal for fifth- and sixth-grade students, including units of measure and their coordination in three dimensions, and the volume formulas for volume measure of a rectangular prism (Taiwan Ministry of Education [TME], 2010; National Council of Teachers of Mathematics [NCTM], 2000).

An understanding of 3-D measurements requires plenty of physical-manipulation experience with concrete objects for 1-, 2-, and 3-D attributes, and a greater degree of mental adeptness in terms of manipulating 2-D and 3-D geometric shapes (Ben-Haim, Lappan, & Houang, 1985). Seeing that the properties and feature of volume are more complex than those of linear and area measurements, mathematics educators have suggested that providing learning activities that encourage students to engage in constructing connections between number and geometry should be taken into account for developing conceptual understanding of volume concepts (Owens & Outhred, 2006).

In more recent studies, Huang and Witz (2011) and Huang (2011) argued that connecting geometry with area measurement and numerical calculations may facilitate students' conceptual understanding of area measurement and their ability to solve area measurement problems. In those studies, children who were exposed to a context that integrated 2-D geometric knowledge with area measurement appeared to show higher competency in explaining mathematical thinking about area measurement in tasks for which conceptual understanding was required. The current study aims to extend ideas about curriculum design that integrate concepts of measurement underlying spatially-organized quantities with geometry to develop instructional activities for the purpose of teaching children volume measurement.

Furthermore, for developing students' conceptual understanding of mathematical knowledge, in addition to physical manipulations, it is suggested that teachers take advantage of utilizing appropriate dynamic computer environments (TME, 2010; NCTM, 2000). In the current study, the sets of instructional treatments that employ computer softwares as tools have been designed to help students comprehend concepts of volume measurement and the formula, Volume (V) = Length (L) x Width (W) x Height (H) [or Volume (V) = Base (B) x Height (H)]. Moreover, the instructional approach, guided question-and-answer instruction, was adopted in accordance with Huang and Witz (2011) and Huang (2011).

In sum, the present study aims at describing ways of improving fifth-grade children construct knowledge of volume measurement and how to use that knowledge as they solve volume measurement problems. The study therefore addresses two questions as listed below.

1. What is the influence of the two instructional treatments on developing children's ability to solve volume measurement problems?
2. What is the influence of the two instructional treatments on developing children's ability to solve volume measurement problems that demand mathematical explanations?

THEORETICAL FRAMEWORK

Generally, middle-and-upper grade students are taught volume measurement by means of a series of instructional activities, including direct measurement, which involves stacking Cuisenaire cubes and counting the number of the cubes constructed in a solid, followed by measuring each length of the three dimensions of rectangular prisms (e.g., Nan I Publications, 2008; The University of Chicago School Mathematics Project, [UCSMP], 2002). Later, fifth-grade students are expected to develop the formula, L x W x H = V, based on their prior knowledge of linear and area measurement (TME, 2010).

When children learn the volume formula of a rectangular prism, they need to see the layer of cubes and count the number of cubes constructed in that layer, and then use multiplication to find the number of cubes needed to a prism. Seeing the layer of cubes requires processes of conceptualizing the structure of units that are obtained from adequate experiences of stacking cubes, as well as discussion about the structure of rectangular arrays and layers of cubes (UCSMP, 2002; Owens & Outhred, 2006). However, children frequently encounter difficulties in conceptual understanding of the volume formula and its related concepts of volume measurement. These obstacles include a failure to see the hidden portion of the pictorially-presented rectangular prisms on a plain text (Ben-Haim et al., 1985), in addition to memorizing the formula without understanding its meaning (Shieh, 2011). These difficulties may result from a lack of spatial visualization activities (Ben-Haim et al., 1985). In particular, children only focus on solving problems that require calculations with representations of 2-D or 3-D figures, which are provided on plain materials (e.g., paper and worksheets). If a

child does not have sufficient experiences in constructing a rectangular prism, s/he is less likely to see the relevance of concepts of volume to the idea that volumes of cuboids can be determined by using the volume formula of a rectangular prism, L x W x H. Therefore, providing instruction in spatial visualization activities should be taken into account while teaching volume concepts.

For instruction of concepts of volume measurement, using physical manipulations (e.g., using concrete cubes to stack and construct solids and measuring the lengths of three dimensions) and technologies as instructional tools seem promising (Huang & Crockett, 2008; Shieh, 2011). The concrete manipulations of stacking cubic units may promote children's construction of a 3-D array of cubes as a series of layers. This method of spatial construction facilitates children's understanding of the formula for measuring the volume of a rectangular prism. At the same time, the processes in terms of identifying attributes of 1-, 2-, and 3-D figures and measuring the length of each dimension of the prism that is constructed help children grasp the idea of dimension, which is an essential concept of volume (UCSMP, 2002).

In addition to physical manipulations, a growing body of research suggests that using dynamic geometry technology (or softwares) effectively helps students develop the generalisation of spatial structures (Owens & Outhred, 2006)— for example, a dynamic PowerPoint® format which is a prevailing program used for demonstrating teaching problems and discussing 3-D figures (Huang & Crockett, 2008). Moreover, Cabri 3D is also recommended as a computer software program that helps students construct their own concepts of geometry (Kordaki & Balomenou, 2006). In particular, one of the strengths of Cabri 3D is the availability of numerical and figural cues, which illustrate the transformation between 2- and 3-D objects (Shieh, 2011). Thus, in the current study, Cabri-3D was used as a tool to link with the instructional materials, which were designed in PowerPoint® format, for the purpose of helping students visualize the 2-D representation of solids and the transformation between 2- and 3-D figures.

As to the instructional approach, the guided question-and-answer instructional approach was adopted in accordance with the studies of Huang's, as mentioned above. The features of this instruction include providing a learning environment in which stress is placed upon measuring objects, representing and communicating the results of measurement, reasoning about evidence and explanation, evaluating measurement claims, and clarifying mathematical thinking of measurement.

Considering that volume measurement and the volume formula, $V = L \times W \times H$, is a conceptual domain of learning that may be taught efficiently if these concepts can be developed within this particular context, in which a set of core concepts of volume measurement is provided along with physical manipulations and discussions about how to see the layer of cubes. Moreover, for solving problems embedded with concepts of volume measurement and problems that required mathematical explanations, it is assumed that children receive more benefit from receiving an

enriched curriculum, one which integrates Geometry with Volume Measurement (GVM), than one that employs a regular Volume Measurement curriculum (VM), which stresses numerical calculations and application of the formula, V = L x W x H, for volume measurement.

Additionally, the effectiveness of the two instructional treatments (GVM and VM) was examined in the study. The considerations that a condition in which children were provided geometry without concepts of measurement was not taken into account included two aspects. First, according to the studies of Huang and Witz (2011) and Huang (2011), knowledge of 2-D geometry is necessary but still inadequate for enhancing children's competency in area measurement. That is, providing a curriculum that solely highlights geometry fails to improve children's performance in solving area measurement problems. Second, since volume measurement is more complex than area measurement, a curriculum that attempts to help children solve problems embedded with concepts of volume measurement should include the concepts and skills of numerical computations for measurement.

METHOD

This study began by analysing the elements of essential mathematical subject matter that underlies volume measurement, and then proceeded to develop two computer-based curricula, GVM and VM, for providing children various activities that consist of the core concepts of volume measurement, based on a theoretical framework and curriculum guidelines (e.g., TME, 2010). Moreover, a quasi-experimental design was used to examine the effectiveness of these two instructional treatments on children's ability to solve volume measurement problems. Each curriculum was conducted in five class periods using guided question-and-answer instruction. Each class period lasted about 40 minutes. The children's learning was assessed prior to and after these instructional treatments.

Participant

Fifty-one fifth-grade children, 29 boys and 22 girls, were recruited from two classes in a public elementary school that serves middle-class communities in Taipei, Taiwan. The children were 10.64 years old (M = 127.67 months, SD = 3.11). A *t*-test revealed no significant differences among the two classes for mathematics achievement scores from the semester prior to the experiment, t (49) = -.49, p = .62. All the children had been exposed to knowledge about standard volume unit, 1 cubic centimetre and the properties of cubes and rectangular prisms before participating in these treatments.

