PROCEEDINGS

of the
36th Conference of the International Group for the Psychology of Mathematics Education

Opportunities to Learn in Mathematics Education

Editor: Tai-Yih Tso

Volume 4
Research Reports [Rez – Ye]
Short Oral Communications

PME36, Taipei – Taiwan
July 18-22, 2012

PME 36
TAIWAN
2012

Taipei – Taiwan
July 18-22, 2012

Cite as:

Tso, T. Y. (Ed.), (2012). Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education, Vol 4, Taipei, Taiwan : PME.

Website: **http://tame.tw/pme36/**

The proceedings are also available on CD-ROM

ISSN 0771-100X

Logo Concept & Design: Ai-Chen Yang
Cover Design: Ai-Chen Yang, Wei-Bin Wang & Chiao-Ni Chang
Overall Printing Layout: Kin Hang Lei
Production: Department of Mathematics, National Taiwan Normal University;
Taiwan Association of Mathematics Education

TABLE OF CONTENTS
VOLUME 4

Research Reports

Short Oral Communications

RESEARCH REPORTS

PME 36

TAIWAN

2012

FUNDAMENTAL IDEAS: A MEANS TO PROVIDE FOCUS AND IDENTITY IN DIDACTICS OF MATHEMATICS AS A SCIENTIFIC DISCIPLINE?

Sebastian Rezat

Justus-Liebig-University of Giessen (Germany)

This paper contributes to the self-reflection of didactics of mathematics as a scientific discipline in terms of its focus and identity and the related discussion about the diversity of theories. Since the call for a grand theory of didactics of mathematics seems to fade and the diversity of theories is accepted as an inevitable feature of the discipline the question remains how we cope with this diversity. Fundamental ideas (Bruner) are introduced as a means to foster the development of a focus and identity of the discipline.

INTRODUCTION

In a broad meaning didactics of mathematics can be defined as

> the 'sum' of scientific activities to describe, analyze and better understand peoples' joy, tinkering and struggle for/with mathematics (Sträßer, 2009, p. 68).

Therefore, the object of didactics of mathematics is the relation between humans and mathematics which implies a clear understanding of what mathematics is about. The importance of an understanding of the nature of mathematics in didactics of mathematics has been paradigmatically expressed by Thom:

> In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. (Thom, 1973, p. 204)

Accordingly, contemplating about the nature of mathematics is an important branch in didactics of mathematics. However, within the scientific discipline didactics of mathematics it is also important to contemplate about the nature of the discipline itself in order to position the discipline in relation to neighboring disciplines and to foster its own development:

> Taken as the scientific endeavor to describe and analyze the teaching and learning of mathematics, didactics of mathematics has to organize its own approach to the problem and exploit the knowledge available in neighboring disciplines. The systematic self-reflection of didactics of mathematics is a necessary element of its further development. (Biehler, Scholz, Sträßer, & Winkelmann, 1994, p. 5)

One important aspect in this context is the reflection about theories in and about didactics of mathematics. This paper contributes to the discussion about the diversity of theories in didactics of mathematics and introduces fundamental ideas as a means to foster the formation of focus and identity in the field.

THEORIES IN DIDACTICS OF MATHEMATICS

The diversity of theories in field of didactics of mathematics has been an issue of discussion ever since the foundation of the discipline. This is documented in the *Theory of Mathematics Education Group* (TME) founded by Steiner and regular study groups at the *International Congress on Mathematics Education* (ICME) and the annual conference of the *International Group for the Psychology of Mathematics Education* (IGPME). The current significance of this issue as well as the controversy about it can be seen in the comprehensive volume "Theories of Mathematics Education" (Sriraman & English, 2010). The tenor of the contributions is that diversity of theories is an inevitable and even welcome hallmark of didactics of mathematics.

By many researchers the theoretical manifoldness is traced back to the vast variety of goals and research paradigms, which are recorded in Volumes such as "Didactics of mathematics as a scientific discipline" (Biehler, et al., 1994) or the Study of the *International Commission on Mathematical Instruction* (ICMI) "What is research in mathematics education, and what are its results" (cf Sierpinska & Kilpatrick, 1998). Critics such as Steen (1999) argue that a lack of focus and identity pervades the foundations of the discipline:

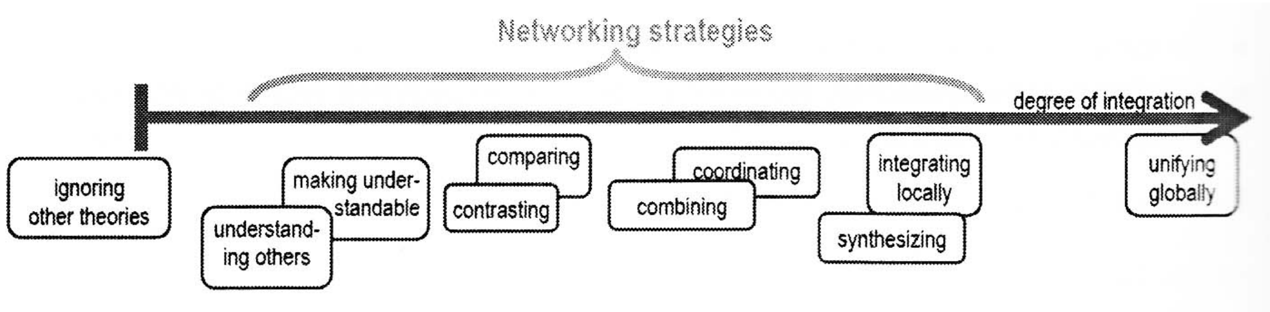
> there is no agreement among leaders in the field about goals of research, important questions, objects of study, methods of investigation, criteria for evaluation, significant results, major theories, or usefulness of results. (Steen, 1999, p. 236)

This observation even leads him to question the scientific nature of the field which he describes as

> a field in disarray, a field whose high hopes for a science of education have been overwhelmed by complexity and drowned in a sea of competing theories (Steen, 1999, p. 236)

This criticism is often encored by the call for a grand theory of mathematical thinking. While a growing number of convincing arguments is presented to support the necessity of multiple theories (e.g. Bikner-Ahsbahs & Prediger, 2010; Lerman, 2006) the related problems of the discipline's missing focus and identity persist. The questions are how we deal with this variety and if there are other ways to promote the development of focus and identity of the discipline than a grand theory of mathematics education.

Bikner-Ahsbahs and Prediger (2010) argue that "the diversity of theories and theoretical approaches should be exploited actively by searching for connecting strategies" in order to "become a fruitful starting point for a further development of the discipline" (p. 490). Based on a meta-analysis of case studies about connecting theories they suggest different strategies for connecting theories, which they call "networking strategies" (Bikner-Ahsbahs & Prediger, 2010, p. 492). These networking strategies are organized according to their degree of integration between the two extremes "ignoring other theories" and "unifying globally" as shown in fig. 1.



A landscape of strategies for connecting theoretical approaches (Bikner-Ahsbahs & Prediger, 2010, p. 492)

While this overview of strategies for networking theories in didactics of mathematics provides a fruitful approach to deal with multiple theories it seems hardly capable of contributing to the discipline's search for focus and identity, because it does not say anything about the phenomena these theories are related to. The networking strategies can be understood as heuristics to connect given theories. But, how to find theories that are worthwhile connecting? Which theories relate to a certain phenomenon?

In order to answer these questions I suggest reflecting upon fundamental ideas of didactics of mathematics as a scientific discipline. Pointing out fundamental ideas could help to focus on the core issues of the discipline and could provide a means to organize theories in terms of being related to a similar idea.

FUNDAMENTAL IDEAS

In his book „The Process of Education" (1960) Bruner introduced fundamental ideas as a means for curriculum development. For him they provide an answer to the basic problem that learning should serve us in the future which is at the heart of the educational process and therefore a fundamental problem of curriculum development. Since students only have limited exposure to exemplary materials they are to learn, how can they learn something that is relevant for the rest of their lives? He argues that this "classic problem of transfer" can be approached by learning about the structure of a subject instead of simply mastering facts and techniques. "To learn structure" for Bruner means "to learn how things are related" (Bruner, 1960, p. 7). According to him transfer is dependent upon the mastery of the structure of a subject matter in the following way:

> in order for a person to be able to recognize the applicability or inapplicability of an idea to a new situation and to broaden his learning thereby, he must have clearly in mind the general nature of the phenomenon with which he is dealing. The more fundamental or basic is the idea he has learned, almost by definition, the greater will be its breadth of applicability to new problems. Indeed, this is almost a tautology, for what is meant by 'fundamental' in this sense is precisely that an idea has wide as well as powerful applicability. (Bruner, 1960, p. 18)

Ever since Bruner, fundamental ideas of mathematics have been discussed in mathematics education as a didactical principle to organize curricula, and various catalogues of fundamental ideas of mathematics have been suggested (for an overview see Heymann, 2003; Schweiger, 2006). I will not discuss these in detail, because it would not support the central claim made here.

In his attempt to characterize mathematics as a cultural phenomenon Bishop (1991) also arrives at something similar to Bruner's notion of fundamental ideas which he calls "similarities" (Bishop, 1991, p. 22). 'Similarities' are similar mathematical activities and ideas that occur in different cultural groups. They are supposed to be a means to overcome the culturo-centrism by focusing on mathematical similarities between different cultural groups rather than on the differences in order to acknowledge that all cultures engage in mathematical activity. Therefore Bishop's similarities might be understood as a cross-cultural approach to characterize the structure of mathematical activity whereas Bruner's view is limited to a Western / American perspective. Nevertheless, fundamental ideas or similarities both are means to think about the inner structure of a discipline.

Schweiger as opposed to Bishop does not speak of one mathematical culture, but of several mathematical cultures, e.g. "mathematics in every day life or social practice, mathematics as a toolbox for application, mathematics in school, and mathematics as a science" (Schweiger, 2006, p. 63). He claims "it is more fruitful to acknowledge these facts than to try in vain to reconcile these different cultures" (Schweiger, 2006, p. 63). Interestingly for him also, fundamental ideas are a way of dealing with this diversity of mathematical cultures by providing an understanding of what mathematics is about (Schweiger, 2006, p. 64).

To summarize these reflections on the functions of fundamental ideas I want to distinguish epistemological functions of fundamental ideas on the one hand from pragmatic functions on the other. From an epistemological point of view fundamental ideas are a means to elicit the structure of a discipline and build up semantic networks between different areas. Furthermore they are supposed to elucidate the practice and the essence of a discipline. In doing so their pragmatic functions are to support the design of curricula and to improve memory.

While fundamental ideas are discussed in didactics of mathematics to serve these functions with respect to mathematics it is important to remember that Bruner's introduction of the notion of fundamental ideas was not limited to mathematics, but related to any discipline. Therefore it seems legitimate to broaden the perspective and to not only discuss fundamental ideas of mathematics in didactics of mathematics, but to contemplate on fundamental ideas of didactics of mathematics itself as a scientific discipline. From the epistemological functions of fundamental ideas it follows that fundamental ideas could serve as a means to overcome the criticism based on the diversity of theories in the field and to promote the formation of a focus and an identity of the scientific discipline didactics of mathematics.

While I expounded the functions of fundamental ideas in this section it remains vague what fundamental ideas are and how they can be identified. Or, as Schweiger put it "one has the uneasy feeling there is no agreement about fundamental ideas" (Schweiger, 2006, p. 68).

Towards a definition of fundamental ideas

Bruner simply leaves it to specialists in every discipline to identify the fundamental ideas of the discipline:

> It is that the best minds in any particular discipline must be put to work on the task. The decision as to what should be taught in American history to elementary school children or what should be taught in arithmetic is a decision that can best be reached with the aid of those with a high degree of vision and competence in each of these fields. (Bruner, 1960, p. 19)

But, even the specialists need to know what they are looking for. Bruner himself does not provide a clear definition of fundamental ideas. Revising the relevant literature on fundamental ideas Schweiger (2006) offers four descriptive criteria in order to characterize fundamental ideas of mathematics:

> Fundamental ideas
>
> - recur in the historical development of mathematics (time dimension)
> - recur in different areas of mathematics (horizontal dimension)
> - recur at different levels (vertical dimension)
> - are anchored in everyday activities (human dimension) (Schweiger, 2006, p. 68)

Although these dimensions all relate to fundamental ideas of mathematics they seem to be of a general nature which allows applying them to other disciplines as well. The time dimension and the horizontal dimension can be easily transferred to any other discipline. However, it is not obvious at the first sight what could be conceived of as a vertical dimension and a human dimension in didactics of mathematics. I suggest that different contexts of the disciplines involvement could be regarded as the vertical dimension: mathematics didactics is concerned with scientific inquiry of issues related to the people's involvement with mathematics, but also with issues of teacher education and development. Therefore, ideas recurring as objects of inquiry and as relevant themes for teacher education and development could be conceived of as fundamental in a vertical sense. Finally, I suggest that important ideas teachers are concerned about in their daily practice could be conceived of as the human dimension of didactics of mathematics.

The question remains how fundamental ideas can be found. It would be easy to just follow Bruner and leave it to "the best minds in any particular discipline". But how will they be able to find fundamental ideas?

Bishop's focus on similarities between different cultural groups in terms of mathematical activities and ideas offers a method to identify such similarities: cross-cultural comparison of mathematical ideas and activities. Accordingly, cross-cultural comparison of ideas informing research in mathematics didactics could be one way of approaching fundamental ideas of the discipline.

According to Schweiger's characterization, cross-cultural comparison ought to be complemented by historical, horizontal and vertical analysis of the disciplines areas of study and activity in order to link to the time dimension, the horizontal dimension and the vertical dimension of fundamental ideas.

AN EXAMPLE

The main aim of the paper is to introduce the notion of fundamental ideas as a means to foster the development of didactics of mathematics as a scientific discipline. It is not possible within the scope of this paper to discuss a list of fundamental ideas of didactics of mathematics. But, in order to substantiate my theoretical deliberations I want to provide an example of the kind of fundamental ideas I have in mind.

From my point of view, *transformation* is a candidate for a fundamental idea of didactics of mathematics. Transformation is not only a fundamental mathematical concept, but it is also a central issue in didactics of mathematics. Many areas, e.g. semiotics, learning of mathematical concepts, curriculum development, deal with transformations of various kinds (horizontal dimension). Here I can only hint at some transformations related to these areas:

An important aspect of doing mathematics is the construction and transformation of signs in different contexts. Hoffmann (2006) describes the major mathematical activities *mathematization*, *calculation*, *proving*, and *generalization* in terms of related transformations of signs and concludes that the "essence of mathematics consists in working with representations" (p. 279). On the one hand mathematics is embodied in different representations. This raises the question if the representations themselves are likely to transform mathematics (cf. Sträßer, 2001). On the other hand humans can only act upon the mathematical reality via representations. Consequently, transformations of mathematical representations construct and transform mathematical reality (Steinbring, 2005) and are therefore at the heart of the mathematical activity. Another aspect is that mathematics is also transformed in relation to its context. According to Chevallard (1985) mathematical knowledge is transformed in the transition from scholarly knowledge "savoir savant" to taught knowledge "savoir à enseigner".

These glimpses already show that transformation of mathematics is a central issue in didactics of mathematics. Furthermore, they draw attention to theories which relate to transformations: e.g. semiotics, transposition didactique. Since these theories are relating to the same fundamental idea it might be worthwhile to ask further how transformation is conceptualized in these theories.

FURTHER APPLICATIONS OF FUNDAMENTAL IDEAS

So far, fundamental ideas have been discussed as a means to foster the development of focus and identity in didactics of mathematics as a scientific discipline. In the preceding discussion it has already been mentioned but more or less neglected that curriculum development is the main pragmatic function of fundamental ideas.

Therefore, I would like to raise the question if we need a curriculum of didactics of mathematics / mathematics education.

First of all, it must be noticed that efforts for a joint curriculum of teacher education are made already in the European Union (EU) (JoMiTE Group, 2009). The main aim is to provide a cross-cultural curriculum of teacher education in order to "increase the cultural awareness of students, improve their social and cultural competences and increase their knowledge about different pedagogies and educational methods" (JoMiTE Group, 2009, p. 1) and not least to support mobility of teachers within the EU.

These efforts are also to be seen related to critic voices of teacher education. Based on the observation that each teacher education program "seems to feel that it needs to be special, singular or unique" Shulman – well known for his categorization of teacher knowledge – once provocatively concluded that "there is no teacher education" (Wood, 2006). According to Shulman teacher education unlike law or medicine lacks a "signature pedagogy". Shulman defines "signature pedagogies" as

> types of teaching that organize the fundamental ways in which future practitioners are educated for their new professions. In these signature pedagogies, the novices are instructed in critical aspects of the three fundamental dimensions of professional work – to think, to perform, and to act with integrity. (Shulman, 2005, p. 52)

Since fundamental ideas always have been regarded as a means to structure curricula an additional benefit from reflecting upon fundamental ideas of didactics of mathematics as a scientific discipline might be that they could contribute to the development of curricula for teacher education which emphasize commonly accepted fundamentals of the discipline a therefore provide teacher education in mathematics education with a more homogeneous "signature".

EPILOG

The ideas presented in this paper are only glimpses on a new way of approaching the self-reflection of the discipline's focus and identity. They are just the starting point for a discussion about fundamental ideas of the discipline which will need to be continued in the future.

References

Biehler, R., Scholz, R. W., Sträßer, R., & Winkelmann, B. (Eds.). (1994). Didactics of Mathematics as a Scientific Discipline. Dordrecht: Kluwer.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of Theories - An Approach for Exploiting the Diversity of Theoretical Approaches. In B. Sriraman & L. D. English (Eds.), Theories of Mathematics Education. Seeking New Frontiers (pp. 483-506). Berlin: Springer.

Bishop, A. J. (1991). Mathematical Enculturation. A Cultural Perspective on Mathematics Education. Dordrecht: Kluwer Academic Publishers.

Bruner, J. S. (1960). The Process of Education. Cambridge: Harvard University Press.

Chevallard, Y. (1985). La Transposition Didactique. Du Savoir Savant au Savoir Enseigné. Grenoble: Pensées sauvages.

Heymann, H. W. (2003). Why Teach Mathematics? A Focus on General Education. Dordrecht: Kluwer.

Hoffmann, M. (2006). What Is a “Semiotic Perspective”, and What Could It Be? Some Comments on the Contributions to This Special Issue. Educational Studies in Mathematics, 61(1), 279-291.

JoMiTE Group. (2009). JoMiTE Curriculum Framework 2009/2010 Available from http://www.rug.nl/jomite/jocite/framework20092010.pdf

Lerman, S. (2006). Theories of Mathematics Education: Is Plurality a Problem? ZDM, 38(1), 8-13.

Schweiger, F. (2006). Fundamental Ideas. A Bridge between Mathematics and Mathematics Education. In J. Maaß & W. Schlöglmann (Eds.), New Mathematics Education Research and Practice (pp. 63-73). Rotterdam: Sense.

Shulman, L. S. (2005). Signature Pedagogies in the Professions. Daedalus, 134(3), 52-59.

Sierpinska, A., & Kilpatrick, J. (1998). Mathematics Education as a Research Domain. A Search for Identity. An ICMI study. Dordrecht: Kluwer.

Sriraman, B., & English, L. D. (Eds.). (2010). Theories of Mathematics Education. Seeking New Frontiers. Heidelberg: Springer.

Steen, L. A. (1999). Review: Theories That Gyre and Gimble in the Wabe. Journal for Research in Mathematics Education, 30(2), 235-241.

Steinbring, H. (2005). Do Mathematical Symbols Serve to Describe or Construct "Reality"? Epistemological Problems in Teaching Mathematics in the Field of Elementary Algebra. In M. H. G. Hoffmann, J. Lenhard & F. Seeger (Eds.), Activity and Sign. Grounding Mathematics Education (pp. 91-104). New York: Springer.

Sträßer, R. (2001). Cabri-géomètre: Does Dynamic Geometry Software (DGS) Change Geometry and its Teaching and Learning? International Journal of Computers for Mathematical Learning, 6(3), 319-333.

Sträßer, R. (2009). Instruments für Learning and Teaching Mathematics. An Attempt to Theorise about the Role of Textbooks, Computers and other Artefacts to Teach and Learn Mathematics. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.), Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 67-81). Thessaloniki: PME.

Thom, R. (1973). Modern Mathematics: Does it Exist? In A. G. Howson (Ed.), Developments in Mathematical Education. Proceedings of the Second International Congress on Mathematical Education (pp. 195-209). Cambridge: Cambridge University Press.

Wood, T. (2006). “Teacher Education Does Not Exist”. Journal of Mathematics Teacher Education, 9(1), 1-3.

"THE MAIEUTICAL DOGGY": A WORKSHOP FOR TEACHERS

Mirela Rigo Lemini — Bernardo Gómez

Center for Research and Advanced Studies, Mexico. — University of Valencia, Spain.

This paper deals with the contents, application and results of a workshop designed for elementary education teachers, which was structured around meta-cognitive principles taken from Socratic maieutics. We argue that the open-type mathematical task, around which the workshop revolved, was crucial in allowing its participants to perform a maieutical activity, but it also acted as an obstacle to that.

BACKGROUND

Since Flavell's work, experts in mathematical education have expressed a positive view towards the efficacy of meta-cognition (mcn) for the purpose of learning Mathematics, and have developed teaching proposals based upon meta-cognitive (mc) practices. These proposals rest on the assumption “that mcn demands to be taught explicitly" (Desoete, 2007, p. 709). Two decades earlier, Schoenfeld (1992) had already pondered the centrality of teaching and teachers in this kind of instructive processes. This position was later taken, among others, by Hartamn and Sternberg (1993, quoted in Desoete, 2007), who believe that in mcn-based teaching teachers have a central role to play, going as far as setting themselves as an example of the way mc tools are to be implemented. Despite this, "[teachers] still pay little attention to explicit mcn teaching" (Ibid, p. 709). This is probably due to the scarcity of offerings in support of Math teachers in order to acquire the skill and mastery needed for the deployment of mc tools (Kozulin, 2005).

In this regard, this paper presents the contents, application and results of a workshop designed for active elementary education teachers, which was structured around mc principles taken from Socratic maieutics. The workshop, given in the context of a program for professional development, was constituted by two three-hour sessions, supplemented with individual meetings for every individual who took the workshop; this paper deals with the first session.

While there are reports regarding proposals for the professional development of teachers, which are based on several pedagogical methodologies and epistemological points of view, few of them (among them, Sowder's, in 2007, who uses 'the Socratic model') expound upon and analyze ways of offering professional training that are based on the use of mc practices, such as the one approached here.

THEORETICAL AND INTERPRETIVE FRAMEWORK

Mc activities in the Math classroom. A classification

For analyzing the mathematical activities proposed herein, the following mc categories have been used (described and typified in Rigo, Páez & Gómez, 2010):

	In reference to the Task (T)	*In reference to the Person (P)*
Specific Type (S)	How did I solve it? What did I base it on? What is its degree of difficulty?	How confident am I in the solution I propose? What do I base it on?
Generic Type (G)	Processes for transfer to other tasks	Awareness of what I do not know about the subject

Table 1. Examples of mc activities. Variables (T and P) and types (S and G).

Maieutics: "giving birth to truth"

It is a pedagogical method conceived by Socrates and expounded in Plato's *Meno* dialogue. This study has identified three moments of maieutics (Rigo, 2011):

Construction moment. A task is put forward with the foreknowledge that the student will solve it incorrectly or limitedly, but also that he/she will feel a high degree of confidence about the resolution proposed. *De-construction moment.* The teacher confronts the student with cognitive conflicts which the student then uses for reconsidering his/her resolution (S, T mc) and understand his/her mistake. *Re-construction moment.* The teachers guide the student in the building of a new solution, one which allows him/her to understand what he/she does not know about the subject (G, P mc). Within this process, two types of conflict can be distinguished: a *cognitive* one, when the student must confront the contradictions that emerge from his/her wrong answer, and an *mc* one, which emerges when he/she is constrained to acknowledge his/her ignorance about a subject he/she thought he/she knew about.

THE "MAIEUTICAL DOGGY": A WORKSHOP FOR TEACHERS

About the design of the workshop and its application

At the workshop it was expected that, starting out from an open task, from responses to a written questionnaire (Q), from collective discussions, and from talks delivered by researchers (R), participants would construct a mathematical definition of an intuitive notion (which is not defined) and that this cognitive process may serve as a reference to do some mc (T, P) reflecting, specifically of a maieutical nature. For maieutical purposes, it was essential to promote among students the emergence of cognitive and mc conflicts.

About the task

For the workshop's purposes, the task was essential. In the case dealt with here, a task called "the doggy" was chosen, in which a line figure in the shape of a dog is drawn upon a grid (Fig. 1). Data are given in graphic form and the solution must be presented in the same way:

"We found a pill that causes things to grow to twice their size. The dog drawn here is going to eat the pill. What will he be like after eating it? Draw him". This is an open task of an exploratory nature (Ponte, 2005) in which students, adopting an autonomous attitude (Ibid.), must deploy their intuitive and mathematical knowledge in order to signify the idea of size (not defined in the statement) and make its meaning concrete in the graphic register.

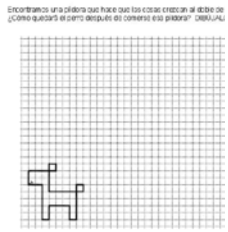


Fig. 1. The doggy task

The doggy is an application of the problem Socrates poses to the slave in *Meno*, concerning the duplication of areas. It involves several notions having to do with the idea of size: similarity, shape, area, reason, proportion, Pythagoras' theorem, $\sqrt{2}$, which are part of the arithmetic and geometry concepts that are studied in elementary and secondary schools. Students from different educational levels, including higher education students and active teachers, have been asked to solve the doggy, so many of the problem's possible resolutions have already been identified and systematized in previous studies (Gómez, 2007). This permitted anticipating possible answers from participants and planning which cognitive and mc conflicts to promote among them.

About the workshop's structure and contents

As per the maieutical method, the Workshop was divided in three moments:

Construction moment. The first solution

The student is asked to solve the task in the Q, to justify his solution and to meditate about the degree of certainty he/she has in his/her solution, about his/her knowledge of proportionality and about the task's degree of difficulty (S, P mc).

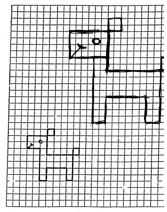
De-construction moment. The cognitive conflicts

Based on their previous analytical work, researchers organize the presentation of various resolutions to the task, showing first those centered on shape, followed by those centered on area and finally those resolutions in which harmonizing shape and area duplication is sought. With this process is expected to promote the generation of cognitive and mc conflicts that allow teachers to gain awareness of the limits of their resolution(s) and of their ideas regarding the duplication of size.

- Resolutions centered on shape. Two types stand out:

-- Duplication of the length of the sides of the original figure (Fig. 2). R: In previous studies, the higher the educational level, the greater the frequency of this figure, known as the Big Dog (BD). But isn't BD too big? And what about BD's ear and tail?

Aren't they too large? Some students thought about other options. One of them is shown in Figure 3.



--A doubling of the perimeter, keeping an eye on the unit square.

R: The ear and the tail of the dog in Figure 3 have an area of two. Is this a reasonable answer to the task?

Fig. 2. Big Dog

Addressing those who continue to defend the BD solution, despite the fact that it has four times the area: Why pay attention only to the shape and not to the area? Consider the case of circles. What would the criterion be for choosing a circle with twice the area, if all circles have the same shape?

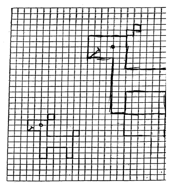


Fig. 3

- Resolutions based on area. The following is presented, among others:

-- One dimension increases, in order to arrive to a figure with twice the area (Fig. 4). R: The design has taken into consideration that, in order to obtain a rectangle with twice the area, it suffices to increase one of its sides to twice the length. But, must it maintain its similarity; i.e., the proportions between its sides? Or is retention of a dog shape enough? Could doubling the size be synonymous with doubling the area? What happens with the segments?

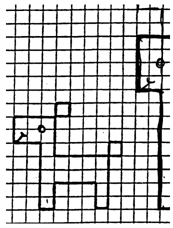


Fig. 4

- Resolutions that harmonize doubling of the area with preservation of the shape. Among other solutions, one that harmonizes area and shape is presented, even though it entails working outside the metrics induced by the grid (v. Fig. 6 and explanation on p. 6).

Re-construction moment. Construction of a mathematical solution and identification of mathematical contents that were considered to be known

At this point in the Workshop, the visual 'demonstration' that Socrates presents in the *Meno* Dialogue, in which he makes use of the diagonal to build a square that is twice the area of a given square, is introduced. Considering this construction, the teacher is asked to attempt a new solution to the task.

-- Resolution that formally harmonizes the doubling of the area with the preservation of the shape: "The Socratic dog". The teacher is expected to draw a dog such as the one presented in Fig. 5 and that he/she justifies his/her solution mathematically. Finally, the teacher is asked to draw a circle that is twice the area of a unit circle and to write down (in Q) his maieutical reflections (G, P mc).

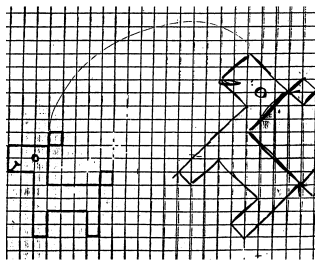


Fig. 5. The Socratic dog

Empirical results encountered in the Workshop

Some of the empirical findings are presented in this part of the paper. The analysis (carried out using the two video recordings of the session, their written transcripts and the results of the Q) is made taking into account the mc and maieutical categories mentioned in the theoretical framework, and is based upon qualitative case studies, as per the recommendations made by experts (Sowder, 2007).

Of the thirteen participants in the Workshop, eleven proposed BD as a first solution. Another person proposed the Socratic dog and one a 'Cubist dog' with twice the area.

Those who opted for BD based their solution on several beliefs and considerations. Some of these are:

- An erroneous belief, which more than half the teachers tacitly maintained, was that the doubling of the segments would result in a doubling of the area (or 'size'); i.e., they thought that the growth of area in a square is linear or proportional to the growth of its sides. We call this here the "spontaneous idea of proportionality", because it coincides with the one that guided the immediate response that the slave gave Socrates in *Meno*. Pedro, as many of his companions, after doubling the sides of the figure, asserts that "each square in the original is equal to two in the enlarged drawing"; he does not realize that in doubling the figure's perimeter, the area ('the square') is multiplied by four (and not by two as he suggests); i.e., he does not conceive that bi-dimensional magnitudes (area) behave differently than uni-dimensional ones (segments) and that 'size' is related to the former.

- Other teachers responded thinking that the task was a routine scaling exercise: "I thought the activity called for doing what is proposed in the Secondary school curriculum: the application of scales", commented José, as did other teachers.

- Another consideration, defended by three quarters of the group was that the task as a problem was not correctly enunciated, since the idea of size is not defined: "the statement is ambiguous and the parameters within which the student is expected to provide a solution have not been well established. How awful!", said Rita.

The truth of these beliefs was brought into question (and some of them even resulted in mathematical contradictions) as the session progressed and different solutions and meanings of size were produced and pondered. The mathematical activity displayed by the teachers for the purpose of responding to such questionings and contradictions, together with the mc ponderings they carried out in connection with that activity, determined different patterns of participation in the Workshop. In the section that follows, three of these patterns are described and illustrated by some cases.

Pattern of participation with maieutical activity (MA)

Teachers who displayed MA participation developed:

Cognitive activities: teachers were involved in the analysis and appraisal of the different resolutions and ideas concerning size that were produced in the course of the workshop session; they were sensitive of the cognitive conflicts that derived from analysis and drew mathematical challenges to solve them. This allowed their first-offered resolution to evolve as the session progressed.

An autonomous attitude: teachers defined and responsibly assumed a characterization based upon what was meant by doubling size, a characterization which they then attempted to represent graphically.

Meta-cognitive activities: The cognitive activities described above allowed the teachers to become aware of some of their conceptual problems with regards to the notions involved and the difficulty of the task.

Lino, one of the teachers who had an MA participation, recounts: "when I finished [BD], I realized that the ear and the tail had grown by four… The problem asks to double the size of the figure and I made it four times bigger". The ear and the tail, which are both one square in area in the original figure, were the trigger that revealed the contradiction which he unconsciously incurred; in order to solve the conflict and self-regulate his solution, he set for himself the challenge to "draw a figure that was proportional [to the original one] and then make it twice as big as the original… [that is, if] the original area is one, now it must be two across the figure".

With an autonomous and precise idea of what it meant to 'grow to twice the size', that challenge brought him to another: in order to arrive to twice the area while preserving the shape, he had to transcend the $\mathbb{N}$ domain metrics induced by the grid; thinking perhaps only in the $\mathbb{Q}$ domain and possibly ignoring $\sqrt{2}$, he arrived to the idea that "the grid got in the way".

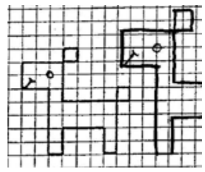


Fig. 6

He then made a figure that was a qualitative approximation of the expected response (Fig. 6). In his final reflection, he clarifies: "I said I was 50% convinced by my own solution, because I was not sure; now I see I was right, because my solution lacked arguments. The problem is not as straightforward as I initially thought it to be".

It is significant that all the teachers who had an MA participation doubled the unit circle by resorting to the use of the diagonal line, successfully transferring the diagonal method in rectilinear figures to the circular figure.

Pattern of participation with incipient maieutical activity (IMA)

Most teachers did not understand and even rejected the open character of the task. This is possibly due to the presence of a school sub-culture that is firmly entrenched among teachers, which is shaped around beliefs of what math tasks ought to be. Some of these beliefs, which Rita expressed with great emphasis (v. p. 5), prevented them from carrying out fully maieutical activities: by not assuming an autonomous position regarding the concept of size, they did not set the challenge for themselves to solve it graphically, neither did they fully involve themselves in discussing the different solutions, nor were they sensitive to the questionings that emerged. In this context of scarce autonomy, their cognitive activity was merely incipient; as a result, their mc activity was also incipient, thus allowing them to have just IMA participation. José, for example, after proposing BD in his first attempt, posited in his next intervention that, for the purpose of solving the problem, "he would have to consult a dictionary in order to determine what size means". With that approach, he systematically considered that every solution to the task was valid, since it depended on what 'size' was understood to mean. Even though he carried out some mathematical tasks, such as the Socratic dog, he developed them solely as a school exercise and they were not useful to him as a reference for appraising his previous mathematical activities and for his mc ponderings.

Pattern of participation without maieutical activity (WMA)

Two teachers assumed that the doggy task was a routine school exercise, one that was also badly formulated. These beliefs, which they held without the possibility of negotiation, really weighed them down because, under such a position, they became refractory to all questionings that emerged in the course of the session, which resulted in a severe reduction in their mathematical activity and, therefore, in their mc activity as well, both being features that define WMA participation. Juan, for example, at the beginning of the Q states that he feels great assuredness concerning his knowledge of proportionality and area; he further considers that the problem is very accessible and is 100% sure of his solution. In his interventions, he maintains that “BD is correct, despite the fact that it is too big… [since] all the area grows exponentially". He does not allow himself to look at other solutions in order to reflect about his, nor in order to identify what he ignores about the topic. It is possible that Juan, feeling insecure about his mathematical ability, was afraid to find himself exposed; in order to avoid this, he held tightly to his beliefs, something which probably made him feel secure, a meta-affective context that stabilized such beliefs (see Goldin, 2002). Sowder (2007) correctly remarks that, in these processes, teachers are often anxious about and reluctant to change, something that professional developers need to be aware of.

FINAL CONSIDERATIONS

A little over half of the workshop's attendants had a maieutical participation. The task was a key point: on the one hand, because the empirical and analytical work that one of the authors had previously carried out on it made it possible to plan for the cognitive and mc conflicts that characterize maieutical education; on the other hand, because of its exploratory, open nature. But this, in turn, was the source of conflict that prevented other participants from getting involved in it, not just because of the relatively difficult mathematical challenges demanded by its solution, but because it went against the grain of teaching beliefs firmly entrenched in them and because it forced them to make autonomous mathematical decisions which they do not seem accustomed to making. Several authors, among them Nelson and Hammerman (quoted in Sowder, 2007), point out the need to increase knowledge in preparing and training math teachers. The contents of this paper may contribute to this end.

References

Desoete, A. (2007). Evaluating and improving the mathematics teaching-learning process through meta-cognition. *Electronic Journal of Research in Educational Psychology, 13,* 5 (3), 705-730.

Gómez, B. (2007). La razón en semejanza: El caso del perrito. In E. Castro y J. L. Lupiáñez (Eds.), *Investigaciones en educación matemática: Pensamiento numérico* (pp. 237-257). Granada, Spain: Editorial Universitaria de Granada.

Goldin, G. (2002). Affect, Meta-Affect, and Mathematical Belief Structures. In G. Leder, E. Pehkonen, G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education* (pp. 59-72). The Netherlands: Kluwer Academic Publishers.

Kozulin, A. (2005). Who needs meta-cognition more: students or teachers? Paper presented at the *Annual Meeting of American Educational Research Association*. Canada. http://www.icelp.org/files/research/AERA2005paper.doc

Plato. (1981). Menón o de la Virtud (P. Samaranch, Translator). *Platón, Obras Completas* (pp. 435-460). Madrid, Spain: Aguilar S. A. de Ediciones.

Ponte, J. P. (2005). Gestão curricular em Matemática. In GTI (Ed.), *O profesor e o desenvolvimiento curricular* (pp. 11-34). Lisbon: APM.

Rigo, M., Páez, D. & Gómez, B. (2010). Prácticas meta-cognitivas que el profesor de nivel básico promueve en sus clases ordinarias de matemáticas. Un Marco Interpretativo. *Revista Enseñanza de las Ciencias*. *28*(3).

Rigo, M. (2011). La Mayéutica y su aplicación a un cuestionario dirigido a docentes. In M. Rodríguez, G. Fernández, L. Blanco, M. Palarea (Eds.), *Investigación en educación Matemática XV* (pp. 523-532). España: Universidad de Castilla-La Mancha.

Schoenfeld, A. (1992). Learning to Think Mathematically: Problem Solving, Meta-cognition, and Sense Making in Mathematics. In D.A. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning (*pp. 334-370). New York: MacMillan

Sowder, J. (2007). The Mathematical Education and Development of Teachers. In F. K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. United States of America: Information Age Publishing Inc.

FIRST GRADE STUDENTS' EARLY PATTERNING COMPETENCE: CROSS-COUNTRY COMPARISONS BETWEEN HONGKONG AND THE UNITED STATES[i]

F. D. Rivera — San Jose State University, USA

Chi Keung Eddie Leung — Hongkong Institute of Education, HK SAR

24 US Grade 1 students and 24 HK Primary 1 students (mean age of 6 years) were clinically interviewed on five figural and two numerical patterning tasks. In most cases of figural and numerical tasks, the US sample consistently produced full structural extensions that were mostly repeating, which could be attributed to an instruction effect. Since patterns are not explored in the HK primary curriculum, the HK sample produced growth structures that were mostly linear and relatively few quadratic. Both samples, however, found the numerical task involving growth difficult.

CONTEXT OF THE STUDY

Early patterning activity provides entry into structural thinking, which is central to algebraic learning. For Clements and Sarama (2009), *patterning* basically involves constructing and being predisposed to establishing mathematical regularities and structures in both ordered and unorganized data. They write: "*Patterning is the search for mathematical regularities and structures*. Identifying and applying patterns help bring order, cohesion, and predictability to seemingly unorganized situations and allows you to make generalizations beyond the information in front of you. Although it can be viewed as a 'content area,' patterning is more than a content area; it is a process, a domain of study, and a habit of mind" (Clements & Sarama, 2009, p. 190). For Rivera (2010), a *structure* for figural patterns with several known stages could be any of the following three kinds: (1) *no structure*, which means there is no exact, predictable, coherent, and consistent count and shape from one stage to the next; (2) *partial structure*, which means either shape or count is preserved but not both; and (3) *full structure*, which means both shape and count are preserved.

Patterning is also associated with the important task of constructing and justifying a *generalization*. Here, we assume the following definition of generalization offered by Peirce (1960): "Generalization, in its strict sense, means the discovery, by reflection upon a number of cases, of a general description applicable to all of them….. So understood, it is not an increase in breadth but an increase in depth" (p. 256). A much desired mathematical depth involving generalization is the quality of *algebraic usefulness*, which is a generalization that could be conveyed algebraically in terms of a direct expression (i.e. a closed formula in function form; cf. Rivera, 2010).

In this research report, the term *early patterning competence* refers to young children's capacity to infer and impose regularities and structures on figural and numerical sequences of objects without the benefit of a sustained and formal instruction on the topic. The constructed generalizations indicate cognitive transformations in the way the learner thinks about the sequences together, which also signal depth in the manner they understand the sequences via the mathematical relationships that they find meaningful to construct. Examples of figural sequences (or patterns) are shown in Figure 1. Figure 2 is an example of a numerical sequence (or pattern) with a context. *Near generalization (NG) tasks* refer to stages 9 and below, while *far generalization (FG) tasks* refer to stages 10 and above. So, for example, items (i) through (vii) are NG tasks, while item (viii) is a FG task.

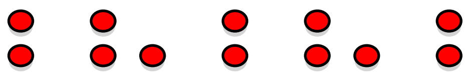
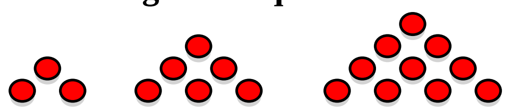
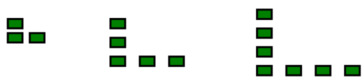
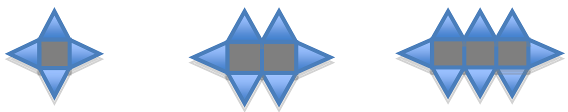
2 3 2 3 2 3 2 Sequence	**Triangular Sequence**
	
L Shape Sequence	**Square Shape Sequence**
	
House Sequence	**Flower Sequence**
	

Figure 1: Six Figural Sequences

i. This is a monkey. How many ears does it have?

ii. Now I have two monkeys. How many ears do they have in all? How did you get your answer? Show or tell me how. (Code: “Explain.”)

iii. I have three monkeys. How many ears do they have altogether? Explain.

iv. Now I have four monkeys. How many ears do they have in all? Explain.

v. I don’t have any more monkeys to show you. Suppose there are five monkeys. How many ears do they have in all? Explain.

vii. What if you have six monkeys, how many ears do they have altogether? Explain.

viiii. How many ears do 10 monkeys have in all? Explain.

Figure 2: A Numerical Sequence (Monkey Ears) with a Context

Since patterns are prevalent in one way or another in any school mathematics curriculum across the globe, it is possible to conduct cross-country comparisons of patterning competence among children despite being situated in different locations around the world. Recent international mathematics assessments have shown Hongkong (HK) SAR elementary students to be consistently performing at the top of the achievement ladder, and general differences have been oftentimes attributed to certain cultural practices (e.g. home expectations regarding knowledge of addition facts before coming to school (Siegler & Mu, 2008) and artifacts (e.g. visual-based inscriptions of Chinese characters, some weighing more than others.

The study is premised on the consistent finding that shows the superior mathematical performance of East Asian elementary children in a variety of mathematical contexts. HK SAR Primary 4 students, in particular, were ranked first in the 2007 Trends in Mathematics and Science Study, while the US Grade 4 students were ranked eleventh (Olson, Martin, & Mullis, 2007). Hence, it would be interesting to compare and analyze similarities and differences between HK and US Grade 1 students (mean age of 6 years) on grade-appropriate patterning tasks. We note briefly the Kindergarten and Grade 1 (or Primary 1) mathematics curricula in both countries that might have a bearing on how our Grade 1 samples might approach and analyze patterns. In the US, kindergarten students at least in California learn to “identify, describe, and extend simple patterns (such as circles or triangles) by referring to their shapes, sizes, or colors” (California Department of Education, 1999, p. 3). Further, in Grade 1, they begin to “describe, extend, and explain ways to get to a next element in simple repeating patterns (e.g., rhythmic, numeric, color, and shape)” (p. 6). In HK SAR, patterns appeared as a topic in the 1993 kindergarten curriculum with an explicit focus on simple repeating patterns limited to 2 colors or shapes (Curriculum Development Institute, 1993, pp. 121-122). However, in the 2006 kindergarten curriculum, there was no mention of the topic in either the mathematics-related section (Curriculum Development Council (CDC), 2006, pp. 31-32) or the explicitly recommended “exemplar” activities (ibid, pp. 91-92). Further, the fact that the “algebra dimension is not included at the (Primary 1 through 3) key stage” (CDC, 2000, p. 10) meant that patterns as a topic has also been excluded in the Primary 1 mathematics curriculum. The topic of “number patterns” appeared only as a suggested enrichment topic for Primary 6 students (ibid, p. 46).

METHODOLOGY

Participants 24 US Grade 1 (mean age of 6 years; 12 girls) and 24 HK Primary 1 (mean age of 6 years; 12 girls) participated in the study. All children were students from public schools. The US sample was the second author’s entire class, which consists of students that have been randomly assigned to available first-grade classes in the district. The HK group was a random sample of Primary 1 students from two schools that volunteered to participate in the study. All the subjects were given tokens for participating in the clinical interviews, and each interview lasted between 20 to 30 minutes. In both samples, the clinical interviews took place one month after the start of classes.

Clinical Interview Protocol and Tasks Seven patterning tasks were presented to each student one at a time. Figures 1 and 2 show the sequences that were used during the interviews. Manipulatives were provided for all six tasks under Figure 1, while a construction paper was provided in the case of the numerical task in Figure 2. Each student was asked think aloud throughout the interview process. For each figural task in Figure 1 presented to a student, the following process was followed: (1) The student was asked to either close his or her eyes or turn around in order to avoid any indirect influence from the interviewer who laid down each stage in a sequence one at a time; (2) The interviewer then pointed out to the student each stage in a given sequence ("here's my stage 1; stage 2; stage 3"); (3) The interviewer posed the question "How might stage 4 look to you?;" (4) The interviewer repeated the same questioning style in (3) several more times until the student arrived at stage 5 or 6; (5) When there was evidence of an emerging structure, the interviewer prodded the student to extend the pattern to stage 10. The numerical task in Figure 2 used 4 puppet monkeys that were shown to the student one by one. In the case of the House Sequence in Figure 1, the interviewer began with a story that involved building houses each day with the three stages corresponding to the number of houses built on three successive days. The first author conducted the interviews with the HK sample, while the second author conducted the interviews with the US sample. All the interviews were videorecorded. Both authors have copies of each other's videorecorded interview data set.

DESCRIPTION OF DATA ANALYSIS AND SUMMARY TABLES

The second author independently developed a coding template based on his initial data analysis of the US sample. The coding template used the structure shown in Table 1. Since there were 7 patterning tasks altogether, the second author developed a 7 x 7 array and tallied the students' responses accordingly. In Table 1, NG Tasks 1 and 2 refer to the two succeeding unknown stages following the known stages, that is, stages 4 and 5 in all the figural sequences except for the first one in Figure 1 and items (v) and (vi) in Figure 2. Based on the individual student responses on the NG tasks, their extensions were then categorized by kind, that is, no structure, partial structure, and full structure. If a full structure was evident, it was further categorized in terms of whether it was a repeating pattern or a growth pattern. Also, in Figure 3, their modes of extensions were noted in terms of whether the extensions were expressed verbally, gesturally, concretely with the blocks, or pictorially (i.e. if they drew pictures).

Since the second author was present during all the interviews with the HK students, both authors managed to engage in a separate-then-joint analysis of the HK data following the Table 1 template. Further, in the case of the numerical sequence in Figure 2, the students' arithmetical strategies (e.g. skip counting by 2; using addition facts; using doubles; counting all; counting on; employing mixed strategies) were noted and tallied accordingly.

Pattern Task (PT)	NG Task 1 (NG 1)	NG Task 2 (NG 2)	No Structure (NS)	Partial Structure (PS)	Full Structure (FS)	Mode of Extension

Table 1: Beginning Data Analysis Coding Template

Tables 2 and 3, which will be discussed in the next section, provide summaries of the student responses from the US and HK samples, respectively. To better understand the numbers, we note the following coding scheme: (1) A NS response meant incorrect NG 1 and NG 2, which meant that the NG 1 and 2 responses were not tallied but a single tally was assigned under NS; (2) A PS response meant either NG 1 or NG 2 was correct, which meant marking a tally for the correct one and another tally under PS; (3) A PS response could also have meant incorrect NG 1 and NG 2 in which case none of these responses were counted but a tally was counted under PS; (4) A FS response meant correct NG 1 and NG 2, which meant marking a tally for each response and another tally under FS. Since a PS accounts for consistency in either count or shape, having consistent shape does not necessarily imply consistent count. We had cases, as an example of (3), where constructed stages 4 and 5 of the Triangle Sequence both resembled triangles but the count was not consistently done within and across the stages. We also had cases, as an example of (2), where the counts corresponding to constructed stages 4 and 5 of the Square Sequence were both correct but the shapes they took were inconsistent (e.g. stage 4 looked like a square and stage 5 resembled a rectangle). We should point out that since the most frequent mode of extending the patterns in Figure 1 involved the use of the manipulatives on the table, we decided to eliminate the last column in Table 1. In the case of Figure 2, the most frequent mode of extension involved the use of a brief verbal explanation, which also meant doing away with the last column in Table 1.

RESULTS

The summary frequencies in Tables 1 and 2 indicate marked differences in the students' early patterning competence on seven tasks. In the case of the figural tasks shown in Figure 1, the US Grade 1 sample produced full structural extensions that were mostly repeating, while the HK Grade 1 sample had both repeating and growth responses. In the HK sample, in particular, while the growth structures for the House and Flower tasks were all linear (i.e. had a constant slope), all four Triangle Sequence extensions reflected growing triangular numbers (at least considering stages 1 through 5 together). In the US sample, quite a small number of students produced linear structures in comparison with the HK sample and none produced a growth sequence relative to the Triangle Sequence task.

With respect to the 23232 Sequence task, the small number of partial structural responses in both samples preserved the count from stage to stage and not the shapes in which the stages have been initially laid out to the students. Further, while the US sample shows successful extensions relative to the task, the HK sample found the task difficult to manage with 13 of the students failing to provide either a partial or full structure. Among the 10 HK students who provided a full structural response, 5 of them interpreted the sequence differently, as follows: 1 saw a 2122-2122-2122 sequence (Figure 3); 3 saw a 221-221-221 sequence (Figure 4); and 1 interpreted the initial sequence 23232 as one entire stage in which case his stage 2 was 23232.