Material

The set of teaching problems used in the two instructional treatments and the questions used in the pre-learning assessment and post-learning assessment were referenced from the materials developed by Huang and Crockett (2008) and Shieh (2011). The mathematical concepts underlying the teaching problems included eight mathematical subject-matter elements in different combinations for the two sets of teaching

problems. The subject matter elements were (A1) Introduction of the standard unit, a cubic unit and its volume (1 cubic centimetre, cm^3); (A2) Geometric properties of a 2-D shape and a solid, transformation of 2- and 3-D figures, as well as differences between measuring area and volume; (B1) Direct and indirect comparisons of the size of various solids; (B2) Use of centimetre cubes to build rectangular prisms and measure the dimensions (length, width, and height) of a rectangular prism, and report the measures of the volume of the prisms; (B3) Use of centimetre cubes to build irregular solids and report the measures of the volume of the solids; (B4) The structure of rectangular arrays and layer of cubes in a rectangular prism as illustrated by means of animations on a computer; (B5) Discovery of the volume formula of a rectangular prism and its meaning by using cubic blocks; (B6) Use of multiplication to express the volume of a rectangular prism and numerical calculations. The specific features of each computer-based curriculum were presented as follows.

(a) The VM curriculum, which stresses using physical manipulations (stacking units), the volume formula of a rectangular prism, and numerical calculations for volume measurement, was composed of 26 teaching problems. The mathematical subject elements embedded in the teaching problems included A1, B1, B2, B3, B5, and B6.

(b) The GVM curriculum, which is a spatial-and-volume-measurement connection curriculum, involves examining geometric attributes of 2-D and 3-D figures, visualizing structures in two- and three-dimensions by means of the dynamic geometric software (Cabric 3D), and considering the concepts of volume measurement. This curriculum was composed of 24 teaching problems embedded with A1, A2, B1, B2, B4, B5, and B6.

Compared to the GVM curriculum, the VM curriculum involves more activities of physical manipulations of stacking units and measuring the lengths of the three dimensions of the solids constructed, in addition to application of the volume formula. Also, such physical manipulations as provided in the VM curriculum, exceeded those provided in the unit of volume measurement of current mathematics textbooks that had been adopted in elementary schools in Taiwan (e.g., Nan I Publications, 2008). Conversely, in the GVM curriculum, knowledge of geometry and spatial visualization related to the spatial-organization of volume measure was demonstrated primarily by the use of dynamic geometric figures.

There were three types of problems, all of which required different levels of thinking and responses, as contained in the blocks of problems presented. (a) Calculation (CAL) problems. The CAL problems could be solved by means of either counting the grids provided with the figure or by directly applying the formula for volume measurement of a rectangular prism and computations. According to Kenney and Lindquist (2000), counting and doing simple computations require lower-level conceptual understanding. (b) Mathematical judgement (MJ) problems. These problems were short constructed-response items that required judgement for the accuracy of a

solution statement regarding volume measurement. (c) Explanation (EXP) problems. These problems required a written explanation of the reason for justifying the judgement given to the corresponding MJ problem or of explaining their reasons for the way they solved the problems. The ability to explain the mathematical reasoning while solving problems represents high-order mathematical thinking (Kenney & Lindquist, 2000). Thus, EXP problems can be applied to evaluating students' conceptual understanding.

All three types of problems were included in the pre-learning assessment and post-learning assessment. Each assessment contained 10 CAL, 3 MJ, and 7 EXP items. The rubric schemes for the CAL, MJ, and EXP problems were refer to in the context of previous studies by Huang and Witz (2011) and Huang (2011). The maximum total score of each assessment was 46 points. In each assessment, the maximum total score of EXP problems was 14 points. The coefficient of equivalence of the pre-learning assessment and post-learning assessment was .79, $p < .001$.

RESULTS AND DISCUSSION

To determine whether children benefited from the instructional treatment that they received, a 2 (groups: GVM, VM) x 2 (problem-solving phase: pre-learning, post-learning) analysis of variance (ANOVA), with problem-solving phase as the repeated factor was conducted for analysing the total scores and EXP scores, respectively. Table 1 shows descriptive statistics for the analyses in terms of Means of the total scores and scores of the EXP problems for the two groups at two problem-solving phases (pre-learning and post-learning assessments).

Table 1. Means and SD of the pre-learning assessment and post-learning assessment total scores, and scores of the EXP problems, by instructional treatment

Group	n	Pre-learning assessment M (SD)	Post-learning assessment M (SD)
Total Scores			
GVM	26	21.20 (11.35)	31.17 (10.41)
VM	25	20.67 (9.49)	29.64 (8.09)
EXP problems			
GVM	26	4.37 (2.96)	7.14 (2.70)
VM	25	4.08 (2.29)	5.06 (1.78)

For the total scores, the results shown that there was no main effect for group (GVM, VM), $F (1, 49) = .17$, $p = .68$, nor an interaction effect between the group and problem-solving phase, $F (1, 49) = .17$, $p = .68$. There was a significant main effect for problem-solving phases, $F (1, 49) = 59.85$, $p < .001$, suggesting that both groups obtained score gains in the post-learning phase.

Using repeated measures ANOVA for the EXP scores, the analysis yielded a significant interaction effect between the group and problem-solving phase, $F (1, 49) = 8.22$, $p < .01$, and a significant main effect for the problem-solving phase, $F (1, 49) = 36.07$, $p < .001$, whereas the main effect for the group reached marginal significance,

F (1, 49) = 3.62, p = .06. Results of the follow-up analysis of the interaction are presented as follows: a. For the GVM group, the children showed increased scores between the pre-learning phase and the post-learning phase, F (1, 49) = 40.15, $p < .001$. Similarly, the children in the VM group obtained increased score gains between the two problem-solving phases, F (1, 49) = 4.83, $p < .05$. b. The significant difference in children's performance in solving volume measurement problems between the two groups was shown at the post-learning phase, F (1, 98) = 8.92, $p < .01$ rather than in the pre-learning phase, F (1, 98) = .17, $p = .68$.

In sum, the results revealed that both the GVM and VM instructional treatments facilitated the children's acquisition of the idea of volume measurement and their ability to solve different types of problems embedded with volume measurement concepts. The findings show that overall in the post-learning assessment, the children in the GVM group performed equally well with those in the VM group. Moreover, this finding shows that children obtained increased scores in solving EXP problems that require a conceptual understanding of volume measurement— if the instructional treatment is one that highlighted spatial geometry integrated with volume measurement, though the differences between the two instructional treatments only reached marginal significance. These findings are consistent with those findings of Huang and Witz's (2011) and Huang's (2011) studies in the domain of area measurement. It suggests that children can benefit from instructional treatments as long as activities involve essential concepts of volume measurement and are conducted by the guided question-and-answer approach— for example, physical manipulations and discussion of the concepts of layering, which is the heart of volume measurement (UCSMP, 2002). Moreover, it also supports Owens and Outhred's (2006) perspective that more spatial relationships, integrated with measurement, potentially enhance children's conceptual understanding of approaches to measurement that are related to spatially-organized quantities.

Although the current findings did not strongly support the hypotheses that the enriched curriculum (GVM) is more effective than the regular curriculum (VM) in promoting children's overall performance in solving volume measurement problems and EXP problems, the enriched curriculum nevertheless seems to be a promising approach. Further studies are needed to explore how fifth-grade children construct their understanding from 3-D figures which represents spatial relationships in the case of dynamic geometry figures shown on a computer screen. Moreover, how children construct the relationships within and between different representations needs further studies— for example, connecting 2-D and 3-D geometric representations with their physical manipulations to develop an understanding of layering in the 3-D structure of units.

Implications of the study

This study represents one step in investigating the effectiveness of a curriculum that incorporates more geometry with volume measurement, from building with cube units

to computer-based instruction to develop children's conceptual understanding of the idea of volume. The evidence presented in the current study explicitly suggests that instructional activities for volume measurement, which consist of a set of core concepts of volume based on physical manipulations and discussions of layering, are valuable for constructing spatial relationships and numbers.

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A PRELIMINARY STUDY ON THE INSTRUCTIONAL LANGUAGE USE IN FIFTH-GRADE MATHEMATICS CLASS UNDER MULTI-CULTURAL CONTEXTS

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This study aimed at mathematics instructional language culture of a teacher and explored occurrence of instructional language use, narration, and communication. Naturalistic observations were used to explore the practice of an elementary school teacher in mathematics class, when he faced students in multi-cultural contexts with local characteristics. Data source are video, classroom observations notes and interviews; they were analyzed qualitatively. *It was found that the teacher did not believe that there was a need to attend to such cultural differences. His observed language culture use might result in negative effects on students' mathematics learning. Specific teachers' utterances that deserved attention were reported, together with recommendations for further studies.*

INTRODUCTION

Mathematics has been considered as the basic science and an international course with the learning effects on individuals, societies, and countries (Nath, & Vineesha, 2009). Nonetheless, Swetz (2009) indicated that mathematics learning was not the culture-neutral and culture-free process; the teaching method and the focus (specific posture) as well as the behaviours of gaze, eye contact, or constant presentation as expected would reflect certain values. Gay (2010) indicated that people often considered education unrelated with culture and tradition, teachers were lack of understanding of how mathematics instructions reflected the European and American culture and values and considered their instructions being consistent and transcendent because of insufficient culture knowledge. However, the knowledge and the formation of mathematics were the process to construct a society which appeared strong correlations with language culture and cognitive process (Banks & Banks, 2001; Gay, 2010).

In Taiwan societies for the past decade, cross-national marriages prevailed. The social group and the family appearance have become diverse, multiple, and abundant. The group culture among pupils in a class appears great differences. The effects of culture on learning have become emphasized (Wang, 2004) and research studies on culture and mathematics learning were also reported (e.g. Yao, 2008; Yang & Hung, 2009). Nevertheless, most research on mathematics classes with multi-cultural groups focused on the children in cross-national marriages; few of them studied the

instructional behaviours of mathematics teachers. The instruction and the implementation manners of mathematics teachers in multi-cultural classrooms were rarely challenged.

In fact, the effects of language culture on mathematics learning included rest not only on students' learning but also on teachers' language use in instructions (Nath & Vineesha, 2009). Most mathematics teachers tend to use discourse in instructions and it is also a main teaching activity during mathematics instruction (Chang, 2000). When teaching students the same materials, the same grade, and the same school, researchers did find significant differences among teachers (Berry & Kim, 2008). Unfortunately, there was little research on the nature of instructional languages of mathematics teachers.

With language use in classes, the delivery, exchange, and learning of mathematics knowledge allow to understand the problems and thoughts of students in mathematics learning as well as to link with the key success factors in mathematics instructions (Ongstad, 2006). Nevertheless, because of distinct culture background of learners, it is likely to cause differences and difficulties in language use, interpretation, and understanding (Stathopoulou & Kalabasis, 2006). In this case, multi-cultural education/view has been promoted, which provides students with equal education opportunity and emphasizes better learning and success opportunity for students with various culture traits. Besides, it is an educational reform to change the school and the educational environment as well as a constant process emphasizing the integration of education contents and the process in knowledge construction, reducing prejudice, and equally instructing students to enhance the adjustment of school culture and society structure (Banks, & Banks, 2004).

The aim of this study is on mathematics instructional language use of teachers, which refers to the narration of mathematics concepts, exercises, and algorithms and the contents exchanged with students. The research questions are two. 1. What types of instructional language use, narration, and communication were found in this class with multi-cultural student groups? 2. What language culture observed might result in negative effects on students' mathematics learning?

LITERATURE REVIEW

Culture in mathematics teaching and learning

As Swetz (2009) mentioned that learning was not the culture-neutral and culture-free process. The belief and the implementation of traditional education was full of "blaming the victim" and "defects" (such as poor family environment, parental problems, and lack of passion), similar to "correction or treatment" (Gay, 2000). Mathematics learning appeared the similar situation. Gay (2010) indicated that a teacher who did not believe in cultural principles and culture blindness was likely to ignore the negative attitudes, anxiety, and fear from different people and would tend to change the students. As a teacher might not distinguish the relations among ethnic

groups, culture, and personality, he/she was likely to impose personal concepts on students and result in the educational process of insulting other culture. Besides, he/she would educate students and explain the bad performance of students with the idea of "blaming the victim" in "defect syndrome" (e.g. lack of talent, ability, and esteem, poor language expression, parents not participating in education, high absence rate). In addition, regarding mathematics narration, the distinct cultural background and concerned aspects between teachers and students or the understanding and expression differences of the teacher might have teachers regard their instructions being clear, which, indeed, could not be understood by students (Davis, Hauk, & Latiolais, 2009).

Some domestic research discovered the lack of cultural stimulation and language has resulted in immigrant children not understanding mathematics questions and further blocked the mathematics learning (Yang & Hung, 2009). Besides, most teachers would not utilize examples related to the living experiences to explain the mathematics concepts for the children in different groups (Yao, 2008).

Language in mathematics teaching and learning

According to the socio-culture theories, Berry and Kim (2008) indicated that learning is thought to be both individual and social, for the internalization of learning and knowledge, social discourse functions is useful tool for teacher to use in constructing effective teaching strategies and developing active learning roles. They also identified six categories of teacher utterances containing: questioning/eliciting, responding to students' contributions, organizing/giving instructions, presenting/explaining, evaluating and sociating. However, learners seldom understand the mainstream language coding in mathematics instructions (such as the transformation of mathematics concepts, languages and symbols) that learning inequality is likely to appear (Stathopoulou, & Kalabasis, 2006). Teachers therefore play a critical role in mathematics classes who use general languages, pay attention to learners' language positions, and interpret mathematics symbols and knowledge in textbooks, and generate understanding and learning with the delivery and the exchange of languages (Ongstad, 2006). In this case, languages of mathematics teachers in the teaching activities, as a moderator, appear as the communication bridge between learners and mathematics as well as the demonstration of mathematics language. How a teacher express the appearance of culture, narration, and explanation is therefore important for mathematics learners.

To summarize the above review on culture and language, different cultural experiences would reveal distinct understanding and interpretation, influence the description and the use of languages, and further affect the understanding and the application. In terms of culture and languages, Moschkovich and Nelson-Barber (2009) considered to understand the understanding of students and utilize student languages to express the mathematics cognition in order to effectively enhance students' mathematics learning. Among the differences, teachers should become the bridge or

utilize local cultural contexts and mathematics-related materials for the instructions or the materials of mathematics. With easy and comfortable methods to support the participation of students and to integrate community, family, and student knowledge and life experiences into the lesson, local environment, weather, and unsolved problems are the favourable teaching strategies for multi-cultural groups. It is therefore worth-concerning whether instructors could build the bridge to reduce the obstacle in mathematics learning.

RESEARCH METHOD AND PROCESS

According to the research purpose and the literature review, the investigator aimed a classroom with diverse groups, a school in a community with multi-cultural characteristics, and, a teacher with considerable amount of teaching experience willing to accept being observed. In this school, they assigned the same class teacher for grade 1 and 2 (grade 3 & 4; grade 5 & 6). We observed class during the second semester of a fifth grade class. There were 29 students in the class and seven of them were immigrant children of Chinese, Indonesian, and Vietnamese mothers. The school was near to the sea that the inhabitants made a living with tourism and fishery. There are language courses afterschool to encourage immigrants to learn Mandarin (the official language) and to integrate Taiwan culture. The observed teacher was the fifth-grade male teacher with ten years teaching experiences. He was also the master teacher and mathematics teacher of the class. The teacher reported that the performance of children from immigrant families could be high or low. He believed that there was no need to teach by attending to the diverse cultural background because the children could incorporate Taiwan culture.