PT	NG 1	NG 2	NS	PS	FS Repeating	FS Growth
2 3 2 3 2	24	24	0	3	21	0
Triangle	19	19	5	11	8	0
L Shape	20	20	4	3	17	0
Square*	19	19	3	1	19	0
House	22	21	1	4	17	2
Flower	20	20	3	1	15	5
Monkey	21	21	3	0	0	21

Table 2: US Data (n = 24; *1 student did not do, so n = 23 for this row)

PT	NG 1	NG 2	NS	PS	FS Repeating	FS Growth
2 3 2 3 2	11	9	13	1	10	0
Triangle	12	9	9	5	6	4
L Shape	8	7	11	4	4	5
Square	6	6	12	5	4	2
House	9	9	7	7	3	7
Flower	13	11	5	5	3	11
Monkey	16	16	0	0	0	25*

Table 3: HK Data (n = 24; *There is one double count.)

With respect to the Monkey task in Figure 2, the two samples were very similar in terms of their inability to deal with item (viii), a FG task that asked them to obtain the total number of ears in the case of 10 monkeys. The 4 students in the US sample who correctly produced the answer used two different strategies, as follows: 3 of them used all ten of their fingers, counted on from 2 all the way to 20; 1 student consistently used a doubles strategy ("1 monkey has 1 + 1 ears; 2 monkeys have 2 + 2 ears; 3 monkeys have 3 + 3 ears; 10 monkeys have 10 + 10 ears; 25 monkeys have 25 + 25 ears; and 50 monkeys have 50 + 50 = 100 ears in all"). In the case of the 2 students in the HK sample who produced the correct answer, they both verbally expressed their responses in terms of multiplication ("2x10 = 20" and "2 x 50, 50 + 50 = 100"). When asked how they learned to express them using multiplication, both noted their training in Kumon classes.

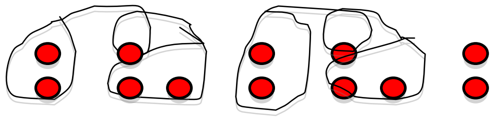


Figure 3: 2122-2122 Pattern

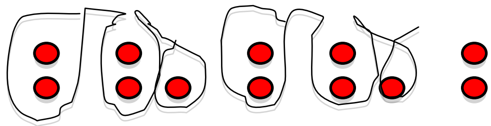


Figure 4: 221-221 Pattern

Still on the Monkey task in Figure 2, both the US and HK samples competently dealt with all NG tasks using a variety of arithmetical strategies. The HK sample provided the following NG strategies: 15 skipped counted by 2; 2 used multiplication; 2 counted all; 1 counted on from 2; and 6 employed mixed strategies. The US sample provided the following NG strategies: 14 counted all; 3 skipped counted by 2; 1 counted on; 2 used adding doubles; 2 used addition facts; and 2 employed mixed strategies. Figure 5 illustrates several of the mixed strategies that the students expressed verbally.

HK Grade 1	US Grade 1	HK Grade 1
Skip count by 2: 2, 4, 6, 8.	1 monkey: 1, 2.	1 monkey: 2.
5 monkeys have 8 + 2 = 10 ears.	2 monkeys: 1, 2, 3, 4.	2 monkeys: 2 + 2 = 4.
6 monkeys have 10 + 2 = 12 ears.	3 monkeys: 1, 2, 3, 4, 5, 6.	3 monkeys: 2 + 4 = 6.
10 monkeys have 2 x 10 = 20 ears.	4 monkeys: 1, 2, 3, …, 8.	4 monkeys: 4 + 4 = 8.
	5 monkeys: 2, 4, 6, 8, 10.	5 monkeys: 8 + 2 = 10.
	6 monkeys: 2, 4, 6, …, 12.	6 monkeys: 10 + 2 = 12.

Figure 5: Mixed Strategies in the Case of the Monkey Ears Task in Figure 2

Some of the NS responses on the figural tasks were quite interesting. While a NS response meant that both count and shape were not preserved from stage to stage, a few of the children's responses reflected justifications that have been drawn on their experiences in daily life. For example, in thinking about the Flowers sequence in Figure 1, 1 HK student (1) drew 2 long rows of squares and added a few squares diagonally on the bottom row for stage 4, (2) added 2 triangles on the diagonal for stage 5, and (3) finally gathered several squares for stage 6. For him, the Flowers sequence represented how a flower grew, gained roots, became firmly grounded on the soil, and then stepped upon leaving a few petals in the process. Several HK students extended the House Sequence in Figure 1 as a result of projecting their view of a house that gets bigger and taller. Since their actual houses were typically small, they interpreted the House Sequence in terms of how they wanted the stages to grow.

DISCUSSION

With respect to the figural patterns presented to the students, the US Grade 1 sample was definitely influenced by their earlier kindergarten experiences with repeating patterns. We are unable to make any definitive assessment in the case of the HK Grade 1 sample. What we found remarkable with the HK sample was the disposition toward growth patterns, both linear and quadratic. While most constructed repeating patterns could be classified as being fully structural at least from a developmental perspective, the constructed fully structural growth patterns conveyed incipient algebraically useful generalizations that would have positive long-term effects on the students' mathematical experiences.

Partial structural extensions are likely to lead to generalizations that either have a more global character, which, consequently, are not algebraically useful in most cases, or might eventually support and favor numeric-driven algebraically useful generalizations that could not be justified other than the fact that they fit the counts in which they have been drawn in the first place. Certainly, from a developmental perspective, shape-driven partial structural extensions are reasonable, but they are oftentimes more approximate than exact unlike algebraically useful generalizations that have an exact nature.

References

California Department of Education. (1999). Mathematics content standards for California public schools. Sacramento, CA: Author.

Clements, D. & Sarama, J. (2009). Learning and teaching early math: The learning trajectories approach. New York, NY: Erlbaum.

Curriculum Development Council. (2006). Guide to the pre-primary curriculum. Hongkong: Author.

Curriculum Development Council. (2000). Mathematics curriculum guide (P1-P6). Hongkong: Author.

Olson, J.F., Martin, M.O., & Mullis, I.V.S. (Eds.). (2008). TIMSS 2007 technical report. Chestnut Hill, MA: TIMSS & PIRLS Center.

Peirce, C. (1960). Collected papers of Charles Saunders Peirce: Volumes I and II. Cambridge, MA: The Belnap Press of Harvard University Press.

Rivera, F. (2010). Second grade students' preinstructional competence in patterning activity. In M. Pinto & T. Kawasaki (eds.), Proceedings of the PME 34 (Vol. 4, pp. 81-88). Brazil: PME.

Siegler, R. & Mu, Y. (2008). Chinese children excel on novel mathematics problems even before elementary school. Psychological Science, 19(8), 759-763.

---

[i] This research has been supported by a grant from the National Science Foundation awarded to F. D. Rivera under grant #DRL 044849. All the views and opinions expressed in this article are those of the authors and do not in any way imply the endorsement of the foundation.

ENRICA'S EXPLANATION: MULTIMODALITY AND GESTURE

Ornella Robutti

University of Torino

Laurie Edwards

Saint Mary's College of California

Francesca Ferrara

University of Torino

The perspective of embodiment and multimodality is applied to the analysis of a 10-year-old girl's mathematical discourse, which is part of a group discussion deciding on an appropriate input to a pedometer. The complementarity of several modalities used by the student is examined as well as the role of her gestures in constructing and expressing phenomenological and mathematical meanings.

INTRODUCTION

The goal of this paper is to consider the nature of multimodality in mathematical practice, and to use this perspective to analyse the discourse of a young mathematics student. Recent work on embodied cognition in mathematics is based on a model of thought and communication in which perception and action are intertwined, and which involves multiple bodily-based modalities. These modalities include speech and other verbalizations, written inscriptions (including words, symbols and graphics), gesture, and physical interaction with objects in the world or with virtual "objects" on computer screens. From the perspective of multimodality and embodiment, mathematical thought is not disconnected from physical experience, but ultimately based on it. This is not a new idea; nearly 30 years ago, the mathematician and physicist Struik (1986) stated the following about mathematics:

> Its abstract symbolism can blind us to the relationship it carries to the world of experience. Mathematics, born to this world, practised by members of this world with minds reflecting this world, must capture certain aspects of it—e.g., a "number," expressing correspondences between sets of different objects; or a "line," as the abstract of a rope, a particular type of edge, lane or way. The theorem you discover has not been hauled out of a chimerical world of ideas, but is a refined expression of a physical, biological, or societal property. (p. 286)

THEORETICAL FRAME: MULTIMODALITY & GESTURE

Kress (2001) describes multimodality as "the idea that communication and representation always draw on a multiplicity of semiotic modes of which language may be one" (p. 67-68). Because Kress's definition focuses on the role of multiple modalities only in communication, and not in perception and cognition, we propose a broader definition for modality: modalities are the cultural, social and bodily resources

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 27-33. Taipei, Taiwan: PME.

available for receiving, creating and expressing meaning. In addition to sensory modalities, which receive information, this category would also include motor modalities, such as gesture, bodily stance, touch and so on – essentially anything that humans can do with their bodies.

Gesture is an important modality for understanding mathematical thinking due to its pervasiveness in discourse. McNeill (1992) defined gestures as "movements of the arms and hands" that "are closely synchronized with the flow of speech" (p. 11). He further proposes that gestures, "belong, not to the outside world, but to the inside one of memory, thought, and mental images" (p. 12). McNeill identifies different types of gestures: beats (rhythmic, for emphasis), iconic (with concrete referents) and metaphorical (with abstract referents). We utilize gesture to study the continuum of thinking processes, examining not only the type of gesture, but also its semiotic power with respect to the construction of knowledge (Arzarello, & Robutti, 2009; Ferrara, 2006; Radford, 2009). Furthermore, our interest is on the interplay between different modalities, including gestures and speech, and the affordances and complementarities that have led to their simultaneous use in so many human contexts.

METHODS

The current study was carried out with 25 Grade 5 students (ages 10-11), in a teaching experiment carried out during the second part of the school year as part of the regular mathematics class. In addition to the classroom teacher, the researcher (first author) participated actively in the lessons. The goal of the teaching experiment was to assist students in understanding direct proportionality, starting from perceptuo-motor experiences. The main activity of the students was constructing symbolic and visual models of motions they performed. The students worked together in small groups, using a pedometer, which detects the number of steps taken by a person, and, given an input of step length, calculates total distance covered. The example discussed here is taken from a videotape of a class discussion about what specific input to provide to the pedometer.

DATA: ENRICA'S EXPLANATION

As background to the excerpt below, the researcher has modelled taking three steps, each intentionally different in length from the others, to make the students aware of the fact that they should consider an average value rather than simply choosing an arbitrary step to measure. Then, the students discuss how to calculate this average value. One student in particular, whom we have called "Enrica", has clear ideas on how to do it. In the excerpt below, she explains her ideas to her classmates, the teacher and the researcher:

Enrica:	You have made three steps, so one of 54, one of 57, one of 75 centimetres, you made them different, then we sum them up and we do the average value and we obtain, as average, the length of the steps.
Researcher:	That's a good idea.

Antonella: You take a step and then you multiply by three.

Researcher: I take a step and then I multiply it by three, yes, this is another idea. And there is one method more … say, can we try to make all the steps equal?

Someone: No!

Enrica: We can try.

Researcher: […] so I suggest you to try to make some measurements in every group, of your steps when walking and then we can use Enrica's idea of … what did she tell us, Sara?

Sara: To do the average value.

In Table 1, we present Enrica's verbal explanation again, showing its interplay with the modalities of gesture and head movement. The words spoken during the stroke (emphasized) phase of each gesture are underlined.

<table>
<tr><th>*Speech*</th><th colspan="2">*Gesture*</th><th>*Head Motion*</th></tr>
<tr><td>***Enrica**: You have made <u>three steps</u>*
(Figure 1)</td><td></td><td>palm facing right</td><td>Head goes down (chin closer to chest)</td></tr>
<tr><td>*so one of <u>54,</u>*
(Figure 2a,b)</td><td></td><td>palm up, hand dropped to table</td><td>Head nodding</td></tr>
<tr><td colspan="3">(Gestures in Figure 2a,b repeated for each number Enrica states)</td><td></td></tr>
<tr><td>*one of 57, one of 75 <u>centimeters</u>*
(Figure 3)</td><td></td><td>palm down on centimeters</td><td>Head stops (chin almost touching chest)</td></tr>
<tr><td>*you made them <u>different</u>*
(Figure 4)</td><td></td><td>palm up, hand spread</td><td>Head stops (chin almost touching chest)</td></tr>
</table>

then we sum them up (Figure 5)		palms toward chest, fingers touching	Head up, starting to move down
and we do the average (Figure 6)		both palms open at an angle	Head stops in forward position
And we obtain, as average (Figure 7a,b)	 	right palm moves over left, then both palms open at an angle	Head is lifted
the length of the steps (Figure 8a,b)	 	fingertips touching, then hands spread, palms facing	Head up

Table 1. *Enrica's explanation*

ANALYSIS OF ENRICA'S DISCOURSE

In terms of the mathematics of the activity, Enrica immediately proposes an appropriate solution: to use the lengths of each of the three steps taken by the researcher, sum them, and determine the average step size. Enrica's explanation of her solution is rich with multimodality. The gestures that accompany Enrica's speech and her tone of voice are meaningful: at almost every word there is a gesture with a particular shape, function, intentionality and affordance. Enrica also uses head movements and voice intonation as modalities in constructing and expressing her knowledge.

As Enrica states the numerical measurement of each step, she uses a beating gesture with the left hand moving down to the desk, palm up, with a rhythm insisting on the measures of the three steps (Figures 2a,b). The unit of measurement, centimeters, is then emphasized by a palm-down gesture that closes the series of words and gestures (Figure 3). This gesture, we propose, is different in form from the previous ones because it has a different status in the process Enrica is explaining. It is used to convey the unit of measure, which is important because it needs to be used when inserting the value in the pedometer, and it is not one of the sequence of specific numbers emphasized by the rhythmic beating gestures.

Enrica then depicts the operation of the sum of these three measures by bring both hands together, palms inward and tips of corresponding fingers touching. (Figure 5). In expressing the idea of addition, this gesture has a metaphorical dimension, expressed through the iconicity of physically “bringing together” or collecting. Thus, it reflects the “Arithmetic is Object Collection” metaphor, one of the basic metaphors of arithmetic identified by Lakoff and Núñez (2000).

She next offers a gesture coordinated with her words about finding the average, opening her arms and hands, palms up (Figure 6). This gesture, rather than being an attempt to communicate the idea of mathematical average, is more likely an interactive gesture (Bavelas, Chovil, Coates, & Roe, 1995). It belongs in a family of gestures used to metaphorically “offer” an idea to interlocutors. The simplest of this type of gesture is putting the palm up and toward the interlocutor, as if offering an object. These gestures are based on the metaphor that communication is transfer of a material object, a ubiquitous though culturally specific metaphor (McNeill, 1992; Bavelas *et al*., 1995). One interpretation is that at this point Enrica is “offering” the idea of using the average to her audience; the two-handed, open gesture also carries the suggestion (again, culturally-specific) that this idea should be accepted as obvious. This is reinforced by Enrica’s head movement and posture (head forward and shoulders up a little) (Figure 6), and by the repetition of the same word (“average”) with the same gesture (Figures 6 and 7b). Finally, once the idea of taking the average is introduced and shared within the class, Enrica proposes that it be used as the step length input to the pedometer, with “length” indicated by bringing together the finger tips of the two hands (Figure 8a), and “step” indicated with an iconic “measuring” gesture (palms facing and parallel, Figure 8b).

The expressivity of Enrica’s gestures is very high. However, the gestures displayed during this sequence have two different kinds of referents. The beating and iconic gestures at the beginning of the sequence characterise the process of movement (taking steps, measuring, and counting). On the other hand, later gestures in which Enrica opens and closes both hands seem to be metaphorical gestures referring either to mathematical objects (sum, length, average), or to be interactive gestures of communication. During the first group of gestures, Enrica is recalling the action of the researcher as she took the three steps and measured them, a bodily, phenomenological experience. However, the second group of gestures are directed to the construction of meanings that are new at that moment for the students. With these gestures, Enrica moves towards abstraction and proposes the concept of the average step length in order to express the appropriate input for the pedometer.

Gestures, words and head movements generally match each other in this brief excerpt; this redundancy across modalities suggests that Enrica fully understands the ideas that she wants to depict and explain to the teacher and her classmates.

We can call Enrica's use of different modalities simultaneously *multimodal expressivity*; as noted earlier, this kind of expressivity is typical in human communication. Each modality has different characteristics, as noted by McNeill (1992): the speech is linear and analytic; the gestures and the head movements are global and synthetic. Furthermore, their diverse characteristics let them overlap and integrate with each other to express the same meaning. These three modalities overlap each other, but each one of them has its particular affordances. The words (speech) provide the exact measurements of the steps; the gestures give emphasis to the discrete values attached to each measurement as well as identifying their individuality, and the head amplifies this emphasis.

DISCUSSION

The analysis of Enrica's discourse shows the complex dynamic interplay of different modalities and how they intertwine to bridge the gap between the students' personal senses (stimulated by the use of the tools) and the mathematical knowledge, derived from culture and community, related to the concept of average step length. Each of the modalities Enrica utilizes in expressing herself participates in the communication process in a complementary way, while at the same time "conveying" the same general information. This multimodality is demonstrated by the simultaneous talking, gesturing, and motions of the head, as well as posture and other bodily expressions not analyzed here. Each of these modalities has different affordances and play different roles in the communication process. At the beginning of the episode, Enrica's words have the specific function of recalling the walking experience; the numbers she states allow the specification of the three steps in terms of their lengths. The language used is linear and analytic, and draws on semiotic systems well known to her interlocutors (the spoken word and whole numbers). At this point, Enrica is expressing mostly the phenomenological side of the experience: the taking of steps and measuring them. In the second phase of the episode, we note a change of perspective, in which Enrica moves from the personal, phenomenological aspect to focus more on the mathematics. In addition to evoking the sum of the step lengths, she also proposes the use of the average as the input value for the pedometer. The change in perspective is evident in the new gestures that enter the scene. Instead of only beats and iconic gestures, we now see gestures with a metaphorical dimension (e.g. for the sum, the average value, etc.) and an abstract reference.

In this second phase, the character of the spoken language is again linear and analytic, while that of head movements and gestures is global and synthetic; each modality reinforcing the meaning expressed in the other. For example, when discussing the sum of the measurements, Enrica brings her hands together while leaving an open space between her arms, in a kind of "gathering" motion. This global-synthetic gesture provides an instantaneous image of an idea that is communicated linearly, one word at a time, via speech, that summing is like bringing together or collecting.

The complementarity of the modalities in play can be appreciated through a thought experiment. If, while watching the video of this brief class episode, we turned off the audio (speech), we would be able to identify the appearances of many gestures and head movements. We would be able to grasp that some gestures are repetitive and that some head movements appeared to be used as a means to emphasize certain elements of the discourse. But, we would not be able to understand the full meaning or the depth and richness of Enrica's reasoning without access to the modality of speech. A similar argument may be used with respect to the affordances of the other modalities. For example, if we turned off the video and listened only to the audio, we would lose the visual markers of the intensity of the communication, as well as the imagery displayed in Enrica's different gestures.

Thus, it is important to be aware that single modalities, analyzed in isolation, cannot provide a full picture of what is going on within the classroom. The goal, when engaging in an analysis of mathematical activity from the perspective of multimodality, is to build an understanding of learning, teaching and thinking that both utilizes the various kinds of information communicated by all the modalities present, and also takes account of their roles in the construction of knowledge.

References

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*. 97-109.

Bavelas, J. B., Chovil, N., Coates, L., & Roe, L. (1995). Gestures specialized for dialogue. *Personality and Social Psychology Bulletin, 21*, 394-405.

Ferrara, F. (2006). Remembering and imagining: Moving back and forth between motion and its representation. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, 3* (pp. 65-72). Prague, Czech Republic: Charles University, Faculty of Education.

Kress, G. (2001). Sociolinguistics and social semiotics. In P. Copely (Ed.), *The Routledge companion to semiotics and linguistics*, 66-82. London: Routledge.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: Chicago University Press.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(3), 111-126.

Struik, D. J. (1986). The sociology of mathematics revisited: A personal note. *Science and Society, 50*, 280–299.

HEURISTICS IN THE PROBLEM SOLVING PROCESSES OF FIFTH GRADERS

Benjamin Rott

Leibniz University of Hanover

This paper reports on an exploratory study which investigates (mathematical) problem solving processes of fifth graders (aged 10 to 12) from German secondary schools. 64 processes of pupils working on a geometry and an arithmetic task are analyzed with regard to the use of heuristics. It can be shown that the children – though untrained as problem solvers – do use several different heuristics and that their usage of heuristics is significantly related to success in solving the assigned mathematical problems.

BACKGROUND

Problem solving is a very important part of mathematics and thus also fundamental for school mathematics (cf. Schoenfeld 1992, p. 2 ff.). The term "problem solving" has different meanings ranging from solving routine tasks to solving perplexing or difficult situations (ibid. p. 10 ff.). I refer to the latter interpretation which is also the position of Mayer & Wittrock (2006, p. 287):

> "When you are faced with a problem and you are not aware of any obvious solution method, you must engage in a form of cognitive processing called *problem solving*. Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver […]."

It is important to note that the attribute "problem" depends on the solver, not on the task. A difficult problem for one student can be a routine task for another (e.g. more experienced) one. Thus, research on problem solving should focus on the problem solving *process*.

A major role in those processes is played by *heuristics* (cf. Schoenfeld 1985, p. 44 f.), which are "rules of thumb for successful problem solving" (ibid., p. 23) or "methods and rules of discovery and invention" (Pólya 1945, p. 112). There is a multitude of examples of heuristics like *drawing a figure* or *working backwards*, but there is no precise definition of this term. Kilpatrick (1967, p. 19) therefore paraphrases "a heuristic as any device, technique, rule of thumb, etc. that improves problem-solving performance." Koichu et al. (2007, p. 101) specify it this way:

> "We refer to the concept of *heuristics* as a systematic approach to representation, analysis and transformation of *scholastic mathematical problems* that actual (or potential) solvers of those problems use (or can use) in planning and monitoring their solutions. […] Heuristics at large can be seen as a cognitive tool used to approach problems, effectiveness of which is never known in advance."

Heuristics can be narrow and domain-specific – like *reduce fractions first* – or universal – like *looking for a related problem* (Koichu et al. 2007; Schoenfeld 1992).

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 35-42. Taipei, Taiwan: PME.

Several studies show that "students' use of heuristic strategies was positively correlated with performance on ability tests, and on specially constructed problem solving tests; however, the effects were relatively small." (Schoenfeld 1992, p. 52) Komorek et al. (2007, p. 193) report on significant positive correlations between "the use of heurisms and achieved attainment in the maths post test" (Pearson: r=.61, p<.01) in an intervention study for 7th and 8th graders in German secondary schools; but they only evaluated written results and did not look at videos of the problem solving processes. The research group of Stein (Stein 1996; Burchartz & Stein 1999) showed that children in grades 3 – 4 are able to use mostly the same problem solving techniques (like *backtracking* or *combinatorial exhaustion*) as students in grades 8 – 9 working on unsolvable arithmetical tasks and geometrical puzzles.

At present, there is a lack of research regarding the development of problem solving abilities, especially within younger children (cf. Heinze 2007, p. 15). And – apart from the work of Stein – there are only a few results for younger children related to their genuine problem solving abilities. In Germany, 5th grade marks the transition between primary school and different types of secondary schools, to which pupils are assigned to with regard to their marks and abilities. Therefore, it is interesting to research the problem solving abilities and the usage of heuristics of children in grade 5.

For this paper, I want to address the following *research questions*:

- Do 5th graders, who are untrained as problem solvers, use heuristic strategies?
- To what extent is the use of heuristic strategies related to success in problem solving?

DESIGN OF THE STUDY

From November 2008 till June 2010, the first four terms of our support and research program MALU[1] took place, which is the basis for my research (see Rott 2011 for additional results). Each term, a new group of 10 – 16 (45 altogether) interested fifth graders (aged 10 to 12) from secondary schools in Hanover was formed.

These pupils came to our university once a week and the sessions usually followed the following pattern: After some initial games and tasks, the students worked in pairs on 1 – 3 mathematical problems for about 40 minutes and were videotaped thereby. They eventually presented their results to the whole group. The pupils worked on the problems without interruptions or hints from the observers, because we wanted to study their uninfluenced problem solving attempts. We decided not to use an interview or a think-aloud method, so as not to interrupt the students' mental processes. To get an insight into their thoughts, we let the children work in pairs to interpret their communication in addition to their actions.

---

[1] ***M**athematik **A**G an der **L**eibniz **U**niversität* means to *Mathematics Working Group at Leibniz University*.

Overall, we used more than 30 different tasks, selected to represent a wide range of mathematical areas and to allow the use of different heuristics. For the analyses presented in this article, I chose two of those tasks with all related pupils' processes[2] (see Table 1).

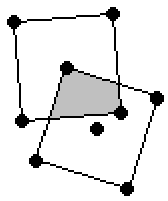
Beverage Coasters	**Marco's Number Series**
The two pictured squares depict coasters. They are placed so, that the corner of one coaster lies in the center of the other.  Examine the size of the area covered by **both** coasters.	Marco wants to arrange the numbers from 1 to 15 into the caskets so that the sum of every adjoining pair is a square number: □□□□□□□□□□□□□□□ For instance, if there are the numbers 10, 6, 3 in three consecutive caskets, the 6 adds up to a square number with its left (10+6=16) and its right neighbor (6+3=9). How could Marco fill-up his 15 caskets?
Idea: Schoenfeld (1985, p. 77)	Source: Fürther Mathematikolympiade, 2005/06, 1. round (www.fuemo.de)

Table 1: The two tasks selected for the analyses in this paper (translated by the author)

METHODOLOGY

Product Coding: To determine the pupils' success in problem solving, their work results were graded in four categories: (1) *No access*, when the pupils showed no signs of understanding the task properly or did not work on it meaningfully. (2) *Basic access*, when the pupils mainly understood the problem and showed basic approach. (3) *Advanced access*, when they understood the problem properly and solved it for the most part. And (4) *full access*, when the pupils solved the task properly and presented appropriate reasons.

This grading system was customized for each task with examples for each category. Then, all the products were rated independently by the author and a research assistant. After calculating Cohen’s kappa (Bakeman & Gottman 1997, p. 62), which was κ=.87 for the Coasters and κ=.91 for the Number Series task, we discussed the few products with different ratings, reaching consensus every time. It is important to note, that the two members of a pair could gain diverse ratings, when their written results differed.

Process Coding: As there is no precise definition of the term “heuristic” and as heuristics can be domain- and task-specific (see above), we chose qualitative methods to start our analysis. In company with a research assistant, I watched all the videos of our pupils, who were trying to solve the selected tasks (about 800 minutes of video).

[2] We videotaped 43 pupils working on the two tasks; 32 working on each task with 21 of those pupils working on both.

We took notes of every pupils' action that looked like a rule of thumb, problem-solving technique, etc. (*inductive* approach). We also looked-up descriptions of heuristics in the literature and tried to find accordant behavior in the videos and wrote it down (*deductive* approach). In conjuction with the research assistant, I then characterized the heuristics from both lists by adding names (like *examining special cases*) as well as general and task-specific descriptions of the pupils' behavior (*consensual validation*).

After this qualitative process, these characterizations served as a coding manual (see Table 2) and all the processes were coded independently by three other research assistants to determine the points of time and the names of the heuristics reliably. We applied the “percentage of agreement” (Bakeman & Gottman 1997, p. 59)[3] approach to compute the interrater-reliability as described in the TIMSS 1999 video study (cf. Jacobs et al. 2003, p. 103 f.) for randomly chosen videos (40 % of the processes). We achieved more than $P_A = 0.7$ for identifying points of time in the videos with heuristics and more than $P_A = 0.85$ for characterizing the heuristics. After calculating the reliability, we attained agreement by recoding together all differing codes.

Code	**Description**	**Examples**
Drawing a figure	Drawing a figure, a graph, or a diagram.	**Coasters**: a drawing of possible positions of the two squares. **Number Series**: drawing a diagram of numbers with possible neighbors.
Special cases	Assigning special values (like 0 or 1) to algebraic problems or examining special positions in geometric problems.	**Coasters**: positions of the two squares which make it evident that the marked area amounts to one forth of a square.
Back-tracking	Working forwards until being stuck; then tacking back steps to try alternate ways.	**Number Series**: working till getting stuck (e.g., at 8 or 9) and then deleting numbers till being able to proceed with another number.

Table 2: An extract of the heuristics coding manual (on the basis of Koichu et al. 2007)

RESULTS

To answer the first research question: Even though our pupils were inexperienced as problem solvers,[4] at large they did use a multitude of different heuristics working those two tasks. Here are descriptions of the ones with more than one or two occurrences:

Working on the **Beverage Coasters** task 22 of the 32 pupils *measured* the side lengths of the marked area, but most of them realized that this is better suited for calculating

[3] Chance-corrected measures like Cohen’s kappa are not suitable for this calculation, as there is no model to calculate the agreement by chance for a random number of heuristics distributed randomly over the course of the process.
[4] An explicit training of heuristic usage is not part of the school curriculum in grades 1 to 4 in Germany. And they didn’t get an appropriate training in our support program either.

the circumference than for examining the area. 4 of those pupils tried to *decompose* the marked quadrilateral into rectangles and triangles to calculate its' areas.

15 pupils *drew figures* of squares in different positions and 11 pupils considered *special cases* (only 3 pupils drew both special cases, cf. Table 2) (see Figure 1). *Auxiliary elements* (prolongations of the sides of a square or other auxiliary lines) were used by 10 pupils (see Figure 1).

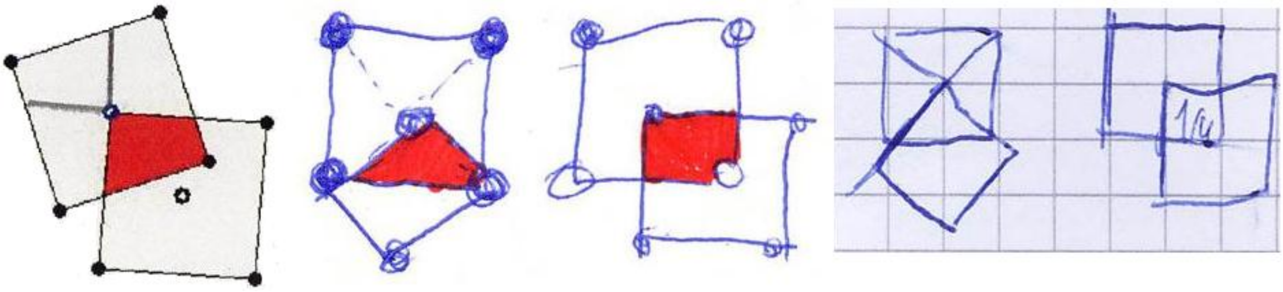


Fig 1: Heuristics of the Coasters task, figures of special cases partly with auxiliary lines

For **Marco's Number Series**, several of the 32 pupils used *tools of systematization*: 17 pupils made lists of the numbers from one to fifteen to discard the ones already used; and 12 pupils compiled a list of square numbers (and in doing so often realizing that square numbers greater than 25 cannot be reached) (see Figure 2).

Seven pupils *searched for patterns* by looking for possible combinations of adjoining numbers (e.g., 4 = 1+3; 9 = 1+8 = 2+7 = …); in the process, many pupils noticed that 8 and 9 have only one possible neighbor each and have to start or end the line (see Fig 3).

Actions that were labeled as *backtracking* (cf. Table 2) were used by 4 pupils. Also 4 pupils *tried systematically* by starting their number series with ascending numbers one after another (the first series started with “1”, the second with “2”, and so on).

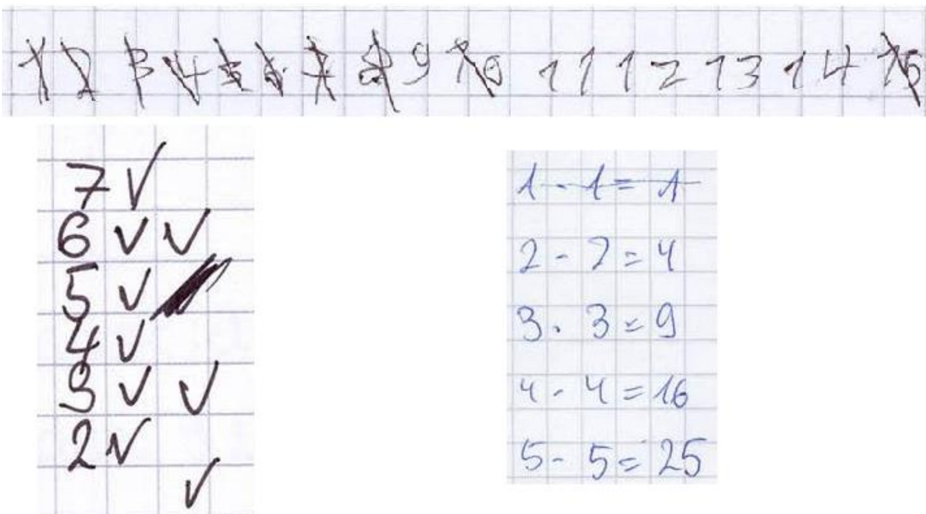


Fig 2: Heuristics of the Number Series task, tools of systematization

One girl *drew a figure* aiding her finding possible combinations of numbers (see Fig 3). However, this figure differs explicitly from the ones drawn for the Coasters task. This is in accordance with the reasoning of Schoenfeld (1992, p. 53 ff.) that heuristics can show very diverging characteristics for different problems (and are therefore hard teach or learn).

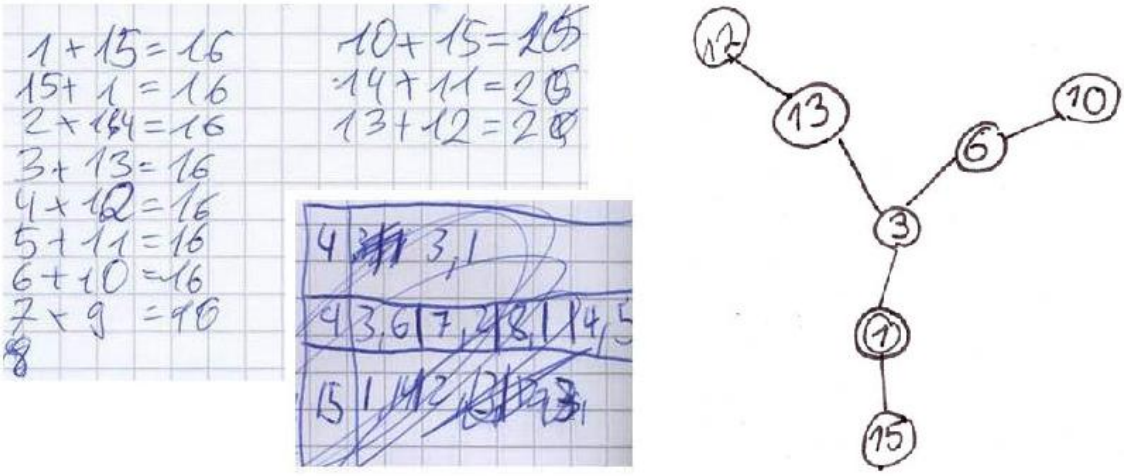


Fig 3: Heuristics of the Number Series task, search for patterns & informative figure

The second research question deals with success / failure in solving the problems and to what extent the heuristics are related to it.

Only 3 of 32 pupils scored product category 4 (full access) at the **Beverage Coasters** task. And another 6 pupils scored category 3 (advanced access). Most of these 10 pupils considered *special cases* and 4 of them used *auxiliary lines* to show that the marked area is always one fourth of a square. But these heuristics were also used by pupils who weren't that successful. 18 of the 22 pupils that *measured* reached only category 1 or 2, so this heuristic seems to be inefficient for this task.

The **Number Series** task was easier for our pupils; only 12 of 32 scored category 1 or 2 and only 3 of these unsuccessful problem solvers showed signs of heuristic usage. All other heuristics described above were used by the successful pupils – with no heuristics that were used solely by the 13 pupils in category 3 or the 7 pupils in category 4.

Figure 4 displays the number of heuristics used by the pupils in comparison to their product grades. These scatter plots show that there are at least weak linear correlations – the more heuristics a pupil used the more likely s/he was successful in solving the task (or vice versa). Spearman rank-order correlations[5] of these data show significant ($p<.01$ each) correlations for the Coasters ($\rho=.54$) and the Number Series ($\rho=.68$) tasks. These values match the correlations reported by Komorek et al. (2007) and meet the expectations as the use of heuristics should be helpful in solving problems.

[5] As the product categories yield only ordinally scaled data, no Pearson correlation coefficient was calculated.

However, there are successful processes with only one or two heuristics as there are unsuccessful ones with three or four. Some pupils picked a goal-oriented heuristic and used it to solve the task outright; otherwise heuristics do not seem to help everytime. As expected, there is no straight “the-more-the-better” rule for the use of heuristics.

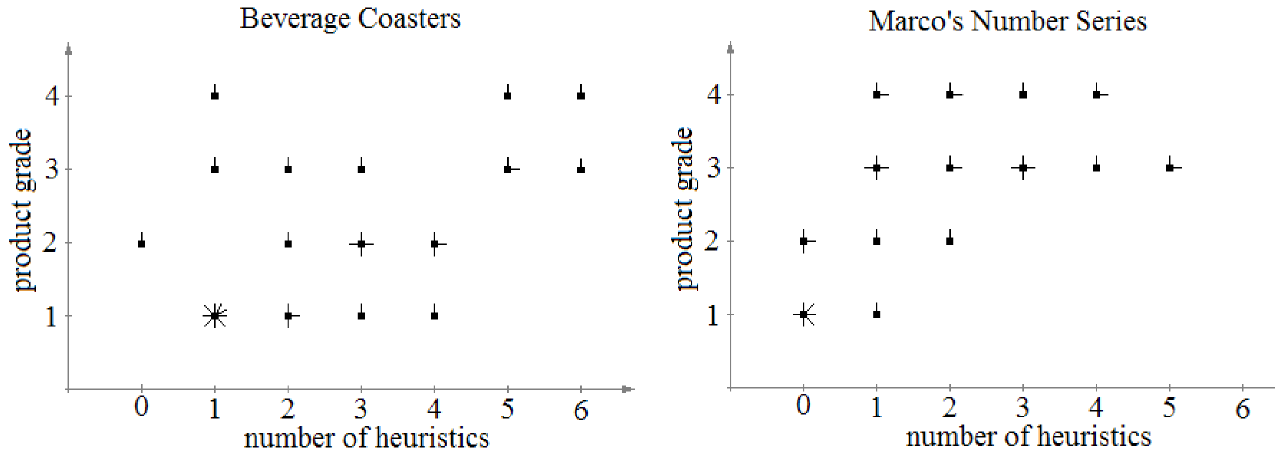


Fig 4: The number of heuristics compared to the product grades for both tasks (the number of the bars indicates the number of the pupils for each entry)

DISCUSSION AND CONCLUSIONS

These results are important as they show that even children at the age of 10 – 12 are able to and do use heuristics in problem solving – and that they benefit from doing so. The teaching of problem solving in schools does not need to start from scratch but can be based on the already existing skills and abilities of the pupils. Therefore, it is of importance to know more about the genuine problem solving abilities of children. In the coming months, the processes of our pupils working on other tasks will be analyzed with the same methods. A first coding of additional data shows that the results presented in this paper appear to hold true for a combinatorial task as well.

But, of course, the presented correlations do not explain success or failure in problem solving entirely as there are other factors that contribute to problem solving (cf. Schoenfeld 1985). A very important factor is self-regulation (cf. Schoenfeld 1992, p. 57 ff.), which is also intertwined with the use of heuristics, as “the successful use of such strategies calls not only for ‘knowing’ the strategies, but for good executive decision-making [control / self-regulation]” (Schoenfeld 1985, p. 95). Thus, I intend to analyze our processes with regard to metacognition / self-regulation.

Finally, given that all processes have been parsed into *episodes* using an adapted version of the protocol analysis framework from Schoenfeld (1985, ch. 9) (see Rott 2011 for details), I plan to look at the positions of the coded heuristics in the processes. Do they mainly occur during Analysis-/Understanding episodes or during Exploration-/Planning episodes (cf. Pólya 1945; Schoenfeld 1985)? Do they help in devising a plan?

References

Bakeman, Roger; & Gottman, John M. (1997). *Observing interaction: An introduction to sequential analysis*. New York: Cambridge University Press. 2. edition.

Burchartz, Birgit; & Stein, Martin (1999). Reasoning and Problem Solving Processes of Primary and Secondary Students working on Tasks with a Goal that cannot be reached. In Weigand, Hans-G.; Neill, Neville; Peter-Koop, Andrea; Reiss, Kristina; Törner, Günter; & Wollring, Bernd (Eds.). *Developments in Mathematics Education in German-speaking Countries – Selected Papers from the Annual Conference on Didactics of Mathematics,* Bern. http://webdoc.sub.gwdg.de/ebook/e/gdm/1999/stein_99.pdf (20.12.2011)

Heinze, Aiso (2007). Problemlösen im mathematischen und außermathematischen Kontext. Modelle und Unterrichtskonzepte aus kognitionstheoretischer Perspektive. In *Journal für Mathematikdidaktik (JMD), 28(1)*, p. 3 – 30.

Jacobs, Jennifer; Garnier, Helen; Gallimore, Ronald; Hollingsworth, Hilary; Givvin, Karen B.; Rust, Keith et al. (2003). *Third International Mathematics and Science Study 1999 Video Study Technical Report. Volume 1: Mathematics*. Washington: National Center for Education Statistics. Institute of Education Statistics, U. S. Department of Education.

Kilpatrick, Jeremy (1967). *Analyzing the solutions of word problems in mathematics: An exploratory study*. Unpublished doctoral dissertation, Stanford University. Dissertation Abstracts International, 1968, 28, 4380-A. (University Microfilms, 68-5, 442).

Koichu, Boris; Berman, Abraham; & Moore, Michael (2007). Heuristic literacy development and its relation to mathematical achievements of middle school students. In *Instructional Science, 35*, p. 99 – 139.

Komorek, Evelyn; Bruder, Regina; Collet, Christina; & Schmitz, Bernhard (2007). Contents and results of an intervention study in mathemtics lessons at secondary level I with a teaching concept to support mathematical problem-solving and self-regulative competencies. In Prenzel, Manfred (ed.). *Studies on the educational quality of schools: the final report on the DFG priority programme*. Münster: Waxmann, 2007, p. 175 – 197.

Mayer, Richard E.; & Wittrock, Merlin C. (2006). Problem Solving. In Alexander, Patricia A. & Winne, Philip H. (Eds.): *Handbook of Educational Psychology*. Routledge, p. 287 – 304.

Pólya, George (1945). *How To Solve It*. Princeton Paperback Printing, 2. Printing 1973.

Rott, Benjamin (2011). Problem Solving Processes of Fifth Graders: an Analysis. In Ubuz, B. (Ed.). *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4. Ankara, Turkey, p. 5 – 72.

Schoenfeld, Alan H. (1985). *Mathematical Problem Solving*. Orlando, Florida: Acad. Press.

Schoenfeld, Alan H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In Grouws, Douglas A. (Ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan, p. 334 – 370. http://www-gse.berkeley.edu/faculty/AHSchoenfeld/Schoenfeld_MathThinking.pdf (online edition, 01.12.2011).

Stein, Martin (1996). Elementare Bausteine der Problemlösefähigkeit: Problemlösetechniken. In *Journal für Mathematik-Didaktik, Jg. 17 (1996), Issue 2*, p. 123 – 146.

DYNAMIC GEOMETRY, IMPLICATION AND ABDUCTION: A CASE STUDY

Carmen Samper, Leonor Camargo, Patricia Perry, & Óscar Molina

Universidad Pedagógica Nacional, Bogotá, COLOMBIA

In this paper we illustrate the role of dynamic geometry as an environment that propitiates the use of empirical explorations to favor learning to prove. This is possible thanks to abductive processes, related to the establishment of implications that university students of a plane geometry course carry out when, supported by a dynamic geometry program, they solve a problem in which they must discover a geometric fact, formulate a conjecture and prove it.

INTRODUCTION

The potential of dynamic geometry programs to favor the connection between empirical exploration of geometric figures, the formulation of conjectures and the production of deductive chains is widely recognized (Laborde, 2000; Olivero, 2002; Cerulli & Mariotti, 2003; Mariotti, 2007; Arzarello, Olivero, Paola & Robutti 2007, Fujita, Jones & Kunimune, 2010). In such a connection, as our study reveals, both the establishment of implications and abductive argumentation play a primary role. From our point of view, the frequent use of the program permits students to recognize that any result obtained with it is possibly valid in the theory that the program models, and to take advantage of this circumstance to look for its justification in that theory.

In the problem solving process analysis, the above mentioned authors have signaled out the role of abductive argumentation as the contact point between conjecture production and proof construction. Yet, from our point of view, there is a need for greater research evidence of the role dynamic geometry plays to orient the search of a specific thematic core within a theory and to identify those properties that can be used to justify a conjecture. This deficiency leads us to pay close attention to the student's arguments in the different moments of a problem solving process, to analyze the effect of the use of dynamic geometry in those arguments and link between the establishment of implications and the abductive processes.

THEORETICAL REMARKS

We consider *exploration*, in general, as a heuristic type of activity that can be carried out in the world of phenomena and in the theoretical world. In the world of phenomena, exploration is realized on geometric figure representations and it has an empirical character. We therefore refer to it as *empirical exploration*. When it is carried out in a dynamic geometry environment, the objective is to detect invariants and formulate them as regularities as properties, through inductive arguments. We name this activity *dynamic exploration*. In the theoretical world, the exploration is realized on the statements that make up individual knowledge. We refer to it as *theoretical exploration*. It is carried out with the purpose of recognizing or finding statements that permit justifying an affirmation or making decisions about where to direct empirical exploration.

From a mathematical point of view, an implication is a narrative which expresses that a statement is a logical consequence of a theory (Arzarello, 2007) and therefore, if such theory, or the part of it that is of interest, is admitted to be valid, then the statement is also, once a proof of its validity is produced. In other words, a conditional statement, $p \rightarrow q$, is a logical consequence of a theory if q can be obtained, from using the theory. In the educational realm, we are interested in identifying possible *p*, implication manifestations linked to the recognition of a work space in which efforts in finding a path to justify the conditional and being able to affirm that it is a logical consequence of the theory are concentrated, even if there is no clear way to construct the justification.

Once placed in a work space, we consider as an *abductive process* the act of evoking specific conditional statements with the same consequent as the formulated conjecture that is going to be proven, to obtain a possible antecedent which leads to the consequent. This notion is compatible with Peirce's abduction (Arzarello, Olivero, Paola & Robutti 2007); evidencing this lets us assure that the evocation of that theoretical work space actually gave place to the establishment of an implication. We differentiate the process of establishing an implication from that of formulating an abduction because in the first case a theory is referred to, or part of one, and in the second case, the reference is to one or more rules or specific statements that can be later connected with the consequent found.

RESEARCH CONTEXT

In our research study we adopted a qualitative methodology situated within the descriptive-interpretive tradition. We gather information in the natural classroom context that is interpreted through analysis categories that arise from the data study and from the conceptualization we develop. Guided by the framework, we search for evidence of the connections students make between their experimental activity and the recognition that a property is logical consequence of a theory, giving way to possible implications throughout the problem solving process.

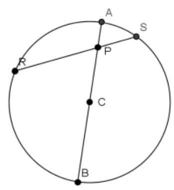
The students, who in the first semester of 2008 were enrolled in a Euclidian geometry course that is part of the mathematics requirements of the pre-service teacher program of the Universidad Pedagógica Nacional (Bogotá, Colombia), were invited to solve the following problem: Using dynamic geometry, construct $\odot C$ and a fixed point P in its interior. For which chord AB of the circle, containing point P, is the product $AP \times BP$ maximum?

The problem requires that students recognize that to theoretically be able to establish that there is no maximum value it is necessary to use the relationships between the lengths of the segments determined by point *P* in two chords that contain it, and to discover that angles subtending the same arc are congruent, a fact they did not know. We consider that the problem has optimal characteristics for our study: (i) it is similar to other problems proposed to the students in previous courses, in which they are asked to find the conditions so that a certain property is satisfied; (ii) the students had a theoretical knowledge in geometry that permits interpreting the representation phenomena, due to the construction of geometric figures and their exploration; (iii) the students had sufficient experience in the dynamic geometry software management, reason why we supposed they would use it ideally as an exploration environment to establish a regularity that would become a conjecture; (iv) a first approximation to the problem generally favors an anticipating a result that is later discarded or ratified with exploration, and therefore, the interest to justify their findings is impulse.

CASE STUDY

Due to space limitations, to illustrate the analysis we made we will present only some moments of the work carried out by the group conformed of Susana (S), Juan (J) and Felipe (F) who established a correct implication, developed work rich in abductions, and were able to progress in the proof of their conjecture.

S initially represented their anticipation using dynamic geometry, constructing a chord that contained both *P* and *C*, the center of the circle, and calculating the product of the measurement of the two segments determined by *P* on the chord. As soon as she obtained the product, S expressed that they should have constructed "any" chord containing *P* so that they could compare results. She then constructed another chord without erasing the one she already had, mechanism that she used to carry out the dynamic exploration of the situation. Immediately, J realizes the invariant: "Constant!" and asks himself why.



143. J: I don't know. Maybe it is because each time one moves, this one diminishes and this one increases. [Shows $\overline{PR}$ and $\overline{PS}$.]

148. J: But it diminishes proportionally.

153. S: […] [Speaking to J] But why does it diminish proportionally? That is the doubt. Because then it always is, that is they diminish and increase the same amount. [She refers to the lengths of the two subsegments.]

154. J: Let's look at the ratio to see what happens. Ahh, which is which?

[…] [Since they can't find the adequate combination to obtain equal ratios, they decide to analyze algebraically the factors of the constant product.]

270. J: No, because no, uhmmm, up to now, what do we have? Of what S has there, we found that *AP* times *BP* is a constant, right?