Data source are three: 1) video; 2) classroom observations notes; and, 3) interview one hour right after the lesson. According to Strauss and Corbin (1998), observation could help acquire educational activities or the relevant data, and observing in natural situation was likely to receive the authentic data, rather than structural observations for exploratory research. With the camera recording in the class and the professional trainings of observers, it would benefit the representative and the objectiveness of data. The classroom observation was regarded as the optimal method for this study. Without affecting the regular schedule (e.g. monthly examinations and school activities) and under teacher's suggestion, three classroom observations were arranged for this class. The contents of the 3 separate and independent mathematics lessons are 【Area】, 【Time】, and, 【The unit converts】. The observers would remain quiet in each session and set up the camera to record the instructions. The teacher was observed between 10-11 o'clock on March 17th and 31st and May 19th, 2011. The observation video was transcribed into text transcripts and used content analytical coding methods by Berry and Kim (2008).

Data coding of teacher utterance was done by referring to Berry and Kim (2008): (a) questioning/eliciting, a response on questioning techniques to elicit and manage student involvement, (b) responding to students' contributions, a response other than

evaluating, to students' contributions, (c) organizing/giving instructions, this category included starting the lesson, directing its procedural aspects, directing students' attention, and regulating students' behaviour, (d) presenting/explaining, it interchanged that dealt with lesson content, (e) evaluating, including overtly evaluative remarks, such as "very good", and (f) sociating, strategies designed to draw students into the lesson dialogue as well as to manage and maintain the social relation of the lesson. In the data set, VW in the code of "" stood for transcripts, (2011.03.17) for the date of March 17th, 2011, and '' for Taiwanese languages. Data were coded by two independent raters and reliability was checked. Disagreements were discussed and resolved through e-mails or telephone discussions.

RESULTS AND DISCUSSIONS

Based on the 6 categories, the instructions were analysed as follows.

(a) Questioning/eliciting: The teacher utilized Mandarin and Taiwanese (a common dialect in south Taiwan) for asking, questioning, and explaining students. Direct enquiry was mostly applied, such as

> "How about 100 hectares? Have you heard of it? Never heard of it? What is meter? 'Umm… what about a square meter? You tell me?' What is a square meter? 'You got the highest score in the test, so you tell me.' You don't know about a square meter? Area units? Volume units?" (2011.05.19VW)

The direct enquiries (e.g. "You don't know…") were obviously different from the utterances of "Who can tell me…" and "Tell me what that coin is, Tina", found by Berry and Kim (2008, p.367). It intimidated students and discouraged attempts.

(b) Responding to students' contributions: When students explained and rephrased problems, the teacher would ask questions and insisted them to reply. When the students had correct answers, the teacher would continuously propose questions related to the solution to help student complete the questions. For instance, "Why do you do it this way? " "And then?" "Why is it not like this? " When students stopped, the teacher would give hints and continuously ask. Sometimes, the teacher would become impatient. "How come? You tell me. You tell me." (2011.03.17VW) Stathopoulou and Kalabasis (2006) proposed that learners seldom understood the mainstream language coding in mathematics instructions; teachers therefore needed to play the connection role.

(c) Organizing/giving instructions: When starting a new unit, the teacher would paste the posters on the blackboard and turn to students "OK, let's take a look…" "Open the book…"and start the lesson "The point is…" and then continuously guide and explain the lesson. For new concepts, the teacher mainly explained with narration, when no other teaching aids or activities would be utilized. For example, 【The unit converts】 was continuously explained for about 20 minutes; and about 8 minutes was utilized for explaining how the ancestors judge the time in 【Time】. The utterances were comparatively imperative (e.g., "You tell me first"; "I ask you…", "Take your

book to…", "What, louder!"). In other words, the teacher preceded the instruction with self-control which was similar to the research of Chapman (2001). The masculine languages were established in the instructions.

(d) Presenting/explaining: The three sessions were textbook-based that no local characteristics were presented. When students appeared misunderstanding, the teacher would translate the mathematics symbol into a language symbol and repeatedly explain it. For example, a female student conversed 6.4 tons into 6400 tons, the teacher dropped into a demonstration without explanation. He wrote only 〝**6.4t=6400kg 6.4×1000kg**〞 and said

> "How heavy is a dinosaur? 6.4 tons equal to 6400 kilos, why? 6.4 multiplied by 1000 is 6.4 tons and multiplied by 1000 is 6400 tons. 6.4 multiplied by 1000. 6.4 multiplied by 1000, you have to converse the units. 6.4 multiplied by 1000, not 6.4 tons multiplied by 1000. If 6.4 tons multiplied by 1000, you appear a myth." (2011.05.19VW)

Such an outcome corresponded to the teaching field indicated by Davis, Hauk, & Latiolais (2009) that teachers tended to understand with personal understanding, and ignored the importance of mathematics symbols, mathematics languages, and living languages as well as the understanding of listeners/learners (Ongstad, 2006). In other words, teachers should interpret personal understanding and language use to ensure the understanding of students, rather than to dodge responsibilities by "blaming the victim" (Gay, 2010).

(e) Evaluating: When students' responses were expected or correct, the teacher mostly said "Yes" , "ok" or continued the next question as a reply. When the students gave unexpected response or wrong answers, the teacher would directly deny, neglect, or criticize. (e.g. "What are unit?" Students: "kilogram." Teacher: "Kilogram? 'It is ridiculous.'" (2011.05.19VW) another e.g., Teacher: "What? Area units and volume units are replaceable? 'Is there something wrong with your brain?'"(2011.05.19VW) The students did not have further chance to explain, as the teacher did not offer the students chance to clarify their languages and ideas. It therefore became the teacher unilaterally uttering and controlling the class. It was similar to the research of Erchick (2001) that students did not easily utter in class.

(f) Sociating: Though the teacher did not treat immigrant children differently he tended to ask high performers to come to the chalkboard to show solutions. When female students' performance did not bring textbook, he presented gender stereotyped responses 〝You know the regulations, no textbook-stand in front of class" (the female student then stood up). "You have the book but you don't bring it in. Do you do it on purpose so that others would pay attention to your beauty, right? Is that it? Class, let us look at her…" (2011.05.19VW) From this teacher's response, "without a textbook" was connected to the gaze on a female body, rather than the responsibility on learning. In other words, because of the gender cultural blindness, the teacher imposed the personal concept on the students or even an insulting education (Gay, 2010). In the entire instruction of all observed lessons, students seldom needed to express their

opinions about problems; and, there was no teaching skills or strategies geared for multi-cultural characteristics in the class.

CONCLUSION AND RECOMMENDATIONS

The teacher's languages appeared "cultural blindness" (or the bias of culture), contain: 1. The teacher utilized only languages common for local students for asking, questioning, and explaining in the mathematics class, and most of them were direct enquiries. 2. When students were confused, the teacher tended to understand it with personal languages and neglect the importance of mathematics symbols and languages. 3. When students correctly answered the question, the teacher did not use language to praise, but used a simple "ok" and "Yes". The teacher would unilaterally utter and controlled the class atmosphere. 4. In terms of sociating, the teacher exhibited gender stereotyped utterances. For negative communications, the teacher kept silent.

As a whole, the students seldom discussed their opinions on the questions, and there was no teaching skills and strategies specially prepared for the multi-cultural group. The finding of this study is similar to Berry and Kim (2008) in that the strategies for students talk were missing. On the other hand, what was different was that the teacher in this study was authoritative and complained about students.

According to the research outcomes, it is suggested that: 1.The teachers of mathematics should reinforce the language use and emphasize mathematics symbols and the use, which includes the personal language use and the learners' language understanding. 2. In addition to understanding the problem-solving strategies for mathematics, multi-cultural sensitivity should be adopted. 3. to introduce multi-cultural viewpoints into mathematics classes (Banks and Banks, 2001) so that teachers were no longer the narrator and the copier of mathematics knowledge, but a participant and a constructor of mathematics learning who brought the student voice into the class.