271. S: Aha.

272. J: And that *RP* and *SP* gave us the same constant, therefore they are the same.

273. S: *PA* over *PS* …*PR*? [Writes in the notebook: $PA \cdot PB = PR \cdot PS;$

PA/PS = PR/PB]

317. J: We are assuming that this ratio is always the same.

318. S: Well, in theory it must be so.

J observes that as he drags one of the ends of chord $\overline{RS}$, the length of $\overline{RP}$ increases when that of $\overline{PS}$ diminishes. This observation is bizarre for S reason why J adventures a first explanation: “it diminishes proportionally” [143]. Initially, J is referring to the length reduction and increase of the two segments in one chord and it seems that mentioning proportion is not, at that moment, because he is thinking of the theory of proportions. However, the action that they carry out with dynamic geometry leads J to connect the idea of constant product with constant ratio. Without explicitly mentioning a definition, a theorem or a postulate from which the fact can be derived, J invites his partners to find reasons that lead to a proportion from which the constant product is derived [154].

His proposal to examine proportions is for us an implication especially because, in what follows, the students devote their time to form ratios between the segment measurements they have found. Recurrently, they mention that their conclusion must be a logical consequence of the theory, something we consider as another factor to assure that this is a manifestation of implication. Later, dynamic geometry becomes an instrument to determine which ratios permit establishing the evoked theory as the theoretic foundation of the result they have established. The manifestation of implication can be schematized as follows:

Theory A:	Proportions
Concluded fact:	The product of the measures of the lengths of the segments in which the chord is divided is constant (*q*)

When the students are able to correctly establish the ratios, a process of geometric implication begins, because they evoke triangle similarity.

324. S. Do we have it? In theory, we should have similar triangles, right?

325. J: Triangles?

326. S: Similar triangles.

327. J: Yes, yes, yes.

328. S: That is, what we have drawn are similar triangles.

329. J: In theory, yes.

330. S: Well, we haven’t drawn the triangles as such, right? But implicitly, there are similar triangles.

[…] [In what follows, they discuss about how to express in their conjecture that the product is the same regardless of which chord containing *P* is chosen.]

529. S: Products and ratios… Well construct the triangles because …

Without having represented triangles in the constructed figure, S alludes to the possibility of having similar triangles. So again, there is a manifestation of an implication. It is not an abduction because she does not evoke a specific rule but a theory and they do not know yet what to use from it. Such theory becomes the space for their future work. Once they have finished the process of writing their conjecture so that it clearly expresses the generality found, S again evokes the theory of similar triangles, stressing that it is where they can find an explanation. In intervention [529], S explicitly says why she establishes the implication with similar triangles. The scheme represents the above:

Theory B:	Similar triangles
Fact that wants to be concluded:	Ratios between the measures of the lengths of the segments in which the chord is divided are equal (*q*)

The students look for arguments that in the Theory of similar triangles guarantee that this relation exists for a pair of triangles. They begin an eminently theoretic search (in an abductive way) to guarantee the similarity and to be able make a deduction. They allude to more specific elements to assure the similarity, as the Angle-angle Criteria, and they propose an auxiliary construction for that effect: the construction of a line parallel to one of the sides of a triangle, since they must establish the congruence of another pair of angles that are not vertical angles.

514. J: For similarity, what must we do?

515. S: For similarity we have the criterion…

516. J: Yes.

517. S: The theorem…

518. J: But, what do we have here? That is …

519. S: Here we have only two congruent angles, and that is all. [They have marked, in a paper representation, the congruence of the vertical angles.]

543. S: We need another angle at least.

[…] [In this interval, the students theoretically explore possible auxiliary constructions to obtain the congruency of another pair of angles, all of them unsuccessful.]

Even though the students do not mention the Angle-angle criteria explicitly, they are referring to it, since S alludes to the pair of angles they can already assure as congruent [519] and she mentions the need to establish that another pair of angles are congruent [543]. This is an abduction process that can be represented as:

Fact they want to justify:	Similar triangles (*q*)
Theoretic backing:	Angle-angle similarity criteria

Abduction product:	The necessity of two congruent pairs of corresponding angles (*p*)

544. J: But I do not see anything.

545. S: No, there isn't … [A few seconds of silence go by.]

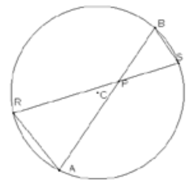
546. J: The angle…

547. F: Construct a parallel line…

548. J: But, parallel to whom? To this? [the short side of the triangle that does not contain *P*.]

549. S: We would need a parallel line to this one [the same side already mentioned] through this point [point *P*].

551. J: If we construct a parallel to this through here? [It can't be seen what he is referring to.] There, what would we get? We would have only this angle with this one [angles *ARP* and *BSP*], right? [He is referring to the graph on the paper.]



The students clearly know what the consequent of the conditional statement they want to establish is but they have not identified the specific angles. F suggests constructing a parallel line [547]; S and J accept the idea, and specify the point the line must contain and to which line it must be parallel. It is an abductive process because they mention a possible antecedent, referring to alternate interior angles [651], to be able to conclude the consequent they have established. This process is schematized:

Fact they want to justify:	The existence of another pair of congruent angles (alternate interior) (*q*)
Possible theoretic backing:	Paralelism
Abduction product:	Existence of a line parallel to a side of the triangle (*p*)

CONCLUSIONS

Our research interest is centered mainly in the student's search for the nexus between the theory they count on, the information dynamic geometry provides, the explicit establishment of implications and the abductive processes that ultimately lead to a proof. We presuppose that the evocation of a work space directs the exploration in search of an explanation of why a statement is true, and that the proof construction can include a mixture of deductions and abductions through which they advance towards finding the correct path.

When one is learning to prove, generally the argumentation on which lies the construction of a justification is of an abductive character. In a dynamic geometry environment, the students carry our empirical explorations, not only to establish a conjecture, but also to work within the frame of a theory evoked through their implication processes and to determine the viability of the ideas that emerge from their abductive processes. It is worth noting that this type of environment plays an important role in the evocation of theories; the first explorations carried out by the students induced them to frame themselves within the theory of ratios, fact that lead them to the theory of similar triangles.

This study well portrays the usually hidden nature of genuine and creative mathematical activity in which a mathematician is involved when he wants to justify a statement he believes to be true. Even though the expected product is a deductive chain that shows the statement's validity, the path to attain that involves other type of processes related to the search of generalizations and of ideas or rules that can be warrants in the justification of the induced property.

References

Arzarello, F. (2007). The proof in the 20th century. From Hilbert to automatic theorem proving introduction. In P. Boero (Ed.), *Theorems in school. From history, epistemology and cognition to classroom practice* (pp. 43-59). The Netherlands: Sense Publishers.

Arzarello, F., Olivero, F., Paola, D. & Robutti, O. (2007). The transition to formal proof in geometry. In P. Boero (Ed.), *Theorems in schools: From history, epistemology and cognition to classroom practice* (pp. 305-323). The Netherlands: Sense Publishers.

Cerulli, M. & Mariotti, A. (2003). Building theories: Working in a microworld and writing the mathematical notebook. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 181-188). Honolulu: University of Hawaii.

Fujita, T., Jones, K. & Kunimune, S. (2010). Students'geometrical constructions and proving activities: A case of cognitive unitiy? In M.F. Pinto & T.F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (vol. 3, pp. 9 - 16). Belo Horizonte, Brasil: PME.

Laborde, C. (2000). Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics, 44*(1-3), 151-161.

Mariotti, M.A. (2007). Geometrical proof. The mediation of a microworld. In P. Boero (Ed.), Theorems in school. From history, epistemology and cognition to classroom practice (pp. 285-304). The Netherlands: Sense Publishers.

Olivero, F. (2002). The proving process within a dynamic geometry environment. Doctoral thesis not published. University of Bristol, Graduate School of Education.

THE TEACHER'S ORAL COMMUNICATION DURING WHOLE-CLASS DISCUSSIONS

Leonor Santos, Sílvia Semana

Institute of Education; Ph.D. of Institute of Education, University of Lisbon

In this paper, we study the oral communication of a mathematics teacher from middle school during whole-class mathematical discussions over four lessons. The analysis of the classroom episodes reveals a general trend in the teacher's oral communication, concerning dynamics, focus, pedagogical direction and intention, as well as some specific characteristics for each lesson. The results reinforce the importance and the demanding of the teacher's role in classroom communication.

INTRODUCTION

Communication is an essential tool for learning mathematics (Lampert & Cobb, 2003; NCTM, 2000). How the teacher questions, listens and answers to the students and how he/she promotes and orchestrates mathematical discussions in the classroom are crucial to characterize the oral communication that takes place in the mathematics classroom and reflect, to some extent, the students' role in the communication process. However, these classroom practices often aren't developed in order to promote learning (Black, Harrison, Lee, Marshall & Wiliam, 2003).

This study is part of a broader research which main objective is to understand assessment practices of mathematics teachers, designed in a context of collaborative work, aimed at promoting self-assessment of students. In this paper, we focus in a particular assessment practice, an intentional oral communication of a mathematics teacher. Our aim is to understand the main characteristics (dynamics, focus, pedagogical direction and intention) of that communication during whole-class mathematical discussions and how, eventually, they evolve over a set of four lessons.

THEORETICAL FRAMEWORK

Oral communication in the classroom is an essential dimension of the teaching and learning process (Voigt, 1995). Depending on the role that the teacher plays in the communication, it is possible to identify several forms of communication, from an uni-directional communication - strongly associated with the transmission of knowledge and in which prevail IRE and funnel patterns of interaction (Wood, 1998) – to a reflective and instructive communication (Brendefur & Frykholm, 2000). To promote a classroom environment with mathematical quality involves the adoption of practices to orchestrating productive mathematical discussions (Stein, Engle, Smith & Hughes, 2008), which encourage students to explain, justify and assess their ideas and the ones of their colleagues (Forman, 2003). Taking into account that knowing mathematics is to be able of doing mathematics and students must learn mathematics with understanding (NCTM, 2000), mathematical discussions have to focus in conceptual meaning of mathematical objects, in processes and in final products.

The oral questioning is recognized as an important strategy to encourage and support the processes of learning (Santos & Pinto, 2009). However, often this practice is not planned or conducted in order to contribute to learning, since the teacher doesn't give students enough time to think about the question (Black & Wiliam, 1998) or asks direct close questions, that tends to promote superficial and unreflective answers. Students lead to change their opinion quickly, looking only to find the expected answer by the teacher (Gipps, 1999). An oral communication with potential for the regulation of learning should be intentional and participated by the several actors; consider the error as natural; privilege and respect different ways of thinking; and recognize the class as a legitimate field for validation or correction (Santos, 2011). In addition, questions should be open and adapted to the students' thoughts, helping them to improve their way of thinking and find new answers in a more comprehensively way. However, these practices demand from the teacher a solid professional knowledge, since they bring changes in the classroom management and the need of a deep knowledge about both the scientific area of teaching and the learning process (Moyer & Milewicz, 2002).

METHODOLOGY

The methodology used was qualitative and interpretative in nature (Goetz & LeCompte, 1984) with the design of case study (Yin, 2009). We study the oral communication of a math teacher during whole-class discussions over four lessons of an 8th grade class (19 students, aged 13 or 14). These lessons focus the mathematical topic "Sequences and Regularities" and took place at the end of 2010. They were planned by a collaborative group according to the following guidelines: (i) small-groups work followed by whole-class discussion, having as starting point semi-formal presentations of the groups; (ii) the teacher as orchestrator of productive mathematical discussions (Stein *et al.*, 2008) and responsible for promoting an oral communication in which students interact with the teacher and with each other in a reflective and instructive communication (Brendefur & Frykholm, 2000), following a discussion pattern (Voigt, 1995).

Data was collected mainly through participant observation of the four lessons (audio and video recorded) and an informal conversation (IC) held with the teacher (audio recorded). The analysis grid was developed from the one proposed by Santos and Pinto (2008) and refined through its application to the classroom episodes, regarding the whole-class discussions of the observed lessons. The last version of the grid includes four dimensions: *dynamics*, *focus*, *pedagogical direction* and *intention*. For each one, we consider several categories (Table 1). The category *question* contains the sub-categories opened or closed (OQ/CQ). We tried to place each intervention in only one category or sub-categories of each dimension, privileging the dominant sense. However, for the *pedagogical direction* and *intention*, sometimes it was necessary to combine two of them (for example, there are questions that simultaneously ask for justification /explanation and provide orientation about how to do it).

Dynamics		Focus	Pedagogical direction	Intention
Who produce it?	**To whom it's**			
Teac Stude Grou p of students	Teac Stude Grou p of students	Task Process Product Conceptualiza Self-regulatio Classroom management External to the teaching and learning	Question (Q) Answer (A) Presentation Additional Interjection Gesture Laugh Silence	Repeat (Rpt) Correct (Cor) Validate (Val) Solve (Sol) Justify/Explain Orient (Or) Appreciate Compare (Comp) To show that

Table 1: Dimensions and categories of analysis

RESULTS

Dynamics

Through the four lessons, the teacher's interventions tend to be less frequent (Table 2), suggesting a concern from the teacher in focusing the discussion on the students, which apparently had been achieved, especially in the 2nd and 4th lessons. At the same time, interactions between students become more frequent, which is suggested by the increasing on the number of students' interventions between two teacher's interventions (Table 3).

Lesson	Discussion duration	Total of	Teacher's
1	30 min	340	50 %
2	28 min	318	25 %
3	10 min	156	44 %
4	55 min	682	24 %

Table 2: Generic characterization and frequency of the teacher' interventions

Less on	Number of students' interventions between two teacher's interventions					
	0 - 1	2 - 4	5 - 9	10 -	20 or	Maximum
1	94 %	5 %	1 %	0 %	0 %	6
2	59 %	26 %	9 %	1 %	5 %	27
3	81 %	18 %	1 %	0 %	0 %	6
4	64 %	24 %	7 %	2 %	3 %	85

Table 3:Number of students' interventions between two teacher's interventions

(* Maximum number of students' interventions between two teacher's interventions, for each lesson)

We see that in 6%, 41%, 19% and 36% of the cases (from the 1st to the 4th lesson) there are two or more students' interventions, between two teacher's interventions. The 4th lesson even registered 85consecutive interventions of students, without any teacher's intervention (Table 3). However, data from the 3rd lesson seem to indicate a backward when compared with the one from the 2nd lesson. This may be associated, on the one hand, to the short duration of the whole-class discussion (10 minutes) and to the pressure felt by the teacher in order to end the discussion in that lesson – “I was seeing the time passing by and wanted to push to end the discussion, I didn’t want to prolong it to the next lesson” (IC) - and, on the other hand, to the fact that the proposed task proves to be unchallenging for students and didn’t originate different solving strategies: “there was no difficulty (...) there was no different resolutions” (IC).

Focus

The mains focuses of the teacher’s interventions are the process and the product, but we also find self-regulation and classroom management ones (Table 4).

Lesson	Task	Process	Product	Conceptualization	Self-Regulation	Classroom Management	Self
1	[illegible]	4	1	0	13	23	0
2	[illegible]	3	1	0	23	21	1
3		3	5	0	4	4	0
4	[illegible]	3	3	2	6	22	0

Table 4: Percentage of the focus of the teacher’s interventions

The relationship between the focus on the process and on the product throughout the four lessons can be explained by the type of task at hand. The task in the 3rd lesson was unchallenging for students and didn’t originate different solving strategies, what explain the high number of interventions in the product (57%). The tasks solved in the 1st and 2nd lessons, in general, led to different solving processes but similar products, leading to a focus mainly on the process (42% and 35%, respectively). The task solved in the 4th lesson, besides allowing different solving processes, also led to different products, justifying closes percentages between the interventions focused on the process and on the product (32% and 38%, respectively). The interventions focused on the classroom management have a frequency close to 20% in almost all lessons, except the 3rd, in which this frequency is significantly lower (4%). This might be associated to the fact that the 3rd lesson was the only one in which the students weren’t requested to make a semi-formal presentation of their work on the board, but they answered directly to the teacher's questions from their places. Concerning interventions focused in self-regulation, they increase significantly from the 1st to the 2nd lesson, and almost disappear in the last two lessons. A deeper analysis is done in the next sub-section in order to understand this evolution.

Pedagogical direction and intention

The *teacher’s interventions focused on the process* (Table 5) mostly assume the form of questions (58%, 64%, 39% and 78%) or presentations (35%, 36%, 52% and 20%). The teacher’s questions are in general for students to solve (18%, 21%, 13% and 39%) or to justify/explain (32%, 21%, 9% and 30%) and may include guidelines to orient students (4%, 4%, 9% and 11%). The following example is a teacher’s intervention that both question students to better explain the solving process and includes some orientations about how students should do it: "May you draw there, for example, the seventh figure, and the seventh figure? You are going to use the process that you are explaining" (L4).

	Q (CQ)				A				P			
	L1	L2	L3	L4	L1	L2	L3	L4	L1	L2	L3	L4
Rpt				2 (0)					11	11	9	4
Cor	3 (1)	4 (4)	4 (4)									
Val	1 (0)			4 (4)					1		4	2
Sol	18 (0)	21 (0)	13 (0)	39 (2)	1				3		9	
Jus/Ex	32 (0)	21 (0)	9 (4)	30 (6)	1			2	15	7	4	6
Or	4 (0)	4 (0)	9 (0)	11 (0)	7		4		3	11	9	6
Aprc	1 (1)	11 (0)	13 (4)	9 (6)			4		3	25	26	6
Comp	6 (6)	7 (7)	4 (0)	2 (2)					4			
SL											13	2
Total	58 (8)	64 (11)	39 (13)	78 (19)	8	0	9	2	35	36	52	20

Table 5: Percentage of the pedagogical direction and intention of the teacher's interventions focused on the process

The answers given by the teacher, either to students questions or to her own questions, are less frequent, although they still happening (8%, 0%, 9%, 2%). In the 1st lesson, it even happens, once, the teacher asking a question and answering it, partially, in the same intervention: "After... try to draw the diagram here (points to the 4th figure). How did you think? It's the 4th figure, then in the horizontal direction there are twice...".

The mains intentions of the teacher's presentations differ according to the lessons. Solving and validating presentations are relatively rare in almost all the lessons (3%, 0%, 9%, 0% and 1%, 0%, 4%, 2%, respectively), while justifying/explaining and repeating seem to follow a tendency of decrease (11%, 11%, 9%, 4% and 15%, 7%, 4%, 6%, respectively). Sometimes, both intentions (justify/explain and repeat) are associated to the same intervention, mainly when teacher repeats something said by a student to systematize and adds an explanation of her own. In the following example, the teacher talks to all the class after a student saying "we did 649 divided by 2 which gave 334.5": "They tried to divide the remaining points into two parts. That would be for each of the parts of the figure" (L1). Presentations to appreciate are specially common in the 2nd and 3rd lessons (25% and 26%), as illustrated: "Oh, I get it! Right! When you started saying, I thought you had added the general expression, which is something simpler. But no, you went to the expression" (L3).

Regarding *teacher's interventions focused on the product* (Table 6), they mostly assume the form of questions or presentations, while answers are absent in almost all lessons (0%, 0%, 3%, 0%). During the four lessons, questions tend to increase (70%, 69%, 95%, 92%) and presentations go in the opposite direction (30%, 31%, 5%, 8%).

This reveals the teacher intention of focusing the discussion in the students. In particular, questions to solve and compare products are the most frequent (43%, 31%, 33%, 25% and 13%, 23%, 23%, 48%, respectively), followed by the ones to validate (0%, 8%, 26%, 8%).

	Q (CQ)				A				P			
	L1	L2	L3	L4	L1	L2	L3	L4	L1	L2	L3	L4
Rpt			13 (3)	2 (0)					9	23	3	
Cor	9 (9)			5 (2)					4			
Val		8 (8)	26 (26)	8 (8)							3	2
Sol	43 (13)	31 (0)	33 (0)	25 (2)					9	23	3	2
Jus/Ex	4 (0)	8 (0)		2 (2)								
Or	9 (9)		3 (3)	3 (3)					4			
Aprc				3 (3)					9		3	3
Comp	13 (13)	23 (23)	23 (18)	48 (41)			3			8		2
SL												
Total	70 (35)	69 (31)	95 (46)	92 (57)	0	0	3	0	30	31	5	8

Table 6: Percentage of the pedagogical direction and intention of the teacher's interventions focused on the product

Still focusing on the product, the teacher's presentations to solve and to repeat seem to be quite frequent in the 2nd lesson (both 23%). However, they correspond to only four in a total of 13 interventions with focus on the product, in which the teacher repeats the result presented by the student(s), but improves its formulation.

With regard to the *teacher's interventions focused on the self-regulation* (Table 7) they are mostly questions asking students to make an appreciation (55%, 33%, 33%, 30%) or are presentations (41%, 67%, 33%, 60%).

	Q (CQ)				A				P			
	L1	L2	L3	L4	L1	L2	L3	L4	L1	L2	L3	L4
Or	5 (0)				14		33	10	41	67	33	60
Aprc	55 (50)	33 (33)	33 (33)	30 (30)	14				5			
Total	55 (50)	33 (33)	33 (33)	30 (30)	5	0	33	10	41	67	33	60

Table 7: Percentage of the pedagogical direction and intention of the teacher's interventions focused on the self-regulation

Questions requesting students to appreciate their performance or understanding are quite frequent, but generally they are also quite poor, since they tend to be close questions (50%), many of the times, only asking students if they understood something that was presented, as the following example illustrates: "Did you understand how they did it? How they think, or not? Did you realize this explanation that they gave now at the end?" (L1). Such questions are dominant in the 1st lesson and may explain the high frequency of interventions focused on self-regulation, when compared with the ones from the 3rd and 4th lessons. In the 2nd lesson, there is the higher frequency of interventions focused on self-regulation, but these interventions are mostly presentations to guide (67%) rather than questions to lead students to appreciate (33%). In fact, orienting presentations are more frequent in the 2nd and 4th lessons and they provide guidance for students to regulate their activity and their learning: “Oh T., you aren’t explaining to me. Your colleagues aren´t understanding, you must speak to them” (L2); “Shouldn’t be me asking” (L2); “Oh B., we must make an effort to understand the colleagues, okay?” (L4).

CONCLUSIONS

It’s possible to identify a general pattern in the teacher’s oral communication during whole-class discussions, despite the specificities of each lesson. The process and the product are the mains focuses of the teacher’s interventions, but the self-regulation and the classroom management are also significant areas of activity for the teacher’s interventions. It emerges a strong relationship between the interventions focused on the process and on the product and the type of tasks proposed to students. The richer the task (in terms of diversity of processes and solutions), the more balanced are these two types of intervention.

Whatever the focus of the interventions (processes, products or self-regulation), questions and presentations are the mostly used forms. However, the pedagogical intention and direction differ, emerging a clear relationship with the students' mathematical activity. While interventions focused on the processes emphasize justifying/explaining, those ones focused on the product refer to solve and compare solutions and the ones focused on the self-regulation encourage appreciation and give students guidelines. Answers and solving and validating presentations are rare in almost all the lessons. But there is a limitation in the teacher’s communication, frequently refer in others studies (Black & Wiliam, 1998). A high number of close questions is used, restraining the potential for leaning (Black *et al.*, 2003).

Over the four lessons, the teacher’s oral communication tend to be less frequent, increasing the number of interactions between students, questions tend to increase and presentations go in the opposite direction. The teacher, increasingly, gives voice to students, encourages them to present, justify, validate and compare results and processes, and guides them in the regulation of their learning process, into the direction of a reflective and instructive communication (Brendefur & Frykholm, 2000). This trend isn’t linear. The nature of the mathematical task as well as aspects related to the management of the different moments of the lesson, namely the time devoted to each one, seem to have a great influence in the teacher's role during whole-class discussions. Despite of being aware of what she wants, the teacher faces difficulties and her action isn’t always adequate in a perspective of regulation of learning (Black *et al.*, 2003).

References

Black, P. & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education, 5*(1), 7-74.

Black, P.; Harrison, C.; Lee, C.; Marshall, B. & Wiliam, D. (2003). *Assessment for learning. Putting into practice*. London: Open University Press.

Brendefur, J., & Frykholm, J. (2000). Promoting mathematical communication in the classroom: two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education, 3*, 125-153.

Forman, E. (2003). A sociocultural approach to mathematics reform: Speaking, inscribing, and doing mathematics within communities of practice. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 333-352). Reston, VA: NCTM.

Goetz, J. & LeCompte, M. (1984). *Ethnography and qualitative design in educational research*. New York: Academic Press.

Gipps, C. (1999). Socio-cultural aspects of assessment. *Review of Research in Education, 24*, 355-392.

Lampert, M., & Cobb, P. (2003). Communication and language. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 237- 249). Reston, VA: NCTM.

Moyer, P. & Milewicz, E. (2002). Learning to question: categories of questioning used by preservice teachers during diagnostic mathematics interviews. *Journal of Mathematics Teacher Education, 5*(4), 293-315.

NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Santos, L. (2011). Evaluar para aprender matemáticas, *UNO, Evaluación desde el centro escolar y matemáticas, 57*, 32-42.

Santos, L. & Pinto, J. (2008). The teacher's oral feedback and learning. *ICME11* (available in http://tsg.icme11.org/tsg/show/37)

Santos, L. & Pinto, L. (2009). Lights and shadows of feedback in mathematics learning. *PME 33, International Group for the Psychology of Mathematics Education*, (vol. 5, pp. 49-56). Thessaloniki, Greece.

Stein, M., Engle, R., Smith, M. & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyong show and tell. *Mathematical Thinking and Learning, 10*, 313-.340.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In H. Bauersfeld & P. Cobb (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 163 – 201). Hillsdale, NJ: Lawrence Erlbaum Associates Publishers.

Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing? In H. Steinbring; M. Bartolini Bussi & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 167-178). Reston, VA: NCTM.

Yin, R. K. (2009) *Estudo de caso: Planejamento e métodos* (4.ª ed.). Porto Alegre: Artemed Editora S.A.

EFFECTS OF TREATING MULTIPLE SOLUTIONS WHILE SOLVING MODELLING PROBLEMS ON STUDENTS' SELF-REGULATION, SELF-EFFICACY EXPECTATIONS AND VALUE

Stanislaw Schukajlow, University of Paderborn, Germany, schustan@math.upb.de

André Krug, University of Kassel, Germany, akrug@mathematik.uni-kassel.de

In the project MultiMa (Multiple solutions for mathematics teaching oriented towards students' self-regulation) the effects of treating multiple solutions while solving modelling problems on students' learning are investigated. In the quasi-experimental study we report on the comparison of two groups of students. In one group modelling tasks, where the solutions do not demand making assumptions about missing data were treated. In another group students solved similar modelling problems, where different assumptions are possible, and students had to develop two and more different solutions each. About 120 9th graders from six middle track classes took part in this study for five lessons. Before and after a teaching unit students' self-regulation, self-efficacy expectations and value were tested.

INTRODUCTION

Although there are a number of results about some aspects of modelling, the influence of different treatments of modelling on student's self-perceptions is still an open question. One way to answer this question is an investigation of different learning environments and their effects on non-cognitive variables. In the recent study we focus on the improvement in self-regulation, self-efficacy and value in learning environments with und without a possibility to solve modelling problems in different ways.

THEORETICAL BACKGROUND

Self-regulation, self-efficacy expectations and value

Boekaerts (2002) distinguishes three main parts in the process of self-regulation: (1) students' orientation toward the attainment of their own goals, (2) the thoughts, feelings, and actions that can help them to attain these goals, and (3) working toward the attainment of their goals. Self-regulation is an essential aim of teaching preparing students for lifetime learning. Training in self-regulation and influences positive students` achievements and affect (Marcou & Lerman, 2007). According to Bandura (2003) self-efficacy expectations are "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments". Self-efficacy is connected with achievements in mathematics and with learning in general. Students with a high level of self-efficacy regulate their learning process more intensively and show better performances in mathematics (Malmivuori, 2006). "Value" characterizes the perceived importance attributed to objects, contents and actions (Eccles & Wigfield,

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 59-66. Taipei, Taiwan: PME.

2002). Values play an important role in theories of human motivation, which assume that students' motivation to learn is influenced by the importance attributed to learning and its objects. Students with high value beliefs in mathematics are described by teachers as better learners in the cognitive, metacognitive and motivational domains (Metallidou & Vlachou, 2010). The investigation of teaching methods for fostering students' self-regulation, self-efficacy expectations and value are important goals of mathematics education.

Multiple solutions while problem solving

Constructivist theories argue that developing different solutions and representations helps students acquire a multiple representation of the subject matter. Due to a multiple representation, students have a procedural flexibility in the respective domain and are able to solve unfamiliar problems. The crucial point while fostering a multiple representation in the classroom is a link between single representations and solutions. Thus teaching problem solving should stimulate the development of different solution methods, improve the connected mathematical knowledge and competencies as well as include a presentation of the individual solutions of students in the classroom (Leikin & Levav-Waynberg, 2007). Recently some experimental studies were carried out to identify the influence of treating multiple solutions on students' learning in mathematics (Rittle-Johnson & Star, 2007). Students that developed two solution methods for the same task outperform students that developed one solution at a time. Although investigation of emotions, attitudes, beliefs and other affective measures has been an important part of mathematics education research for decades (Zan, Brown, Evans, & Hannula, 2006), there is still a lack of studies that investigate the impact of different learning environments on students' self-perceptions. Moreover, we found no study that investigates the connection between developing multiple solutions and students' affect.

Multiple solutions and modelling

The important activities while modelling are simplifying a complex situation that is presented in the task, mathematizing and working mathematically to reach a mathematical result. While solving a modelling problem a problem solver can often choose among several possibilities to simplify a problem, to mathematize or to work mathematically. Different solution paths or methods can be chosen and sometimes there are different outcomes as result. To illustrate some of these activities we analyse the solution of the task "Parachuting", developed in the Framework of MultiMa-Project. First the problem solver has to understand the problem "Parachuting" and construct the situation model. Then the situation model should be simplified and structured. In order to do so, the problem solver has to make assumptions. The main assumptions while solving this problem are (see Schukajlow & Krug, in press):

- "the deviation remains constant at the different stages of the jump,
- the wind speed is light, middle or strong during the respective stages,

- the parachute opened e.g. at 1000 m above the ground."

Parachuting

When "parachuting", a plane takes jumpers to an altitude of about 4000 m. From there they jump out the plane. Before a jumper opens his parachute, he makes free fall of about 3000 m. At an altitude of about 1000 m the parachute opens and the sportsman glides to the landing place. While falling, the wind carries the jumper away. Deviations at different stages are shown in the table below.

Wind speed	Side deviation per thousand meters during free fall	Side deviation per thousand meters while gliding
Light	60 m	540 m
Middle	160 m	1440 m
Strong	340 m	3060 m

What distance does the parachutist cover during the entire jump?

Figure 1: Modelling task "Parachuting"

Next right-angled triangles should be identified, in these triangles the hypotenuses have to be calculated (e.g. with Pythagoras' theorem) and added in order to find a mathematical result. This result will be interpreted and validated. This analysis of solving the task "Parachuting" shows different ways to solve a modelling problem. Particularly changing assumptions lead to the different solutions and cause changes in the results of modelling. An important research question is: How does dealing with multiple solutions influence the modelling competency of students and their self-perceptions of mathematics? In the recent study we focus on the solutions that are developed as a result of the different assumptions students make and analyse the changes in self-perceptions of self-regulation, self-efficacy expectations and value. In the task "Parachuting" the assumptions are the wind velocity and the distance of the free falling stage among others.

Learning environments for treating modelling

In recent decades some key features of learning environments were identified as efficient classroom management, cognitive challenging activities, well-structured instructions and learning support of students (see e.g. Baumert et al., 2010). Special for treating modelling the sense making and modelling eliciting problems have to be used (Blum, 2011). The instructions in modelling were developed and evaluated in the Framework of the DISUM Project, where two teaching methods – "student oriented, self-regulated teaching method" and "teacher centered, directed instruction" were compared. Although the evaluation of this teaching unit showed positive results concerning students' achievements and affect (Schukajlow et al., in press), the students' progress is still disappointing from the normative point of view. One possibility to optimize this teaching unit is to integrate directive teaching elements like strong guidance of students at the beginning (Blum, 2011) and acquire the development of multiple solutions.

RESEARCH QUESTIONS

1. How many solutions do students develop in the group where multiple solutions were treated and are there differences in the number of developed solutions between this group and the group, where the development of only one solution was treated?
2. Are there differences in students' self-regulation, self-efficacy and value in mathematics due to the applied self-regulated teaching method of modelling problems?
3. Do students' self-regulation, self-efficacy and value differ according to the possibility to develop multiple solutions? In particularly, whether students that make different assumptions and develop multiple solutions while solving modelling problems report on other self-perceptions of self-regulation, self-efficacy and value than students in the group, where the development of only one solution was treated?

METHOD

Design and sample

138 German ninth graders (42.8% females; mean age= 15.2 years) were asked about their self-regulation, self-efficacy expectations and value before and after a five lesson period teaching unit (see Figure 2). Three schools with two middle track classes each took part on this study. Each of six classes was divided into two parts with the same number of students in each. The way the average achievements in the both parts did not differ and there was the approximately same ratio of males and females in each part. In the one part of each class multiple solution of modelling problems (group "multiple solutions") and in the other part one solution of modelling problems (group "one solution") were treated.

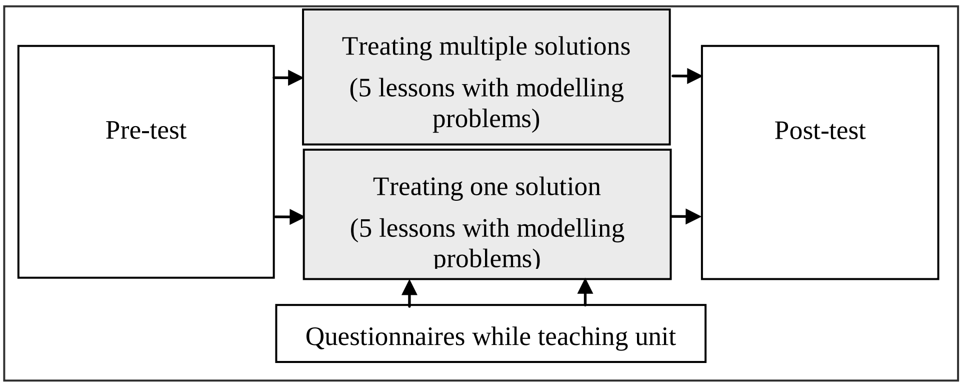


Figure 2: Overview of the study design

To implement the treating of modelling with and without multiple solutions two teaching scripts were developed. Four teachers that participated in this study received these scripts with all tasks to be treated and a detailed plan of the teaching unit. Further they were instructed about specific ways to promote modelling competency in both groups. Each teacher taught the same number of student groups in the group “multiple solutions” as in the group “one solution”, so the influence of a teacher on students’ learning did not differ between both groups. In each lesson one member of the research group was present to videotape and to observe the implementation of the treatment.

Treatment

The student-centered learning environment from DISUM Project was complemented by a directive instruction for the teaching unit used in the recent study. In both experimental groups the same methodical order was implemented. A teacher first demonstrates how modelling problems can be solved and multiple solutions can be developed. Students solve a modelling task according to a special kind of group work (alone, together and alone) and then discuss their solutions in the whole group in the classroom. A solution (or different solutions) of the first modelling task are presented by the students. The teacher has to summarise and to reflect on the key points of each group. In the group “multiple solutions” the teacher has to emphasise the development of different outcomes by estimating missing data. To foster the development of different solutions in one group and to prevent the development of more than one solution in the other group, two similar versions of each treated task were developed. The tasks in the group “multiple solutions” require the development of two solutions as, for example, in the task “Parachuting” (see Figure 1), where the question was “What distance does the parachutist cover during the entire jump? Find two possible solutions”. In the group “one solution” students solved a version of this task where the main data needed to solve the task (the wind velocity and parachute altitude) were specified.

Measures

After every lesson the students were asked about a number of solutions they developed for the respective modelling problem. For example: “While solving the problem “Parachuting” I developed today (0: no solution; 1: one solution; 2: two or even more solutions)”. Other students’ self-perceptions were asked using a 5-point Likert scale (1=not at all true, 5=completely true) before and after a teaching unit (see Figure 2). The sample items were for self-regulation (6 items) “While learning mathematics I set my own goals which I would like to achieve”, for self-efficacy (4 items) “I’m confident that I can understand the most difficult topics in mathematics” and for value (3 items) “I attach great importance to mathematics”. All scales were adapted from the longitudinal PALMA study (Pekrun et al., 2007). Reliability values (Cronbach’s Alpha) for self-regulation were .76 and .80, for self-efficacy .86 and .87 and for value .70 and .63 in pre- and post-test respectively.

RESULTS AND DISCUSSION

Research question 1

First we investigated how many solutions across all problems were developed in the group “multiple solutions”. The analysis of students’ answers shows that 4% of the students could not find any solution, 38% of the students developed one and 58% two or even more than two solutions. The majority of the students in the group “multiple solutions” developed two and more solutions (mean=1.55, standard deviation SD=0.39) as intended. In the group “one solution” students report on the development of two and more solutions less frequently (mean=1.14, SD=0.33). The analysis with t-Test (T(138)=6.7; p<0.001; effect size Cohen’s d=1.16) indicates that there are significant differences between the numbers of solutions that were developed in the respective groups. These results demonstrate that if you encourage students to find multiple solutions while solving modelling problems in the classroom, the majority of students really do it. In line with the intentions of the recent study there were significant differences in the numbers of developed solutions between students in the groups “multiple solutions” and “one solution”.

Research question 2

Teaching modelling problems for five lessons with a method used in the recent study improves students’ self-regulation, self-efficacy and value as an analysis with t-tests showed. All measured students’ self-perceptions are increased after the teaching modelling problems. The effect size Cohen’s d varies from small for self-regulation to medium for self-efficacy and value. We sum up that teaching modelling with a combination of directive guidance at the beginning of the teaching unit and group work in later parts improve students’ ability to regulate themselves, increase their self-efficacy expectation in mathematics and positively influence the students’ value of mathematics. However, one limitation is that the study design did not include a control group which is why it is not possible to exclude testing effects in interpreting these findings.

	Pre Mean(SD)	Post Mean (SD)	T(df)	p	Cohen’s d
self-regulation	3.62(.70)	3.79(.66)	2.8	<.01	.27
self-efficacy	3.23(.92)	3.57(.91)	5.2	<.01	.47
value	3.26(.90)	3.62(.85)	5.3	<.01	.49

Table 1. Students' self-regulation, self-efficacy and value at pre- and post-test

Research question 3

In order to investigate the impact of treating multiple solutions while solving modelling problems on measured self-perceptions of students we compared self-regulation, self-efficacy and value of both teaching environments in post-tests, taking into account their pre-test measures as covariate. The analysis with ANCOVA showed nearly significant differences between the groups “multiple solutions” and “one solution” in students’ self-regulation ($F(118)=3.5$, $p=.06$, effect size $(\eta)2=.03$). Students that were encouraged to develop multiple solutions while modelling most frequently report on regulation of their learning in mathematics after the teaching unit than students that have to develop only one solution. However, no differences between students in learning environments with and without treatment of multiple solutions were observed in self-efficacy expectations and value (Self-efficacy: $F(118)=0.6$, $p=.42$, $(\eta)^2=.01$; Value: $F(119)<0.1$, $p=.95$, $(\eta)^2<.01$).

	Treatment of multiple solutions		Treatment of one solution	
	Pre Mean(SD)	Post Mean (SD)	Pre Mean(SD)	Post Mean (SD)
self-regulation	3.69(.66)	3.93(.65)	3.52(.75)	3.64(74)
self-efficacy	3.29(.93)	3.64(.84)	3.14(.95)	3.49(.98)
value	3.31(.88)	3.66(.82)	3.15(.96)	3.57(.88)

Table 2. Students' self-regulation, self-efficacy and value before and after a teaching unit with and without treatment of multiple solutions

The results of the recent study show that a learning environment for treating modelling problems, where directive instruction and group work were combined, has positive influence on students’ self-regulation, self-efficacy and value. Moreover, the treatment of multiple solutions that are developed because of the different assumptions while solving modelling problems guide the majority of students to develop multiple solutions and increase their self-regulation. However, the treatment of multiple solutions has no effect on students’ self-efficacy and value. In further studies the effects of treating multiple solutions on affect and achievements need to be investigated.

References

Bandura, A. (2003). *Self-efficacy: the exercise of control* (6th printing ed.). New York: Freeman.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Dubberke, T., Jordan, A., XXX Tsai, Y.-M. (2010). Teachers’ Mathematical Knowledge, Cognitive Activation in the Classroom, and Student Progress. *American Educational Research Journal, 47*, 133-180.

Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In G. Kaiser, W. Blum, R. Borromeo Ferri & G. Stillman (Eds.), *Trends in the teaching and learning of mathematical modelling - Proceedings of ICTMA14* (pp. 15-30). New York: Springer.

Boekaerts, M. (2002). Bringing about change in the classroom: strengths and weaknesses of the self-regulated learning approach. EARLI Presidential Address, 2001. *Learning and Instruction, 12*, 589-604.

Eccles, J. S., & Wigfield, A. (2002). Motivational beliefs, values, and goals. *Annual Review of Psychology, 53*, 109-132.

Leikin, R., & Levav-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies, 66*(3), 349-371.

Malmivuori, M.-L. (2006). Affect and self-regulation. *Educational Studies in Mathematics, 63*, 149-164.

Marcou, A., & Lerman, S. (2007). *Changes in students' motivational beliefs and performance in a self-regulated mathematical problem-solving environment.* Paper presented at the CERME 5, Cyprus.

Metallidou, P., & Vlachou, A. (2010). Children's self-regulated learning profile in language and mathematics: The role of task value beliefs. *Psychology in the Schools, 47* (8), 776-788.

Pekrun, R., vom Hofe, R., Blum, W., Frenzel, A. C., Goetz, T., & Wartha, S. (2007). Development of Mathematical Competencies in Adolescence: The PALMA Longitudinal Study. In M. Prenzel (Ed.), *Studies on the educational quality of schools. The final report on the DFG Priority Programme.* (pp. 17-37) Münster, Germany: Waxmann.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561-574.

Schukajlow, S., & Krug, A. (in press). Considering multiple solutions for modelling problems - design and first results from the MultiMa Project. In W. Blum, J. Brown, G. Kaiser & G. Stillman (Eds.), *Proceedings of ICTMA15*. Heidelberg: Springer.

Schukajlow, S., Leiss, D., Pekrun, R., Blum, W., Müller, M., & Messner, R. (in press). Teaching methods for modelling problems and students' task-specific enjoyment, value, interest and self-efficacy expectations. *Educational Studies in Mathematics*. doi: 10.1007/s10649-011-9341-2

Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in Mathematics Education: An Introduction. *Educational studies in mathematics, 63*(2), 113-122.

CONSTRUCTING THE PERCENT CONCEPT THROUGH INTEGRATIVE MODELLING ACTIVITIES BASED ON THE REALISTIC APPROACH

Juhaina Awawdeh Shahbari and Irit Peled

Sakhnin College, University of Haifa University of Haifa

This report presents learning processes of the percent concept among seventh grade students using modelling activities based on the Realistic and Modelling approaches; moreover promoting the integration of fractions, decimal numbers, and ratio and proportion concepts. The findings are based on students' discussions during their work on modelling activities and are supported by tests conducted at the beginning and end of the learning process. The findings indicate a significant growth in understanding the percent concept and in integrating and reorganizing different components in children's knowledge. Furthermore, these results were significantly better than corresponding results in a control group that used a conventional instructional unit.

THEORETICAL BACKGROUND

In the current study modelling activities were constructed for teaching the percent concept. These activities were based on two design components. The first component is connected to the definition of understanding according to Hiebert & Carpenter (1992), who define understanding in terms of the way information is structured. According to their definition, a mathematical concept is understood if it is connected to other ideas and is a part of a rich knowledge network.

The second component of constructing the activities is based on principles of Realistic Mathematics Education (RME) and of the Modelling Approach. Realistic Mathematics Education involves planning realistic activities for each subject; these activities drive growing of models, which allow the growth of formal mathematics (Gravemeijer & Stephan, 2002). The main RME principles were suggested by Freudenthal, who advocated the teaching of mathematics as a process of constructing mathematical knowledge, by providing an opportunity for students to experience the mathematics through a similar process as mathematicians had when mathematics inventions occurred. The experience of mathematics invites the students to "Reinvention" process which gives them the opportunity to build their mathematical knowledge by themselves (Freudenthal, 1971; Gravemeijer & Terwel, 2000). RME doesn't see in mathematics as only a body of knowledge, or just a task of searching for problems and solutions, but a task of organizing content from realistic or mathematical content (Freudenthal, 1971). That process of organizing mathematical problems was called mathematization by Freudenthal (Gravemeijer & Terwel, 2000).

Mathematization process in the RME approach is developed to particular concept or idea. We mentioned earlier the important of connections between the new concept and the existing knowledge network. So, we adopted the Modelling Approach, in order to make the mathematical experience more richness and to provide the choices to deal with different existing concepts and use several skills. That can allow the integration between new and existing concepts.

The Modelling approach emphasizes the mathematization of realistic activities in a meaningful way for the learner (English & Fox, 2005). it sees that routine problems of school as underwent the process of mathematization, because the interpretations of the students were turned off and required from them only to apply procedures for solution (English & Fox, 2005; Zawojewski & Lesh, 2003). Researchers that use the Modelling Approach offer Model Eliciting Activities (MEA). MEA provides an opportunity for students to mathematize activities and to create mathematical models, through repeated cycles of translation, description, prediction data and deliverables in solution path (Lesh & Doerr, 2003). For effective design of MEA Lesh, Hoover, Hall, Kelly and post (2000) presented six principles: Reality principle, Model Construction principle, Self-evaluation principle, Model Documentation principle, Model Generalization principle and Effective Prototype principle.

Effective MEA provides skills that students need now, in a technological era with complex and dynamic information systems. It allows students to engage in mathematical processes that are not provided enough in the current mathematics curriculum, such as: description, quantification, analysis, construction, explanation, representation, and organization of data (English & Fox, 2005). Our research hypotheses were that the modelling activities invite students to build mathematical models from realistic content. During this modelling process, the percent concept will be constructed and knowledge in fraction, decimals and ratio and proportion would be integrated, emphasized and reorganized. We hypothesized also that as a result of the modelling process, students will develop a higher degree of understanding, by building strong and many connections between the percent concept and other existing concepts.

METHOD

The current study was carried out with 96 seventh grade students (12-13 years old), divided equally into three heterogeneous classes, with similar population distribution. Two classes were assigned to be the experimental groups and the third class was the control group.

Data collection: The data sources include Videotape, students' notes and worksheets during the whole learning process. Three similar questionnaires tested the existing knowledge in common fractions, decimal numbers, and proportion. Three similar questionnaires examined the new knowledge related to the concept of percentage. The questionnaires were administered three times: before the beginning of modelling activities, at the end of the learning process and three months past the end of the learning process.

The Modelling activities:

The "Bedding Set" activity: in this activity the students were required to function as workers in a textile shop. They were required to price the different set items, in order to sell them separately while keeping the sum of all item prices equal to the price of the whole set. The "Bedding Set" activity was considered as a preparation for the next activity.

The "Tableware Set" activity: this activity was similar to the first one but with more complex components. The students were required to function as kitchenware shop workers and they had to price the items. Again, the students had to be aware that the sum of all item prices had to be equal to the price of the complete tableware set. The tableware sets were of different kinds (Porcelain, Kristal…) so they had different prices but were similar in their composition each contained 30 items and was intended for 8 persons.

The "Losing Weight" activity: in this activity the students were required to function as organizers of an animal weight loss competition. They were required to list six animal participants and their weights in three weighing and then to choose the winning animal.

The "Chips Bag" activity: the students in this activity were required to design and build a chips bag. Later they were required to build another enlarged model depending on the paper model that they built in the first stage. In the third stage the students were required to reconstruct the enlarged model to the original one.

The instruction unit for the control group: The control group studied the percent through a conventional mathematics text-book unit. The unit included transforming fractions and decimals to percent and vice versa, tasks that required to present shaded areas in percentages and vice versa and increase–decrease problems. These tasks were similar to tasks that were classified by Parker and Leinhardt (1995) as conventional tasks.

FINDINGS

Before presenting the findings, we will present briefly the progression of the modelling process according to the sequence of events. Initially, the students were engaged in the "Bedding Set" activity, they succeeded in building a general pricing model for the bedding sets. The pricing model was expressed by fractional parting process; each item in the set was expressed by fractional part. The fractional parts were adjusted to each item according to its importance and its size; taking into account that the sum of all fractional parts should be equal to the whole.

Due to the long pricing process which students had to make for each set, it became apparent that a general pricing model was needed. Later, the general pricing model was expanded and implemented in the "Tableware Set" activity, in which the pricing model transforms to decimals. At this point there arose a need for using a new sign or new concept and thus at this stage the percent concept was reinvented. Later, the students were engaged with the "Chips Bag" activity, which handled the quantity referent and the keeping of symmetry in markup-markdown problems. This was handled in terms of fraction, which was expanded to percent terms during the engaging of the last activity "Losing Weight ".

The findings derived from the tests and the students' discussion will be presented according to the events sequence. So firstly we will report about the existing knowledge and then the new one "percent", with handling of specific components.

Reorganizing the existing knowledge

The first test that was conducted before the learning process, showed similar average among the three research groups. It identified difficulties in different components of fractions, decimals and ratio and proportion.

These difficulties were emphasized during experimental students' discussions while they were engaged in the modelling activities. We will exemplify some of students' arguments. The first difficult is connected to fraction's operator meaning as described in the first Episode.

Episode1: Bedding set pricing

Duaa: The price of each cushion cover equals half the total price.

Saleh: I don't understand! Does it mean that the price of the cushion cover is the same in each Bedding set? They must be different!

Duaa: No, the price is different. The first cushion cover is NIS 120, and in the second bedding set is equal to 150, but all of them are equal to half of the total price.

We noted another difficulty, having to do with ignoring that the whole must be equal to the sum of the parts, as described in Episode 2.

Episode2: Pricing items in bedding sets containing 5 items

(The components of the set were listed by students before the pricing process; each set is contains 2 cushion covers, 1 quilt cover, and 2 sheets).

Hussein: We must divide the price by 10.

Riemann: I think so too.

Hussein: Each cushion cover is equal to $\frac{1}{10}$ of the price, and here (pointing to the quilt cover) is equal to $\frac{2}{10}$, and each sheet is equal to $\frac{5}{10}$.

The findings of the second and the third tests point to considerable improvement in the experimental groups compared with the control group. The analyses of repeated measurements indicate a significant difference regardless of the group ($F_{(2,\ 186)}=100.962$, $p<0.001$). The difference can be attributed to the change between the first and the second tests in the experimental groups. Also the difference between the groups regardless of the tests was statistically significant ($F_{(2,93)}=5.20$, $p< 0.01$), When the source of the difference between the experimental and the control groups is the first and the second tests. Distribution of the averages scores by groups and across the three tests described in figure1

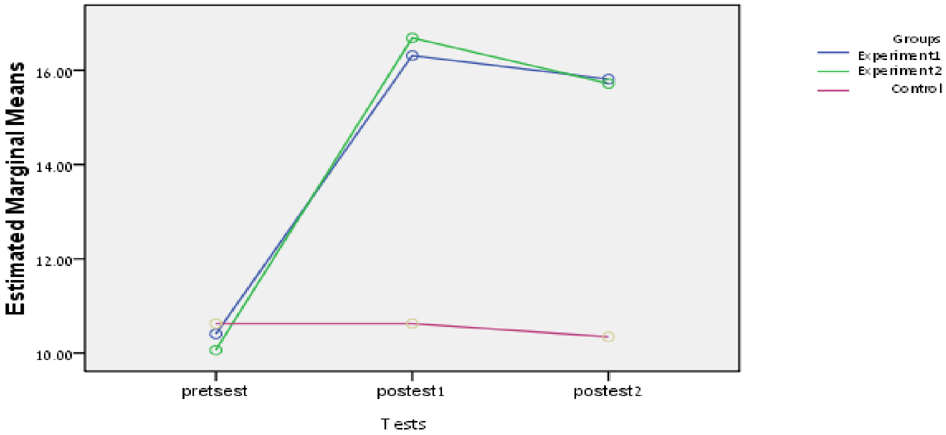


Figure1: Distribution of the averages existing knowledge scores by groups.

The findings above were emphasized by students' discussions, while they were engaged with the different modelling activities. The third episode exemplifies the overcoming of one of the difficulties. It shows that students become aware of the fact that the sum of the parts must be equal to the whole.

Episode 3: Tableware set pricing

Hussein: We can divide the price to 12 parts.

Dima: The template is equal to $\frac{2}{12}$ from the price.

-continuing -

Suliman: Each of the small plates is equal to $\frac{1}{12}$.

Hussein: It can't be, the tray must be more expensive than the small plate. If the tray is equal to $\frac{1}{12}$, so the parting to 12 can't be enough, the sum of the parts will be more than $\frac{12}{12}$.

The process of the engagement in modelling activities gave rise to the students' acknowledgment that a new concept is needed. It occurred when the students needed to transform the pricing model from decimals term to more comfortable one. Accordingly they suggested some signs like * or X which have the same function of the percent concept, and later they remembered the percent sign "%". The growth of the percent concept was connected to its operator meaning.

Constructing the New Knowledge- Percent Concept

The findings of the tests point to a higher average and a significant difference in the experimental groups compared with the control group. The analyses of repeated measurements indicate a significant difference regardless of the group ($F(2, 186) = 345.30$, $p<0.001$). The difference can be attributed to the change between the first and the second tests in the experimental groups. Also the difference between the groups regardless of the tests was statistically significant ($F_{(2, 93)} = 6.30, p<0.01$) , when the source of the difference between the experimental and the control groups is the first and the second tests. Distribution of the averages scores by groups and across the three tests described in figure 2.

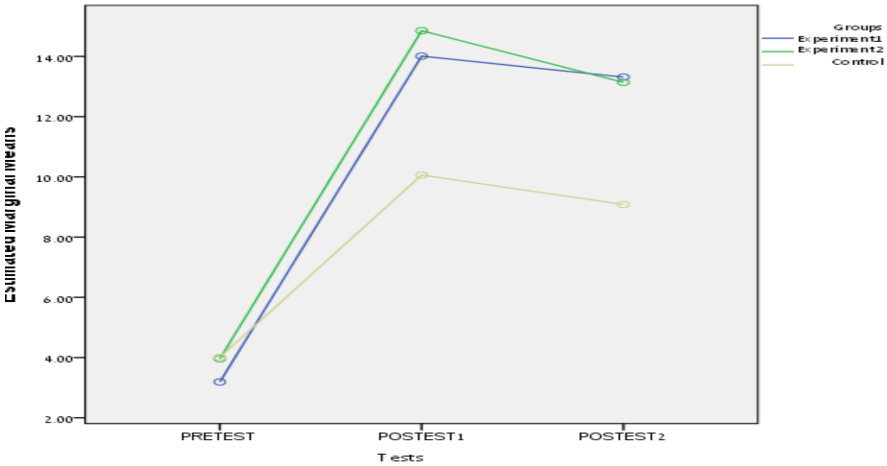


Figure 2: Distribution of the averages percent knowledge scores by groups

The findings above are supported by the students' discussions and their notes, and show the construction of the percent concept with a due emphasis on its different meanings and the understanding of various components that are related to it. We will report about three percent concept's meanings and other components.

<u>Operator function meaning</u>: Percent meaning as operator was not constructed as a new meaning. But it grew from the same meaning in fraction. It was used in order to build a pricing model for the activities of "Bedding Set" and "Tableware Set". Later, the pricing model was expanded to decimals, after that it was connected to the percent sign, when each item had a percent from the whole price. For example, the price of small plate is worth to 2% of the whole price. Besides the computation procedure, the students understood the underlying implicit relationships between percent and its referent quantities. Like the case of Episode 4

Episode 4: The percent as an operator with a consideration of the referent quantities

Iyad: Which is more expensive, glass's bowl or porcelain's bowl?

Shahed: They must not be equal, even if it's the same percent.

Iyad: It depends on what is more expensive, porcelain or glass.

<u>Part - whole meaning</u>: The part- whole meaning is expanded from the same meaning in the fraction. The pricing model was constructed in fraction, each item had a fractional part from the overall price, and the model was developed to the decimal and then to the percent, in which each item in the tableware had a percent from the price, with an emphasis on the fact that the whole price is equal to 100%.

Ratio meaning: Like other meanings this meaning is constructed out of the existing knowledge. While engaging in the "Chips Bag" activity the students used the ratio and the proportion in fraction terms. This meaning was emerged in percent terms in the "Losing Weight" activity.

In additional to the three meanings that we presented above, the engaging of the modelling activities contributed to deal with important components, particularly those which considered as difficult for students. Like, shift from decimals to percent and vice versa, the change of referent quantity, mark up-markdown problems. For example, Episode 5 presents the considering of the whole changing.

Episode 5: overcoming the symmetry keeping

Dima: If it decrease 5% and then increase 5%, it would not return the original weight.

Suleiman: you must write an explanation.

Riemann: If the animal decreases 5%

Suleiman: (continuing) if it decreased 5% then the weight now is less than the original, Then the increase of 5% will be from a smaller weight.

DISCUSSION

The findings indicate that modelling activities enable the construction of the percent concept together with some of its different meanings while at the same time also promoting the understanding of related knowledge and earlier learned concept. The Construction of the percent concept occurred in a modelling process of fractions, which transferred to a decimal representation and then to percents. Throughout the model building process, the percent meanings were built, even before the sign "%" emerged. This happened as a result of the links between the various meanings of the percent concept and parallel meanings in the existing knowledge. For example, dealing with pricing items in the "Tableware Set" activity, allowed the construction of the operator meaning of percent together with links to the operator meaning of fraction. The activity also contributed to the construction of part-whole meaning of percent, by determining the price as a whole, while the items prices were the parts. Similarly, the "Losing Weight" activity contributed to building the ratio meaning of the percent concept.

Together with constructing the various meanings of the percent concept, the modelling activities contributed to understanding different components related to it. This understanding was general and not constraining and it related to the modelling activities from which the percent concept had grown. These findings, were inferred from the discussions and the notes of the students during their modelling activities, and were supported by findings from the tests conducted at the beginning and end of the learning process, pointing to a higher average and a significant difference between the experimental groups and the control group. In addition, the modelling activities provided the opportunities to overcome the difficulties encountered in related existing knowledge. This was also evident in the test results showing a considerable improvement among the experimental groups with no significant changes detected in the control group.

Putting together the various findings of this research, those relating to the existing and those dealing with the construction of new knowledge, supports our suggestion to combine the RME and modelling approaches with an emphasis on conceptual integration with earlier learned knowledge. This integrated approach provided the possibility to reinvent the percent concept through integration with the existing knowledge. This integration strengthened the network knowledge network that defines the degree of new concept understanding according to Hibbert and Carpenter (1992).

Moreover, while constructing the percent concept effectively and reorganize existing knowledge, the modelling activities also contributed to the development of various modelling skills, such as constructing, organising, representing, ranking, quantifying, reasoning and explaining.

Reference

English, L. D., & Fox, J. L. (2005). Seventh-grader's mathematical modeling on completion of a three-year program. In P. Claarson et al. (Eds.), Building connections: Theory, research and practice (vol.1, pp. 321-328). Melbourne: Deakin University Press.

Gravemeijer, K. & Stephan, M. (2002). Emergent models as an instructional design Heuristic. In Gravemeijer, K. Lehrer, R. Oers, B. & Verschaffel, L (Eds). Symbolizing, Modeling and tool use in mathematics education (pp. 145-169). Netherlands: Kluwer.

Gravemeijer, K & Terwel, J. (2000). Hans freudenthal: a mathematician on didactics and curriculum theory. Curriculum studies, 32(6), 777-796.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. Educational studies in mathematics, 3, 413-435.

Hiebert, J., &Carpenter, T.P. (1992). Learning and teaching with understanding. In D.A.Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp.65-97). New York, NY: Macmillan.

Lesh, R. A. & Doerr, H.M. (2003). Beyond Constructivism: Models and Modeling Perspectives in Mathematics Teaching, Learning, and Problem Solving. Mahawah, N.J: Lawrence Erlbaum.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principle for developing thought revealing activities for students and teachers. In R. Lesh & A. Kelly (Eds), Handbook of research design in mathematics and science education. (pp. 591- 644). Mahwah, N.J.: Lawrence Erlbaum Associates, Inc.

Parker, M., & Leinhardt, G. (1995). Percent: A privileged proportion. Review of Educational Research, 65(4), 421-481.

Zawojewski, J., & Lesh, R. (2003). A models and modeling perspective on problem solving. In R. Lesh & H. M. Doerr (Eds.), Beyond constructivism: Models and modeling perspectives on mathematics teaching, learning, and problem solving. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

EMERGENCE OF STUDENTS' VALUES IN THE PROCESS OF SOLVING THE SOCIALLY OPEN-ENDED PROBLEM

Isao Shimada Takuya Baba

Seijo Gakuen Elementary School Hiroshima University, Japan

Values aspect of mathematics education has been explored (e.g. Bishop,1991). Iida et. al. (1994) pointed out that problem solving activities such as dividing melons and sharing rooms cannot help but avoid values. On the other hand, mathematics learning is interpreted as unreal and value-less by students. This perception deprives them of the willingness and positive attitudes towards learning. In this paper we focused on the value aspect of problem solving process. The particular type of problems called socially open-ended problem (Baba, 2009, 2010) have been developed to elicit students' values by extending the traditional open-ended approach (Shimada, 1977). We developed three lessons for the fourth graders and identified the four characteristics of interaction during the class.

BACKGROUND OF RESEARCH

The values have the most fundamental components influencing and controlling human acts. We live to aim at realizing individual values according to personality under the influence of social norms. This is the fundamental form of human existence. We choose some favorable acts and avoid unfavorable ones, and thus our acts are somehow directed by values and norms. The theory of value is the one of the important flow of philosophy since the late 19th century philosophy (Kuroda 1992).

Thus we think that it is important to understand and appreciate the students' values during the mathematics lesson.

LITERATURE REVIEW ON VALUES IN MATHEMATICS EDUCATION

Three types of values in mathematics education can be identified according to the references. The first is directly related to the values in mathematics and the second one is related with the society or societal events. The third one, closely related to social values, is the values which correspond with individual preference.

Mathematical values

The UNESCO report (ICMI, 1978) is one of the earliest researches, which cite values in the mathematics education. The five mathematical values are intelligibility, brevity, accuracy, relevance, and normality. Milestone in the research on values is Bishop (1991), which advocated importance of the theme from cultural perspective. It stated three pairs of values such as Rationalism and Empiricism, Control and Progress, and Openness and Mystery.

Open-ended approach was the teaching approach in Japan to enhance mathematical thinking abilities of generalization and application through various solutions in open-ended problems (Shimada Eds., 1977, The English translation is Becker, Shimada Eds. 1997). “The mathematical values as a basis of three characteristics (abstractness, logicalness and formality) ... are related with conciseness, clarity and integration as a driving force” (Nakajima, 1981).

Social (human) values

Brown (1984) has stated that real world problem unavoidably involves the value in the decision-making and the issues of ethics and values become a central component in this. Iida et. al. (1994), in the process of researching on open-ended approach, found out that moral problems or ethical values may occur beyond mathematics discussion when students deal with melon division and room assignment.

More recently, Greer (2007) points out that the mathematical modeling for proportion problem prompts recognition of social equity. Mechanistic division in a real world context of proportion may create issues related to fairness thus equity. Nishimura et. al. (2011) classified social values into fairness, diversity and cooperation, responsibility and autonomy, sustainability, efficiency and so on.

In the mathematics lessons so far, only mathematical values have been cherished and social (human) values were not taken up so much. On the other hand, when dealing with real world problems, the student may express social values and we cannot simply ignore them in order to ensure their meaningful learning. In this paper these moral and ethical values are to be taken up as social values.

Personal values

The third category is values related to the individual taste. For example, when selecting a new car, you may choose it based on the price, color, design, and functionality. These economical reason, design, and functional importance are referred to as personal values. Certainly they are not just personal but also under social influence at the time. In that sense, the second and third categories may be interrelated to each other.

RESEARCH OBJECTIVE AND METHODOLOGY

From these, the objective of this paper is set to analyze emergence of students’ values, especially social values in the mathematics lesson. The method is to develop three socially open-ended problems, which elicit students’ ideas with some social values, and to conduct one of them once per month from November 2009 (Shimada 2009). There are 36 grade four students in a classroom. The lesson protocol was analyzed to clarify the characteristics of the students’ values emerging during the lesson.

The developed problems are as follows:

The first problem [Hitting the target] Date: November 13th, 2009, 3rd period

In a school cultural festival, your class offered a game of hitting the target with three balls. If the total score is more than 13 points, you can choose three favorite gifts. In case of 10 to 12 points, two prizes and in case of 3 to 9 points one prize only. When a first grader threw a ball three times and hit the target as 5-point area, 3-point area, and border between 3-point and 5-point areas. How do you give the score to the student?

The second problem [Room assignment] Date: December 14th, 2009, 2nd period

There are 19 boys and 19 girls in your class. When your class goes for excursion, you have to share 12 rooms in the hostel. How do you assign the rooms to them?

The third problem [Cake division] Date: January 14th, 2010, 2nd period

I have six family members all together. They are my grandfather, grandmother, father, mother, younger sister and me. We are given 5 cakes. How can we share?

ANALYSIS AND CONCLUSIONS OF TEACHING

The protocol analysis of classroom interaction reveals characteristics of students' cognition and values.

(1) Two types of assumptions at the formulation stage of problem solving

The first characteristic is that there are two types of assumptions at the formulation stage of the problem solving, base on Nagasaki, Shimada, Nishimura (2008), which has elaborated on setting the assumptions (hypothetical qualitative and quantitative assumptions). One of them is setting value-laden conditions such as equality among the whole class and priority to specific person. The other is some mathematical interpretation based on the above conditions. The following protocol reveals this.

(Protocol [1] of Hitting the target)

1	T4:	So, please think about this problem and also write the reason.
10 minutes later, discussion on different ideas		
2	S4:	The first grader might be happy to get the bigger score because the ball is between 1-point area and 3-point area. They can get two prizes because 5 +3 +3 = 11.
3	S5:	5 +3 + (3 +1) = 12. Since they are the first grader, both points would be given.
4	T5:	That's a great service to give both points when the ball is on border line between two areas. It is very kind of you to a small child.
5	S6:	The first grader may complain if we give only one point. It is not fair if give three points. I would like them to slow once again.

S4 made a decision to give the higher score and S5 to give both scores, when the ball came on the border of two areas in the target. They cared about the small child and developed the models such as giving the higher the score or adding both points. As you can see, different mathematical models can be made based on the same value of care about the first grader. In the course of this formulation, there are two types of assumptions. One is related to the social value in the daily life that we have to be kind to a small child. And the other one is the mathematical interpretations and models.

(2) Values against the whole class or the specific person

The second characteristic is related to social values such as equality and fairness and care about the specific person or the weak person. That means the students sometimes value fairness against the whole class and sometimes care about specific person such as friends and family members. We can analyze this in the following interaction.

(Protocol [1] of Room assignment)

1 T3: So, please consider and also write the reason.

10 minutes later

2 S4: $19 \div 6 = 3 \cdots 1$ Then 3,3,3,3,3, and 4. It is lonely to stay alone in a room, and so we can make one room with four students, by using all the rooms.

3 S5: $19 \div 6 = 3 \cdots 1$ Then 3,3,3,3,3, and 4. Since we want to use the rooms comfortably, we limit the number of persons in a room.

4 S6: We put four or five students in one room. $5 \times 4 = 20$, 20-1 = 19 Then 5,5,5,and 4. Teacher use the remaining rooms.

Looking at the students' reactions, we can observe the value related to affection to specific person (S4) and (S6), and the value related to fairness for the whole class (S5). We can extend the findings to all three lessons of Hitting the target, Room assignment, and Cake division revealed two types of values (Table 1). The first type is that they give affection to the specific person, and this corresponds with both expected values of a "thinking about the remaining one" and b "thinking about teacher or sick person" in the Room assignment problem. And the second type corresponds with c and d with some varieties. They are related to fairness and equality, having the larger group in mind. The target in the latter one is broader and a little more abstract, because we cannot see it unless we are being conscious about it.

(3) Explicitness and implicitness of values

The third characteristic obtained is that certain values can be observed more explicitly and other values are rather implicit. For example, values of affection to the specific person are seen more easily, but the values of fairness and equality for the larger group are rather indirect.

(Protocol [2] of Hitting the target)

1 S9: We can give 2 points which are between 3 points and 1point. With the 2, the total score is 2 +3 +2 = 10. Thus the student can get two gifts.

2 T6: Why did you think about giving 2 points?

3 S10: It is because it is between 3 points and 1 point.

In this scene, due to care about the specific person, the first grader in this case, they can say "It is because it is between 3 points and 1 point." However, they cannot say that they should treat everyone equally which may contradict or may support it, depending upon the interpretation of equality. This explicitness of values related to specific person and implicitness of values related to the larger group applies also to room assignment and cake division problems in the Table1. In the last problem of the cake division, both values become almost equivalent in frequency.

Problem	Type	Expected value	Percentage of value description	Percentage of type-wise value description
Hitting the target	Specific person	a. Kind to the first grader(1)	88.2 (15/17)	75.0
		b. Kind to the first grader(2)	40.0 (2/5)	
		c. Rigid to the first grader	50.0 (1/2)	
	Whole class	d. Fairness (1)	6.3 (1/16)	16.7
		e. Fairness (2)	0.0 (0/2)	
		f. Fairness (3)	27.3 (3/11)	
		g. Fairness (4)	100.0 (1/1)	
Room assignment	Specific person	a. Care about remaining person	75.0 (9/12)	81.3
		b. Care about teacher and sick person	100.0 (4/4)	
	Whole class	c. Fairness, all rooms used (1)	29.6 (8/27)	47.5
		d. Fairness, not all rooms (2)	84.6 (11/13)	
Cake division	Specific person	a. Priority to the child	66.7 (4/6)	69.6
		b. Priority to the adult	50.0 (3/6)	
		c. Priority to the grandfather and grandmother	75.0 (3/4)	
		d. Own preference	85.7 (6/7)	
	Whole class	e. Fairness (1)	52.2 (12/23)	60.5
		f. Fairness (2)	100.0 (2/2)	
		g. Fairness (3)	69.2 (9/13)	

Table 1: Students' Reaction and Percentage

Note: All numbers are in % except those inside the parenthesis. In the column of the percentage of value description, the fractions in the parenthesis show (the number of students who expressed the values explicitly)/(the number of students who are expected to have values), depending upon the mathematical models.

Table 1 shows the percentage of description of each value by the students who explicitly displayed the value against those who made a certain mathematical model connected to a certain value. In the table, there are two types of values such as “care about specific person” and “fairness and equality in the whole class” [Second characteristic]. The table also shows the percentage as per type. In each of the three lessons, the percentage of care about the specific person is 75.0%, 81.3%, and 69.6% correspondingly. On the other hand, the percentage of fairness for everyone is 16.7%, 47.5%, and 60.5%. The latter are much lower than the former, specific person. The higher the percentage, the more students display those values [Third characteristic]. For example, the mathematical model of 5 +3 +3 (Expected value in the Hitting the target “a”), 88.2% of students wrote words representing social values by recalling the days when they were the first grader. They can be interpreted as being conscious about those values. On the other hand, the mathematical model of 5 +3 +2 (Expected value “d”), only 6.3% of students wrote some words representing values of fairness and thus only minority are interpreted as being conscious about those values.

The percentage of fairness values has risen as the lesson proceeded, because the teacher’s intervention may have influenced the students. For example we can observe teacher’s intervention it in T7 to T10 in the protocol [3]. The teachers’ words may have enabled students to be aware of the fairness and thus to express its values.

(Protocol [3] of Hitting the target)

1	T7:	In this (S4), the first graders are so happy. I believe. In S5, I believe that it thought of the first grader. S6 cares about the first grader as well but at the same time, is concerned about being impartial. This not only think about the first grader. And whom did he think?
2	S11:	It thinks all the people who do the game.
3	S12:	He thinks the other grades as well.
4	T8:	I think he wants to be impartial to all people.
5	S13:	In relation with the idea of 2 point, what will happen when the ball comes between 0 points and 1 point?
6	S14:	We can give 0.5 point.
7	T9:	Whom does she think about in this idea?
8	S15:	Being impartial to everybody.
9	S16:	In rock-scissors-paper, 3 points when you win and 1 point when you lose.
10	S17:	What about when we cannot decide a winner or loser?
11	S18:	We can give 2 points.
12	T10:	This (S16) was thinking about everybody? Or is he thinking about the first grader?
13	S19:	All. He wants to be impartial to everybody.

14 S20: We can give the lower score. So the total is 5 +3 +1 = 9

15 S21: Why?

(The student was not able to answer. Then another boy with the same idea presented his idea.)

16 S22: We will not be lenient to the first grader, thinking about the future. (All roar with laughter.)

On the other hand, despite of teacher's intervention, the percentage can only reach 60 %. Many students are hardly conscious about the value.

The reason why fairness and equality are rather latent may be that it is too obvious for the students to think about the whole class. So surprisingly, unless some critical issues occur, we are not even conscious about those values related to the larger group. On the other hand, when the students think about the specific person, of course you have to mention it so that others can understand your thinking. We however, think that it is important to nurture consciousness about those values in a democratic society and at the same time coexistence of different values.

(4) Existence of combined values

Existence of combined values was observed in the room assignment problem but not in the problems of cake division and hitting the target. In the room assignment, thinking about the specific person can be connected to thinking about the whole class because the rooms are limited in number and we have to allocate this limited resource among the whole class. In this sense of limited resource and sharing among the large group, the problem becomes more public. On the other hand, we don't feel much limitation of resource (i.e. total number of gifts) in the Hitting the target nor much public situation (i.e. sharing among the family members) in the cake division.

The following protocol reveals combination of both values against the whole class and the specific one.

(Protocol [4] of Assignment rooms)

1 S16: (6,6,6,1) 19 ÷ 6 = 3 · · · 1 It is a fun to have more people, and the remaining one can occupy a room alone. It is because that student can feel relax alone.

FUTURE ISSUES

In this paper, we developed three socially open-ended problems and analyzed the classroom interaction with those problems. We described the emergence of students' values and clarified four characteristics through the analysis of interaction.

As stated in the beginning, we would like to enable students to learn mathematics more meaningfully. In next analysis, we would like to analyses how students appreciated others' values and transformed their values through iteraction.

References

Baba, T. (2009). Analysis of Socially Open-end Problems in Mathematics Education from the Perspective of Values. *Journal of JASME Research in Mathematics Education,16*, 247-252 (in Japanese).

Baba, T. (2010). Socially Open-End approach and Critical Mathematics Education. A paper presented at the EARCOME 5, Tokyo.

Bishop, A. (1991). *Mathematical Enculturation : A Cultural Perspective on Mathematics Education*. Dordrecht, Kluwer Academic Publishers.

Becker, J.P., Shimada, S. (Eds.) (1997). *The Open-ended Approach in Mathematics Education: A New Proposal for Teaching Mathematics*. Reston, National Council of Teachers of Mathematics (The original version is in Japanese Shimada, S. (Eds)(1977)).

Brown, S.I. (1984). The Logic of Problem Generation: from Morality and Solving to De-Posing and Rebellion. *For the Learning of Mathematics*, 41, 9-20.

Greer, B. (2007). "A sense of proportion for social justice", *Philosophy of Mathematics Education Journal*, 21.(http://people.exeter.ac.uk/PEenest/pome21/)

ICMI. (1978). *New Trends in Mathematics Teaching*. Paris, Unesco.

Iida, S. et al. (1994). Research on Recognition of Values in Open end Problem in Mathematics Education (1): Summary and Preliminary Report. *Research Journal on Kyushu Mathematics Education, 1*, 32-43 (in Japanese).

Kuroda, W. (1992). *Acts and norms*. Keiso Shobo (in Japanese).

Nakajima, K. (1981). *Mathematical Thinking and Mathematics Education*. Kaneko Shobo (inJapanese).

Nagasaki E., Shimada, K., Nishimura, I. (2008). Study on the Refinement of the Structure of the Power to Connect Mathematics and Society. *Proceedings of JSME 41st Annual Confrence*, 231-236 (In Japanese).

Nishimura et. al. (2011) Research on Formation of Mathematical Judging Ability: Its Significance and Framework of Lesson, *Proceeding of JSME 44th Annual Conference*, 237-242

Silver, B.A. (1993). On Mathematical Problem Posing. *Proceedings of 17th Psychology of Mathematics Education*, Vol.1, 66-85.

Shimada, I. (2009). Research on Lesson to Grow Ability of Decision Making in Mathematics. *Journal of Japan Society of Arithmetic Education, 91(12)*, 20-30 (in Japanese).

Shimada, S. (Eds.) (1977). *Open end Approach in Mathematics Education ∶ A Proposal for Lesson Improvement.* Mizuumi Press (in Japanese).

CHARACTERIZING THE REIFICATION PHASE OF VARIABLES IN FUNCTIONAL RELATION THROUGH A TEACHING EXPERIMENT IN A SIXTH GRADE CLASSROOM

Shinno, Yusuke

Osaka Kyoiku University

This paper reports part of an ongoing developmental study into conceptual change in the teaching and learning of mathematical expressions. It focuses on sixth graders' conceptions of variables and their reification processes. The purpose of this paper is to characterize the conceptions these students have of variables in functional relation in a mathematics lesson. To attain this objective, classroom activities in a teaching experiment are designed, described, and analyzed. The results suggest that recognizing the functional expression as a whole relation in the numerical table, which is an essential aspect of the reification of variables, is an inevitable part of conceptual change in the evolution from the operational to the structural conception.

THEORETICAL PERSPECTIVES

The *theory of reification* invented by Anna Sfard is well known as a model of mathematical concept formation. In this theory, the same mathematical notion can be conceived in two fundamentally different ways: *operationally*, as a process, and *structurally*, as an object (Sfard, 1991). A basic tenet of the theory is that the operational conception emerges first and the structural conception develops afterward in three phases: *interiorization*, *condensation*, and *reification*. This transition is a long-term process, and reification in particular can be a rather complex phenomenon. In this report, we will illustrate the reification phase with the help of a teaching experiment in a sixth grade classroom. Although this attempt might reveal just a part of this complex phenomenon, it presents an essential introductory situation in the promotion of transition from elementary to secondary school mathematics.

Another theoretical concern in this paper is reification as key to conceptual change in mathematics. Since Posner *et al*. (1982) introduced the *theory of conceptual change* to the field of science education, with the constructivist view of learning, several theoretical models have been proposed to explain student conceptions (diSessa, 2006). They have mainly been used to explain knowledge acquisition, in particular to characterize the drastic reorganization of existing knowledge in processes of learning. However, in mathematics education, only a few attempts have so far been made to explain conceptual change based on the nature of mathematical knowledge (Vosniadou, 2006; Vosniadou & Vamvakoussi, 2007). Since reification involves an ontological shift or a qualitative jump from operation to structure, Sfard (1991) states that "the ability to see something familiar in a totally new way is never easy to achieve. The difficulties arising when a process is converted into an object are, in a sense, like those experienced during transition from one scientific paradigm to another" (p. 30). In this way it seems that there is a common foundation between the theory of reification and that of conceptual change (cf. Merenluoto & Lehtinen, 2002). Thus, we would like to examine the reification of the notion of a "variable" from a conceptual change perspective.

METHODOLOGY

In the present study, we attempt to design a particular conceptual change situation with the help of the teaching experiment methodology presented in Cobb (2000). The particular type of classroom teaching experiment with which we are concerned here developed from the constructivist teaching experiment (e.g., Steffe, 1991). According to Cobb, the basic tenets of this methodology are collaboration between the researcher and the teacher, coordination of psychological and social perspectives, and conducting of developmental research cycles. Although the teaching experiment methodology has a long history in mathematics education research (Steffe & Thompson, 2000; Confrey, 2006), there is space here only for the central three phases of the recent approach:

- Instructional design and planning;
- Ongoing analysis of classroom events; and
- Retrospective analysis.

The first phase involves the formulation of goals, content, activities, and processes for teaching and learning that might be realized in the classroom. This phase can be guided by discipline-specific instructional theory. In this respect, Confrey (2006) refers to the *conceptual corridor* and *conceptual trajectory*, as follows:

> The aim of a design experiment is the articulation of two related concepts: a conceptual corridor and a conceptual trajectory. The *conceptual corridor* is a theoretical construct describing the possible space to be navigated successfully to learn conceptual content. During any particular set of episodes of teaching, that is, a design experiment, student will traverse a particular *conceptual trajectory* though the corridor. (Confrey, 2006, p. 145)

The second phase requires capturing data that document the descriptions of the activity's actual trajectory in the classroom. Cobb (2000) mentions that "these ongoing analyses of individual children's activity and of classroom social process inform new thought experiments in the course of which conjectures about possible learning trajectories are revised frequently" (p. 320). In this sense, the conceptual corridor is never completely specified but rather permitted to modify and evolve.

The third phase concerns the analysis of all the data sources generated in the course of experimentation in the classroom. The aim of this analysis is "to place classroom events in a broader theoretical context, thereby framing them as paradigmatic cases of more encompassing phenomena" (ibid., pp. 325-326). In other words, the aim of this phase is to characterize the conceptual corridor and articulate it with the description of conceptual trajectories. In the context of developmental research, these three phases undergo micro and/or macro cyclic processes.

A TEACHING EXPERIMENT IN A SIXTH GRADE CLASSROOM

Epistemological orientation towards instructional design

In preparing for the teaching experiment, we took account of some basic ideas for designing lessons that derive from the epistemological orientation. In the transition from elementary to secondary mathematics, the teaching and learning of variables can be a crucial didactic issue in our field. One of the most significant issues in learning variables is making sense of the mathematical usage of letters. Here we shall accept the sense of letters as variable, defined by Küchemann (1981) as follows: "the letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values" (p. 104). However, on the semantic level, the notion of "variable" has a plurality of conceptions, depending on the particular way it is used in problem situations (Malisani & Spagnolo, 2009). Therefore, based on some historical and epistemological analyses (Katz, 2007; Sfard & Linchevski, 1994), we can identify the following transitions as conceptual change situations in the teaching and learning of variables: 1) *from arithmetic expression as a computational process to algebraic expression as a computational object*; 2) *from dynamic awareness of proportional relation to static awareness of functional relation*. In order to facilitate these interrelated transitions, we should take account of a long-term perspective (cf. Greer, 2006). The target of this experimental study is the latter route, at the elementary school level in particular. Thus, the research question can be stated as the following: *How can we characterize sixth graders' conceptions of variables in functional relation in a mathematics lesson?* To address this question, there are three basic ideas for designing lessons presented in this study:

- Linking different operational models/representations that describe a quantitative relation in a given word problem;
- Becoming aware of the restrictions of operational models in the process of problem-solving activities; and
- Recognizing the expression $y = f(x)$ structurally in order to see the quantitative relation as a whole.

Instructional tasks and aims

The teaching experiment was conducted in a sixth grade classroom at an elementary school attached to a national university in Japan. In order to design and plan the lesson, one teacher collaborated with the author, and the above ideas were shared between them. As a result, the experimental lesson was set as the final session in a teaching unit that was ordinarily planned for three sessions. All three lessons were videotaped.

In the experimental lesson, "literal symbols" such as x and y are introduced—these have been primarily used as placeholders in sixth grade. According to a sixth grade mathematics textbook (Shimizu & Funakoshi *et al*., 2008), "exploring the rule of changing quantities" is done in the teaching unit that deals with the so-called "*cranes and tortoises problem*." This is an application problem using four operations, in which students apply the operations and solve them through arithmetical method. It can be described with the following word problem:

There are 8 cranes and tortoises altogether. The sum of the number of their legs is 20. How many cranes and tortoises are there, respectively?

The numerals in a given problem can of course be variable. We expect that one possible problem situation is for students to become aware of variables in functional relation through problem-solving activities (cf. Hirabayashi, 1994). In the process of problem-solving, students can create different solutions using different models or representations, and might discuss the effectiveness of each model. Therefore the aims of the experimental lesson are to ensure that (1) students are able to solve the problem by means of pictorial figures and numeral table; and (2) students appreciate how to represent the (invariant) rule of changing quantities.

Describing the actual classroom activities

In the beginning of the lesson, the teacher posed the "cranes and tortoises problem" and students solved its problem by means of a numeral table or the arithmetic method. Since students had used these of solutions in the previous two sessions, the problem itself was not a new situation for them. In this context, the teacher suggested that "someone has solved the problem by using a figure," and drew the figure on the backboard without any explanation of it (see Figure 1).

Figure. 1: The solution by the figure

Many of students were surprised and interested in this solution; some were able to understand the meaning of the drawings and explain the procedures thus: i) there are eight circles because there are eight animals altogether, ii) one can distribute two sticks each if one supposes all the animals are cranes, and iii) if one can distribute four more sticks, it means there are twenty legs in total (cf., Fischbein, 1977).

In the next phase of the lesson, the teacher revised the original problem as follows:

*There are **57** cranes and tortoises altogether. The sum of the number of their legs is **178**. How many cranes and tortoises are there, respectively?*

Then, the teacher said, "today, I would like you to consider how to represent the relationship at hand." After about five minutes, a student wrote down the numerals in a given table. Then, in the next phase, the teacher and students interactively recognized the (invariant) rule of changing quantities.

1 T: How can you find the answer in this table?

2 S: If the sum of legs is 178 legs, we can find the answer.

3 T: It is the long way, isn't it?

4 T: So, can you find a rule?

5 S: As the number of tortoises increases by one, the sum of legs increases by two.

6 T: If you use this rule, can you find the answer by calculations?

In addition, one student answered the solution using the arithmetic method (see Figure 2: the table consists of three rows, denoting the number of tortoises, the number of cranes and the sum of the number of legs, respectively), and another student explained the meaning of calculations such as $178-114=64$, $64\div 2=32$, and $57-32=25$, by referring to the procedural aspects of the figure (see Figure 1) and the horizontal relations in the table (see Figure 2).

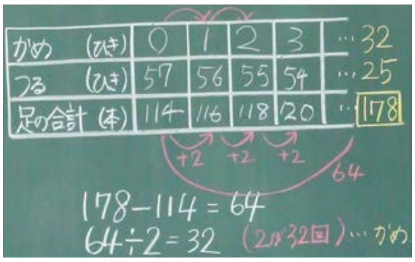


Figure 2: The solution by table with calculations

Some students tried to solve the revised problem with the figure as well. In the next episode, the teacher encouraged the students to look back at their solutions and appreciate the different solutions.

7 T: Some students solved the problem by the figure. It is interesting … but do you always use the figure?

8 S: … [No]

9 T: Why not?

10 S: It takes a lot of time if I draw 57 animals. And the table is better to see.

11 T: Ok. What kind of effectiveness does the figure have? What about the table? How do you appreciate them?

12 S: The table is useful for finding the rule

13 S: In the case of the figure, it is easy to see how many animals are there.

14 T: Do you have any idea about the disadvantages?

15 S: In the both cases, if the quantities increase, they are difficult to draw.

Eventually, the teacher showed a quasi-algebraic expression that describes quantitative relations in a given problem (see the right frame in Figure 3; *x* stands for the number of tortoises and *y* stands for the sum of the number of legs). The teacher and students inductively recognized the fact that this expression is consistent with the vertical relation in the table (see the left frame in Figure 3).

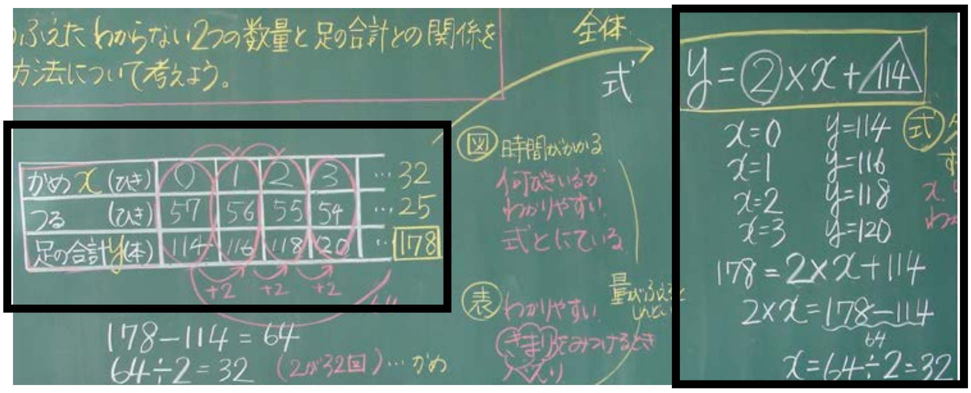


Figure 3: The relationship between the table and the quasi-algebraic expression

In this context, the students came to procedurally understand this expression as a kind of formula for finding the answer. So, when the teacher asked "How about the effectiveness of the expression, compared with the table?" the students responded as follows:

16 S: It takes a lot of time, if I use the figure or the table. But I can find the answer quickly, if I use the expression.

17 S: In the case of the expression, it is difficult to explain what *x* and *y* represent.

Following this episode, in order to facilitate students' conceptual understanding of the expression, the teacher posed a very challenging question to students:

18 T: Ok, you know, if you substitute a number for *x*, then you can get the number for *y*. So, how about this expression—without any calculation? To what extent is this expression represented in the table? What do you think?

19 S: It is … all [of it].

20 T: How about you?

21 S: The table as a whole.

Characterizing students' conceptions in classroom activities

We shall now analyze the classroom activities. Since the research concern of the study is with students' conceptions of variables in functional relation, we attempt to characterize them in terms of the reification phase. As the notion of a "variable" has a plurality of conceptions depending on the problem situation, in the instructional design we took account of different models or representations that describe the quantitative or functional relation in a given word problem. In this teaching situation, the development from an operational model such as a table model to a structural model such as an expression model can be essential to the reification of variables in functional relation. In Confrey's terms (2006), this situation can be seen as a conceptual corridor in the teaching experiment. On the other hand, the processes of teacher and student activities we described in the previous section can be seen as an actual conceptual trajectory.

There are three main models for solving word problems: the figure, the table, and the expression. In finding the answer by means of a figure model, students' conceptions can be interpreted as *pre-operational*, or that based on an enactive-pictorial solution.

As one student pointed out, in the 10th line of the transcript, most students recognized a limitation to the enactive-pictorial solution. Because of this realization, as seen in the 12th line of transcript, students could appreciate the solution by the table as a heuristic approach compared with the solution by the figure. In this phase, students' conceptions can be referred to as *operational*, based on a computational solution. Most students in the classroom were very familiar with the drawing and with how to use the table. In Sfard's (1991) term, it seems that these students have undergone interiorization and condensation of prior learning. As a matter of fact, to find a(n invariant) rule in changing quantities by means of the table is relevant to the operational conception in this situation. However, the table as an operational model has numerical restrictions. Therefore, it is important to note that some students were aware of the disadvantages of the table as well as those of the figure. Because the table itself cannot refer to the quantitative relation as a whole, awareness of this restriction can be crucial for the evolution from operational to structural conception.

In order to see the quantitative relation as a static object, that is, as the reification of a variable, one has to manipulate the horizontal-vertical relations in the table as a unified entity. Although the solution by the expression $y = 2x + 114$ was posed by the teacher, students were able to understand procedurally with the help of the teacher's guidance; for example, if $x = 0$ then $y = 114$, if $x = 1$ then $y = 116$, etc. Here, it is important to mention that when teacher and students were discussing the effectiveness of the models, a student referred, in the 17th line of the transcript, to a conventional-symbolic aspect of the expression. In addition, some students' conceptions can be interpreted as *structural*, because they recognized the table in totally new insight; i.e., the expression represents the table as a whole.

FINAL REMARKS

By way of conclusion, let us restate that we have analyzed and characterized reification as key to the conceptual change from an operational to a structural conception in an experimental lesson. In our teaching experiment, we did not intend students themselves to construct the expression, because the linear function is not introduced until seventh grade in the course of study. Nevertheless, in this transitional situation, we observed the emergence of students' structural conceptions with the teacher's support. As a future task, we need to plan an experimental lesson in the seventh grade and analyze this teaching experiment from a long-term perspective.

References

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. Kelly & R. Lesh (eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 307-333). Mahwah, New Jersey: LEA.

Confrey, J. (2006). The evolution of design studies as methodology. In R. K. Sawyer (ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 135-151). Cambridge: Cambridge University Press.

diSessa, A. A. (2006). A history of conceptual change research. In Sawyer, R. K. (ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 265-281). Cambridge: Cambridge University Press.

Fischbein, E. (1977). Image and concept in learning mathematics. *ESM*, *8*(2), 153-165.

Greer, B. (2006). Designing for conceptual change. In Novotná, J., *et al*. (eds.), *Proc. 30th Conf. of the IGPME, vol. 1*, 175-178. Prague: PME.

Hirabayashi, I. (1994). *Mathematical Experience for Elementary School Teachers (Syougakkou-kyoushi no suugaku taikenn)*. Nagoya: Reimei-Syobo. (In Japanese)

Katz, V. J. (2007). Stages in history of algebra with implications for teaching. *ESM*, *66*(2), 185-201.

Küchemann, D. E. (1981). In K. M. Hart (ed.), *Children's Understanding of Mathematics: 11-16*, 102-119. London: John Murray.

Malisani, E. & Spagnolo, F. (2009). From arithmetical thought to algebraic thought: The role of the "variable." *ESM*, *71*(1), 19-41.

Merenluoto, K. & Lehtinen, E. (2002). Conceptual change in mathematics: Understanding the real numbers. In M. Limón & L. Mason (eds.), *Reconsidering Conceptual Change: Issues in Theory and Practice* (pp. 233-257). The Netherlands: Kluwer.

Posner, G. J. *et al*. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education*, *66*(2), 211-227.

Sfard, A. (1991). On the dual nature of mathematics conceptions: Reflections on processes and objects as different sides of the same coin. *ESM*, *22*(1), 1-36.

Sfard, A. & Linchevski, L. (1994). The gains and pitfalls of reification: The case of algebra. *ESM*, *26*(2-3), 191-228.

Shimizu, S. & Funakoshi, S. *et al*. (eds.) (2008). *Fun with Math 6B for Elementary School (Wakuwaku Sannsuu 6-ge)*. Osaka: Keirinkan. (In Japanese)

Steffe, L. P. (1991). The constructivist teaching experiment: Illustrations and implications, In E. von Glasersfeld (ed.), *Radical Constructivism in Mathematics Education* (pp. 177-194). Boston: Kluwer Academic Publishers.

Steffe, L. & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. Kelly & R. Lesh (eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-306). Mahwah, New Jersey: LEA.

Vosniadou, S. (2006). The conceptual change approach in the learning and teaching of mathematics: Introduction (RF3). In Novotná, J., et al. (eds.), *Proc. 30th Conf. of the IGPME*, *vol. 1*, 155-184. Prague: PME.

Vosniadou, S. & Vamvakoussi, X. (eds.) (2007). *Reframing the Conceptual Change Approach in Learning and Instruction*. Oxford: Elsevier.

TEACHERS' PERCEPTIONS OF MATHEMATICAL CREATIVITY AND ITS NURTURE

Atara Shriki Ilana Lavy

Oranim Academic College of Education Emek Yezreel Academic College

In order to design a graduate course for upper-elementary mathematics teachers that focuses on creativity, its nurture and evaluation, we asked teachers to respond a questionnaire aimed at uncovering their beliefs and conceptions regarding various issues that concern with creativity. Part of the teachers was also interviewed. The findings indicate that teachers' conceptions are narrow, and that nurturing students' creativity is perceived as impossible in the context of regular classes due to curricular constraints. Following the results we designed a learning environment that exposes teachers to the multifaceted nature of creativity, and engage them with discussing appropriate approaches to nurture and evaluate students' creativity as an integral part of daily teaching.

INTRODUCTION

Interest in nurturing creativity in the context of education started in the 1950’s (Craft, 2001). Perceiving creativity as a catalyst for social and economic change (Linn, 2011) and as essential for future success in life (NACCCE, 1999), research has acknowledged the central role of education in nurturing students' creativity as part of its wider responsibility to impart content knowledge (Lin, 2011). Consequently, educators are showing interest in exploring and enhancing creativity (Henry, 2009), assuming that all students possess an innate sense of creativity (Feldman & Benjamin, 2006) and that creativity can be developed through explicit instruction (Fryer, 1996). Specifically to mathematics, developing students' mathematics creativity ought to be one of the primary goals of mathematics education in general (NCTM, 2000). However, creativity is normally not encouraged in schools (Sriraman, 2005), and as a result, most students are provided with few opportunities to experience creative learning and thinking (Silver, 1997). Teachers' abstention from nurturing students' creativity is a result of various circumstances, one of which is insufficient knowledge about the subject, and holding beliefs that prevents the implementation of learning environment that might support the development of students' creativity (Bolden, Harries & Newton, 2009).

As part of our work with teachers we engage them with various activities aimed at developing their awareness of mathematical creativity and strengthening their ability to nurture and evaluate it. These activities vary from one group of teachers to another, depending on their previous experience and beliefs they hold about mathematical creativity. In this paper we present initial beliefs about creativity in mathematics of one group of graduate students, experiences upper-elementary teachers who were studying towards M.Ed degree in mathematics education, and introduce some of the guidelines we followed for generating a learning environment that is based on these beliefs and perceptions.

LITERATURE BACKGROUND

Creativity. The nature of creativity and its development have been explored by educational and psychological researchers for more than a century (Plucker, Beghetto, & Dow, 2004), and yielded over one hundred definitions of creativity (Mann, 2006). Due to the multifaceted nature of creativity, most of these definitions are vague or inadequate (Sriraman, 2005), and none are universally accepted (Treffinger, Young, Selby & Shepardson, 2002). Moreover, creativity has been explored through at least four different perspectives: the creative process, the creative person, the creative environment, and the creative product (Henry, 2009). Therefore, while discussing creativity, one should explicitly state the preferable definition and perspective. For our educational purposes, while working with teachers we draw on Torrance's (1974) definition of creativity, relating to four of its aspects: fluency, flexibility, novelty, and elaboration, and took the perspective of the creative product and the creative process. With respect to these aspects of creativity "*Fluency relates to the continuity of ideas, flow of associations, and use of basic and universal knowledge. Flexibility is associated with changing ideas, approaching a problem in various ways, and producing a variety of solutions. Originality is characterised by a unique way of thinking and unique products of a mental or artistic activity. Elaboration relates to the ability to describe, illuminate, and generalise ideas*." (Lev-Zamir & Leikin, 2011, p. 19). Of the four aspects listed in Torrance's definition of creativity, novelty (or originality) is widely acknowledged as the most appropriate aspect because creativity is generally viewed as a process related to the generation of original ideas, approaches, or actions (Leikin, 2009).

Teachers' conceptions of creativity. Students' mathematical creativity can be nurtured and developed only if it is part of students' educational experience (Mann, 2006), and therefore it is teachers' responsibility to initiate proper experiences in their classes (Aiken, 1973). However, in order to be able to foster students' creativity, teachers need to be capable of identifying characteristics of the creative personality and creative production, recognize the cognitive processes involved, and initiate learning environments that promotes creativity (Hill, 1992). Unfortunately, although most teachers tend to agree that creativity could be developed in school settings (Lynch & Harris, 2001), and believe that nearly every student is potentially creative (Aljughaiman & Mowrer-Reynolds, 2005), they often show a general lack of understanding about the nature of creativity and it's nurturing (Fleith, 2000).

Teachers' conceptions of creativity are very similar in various parts of the world. Their prevailing approach to creative product is expressed in terms such as 'original', 'novel', 'valuable', and 'unpredictable'. Relating to the creative process they use terms such as 'independent work', flexible thinking', and 'breaking mindsets' (Bolden et al., 2009).

THE STUDY

The issue of creativity was neglected in mathematics education research (Lev-Zamir & Leikin, 2011), and started to attract attention only in recent times (Treffinger et al., 2002). Within this context, little attention has been given to teachers' beliefs about creativity (Diakidoy & Kanari, 1999), and research points to an unexplained inconsistency between teachers' beliefs regarding the importance of promoting creativity and their actual practices (Fleith, 2000). Given the above and in order to develop teachers' capability of teaching for creativity, it is suggested that teacher educators better understand teachers' conceptions of creativity (Lev-Zamir & Leikin, 2011). Therefore, this study is aimed at exploring mathematics teachers' beliefs and conceptions of mathematical creativity, its development, nurture and evaluation.

Research aims. Realizing the importance of nurturing students' creativity, and being aware of the fact that the beliefs teachers hold about creativity are likely to influence how they define, operationalize, and evaluate students' creativity (Andiliou & Murphy, 2010), the aim of the study was to explore teachers' perceptions and beliefs about mathematical creativity, in order to be able to design a learning environment that considers these views. The learning environment was designed based on responses received from a questionnaire and interviews aimed at uncovering the teachers' belief about mathematical creativity.

Participants. Twenty-two graduate students, experienced teachers (having, on average, 16 years of teaching mathematics in upper-elementary school) in their first year (out of two) of studying towards M.Ed degree.