Finally, there is limitation as results were from only a single case. For future research, it is suggested to discuss and evaluate from the aspect of learners and to reflect the feelings and the ideas of instructional language use. We also need to compare the results of this teacher to a teacher who attended to multi-cultural background in instructional language use. Alternatively, if this teacher changes his belief in attention to multi-cultural background over time, it is worth study his future actions and compare to results in this exploratory study.

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MATHEMATICS AND ECONOMIC ACTIVITY IN PRIMARY SCHOOL CHILDREN

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This paper reports a study informed by the debate between situated learning theorists and cognitivists, and by research on the importance of realistic contexts in the teaching of mathematics. Twenty-three children aged between 10 and 11 years were asked to participate in a series of activities designed to help us understand the kinds of mathematical thinking that takes place outside of school. These included a diary study, a documenting activity and a set of focus groups. We describe some of the examples of mathematical activity that children participate in outside of school, along with evidence to suggest that these children found it difficult to make connections between the mathematics of the classroom and the mathematics of their lives outside of school.

INTRODUCTION

There is increasing interest in understanding the relationship between classroom mathematics and the mathematics that children learn and use out of school. Although everyday situations are recognised as potential sources of mathematical thinking, there is a debate in regards to both the extent to which learners' usage of mathematics in real contexts can be transferred to the classroom, and to which children can make use of classroom mathematics learning in their lives outside of school.

Here we report a study that explores the mathematics involved in the economic activities that a group of primary school children engage with, both on their own and whilst socialising with their peers and family out of the classroom. We chose to focus of economic activity due to the available literature on its range and prevalence amongst our target age group and due its potential as a source of mathematical thinking. Economic activity here is to be thought of in a broad sense, including activities involving money such as spending and saving, as well as non-monetary activities such as swapping, collecting and game-playing.

BACKGROUND

In this section, we review two areas of relevant research literature, that relating to the debate between situated learning and cognitivist perspectives and that relating to children's economic activity.

THE SITUATED LEARNING DEBATE

The debate of Anderson, Reder & Simon (1996) and Greeno (1997) hinges in part on the question of transfer. Since Carraher, Carraher & Schliemann (1985), there has been a position amongst educational researchers that learning is situated in the context in which it occurs – this position is referred to variously as situated learning, situated cognition and so on. The main finding of Carraher et al. (1985) is referred to by both Anderson et al. (1996) and Greeno (1997), but both have very different interpretations. Carraher et al. (1985) found that Brazilian street vendors could not resolve schooled versions of the arithmetical operations that they mastered on the streets. This and other studies seem to be in line with the Situated Learning Theory (e.g., Lave & Wenger,

1991), which claims that learning is bounded to the context in which it occurs. This claim has been challenged with evidence of the transferability of knowledge across contexts. Anderson et al. (1996) claim that findings such as that of Carraher et al. (1985) show only that mathematics that is learned in concrete contexts does not always transfer easily to other contexts.

Alongside this debate, there is evidence that incorporating realistic elements can be beneficial for learning mathematics in school (e.g., Van Den Heuvel-Panhuizen, 2003), although more research is needed to understand the ways in which children's out-of-school experiences could impact their formal learning of mathematics. González, Moll, & Amanti (2005) coined the term *funds of knowledge* in reference to the knowledge that children gain in their household. Classroom practices often fail to exploit this knowledge, provoking a discontinuity between home numeracy practices and classroom mathematics (D. D. Anderson & Gold, 2006; Moll et al., 1992).

ECONOMIC ACTIVITY

The literature about children's economic reasoning suggests a number of developments and practices that are likely to involve mathematics. Children as young as six years old understand economic concepts such as supply and demand (Leiser & Beth Halachmi, 2006), and parents actively teach their children to handle money autonomously (Furnham, 1999, 2001; Lewis & Scott, 2000). It is also known that children are capable of developing strategies to make effective economic decisions when saving (Otto, Schots, Westerman, & Webley, 2006). A small number of studies have addressed the link between these forms of economic reasoning and behaviours and children's mathematical thinking. Taylor (2009) analysed the arithmetic operations involved in real-life monetary practices, and Guberman (2004) described various ways in which children perform multi-digit operations whilst shopping. Children's economic activities are not restricted to handling money and, moreover, non-monetary economic activities are also likely to involve mathematical thinking. For example, the exchanging of food in the playground (Nukaga, 2008), the negotiating of labour whilst playing (Webley, 1996) and the collecting of trading cards (Cook, 2001) all involved mathematical thinking. The study presented here adds to this literature by describing a number of ways in which mathematics might be involved in children's understanding of the norms that rule their economic activity. It also investigates children's perceptions of any connection between this informal, out-of-school mathematical thinking and children's experience of classroom mathematics learning.

This study was designed in order to investigate the economic activity that children are engaged in outside of school, and explore the mathematical thinking and dialogues that are involved in this activity. We are interested in finding:

- ⚰ What kinds of economic activities children typically participate in
- ⚰ What mathematical thinking these activities involve

⚔ What correspondences are there between the mathematical thinking that is involved in children's out-of-school economic activity and the mathematical thinking that is involved in children's classroom activity

We are working with the motivation that there is evidence that children are engaged in sophisticated mathematical activity outside of school that has the potential on which to build formal mathematical thinking.

A previous study consisted of surveying the incidence of economic activity amongst children in primary and secondary school. The sample consisted of 484 boys and girls aged 10, 12 and 14 (Year 5, Year 7 and Year 9 in the UK education system), students in a representative range of primary and secondary schools in a middle-size city located in Southwest England. The survey covered a range of topics such as the things that children value, their usage of pocket money, as well as the incidence of monetary and non-monetary economic activities. The complete set of results has been presented elsewhere (Xolocotzin Eligio & Jay, 2011), and was used to inform the development of the present study.

RESEARCH DESIGN AND METHODS

RESEARCH SITE AND PARTICIPANTS

The work presented here was conducted in a primary school situated in a central urban area, which draws pupils from a wide range of social and economic backgrounds and whose standards on mathematics achievement are above the national average. This school had participated in the survey of economic activity conducted during the prior year. Participants in this study formed part of one Year 6 class, and their average age was approximately 10.5 years. There were 26 children in the class, and every one of them was invited to participate. In total, 23 children participated with the informed consent of their parents.

ACTIVITIES

The activities of the main study were conducted over three weeks. In week one the participants were introduced to the study, and took home a questionnaire for their parents and a diary booklet for recording their economic activities over 1 week. In week two we conducted a self-documentation activity. Children were presented with two sets of cards. One set included six cards of monetary activities: *Working, Selling, Borrowing money, Lending money, Spending, and Saving*. The other set included five cards of non-monetary activities: *Borrowing things, Lending things, Giving gifts, Swapping things* and *Collecting*. Children selected one card of each set (one monetary and one non-monetary) according to what they wanted to, or thought they would be able to, document during the following week. Then we lent them a digital photo camera and asked them to take pictures to document their chosen activities. Finally, in week three we conducted group interviews where we asked children to talk about their

pictures. These activities produced a large amount of data. For the sake of space, here we focus on the group interviews.

GROUP INTERVIEWS

We conducted six group interviews, each with three or four children. The class teacher helped us to organise the groups according to the children's mathematics ability so we had two groups of each level: higher, medium and lower. This mirrored the typical organisation of children's mathematics lessons. Prior to the interview we collected the cameras so that during the interviews we could use a laptop to display the pictures.

In the first part of the interviews, each selected one of his or her pictures and was prompted to talk about it with questions such as '*What's in the picture?', 'Why did you take it?'*. Follow-up questions retrieved details about the practices involved in the activity. For example, when the picture showed a monetary activity such as spending, questions addressed issues such as the origins of the money or the amounts of money involved. Children were asked to comment about each other's pictures, in order to search for common or contrasting experiences. In the second part of the interviews children were prompted to talk about mathematics in terms of the concrete situations that they described; for example, explaining how they used mathematics to make decisions about spending or saving money. We also explored the connections between the reported activities and the learning of mathematics in the school. For example, we usually asked children whether the examples used in maths lessons resembled the experiences they were telling us about.