Research tools. For uncovering participants' beliefs, conceptions and perceptions regarding mathematical creativity we used two research tools: a questionnaire and interviews. The participants were asked to complete a questionnaire, anonymously, aimed at eliciting their conceptions about creativity, its nurturing and evaluation. The questionnaire was designed following questionnaires appear in previous studies that examined pre-service and in-service elementary school teachers' conceptions of creativity (Bolden et al., 2009; Aljughaiman & Mowrer-Reynolds, 2005). The questionnaire used both closed (Likert-type) and open-ended items. In addition, five teachers were interviewed for about 30 minutes. The interviewees were chosen randomly, and agreed to be interviewed on a voluntary basis.

In this paper we discuss some of the participants' responses to the open-ended questions, and their related responses to the interview's questions.

Data analysis. Due to the relatively low number of participants, the participants' responses to the closed items included in the questionnaire were analysed using basic descriptive statistics, and Pearson correlation. Participants' responses to the open-ended questions included in the questionnaire as well as the interviews were analysed using open coding methods (Strauss & Corbin, 1990).

RESULTS AND DISCUSSION

In this section we refer to the participants' responses to three of the seven open-ended questions included in the questionnaire, followed by relevant quotations taken from the interviews (quotations appear in Italic). The three aspects we wish to illuminate are participants' perceptions of mathematical creativity, its nurture and evaluation. We believe these three aspects are interconnected, as beliefs teachers hold about creativity influence what they do in class. It also implies that changing teachers' practical approaches to creativity should start with changing their beliefs.

Before delving into teachers' responses, we wish to note that, similar to other studies (e.g. Aljughaiman & Mowrer-Reynolds, 2005), all the participants believed that mathematical creativity can and should be nurtured in school. However, most of them were not sure whether this was their responsibility or not, neither they were not conclusive as how to do it. Half of them believed that not every student is capable of demonstrating creative behaviour. This stands in opposition to findings of previous studies (e.g. Isaksen, Dorval, & Treffinger, 2000) in which most teachers believed that creativity exists in every child at different levels.

Participants' perceptions of mathematical creativity. In response to the open-ended question: "To your opinion, what is the meaning of 'mathematical creativity'?", eighteen (82%) teachers wrote that it has to do with finding unusual or original solution/explanation/proof to a problem. Similar to the prevalent image of creativity (Leikin, 2009), most of the study participants view mathematical creativity in term of the product, relating to its originality. When asked in their interviews to explain what they mean by "unusual or original solution/ explanation/ proof to a problem", the teachers maintained that "*it is a proof I have never seen before*"; "*a solution that surprises me*"; or "*explanation I haven't thought of*". These perceptions of mathematical creativity are noteworthy. The view of personal creativity, as a quality that can be developed in school students, requires a distinction between relative and absolute creativity (Sriraman, 2005; Leikin, 2009). While absolute creativity is related to outstanding historical works of mathematicians, relative creativity refers to discoveries made by a specific person within a specific reference group. As can be seen, the teachers indeed referred to the creative product in terms of relative creativity; however, they addressed themselves as the reference group. This view of students' mathematical creativity prevented the teachers from perceiving creativity as a trait that exists in every student at different level: "…*The fact that I rarely encounter a solution I haven't seen before indicates that only few students are capable of being creative*".

Participants' perceptions of nurturing mathematical creativity. In response to the open-ended question: "Is it important to develop students' mathematical creativity? Why?", all the participants argued that it was important to develop all students' mathematical creativity. Ten teachers believed it projects on the development of students' mathematical thinking in general, and ten teachers believed it might change students' image of mathematics. Furthermore, when asked: "Do you develop you students' mathematical creativity? If not- why? If yes- explain how you do it (or give examples)", all the teachers maintained that they do develop students' mathematical creativity, although four of them admitted they do not do it "in a satisfactory manner". Most of the examples (73%) the teachers provided as demonstrations to developing students' creativity referred to "encouraging students to solve problems in various ways". They did not, however, exhibit any specific problem in order to exemplify what they meant.

However, the interviews with the teachers revealed a totally different picture: "*It is not that I tell them 'O.K., today I expect you to solve this problem in various ways'. If someone does it, I reward him by giving a positive feedback*"; "*My students know that if they suggest an unusual solution I mention it in the school data base…I believe it encourages them to think creatively, but it is not something that I remind them every lesson, they just know it*". The excerpts of these two teachers indicate that although they refer to mathematical creativity in term of solving problems in various ways, similarly to other researchers (e.g. Leikin, 2009), they do not, however, initiate any explicit instruction in order to encourage students' expressions of creativity neither perceive their role as instructing students to do so on a regular basis as in integral part of their teaching.

The other three teachers related differently to the issue of solving a problem in various ways: "*From time to time I show them more than one solution to a problem. I admit I don't do that systematically, but I believe it opens their eyes to see some interconnections between mathematical topics, and I think it also develops their creativity because they see that one can approach a problem from different perspectives*". Namely, being aware of the importance of solving problems in various ways, these teachers believe that from time to time they have to present this idea to students. Nonetheless, they do not initiate learning situations in which they ask students to solve a problem in different ways and discuss it.

All the five interviewees justified the fact they were not encouraging students to solve a problem in various ways on a regular basis by relating to time constraints: "*I have to prepare my students to matriculation tests, where unconventional solutions are not always welcomed. I don't have time for making experiments with the students*"; "*Teaching is like running against the clock. You have to teach the curriculum at a certain level, and teaching for creativity is actually a luxury*"; "*If I ask students to solve a problem in several ways it means that I also have to give students time to present all these solutions, and I don't have this extra time*".

The utterances indicate that these teachers perceive their role mainly in term of "*covering the curriculum*", and therefore, despite recognizing the importance of encouraging students to solve problems in various ways and its relation to mathematical creativity, still they avoid it due to the lack of available time. Moreover, it should be mentioned that in recent years students are allowed to solve problems in the matriculation tests using any mathematical tool. Nonetheless, the teachers' responses indicate that they keep teaching conservatively, namely-solving problems in merely a single way.

Participants' perceptions regarding evaluation of creativity. In response to the open-ended question "To your opinion, how should students' mathematical creativity be evaluated?", the teachers repeated their previous responses, and related to the originality of students' solutions. However, during the interview they referred mainly to technical aspects of evaluation, namely-grades in tests and end-of-term certificate, rather than to the manner in which students' creative product and process should be examined and evaluated. Eight teachers maintained the creative solutions should be given scores in tests, among them six believed that this should only be in a format of "a bonus question" to which the score should not exceed 5 points. Twelve teachers argued that creative students should be publically praised, in order to encourage them to keep developing their creativity. However, during the interviews it turned out that these apparently technical aspects are rooted in emotional aspects: all the five teachers referred to the assessment of creativity rather than to its evaluation: "*Only few students are creative. It wouldn't be fair to discriminate between students based on their special gift*"; "*It is not the fault of uncreative students that they cannot present unusual solution...I don't want them to feel underprivileged*"; "*The aim of tests is primarily to examine students' knowledge…so if someone knows how to solve a problem, it is enough. The aim of regular school tests is not to identify creative students*".

It seems that the teachers have only a vague idea about evaluation of creativity, and as a result they tend to link between evaluation and assessment, recoiling evaluation of creativity in order not to discriminate between students on the basis of a unique trait instead of knowledge.

SUMMARY AND IMPLICATIONS

Creativity can be developed through explicit instruction (Fryer, 1996). However, in order to be able to foster students' creativity, teachers need to be capable of providing students with appropriate assignments (Mann, 2006). This can be done only if teachers are aware of the various aspects concern with creativity (Hill, 1992). Discussing issues that relate to creativity should be based on what teachers already know and believe (Dyer et al., 2004). Therefore, our aim was to examine the teachers' beliefs about mathematical creativity, its nurture and evaluation, in order to design learning environment that considers these beliefs and perceptions.

What did we learn from the teachers' responses to the questionnaire and interview? Apparently, all the participants believe that mathematical creativity can and should be nurtured in school. However, similarly to Fleith's (2000) findings, there were inconsistencies between their statements and actions. For example: on one hand they argue that they develop students' creativity by encouraging them to solve problems in various ways, but on the other hand they maintain that they do not operate this way due to time constraints. It appears that the teachers' insufficient knowledge about creativity, its nurture and evaluation might result in their abstention from implementing learning environment that support the development creativity.

What were the derived guidelines for designing a learning environment that would increase teachers' awareness of creativity? Given the above findings the learning environment we implemented was based on the following guidelines: (i) exposing the teachers to the multifaceted nature of creativity in general, and mathematical creativity in particular, through emphasizing issues such as: relative creativity, creative product, creative process, and creative person; (ii) discussing approaches that have the potential to nurture students' creativity in mathematics and ways to implement them in class; (iii) introducing practical tools to evaluate students' creativity, and how to respond to expressions of creativity in class. And above all, the learning environment we designed puts emphasize on how to turn the nurture of students' creativity into an integral part of daily teaching, considering time constraints imposed by the curriculum, believing this would increase teachers' readiness and ability to nurture students' mathematical creativity regularly.

References

Aiken, L. R. (1973). Ability and creativity in mathematics. Review of Educational Research, 43(4), 405–432.

Aljughaiman, A., & Mowrer-Reynolds. E. (2005). Teachers' conceptions of creativity and creative students. *Journal of Creative Behavior, 39*, 17- 34.

Andiliou, A., & Murphy, K. P. (2010). Examining variations among researchers' and teachers' conceptualizations of creativity: A review and synthesis of contemporary research. *Educational Research Review, 5*, 201-219.

Bolden, D. S., Harries, T. V., & Newton, D. P. (2009). Pre-service primary teachers' conceptions of creativity in mathematics. *Educational Studies in Mathematics, 73*(2), 143-157.

Craft, A. (2001). Little C creativity. In A. Craft, B. Jeffrey, & M. Leibling (Eds.), *Creativity in education*. London: Continuum.

Howe, A., Davies, D., Rogers, M., & Fasciato, M. (2004). How do trainee primary teachers understand creativity? *Innovation Education Conference,* University of Leeds, 8 – 10.

Dyer, C., Choksi, A., Awasty, V., Tyer, U., Moyade, R., Nigam, N., Purohit, N., Shah, S., & Sheth, S. (2004). Knowledge for teacher development in India: the importance of local knowledge for in-service education. *International Journal for Educational Development, 24*(1), 39-52.

Henry, J. (2009). Enhancing creativity with M.U.S.I.C. *The Alberta Journal of Educational Research, 5*(2), 199-211.

Hill, R. (1992). Finding creativity for children. Paper prepared for the *Leadership Accessing Symposium*, Lafayette, IN. (ERIC DocumentReproduction Service No. ED348169).

Feldman, D. H., & Benjamin, A. C. (2006). Creativity and education: An American retrospective. *Cambridge Journal of Education, 36,* 319-336.

Fleith, D. (2000). Teacher and student perceptions of creativity in the classroom environment. *Roeper* Review, 22(3) 148-53.

Fryer, M. (1996). *Creative teaching and learning*. London: Paul Chap-man Publishing Ltd.

Isaksen, S., Dorval, K., & Treffinger, D. (2000). *Creative approaches to problem solving*. Dubuque, Iowa: Kendall/Hunt Publishing Company.

Lev-Zamir, H., & Leikin, R. (2011). Creative mathematics teaching in the eye of the beholder: focusing on teachers' conceptions. *Research in Mathematics Education, 13*(1), 17-32.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students*, 129–145.

Lin, Y-S. (2011). Fostering creativity through education- A conceptual framework of creative pedagogy. *Creative Education*, *2*(3), 149-155.

Lynch, M., & Harris, C. (2001). *Fostering creativity in children, K-8: Theory and practice*. Needham Heights, MA: Allyn and Bacon.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, *30*(2), 236-260.

NACCCE (1999). *All our futures: Creativity, culture and education.* National advisory committee on creative and cultural education. London: DFEE. Retrieved July 2011 from http://www.cypni.org.uk/downloads/alloutfutures.pdf

NCTM - National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Plucker, J. A., Beghetto, R. A., & Dow, G. T. (2004). Why isn't creativity more important to educational psychologists? Potential, pitfalls, and future directions in creativity research. *Educational Psychologists*, *39*, 83–96.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing, *ZDM-The International Journal on Mathematics Education, 29*(3), 75-80.

Sriraman, B. (2005). Are giftedness and creativity synonyms in mathematics? *The Journal of Secondary Gifted Education*, XVII(1), 20–36.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications, Inc

Torrance, E. P. (1974). *The Torrance tests of creative thinking: Technical-norms manual*. Bensenville, IL: Scholastic Testing Services.

Treffinger, D. J., Young, G. C., Selby, E. C., & Shepardson, C. (2002). *Assessing creativity: A guide for education*. The National Research Center on the gifted and talented, Sarasota, Florida.

EQUIVALENCE OF RATIONAL EXPRESSIONS: ARTICULATING SYNTACTIC AND NUMERIC PERSPECTIVES

Armando Solares — Universidad Pedagógica Nacional, México

Carolyn Kieran — Université du Québec à Montréal, Canada

Our study concerns the conceptual mathematical knowledge that emerges during the resolution of tasks on the equivalence of polynomial and rational algebraic expressions, by using CAS and paper-and-pencil techniques. Our findings highlight the mathematical knowledge (technological discourse) constructed in the process of confronting, differentiating, and articulating the several mathematical praxeologies (pertaining to the numeric perspective or the syntactic perspective in algebraic equivalence) that arose in the solution of the designed equivalence tasks.

INTRODUCTION AND BRIEF LITERATURE REVIEW

While a substantial amount of research has been carried out with respect to the equivalence of algebraic expressions, very little of it has dealt explicitly with either the comparison of polynomial and rational expressions or the bridging of syntactic and numeric perspectives. Some researchers have highlighted the importance of being able to work flexibly with algebraic expressions in various forms and of recognizing equivalent expressions; other studies have focused on the difficulties that students encounter with understanding algebraic equivalence (see for example Nicaud et al.; 2004, Sackur et al., 1997; or Ball et al., 2003).

With respect to the different perspectives, Cuoco (2002) distinguishes between polynomial functions and polynomial forms as follows. Polynomial functions involve thinking about the letter in a polynomial as a variable and about the polynomial as an input-output machine that can yield a table or a graph, and has all the attributes of real-valued functions of a real variable. In contrast, polynomial forms are viewed as formal expressions with the letter considered as an indeterminate and which involve operations such as factoring, adding, multiplying, and so on. However, according to Cuoco “the distinctions between polynomial forms and functions tend to be ignored in school mathematics” (2002, p. 297). Cerulli and Mariotti (2002) make similar distinctions in what they refer to as functional and axiomatic definitions of equivalence. They point out that since, for polynomials in *n* variables, the functional and the axiomatic definitions are equivalent, they do not go into the particularities of the equivalence of these definitions with their learners.

Artigue (2002) has drawn on students’ work involving the passage from one given form of algebraic expression to another to illustrate the difficulties that students experience with equivalence problems, difficulties that come to the fore when they use Computer Algebra Systems (CAS) (e.g., Artigue, 2002; Guin & Trouche, 1999;

Lagrange, 2000). She asserts that the CAS forces students to confront issues of equivalence and simplification in ways that are not so easily achieved in more traditional, paper-and-pencil, treatments. Following Artigue´s considerations, Kieran and Drijvers (2006) and Kieran and Saldanha (2008) have investigated the learning of the technical and theoretical aspects of various topics in high school algebra within CAS environments, including the topic of equivalence. They have found that "students linked the notion of restrictions [within the rational expressions] to the numerical view on equivalence" (Kieran & Drijvers, 2006, pp. 227-231). While the distinctions between form and function are particularly important when students use CAS technology to solve equivalence tasks because the polynomial-form perspective underlies CAS – even if it also deals with polynomials as functions – many crucial questions regarding the articulation of the syntactic and numeric perspectives could not be answered by the classroom-based study reported by Kieran and Drijvers. The present article deals exactly with such articulation, within the methodological frame of individual-based student interviews.

THEORETICAL FRAMEWORK

As in previous studies (e.g., Kieran & Drijvers, 2006), we adopt the Anthropological Theory of Didactics (ATD) developed by Chevallard (1999). As per this theory, the objects of mathematical knowledge emerge from systems of practices whose norms and manners of use define the ways of knowing and understanding these objects and their way of living in specific institutions. These systems are named *mathematical praxeologies* in Chevallard's theory and they are described by: the *types of tasks* in which the objects of knowledge are immersed; the *techniques* or ways of solution of these tasks; the discourse that explains and justifies the techniques, named *technology*; and the *theory* that provides the structural basis of the technological discourse and that can be seen as the "technology of the technology" (Artigue, 2002, p. 248).

In this study we focus on the co-emergence of techniques and conceptual mathematical knowledge when solving equivalence tasks. We share the point of view of Lagrange (2000, p. 16), who affirms that techniques develop mathematical meaning in a double relationship with, on the one hand, the tasks they permit the user to solve, and, on the other, the theorizations they promote. In accordance with Lagrange, Artigue (2002) states that techniques are usually recognized by their *pragmatic value* for task solution, in other words, in terms of their efficacy, cost, and validity domain. However, techniques also have an essential *epistemic value* as they contribute to the understanding of the objects they handle.

METHODOLOGICAL CONSIDERATIONS

The study presented herein is part of a larger program of research whose central objective was to shed light on the co-emergence of algebraic technique and theory within an environment involving novel tasks and a combination of Computer Algebra System (CAS) and paper-and-pencil (PP) media (see Kieran & Drijvers, 2006). With this objective and our theoretical perspective in mind, for this study our research team developed a series of task sequences on equivalence, within an environment involving both paper-and-pencil and CAS (the TI-92 Plus handheld calculator), that would encourage both technical and theoretical development (Chevallard, 1999, Artigue, 2002, and Lagrange, 2000) in 10th grade algebra students.

For the design of tasks, we consider algebraic expressions that are polynomials and polynomial quotients in one indeterminate with coefficients on the set of real numbers *R*. If we see algebraic expressions as polynomials or polynomial quotients, we will say that two expressions are *equivalent from the syntactic perspective* when they have a *common algebraic rewriting* by applying the properties of the algebraic properties of polynomial and polynomial quotients operations. If we see algebraic expressions as polynomial functions, we will say that two algebraic expressions *f* and *g* are *equivalent from the numeric perspective* when have the same values for all *x* in the *common domain.* The two perspectives of equivalence emphasize different aspects. For the numeric perspective, it is essential to include the study of the characteristics of domains and images for the corresponding algebraic expressions being compared. For the syntactic perspective, the rewriting of expressions plays a central role. On the basis of the algebraic properties of the operations of the ring of polynomials and the field of quotients of polynomials, the rewriting of expressions allows for verifying equivalence and for obtaining equivalent expressions.

The research program includes data collection of classroom lessons and videotaped interviews with students. The analysis presented in this article is based on the first of three interviews carried out with one Grade 10 algebra student, Andrew. At the moment of the interview, Andrew had already learned the four basic operations with polynomials, and a few techniques for factoring certain binomials and trinomials, and for solving linear and quadratic equations. He had not yet had any formal school experience with the notion of equivalence, nor with rational algebraic expressions, but he had studied the introductory topics of domain and range, dependence, relation versus function, and modes of representation. Andrew had also been introduced to CAS technology.

In Table 1 we present the expressions used in the interview of this study.

Expression A:	$(x^2+x-20)(3x^2+2x-1)$
Expression B:	$(3x-1)(x^2-x-2)(x+5)$
Expression C:	$\dfrac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$

Table 1: Designed expressions for this study

ANALYSIS

The first part of the interview consisted of evaluating the given algebraic expressions and producing a conjecture regarding the numerical relationships among the obtained values. Andrew evaluated the expressions for x = 1/3, -5, 6, 7, and conjectured: *The results for Expressions B and C would continue to be equal to each other*. In our theoretical terms, we can say that he conjectured the numeric equivalence of Expressions B and C.

When asked to justify his conjecture "for all numbers", he resorted to syntactic techniques: he *expanded* (with paper and pencil and with CAS) and *factored* the expressions so as to obtain "forms" that he could compare. So, if the rewriting of the expressions (obtained by expansion or by factorization) is the same, then they take on the same values for any x (in our theoretical terms: if they are syntactically equivalent, then they are numerically equivalent). Table 2 shows the expanded and factored forms obtained by Andrew just as he wrote them.

Original expression	Expanded form	Factored form
$(x^2+x-20)(3x^2+2x-1)$	$3x^4+5x^3-59x^2-41x+20$	$((x+5)(x-4))((x+1)(3x-1))$
$(3x-1)(x^2-x-2)(x+5)$	$3x^4+11x^3-25x^2-23x+10$	$(3x-1)((x-2)(x+1)(x+5))$
$\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	$3x^4+11x^3-25x^2-23x+10$	$(x-2)(x+5)(3x-1)(x+1)$

Table 2: Andrew's expanded and factored forms

It is important to say that just a few students of the whole research program used syntactic techniques for justifying their conjectures for the numeric equivalence of expressions. Most of them "felt unsure about algebra providing certainty about numerical values, even if their algebraic skills were good" (Kieran & Drijvers, 2006, p. 222).

The interview continued, with Andrew being asked to find the *domain of definition* for Expression C:

Interviewer: Is there any value of x that would not be permissible as a replacement value for x in Expressions B and C?

Andrew concluded that Expression C was not defined for x = -2. Then he was asked about the consequences of this non-definition with respect to the equivalence of Expressions B and C. Andrew answered as follows:

Andrew: Well just, it [the factorized form for Expression C, once the common factors are cancelled: $(x-2)(x+5)(3x-1)(x+1)$] is, like, another form of the expression, which is, I guess, once the expression is factored out, and then they're still equal to each other. So, that makes sense [small laugh]. Basically, Expressions B and C, when they are fully factored, at least to my capability, they're equal to each other, still. So, it just supports my conjecture that, with any x value, excluding negative 2, they would be equal to each other.

In this first contact with the differences between the two perspectives on algebraic equivalence, Andrew incorporated the restriction as an exception to the equality of the values of the expressions: although these expressions are syntactically equivalent (they can be rewritten as the same expression), there is a value of *x* for which they are not equal.

Straightaway, Andrew spontaneously proceeded to evaluate at $x = -2$ the expressions syntactically equivalent to Expression C. Table 3 shows the results that he obtained.

Expression		Value at $x = -2$
Expression C:	$\frac{(x^2+3x-10)(3x-1)(x^2+3x+2)}{(x+2)}$	Undefined
Expression B:	$(3x-1)(x^2-x-2)(x+5)$	-84
Expanded form of expressions B and C, obtained by using CAS:	$3x^4+11x^3-25x^2-23x+10$	-84
Factored form of expressions B and C, obtained by using PP:	$(x-2)(x+5)(3x-1)(x+1)$	(-84) Andrew considered this evaluation, but did not explicitly carry it out during the interview.

Table 3: CAS evaluation of the expressions equivalent to Expression C (at $x = -2$)

This evaluation performed by Andrew was not considered in the original design of the interview. It was a spontaneous confrontation of the different facts that support the numeric equivalence of Expressions B and C versus the fact that these expressions take on different values at the restriction. At the same time, this evaluation was an exploration of the possibility that the restriction is inherited by the syntactically equivalent forms and, thus, if the equality of the values could be kept for $x = -2$. In Andrew's words "it is very possible that if I work this [expression B] out, with minus two incorporated into it, that that would equal zero too, which is based on the fact that they [expressions B and C] have always been equivalent".

The interview continued with some tasks that required Andrew to confront the differences between the two perspectives on algebraic equivalence. For the solution of these tasks we introduced two CAS techniques: the *equivalence test* and the *numeric equality test*. These techniques allowed him to determine the equivalence of two expressions. In the case of expressions that are syntactically equivalent but which have a restriction (like Expressions B and C), the results obtained by applying the *equivalence* and the *numeric equality tests* are "contradictory". For the restriction ($x = -2$), when using the numeric equality test, the result that is obtained is FALSE; whereas when using the equivalence test, the result is TRUE.

The contradictory results obtained for Expressions B and C confirmed what Andrew had already obtained by applying the factoring and expanding techniques, with and without CAS. As Andrew said, the CAS was not considering the domain restriction of Expression C. Andrew explained the results as follows:

Andrew: Yeah, at first [he is referring to the result of the test of equivalence] it's saying that any value of x would be true, that any value of x can be substituted and they would be equivalent. But, like this just proves, that when minus two is incorporated that it's not true [he is referring to the result of the test of numeric equality for $x = -2$], in this form at least [original expression C]. Because once it's expanded, it [the calculator] saw they were still equivalent, and it didn't. I guess in different forms it's not true, but in this particular form [original expression C] it is.

Throughout the interview Andrew managed to construct an articulation between the differences and contradictions of the results obtained through the syntactic and numeric techniques: the numeric equivalence of two algebraic expressions could be established in a general manner (for every value of x, not just for a finite set of values) by means of syntactic techniques, both using paper and pencil and CAS. For example, through *expanding* and *factoring techniques*, Expressions B and C could be rewritten as the same expression. However, numeric equivalence requires considering the restrictions. At the domain restriction for Expression C, these expressions do not have the same value. Table 4 presents a theoretical analysis of the techniques and the technological discourses articulated by Andrew.

	Numeric perspective	Syntactic perspective
Type of task	Establishing the equivalence of rational expressions (with remainder equal to zero) and polynomial expressions.	
Technique	Establish the common domain of the expressions, i.e., determine the restrictions. Compare the expressions over the common domain. Evaluate the expressions for several values of the common domain.	Rewrite the expressions in a common algebraic form, in a factorized or expanded form. Compare the rewritten expressions (term by term or factor by factor).
Technology	If the expressions take on the same values for a set of values, they can take on the same values for any value of the common domain (their *numeric equivalence* is conjectured).	If the expressions can be rewritten as the same expression (in a factorized or expanded form), they are *syntactically equivalent*.

Table 4: Numeric and syntactic perspectives for determining the equivalence of rational expressions (with remainder equal to zero) and polynomial expressions

DISCUSSION

Andrew resorted to techniques belonging to different mathematical praxeologies and to different perspectives on algebraic equivalence. Syntactic techniques and technologies used by Andrew belong to two well-distinguishable praxeologies: the *Expansion praxeology* and the *Factorization praxeology*. The third praxeology involved in Andrew's solutions is determined by the *evaluation technique*, which corresponds to the numerical perspective of equivalence. We call it the *Evaluation praxeology*.

For Andrew the applied techniques do not belong exclusively to a unique perspective. In fact, for him, differentiated perspectives on equivalence do not exist. They are just mathematical knowledge and resources for solving the same type of tasks. Andrew articulated the differences and contradictions by establishing distinctions between the numeric and the syntactic techniques, as well as between their corresponding conceptual elements (technologies): numeric equivalence of algebraic expressions can be "proved" by rewriting them and showing that they are the same (syntactic equivalence). However, at the restrictions (numerical values of x where the rational expressions are not defined), numeric equivalence does not correspond to the syntactic equivalence of the expressions; numerical evaluation is necessary in this case.

The use of CAS was central to the exploration of the values of the expressions; its *pragmatic value* (Lagrange, 2000) was given relevance through this use. CAS was central also for making explicit the differences and their conciliation; it thereby acquired an *epistemic value* through the constitution of the technological discourses for explaining and conciliating the "contradictions" that emerged during the solution of the tasks.

Clearly, the articulation of the two perspectives on algebraic equivalence was not completed by the theoretical and technological explanations generated through the solution of the set of tasks presented in this study. For instance, certain issues related to restrictions, zeros, and factors of polynomials were not explained nor even dealt with. However, the design of tasks that involved confronting and differentiating perspectives of equivalence allowed for the creation of a mathematical arena where mathematical knowledge about algebraic expressions and their equivalence (the mathematical praxeologies involved) is articulated and, in this way, constructed.

Acknowledgments. We express appreciation to our co-researchers A. Boileau, D. Tanguay, F. Hitt, J. Guzmán, and L. Saldanha. We also acknowledge the support of the Social Sciences and Humanities Research Council of Canada (Grant #410-2007-1485) and travel funds from PROMEP-México (PROMEP/103.511/6503).

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*, 245-274.

Ball, L., Pierce, R., & Stacey, K. (2003). Recognising equivalent algebraic expressions: An important component of algebraic expectation for working with CAS. In N.A. Pateman, B.J. Dougherty, & J. Zilliox (Eds.), *Proceedings of 27th PME* (Vol. 4, pp. 15-22). Honolulu, HI: PME.

Cerulli, M., & Mariotti, M.A. (2002). *L'Algebrista: un micromonde pour l'enseignement et l'apprentissage de l'algèbre. Sciences et techniques éducatives* (special issue edited by J.-F. Nicaud, E. Delozanne, & B. Grugeon), *9*(1-2), 149-170.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, *19*, 221-266.

Cuoco, A. (2002). Thoughts on reading Artigue's "Learning mathematics in a CAS environment." *International Journal of Computers for Mathematical Learning*, *7*, 293-299.

Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*, *3*, 195-227.

Kieran, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning*, *11*, 205-263.

Kieran, C., & Saldanha, L. (2008). Designing tasks for the co-development of conceptual and technical knowledge in CAS activity: An example from factoring. In K. Heid & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Syntheses, cases, and perspectives* (Vol. 2, pp. 393-414). Greenwich, CT: Information Age Publishing.

Lagrange, J.-B. (2000). L'intégration d'instruments informatiques dans l'enseignement: une approche par les techniques. *Educational Studies in Mathematics*, *43*, 1-30.

Nicaud, J.-F., Bouhineau, D., & Chaachoua, H. (2004). Mixing microworld and CAS features in building computer systems that help students learn algebra. *International Journal of Computers for Mathematical Learning*, *9*, 169-211.

Pomerantsev. L., & Korosteleva, O. (2003). Do prospective elementary and middle school teachers understand the structure of algebraic expressions? *Issues in the Undergraduate Mathematics Preparation of School Teachers: the Journal*, Vol. 1: Content Knowledge, December 2003 (electronic journal: http://www.k-12prep.math.ttu.edu/journal/contentknowledge/volume.shtml).

Sackur, C., Drouhard, J.-Ph., Maurel, M., & Pécal, M. (1997). Comment recueillir des connaissances cachées en algèbre et qu'en faire? *Repères – IREM*, *28*, 37-68.

FROM EUCLID TO GPS : DESIGNING AN INNOVATIVE SPATIAL COORDINATION ACTIVITY WITH MOBILE TECHNOLOGIES

Håkan Sollervall

Halmstad University and Linnaeus University

Guided by the notion of design research we develop a learning activity for 12 year old students, who are asked to coordinate themselves physically in terms of distances with respect to two given points in an outdoor setting. The outdoor activity, as well as its continuation into the mathematics classroom, involves mobile software applica-tions specifically developed to support this activity. In this paper, we argue that the design of innovative learning activities is enhanced by the coordination of expertise and knowledge from several research domains, whose collaboration is facilitated by using affordances for representation and communication as design instruments. We present a case where ancient Greek mathematics, modern psychology and techno-logical affordances guide the design of an innovative spatial coordination activity.

Introduction

This paper reports from a project regarding the design of technology-enhanced learning activities developed in collaboration between researchers in mathematics education and media technology. Several members of the current design team have previously collaborated in projects involving outdoor mathematics and information technology (Nilsson et al., 2009; Sollervall et al., 2010; Nilsson et al., 2010). These efforts have been inspired by Cobb, Confrey, diSessa, Lehrer, and Schauble (2003), who formulate a mission for research in mathematics education as striving to develop, test and revise learning activities which are designed in order to support envisioned learning processes of an object of study. The current learning activity, which will be presented and discussed later in this paper, emerged as an idea during a design team meeting where a selection of available mobile technologies and their didactical potential for the learning of mathematics were discussed.

The discussions and negotiations during team meetings are guided by scenario-based design (SBD) where narratives about students' hypothesized actions during a proposed activity enable communication about usage possibilities and concerns among different stakeholders (Penuel, Roschelle, and Shechtman, 2007) who interpret these narratives according to their different expertise and interests, respectively students' learning of mathematics and technological specificities. This common understanding, which relates to the opportunities for action and learning that the students are offered in the implemented activity, serves as a catalyst for enhancing the respective considerations of didactic and technological issues.

research objective

The research objective for the current study is to argue for the value of using *affordances*, defined as *opportunities for action* (Bower, 2007), to facilitate the collaborative design of innovative mathematical learning activities and enhancing the opportunities for learning that are offered during such activities. We present a case of a collaboratively designed mathematical learning activity that involves mobile technologies in an outdoor setting. This particular design effort involved expertise and knowledge from several research domains, including mathematics education, media technology and psychology. We will show how careful collaborative considerations of affordances, either to be offered or denied, made it possible to provide opportunities for students' experiencing spatial orientation as a specific aspect of spatial ability seldom explored in a regular school setting. We argue for the necessity of involving both an outdoor setting and mobile technologies to support these unique opportunities for learning.

Design research as a methodological framework

Design research may be considered as a reaction against the dominating educational research tradition of "outcome evaluation", which does not explicitly consider the provision of opportunities for learning (Glass, 1976; Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003). Two key aspects of design research are the central position of the design of learning activities and the cyclic character that allows adjustment and improvement of the activities in three consecutive phases; the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis. (Drijvers, 2003). In comparison, Cobb and colleagues (2003) put forward five crosscutting features to summarize their notion of *design experiments* (DE). Their first feature, *develop theories*, concerns understanding processes of learning and the means that are designed to support that learning. Theories are considered both as a 'theory-guided bricolage' by combining and integrating global and local theories (Drijvers, 2003) and further as theories emerging as consequences of the teaching experiment. The second feature, which relates to *develop theories*, concerns *control*: "The intent is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them" (Cobb et al., 2003, p. 10). The third feature, *prospective* and *reflective* analysis, supports the analyses and development of theories and exerts control of the learning processes. The prospective and reflective aspects come together in the fourth characteristic of DE, *iterative design,* which correlates with the cyclic aspect in Drijvers' model. The activity as well as the supporting theoretical constructs are successively developed and refined during the cyclic process. Furthermore, the execution of the teaching experiment has to negotiate factors of implementation such as teachers, curriculum, students, time, and classroom arrangement (Drijvers. 2003, p. 31-36) corresponding to the *pragmatic roots* of the teaching activity, as the fifth feature of DE.

Following the previously described models, we negotiate the design of a specific learning activity with prospective analysis in a local cycle. When focus is on designing the activity, there is a parallel process of informal prospective analysis that draws on the researchers' theoretical pre-knowledge and intuitions. When focus shifts to the prospective analysis, the theoretical base is formally strengthened and further developed through coordination of theories as a 'theory-guided bricolage' implying further improvements of the activity.

affordances as an instrument for collaborative design

The current design research project is carried out in an environment which includes researchers in mathematics education and media technology. The design team also includes technical developers and experienced mathematics teachers. During the five years we have been working together in the direction of design research, technologies have been developing much more rapidly than our initiated research efforts. Currently, there is an abundance of available technologies within our research environment which are waiting to be explored for didactic purposes. The researchers and developers in media technology provide expert knowledge about the technologies as tools and how they may or may not be instrumented to support certain actions. On the other hand, identifying and proposing actions that are desirable from a didactic perspective requires contributions from educators, in our case mathematics education.

The matching of learning tasks with learning technologies has been explored by Bower (2007) who suggests a design methodology involving analyses and coordina-tion of 1) the affordances that can be deployed on specific technological resources, and 2) the affordance requirements of the tasks with respect to educational goals. The interplay between affordances for representation and communication has been explored for educational purposes in the development of the particular dynamic mathematics software SimCalc MathWorlds® (Hegedus and Moreno-Armella, 2009). Hegedus and Moreno-Armella argue that the interplay between these communicational and representational affordances results in the emergence of *representational expressivity*, which enables the users to express themselves through speech acts and physical actions and allows for the linking of individual students' cognitive efforts to public social displays.

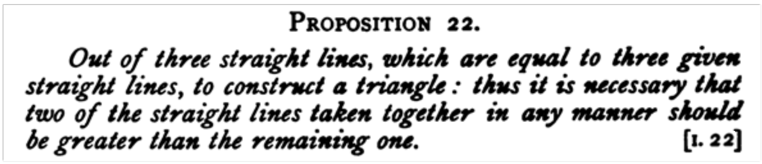
Our use of affordances differs from Bower (2007) in that we do not perform affordance analyses. Instead, we draw on the expertise in media technology and their knowledge about technological affordances. Furthermore, the learning tasks are not pre-defined but emerge in the creative process facilitated by collaborative design. In comparison with Hegedus and Armella (2009) we are not committed to specific technologies. The appropriation of technologies is a central aspect of our work where the appropriated technologies serve as mediators of specific affordances. As a consequence we often involve traditional as well as digital technologies in our activities, where we choose the technology that best mediates the desired affordance.

A central feature within the current activity is a specifically designed application for a mobile device to be used in an outdoor environment. Based on this application, an activity for the learning of mathematics has been developed and initially tested by the design team. The process of design, technical implementation, and testing the activity with students, required a time frame of less than three months. During a meeting prior to testing the activity, the issue of pragmatic roots was addressed by discussing and negotiating the activity with the students' teacher (and the school principal).

Bricolage of pure mathematics and modern psychology

The activity described in the next section was initiated during a design team meeting where a selection of available mobile technologies and applications were introduced by researchers from media technology. The team readily agreed to design an activity that involved outdoor constructions of large-scale triangles. The design was enhanced through identification and coordination of relevant theories and research in order to refine and support hypothetical learning trajectories for the learning of mathematics.

From a mathematical perspective, the activity involves the construction of a triangle with three given sides. A while after the task was proposed we felt confident that this geometric construction must have been considered by Euclid (~300 BC). Indeed, we found such a proposition in Book 1 in Euclid's Elements (Heath, 1908).


PROPOSITION 22.

Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [I. 22]

The construction may be illustrated by placing the three line segments next to each other on a straight line. We may name the line segments AB, BC, and CD, respectively, and draw two circles; one with centre B and radius AB, the other with centre C and radius CD (Fig. 1). The condition at the end of the proposition ensures that the two circles intersect at two points E and F. The two triangles BCE and BCF in the figure below fulfil the requirements of the proposition.

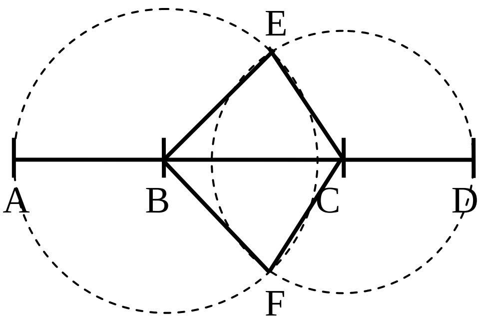


Figure 1: Illustration of Proposition 22 in Book 1 of Euclid's Elements.

The construction above can be regarded as a traditional school task related to geometrical constructions, problem solving and visualisation. Students may readily solve the task on a piece of paper by using a ruler and a protractor. In that case, they make use of a spatial ability which is sometimes referred to as *object manipulation*. This ability includes abilities for spatial visualisation and spatial relations and concerns manipulation of spatial forms from a fixed perspective, involving an object-to-object representational system (Kozhevnikov & Hegarty, 2001). Within the psychometric research tradition, spatial visualisation and spatial relations are contrasted with a third spatial ability, namely spatial *orientation*, which involves "movement of the egocentric frame of reference" (ibid., p. 745) and a self-to-object representational system. The self-to-object system activates another part of the brain than does the object-to-object system (ibid., p. 745-746), which implies that object manipulation and spatial orientation should be considered as separate spatial abilities.

Steering documents for compulsory school in Sweden have a one-sided focus on object manipulation and consider spatial orientation explicitly only in pre-school. This fact may be contrasted with the claim by Bishop (1980, p. 260) that "insofar as we are concerned with spatial ideas in mathematics as opposed to just visual ideas, we must attend to large, full-sized space, as well as to space as it is represented in models, and in drawings on paper". Activities taking place in full-sized space may be related to Bruner's (1966) enactive mode of action and corresponding mode of thinking, as one out of three modes – enactive, iconic, symbolic – characterising an individual's interaction with the world. We find it reasonable to claim that these different modes, which Bruner considers as emphases (rather than stages) in a child's development (ibid., p. 28), may be fruitful to draw on during learning activities also for older children, especially with respect to learning subject matter of abstract nature, such as mathematics.

Our ambition has been to design a learning activity that stimulates students' enactive mode of action by putting special focus on spatial orientation while minimising features related to spatial visualisation. Furthermore, we attend to the pragmatic roots with respect to the mathematics curriculum by stimulating learning trajectories of relevance for school mathematics.

Design of the activity

The learning activity described in the current paper draws on the innovative use of mobile technologies and it may be considered as a prototype, as it is not easily reproduced by an individual teacher outside the research environment where it has been developed. However, we feel confident that the activity could be readily implemented in a user-friendly version by a developer with technical expertise.

The activity draws on GPS technology available in a mobile device. The design team has developed a mobile application, which allows the user to measure distances between her own device and other (user-defined) mobile devices. Based on this feature, several possible learning activities were readily proposed during the discussions within the design team. One activity focused on one user coordinating two given distances with respect to two fixed points, which were respectively marked by a triangle and a square (Fig. 2). Both standing markers could be readily identified from a large distance (compare Fig. 4).

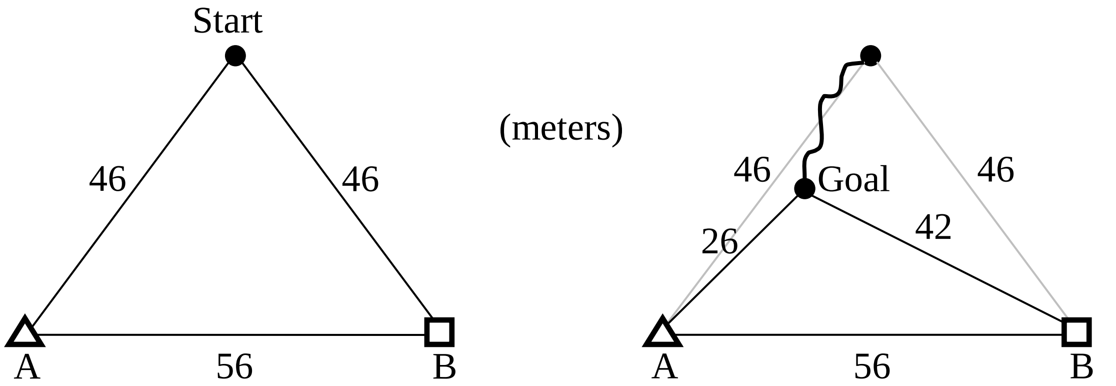
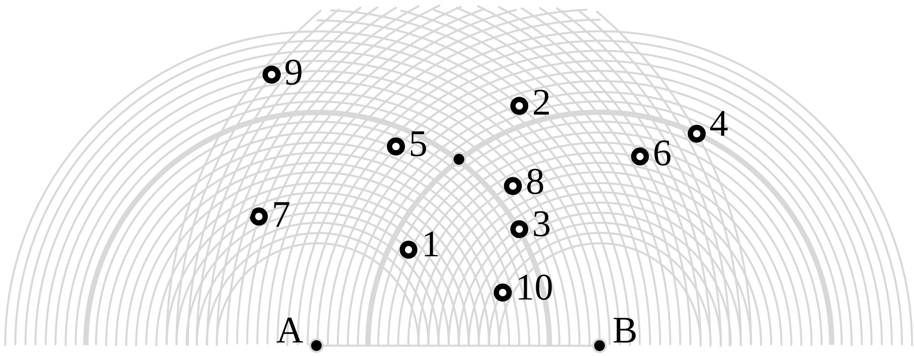


Figure 2: Visual representation of the activity

We chose to include ten tasks in the activity based on a diagram of level curves used to secure variation between longer and shorter distances (Fig. 3).



Point	A	B	dev.
1	26	42	– –
2	62	50	+ +
3	46	28	0 –
4	86	46	+ 0
5	42	56	– +
6	74	38	+ –
7	28	72	– +
8	50	36	+ –
9	54	84	+ +
10	38	22	– –

Figure 3: Distribution of points for the tasks (unit: meter).

The rather large distances, 22-86 meters, were chosen for two reasons. Firstly, we wanted the students to move substantially (and reasonably far) within the chosen field. Secondly, inaccuracy of the GPS values resulted in an error for the computed distances. The design team tested both these aspects outdoors and found that a tolerance of two meters was enough to compensate for the inherent inaccuracies of the GPS technology. At the final stage of implementation (December 2010) it was decided that twelve students in grade 6 should participate in the experiment and that they should work simultaneously with the activity on the same field, which was covered with 30 cm snow. The students were randomly organised into six groups. To avoid having the groups follow each other (in order to complete their ten tasks) six variations of the initial sequence of points were constructed based on symmetry (interchanging distances for A and B) and taking the points in individual order (1-10, reverse order 10-1, and 3-10 followed by 1-2). A reference point was marked on the field with the two starting distances 46, 46 (meters). Between the points A and B, we provided distance markers for 5 and 10 meters which the students could use as references either before or during the activity. In order to put focus on the spatial orientation ability, we decided not to provide visual references on the mobile device although this was technically possible (such as maps with marked attempts). They were instructed in the classroom about the activity and functionality of the mobile device. To promote students' reflections during the activity, their new distances were shown on the display of the mobile device only when so prompted by the students.

Figure 4: Picture from the outdoor activity and the display of the mobile device

They were instructed to try to minimize the number of prompts/tries for each task and were also asked to record messages (on the same device) to be used for reflection in the classroom. The activity took less than one hour to complete by the six groups.

learning trajectories and affordances

The prospective analysis from the first activity has led us to identification of two main hypothetical strategies, each with two sub-strategies. We refer to these as the comparative strategy and the circle strategy, respectively. Preliminary results show that these strategies have been applied successfully by the students as they managed to identify the correct positions readily, most tasks requiring only a few attempts. The *comparative strategy* puts focus on comparing current distances with the distances required in the task and moving to reduce or increase distances. This accommodation of distances is possible to execute one distance at a time or both distances simultaneously. The *circle strategy* draws on moving in circular patterns, basically following the previously described construction in Euclid's Elements and executed either with one circle at a time or both simultaneously.

A crucial decision regarding affordances was the decision not to afford iconic visualization through the mobile device. Instead, we decided to promote an enactive mode of physical interaction in the outdoor environment with focus on spatial orientation. This singular activity may serve as a general frame of reference regarding the students' future geometric constructions on paper, using ruler and protractor, where the outdoor activity may provide a connection between iconic constructions on paper and constructions imagined to be enacted in an outdoor setting.

References

Bishop, A.J. (1980). Spatial abilities and mathematics education – a review. *Educational Studies in Mathematics, 11,* 257-269.

Bower, M. (2008). Affordance analysis – matching learning tasks with learning technologies. *Educational Media International, 45(1),* 3-15.

Bruner, J.S. (1966). *Toward a Theory of Instruction*. Harvard University Press.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32 (1),* 9-13.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment.* Doctoral Dissertation. Utrecht: Freudenthal Institute.

Glass, G.V. (1976). Primary, secondary, and meta-analysis of research. *Educational Researcher, 5,* 3-8.

Heath, T.L. (1908). *The thirteen books of Euclid's Elements*. Cambridge: Cambridge University Press.

Hegedus, S., & Moreno-Armella, L. (2009). Intersecting representation and communication infrastructures. *ZDM, 41(4),* 399-412.

Kozhevnikov, M., & Hegarty, M. (2001). A dissociation between object manipulation spatial ability and spatial orientation ability. *Memory & Cognition, 29 (5),* 745-756.

Nilsson, P., Sollervall, H., & Milrad, M. (2009). Collaborative design of mathematical activities for learning in an outdoor setting. *Proceedings of the 6th Conference of European Research in Mathematics Education.* Lyon, France.

Nilsson, P., Sollervall, H., & Spikol, D. (2010). Mathematical learning processes supported by augmented reality. *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education.* Belo Horizonte, Brazil.

Penuel, W.R., Roschelle, J., & Shechtman, N. (2007). Designing formative assessment software with teachers: An analysis of the co-design process. *Research and Practice in Technology Enhanced Learning, 2 (2),* 51-74.

Sollervall, H., Nilsson, P., & Spikol, D. (2010). Augmented reality as support for designing a learning activity concerning the mathematical concept of scale. *Proceedings of the 7th Swedish Mathematics Education Research Seminar.* Stockholm, Sweden.

"WAYS OF LOOKING" AT QUOTIENT SPACES IN LINEAR ALGEBRA. HOW TO GO BEYOND THE MODERN DEFINITION?

Blanca Souto- Rubio and Inés Mª Gómez- Chacón.

Universidad Complutense de Madrid

What makes the concept of Quotient Vector Space so hard to teach and learn? What kind of different representations and conceptions do teachers need to help students to make sense of this concept? We reflect on these questions from an epistemic point of view offering a panoramic of the advance and interactions of different conceptions of quotients, from school to last courses of a Degree of Mathematics, and the relations of these conceptions with the kind of representations used and with other notions needed for the development of the concept of quotient such as partition, equivalence relation or well-defined operation. We make this reflection thinking on Linear Algebra teachers interested in improving their students' "cognitive flexibility" but also as a framework for further research.

MOTIVATION AND FRAMEWORK

The starting point of this paper is a bigger project whose final aim is the improvement of the teaching and learning of Linear Algebra (LA) in a first-year course of a four year-long Degree of Mathematics at the School of Mathematics of the Universidad Complutense de Madrid (UCM), which is one of the most important universities in Spain. We focus on the use of visualization as a "pedagogical device" to enhance meaning and to facilitate students acquiring "cognitive flexibility", which is one of the causes of difficulties in learning LA (Dorier, 2000).

In this study we do not understand visualization in LA as the introduction of concepts from a geometrical point of view (Gueudet, 2004) but as "a way of looking at things" (Davis, 1993). The more enriched a person's "way of looking" at a mathematical object is, the more meaningful this object and the better understanding obtained will be. Teaching should facilitate this enrichment of the "way of looking" by offering something beyond the formal definition as: diverse concept's mathematical properties; different languages (Hillel in Dorier, 2000); representations in different registers and their corresponding transformations (Pavlopoulov in Dorier, 2000); and finally different modes of thinking (Sierpinska in Dorier, 2000) and points of views, to which we often refer as *conceptions*. In this paper, we intend to make explicit some of these elements which are part of teacher's Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK) (Ball, Thames, & Phelps, 2010).

The concept of Quotient Vector Space (QVS) attracted our attention when we were taking part as participant observers in the LA course, with the goal exposed above. In the course, the teacher motivated the formal definition using several examples, complemented the definition with an example of a geometric interpretation and, finally, illustrated some properties involving the concept with different metaphors. Moreover, QVS was one of the few concepts graphically represented (Fig. 2) in the textbook (Fernando, Gamboa, & Ruiz, 2010). Despite this special teaching effort, students and teacher admitted to have difficulties making and helping to make sense of QVS, respectively. Hence, there is the need for deeper reflection on the following questions and this is the aim in this paper:

- What makes QVS so hard to teach and learn?
- What kind of different representations and conceptions (SCK) does the teacher need to help students to make sense of the concept of QVS?

The chosen orientation is different from previous related research, which can be found mostly on Group Theory instead of on LA (Asiala, Dubinsky, Mathews, Morics, & Oktaç, 1997): it is not oriented to cognitive development per se but to the epistemic question. We agree with Steinbring (1998), who insists on the necessity of working on epistemological knowledge to improve teaching and proposes to transform current practices by changing teachers' attitudes and understanding of Mathematics. Thus, we firstly review the QVS modern definition and analyze the ideas needed to understand it, having History as our reference. Secondly, we explain other conceptions found in LA classic books and textbooks, paying special attention to the kind of language and the use of Geometry they make. Finally, we briefly explore the role of quotients in more advanced subjects of the Degree of Mathematics, in order to show the importance of knowing different conceptions of QVS for mathematical thinking development. A conceptual map is included at the end to summarize conceptions, notions and representations exposed along the paper.

THE MODERN DEFINITION. WHAT IS NEEDED TO UNDERSTAND IT?

First students' contacts with the concept of quotient are in relation to the operation of division as opposite of multiplication and to problems about comparisons and sharing in primary school, including concepts such as ratios, proportionality, fractions and remainders. The study of congruences and groups as Z_n at University extends these notions but remains in a concrete context. The first appearance of abstract definitions of the concept of quotient is within Set and Group theory, which is covered in two subjects of the first year of the Degree and provides the main ideas needed in order to understand the modern QVS definition (Fig. 1).

As a starting point, QVS definition is built on the notion of *partition* which is a set of sets obtained by an *equivalence relation*. Understanding that the elements in a partition have a different nature from elements in the initial set is one of the main issues together with the notion of *equivalence*. At the end of the 19^{th} century the usefulness of the concept of *equivalence* started to be understood in different contexts (Nicholson, 1993, p. 76): the equality of numbers seen as a 1:1 correspondence by Frege; the study of foundations of analysis by Dedekind and his idea that two *similar* systems of elements allow separating them into *classes*. By extending the idea of equality to *equivalence relation* it was possible to establish a valid criterion to make a classification into any set, leading to a new set of sets called *partition* or *quotient set*. Conversely, a partition in a set defines an equivalence relation, as Van der Waerden proves in the first chapter of his well-known Modern Algebra (1950).

(8.6) Quotients modulo subspaces

Let L be a vector subspace of E. The following equivalence relation is defined in E : two vectors $v, w \in E$ are related if $v - w = u \in L$. The equivalence class of a vector $v \in E$ is

$$[v] = v + L = \{v + u : u \in L\} \text{ (in particular } [0] = L),$$

as $w \in [v]$ means that $w - v = u \in L$, that is $w = v + u \in v + L$. The corresponding quotient set is denoted E/L, and operating equivalence classes by using their representatives is enough to define a vector space structure:

$$[v] + [w] = [v + w], \ \lambda[v] = [\lambda v].$$

Figure 1: Definition of QVS in the textbook (Fernando Galvan et al., 2010, p. 175)

The second idea to be understood in QVS modern definition is that the new set inherits a vector space structure from *E*, given by operations on *equivalence classes*. Therefore it is important to be familiar with the notion of abstract vector space previously. This could explain the late development of the concept (Nicholson, 1993). The modern definition of QVS cannot be previous to the end of the 19th century since the general theory of vector spaces appeared with Grassmann on 1844 but did not start to be accepted until 1888 with Peano’s work (Dorier, 2000:18,19).

Finally, since the definition of the inherited operations depends on the election of *representatives* of the *equivalence classes* it is important to check whether the operations are *well defined*. In the case of Group Theory, *L* is asked to be a normal subgroup. In QVS case, *L* is just to be a vector subspace in *E*. Thus, in order to have a quotient structure from a previous one, we have needed to separate the elements into equivalence classes and to be sure of producing well- defined operations among these classes. This idea can be generalized to any mathematical object with an algebraic structure such as rings, ideals, modules, fields, algebras, etc.

Different conceptions and representations of qvs in LA

QVS in Ausdehnungslehre of 1844 by Hermann Grassmann

With this work on Extension Theory, Grassmann tried to create a new branch of Mathematics not limited by the geometrical context but closely related to it and strongly influenced by Philosophy. Thus, he considered the quotient as the result of a new “conjunction” (operation) called *outer division,* which is the opposite of *outer multiplication* and is defined as follows: *“Outer division thus consists in seeking one factor in terms of the outer product and the other factor”* (Grassmann, 1995, p. 113). Grassmann’s approach to quotient is based on contrasts, being slightly different from the modern one but proving similar properties:

> “If the divisor (*B*) is subordinate to and of lower order than the dividend (*A*), the quotient is only partially defined, and in fact, if one knows a particular value (*C*) of the quotient, one finds the general value by adding to that particular value one of the magnitudes dependent on the divisor (*B*) in the undefined expression, or *A/B.=C+0/B.*” (p. 116).

Geometry appears in Grassmann's work as an application of the general theory for two reasons: (1) to make the content more familiar to the reader; (2) because it is the basis of his science and a source of inspiration for new concepts. However, there is no loss of generality because he presents the abstract concept without relying on any geometrical truth (p. 45-46). Quotients are not an exception and, after an abstract introduction including the previous theorem, there is an application to Geometry.

> "Applied to the theory of space, this theorem expresses, first, that if the base and the area (including the plane to which it belongs) of a parallelogram are given, then the other side, which we have called the elevation, is only partially defined, and that if its initial point is fixed, the position of its final point is on a straight line parallel to the base."(p. 116)

This quotation is followed by a similar reasoning on second and third order extensions, showing Grassmann's idea of an *n*-space built step by step by the evolution of an element (Dorier, 2000, p. 20). In relation to our aim of finding different conceptions of QVS, this quotation introduces the next two. First, the "elevation" refers to a different direction from that of the base, reminding the notion of *complementary space* present in Halmos' approach. Second, this "elevation" is not completely determined, but if the initial point is fixed, then the final should lie on a line parallel to the base. Each *parallel line* defines a class of equivalent vectors, where the equivalence is determined by the volume of the parallelograms the vectors define with the basis. This idea is close to Dieudonné's approach.

QVS in Algèbre Linéaire et géometrie Élementaire by Dieudonné.

As Dieudonné (1971) exposes in his introduction, he wrote his textbook thinking on secondary school teachers and considers not necessary to mention the axiomatic theory of vectors spaces for this level. Instead, he points out that teaching should consist of many wisely chosen "geometric experiences" accompanied by partial reflections on the obtained results. For that reason, his textbook is mostly limited to dimensions two and three, in which geometrical intuition can be used, though it is not possible to find any figure. He argues that figures are not needed and, in any case, the reader can supply this lack (Dieudonné, 1971, p. 10). On the other hand, he defends teaching LA as soon as possible because of the importance of "linear thinking" and its power of unification. Affine spaces have a distinguished role in the book and vector spaces are sometimes presented as those which contain the zero vector. The link between affine and vector spaces are *translations*, context in which notions close to the QVS modern definition emerge naturally:

> [3.1.10] The image by a translation t_a of a vector subspace V of E is the set of vectors $a+x$, with $x \in V$; it is denoted by $a+V$ and it is called a linear variety of E. It is evident that if $a \in V$, $a+V=V$ holds, since for all $y \in V$, $y-a \in V$. (Dieudonné, 1971, p. 34)

In the first section of the chapter about vector spaces, Dieudonné studies the properties of these new sets, $a+V$, proving most of the properties of *partitions*, though there is no direct reference to them. The set of parallel linear varieties to a vector subspace V forms a partition that results to be the same as in the modern definition. The *equivalence relation* can now be interpreted as follows: two vectors are related if their final points lie on the same linear variety from the partition; and the *equivalence classes* can be identified with each parallel variety. Therefore QVS are the *collections of translations* of a given vector subspace. This approach to the notion of QVS appears in the textbook of the course we observed just after the modern definition (Fig. 1), as an example, and a graphic representation accompanies it (Fig. 2).

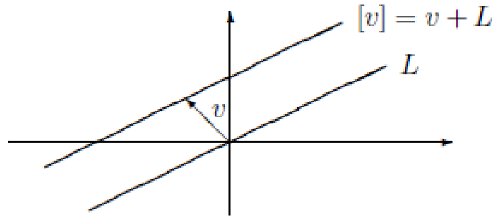


Figure 2. Textbook Graphic representation for QVS (p. 176)

QVS in Finite- dimensional vector spaces by Halmos.

Halmos (1974) uses a similar mental picture in his textbook – with horizontal parallel lines – to illustrate differently the construction of QVS. They are an alternative to the nonexistence of a natural way of choosing, among all the possibilities, a *subspace complementary* of another (p.33). These ideas lead him to the following theorem:

> Theorem 1. If *M* y *N* are complementary subspaces of a vector space *V*, then the correspondence that assigns to each vector *y* in *N* the coset *y+M* is a isomorphism between *N* and *V/M*. (Halmos, 1974, p.34)

Moreover, quotients have the theoretical advantage of not being dependent on any particular choice of basis or anything else. This generality is very important to Halmos as he recognizes in the introduction:

> "My purpose in this book is to treat linear transformations on finite- dimensional vector spaces by the methods of more general theories. The idea is to emphasize the simple geometric notions, and to do so in a language that gives away the trade secrets and tells the student what is in the back of the minds of people proving theorems about integral equations and Hilbert spaces" (Halmos, 1974, p. v).

We note how different this spirit is from Dieudonné's and how it affects to the kind of representation and language Halmos prefers in his book. He chooses "*algebraic, coordinate-free methods*" because they "do not lose power and elegance by specialization to a finite number of dimensions, and they are, in my belief, as elementary as the classical coordinatized treatment" (p. v), idea we do not completely agree with. For a beginner LA's student, coming upon the idea of QVS as a *kind of complementary space* of *V* in *E* may be more natural by studying the construction of a basis for the quotient following these steps: taking a basis of *V*, extending it to a basis in *E*, taking cosets of the elements of the basis and, finally, realizing that each vector in *V* leads to the coset *0*. Thus, only the cosets of vectors added to the basis of *V* remain. These are the cosets of the elements of a basis of the complementary of *V* in *E*. This reasoning is explained in the textbook of the observed course, though too briefly, as part of the proof of the proposition about the dimension of QVS. Halmos' theorem is easier to generalize to infinite dimensions, as there are no bases involved in its proof. However, we believe that this approach hides partially why the relation between the quotient and the complementary space exists. Instead, this relation is clear in the *coordinate-dependent approach*. Therefore, the two arguments are not equally elementary for us, though both can be suitable for a different objective.

Other conceptions of quotients in and BEYOND LA

Closely related to the conception described above of the QVS as a complementary space, it is the conception of the QVS as a *projection*. Actually, the map that assigns each element to his coset is known as the *canonical projection*. This idea of using quotients *to "make disappear" what disturbs* is very useful in other parts of Mathematics such as Algebraic Geometry and Commutative Algebra, for example when defining the localization of a ring. Similarly, in Projective Geometry, subject of the second year of the Degree of Mathematics, we can find quotients as a *method to remove extra information*: only directions matter and therefore vector lines can be considered as points, defining a quotient space called the *projective space*.

On the other hand, quotients appear in LA when classifying matrices or, equivalently, linear and bilinear forms by similarity or congruence. Precisely, this is one of the goals of the course: to decide whether two matrices represent or not the same linear or bilinear form. Quotients are similarly used in other areas of Mathematics to *classify* any kind of objects such as compact surfaces, tessellations, singularities, etc.

Euclidean spaces are studied in the first course of LA too. In this case, talking of *orthogonality* makes sense and it provides a distinguished direction to build a complementary space and to project along. QVS are commonly though in this context as a kind of *orthogonal complement* or *orthogonal projection*. This is important to be known in order to understand a similar result holds for Hilbert spaces but not for arbitrary Banach spaces. Hilbert spaces, which generalize Euclidean spaces to infinite dimensions, together with Banach spaces are studied in Functional Analysis, subject of the fourth year of the Degree. They are simultaneously vector and topological spaces. Thus, both points of view of the notion of quotient are found together.

The topological point of view for quotients is introduced in the third year of the Degree and is different from approaches seen above, though some links with previous notions such as the First Isomorphism Theorem can be established. The motivation is geometrical and responds to the idea of formalizing *"gluing and cutting"- type of transformations*, commonly used to build objects such as surfaces. It is based on the idea of quotient as the collection of the *preimages of a surjective function*.

To sum up, this conceptual map (Fig. 3) represents the advance (by columns, from school on the left to last courses of the Degree on the right) of different conceptions of the quotient (in dark rounded boxes), their interactions (arrows) and the relations with other notions and representations needed for the development of the concepts (white squared boxes).

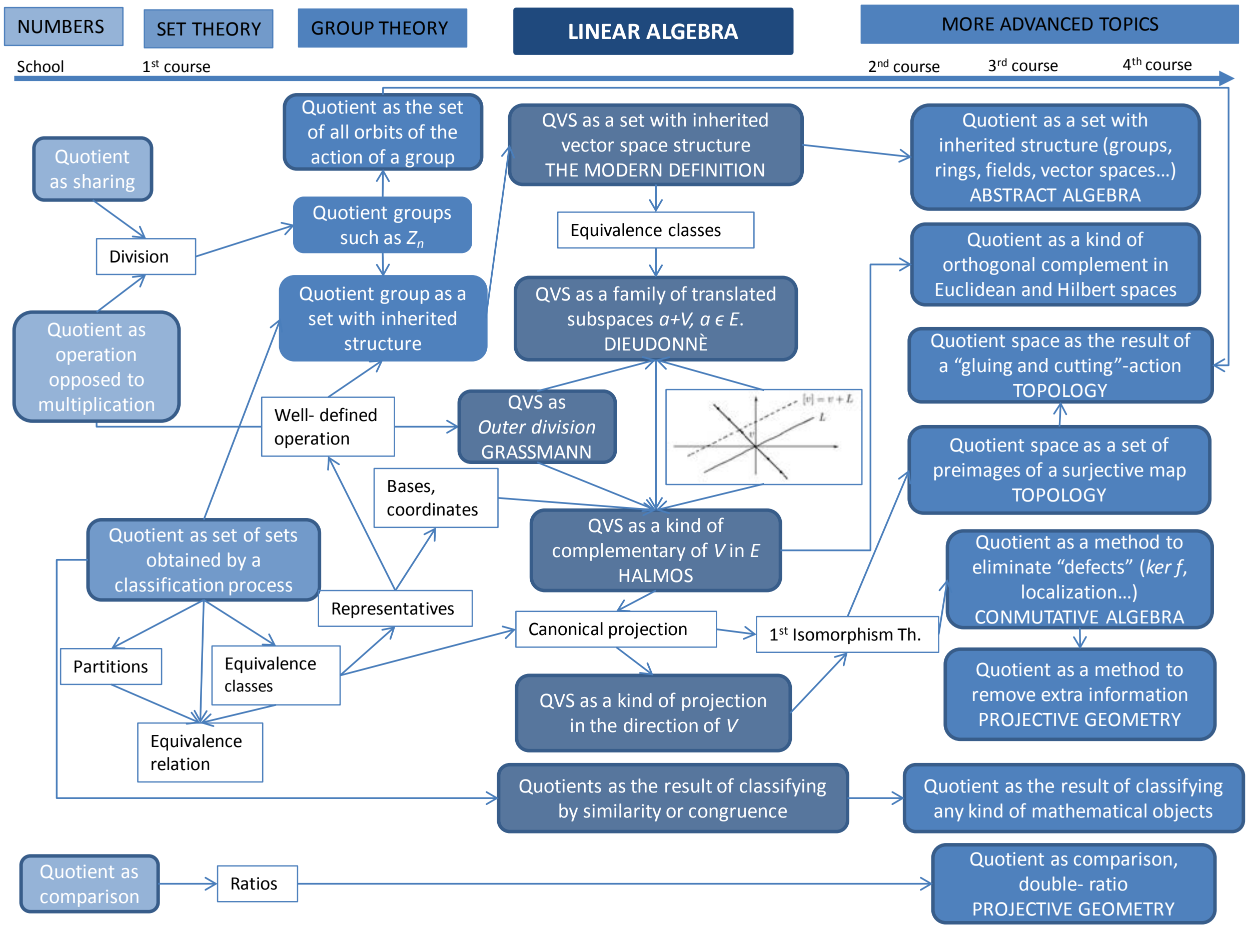


Figure 3: Conceptual map of the different "ways of looking" at quotients

DISCUSSION

The dependence of the notion of quotient on other notions such as equivalence, partitions, abstract algebraic structures (group, vector space) is a reason for the late development of the concept, at the end of 19th century. This dependence could also be a source of students' difficulties. Another source of difficulties may be the big diversity of conceptions of quotient found along the different subjects (Fig. 3). In the case of LA, the modern definition does not necessarily encapsulate all these points of view, which have been proved to be crucial to understand more advance mathematics (Fig. 3). Thus, it is important to teach different "ways of looking", but also how to choose and switch from one to another, that means to teach 'cognitive flexibility'.

In order to teach this 'cognitive flexibility' the comparison between Halmos' and the textbook's approaches shows that is necessary to be careful with the kind of representation used. Halmos' approach was coordinate-free and easier to generalize to infinite dimension while textbook's approach was coordinate-dependent and easier for a beginner student to understand and believe. About the graphic register, which not always appears explicitly, we have seen how almost the same representation seemed to be useful for two different aims: to illustrate the conception of quotient as a family of translated subspaces; and to inspire Halmos to formalize the quotient as a kind of complementary space. In relation to the use of Geometry for teaching LA there are opposing opinions which are both found in Grassmann. On the one hand and according to Dieudonné, Geometry is considered as an inspiration for new concepts and it is defended as a way to make the content more familiar to the reader. On the other hand, Geometry is seen as a limitation or just as a first step for a more general reasoning. Thus, in tune with Halmos, Grassmann (1995) is proud to say that the content of his science is "pure and independent of geometry" (p. 45).

To conclude, from the epistemic approach LA seems to be a good subject to explain QVS and their different "ways of looking". In order to teach them, teachers interested in improving their students' "cognitive flexibility" through visualization need some SCK and HCK that we have tried to make explicit in here. However, further research on how to mobilize this knowledge in the classroom is needed since the course observed seems to include all "ways of looking" but both teacher and students still have difficulties. Our reflection provides a framework from which to start to analyze classroom episodes but probably more cognitive approaches will be needed too.

Acknowledgements

This work has been supported by grant AP2007- 00866. We also express our thanks to Javier Gallego and Carlos Fernández for their disinterested and helpful comments.

References

Asiala, M., Dubinsky, E., Mathews, D., Morics, S. & Oktaç, A. (1997) "Development of Students' Understanding of Cosets, Normality and Quotient Groups", *Journal of Mathematical Behaviour, 16*(3), 241-309.

Ball, D. L., Thames, M. H., & Phelps, G. (2010). Content Knowledge for Teaching. *Journal of Teacher Education, 59*(5), 389 -407.

Davis, P. J. (1993). Visual theorems. *Educational Studies in Mathematics, 24*(4), 333-344.

Dieudonné, J. (1971). *Algebra lineal y geometría elemental.*Madrid :Selecciones Científicas.

Dorier, J. L. (2000). *On the teaching of Linear Algebra.* Dordrecht: Kluwer Academic Publishers.

Fernando, J. F., Gamboa, J. M., & Ruiz, J. M. (2010). *Álgebra Lineal (Vols. 1-2).* Madrid: Sanz y Torres.

Grassmann, H. (1995). *A new branch of mathematics : the "Ausdehnungslehre" of 1844 and other works.* Chicago : Open Court.

Gueudet-Chartier, G. (2004). Should we teach linear algebra through geometry? *Linear Algebra and its Applications, 379,* 491-501.

Halmos, P. R. (1974). *Finite-Dimensional Vector Spaces.* Springer.

Nicholson, J. (1993). The development and understanding of the concept of quotient group. *Historia Mathematica, 20*(1), 68-88.

Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1*(2), 157–189.

COMPARING REALISTIC GEOMETRIC PICTURES

Anna Stenkvist

The Department of Philosophy

Royal Institute of Technology, Stockholm, Sweden

Pictures, obeying geometric laws and depicting concrete physical objects, are increasingly important in education and working life in for example computer applications. But how do we chose among the wide range of modelling tools to get as realistic pictures as possible of the object? We are here trying to contribute to the discussion of this question. It is argued that realistic geometric pictures, which depict the same object under almost the same circumstances, must, apart from resembling the depicted physical object, also resemble each other. It is also claimed that one picture is more realistic than another, not primarily if it contains more information, but if the information is more accurate and representative. Arguments from the theory of aesthetic are here applied on the discussion of resemblance of geometric pictures.

INTRODUCTION

The spread of geometric pictures in school, university and working life needs knowledge in mathematics to read and comprehend the enormous range of geometric pictures within for example science and and technology. The knowledge is also needed to design applications and tools. But the performance among students and interest in mathematics is unfortunately low and decreasing in the western world (TIMMS 2007; TIMMS Advanced 2008). Besides the results, investigations show that school students' ability to comprehend, interpret and reflect, rather than just solving problems, also has weakened (PISA, 2006; SOU 2010:28).

The handling of representations, including realistic geometric models of the physical world, has been pointed out as critical when it comes to deeper comprehension in mathematics (Duval, 2004, 2006; Lesh , Post, & Behr, 1987; Niss, 2002). It is valid both for working within a certain representation system, such as the pictorial geometric, and when changing systems, from a geometric to a symbolic for example (Duval, 2004, 2006).

In this paper we will focus on one representation: the realistic geometric picture. The paper is meant to contribute to a philosophical and fundamental ground for discussions and decisions when for example choosing modelling tools for educational purposes. Arguments from discussing artistic realistic pictures in the theory of fine art and aesthetic are here applied on geometric pictures. We will try to answer two questions.

First, to what extent do realistic geometric pictures resemble each other? It is argued that realistic geometric pictures, which depict the same object under almost the same circumstances, must resemble the physical object as well as each other. This changes the transitivity property for resemblance between two compared objects. The arguments are partly based on Sartwell's (1994) definition of realistic pictures.

Secondly, when is it appropriate to say that one picture is more realistic than another? It is claimed that this is the case if and only if the two compared pictures contain the same number of relevant aspects and that the information of the more realistic picture is more accurate and representative for the purpose of the application intended. The accuracy is measured in terms of the distance between relevant measurable values in the depicted object and the pictures. Partly the arguments are an implication of Sartwell's definition and partly a contribution to Goodman's (1968/1976) claim that a picture is not more realistic than another if it contains *more* information.

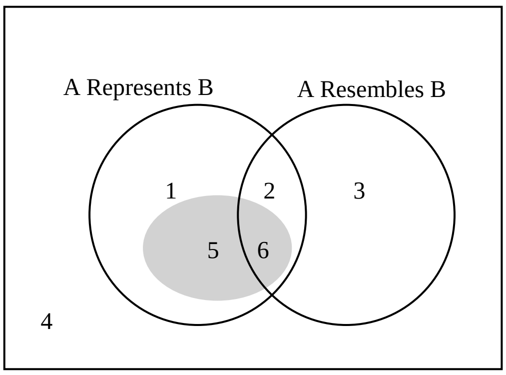
The disposition of this paper is as follows. First in this text the concept *picture* is defined and classified in order to demarcate the scope of the research issue. Then the resemblance is examined, between realistic geometric pictures alone and between pictures and depicted physical objects. After a debate about when it is appropriate to say that one picture is more realistic than another and some reflecions abou representation. Finally follows conclusions.

Pictures

Some approximate definitions are given in this chapter. A demarcation of the kind of pictures, which will be studied more carefully in this text, is marked out.

A picture *P*, in general, is here defined as a non-linguistic physical object, which is manufactured on a two-dimensional surface. Paintings, maps, plans, geometric figures and sketches presented on paper, cloths or screens are all example of pictures. If A is a picture and B another object, we say that *A represents B* if and only if A stands for or denotes B. We say that *A resembles B* visually if and only if whenever looking at A, B is visually recognized, and vice versa. The different cases when resemblance and representation between A and B exist are shown in *Figure 1*.

Figure 1. *Let A be a picture and B another object. The numbers 1-6 refer to areas in the diagram. (5) is the shaded part of*



(1) and 6 is the shaded part of (2) . The numbers denote the following relations between A and B: (1) A represents B, but A does not resemble B visually (2) A represents B and resembles B visually (3) A does not represent B but resembles B visually (4) A does neither represent B nor resemble B visually (5) A is a geometric picture which represents B but does not resemble B visually (6) A is a geometric picture which represents B and resembles B visually.

Resemblance is neither a sufficient nor a necessary condition to establish representation, meaning that neither does representation imply resemblance nor does resemblance imply representation. For example, an icon, a picture of the American flag, might represent United States but there is no visual resemblance between the real country and the picture, as in field *1* in *Figure 1*. An example corresponding to *3* is if someone by accident steps on a paper. The footprint he leaves is not a representation of the sole even though it looks like it. A representation requires an intentional act and is usually established by convention.

Field *2* corresponds to realistic pictures, or as they sometimes are called iconic signs (Presmeg, 2008), meaning that the representation and the object resemble each other physically.

Special cases, belonging to field *3*, are unsuccessful realistic pictures, for example the result when someone fails in realistically depicting a horse, so that the picture resembles something else, a cow for example (Goodman, 1968/1976). The same kind of failure can also result in an object belonging to field *1*, a picture representing something, a horse for instance, that it does not look like.

Field *4* could be accidently created pictures, which do not at all resemble a specific object.

Geometric pictures can represent two kinds of objects marked with *5* and *6* in the shaded area in *Figure 1*. Field *5* denotes pictures representing *abstract* geometric objects. The other field, denoted by *6* in the shaded area, corresponds to the scope of this text: the geometrical representation of physical objects.

REALISTIC GEOMETRIC PICTURES

Geometric pictures do, apart from denoting abstract geometric objects, often represent the physical reality. In many applications, for example in industrial design and computer programming, geometric pictures, are used in the process of designing products or simulate scientific course of events. It is here appropriate to say that the geometric picture is a *model*, in the sense human made copy of the world (Müller, 2009, pp. 647). Let us more carefully investigate the properties of realistic (geometric) pictures as models of the physical objects.

Two mayor streams during the past decades have been prevailing about pictorial realism in aesthetic field. They are applicable for geometric pictures as well. Very simplified one could say that, in aesthetic, there are on the one side the defenders of a natural resemblance between picture and object (Gosselin, 1984; Gilman, 1992). For them the cultural factors of representation are secondary to biological and psychological considerations when deciding the degree of realism of a picture. On the other side there are those who claim that cultural factors, convention and habit are decisive regarding the realism of pictures (Goodman, 1968/1976; Walton, 1984). In this view realism is foremost contextual and relative the current system.

The representational context of aesthetic and artistic pictures is different than the geometrical pictures in education and manufacturing design, but the principles for realism are similar. Both artistic and geometrical realistic pictures have to resemble and represent what they depict. The concept geometric is therefore in the following left out for the most part of this chapter.

Below transitivity is discussed for pictorial resemblance, in general as well as for realistic pictures and further what it means for a picture to be more realistic than another.

To what extent do realistic (geometric) pictures resemble each other?

Pictorial resemblance in general between two arbitrary objects, A and B, is a relation which is reflexive, meaning that each object resembles itself and symmetrical, meaning that A resembles B if and only if B resembles A and non-transitive, meaning that it is not the case that if A resembles B and B resembles C then A necessarily must resemble C (Goodman, 1968/1976). Examples of resemblance where transitivity is not obtained are shown in the cases below:

(i) if some, but not necessarily all, aspects of the compared objects are similar. For example if I resemble my mother in the aspects length and hair colour and she resembles my grandmother in eye colour, then I do not have to resemble my grandmother in any of the aspects length, hair or eye colour.

(ii) if one aspect is gradually changing in a sequence of objects which are order by some value. For example if pictorial resemblance of lightness is compared in a sequence of photographs, ordered by degree of light, two adjacent pictures might resemble each other while the ones in both ends can be each others' counterparts considering lightness.

For pictorial resemblance between a realistic picture and another object, the non-transitivity, exemplified in (i) and (ii) above, might have the implication that two realistic pictures manufactured in the same medium, of the same object, seen from the same angle in the same light et cetera, do not resemble each other according to the general definition. This consequence contradicts our intuitive understanding of pictorial realism.

Let us try to enable transitivity when regarding resemblance between realistic pictures, so that two realistic pictures of the same object resemble each other when they are watched under almost identical circumstances.

Case (i)

Two arguments must be considered.

First, the picture and the depicted object are not compared in every conceivable aspect to assure realism. One of Goodman's (1968/1976) arguments against natural resemblance between the realistic picture and the depicted object has therefore been criticized. Goodman claimed that a painting does not in any natural way resemble the depicted object. A man does for example not in any natural way resemble a depiction of himself, because the picture is flat, immobile and hangs on a wall, unlike the three dimensional man of flesh and blood.

Sartwell (1994), one of the defenders of natural realism, is arguing against Goodman's view above. Resemblance should according to Sartwell be demarcated, so that not every aspect of picture and depicted object are compared. Only the aspects, which are determined by the medium and used to manufacture the picture, are considered. For example, a sculpture is not more realistic than a painting because of the spatial properties, according to this view. The painting is only compared with the depicted object and other paintings concerning the degree of realism. In a similar manner a computer simulator might be realistic both in 2D and in true 3D, even though true 3D picture might be compared in more aspects than the 2D picture. A proposed alternative definition of *realistic representation* is then the following in accordance with Sartwell's (1994, p.10) view: Given that a picture *p* represents an object *o*, *p* represents *o* realistically to the extent that *p* resembles *o* in all relevant aspects.

The pictures and objects are accordingly compared concerning resemblance assuming an array of relevant aspects, each of which is defined for picture and object. When pictorial realism is considered only pictures with the same number and kind of relevant aspects are compared.

Case (ii)

Suppose that only one aspect is relevant for comparison. Let the relevant aspect be a geometric property like the gradient of a drawn stroke, depicting a hand of a clock. In a sequence of pictures the stroke is turned, gradually, little by little from 0 to 90 degrees. Two adjacent pictures can resemble each other while the ones in both ends are opposites considering the gradient and with totally different meaning. This is unreasonable for realistic pictures which we think should resemble each other.

Given that one relevant aspect, the gradient of a line for example, is measured somehow and that the aspect is assigned the measured values, *o* for the depicted object and *a* for the picture, then if the distance between the values is *close enough* the picture is said to (realistic) resemble the object, concerning for instance the gradient. Another picture of the same object, manufactured and looked upon under the same conditions, has a measured value *b* for the same aspect. The two pictures that resemble the same object under almost identical circumstances and with the same relevant aspect, must then resemble each other to some extent. The distance between the values is as most the *double distance* of the admitted distance between picture and object. This is if no constrains are given on the distance between the values *a* and *b* of the two pictures. Formally this is shown as follows:

(i.i) $|a-o| \le d, |b-o| \le d$

(i.ii) $-d \le a-o \le d,\ -d \le b-o \le d$

(i.iii) $-d-d \le a-o-b+o \le 0$

(i.iv) $-2d \le a-b \le 0$

(i.v) $|a-b| \le 2d$

But it might be more relevant to have the same limit on all the distances. That is for two pictures to *realistic resemble* the same object under almost identical circumstances and regarding the same relevant aspect the distance between each pair of compared values should not exceed a certain value *d*:

(ii.i) $|a-o| \le d, |b-o| \le d, |a-b| \le d$

When considering more aspects, the degree of resemblance might be defined as a function measuring the distance somehow. This means that if for instance $\langle x_1,...,x_n \rangle$ is an array of relevant comparable aspects, then the degree of resemblance between two objects could be computed by measuring the distance between the arrays with some function $f(<x_1...x_n>,<y_1...y_n>)$, where each array belongs to one of the compared items, pictures alone or picture and depicted object. The same relevant aspects are compared, but assigned different values by measurement.

When is it appropriate to say that one picture is more realistic than another?

Goodman (1968/1976) claimed that it is not appropriate to call a picture realistic on the bases of how much information it contains. To illustrate the unreasonable in such a criteria for realism he gives the following example. Imagine a motive drawn according to the laws of perspective, which is some kind of established geometric projection of the three-dimensional reality to the two-dimensional picture plane. Then, in a different picture depicting the same motive, the perspective is reversed, so that for example a small figure looks big. The two pictures contain the same amount of information but are they equally realistic?

Concerning the pictorial resemblance, between realistic pictures alone and between realistic pictures and objects, we have already stated that they are all compared only by means of their relevant aspects. These aspects are all considered for the same kind of pictures.

We have also stated that the distance between the values of those measureable aspects in motive and pictures matters when deciding if a picture is realistic at all, and accordingly if it is more realistic than another one.

So the information would reasonably matter, in our lines of thought, when examining which picture that is the most realistic one. But it is not the amount of information that matters, measured by for instance the number of pixels in a digital image. It is rather the quality, the values, of certain aspects in the motive and in the pictures that is important. For example to reverse a perspective would reasonably give values out of the range of the permitted distance between picture and motive.

Realistic Representation

In this text, so far, *the person* comparing pictures and motive has almost been neglected. When discussing resemblance, we have presupposed an almost machinelike registration of all the compared aspects. We have just tried to focus on some of the necessary conditions for realism apart from the human factor. We are aware of that we have not been describing human perception in an authentic way. The human perceptual system is very sophisticated, but it is far beyond the scope of this examination to allow for it.

Apart from the biology of the human there are of course also social factors depending on the prevailing society and culture. Recalling from above, a picture is realistic when it both resembles *and* represents the motive. Representation is an act based on convention. It is also strongly contextual and requires an active choice of *the relevant aspects* for a certain purpose. Is it, for instance, appropriate to use a three-dimensional data program when showing the chemical reaction, or is it more appropriate with a two-dimensional representation?

The concept *model* can, by considering representation, be given a slightly different meaning than Müller's (2009) description of it as a human made copy of the world. According to Føllesdal, Walløe and Elster (1993) a model is instead a representation where some properties, here aspects, essential for a certain aim, are accentuated while others are disregarded. A realistic picture would then represent a motive realistically if all the aspects, which are relevant for visual resemblance, are reckon with.

SUMMARY AND CONCLUSIONS

We have stated some reasonable properties of pictorial realistic depiction, concerning especially pictures obeying the laws of geometry. We have based our discussion on the similar reasoning for artistic pictures. We have tried to define properties of realistic pictures, manufacturing the same motive under almost identical circumstances, so that they resemble each other. In acheiving this objective the transitivity property has changed.

First, we have in accordance with Sartwell's (1984) lines of thought stated that pictorial resemblance is limited by certain relevant aspects in comparison to a physical object. The relevant aspects are chosen in accordance with the style of the pictures and the medium which were used when producing them. We have found it reasonable that only pictures defined for the same relevant aspects can be compared regarding realisitic resembalnce. Secondly, we also found that it is reasonable that a realistic picture must have assigned values to all the relevant aspect within a certain range. In that way accuracy of information rather than an extensive amount of information matters for a picture to be realistic.

As an implication we have also proposed conditions when one picture is said to be more realistic than another one, in favour of natural resemblance and as an answer to Goodman's (1968/1976) arguments against it.

References

Duval, R. (2004). *A Crucial Issue in Mathematics Education: The Ability to Change Representation Register*, ICMI (International Commission on Mathematical Instruction) the 10th International Congress on Mathematical Education, ICME-10, Copenhagen, Denmark 2004

Duval, R. (2006). *A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics*. Educational Studies in Mathematics (2006) 61: 103–131 DOI: 10.1007/s10649-006-0400-z. Netherlands: Springer (2006).

Føllesdal, D., Walløe, L., and Elster, J. (1993). *Argumentationsteori, språk och vetenskapsfilosofi* [Argumentation theory, language, and philosophy of science]. Stockholm: Thales.

Gilman, D., (1992). A New Perspective on Pictorial Representation, *Australian Journal of Philosophy*, 70 (2), pp. 174-186.

Goesselin, M., (1984). Conventionalism Verus Realism in Pictorial Art. Is Perception Basically Particular or General?, *Communication and Cognition*, 17(1), pp. 57-88.

Goodman, N., (1968/1976). *Languages of Art*, Indianapolis: Hacket Publishing Company Inc. ISBN 0-915144-34-4.

Lesh, R., Post, T., & Behr, M. (1987). *Representations and Translation Among Representations in Mathematics Learning and Problem Solving*. In C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics (pp. 33-40). Hillsdale, New Jersey: Lawrence Erlbaum Associates, Inc.

Niss, M. (2002). *Mathematical Competences and the Learning of Mathematics: the Danish KOM Project*. Technical report, IMFUFA, Roskilde University, 2002.

Müller, R., (2009). The Notion of a Model: a Historical Overview, In *Philosophy of Technology and Engineering Sciences.* A. Meijers (ed.), pp. 861–879. Burlington, MA: North Holland.

OECD Programme for International Student Assessment (PISA), URL http://www.pisa.oecd.org/pages/0,2987,en_32252351_32235731_1_1_1_1_1,00.html Visited the 26th of December 2011.

Presmeg, N., (2008). An Overarching Theory for Research in Visualization in Mathematics Education. ICME 11.

Sartwell, C. (1994). What Pictorial Realism Is. *British Journal of Aesthetics*, 34 (1), pp. 2-12.

SOU 2010:28, Swedish Government Official Report. (2010). p. 69-72

Trends in International Mathematics and Science Study (TIMMS). (2007). URL http://nces.ed.gov/timss/ Visited the 26th of December 2011.

Trends in International Mathematics and Science Study (TIMMS Advanced 2008). (2008) URL http://timss.bc.edu/timss_advanced/ir-release.html Visited the 15th of January 2012.

Walton, K. L. (1984). Transparent pictures: On the Nature of Photographic Realism. *Critical Inquiry*, pp. 246-276.

A REGIONAL SURVEY OF TAIWAN STUDENTS PERFORMANCE IN GEOMETRIC CONSTRUCTION

Hak Ping Tam and Yu-Liang Chen

National Taiwan Normal University

It is common knowledge that students encountered much difficulty in learning geometric construction. The purpose of this study is to find out the types of difficulties they actually have. A sample of 150 ninth graders' from the Taipei area was asked to work on a test instrument on geometric construction. Their performances were then analysed. It was found that some students did not know the rules imposed on the use of the straightedge and compass. In addition, not many students mastered the basic geometric construction procedures. Even if they did, some of them did not know why carrying out the construction steps would satisfy the requirement. They tended to be empiricists in this aspect. Furthermore, a group of arc drawers were identified and their behaviour studied. Educational implication from this study will be discussed to round up this paper.

INTRODUCTION

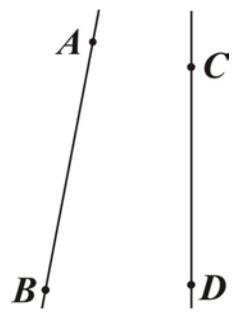
It is common knowledge that geometry is one of the most difficult school subjects to many students (Senk, 1985). Within geometry, proof and proving as well as straightedge and compass construction are among the most difficult topics for students to tackle. For the past two decades, a sizable amount of studies that focused on the understanding of students' difficulties in relation to proof and proving has been steadily accumulated (Recio & Godino, 2001; Selden & Selden, 2002). Moreover, several specialized conferences have been organized with proof and proving as its topic of sole interest. In comparison, there is much less attention towards understanding the kind of difficulties students may encounter in solving straightedge and compass construction problems. One valuable discussion can be found in Schoenfeld (1985), which partly reveal how difficult construction problems can be even to college students. For example, he detailed the performances of several participants in their construction of a circle that was required to be tangential to two intersecting straight lines while at the same time passing through a given point on one of the lines. After taking into consideration that these college students could have forgotten part of what they learned in geometry classes at school, it was nonetheless surprising to find that the participants possessed the necessary skills yet failed to apply them to the given task.

Taiwan's students are no exemption from having difficulties in construction problems even though their eighth graders ranked first in TIMSS 2007 with respect to overall performances in mathematics. For evidence, let us inspect how their students performed on construction problems in the Basic Competence Test (BCTEST), a high

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 131-138. Taipei, Taiwan: PME.

stakes national examination for about 300,000 ninth graders. It can be safely assumed that most examinees will take the exam seriously and put in their best effort because which senior high school they can attend will largely be determined by their performances in the BCTEST. A multiple-choice item on geometric construction was included in three recent rounds of administration of the exam. The items were all framed in a format with two persons solving a given problem in two different approaches. The examinees were asked to determine if they were both correct, both wrong, or only one of them was correct. The percentages correct for the three examinations were 30.9%, 36.7% and 27.3%, respectively. The item adopted in the first assessment of 2009 was reproduced in figure 1 below.

In the figure to the right, $\boldsymbol{AB}$ and $\boldsymbol{CD}$ are two straight lines on the same plane that are not parallel to each other. How could one construct a circle $\boldsymbol{O}$ that is tangential to both $\boldsymbol{AB}$ and $\boldsymbol{CD}$? The following are the steps produced by Cindy and Emily, respectively:



(Cindy)
1. Construct a line L that passes through D and perpendicular to line $\boldsymbol{AB}$. Let it meet line $\boldsymbol{AB}$ at E.
2. Find the midpoint of $\overline{\boldsymbol{DE}}$ and call it O.
3. With point O as the centre and $\overline{\boldsymbol{OE}}$ as the radius, draw the circle O which will serve the purpose.

(Emily)
1. Let lines $\boldsymbol{AB}$ and $\boldsymbol{CD}$ meet at point P.
2. Construct the angle bisector L of $\angle \boldsymbol{BPD}$.
3. Construct a line M that passes through C and perpendicular to line CD. Let it meet line L at O.
4. With point O as the centre and $\overline{\boldsymbol{OC}}$ as the radius, draw the circle O which is what is required.

Which is true about the two procedures?
(A) Both are correct
(B) Both are incorrect
(C) Cindy correct, Emily incorrect
(D) Cindy incorrect, Emily correct

Figure 1: Construction problem in the first assessment of BCTEST in 2009

The results to this item are shown in table 1 below. Though this item does not directly assess students' abilities in actually performing geometric construction, it can, nevertheless, reflect how difficult construction problems could be to Taiwan's students. Notice that about half of the examinees had chosen to agree with Cindy's approach, i.e. options (A) and (C). Given the fact that participants are allowed to bring their compasses to take the exam, one would expect that they could have, if necessary, followed Cindy's steps and verified if her procedures were correct. As subsequent interviews with some ninth graders revealed that just by repeating her procedures did not quite help the discernment because the two given lines were quite close to each other, rendering it quite difficult to tell if the circle touched the line $\boldsymbol{CD}$ at two points or one. In other words, even the verification of the construction procedures for students is not as easy as it may appear to some instructors.

Options	N	%
(A) both are correct	89,862	28.75
(B) both are incorrect	44,275	14.17
(C) Cindy is correct, Emily is incorrect	91,761	29.36
(D) Cindy is incorrect, Emily is correct	85,459	27.34

Table 1: Students performances with respect to the BCTEST item

In view of such difficulty, a survey study was launched to find out the kinds of difficulty in geometric construction as encountered by ninth graders in Taiwan. Preliminary result has been reported in Tam & Chen (2010). The present paper reported some major findings plus extended results. It is hoped that the results reported here can shed some light on how geometric construction should be taught.

CONCEPTUAL FRAMEWORK

In view of students' performance on the BCTEST item, it is obvious that validity discernment of construction procedures is not an easy task for students, to say nothing about coming up with one's own construction procedures. In our opinions, there are three kinds of knowledge that students must master in solving Euclidean construction problems. First of all, they should know the basic rules regarding the use of the construction tools, namely, straightedge and compass. For example, they should know that they cannot use tools other than compass and straightedge, and that they should either use an unmarked straightedge or else abstain from utilizing the markings on a regular ruler for measuring purposes. This kind of rules are basic requirement to geometric construction but are not emphasized often enough by teachers in geometry classes. As a result, many students forgot about their existence when attempting construction problems. Secondly, they must master a set of basic construction procedures. For example, they should know how to duplicate a given segment or copy a given angle. Thirdly, they should know how and when to apply the basic construction rules and procedures to solve various application problems. Straightedge and compass construction problems are difficult because students have to be flexible enough to apply a few basic construction procedures to a multitude of problems under a number of restrictions imposed on the way the tools should be used. Careful analyses are necessary before such problems can be solved.

METHOD

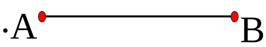
In order to find out the kind of difficulty students encountered in geometric construction problems, a survey study was devised in which a test instrument was designed and validated. Towards this end, an analysis was conducted concerning the coverage of geometric construction in the official Grade 1-9 Curriculum. Based on the specification in the curriculum and the content of various textbooks, it was found that eighth graders should be able to construct a line segment, an angle, an arc, a sector and replicate a given angle. They should be able to bisect a segment or an angle. Moreover, they should know how to construct the following: a right angle, a perpendicular to a given point on a line, and a perpendicular from a given point to a line. They should also master how to construct the perpendicular bisector of a line segment, and a parallel to a line through a given point not on the line (see table 2).

Construct		Bisect
an angle (eg. 90º)	a vertical from an exterior point	an angle
replicate an angle	a vertical at a point on line	a line segment
a line segment	a line via a point parallel to a line	
an arc or sector		

Table 2: Construction skills required in the Grade 1-9 Curriculum

Test items were written based on these contents and in relation to the three types of construction knowledge mentioned above, namely, basic rules, basic procedures, and application. The items had gone through several rounds of updates under the supervision of a team of teachers and math educators. Expert opinions from mathematicians were then sought and the instrument was pilot tested. The final test instrument had four items on the basic rules with the construction tools, five on the basic construction procedures, and eight on discerning the validity of a given set of construction procedures, together with five application items. The respondents being surveyed were a representative group of 150 ninth graders from five different classes in three junior high schools in northern region of Taiwan. Their mathematics abilities ranged from low to high as judged by their teachers and from their regular performances. The test was conducted in the middle of April in 2009. The students first learned straightedge and compass construction in the second semester of eighth grade, which was about one year prior to the time the test was administered.

Evidences of test validity were established by way of expert, content and factorial validity. A strong two factor solution (knowing rules and construction legitimacy, and knowing basic procedures and application) was obtained which accounted for 80% of the total eigenvalues. The Cronbach's alpha coefficients for various subscales ranged from .70 to .86. The item difficulties ranged from .09 to .87. Their point biserial indices ranged from .29 to .78. A sample item is shown in Figure 2 below.

In the diagram to the right, let $\overline{\boldsymbol{AB}}$ =10 cm.


The following steps were used to construct the perpendicular bisector of $\overline{\boldsymbol{AB}}$.
Step 1. Use point *A* as centre and length *a* as radius and draw an arc.
Step 2. Use point *B* as centre and length *b* as radius and draw an arc.
Step 3. Let the two arcs intersect at points *P* and *Q*. Join the two points.

Which of the following could be the length(s) of *a*? Check the appropriate □.
□ 2.5 cm □ 5 cm □ 7.5 cm □ 10 cm □ 12.5 cm

Figure 2: A sample item on knowing rules and construction legitimacy

RESULTS

It was found that only about 40% of the participants knew that they could not use the marking on the ruler to bisect a line segment (cf. Fujita, Jones, & Yamamoto, 2004). Moreover, 35% knew that they could not use the two edges of a ruler to draw parallel lines, 40% knew that they could not use the two adjacent edges of a ruler to draw a right angle, and 36% knew that they could not use a contractor to draw an angle bisector. Nevertheless, only 16% managed all the four items. Apparently, many students thought that anything done with tools at hand was permissible in geometric construction. Besides, participants did not fair too well on items that required discernment of right and wrong. For example, in an item that inquired constructing the perpendicular bisector of a given line segment (see figure 2), some students deemed drawing an arc half of the length of the line segment was permissible whereas an arc longer than the length of the line segment was not.

As regards the basic construction procedure problems, most students could manage them except for the construction of a parallel to a line through a given point not on the line. The percentage correct amounted to only 28%. When prompted if they knew the reason behind the construction procedures, the percentages of those who claimed they knew the reasons ranged from 29% to 49% (see table 3). This finding may reflect that quite a number of the participants merely resorted to memorizing the basic construction procedures without relational understanding (Skemp, 1976).

Content	Perpendicular bisector	Angle bisector	Perpendicular from a point	Perpendicular at a point	Parallel to a line via a point
Number Correct	114	105	88	82	42
Know reason	73(49%)	71(47%)	48(32%)	48(32%)	29(19%)
Don't know reason	41(27%)	33(22%)	38(25%)	32(21%)	13(9%)

Note: Some students constructed correctly but did not check if they knew the reason for the steps. The percentages of those who knew or not knew the reason behind the steps are reported in brackets.

Table 3: Students' performances on the five basic procedure problems

This interpretation is further supported from observations on some participants' performances in constructing a perpendicular from a given point to a line. For example, figure 3 showed how a student solved the problem by imitating the standard procedure. Using point C as centre, the student drew an arc that cut $\overline{AB}$ at two points. Then he used the same radius and the two points on $\overline{AB}$ to draw two arcs that met at point C. He finished up by estimating a perpendicular from point C to the line given segment.

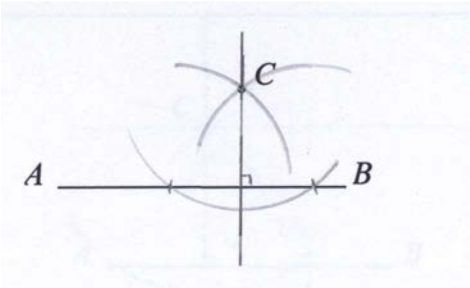


Figure 3: An example of constructing a perpendicular from a point to a line

As regards the application problems, the overall performances are not quite satisfactory as can be seen in Table 4 below. There are evidences that performance on a basic procedure problem is strongly related to the performance on an application problem requiring the same basic construction procedure. More precisely, doing well on the basic procedures was found to be a necessary condition for doing well on the corresponding application problem requiring the same construction skill. For example, for the 114 participants who could correctly construct the basic perpendicular bisector problem, 41 of them could also answer the application problem on perpendicular bisector. However, of the 36 participants who were unsuccessful on the basic perpendicular bisector problem, only 1 of them could answer the corresponding application problem.