RESULTS

Here we present examples of three activities: *Selling*, *saving*, and *collecting.* The extracts below illustrate the contexts of these activities, the way in which children understand their usage of mathematics in these contexts, and the links between this usage and their learning of classroom mathematics. We selected these activities because they seem to involve forms of mathematical thinking that appear to be specific to the context, and because they have received less attention than other economic activities that have been traditionally linked with formal mathematics learning, such as spending money.

SELLING

Many children mentioned selling experiences. Their motivations for selling included both individualistic reasons such as getting money to buy something, or for altruistic causes such as raising money for their school. We asked about children's ways of making decisions about issues such as pricing and profit. The answers of some children suggested a link between their usage of mathematics and their understanding of people's buying behaviour. For example Ellie, a girl in one of the higher ability groups:

> Ellie: last year we made a cake stall in the playground and all the money that we got we gave it to the Japanese earthquake

Researcher: Did you have to do any kind of maths thinking for that?

Ellie: Yeah, we had to do all the same for that, all the prices the same… but we did it slightly more expensive than usual because it's for a charity

Saving

Many children documented saving. Children's saving goals were commonly short-term, such as for toys or electronic devices. However, some children mentioned saving for no particular reason, or for unexpectedly long-term goals such as buying a house, or paying for their university. For example, Liz and Jenny, from one of the middle ability groups said that they were saving for the time when they would leave home:

Researcher: [about savings] What kind of things you'll need the money for in the future?

Liz: A house

Jenny: Bills

Researcher: how old you will be when you start needing to pay for these things?

Liz: 17

Jenny: 18

When asked about their usage of mathematics for saving, children typically mentioned counting. However, the answers of some children suggested other forms of mathematical thinking, such as understanding the relationship between saving and time. This relationship seems important to decide whether saving goal is feasible. Children with long-term goals such as buying a house seemed to disregard the role of time; whereas children with short-term goals are probably more likely to consider time in order to discern about he feasibility of their goals. For example, Laura, from one of the high ability groups:

Laura: I don't know what I am saving for…

Kelly: I thought you were saving up for an iPad

Laura: I am trying to, but that's not really going to work [she gets £ 2.50 per week]

Collecting

Many children mentioned collecting something. In many cases, collecting was described as a social activity, shared with others in the school, as in the case of collecting trading cards. In some other cases, collecting was more an individual activity conducted out-of-school and driven by particular preferences, as in the case of children collecting toy cars or earrings. Children did not associate collecting with using mathematics in ways other than counting the money for buying the items they collect. However, different kinds of mathematical thinking are likely to be intertwined with the norms governing their collecting practices. Some children denoted awareness of the strategies used by the card sellers for pushing them to buy more cards. Noticing that collecting a complete set of cards is costly requires an understanding of forces such as demand and supply. For example, Ellie and Kelly from a high ability group:

Ellie: [the cards] keep changing; they want to get more money out of you. Like some years ago there were these cards that were really good and there were loads of them, and then they changed to these new cards and we collected them, and then after about six weeks they changes to another one, and we still have not finished collecting the other ones.

Ellie: But we don't collect them any more, because we realise this is very expensive…

Kelly: I spent more than £50

> Ellie: …likes ages ago, when there were these plastic figures called *gogos,* and in the end I just said I won't collect them any more because they only waste your money, but then I started again… I bet we are going to come back into something…

An understanding of the relationships between attributes and value is another kind of mathematical thinking involved in children's collecting. Cards are given values according to their rarity. One rare card may be worth 3 or 4 more common cards (the cards have an inscription on their backs labelling them as common or rare). However, these rules are flexible. For example, some children mentioned valuing cards they collect on the basis of attributes other than their rarity, such as the character depicted on the card, or whether the card belongs to a newer or an older set. This appropriation of the valuing rules implies that children have to learn a new system for trading the cards. As explained by Ellie:

> Ellie: [talking about the value of a card] it kind of evolves, say that I have a really good common, that has Amy Pond, and she has a rare that is rubbish, then we may just swap

The case of Omar, a boy in a lower ability group, illustrates the richness of collecting as a source of mathematical thinking. This boy is atypical because he receives £20 of pocket money each week. This amount of money gives him the resources to buy the things he collects on eBay. Unlike other children, Omar collects to make money in the future. He has learned and tries to exploit the relationship between attributes, value and time:

> Researcher: How do you know what kinds of things are going to get more expensive in the future?

> Omar: well, condition of the box and the car

Another feature of Omar's collecting is that he is learning to bargain from his mother, who helps him to negotiate the value of an item, both in selling and collecting.

DISCUSSION

This study has allowed us to explore a range of different activities in which we find children participating in some quite sophisticated mathematical thinking. In this section we will discuss an overarching theme that emerged from the data described above. This relates to the fact that the children did not associate the out-of-school mathematical activity that they engaged in with the mathematics that they engage with in the classroom.

Towards the end of each focus group we asked participants about the mathematics that they did outside of school. After the discussion that had preceded this question, we expected to hear answers relating to the economic activity that they had been describing (spending, saving, calculating profit and loss, debating the worth of trading cards and other collectables and so on). Every group talked about only 1 or 2 things – these were checking change after making a purchase in a shop, and measuring ingredients when baking (only the higher and middle achievement groups mentioned baking). We pressed some of the children in the higher achievement groups for more, asking whether they thought that there was any mathematics in the activities that they had been describing to us. One boy said that, "We don't do anything about our own

lives in maths", a statement that was backed up by the other members of the group. We are interested in what these contributions mean in terms of the relationship between the mathematical thinking that children do outside of school and that which they do in the classroom. There is evidence that the children that we spoke to have an implicit understanding of the 'game', or didactical contract (Brousseau, 1997) of classroom mathematics – this is clear from their confident assertion that the mathematics that they do outside of school consists of baking and checking change, activities that are frequently used by mathematics teachers and in mathematics textbooks to provide context for mathematical problems. There is also evidence that children do not associate the mathematical thinking they do outside of school with 'mathematics'. These issues relate closely, in our view, to discussion of 'realistic' mathematical problems – where we often see that real life and mathematics often have different versions of 'realistic' (Verschaffel, Greer & de Corte, 2000; Cooper & Harries, 2002)

These findings bring us to some important questions about what our goals are in teaching mathematics. The evidence presented here is in accordance with the arguments of Anderson & Gold (2006), who claim that the mathematical practices that children bring with them to school are often overlooked by schools and teachers. The main conclusion that we draw from this study is that there is content enough, and motivation enough, in children's out-of-school mathematical activity that we should aim to further explore ways in which we can help and encourage children to build mathematics out of their own experience, rather than always from experience gained within the classroom.

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REPRESENTATIONS AND REASONING IN 3-D GEOMETRY IN LOWER SECONDARY SCHOOL

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A key question for research in geometry education is how learners' reasoning is influenced by the ways in which geometric objects are represented. When the geometric objects are three-dimensional, a particular issue is when the representation is two-dimensional (such as in a book or on the classroom board). This paper reports on data from lower secondary school pupils (aged 12-15) who tackled a 3-D geometry problem that used a particular representation of the cube. The analysis focuses on how the students used the representation in order to deduce information and solve the 3-D problem. This analysis shows how some students can take the cube as an abstract geometrical object and reason about it beyond reference to the representation, while others need to be offered alternative representations to help them 'see' the proof.

INTRODUCTION

The teaching of geometry provides both a means of developing learners' spatial visualisation skills *and* a vehicle for developing their capacity with deductive reasoning and proving (Battista, 2007; Royal Society, 2001). One long-standing issue for research is how learners' reasoning is influenced by the ways in which geometric objects are represented (see Hershkowitz, 1990; Mesquita, 1998). While the term 'representation' can refer to internal (mental) *and* to external (concrete) representations, in this paper the focus is on external representations such as the various representations of a cube in Figure 1. As this figure captures, a particular case of interest is when the geometric object being represented is three-dimensional while the medium of representation is two-dimensional, such as is necessary in this paper.