No. of items correct	0	1	2	3	4	5
N	52	26	34	15	13	10
%	35.33	17.33	22.67	10	8.67	6.67
Cumulative %	35.33	52.67	75.33	85.33	94	100

Table 4: Overall performance on the five application problems

An interesting finding is that some participants were identified as being consistent in solving construction problems by drawing arcs. This paper defined "arc drawers" as those who used the tactic of drawing only arcs to solve at least one problem out of the five items on basic construction procedures and at the same time checked the box of not knowing the reason why their procedure worked. Of the 150 participants, 24 of them could be identified as arc drawers, amounting to about 16%.

One characteristic of the arc drawers is that they would draw arcs to find intersecting point(s) and then used the ruler to draw some lines through the intersecting point(s). Figure 4 reported how an arc drawer attempted to construct an angle bisector. When arc drawing did not help, they might either rely on past experiences to draw lines that approximated the answer or else they would give up. Figure 5 illustrated how an arc drawer draw the perpendicular bisector to a given line segment.

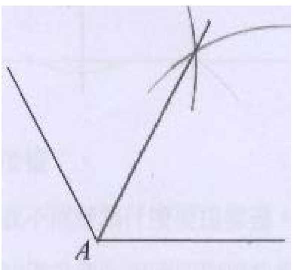


Figure 4: An angle bisector

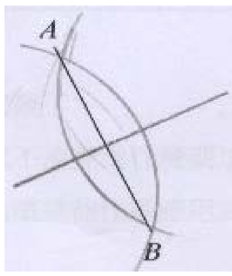


Figure 5: A perpendicular bisector

CONCLUSION AND DISSCUSION

In sum, several major findings were observed. First, some students did not know that geometric construction were restricted to the use of only a straightedge and compass. Second, many participating students were awkward with basic geometric construction procedures. Furthermore, they did not understand the relationship between the construction procedures and their underlying geometric properties. Third, most students were weak in solving application problems.

One possible reason about why they found these items difficult is that they learn construction procedures by memory, rather than through relational understanding (Skemp, 1976). So when a basic procedure involved many steps, it is more difficult to remember, let alone to apply it. For some students, the difficulty is due to their perception that construction problem is all about constructing arcs that intersected with each other. This is rather natural for what can be done with a compass if not for drawing arcs. Thus they will keep on producing arcs and erasing them. Of course, too many arcs and traces will add more confusion rather than help. It will be interesting if arc drawers can be identified in studies that use GSP or Cabri as the construction tools.

Rather than being obsolete, geometric construction is actually an important topic in school mathematics. For example, Djoric & Janicic (2004) has regarded geometrical construction skill as an important stepping stone to such areas as graphics and computability in computer sciences. Yet the findings herein revealed that quite some students were not aware of the rules imposed in construction, nor were they familiar with even the most basic construction procedures. In particular, constructing parallel line can be quite difficult for some students. Others might know the standard procedures but were unaware of the meaning behind them. Hence it is suggested that the rules imposed on construction tools should be explicitly taught and consistently reminded in geometry classes. Teachers should invest more time in delivering clear explanations behind why the construction procedures work.

In addition, teachers should pay more attention to those who construct by trial and error, especially with respect to those who can be identified as arc drawers. One useful tactic that mathematics instructors can employ is to require students to prove whether their constructions have satisfied the requirement specified in the problem statements. It is further suggested that obliging students to provide careful labels can also help in reducing unnecessary errors. At the very least, the students should explain what they have done. This will cut down on the number of students who do construction merely through eyeballing or by estimating. All in all, more efforts should be devoted to help students to learn this difficult yet useful topic.

References

Djoric, M., & Janicic, P. (2004). Constructions, instructions, interactions. *Teaching Mathematics and its Applications*, 23(2), 69-88.

Fujita, T., Jones, K., & Yamamoto, S. (2004). *The role of intuition in geometry education: Learning from the teaching practice in the early 20th century*. Paper presented to Topic Study Group 29 (TSG29) at the 10th International Congress on Mathematical Education (ICME-10). Copenhagen, Denmark; 4-11 July 2004.

Recio, A. M., & Godino, J. D. (2001). Institutional and personal meanings of proof. *Educational Studies in Mathematics, 48*(1), 83-99.

Schoenfeld, A. (1985). *Mathematical problem solving*. Mahwah, NJ: Lawrence Erlbaum Associates.

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher, 78*(6), 448-456.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Tam, H. P., & Chen, Y. L. (2010). *What kinds of difficulty do students encounter in solving geometric construction problems? – A Taiwan perspective*. Paper presented at the Annual Conference of the American Educational Research Association.

USING LARGE-SCALE ASSESSMENT DATA TO GLEAN TEACHER CHARACTERISTICS THAT PREDICT MATHEMATICS ACHIEVEMENT IN LATINOS/ESL[6]

Angela Thompson

University of California Santa Cruz

I use a national,(US) large-scale, longitudinal assessment data set to look for examples of high school mathematics teachers and their Latino and ESL students who are successful in mathematics. Specifically, I find and describe a number of groups of teachers by their characteristics within a data set with over 4000 mathematics teachers and 15,000 tenth grade students. Using cluster analysis, the mathematics teacher profiles are based on teacher survey responses. In describing teacher profiles, I treat each group as a nested case, describing the characteristics and mathematics achievement of the students for the teachers in each group. Finally, I look for examples of success in mathematics achievement for Latino and ESL students, and the characteristics of teachers more likely to have those students.

INTRODUCTION

Although thousands of educational studies are dedicated to identifying, measuring, and explaining the pathways to becoming a high-quality teacher, a solution remains illusive. One reason may be that a one-size-fits-all approach to education does not capitalize on the rich diversity of the students, teachers, schools, regions, languages, cultures, and many other human characteristics that can influence quality teaching. In other words, there may be more than one model of an "ideal" teacher, and more than one way to examine and maximize teachers' individual strengths for the good of their students. Educational policies typically do not take advantage of the strengths of individuals and groups, resulting in a pervasive opportunity gap between the dominant culture and a growing population of underserved students (Flores, 2007).

Unfortunately, national policies tend to motivate educators to treat students and teachers as if they were all the same, which is not the same thing as treating students with equity. Diverse students have diverse needs, so that trying to provide all students with exactly the same resources isn't likely to be helpful for every student. One thing that has become clear from large-scale assessment data is that there is a large and growing group of students in the US that do not perform as well on assessments or in school (Mosqueda, 2007). These US students are most often of non-white and non-Asian ethnicity, they attend schools that are economically and resource poor, and

[6] This research was supported by CEMELA: a Center for Learning and Teaching supported by the National Science Foundation, grant number ESI-0424983. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

many of these students speak a non-English language at home. The score differences between successful students and those sometimes referred to as "underserved populations" are large, significant, and sustained over time. Particularly alarming in the US is the educational disparity for the large and growing population of Latinos, whose opportunity to learn and develop in mathematics achievement has been persistently below all other groups at all grade levels (Llagas & Snyder, 2003).

The abundance of research on teacher quality and the demand for teacher accountability reveals the assumption that teachers and teacher quality have a strong influence on the success of student education. While effective teacher-student interactions may be critical to high-quality education, "teachers are not randomly assigned to schools and students are not randomly assigned to teachers or schools" (Guarino, Hamilton, Lockwood, & Rathbun 2006, p. iv). It is not clear what processes are in place to pair students and teachers, but current methods evidently do not provide Latino and ESL students with the teachers that may be most qualified to teach them (Clotfelter, Ladd, & Vigdor, 2007). Furthermore, there is a scarcity of research specifically linking student achievement in mathematics to teacher qualifications, despite the increasing focus on educational accountability (Guarino et al. 2006). By analyzing a large data set that includes teacher characteristics and student achievement over time, it may be possible to suggest several combinations of teacher and student characteristics that, when paired, increase the likelihood of achievement in mathematics.

Much of the research that tries to link student achievement to teacher attributes has been inconclusive. However, I, like many others, believe that answers are out there. In this study, I will describe a different approach for analyzing a large data set for teacher effectiveness, as well as the need to continue and expand upon quantitative research on teachers. I hope to show that there are a number of "teacher types" that are effective with various populations of students. In other words, I believe that there exists more than one model of effective teaching, and that we might be able to use large data sets to better suggest what models of teachers are more likely to be successful with various students, schools, and environments.

Research questions:

In light of the need for research linking teacher qualities and student achievement in mathematics, I explore the following research questions:

- 1: What are the characteristics of profiles of US 10th grade mathematics teachers?
- 2a: What are the predominant characteristics of students who are assigned to various teacher profiles of 10th grade mathematics teachers?
- 2b: Which teacher profiles, if any, are more likely to have Latinos or ESL students?

- 3: What combinations of matching students with 10th grade teachers might lead to an increased chance of mathematics achievement two years later for Latinos or ESL students?

BACKGROUND LITERATURE

The opportunity gap describes the large and growing difference in educational success between US students of the dominant (white, middle-class) culture, and those whose educational opportunities are reduced or limited as shown by low test scores, a high drop-out rate, and low attendance at post-secondary institutions. Many policies and policy revisions have been enacted in the US in an effort to reduce or eliminate the opportunity gap. These enactments include Title I, Improving America's Schools, No Child Left Behind, and most recently Race to the Top (Miller, Linn & Gronlund, 2009). Through these and other policies, some children have benefited through programs such as Headstart, sheltered instruction for English learners, and special testing accommodations for students with special needs. Our underserved students in US schools represent a very rich and diverse population and therefore, they have diverse educational strengths and needs. However, programs that try to meet some of these needs are often temporary, under-funded, in constant flux, and are implemented by people who may not have a clear idea of how to provide the resources that are needed in the quantities, regions, and qualities that are required (Flores, 2007). The opportunity gap has thus far remained.

Underserved or marginalized populations (of students) are defined as such because of their historic and ongoing poor performance in US public schools (Hunsaker, 1994). Gutierrez (2009) defines them as "African American, Latina/Latino, American Indians, working class students, and English language learners" (p. 9). The following demographics are according to NCES (2004): almost 44% of K-12 students in the US are from a non-white ethnic background. Nearly half of that group is listed as *Hispanic* [1], comprising over 10 million students. "Although the limited availability of disaggregated data often leads researchers to treat Hispanics as if they were a homogeneous group, the US Hispanic population is diverse," including Mexican Americans, Puerto Ricans, Cubans, and others (Llagas & Snyder, 2003). These students include recent immigrants as well as Latinos who have been in the United States for many generations. Some Latino students are bilingual or monolingual English speakers, while others are part of a rapidly growing ESL population: more than 5 million students (1 out of 9) in US classrooms are ESL (Pitoniak et al., 2009). By 2025, it is predicted that 1 out of 4 US students will be ESL. While the ESL population represents over 400 different home languages, 80% of them are native speakers of

[1] For the purpose of consistency, I use the term *Latino* to refer to people in the US who are part of a large ethnic group that includes origins in Spain and Latin American countries such as Mexico, Cuba, Colombia, Dominican Republic, Puerto Rico, El Salvador, and Spanish speaking countries. These people are sometimes referred to in literature as Hispanic or Latina/o.

Spanish. Many Latinos and ESL students are also a part of a large group of students with a low socio-economic status (SES). Twenty million K-12 students in the US are eligible to receive free or reduced lunch (NCES 2004). Given that many underserved students are simultaneously Latino, ESL, and low SES, they are at a much greater risk than students who are either of the dominant culture and language, or only possess one of the risk factors associated with being underserved.

The US Latino student population has been found to have the lowest achievement in mathematics (NCES, 2004), the highest dropout rate (Llagas & Snyder, 2003), and is the least likely to be enrolled in high-tracked, honors, or advanced placement courses in mathematics and science (Mosqueda, 2007). Furthermore, although a number of policies and programs have been initiated with the hope of closing the opportunity gap that separates a large portion of Latinos from ethnic groups who receive more resources and rigorous educational opportunities, four decades of research indicates the gap is stable and in some aspects widening (Lee, 2002). In sum, the Latino population in the US is currently the largest group of underserved students, the most poorly resourced, and as a result the most disenfranchised from their pursuit of education and its culminating benefits.

A topic of heated debate is that of English language learners (ESL) and US policy's (NCLB) state mandated testing. Part of the controversy is over the idea that bilingual students, while they might be proficient in conversational or everyday English, may not have had adequate access to the academic language used on assessments and in school (Scarcella, 2003). If immigrant students have not had adequate access to academic English, it is likely to negatively affect their test scores on a test conducted entirely in English. In order to promote equitable practices and fairness to all students, NCLB requires that ESL students participate in state mandated testing after just one year of sheltered instruction. Contrarily, "research indicates that it takes up to seven years for ELL students to acquire the academic language that is needed in learning academic knowledge from English-based sources" (Young et al., 2000, p. 173). Abedi and Dietel (2004) state that in order "to make the substantial gains required by NCLB, schools will need to identify superlative ELL teaching practices and teachers, using that knowledge to help other schools" (p. 5). Researchers need to look for examples of success, both qualitatively and quantitatively, to better inform policy makers, educators, test designers, and others how to better meet the needs of this rapidly growing population.

A series of studies by Clotfelter, Ladd, and Vigdor (2006) explore the patterns of matching teachers to students and their analysis of student achievement based on teacher quality. They find that "more highly qualified teachers tend to be matched with more advantaged students, both across and within schools" (2006, p. 778), and that this positive matching lends bias to analyses of the effects of teacher quality on student achievement. Their data set of over 3000 North Carolina teachers provides strong evidence that in most of those schools, the assignment of teachers to students is not random, and that teachers with higher teacher licensing scores and more experience are most often matched with high achieving students. In 2007, Clotfelter et al. found that teacher's experience, National Board Certification, and teacher licensing scores all have positive effects on student achievement, especially in mathematics.

While it may not be interesting or surprising that students who have the greatest educational needs are often paired with teachers and schools who are least prepared to teach them, it is interesting to study those patterns to get a clearer picture of what is happening and possibly how to move forward to improve conditions. Crosnoe (2005) looked at relationships between school conditions, student demographics, student mental health (as measured by a teacher's evaluation), and mathematics achievement for kindergartners. Crosnoe found that first- and second-generation Latino immigrants were much more likely to attend a more poorly equipped school, even when compared with other underserved populations of the same SES. His research indicates that "schools are still highly segregated along racial/ethnic lines" (p. 272). Remedial classes often have larger numbers of ESL students, even if the teachers of remedial classes may not have taken courses on bilingual or second-language pedagogy (Mosqueda, 2007). Some research also indicates that schools of lower-than-average performance and lower SES are more likely to have uncertified or alternatively-certified teachers (Goldhaber & Brewer, 2000). As lower performing schools also have a tendency to have a much higher teacher turnover rate, the average number of years of teaching experience is much lower (NCES, 2004).

Several researchers have considered the possible effects on mathematics achievement scores by the qualifications of their teachers (e.g. Goldhaber & Brewer, 2000; Darling-Hammond, Berry, & Thoreson, 2001; Clotfelter, Ladd, & Vigdor 2006, 2007). In this case, "qualifications" are measured in part by the type of teacher certification, the number of years teaching experience, and whether the teacher has a degree in the content area(s) she teaches. However, the amount of variance that can be explained on a large data set of mathematics achievement scores by whether the teacher went through a "normal certification process" is small and contested. It has been shown that teachers with more years of experience and a degree in the content area they teach usually have students with higher test scores. However, I feel part of the explanation for the higher scores is that teachers with more years of experience often get better choices as to what subjects and times of day they teach when compared to their newly-hired peers (Flores, 2007).

METHODOLOGY

Data

The ELS 2002-2004 public release data set includes 7135 English and mathematics teachers, over 15,000 students, and mathematics achievement scores in their 10th and 12th grade years. The teacher data includes responses to two parts of a survey: one on the teacher's own background, and the other an evaluation of each teacher's individual students included in the sample. Survey items for teacher background include basic demographics, years teaching and teaching at that school, certification type, subjects taught, degrees, job satisfaction, professional development, and use of computers for work related tasks. Teacher evaluations of students include questions about each student's work habits and behavior in class, attendance, and the nature and number of phone calls to parents on behalf of that student. Student achievement data also includes high school transcripts and student survey responses. Below I will describe how I analyze the data to address each of my research questions.

Question 1: What are the characteristics of profiles of US 10th grade mathematics teachers?

Cluster analysis is only effective when all data fields are complete. Therefore, my analysis begins with removing cases that contain missing information, or filling empty responses with a non-response code. Next, I perform descriptive statistics to determine the characteristics of the whole sample of mathematics teachers and their students. Although research articles generally describe the one clustering algorithm that was ultimately employed in an analysis, texts recommend running a cluster analysis with two or more models, followed by a comparison to find which algorithm best fits a researcher's data and questions (see Everitt, Landau, & Leese, 2001, for example). I experiment with divisive, k-cluster, two-step, and fuzzy analysis to find a model that creates the most distinctive groups. I then describe the clusters of teachers in their final form by their distinctive features using ANOVA.

Question 2a: What are the predominant characteristics of students who are assigned to various teacher profiles of 10th grade mathematics teachers?

Once I have created and described a number of mathematics teacher clusters, I treat each cluster as a nested case, incorporating student data from the ELS data set for the teachers in each cluster. Using the student and mathematics teacher data that discriminates between clusters, I create profiles of mathematics teachers and their students, describing all the variables that make them unique when compared to the other profiles and when compared to the data set as a whole. With their 10th grade ELS mathematics assessment scores, I also describe the students within each cluster by their mathematics achievement, and how the distribution of mathematics achievement may differ by demographics across groups.

Question 2b: Which teacher profiles, if any, are more likely to have Latinos or ESL students?

Next, I look at profiles to determine which teachers, if any, are more likely to have Latinos, ESL students, and/or students of low SES. It is my hope that I can assign the teacher profiles into one of three loosely-fitting categories: teacher profiles that are *more* or *less* likely to have underserved students, and those that have an average chance of having those students. Using my analysis from question 2a, I look for and describe any differences or similarities in profiles of the same category. With this information, I hope to provide some insights as to what kinds of teachers and students are more or less likely to end up together, and in particular, characteristics of mathematics teachers more likely to have Latinos and ESL students.

Question 3: What combinations of students and 10 grade mathematics teachers might lead to an increased chance of mathematics achievement two years later for Latinos or ESL students?

To address this question, I describe within each cluster and within each of the five base-year mathematics proficiency levels which Latino students are doing well in mathematics, and the features of their teachers, as a part of that cluster, that are somehow related to student achievement in mathematics. The implications may enable me to offer suggestions on how to assign students to mathematics teachers in ways that are predicted to have a higher rate of success. Follow-up research that works towards verifying teacher profiles and their students at the level of mixed methods or case studies may also provide suggestions for teacher professional development.

FINDINGS

Preliminary findings suggest that teachers that give importance to multiple sources of academic success contribute to predicted increase in their Latino and ESL students' mathematics achievement. These sources include home environment, intellectual ability, teacher's attention to the unique interests and abilities of the student, and teacher and/or student enthusiasm for learning. Furthermore, a two-step cluster analysis indicates that teacher qualities that predict higher achievement in mathematics is far more complex than US policy's description of a high-quality teacher based on being certified and having a degree in the subject(s) taught: instead, profiles of teacher characteristics describe more than one model of a high-quality teacher. However, Latino-Latino student-teacher pairings was not a good predictor of mathematics achievement. This is similar to other research, that theorize the very large group of students and educators referred to as "Latino" or "Hispanic" is so rich, diverse, and contrasted, that it may not be appropriate to claim an ethnic or cultural match based on this simplistic label (Darling-Hammond, 1994; Gutierrez, 2002; Martin, 2007). My findings suggest teacher qualities that offer potential for increased success in mathematics achievement, and how schools might make more informed choices about course assignment and student placement policies that could help advance Latino and ESL students in mathematics.

References

Abedi, J., & Dietel, R. (2004). Challenges in the No Child Left Behind Act for English language learners (Policy Brief): CRESST.

Clotfelter, C. T., Ladd, H. F., & Vigdor, J. L. (2006). Teacher-student matching and the assessment of teacher effectiveness. *The Journal of Human Resources, 41(4)*, 778-820.

Clotfelter, C.T., Ladd, H. F., & Vigdor, J. L. (2007). Teacher credentials and student achievement: Longitudinal analysis with student fixed effects. *Economics of Education Review; 26*, 673-682.

Crosnoe, R. (2005). Double disadvantage or signs of resilience? The elementary school contexts of children from Mexican immigrant families. *American Educational Research Journal, 42(2)*, 269-303.

Darling-Hammond, L., Berry, B., & Thoreson, A. (2001). Does teacher certification matter? Evaluating the evidence. *Educational Evaluation and Policy Analysis, 23(1),* 57-77.

Everitt, B.S., Landau, S., & Leese, M. (2001). *Cluster Analysis*. London: Arnold.

Flores, A. (2007). Examining disparities in mathematics education: Achievement gap or opportunity gap? *The High School Journal, 91(1),* 29-42.

Goldhaber, D. D., & Brewer, D. J. (2000). Does teacher certification matter? High school teacher certification status and student achievement. *Educational Evaluation and Policy Analysis, 22(2),* 129-145.

Guarino, C. M., Hamilton, L. S., Lockwood, J. R., Rathbun, A. H. & Hausken, (2006). *Teacher qualifications, instructional practices, and reading and mathematics gains of kindergartners* (research and development report). Washington, DC: National Center for Education Statistics.

Gutierrez, R. (2002). Beyond essentialism: The complexity of language in teaching mathematics to Latina/o students. *American Educational Research Journal 39(4),* 1047-1088.

Gutierrez, R. (2009). Embracing the inherent tensions in teaching mathematics from an equity stance. *Democracy and Education, 18(3),* 9-16.

Hunsaker, S. L. (1994). Adjustments to traditional procedures for identifying underserved students: Successes and failures. *Exceptional Children, 61*, 72-76.

Lee, J. (2002). Racial and ethnic achievement gap trends: Reversing the progress toward equity? *Educational Researcher, 31(1),* 3-12.

Llagas, C., & Snyder, T.D. (2003). *Status and trends in the education of Hispanics* (report). Washington, DC: National Center for Education Statistics.

Martin, D. B. (2007). Beyond missionaries or cannibals: Who should teach mathematics to African American children? *The High School Journal, 91(1),* 6-28.

Miller, M. D., Linn, R. L., & Gronlund, N. E. (2009). Educational testing and assessment: Context, issues, and trends. In *Measurement and assessment in teaching* (10th ed., pp. 1-25). Columbus, OH: Merrill.

Mosqueda, E. (2007). *English proficiency, tracking, and the mathematics achievement of Latino English learners.* Unpublished dissertation, Harvard, Cambridge, MA.

National Center for Education Statistics. (2004). Educational Longitudinal Study (ELS:2002-2004) (CD: Public Release ed.): US Department of Education.

Pitoniak, M. J., Young, J. W., Martiniello, M., King, T. C., Buteux, A., & Ginsburgh, M. (2009). *Guidelines for the assessment of English language learners*. Princeton, NJ: Educational Testing Service.

Scarcella, R. (2003). *Academic English: A conceptual framework* (technical). Irvine, CA: Language Minority Research Institute.

Young, J. W., Cho, C., Ling, G., Cline, F., Steinberg, J., & Stone, E. (2008). Validity and fairness of state standards-based assessments for English language learners. *Educational Assessment, 13*, 170-192.

ASPECTS THAT PLAY AN IMPORTANT ROLE IN THE SOLUTION OF COMPLEX ALGEBRAIC PROBLEMS

María Trigueros[1] Sonia Ursini Convadonga Escandón

trigue@itam.mx soniaul2002@yahoo.com.mx covadonga.escandon@gmail.com

ITAM- Mexico CINVESTAV-Mexico INEGI - Mexico

Research has shown that different aspects concur when students work with complex algebraic tasks. Are these aspects related to each other? Are there any aspects that play a dominant role? To approach these questions 270 Pre-Calculus course students' responses to complex algebra tasks were analyzed by implicative statistic using the program CHIC. The results show that in order to solve successfully complex algebraic problems all the different aspects considered concur and their causal interconnection is shown. Good understanding of the concept of variable and its flexible use are in the core of the capability to correctly solve complex algebra problems.

INTRODUCTION

Research focussing on difficulties students face when working with complex algebraic tasks have stressed difficulties they have mainly in using mathematic concept definitions (Selden and Selden, 1987, 1995); lack of structure sense (Hoch & Dreyfus 2004; Novotná & Hoch, 2008); difficulties to work with algebraic variable using it in a flexible way (Trigueros & Ursini, 2003, 2008; Ursini & Trigueros, 2009); difficulties to distinguish the different cases when faced to inequalities or expressions that include absolute values (Bazzini & Tsamir, 2001; Boero & Bazzini, 2004; Ursini & Trigueros, 2009). All these aspects concur when students work with complex algebraic tasks but, how are they related to each other? Are there any aspects that play a dominant role? The present study aims to approach these questions through an analysis of students' responses to a number of complex algebra tasks.

THEORETICAL FRAMEWORK

Following Hoch & Dreyfus (2004) we consider that to have structure sense implies being able to see algebraic expressions as global entities and find out some structure in them; to use definitions and the restrictions they impose on the solution of a problem and to foresee their implications; as well as to give priority to the different actions that have to be performed in the solution of a problem. Concerning the use of definitions, students' capability to know in which domain a definition might be useful and convenient to be applied (Ball & Bass, 2000) is taken as fundamental. Students' capability to decide the steps to follow while solving a task, the capability to keep in mind the different issues involved in a problem and to keep track of the role of each of them in the solution (Trigueros & Ursini, 2008) are considered. The 3UV Model (Trigueros & Ursini, 2003, 2008) is used to analyse students' capability to work (interpret, symbolize, manipulate) with the different uses of variable: specific unknown, general number and variables in functional relationship. An understanding of variable implies the comprehension of all these aspects and the possibility to flexibly shift between them depending on the problem to be solved. Following Ursini & Trigueros (2008), parameters are viewed as general numbers of second order, that is, they appear when generalising first order general statements (families of equations, families of functions and families of open expressions). First order general statements are derived from generalising statements involving only numbers. When parameters are involved in algebraic expressions their role depends on the context (Bloedy-Vinner, 2001; Furinghetti & Paola, 1994) and this role may change within the same problem, for example, from general number to unknown.

METHODOLOGY

An exam consisting of 6 complex algebraic problems was designed (Table 1). A team (two teachers and researchers) analysed each question in terms of the possible strategies of solution and their implications in terms of algebraic reasoning and solution processes needed in each problem and, based on the theoretical framework, coded them into 8 general categories. The criterion to determine these categories was that each of them should appear in at least half of the problems (Table 2). The categories were: identifies the role played by variables (UV); flexibly changes from on use of variable to other (UFV); identifies parameters (IP); applies definitions and concepts needed (DEF); symbolizes correctly (S); correctly manipulates (M); recognizes restrictions and conditions (CR); recognizes different cases involved (CAS); shows structure sense (EA). Two extra categories were added: correct solution (RES) and ability to interpret the result in terms of the conditions of the problem (IR). The exam was administered to 270 Pre-Calculus course students. Their responses were analysed by the team in terms of the 10 categories and tabulated. Implicative statistics was chosen to analyse the data since it provides conceptual tools to establish both, relations and causal relations among categories. Developed on non-symmetrical indexes, implicative statistic puts forward both, similitude and implicative relations, by using implication rules that can be derived from boolean algebra and artificial intelligence. Its procedures are designed to find or abandon a given rule relating categories by making statistical sense of non strict implications (Gras, 2005).

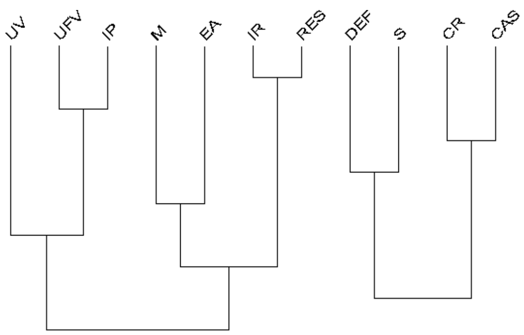
1.- Completely factorize the expression: $(a^2+2a)^2-2(a^2+2a)-3$	4.- For which values of parameter *m* do the roots x_1, x_2 of equation $x^2-3mx+m^2=0$ satisfy that $x_1^2+x_2^2<7$?
2.- Solve the inequality: $\lvert 2x-\lvert x+1\rvert\rvert>10$	5.- Find all the values for de *a* such as the distance between points $P_1=(a,3)$ y $P_2=(5,2a)$ is greater than $\sqrt{26}$
3.- Solve the equation: $\lvert x^2-4x\rvert=2x^2-4$	6.- Given the circumference is $x^2+y^2+4x-10y=16$. If *P* = (1,–1) lies on the circumference find the point which is diametrically opposed to P.

Table 1: Exam questions

According to Gras, learning begins with facts and rules that are interrelated and these progressively form learning structures. Finding rules to describe students´ learning is important in Mathematics Education and it is in accordance to the goals of this study. Following Gras, there are three important rules that can describe the learning process: a*)* *a*➔*b*, where a and b can be categories or rules; b) *a*➔*(b*➔*c)*; and c) *(a*➔*b)*➔*(c*➔*d)*. These rules describe a learning structure that is hierarchic, oriented and not symmetrical. Finding rules to describe students´ learning is important in Mathematics Education and is in accordance to the goals of this study. These rules describe a learning structure that is hierarchic, oriented and not symmetrical. This structure can be obtained with the Clasification Hiérarchique et Cohesitive (CHIC) program (Couturier, 2001). This program produces three diagrams that offer different information: a) Similitude tree: for each pair of variables, it calculates their similitude relation and forms classes of categories. Categories which have the stronger relation to each other are grouped into a set. This tree shows possible classes or categories which are at the basis of the structures obtained. b) Hierarchic tree: a hierarchic and oriented representation where the different levels identify the rules formed according to the strength of an implication index between a pairs of variables or pairs of classes. c) Implicative graph: its construction uses both the intensity index and a validity index. It shows implicative associations which are significant at specific levels.

Results obtained were analysed by the researchers in terms of classes and of implications using logical equivalences, when needed, in order to obtain the rules represented in the diagrams.

RESULTS AND DISCUSSION



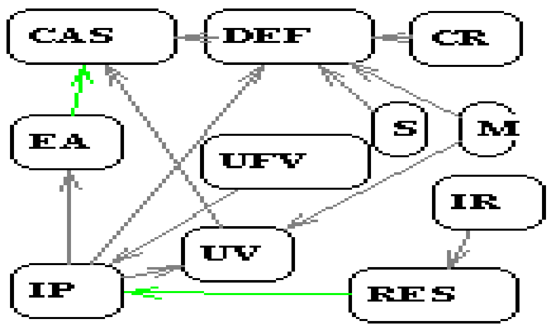


Figure 1: Similitude Tree

Figure 2: Implicative Graph

Similitude Tree Analysis

Figure 1 corresponds to results obtained for the similitude analysis. As can be observed the categories corresponding to the problem under study are organized in two clearly disjoint groups. The first of them corresponds to those related to the correct interpretation of variable, those related to algebraic manipulation and structure sense the possibility to correctly solving problems and interpreting the solution. In the second group categories related to correct interpretation of algebraic definitions in order to produce algebraic expressions in specific problems as well as recognizing restrictions and different cases are shown. Observe that categories corresponding to interpretation and flexible use of variables, structure sense and manipulation belong to the same class. This suggests a relation between a flexible use of variables, structure sense and the ability to manipulate expressions. Leaving aside the fact that a correct interpretation of the result obtained relates strongly to finding the right answer to the problem, which is quite obvious, what is more interesting in this diagram is the strong relationship between the flexible use of variables, including parameters, and the interpretation of the role played by variable in algebraic problems. However, the diagram shows as well that manipulation and structure sense are related to the possibility to solve successfully algebraic problems. This might suggest that it is enough to follow appropriate algorithms, even without understanding, in order to reach a correct answer. But it is worth noting that this group has a lower hierarchy than the one relating correct answers to the interpretation of the uses of variable and its flexible use. This confirms the importance of a deep understanding of the concept of variable in order to correctly solve complex algebraic problems.

Implicative Graph

Figure 2 shows implications at a reliability index of at least 85%. It gives details about the causal relationship between the correct solution of an algebraic problem and the possibility to distinguish different cases and the interpretation of parameters. We observe that correct interpretation of responses to algebraic problems implies a correct solution to the problem and this, in turn, implies the possibility to interpret parameters. It also gives evidence of how the interpretation of parameters is causally related to the correct identification of cases through different variables in this study: on the one hand, to structure sense, on the other hands, to the possibility to apply correctly definitions of concepts, and last but not least, to the interpretation of the use of algebraic variables involved in problems. It is also observed that a correct interpretation of parameters depends on the flexible use of variables; identifying the role of variables depends on the possibility to manipulate symbols; while a correct use of definitions depends on the possibility to symbolize and to manipulate correctly, and on the capability to recognize the restrictions of a problem.

Hierarchic Tree Analysis

From the analysis of the hierarchic tree shown in Figure 3 it is possible to deduce rules according to their implication index (Table 2). As can be observed in Figure 3 the hierarchical analysis connects all the categories. This is not always the case in analysis done with implicative statistics. Although rules 1, 2, 3 and 6 may seem quite obvious for a teacher or researcher, and often appear when qualitative analysis is used, it is important to stress that in this study they were identified by the software from the implicative statistical analysis of data. The other rules are not always so evident when qualitative analysis is used, they could remain hidden, but the statistical analysis we used has made them visible.

These results stress the importance that a correct interpretation of variable, in particular of parameters, the capability of a flexible use of it, together with structure sense and the capability to use definitions, have in order to solve complex algebra problems.

These results are in accordance with those found by research studies mentioned in the introduction which focus on structure sense or on conceptual understanding or on the interpretation and use of variable or on the use of definitions. It is interesting to note that, if in the diagram shown in Figure 3 we do not take into account the last three implications (corresponding to rules 8, 9, 10), the groups of implications can be related with research topics found in the literature just mentioned above. Those studies usually focus only on one or a specific set of the variables studied in this paper, while the analysis presented here shows that all of these aspects play an important role in success when working with algebra and how they are causally interconnected.

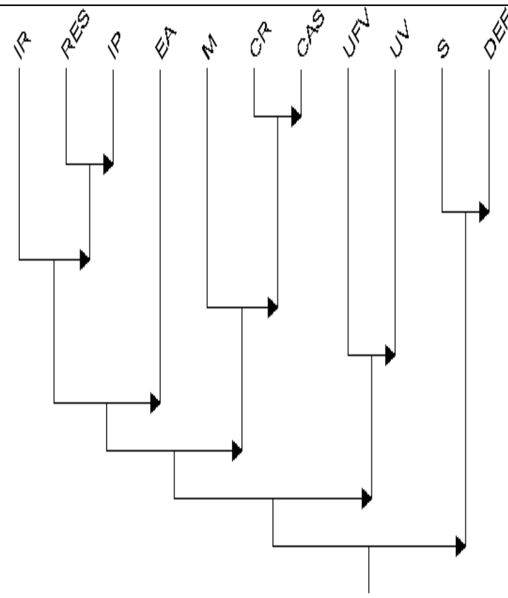


Figure 3: Hierarchic Tree

1. If restrictions are interpreted, the different cases of the problem can be identified.
2. When parameters are not interpreted, a correct solution to the problem is not found.
3. Incorrectly applying definitions implies incorrect symbolization.
4. If a correct result is obtained and interpreted, parameters have been correctly interpreted.
5. If manipulation and identification of restrictions are correct, it is possible to identify cases
6. A flexibly use of variables implies a correct interpretation of the different uses of variable.
7. If a correct result is obtained and interpreted, either parameters have been correctly interpreted or algebraic structure recognized.
8. In order to correctly interpret and solve an algebraic problem it is necessary to interpret parameters, to show structure sense, to manipulate variables correctly, to recognize the restrictions of the problem and to recognize different cases
9. In order to correctly interpret and solve an algebraic problem it is necessary as well to recognize the different uses of variable.
10. In order to successfully solve algebraic problems it is necessary to symbolize definitions, which in turn, requires knowledge of the definitions.

Table 2: Rules from hierarchic Tree

CONCLUSIONS

Qualitative analysis is an invaluable tool in Mathematics Education research. This analysis, however, is difficult to carry out when studying large populations. Although in occasions the value of statistics has been questioned in terms of success with algebraic learning already signalled by different researchers are strongly interrelated, and that a good and flexible understanding of variable, structure sense, distinguishing cases and a correct use of definitions are important for students to be able relate all the aspects considered. The interrelationship among the categories studied is underlined but, more importantly, the rules obtained from the hierarchic tree provide useful information about causality among them. Finally, the role of a flexible use of variables and interpretation of parameters appear to play a dominant role in the implicative graph since they imply significatively the identification of algebraic structure, the identification of possible cases and the correct use of definitions.

References

Ball, D. L. & Bass, H. (2000). Interweaving Content and Pedagogy in teaching and Learning to teach: Knowing and Using Mathematics. In Boaler, J. (Ed.) *Multiple Perspectives on Mathematica Teaching and Learning.* 83-104. Ablex Publishing.

Bazzini, L. & Tsamir, P. (2001). Research based instruction: widening students´ perspective when dealing with inequalities. In Chick, H., Stacey, K. Vincent, J., & Vincent, J. (Eds.) *"The future of teaching and learning of algebra" Proceedings of the 12th ICMI Study,* (Vol. 1, pp. 61-68). Melbourne, AU.

Bloedy-Vinner, H. (2001). Beyond unknown and variables – Parameters and dummy variables in high school algebra. In Sutherland, R., Rojano, T., Bell, A. & Lins, R. (Eds.) *Perspectives on school algebra.* 177-189. Kluwer Academic Publishers.

Boero P. & Bazzini L. (2004). Inequalities in mathematics education: the need for complementary perspectives. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education.* (Vol. 1, pp. 139-143). Bergen, Norway: PME.

Couturier, R. (2001). Traitements de l'analyse statistique dans CHIC. *Acte des Journées sur la Fouille dans les données par la méthode d'analyse implicative*, IUFM. Caen. 33-50. http://math.unipa.it/~grim/asi/asi_00_couturier.pdf

Furinghetti, F. and Paula, D. (1994). Parameters, unknowns and variables: a little difference? In J.P Ponte and J.F. Matos (eds.) *Proceedings of the XVIII International Conference for the Psychology of Mathematics Eduaction,* (Vol. 2, pp. 368-375). University of Lisbon, Portugal: PME.

Gras, R. (2005). Panorama du dévelopment de l'ASI à partir des situations fondatrices. *3ème Rencontres Internationales.* Palermo, Italia, at http://math.unia.it/~grim/asi/asi_05_gras_1.pdf

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49-56). Bergen, Norway: PME.

Novotná, J. & Hoch, M. (2008). How Structure Sense for Algebraic Expressions or Equations is Related to Structure Sense for Abstract Algebra. *Mathematics Education Research Journal*, 20(2), 94-104.

Selden, A. & Selden, J. (1987). Errors and misconceptions in college level theorem proving. In Novak, J. (Ed.) *Proceedings Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics* (Vol. 3, pp.456-470). Itacha, NY, USA: Cornel University.

Selden, J. & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29(2), 123-151.

Trigueros, M. & Ursini, S. (2003). First year undergraduates difficulties in working with the concept of variable. In *CBMS, Research in Collegiate Mathematics Education*, 12, 1-29.

Trigueros, M. & Ursini, S. (2008). Structure sense and the use of variables. In Figueras, O., Cortina, J.L., Rojano, T. & Sepulveda, A. (Eds.), *Proceedings of the Joint Meeting PME 32 and PME-NA XXX.* (Vol. 4, pp. 337 – 344). Morelia, Mexico: PME.

Ursini, S. & Trigueros, M. (2009). In search of characteristics of successful solution strategies when dealing with inequalities. In Tzekaki, M. Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*. (Vol. 5, pp. 265-272). Thessaloniki, Greece: PME.

(1) This research was supported by Asociación Mexicana de Cultura AC

CHILDREN'S DEVELOPMENT OF MULTIPLICATIVE REASONING: A SCHEMES AND TASKS FRAMEWORK

Ron Tzur, Heather Johnson, Evan McClintock | Yan Ping Xin, Luo Si, Rachael Kenney, Jerry Woodward, Casey Hord | Xianyan Jin

University of Colorado Denver | Purdue University | Monash University

We present a synthesis of findings from constructivist teaching experiments—a developmental framework of six schemes that children construct for reasoning multiplicatively and tasks to promote them. The framework is rooted in distinctions of units children seem to use and operations with/on these units—particularly number as an abstract, symbolized composite unit. We provide a task-generating platform game, depictions of each scheme, and tasks supportive of constructing it. We discuss the need to distinguish between tasks and child's cognitive conceptions, and to organize learning situations that (a) begin at and build on the child's available scheme, (b) geared to the next scheme in the sequence, and (c) link to intended math concepts.

INTRODUCTION

In this paper we propose a developmental framework that makes distinctions and links among schemes—conceptual structures and operations children construct to reason in multiplicative situations. We provide a set of tasks (problem situations) to promote construction of such schemes. Elaborating on Steffe et al.'s (Steffe & Cobb, 1998) seminal work, this framework synthesizes findings of our teaching experiments[i] with over 20 children who have disabilities or difficulties in mathematics. This empirically grounded framework contributes to articulating and promoting multiplicative reasoning—a key developmental understanding (Simon, 2006) that presents a formidable conceptual leap from additive reasoning for students and teachers (Harel & Confrey, 1994; Simon & Blume, 1994). In place of pedagogies that focus primarily on multiplication procedures, our framework can inform teaching for and studying of children's conceptual understandings. Such understandings provide a basis not only for promoting multiplication and division concepts and procedures but also for reasoning in place-value number systems, and in fractional, proportional, and algebraic situations (Thompson & Saldnha, 2003; Xin, 2008).

We contrast our stance on children's cognitive change and teaching that promotes it with the *Cognitively Guided Instruction* (CGI) approach (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998)). CGI grew out of research on children's solutions to addition and subtraction tasks. By asserting that "children's solution processes directly modeled the action or relationships described in the problem" (Carpenter, Hiebert, & Moser, 1983, p. 55), CGI researchers seemed to equate children's cognitive processes with tasks. In contrast, we argue for explicitly distinguishing between task features as adults conceive of them and schemes children bring forth for solving tasks. Consider a *Join* task such as, "We had 7 toys and got 4 more; how many toys we then had in all?"

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 155-162. Taipei, Taiwan: PME.

A child may solve such a task by *counting-all* 1s (**1**-2-3-…10-11), by *counting-on* (**7**; 8-9-10-11), or by using a *through-ten strategy* (7+3=10; 10+1=11). The latter two indicate the child understands number as a composite unit, hence preparedness for multiplicative reasoning, whereas the first does not. We concur with CGI's premise of the need to use children's ways of thinking in teaching. However, we disagree that the structure of a task as seen by an adult determines, in and of itself, the way a child makes sense of and acts to solve it. The next section presents the conceptual framework that underlies our synthesis.

CONCEPTUAL FRAMEWORK

Our framework builds on the core notion of *scheme*—a psychological construct for inferring into the mental realms of thinking and learning. von Glasersfeld (1995) depicted scheme as a tripartite mental structure: a *situation* (recognition template) that sets one's goal, an *activity* triggered to accomplish that goal, and a *result* expected to follow the activity. Tzur et al. (Tzur & Lambert, 2011; Tzur & Simon, 2004) further distinguished *effect* from 'goal' and 'result', asserting that effect can more precisely pertain to anticipated and actually noticed outcomes of a mental activity on/with certain 'objects'. As a person's mind 'runs' activities and regulates them by the goal, novel effects can be noticed, differentiated from anticipated ones, and related to the activity. An *activity-effect relationship* (*AER*) is conceived of as a sub-component of a scheme (2^{nd} and 3^{rd} parts). Existing or noticed *AERs* can be linked to a given scheme's situation, transferred to, and linked with other situations.

A mathematical *task* pertains to a pedagogical tool used to promote student learning, that is, advancing from current to intended schemes. Typically, a task consists of depictions of relationships among quantities, some given and some unknown, including a question for figuring out the latter. In recent years, tasks became a primary tool *through* which to foster mathematics learning, as opposed to a way of applying taught concepts *after* learning took place (NCTM, 2000; Watson & Mason, 1998). To solve a task, a child has to (a) assimilate it into an existing scheme's 'situation', (b) identify the quantities (mental objects) involved, (c) set a goal compatible with the question, and (d) initiate mental activities on those quantities that (in the child's mind) correspond to the depicted relationships.

A key construct for distinguishing multiplicative from additive reasoning is number as a *composite unit* (CU) (Steffe, 1992). To reason *additively* requires students to operate with number as a CU. Children establish this in situations that trigger a goal of determining the amount of 1s in a collection of items and the activity of counting, which involves iterating the unit of one to *compose* larger units (e.g, 1+1+1=3). Gradually, the nested nature of the resulting, composed quantity becomes explicit (e.g., [1+1+1]+1=4; +1=5; etc.). When number is conceived of as a CU, children can anticipate *decomposing* units into nested sub-units. For example, a child can think of '11-7=?' as '7+?=11', that is, a CU of 11 ('whole') of which she knows one part (7) and can find the other. Key to additive reasoning is that the referent unit is *preserved* (Schwartz, 1991): 11 *apples* - 7 *apples* = 4 *apples*.

Learning to reason multiplicatively requires a major conceptual shift—a coordination of operations *on* CUs (Behr, Harel, Post, & Lesh, 1994). Consider placing 2 apples into each of 3 baskets; 2 is one CU (*apples per basket*) and 3 is another (*baskets*). Multiplicative reasoning entails distributing one unit over items of another (2 apples per basket) and finding the total (goal) via a *coordinated counting activity*: 1 (basket) is *1-2* (apples), 2 (baskets) are *3-4* (apples), 3 (baskets) are *5-6* (apples). C*oordinated* counting entails *deliberately* keeping track of CUs while accruing the total of 1s based on the distributed CU (2 apples-per-basket). As this example indicates, in multiplicative reasoning the referent unit is *transformed* (Schwartz, 1991), and the product has to be conceptualized as a unit of units of units (Steffe, 1992): here, '6 apples' is a unit composed of 3 units (baskets) of 2 units (apples per basket). The *simultaneous count* of two CUs and the resulting *unit transformation* constitute the conceptual advance from additive reasoning.

A FRAMEWORK OF MULTIPLICATIVE SCHEMES AND TASKS

This section first describes tasks we used to promote students' construction of multiplicative schemes—revolving around the platform game, *Please Go and Bring for Me* (PGBM). Then, a six-scheme developmental framework is presented. This order helps to delineate teaching that can foster construction of the schemes while clearly separating between instructional tasks and children's thinking.

Tasks for Fostering Multiplicative Schemes

PGBM is an example of a task-generating platform game. It fosters multiplicative reasoning by engaging children in tasks conducive to carrying out and reflecting on double-counting activities. The basic form is played in pairs. Each turn partners switch roles—one playing a *sender* and the other a *bringer*. The *sender* begins by asking the *bringer* to produce, one at a time, towers composed of the same number of cubes. Once the *bringer* has produced the needed amount of same-size towers (e.g., 5 towers, 3 cubes each; denoted $5T_3$), the *sender* asks her four questions: (1) How many towers did you bring? (2) How many cubes are in each tower? (3) How many cubes are there in all? (4) How did you figure it out? Questions 1 & 2 orient student reflections on the CUs involved—to distinguish activities of producing/counting a set of CUs from counting 1s to produce each CU. Questions 3 & 4 foster coordinated counting CUs (e.g., raising one finger per tower) while accruing the total of cubes (e.g., 3-6-9-12-15) based on the size of the distributed CU (e.g., 3 cubes per tower).

When students become facile in playing PGBM with tangible objects (cubes and towers), we use two major variations to foster abstraction of coordinated counting. Variation (1) supports students' shift from operating on tangible objects to figural objects in which a substitute item stands for real objects the students attempt to quantify. Variation (2) supports students' shift from operating on figural objects, to abstractly symbolized objects, to mental objects. In (1) partners produce a given set, say $3T_4$, *cover the towers* (Fig. 1a), then answer the 4 questions. Initially, we let children use spontaneous ways of keeping track of CUs and 1s (e.g., count on fingers, tally marks, etc.) Later, we guide them to sketch towers in a gradually more abstract manner. They begin with tower diagrams comprising of single cubes, then sketch tower diagrams with a number indicating the tower's size, then a line-with-number represents tower, and finally just a number (Fig. 1b). Using these diagrams fosters a shift from attending to 1s that constitute a CU to the numerical value resulting from how each CUs was produced. In (2) partners *pretend* as if they were producing towers, but do not actually do so. As in (1), we guide students to sketch increasingly abstract diagrams, beginning with figural objects and progressing to abstractly symbolized 1s and CUs. When a student can anticipate the structure of the 1s and the CUs, this suggests she or he can operate on CUs as mental objects. Like in the Singapore approach (Ng & Lee, 2009), these variations foster students' advancement from acting on CUs as tangible objects, to tangible-but-invisible, to mental objects.

Within Variations (1) and (2) we use different amounts of towers and cubes to support students' productive participation. Initially, children use familiar numbers (2, 5, or 10 cubes per tower) and small sets (up to 6 towers). Then, we guide them to use more difficult numbers (towers of 3-4 cubes, and later of 6, 7, 8, or 9 cubes) and larger sets (up to 12 towers). When students operate on cubes/towers as figural objects, we introduce similar tasks in other contexts (e.g., How many cookies are in 5 bags, if each bag has 3 cookies?). In doing so, we support students' use of coordinated-counting to figure out the total of 1s (e.g., cubes, cookies) across situations constituted by a number of same-size CUs (e.g., towers, bags of cookies).

Building on Xin's (2008) work, we gradually introduce children to a single symbolic structure that ties both multiplication and division. We begin with: *Cubes in each tower x Number of towers = Total of Cubes* (Fig. 1c). As they solve tasks in different contexts, we replace it by: *Items in Each Group x Number of Groups = Total of Items*; and finally by: *Unit Rate x Number of Composite Units = Total of 1s*. This symbolic structure supports students' determination of the needed computation (multiplication or division). In a multiplication situation, the total of 1s is unknown. In a division situation, either the number of CUs or the number of 1s per CU is unknown.