One phenomenon related to learners' understanding of geometric representations is the well-established 'prototype effect' by which a certain representation is judged more representative than another (Hershkowitz, 1990, p82). Due to this 'effect', it seems that learners are much better at recognizing isosceles triangles that are 'standing on their base' than ones that are presented in a different orientation. When representing 3-D geometric objects such as a cube on a two-dimensional medium such as paper (or the classroom board), Parzysz (1988; 1991) reports that not only do learners prefer the parallel perspective (in which parallels are drawn as parallels), but, in particular, they prefer the oblique parallel perspective in which the cube is drawn with one face as a square (in French this is the *perspective cavalière*). Figure 1 shows two orthogonal projections (*a* and *b*) and an example of an oblique parallel projection (*c*). It is the latter,

according to Parzysz, that learners prefer. In many respects, this oblique parallel perspective is the 'classical' representation of a cube in two dimensions. A further convention is the use of dotted lines to show the 'hidden' edges of the cube.
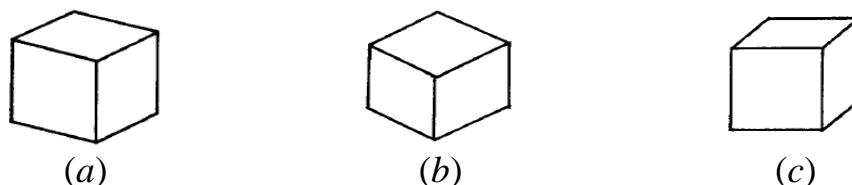

(*a*)   (*b*)   (*c*)

Figure 1: orthogonal and oblique parallel projections of the cube

An important issue that this oblique parallel perspective representation raises for research in geometry education is the way in which learners' reasoning might be influenced by the form of the representation, given the difficulties pupils have with 3-D representations even when 3-D dynamic geometry software is available (Mithalal, 2009). This paper reports on data from lower secondary school pupils (aged 12-15) who tackled a problem involving a cube that was presented using the oblique parallel perspective representation. The research question we focus on is how the students use the representation in order to deduce information and solve the problem.

THEORETICAL VIEWS ON REASONING IN 3-D GEOMETRY

For our theoretical framework we integrate a number of ideas relating to students' reasoning processes in 3-D geometry. In particular, we utilise the ideas of 'productive reasoning processes' (Fischbein, 1987), 'capabilities in 3D geometry thinking' (Pittalis & Christou, 2010) and 'the characteristics of 2D representations of 3D shapes' (Mesquita, 1998).

Fischbein (1987, p. 41) argued that a "productive reasoning process" aims at solving a "genuine problem". In a later article he suggested that in productive reasoning "images and concepts interact intimately" (Fischbein, 1993, p. 144). By this we surmise that Fischbein is referring the notion of 'figural concept' as capturing the combined role of the figural and the conceptual in geometry. Within the context of 3-D geometry reasoning, Pittalis & Christou (2010, pp. 192-4) synthesise various capabilities in 3D geometrical thinking. While all the capabilities they identify are likely to be important, in this paper we refer to the capabilities 'to recognise the properties of 3D shapes and compare 3D objects' and 'to manipulate different representational models of 3D objects'. Both these capabilities, we would suggest, are likely to involve the figural and conceptual aspects of geometrical thinking.

As Mesquita (1998, p184) explains, an external representation of a geometrical problem does not, by itself, enable one to solve the problem, but it may contribute to the definition of the structure of the problem. One way this happens, according to Mesquita (*ibid*), is if the representation gives support to geometrical intuition, which in some situations can be very powerful, by helping individuals "to apprehend relationships among geometrical objects". Yet, Mesquita goes on to show, external representations can lead to some ambiguities with the result that particular geometrical

relationships might appear as 'evident' to students in a way that can prevent geometrical reasoning from developing. What Mesquita (ibid, p186) calls the *double status* of a geometrical representation is that it can represent "either an abstract geometrical object, or a particular concretization". It is this *double status* that impacts on student reasoning.

All this means that, with a particular geometry problem that makes use of a particular representation, students may, or may not, be able to recognise theorems or properties which their teacher might expect them to use to form, and then prove, a conjecture because the representations may, or may not, appear to the students as 'typical'. This is what Mesquita is referring to when she shows that an external representation may become an "obstacle" to student understanding. In this paper, what we are interested in is how a representation which is given intentionally by a teacher might, or might not, lead students to engage in conceptual reasoning to make sense of what they 'see' and what they can deduce from the available information in the problem as presented to them; in other words, how the external representation supports, or not, their reasoning and how it might, or might not, provide an obstacle to their reasoning.

METHODOLOGY

The case of Japan illustrates how geometry teaching plays a role in developing students' ideas about proof and proving, as illustrated by the learning progression generally used in primary and lower secondary schools in Japan (emphasis added, as explained below):

- In primary school (Grades 1-6), basic properties of plane and *solid figures are studied informally, mainly in relation to everyday life objects. Students also start developing their drawing skills to represent 3D shapes on a 2D plane*;
- In Grade 7, students (aged 12-13) study geometrical constructions, symmetry, and selected properties of *solid figures (names of 3D shapes, nets, sections of cube, surface areas and volume)* informally, but logically, to establish the basis of the learning of proof (*note that the measure of the angle between two lines in 3D space is not formally considered*);
- In Grade 8, students (aged 13-14) are introduced to formal proof through studying properties of angles, lines, congruent triangles, and parallelograms, during which they learn the structure of proofs, how to construct proofs, and how to explore and prove properties of triangles and quadrilaterals;
- In Grade 9, students (aged 14-15) study similar figures and properties of circles, drawing on their consolidated capacity to use proof in geometry and Pythagorean theorems with both 2D and *3D shapes*.

As evident in this progression, students in Japanese lower secondary school have relatively limited opportunities to study and explore 3-D geometry (shown in italics above). As a consequence, students in general have difficulties when they are faced with 3-D geometry problems, e.g. when, in Grade 9, they are finding the lengths of a diagonal of a cube by utilising the Pythagorean theorem. Furthermore, it is uncertain what understanding of 3-D representations the students gain during their lessons, and

how they proceed with their reasoning when given such representations when tackling geometric problems. To investigate this issue, and address our research questions set out in the introduction, in this paper we present an analysis of quantitative and qualitative data which were collected through our classroom-based research.

The quantitative data come from a survey that was conducted in 2002. In the survey a total of 570 students in Grades 7-9 in two ordinary lower secondary schools were asked, at the end of their school year, to answer the problem in Figure 2. As can be seen, the representation in the Figure is the oblique parallel perspective the one typically used in the geometry classroom in Japan.
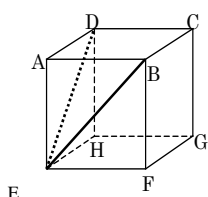
What is the size of the angle BED?

State your reason why.

Figure 2: angle in a cube (survey problem version)

As qualitative data, we analyse an episode taken from a lesson for a class of 46 Grade 8 students in a selective lower secondary school for girls. As such, it should be noted that the students' standards are relatively high. In designing the geometry lessons from which the episode is taken we employed the following process as a 'productive reasoning process' (informed by ideas in Becker & Shimada, 1997 and with some similarities to what Stein et al, 2008, call "reform-oriented lessons"; note that this process is more likely to occur within a classroom in which students can freely share their ideas in geometry, see Fujita, Jones & Kunimune, 2010):

- A problem is introduced, and the students generate conjectures and share ideas that could be used to prove their conjectures.
- Students attempt to prove their conjectures; incorrect proofs might be generated and, if necessary, the conjecture is modified and then proved.
- Students share their reasoning and proofs; incorrect proofs are revisited, and students undertake further proving activities.

In the classroom episode that we analyse, the students are tackling the same 'angle in a cube' problem, but this time the orientation of the triangle inside the cube is different; see Figure 3 (and compare to Figure 2).
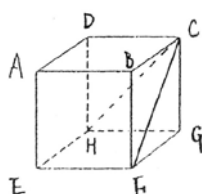
What is the size of the angle FCH?

State your reason why.

Figure 3: angle in a cube (classroom problem version)

In our analysis of the classroom episode, we focus on the interplay between the figural and conceptual aspects of reasoning; in particular on how the students tried to interpret the representations and undertake their reasoning to prove their conjectures.

FINDINGS AND ANALYSIS

Survey results

As mentioned above, in the survey a total of 570 students across Grades 7-9 tackled the problem in Figure 2. We categorised the students' answers as follows: (A) global judgment; e.g. 90°, no reason; (B1) incorrect answer influenced by visual information; e.g. half of angle AEF = 90/2=45°; (B2) incorrect answer with some manipulations of a cube but influenced by visual information; e.g. drawing a net, and then $45° + 45° = 90°$; (C2) incorrect answer by using sections of cube but influenced by visual information; e.g. in triangle BDE, angle B = angle D = 45°, therefore AEF = 90°; (D) correct answer with correct reasoning; e.g. in triangle BDE, EB=BD=DE and therefore AEF = 60°; (E) no answer. Table 1 gives the percentages of student answers in each category.

	A	**B1**	**B2**	**C**	**D**	**E**
Grade 7 (N=146)	13	58	8	4	2	15
Grade 8 (N=204)	26	44	7	4	3	16
Grade 9 (N=220)	19	29	16	7	14	15

Table 1: Categorisation of student responses to the survey (in percentages)

As evident from the results in Table 1, the response of around two-thirds of the students are in A or B categories; that is, they made a global response with no reasoning, or their response was incorrect but clearly influenced in some way by the visual information in the representation of the cube. Even by Grade 9 only 14% of students gave a correct answer with correct reasoning. These results suggest that students in general are not able to manipulate 3-D representations well and that their reasoning, if apparent in their responses, is likely to have been influenced by visual information from the specific representation.

As the students were asked to solve this problem under the restricted conditions of a survey, they did not have an opportunity to engage in the form of productive reasoning processes that can occur in the classroom. To investigate this point, we designed, in conjunction with a teacher, a classroom experiment in which students could exchange their conjectures and reasoning about the problem.

Classroom teaching results

This section reports on an episode taken from a lesson with 46 Grade 8 students. In the first phase of the lesson, the problem in Figure 3 was introduced, and the students generated conjectures. At this point, 28 students considered that angle FCH would be 60°, three said 90°, and 15 said 'I am not sure'. During the lesson, when ideas were shared amongst the class, three of the unsure students opted for the answer of 60°, making a total of 31 students (ie 67%) conjecturing that FCH was 60°.

In the next phase of the lesson, the students engaged in a productive reasoning process by discussing their ideas to deduce the size of the angle FCH. One student (S1) who

considered the size would be 90° explained her reasoning by using a net; see Figure 4 (in the dialogue, T is the teacher):

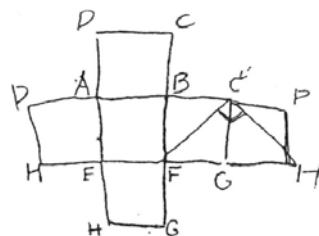| 22 | S1 | I used a net, but it might be wrong? Maybe it is 60°? I don't know. |
| 23 | T | Don't worry; please explain your idea to everyone? |
| 24 | S1 | Because FCG and GCG are 45°, then add them together and get 90°? |

Figure 4: Student use of a net to solve the 'angle in a cube' problem

This type of response was seen in the survey data.

A student (S13) then challenged this reasoning, and the teacher started asking for opinions as to why the angle was 60°.

26	S13	But…, Mr T, is S1's answer only on a plane and we need to fold [to make a cube], so it is different?
27	T	What do you mean? CG will be folded?
28	S13	Yes.
29	T	OK, do you understand what S13 said? [many students nod]
37	T	OK, I would like to listen to opinions about why the angle is 60°
47	S22	Well, in a cube all faces are the same square, and DHGC, BCGF and HEFG are all the same, and HG, CF and HF are all diagonals of the same size squares. So the lengths [of the sides of triangle FCH] are the same, and it is an equilateral.

At this point, the three students who initially said 90° changed their idea.

While students S13 and S22 were using the properties of the cube to construct a reasonable argument, two students showed hesitation in accepting this reasoning.

| 58 | S9 | I do not understand how to join H and F. |
| 59 | S28 | I can accept the explanation (by S22) but in the figure (Figure 4), I cannot see any equilateral triangle. |

At this, student S31 suggested an alternative idea.

| 64 | S31 | I have another idea. We can see the cube from A to C. |
| 65 | T | We can see from A to C? Can you draw a picture? |

Student S32 had a different suggestion, so that teacher asked S31 and S32 to draw their ideas, as shown in Figure 5 (S31 on the left and S32 on the right). The teacher asked the class what they thought of these representations.

| 67 Ss | | It is very clear. It is like an equilateral triangle. |
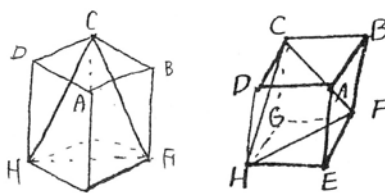
Figure 5: Student's alternative ideas for representing the 'angle in a cube' problem

Hence, by shifting the view of the oblique parallel projection (right-hand of Figure 5) or by shifting to an orthogonal projection (left hand of Figure 5) all the students agreed that triangle CFH is equilateral and hence that angle FCH is 60°. The students could 'see' that this is the case from the representations in Figure 5.

DISCUSSION

In the survey results, even with Grade 9 students, only 14% could give a fully correct response to the 'angle in a cube' problem (ie correct answer with correct reasoning). In the classroom situation, as many as 67% of Grade 8 students could do this (following some sharing of ideas. Such a difference is not altogether surprising and not the point of this paper, though what this comparison does point to is the impact of the "productive reasoning process" during which "images and concepts interact intimately" (Fischbein, 1993, p. 144).

The successful reasoning of many of the Grade 8 students in the classroom teaching experiment shows evidence of the capabilities identified by Pittalis & Christou (2010, pp. 192-4), particularly the capabilities 'to recognise the properties of 3D shapes and compare 3D objects' and 'to manipulate different representational models of 3D objects'. There is also evidence of what Mesquita (1988, p186) calls the *double status* of a geometrical representation in that it can represent "either an abstract geometrical object, or a particular concretization". Students such as S22 were able to take the cube as an abstract geometrical object and could reason about triangle CFH beyond reference to the representation (shown in Figure 3) provided by the teacher. Yet other students in the same class, such as S9 and S28, needed to 'see' that triangle CFH was equilateral. At this, student S31 provided a way to do this by using the orthogonal projection of the cube (left-hand part of Figure 5), while student S32 provided a different viewpoint of the oblique parallel projection (right-hand of Figure 5). Using these representations, then triangle CFH 'appears' to the students to be equilateral. This combination of reasoning and representation convinced doubting students and may help to make proof seem more meaningful (Kunimune, Fujita & Jones, 2010).

CONCLUDING COMMENTS

Some of the student responses to the survey showed them trying to use the net of a cube as a representation to help them solve the 'angle in a cube' problem. Some of the students in the classroom teaching experiment tried the same representation. On the whole, students did not find success when using the net representation. Some of the students who were successful could take the cube as an abstract geometrical object and

reason about it beyond reference to the representation provided by the teacher. Others, who could not 'see' the more abstract reasoning benefitted from being offered alternative representations to help them 'see' the solution. This illustrates how students' reasoning with 3-D geometry problems is influenced by the use of various representations. Of course the situation would be different if 3-D dynamic geometry software (such as *Cabri 3D*) was being used since students could utilise various viewpoints as if the computer representation were a 'concrete' model. Nevertheless, as Mithalal (2009) shows, even with 3-D dynamic geometry software, students need to go beyond visual information in order to solve geometry problems.

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