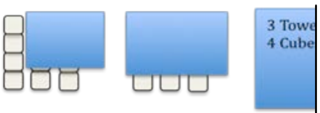


Figure 1a: Covered Towers

Figure 1b: Tower modelling

Figure 1c: Equation modelling

A Six-Scheme Developmental Framework

This section describes each of six schemes that, combined, constitute the framework we propose about children's development of multiplicative reasoning. For each, we indicate what the scheme involves, provide a sample task linked to the scheme, explicate goals, activities, and results associated with developing the scheme, and articulate mathematics that the established scheme supports.

The **first scheme** a child may construct is termed *multiplicative Double Counting* (mDC, Woodward, et al., 2009). It involves recognizing a given number of CUs, each consisting of the same number of 1s. Typical tasks include Variations (1) and (2) of the PGBM platform game. The child's goal is to figure out the total of 1s in this 'set', and the activity is simultaneous (double) counting of CUs and 1s that constitute each CU. When established, mDC includes a child's anticipation that a total of items (say, 24 cookies) is a CU constituted of another CU (4 bags), each of which a CU itself (6 cookies). This scheme provides a basis for the strategic use of known facts to derive unknown ones (e.g., "7x5 is like 5 towers of 7, and I know it is 35 (cubes); so 7x6 is as if I brought one more unit of 7, hence it is the same as 35+7=42").

The **second scheme** is termed *Same Unit Coordination* (SUC). It involves operating additively on CUs without losing sight of each CU being both a unit in and of itself and composed of 1s. Typical tasks linked to this scheme involve *two* sets of CUs and a question to figure out sums of or differences between the sets. SUC tasks may ask: "You brought $7T_5$ and then I brought $4T_5$; How many towers do we have in all?" or "You brought $7T_5$; I brought a few more; Together, you and I have $11T_5$; how many towers did I bring?" The child's goal is to figure out the sum or difference of CUs (not of 1s), and the activity may be any of those a child has constructed for operating additively on 1s (counting-all, counting-on, through-ten, fact retrieval, etc.). Like with units composed of 1s, the key in this scheme is the child's conception of the embedded (nesting) of CU sets within a larger CU (e.g., a CU consisting of 11 units of 10 can be decomposed into 7 units of 10 + 4 units of 10). When established, SUC provides a basis for operating on specific CUs such as 10s, 100s, and 1000s in a place-value system (with contexts including distance, weight, money, etc.).

The **third scheme** is termed *Unit Differentiation and Selection* (UDS, McClintock, Tzur, Xin, & Si, 2011). It involves explicitly distinguishing operations on CUs from operations on 1s, and operating multiplicatively on the *difference* of 1s between two sets of CUs. Typical tasks include, "You have $7T_5$ and I have $4T_5$; how are our collections similar? Different? How many more cubes do you have?" (Note: Sets may differ in number of CUs, or in unit rate, or in both.) The child's goal is to specify the similarities and differences, and to figure out the difference in 1s between the two sets. The child's activity can include (a) operating multiplicatively on each set to find its total of 1s and then find the difference (*Total-First* strategy) or (b) finding the difference in CUs and then multiplying it by the unit rate (*Difference-First* strategy). We promote use and coordination of both. When established, UDS includes a situation recognized as two sets of CUs that can be similar or different with respect to quantities that constitute each set. UDS provides a basis for the distributive property of multiplication over addition (e.g., 7x5+4x5=5(7+4)) and for solving algebraic equations such as 7x-4x=15.

The **fourth scheme** is termed *Mixed-Unit Coordination* (MUC, Tzur, Xin, Si, Woodward, & Jin, 2009). After UDS has enabled distinguishing CUs from 1s, MUC involves operating on 1s to answer questions about CUs in two sets. Typical tasks include, “You have $7T_5$; I’ll give you *10 more cubes*; if you put these 10 cubes in T_5, how many towers would you have in all?” (Note: The question can be, “How many cubes would you have in all?”) The child’s goal is to figure out the number of CUs (or of 1s) in a ‘global’ CU combined of both given quantities. To this end, the child’s activity includes selection and coordination of the unit rate (e.g., 5) from the given set with a segmenting operation on the given number of 1s to yield the additional number of CUs (2 towers), and then adding this newly found set of CUs to the initially given set (2+7=9 towers). MUC includes a situation recognized as one set of CUs and another CU composed of 1s. MUC supports the segmenting of a CU of 1s based on a given unit rate, which is a precursor to partitioning a totality as required for division.

The **fifth scheme** is termed *Quotitive Division* (QD). It involves operating on a given CU of 1s (say, 28 cubes) in anticipation of the count of iterations of a sub-CU (T_4). Typical tasks include, “You have 28 cubes; pretend you’ll take them back to the box in towers of 4 cubes each. How many towers will you take back?” The child’s goal is to figure out how many sub-CUs constitute the given total, and the activity is mDC regulated for stoppage when accruing and given totals are equal. When established, a QD scheme reverses mDC. QD provides a basis for conceiving of division as an inverse operation to multiplication, and thus for using fact “families” of the latter to solve division problems in which the total and the size of each group is given. While playing a game in which children *posed* PGBM tasks, with conditions specified about the fit between the given totality and sub-CUs (e.g., you need to give me a total and a number of cubes in each tower so when I run out of cubes there will still be 2 cubes left), we also fostered a conceptual prerequisite for division with remainders.

The **sixth scheme** is termed *Partitive Division* (PD). Similar to QD, it involves recognizing a situation with a given totality of 1s. However, the other aspect of the situation a child must recognize is that a given number of sub-CUs requires accomplishing the goal of figuring out the equal-size of each. A typical task would be “You want to put 28 cubes in 4 equal towers. How many cubes will you have in each tower?” Initially, children may accomplish the goal by the activity of distributing all given 1s to each group one after another. Given constraints (e.g., “Do you think there would be more than one cube in each tower? Will 3 cubes work? Why?”), children with whom we worked began to anticipate that each round of distribution of 1s would yield a composite unit. They then could double-count to figure out the end result (unit rate) *without* carrying out the distribution—the essence of the PD scheme. PD provides a basis for seeing division as a twofold (QD/PD) inverse of multiplication, and corresponding algebraic operations with equations.

DISCUSSION

The developmental framework of schemes and tasks presented in this paper makes two main contributions. For research and theory building, it demonstrates how the stance that “*the task is not the child’s thinking*” can be applied to children’s learning of a foundational way of reasoning. Thus, studying transformations in schemes can be done via design and use of task sequences that occasion, but do not determine, children's spontaneous and/or prompted thought processes. Reflexively, task design can be guided by conceptual analysis of scheme components to increase the likelihood of promoting, and hence detecting, particular scheme transformations.

For teaching and teacher education, our framework provides general and content specific guidelines for promoting multiplicative reasoning in students and teachers. A key principle indicated by the framework is the need to analyze students' existing schemes. Such analysis supports using tasks that deliberately reactivate those schemes as a means to foster construction of more advanced schemes, while keeping in mind the gradual nature of such advances. For example, two 4th graders with whom we worked solved the task, "*Pretend* you have $9T_3$; together you and I have $14T_3$; how many T_3 do I have?" by counting-up on their fingers ("9; 10-11-12-13-14; so that's $5T_3$"). But when asked a structurally similar task (*adult's perspective*!) with $19T_3$ and $24T_3$, they had no idea how to proceed. One of them could later solve it after drawing the first set of CUs, whereas the other could only do so after producing all towers (tangible objects). Our framework provides a basis for designing tasks, and variations, that address such gradations and individual differences.

References

Behr, M.J., Harel, G., Post, T.R., & Lesh, R.A. (1994). Units of quantity: A conceptual basis common to addivitve and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 121-176). Albany, NY: State University of New York.

Carpenter, T.P., Franke, M.L., Jacobs, V.R., Fennema, E., & Empson, S.B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Carpenter, T.P., Hiebert, J., & Moser, J.M. (1983). The effect of instruction on children's solutions of addition and subtraction word problems. *Educational Studies in Mathematics, 14*, 55-72.

Harel, G., & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: State University of New York.

McClintock, E., Tzur, R., Xin, Y. P., & Si, L. (2011). Engendering multiplicative reasoning in students with learning disabilities in mathematics: Sam's computer-assisted transition to anticipatory unit differentiation-and-selection. In L.R. Wiest & T.D. Lamberg (Eds.), *Proceedings of PME-NA* (pp. 164-172). Reno, NV: University of Nevada, Reno.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Schwartz, J.L. (1991). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M.J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics.

Simon, M.A. (2006). Key developmental understandings in matheamtics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359-371.

Simon, M.A., & Blume, G.W. (1994). Building and understandng multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*(5), 472-494.

Steffe, L.P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L.P., & Cobb, P. (1998). Multiplicative and divisional schemes. *Focus on Learning Problems in Mathematics, 20*(1), 45-62.

Thompson, P. W., & Saldnha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick & G. Martin (Eds.), *Research companion to the NCTM Standards* (pp. 95-113). Washington, DC: National Council of Teachers of Mathematics.

Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as Zone of Proximal Development correlate in constructing counting-on: A plausible conceptual source for children's transitory 'regress' to counting-all. *Journal for Research in Mathematics Education, 42*(5), 418-450.

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2*, 287-304.

Tzur, R., Xin, Y. P., Si, L., Woodward, J., & Jin, X. (2009). Promoting transition from participatory to anticipatory stage: Chad's case of multiplicative mixed-unit coordination (MMUC). In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of PME* (Vol. 5, pp. 249-256). Thessaloniki, Greece.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer.

Watson, A., & Mason, J. (1998). *Questions and Prompts for Mathematical Thinking*. Derby, UK: Anne Watson and John Mason.

Woodward, J., Kenney, R., Zhang, D., Guebert, A., Cetintas, S., Tzur, R., et al. (2009). Conceptually based task design: Megan's progress to the anticipatory stage of multiplicative double counting (mDC). In S. L. Swars, D. W. Stinson & S. Lemons-Smith (Eds.), *Proceedings of PME-NA* (Vol. 5, pp. 1378-1385). Atlanta, GA: Georgia State University.

Xin, Y. P. (2008). The effect of schema-based instruction in solving word problems: An emphasis on pre-algebraic conceptualization of multiplicative relations. *Journal for Research in Mathematics Education, 39*(5), 526-551.

---

[i] This research is supported in part by the US National Science Foundation grant DRL-0822296. The opinions expressed do not necessarily reflect the views of the Foundation.

ARGUMENTATION IN UNDERGRADUATE MATH COURSES :A STUDY ON PROOF GENERATION

Behiye Ubuz — Middle East Technical University
Saygin Dincer and Ali Bulbul — Hacettepe University

The purpose of this study is to analyze the complex argumentative structure in undergraduate mathematics classroom conversations by taking into consideration students' and teacher' utterances in the classroom using field-independent Toulmin's theory of argumentation . The analyses contributed to an emerging body of research on classroom conversations.

INTRODUCTION

Proof is central to university mathematics courses and widely agreed to be central to the activity of mathematicians. It is, however, a notoriously difficult concept for even undergraduate students to learn (Alcock & Simpson, 2004, 2005; Epp, 1998; Jones, 2000; Larsen & Zandieh, 2008; Leron, 1985; Mejia-Ramos & Inglis, 2009; Moore, 1994; Portnoy et al., 2006; Smith, 2006; Segal, 2000; Uhlig, 2002; Weber 2001, 2004). In the early years of research at the university level, proof generation is discussed within a framework of formal logic. Although formal mathematics builds on formal logic, formal logic does not seem adequate to analyse proof generation especially in classroom for two main reasons. First, students are in the process of developing logical thinking patterns, and so the thinking they express in classrooms includes many elements which a logical analysis would simply describe as ''illogical'' but which are nevertheless important to the future development of their thinking (Knipping, 2008). Second, no formal logic captures all of the nuance of natural language because formal logic is the study of inference with purely formal content. Formal logic is inadequate to capture some aspects of students' arguments in proof generation. As a result of these reasons, in recent years many researchers studying proof generation have conducted their studies using field-independent Toulmin model (1958) which has made great contributions to informal logic. As Toulmin's model is intended to be applicable to arguments in any field, it has provided researchers in mathematics education with a useful tool for research, including formal and informal arguments in classrooms (Knipping, 2008). Studies using Toulmin model focused on analyzing students' arguments and argumentations in proving processes in a classroom (Knipping, 2002, 2008; Krummheuer, 1995) and, individual students' arguments in proving processes (Pedemonte, 2007). Toulmin himself noted that his ideas has no finality. Indeed his model has been reshaped in various ways, his claims have been contested by some and in response reformulated by others, and some but not all aspects of his approach have been incorporated in applications in different domains (Hitchcock & Verheij, 2006).

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 163-170. Taipei, Taiwan: PME.

In mathematics education literature the argument concept is used in the sense of justifying a conclusion based on a data (Toulmin 1958; Mejia-Ramos ve Inglis, 2009; Knipping, 2008). On the other hand, argumentation is a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a rational judge' (Eemeren et al, 1996).

Having established these facts, the goal of our research is to study the argumentation generated in undergraduate mathematics classrooms using Toulmin's theory of argumentation while proof generation. Specifically, the aim is to analyze the structure of the argument accomplished in the course of interaction, and the teacher and students involvement in its interactive production. This study is part of a wider study investigating the argumentation generated in undergraduate math classes while proof generation, putting a definition, and problem solving. Here we concentrate only proof generation because of page restrictions. This paper suggests a method by which complex argumentation in proving processes can be reconstructed and analyzed. Analyzing students' and teacher' utterances in the classroom according to Toulmin model allows us to reconstruct argumentations evolving in the classroom talk since arguments are produced by several students together, guided by the teacher.

THEORETICAL FRAMEWORK

In the following sections we will expose some theoretical considerations on the Toulmin model, and the use of it in classroom proving process.

The Toulmin Model

According to Toulmin, an argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one (Toulmin, 2003). He is interested in the finer structure. The Toulmin model is differed from analysis of Arisitotle's logic from premises to conclusion. First, we make a claim(C) by asserting something. For the challenger who asks "What have you got to go on ?", the facts we appeal to as foundation for our claim is called data (D) by Toulmin. After producing our data, we may being asked another question like "How do you get there ?". He notes, at this point we have to show that the step from our data to our conclusion is appropriate one by giving different kind of propositions like rules, principals, inference – licenses or what you will, instead of additional items of information (Toulmin, 2003). A proposition of this form Toulmin calls a warrant (W). He notes that warrants are of different kinds and may confer different degrees of force on the conclusions they justify. We may have to put in a qualifier (Q) such as "necessarily", "probably" or "presumably" to the degree of force which our data confer on our claim in virtue of our warrant. However there may be cases such that the exceptional conditions which might be capable of defeating or rebutting the warranted conclusion. These exceptional conditions Toulmin calls as rebuttal (R). For our challenger may question the general acceptability of our warrant: "Why do you think that?" Toulmin calls our answer to this question our backing (B) (Hitchcock & Verheij, 2006). The diagram of the Toulmin model is as follows :

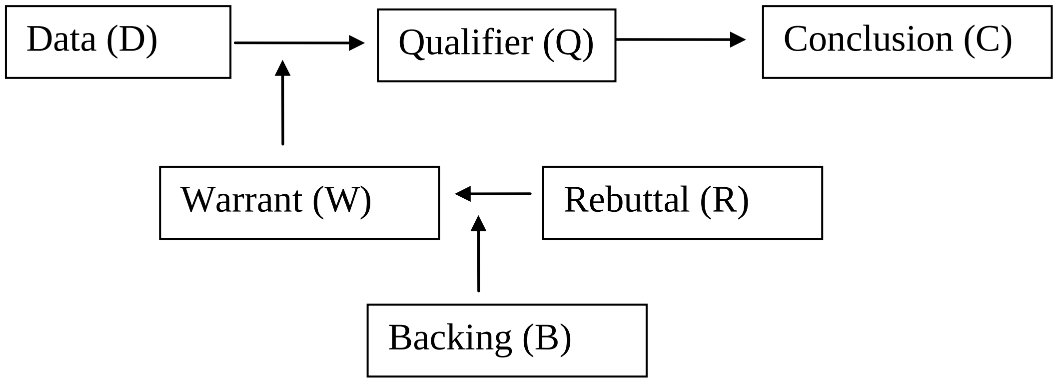


Figure 1: The Toulmin Model

Argumentation in Classroom Proving Process

In the educational literature analysing and documenting how learning progresses in a classroom using Toulmin's model (1958) has been conducted in a primary mathematics classroom (Krummheuer,1995, 2007; Whitenack & Knipping, 2002) , secondary school classroom (Knipping, 2008), and university classrooms (Stephan & Rasmussen, 2002).

Krummheuer (1995) used the Toulmin model in an restricted form. In his observations, rebuttal and qualifier components disappear. He focused argumentations whose structure consists of data, warrant and conclusion. He calls the argument pattern consisting of data, warrant and conclusion as the core of an argumentation, and this core is the minimal form of an argumentation. Krummheuer (2007) confined himself to the four central categories of an argumentation in the sense of Toulmin. They are: data, conclusion, warrant, backing. The teacher deliberately stresses the argumentative aspect of explaining a solution that came up. This opens the scope of abductively based reflection concerning the accounting practice and its impact on the conditions of the possibility of learning mathematics. Whitenack and Knipping (2002) uses the "core" concept of an argumentation proposed by Krummheuer (1995) and backing components of Toulmin model to analyze argumentation structures .

Knipping (2008) analyzed argumentation structures focusing particularly on certain types of argumentation steps that are present in some structures but not in others. Careful analyses of the types of warrants (and backings) that students and teachers employ in classroom situations allowed two distinctions in the justifications: visual and conceptual. The warrants and backings based on conceptual aspect or deductive are mathematical concepts or mathematical relations between concepts, and make reference to theorems, definitions, axioms and rules of logic. The warrants and backings based on visual or figural aspect make reference to figures as part of the argumentation.

Stephan and Rasmussen (2002) analyzed argumentation structures using the core of argumentation in the sense of Krummheuer (1995) and the validity of an argument. The backing component of Toulmin model provides the validity of the core of the argument. Mathematical ideas become taken – as – shared when one of the following two cases occurs : First, when the backings and / or warrants disappears in students' explanations in an argumentation, the mathematical idea based on the backings and/or warrants become taken – as – shared. Second, when the four components (data, warrant, conclusion, backing) of an argument moves within the subsequent argument keeping their positions.

All the above studies reveal that reconstructing and analyzing the complex argumentative structure in classroom conversations follow their own structure.

METHODOLOGY

Data were collected through nonparticipant observations that were videotaped. Observation was conducted 2009-2010 spring semesters in real analysis course for eight weeks, and 2010-2011 spring semesters in advanced calculus course for six weeks, offered to mathematics education student at the third and second years, respectively. These courses were selected as both formal and informal argumentations were at the focus of these courses. In these courses the number of students were 45 and 40, respectively. Formal proof approaches are given to the students at the "Abstract Mathematics I - II" courses provided in the first year. In these courses, students learn what a proof is and how to prove theorems. That is, they learn how to argue mathematically, justify their claims and encounter the cases named "counter example" for the first time which rebuttals their claims.

The analysis of the observations is based on the transcripts. As Toulmin(2003) noted, "an argument is like an organism. When set out explicitly in all its detail, it may occupy a number of printed pages or take perhaps a quarter of an hour to deliver; and within this time or space one can distinguish the main phases marking the progress of the argument from the initial statement of an unsettled problem to the final presentation of a conclusion" (p. 87). Based on this explanation, eleven argumentations were determined and four of them were on proof generation. These four argumentations were observed in real analysis course.

Observations were conducted by the second author. He analyzed the transcripts by marking the progress of the argument from the initial statement to the final conclusion through using Toulmin model components. He noticed that some aspects of observed argumentations were overlooked. He modified the Toulmin model by integrating additional components (guide – backing and guide – redirecting) which were observed in almost all argumentations. We called an approval given by teacher to the warrants, backings or intermediate conclusion as guide – backing and when the argumentation does not start from a right point or students get stuck on an argument point, teacher intervenes with an example, a question or a suggestion to arrange the argument. We called such intervenes as guide – redirecting.

Having discussed with the first author who is a full professor in mathematics education, it was decided that observed argumentations could be considered into three classes: proof generation, putting a definition, and problem solving. Hereafter having discussed with the third author who is a full professor in mathematics, a consensus was reached on the requirement to divide the argumentations into three classes mentioned above. Again having discussed with the first author, she noted that some components could be classified in itself. After re-analyzing observed argumentations, warrant component were divided in two categories: *deductive warrant* and *reference warrant*. Students appeal reasoning like numerical computing, applying a rule to an inequality, creating new ideas from a definition, a theorem or a rule in producing their warrants. We called this kind of warrants as *deductive warrants* as Inglis (2007) did. When a warrant referred to a theorem, a definition, a rule or a problem, we called such a warrant as *reference warrant*. Guide – backing was divided into three categories: *approval*, *reference* and *terminator*. When teacher just approve the students' warrant, backing or conclusion by saying "good, fine, great, well done" and does not use any mathematical phrase, we called this kind of guide backing as *approval guide backing*. When teacher approve the students' warrant, backing or conclusion by referring a definition, a theorem or a problem recently solved, we called this kind of guide backing as *reference guide backing*. Argumentations come to an end when teacher or students reach the final conclusion to be achieved. In case, teacher reaches the final conclusion, students convince that the conclusion is legitimate. In case, students reach the final conclusion, teacher serves a backing. This backing shows the final conclusion and we called it as *terminator guide backing*. One important point that must be noted here is that argumentations were not analyzed according to their mathematically correctness.

Finally, full transcriptions together with analysis model components explanation are provided to an external auditor who is a researcher in mathematics education field. After a week, the auditor completed her analysis and a complete consensus was reached on analysis of argumentations.

RESULT

Four open-ended problems requiring the production of proof constituted four different argumentation context. In this paper only one of these problems is considered as example because of page restrictions. Here we analyze a transcript of a short argumentation in which deductive warrant, reference warrant and terminator guide – backing appear. The following argumentation occurred when proving that any finite set is closed in R^n.

1 Teac: Well…What is your opinion ?

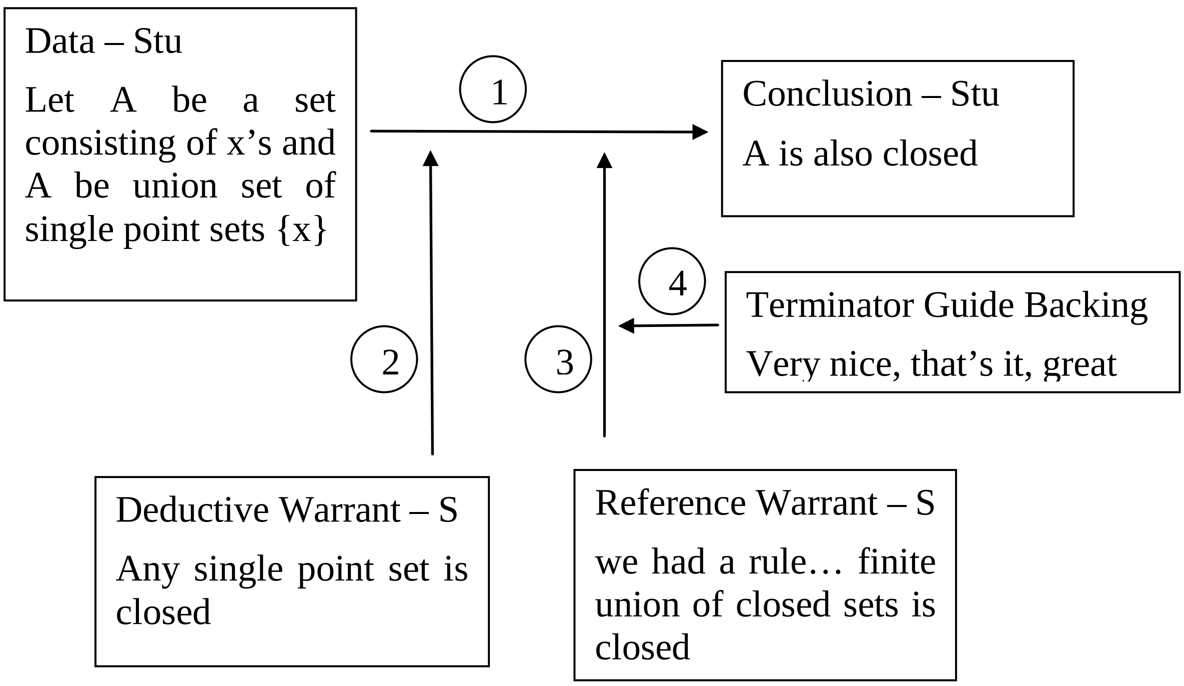
2 Stu: Let's show that any set of single point is closed.

3 Teac: We already did it.

4 Stu: Let A be a set consisting of x's and A be union set of single point sets {x}. Any single point set is closed. Since it is finite, umm, we had a rule. Since any, no, finite union of closed sets is closed, A is also closed.

5 Teac: Very nice, that's it, great!

In line 2, the student intended to show that any single point is closed. She considered it as a deductive argument. Teacher noted that they already did it in line 3. Letting A be a set consisting of x's and taking A as the union of sets {x}, she produced her data. Then, she started to reason in line 4 in which she produced a deductive warrant (any single point set is closed) and a reference warrant (we had a rule… finite union of closed sets is closed). At the end of line 4 , she arrived the final conclusion (A is also closed). In line 5, teacher gives a terminal backing – guide by using phrases "very nice, that's it, great" since these phrases marks the final conclusion. Therefore, the conclusion of her is valid. We observed that student producing deductive and/or reference warrant get easily the final conclusion after getting terminator backing guide. We think that if students have an ability to produce deductive and/or reference warrants and get any kind of guide – backing, then he/she could get easily the final conclusion. We also think, guide – backing and guide – redirecting components prevent emergence of qualifier component. The diagram corresponding to the argumentation above is as follows :



CONCLUSIONS

The model of Toulmin, which is helpful for reconstructing argumentation steps and streams, is not adequate for more complex argumentation structures. Analyzing proving processes in classrooms requires a different model for capturing the global structure of the argumentations developed there. Analyzing students' and teacher' utterances in the class according to the Toulmin model allowed us to reconstruct argumentations evolving in the classroom talk. Argumentations in classrooms are generally teacher guided. Teacher acts as a guide who exactly knows the path to follow i.e. where to start and to end the argumentation. During the argumentation if students follow the wrong path, get a false intermediate conclusion or get stuck in a point , teacher intervene the students to put them on the path in which they have to follow. According to this and based on our observations, roles of teacher played in argumentation like guide – backing and guide – redirecting. Careful analyses of the types of warrants that students and/or teachers employ in classroom situations allowed us to identify and classify the field of justification that applies in that classroom: deductive warrant and reference warrant. Therefore, in our view, analyzing argumentations in classrooms through the Toulmin model needs to be reinterpreted.

References

Alcock, L., Weber, K., 2005, Proof validation in real analysis: Inferring and checking warrants.*Journal of Mathematical Behavior*, 24, 125–134.

Alcock, L., Simpson, A. , 2004, Convergence of sequences and series 1: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, 57, 1 - 32.

Eemeren, F.H. van, Grotendorst, R., Snoeck Henkemans, F. et al, 1996. Fundamentals of Argumentation Theory. A Handbook of Historical Backgrounds and Contemporary Developments. Mahwah, NJ: Erlbaum.

Epp, S., 1998, A unified framework for proof and disproof. *Mathematics Teacher,* 91,8, 708–713.

Fraenkel, J. R., Wallen, N., E., 2006, How To Design and Evaluate Research in Education, McGraw – Hill International Edition.

Hitchcock D., Verheij, B., 2006, Introduction, Arguing on the Toulmin Model New Essays in Argument Analysis and Evaluation, Springer, 1 - 23.

Jones, K, 2000, The student experience of mathematical proof at university level. *International Journal of Mathematical Education in Science and Technology*, 31, 53 – 60.

Larsen, S., Zandieh, M., 2008, Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, 67(3), 205–216.

Leron, U., 1985, A direct approach to indirect proofs. *Educational Studies in Mathematics*, 16(3), 321-325.

Moore R. C., 1994, Making the transition to formal Proof. *Educational Studies in Mathematics*, 27(3), 249-266.

Knipping, C., 2008, A method for revealing structures of argumentations in classroom proving processes. *ZDM -Mathematics Education*, 40, 427–441.

Knipping, C. 2002, Proof and proving processes: teaching geometry in france and germany. In H.-G. Weigand (Ed.), Developments in mathematics education in German-speaking Countries. Selected papers from the annual conference on didactics of mathematics, Bern 1999 (pp. 44–54). Hildesheim: Franzbecker Verlag.

Krummheuer, G., 2007, Argumentation and participation in the primary mathematics classroom : Two episodes and related theoretical abductions. *Journal of Mathematical Behavior*, 60-82.

Krummheuer, G., 1995, The ethnography of argumentation. In P. Cobb, & H. Bauersfeld (Eds.) The emergence of mathematical meaning: Interaction in classroom cultures. Hillsdale, NJ: Lawrence Erlbaum Associates.

Mejia-Ramos, J. B., Inglis, M., 2009, What are the argumentative activities associated with proof ? *Research in Mathematics Education*, 11(1), 77- 78.

Pedemonte, B., 2007, How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66(1), 23–41.

Portnoy, N., Grundmeier, T. A., Graham, J. G., 2006, Students' understanding of mathematical objects in the context of transformational geometry: Implications for constructing and understanding proofs. *Journal of Mathematical Behavior*, 25, 196–207.

Segal, J., 2000, Learning about mathematical proof: Conviction and validity. *Journal Of Mathematical Behavior*, 18 (2), 191-210.

Smith, J. C., 2006, A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior*, 25, 73–90.

Stephan, M., Rasmussen, C., 2002, Classroom mathematical practices in differential equations. *Journal of Mathematical Behaivor*, 21, 459 – 490.

Toulmin, S., 2006, Reasoning in Theory and Practice, Arguing on the Toulmin Model New Essays in Argument Analysis and Evaluation, Springer, 25 – 29.

Toulmin, S., 2003, The Uses of Argument, updated edition. Cambridge, UK: Cambridge University Press.

Toulmin, S., 1958, The Uses of Argument. Cambridge, UK: Cambridge University Press.

Weber, K., 2005, Problem - solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior*, 24, 351–360.

Weber, K., 2004, Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior* , 23, 115–133.

Weber, K., 2001, Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48, 101–119.

Whitenack, J.W. & Knipping, N, 2002, Argumentation, instructional design theory and students' mathematical learning: a case for coordinating interpretive lenses. *Journal of Mathematical Behaivor*, 21, 441 – 457.

PATTERN PROBLEM SOLVING TASKS AS A MEAN TO FOSTER CREATIVITY IN MATHEMATICS

Isabel Vale[1], Teresa Pimentel[1], Isabel Cabrita[2] , Ana Barbosa[1], Lina Fonseca[1]

[1]Polytechnic Institute of Viana do Castelo, [2]University of Aveiro

In this report we explore the implications of some challenging pattern problem solving tasks in the development of mathematical ability and creativity of students (future teachers). Solving challenging tasks usually requires creative thinking and our recent work in a project about patterns in the teaching and learning of mathematics showed that patterns can give a positive contribution to the development of mathematical ability and creativity for all students. So our main concern is to analyze, through some elementary classroom episodes, the contribution of pattern tasks to promote creative solutions by students.

INTRODUCTION

The major purpose of teachers is that students develop an increasing mathematical ability that allows them to solve the different problems they face inside and outside school. Innovation and creativity play an important role, being a dynamic characteristic that students must develop. Creativity begins with curiosity and engages students in exploration and experimentation tasks where they can translate their imagination and originality (Barbeau & Taylor, 2005). Research findings show that mathematical problem solving and problem posing are closely related to creativity (e.g. Pehkonen, 1997; Silver, 1997). So, learning environments with problem solving/posing activity should be used in our classes in order to develop students' creativity. Challenging tasks usually require creative thinking and our recent work in a project about patterns in the teaching and learning of Mathematics showed that patterns can contribute to the development of mathematical ability and creativity of students. So our purpose as mathematics educators is to provide all students (including future teachers) creative approaches for solving any problems and to think independently and critically. This way, future teachers should themselves develop these skills and go through the same type of tasks that they will offer their students.

THEORETICAL FRAMEWORK

Creativity, problem solving and patterns

Mathematical creativity is a rather complex phenomenon. Mann (2006), in an examination of the research about how to define mathematical creativity, found that there is a lack of an accepted definition for mathematical creativity since there are numerous ways to express it. But we can notice that there are some commonalities in

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 171-178. Taipei, Taiwan: PME.

the different attempts to define creativity that are: (1) it involves divergent and convergent thinking; (2) it has mainly three components/dimensions that are fluency, flexibility, and originality (novelty); and (3) it is related to problem solving and problem posing (including elaboration and generalization).

(1) Divergent and convergent thinking are both important aspects of intelligence, problem solving and critical thinking. Convergent thinking is a way of thinking oriented to obtain a single response to a situation. The solver is good at bringing material from a variety of sources to bear on a problem, in such a way as to produce the "correct" answer. It usually involves a thinking process that follows some set of rules or logic, while divergent thinking looks towards the problem, analyzing all the possible solutions and seeking the best solution to the problem. Here the solver is in broadly creative elaboration of ideas prompted by a stimulus. It is the opposite of convergent thinking, a creative process that involves trying to imagine as many possible solutions as one can. In contrast to convergent thinking, divergent thinking is usually more spontaneous and free-flow. People who have divergent thinking try to keep their mind open to any possibilities that are presented to them. The more possibilities they come up with, the better their divergent thinking is. Divergence is usually indicated by the ability to generate many, or more complex or complicated, ideas from one idea (Hudson, 1967).

(2) Components of Creativity: fluency, flexibility and originality. *Fluency* is the ability to generate a great number of ideas and refers to the continuity of those ideas, flow of associations, and use of basic knowledge. Silver (1997) defines it as apparent shifts in approaches taken when generating responses. *Flexibility* is the ability to produce different categories or perceptions whereby there is a variety of different ideas about the same problem or thing. It reflects when students show the capacity of changing ideas among solutions. *Originality* is the ability to create fresh, unique, unusual, totally new, or extremely different ideas or products. It refers to a unique way of thinking. With regard to mathematics classrooms, originality may be manifested when a student analyzes many solutions to a problem, methods or answers, and then creates another one different.

(3) Research has shown that the formulation and solution of problems in mathematics are closely related to creativity (Barbeau & Taylor, 2005; Silver, 1997). Tasks that can promote the above dimensions must be open-ended and ill structured, assuming the form of problem solving, problem posing (including elaboration and generalization) and mathematical explorations and investigations. Rather than closed problems with a single solution, students should be provided open-ended problems with a range of alternative solution methods (Fouche, 1993, as cited in Mann, 2006). Problem posing can be a powerful strategy to develop problem solving skills and to have good problem solvers; on the other hand, to formulate meaningful mathematical problems, it is necessary to be a good problem solver.

Patterns are a powerful tool in the mathematics classroom and can suggest several approaches, as well as they permeate all mathematics, and their study makes possible to get powerful mathematical ideas as generalization and algebraic thinking, where visualization can play an important role. Indeed, according to several authors, patterning tasks have creative potential as they may be open-ended, allow a depth and variety of connections with all topics of mathematics, both to prepare students for further learning and to develop skills of problem solving and posing, as well as communication (NCTM, 2000; Orton, 1999). Figure 1 summarized the ideas above.

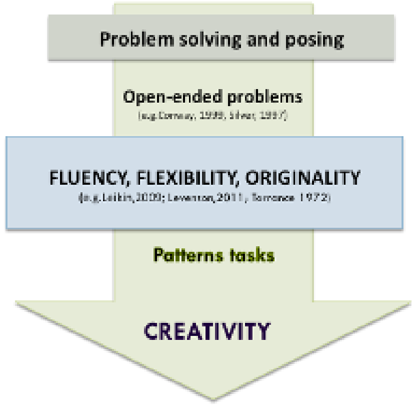


Figure 1: A path to creativity

Teachers and creativity

Learning heavily depends on teachers. One of the major obstacles to reforms is teachers' lack of familiarity with innovative instructional practices and tools. Teachers must have an in-depth understanding of fundamental mathematics and of the mathematical thinking of their students to support the development of their mathematical competence (Hiebert et al., 2007; Ma, 1999). The basic purpose of a math class is that students learn something about a particular topic that was planned by the teacher. Teachers must interpret the curriculum and select *good* curricular materials and strategies to use in the classroom. To achieve this, teachers should propose tasks involving students in a creative form, and also be mathematically competent to analyze their students' resolutions. Research shows that what students learn is greatly influenced by the tasks they are given (e.g. Doyle, 1988, Stein & Smith, 2009). Therefore, it is important to have good mathematical tasks. A task is good when it serves to introduce fundamental mathematical ideas, is an intellectual challenge for students and allows different approaches (NCTM, 2000). Tasks must develop new approaches and creative ideas, so they must provide multiple solutions in order to raise the student flow of mathematical ideas, flexibility of thought and originality in the responses. Teachers must encourage students to create, share and solve their own problems, as this is a very rich learning environment for the development of their ability to solve problems and their mathematical knowledge. Creativity is a dynamic characteristic that students can develop if teachers provide them appropriate learning opportunities (e.g. Leikin, 2009). Creativity is a topic that is often neglected within their mathematics teaching usually because they didn't realize its importance in mathematics and mathematics education. Creativity should be an intrinsic part of mathematics for all programs (Pehkonen, 1997).

METHODOLOGY

We adopted a qualitative exploratory approach with elementary pre-service teachers, to understand in what way a didactical experience through challenging tasks, grounded on figural pattern problems, is a suitable context for promoting creativity in students solutions, in particular in getting creativity ways of expression of generalization. This proposal emphasizes the figurative contexts of patterning as a way to reach generalization, through meaningful representations, in particular the algebraic (or numeric) expressions. The aim of this exploratory study is to note some of the diversified views from the perspectives of pre-service mathematics teachers on improving the creative thinking in solving patterning tasks, in figurative problem solving contexts. Our two main questions were: Did the pattern tasks in figurative contexts promote multiple solutions? How can we characterize creativity when students solve challenge pattern tasks in figurative contexts? The participants in the study were twenty-three elementary mathematics pre-service teachers during the didactics of mathematics classes of the 3rd academic year. The pattern didactical proposal has a sequence of tasks: counting, repeating and growing patterns, and pattern problems. However, in this paper we will only analyze the counting and growing pattern tasks. The data collected in a holistic, descriptive and interpretive way includes classroom observations, notes and documents (e.g. worksheets, tests, individual works).

The major difficulty that we found was how to measure creativity. As we are uncomfortable yet with a psychometric evaluation in this first approach, we chose to follow the basic ideas of the authors (Conway, 1999; Silver, 1997) without assigning a score to each student but making a global analysis. We followed the suggestion to measure creativity of students through the three dimensions according to Silver (1997) and Conway (1999). They suggested that fluency can be measured by the number of correct responses, solutions, obtained by the student to the same task in a process that Silver (1997) describes as multiple solution task. Flexibility can be measured with the number of different solutions that the student can produce organized in different categories or perceptions, whereby there are a variety of different ideas about the same problem or thing; that is, analyzing the number of different categories. And originality can be measured analyzing the number of responses in the categories that were identified as original, by comparison with the percentage of students in the same group that could produce the same solution. This mean can be assessed, as is the statistical infrequency of responses in relation to peer group responses. It is rather difficult to measure mainly the component of originality, which is for many authors the dimension that should out-top (Besemer & O'Quin, 1999). As these authors refer, this category can be highlighted by asking, "how often would this solution be found?" To overcome this difficulty where suggested to hear the opinion from a peer about the resolution. In this exploratory study we didn't measure the solutions but only analyzed them globally: the most common and the most original according to the frequency of the responses.

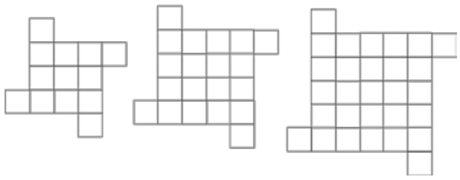
RESULTS AND DISCUSSION

During the didactical experience several tasks of different kinds were used. We present here three of those pattern tasks that require producing various and different responses.

In this paper, we shall highlight the creative responses of the group of students involved in the modeling tasks. The description of the tasks is in Figure 2.

<table>
<tr>
<td>Task 1 - The shells

The sea girl organized the shells she caught yesterday like the figure shows. Can you find a quick process to count them? Discover as much ways as you can.
</td>
<td>Task 2- Squares

Observe the growing pattern.

1. Draw the next figure.
2. Write the expression of the nth term.</td>
</tr>
<tr>
<td colspan="2">Task 3 - Dots

Observe the dots in the figure.

1. Imagine that this is the 1st term of a sequence. Draw the next terms.
2. Write a numerical expression translating a way to calculate the nth number term of the sequence.
3. Imagine that the sequence you draw began with the 2nd term. Draw the 1st term.</td>
</tr>
</table>

Figure 2: Pattern tasks

Task 1. This type of task requires students to see the arrangement in different ways connecting previous knowledge about numbers relationships and their connections with basic geometric concepts. There are different ways to count the arrangement of the shells and each counting can be respectively written through a numerical expression that translates the students' thinking and seeing. Figure 3 illustrates the summary of the most common resolutions, with the expressions corresponding to each way of "seeing".

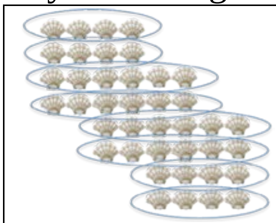
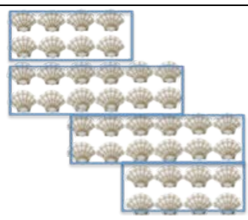
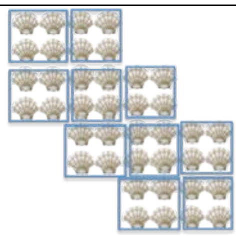
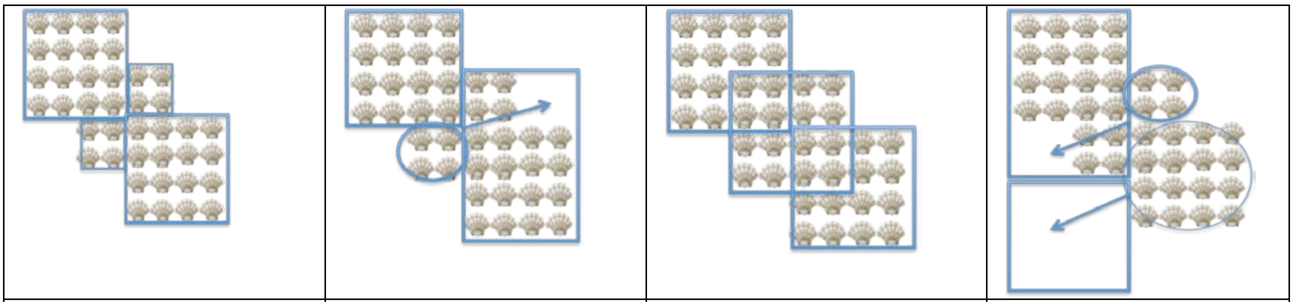
			
4+4+6+6+6+6+4+4 (20)	2x4+2x6+2x6+2x4 (16)	10 x (2x2) (10)	4x2+6x2+6x2+4x2 (9)

Figure 3: Summary of students' most common responses on task1

These expressions can be verbalized as the following: "I see the shells in horizontal rows each one with 4, 4, 6, 6, 6, 4 and 4 shells" or "One rectangle of 2 by 4, another rectangle of 2 by 6, another of 2 by 6 and a last one of 2 by 4" or "I see ten squares of 2 by 2". It is important that teachers allow students to discover that each expression illustrates one way of seeing but they are all equivalent and correspond to the same number of shells, 40. Figure 4 illustrates the most original responses.

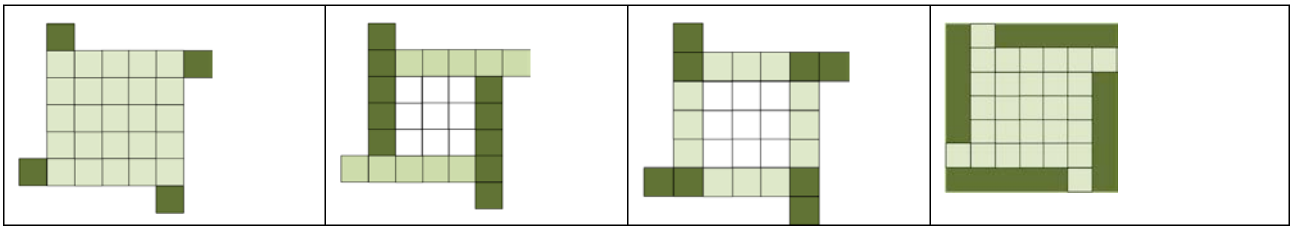


(4X4)+4+4+ (4X4) (4)	4x4+6x4 (2)	3x(4x4) – (4+4) (2)	10 x 4 (1)

Figure 4: Summary of students' most original responses on task1

Our expectations of students' creativity in this task lay in the different original ways of seeing/counting the number of shells. In this class we considered that these four students present the most original solutions, as has the statistical infrequency of responses in relation to peer group of responses. We claim that a previous work with counting tasks in figurative settings can be a particularly good way to develop skills of *seeing* (identification, decomposition, rearrangement) to facilitate similar processes in growing pattern tasks (Vale & Pimentel, 2011).

Task 2. We intended that students look for a pattern in a figurative sequence, describe it, and produce arguments to validate it using different representations. The previous work with visual counting may help to see a visual arrangement that changes in a predictable form and write numerical expressions translating the way of seeing, in order to make possible the generalization to distant terms. Students use different representations, more or less formal, to solve this task. They achieve a general rule through schemes and drawings or tables, but mainly using functional reasoning that allowed them to accomplish far generalization. We will regard only to the different ways of seeing the pattern to get far generalization, as we are convinced that is the most important aspect of solving this tasks in which students can be creative. Figure 5 illustrates different ways of seeing the 3rd term of the sequence.



nxn+4 (10)	nxn+4x(n+2) (3)	nxn+4x2+4xn (3)	$(n+4)^2$–4x(n+2) (1)

Figure 5: Summary of students' responses on task2

The same criterion of the previous task was used to analyze the students' work. The first solution was the most common and to get the general rule students used other representations, mainly a table to relate the number of the figure in the sequence and the number of squares, according to the way they saw the pattern. The last way was used only by one student, that applied deconstructive reasoning (Rivera, 2009).

Task 3. This task has the same objectives of the others but also formulating additional data, according to the solution presented for each student. Figure 6 synthesizes all the answers.

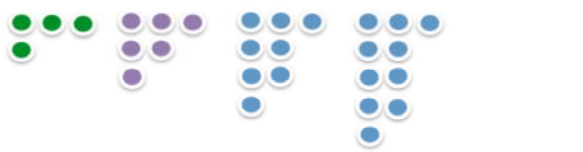
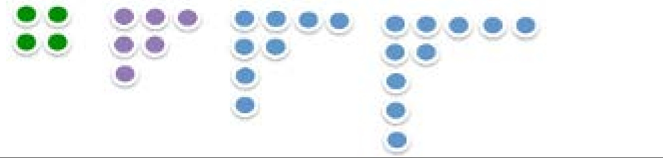
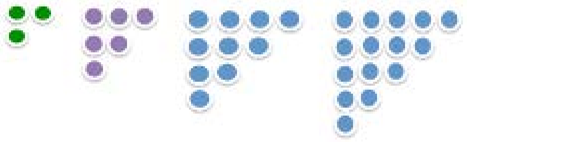
	
$3+1+(n-1)\times2$ (8)	$4+(n-1)\times2$ (7)
	
$(n+1)\,(n+2)/2$ (1)	$2+n^2$ (1)

Figure 6: Summary of students' responses on task3

This task wasn't completely solved by all the students. The table includes only 17 answers (n=21), from the students who completed the task. It was difficult for them not to invent the next terms starting from the given term, but they worked backwards to discover the new first term. The last two solutions were the most original since only two students presented them.

CONCLUDING REMARKS

Creativity is a field that we are just beginning to explore but this allowed us to experience the construction of some tasks that, in addition to the mathematical concepts and processes they involve, mainly generalization, allow students multiple solutions. We observed that two of the components of creativity, fluency and flexibility, were largely identified mainly in the counting tasks. Each task is intentionally not designed to assess only one component of mathematical creativity although, in some cases, one of the components is more relevant. We must look for ways to improve originality that in this class not had high results. Students need to be encouraged to seek unusual and original responses, since this strategy represents a way to get solutions to difficult problems or a path to creative solutions. The most successful problem solver is the individual who can apply diverse approaches (Conway, 1999). It is important that future teachers become themselves creative thinkers and they must be aware to act in the same way with their own students. They need to recognize that both flexibility and originality encourage divergent thinking, which promotes higher-level thinking. Classroom teachers should examine their teaching practices and seek out appropriate curricular materials to develop mathematical creativity. The challenge is to provide an environment of practice and problem solving that stimulates creativity that will enable the development of mathematically competence in all students. Our concern was not to categorize students but to identify potentialities in the tasks to develop creativity in students, detecting their mathematical strengths or weaknesses. This work obviously aimed also to identify potentially creative students.

References

Barbeau, E. & Taylor, P. (2005). Challenging mathematics in and beyond the classroom. Discussion document of the ICMI Study 16. http://www.amt.edu.au/icmis16.html

Besemer, S. P.& Quin, K. (1987). Creative product analysis. Testing a model by developing a judging instrument.. In S. G. Isaksen (Ed.), *Frontiers of creativity research*. Buffalo, NY: Bearly.

Conway, K. (1999). Assessing open-ended problems. *Mathematics Teaching in the Middle School*, *4*, 8, 510-514.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psycologist, 23,* 167-80.

Hiebert, J., Morris, A, Berk, D. & Jansen, A. (2007) Preparing Teachers to Learn from Teaching, *Journal of Teacher Education*, 58, 1, 47-61.

Hudson, L (1967). *Contrary Imaginations; a psychological study of the English Schoolboy* Harmondsworth: Penguin.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students*. (pp. 129-145). Rotterdam, Netherlands: Sense Publishers.

Ma, L. (1999*). Knowing and teaching mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum.

Mann, E. (2006). Creativity: The Essence of Mathematics, *Journal for the Education of the Gifted, 30*, 2, 236-260 .

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teacher of Mathematics.

Orton, A. (1999). *Pattern in Teaching and Learning of Mathematics.* London: Cassel.

Pehkonen, E. (1997). The State-of-Art in Mathematical Creativity, *ZDM, 29*, 3, electronic edition.

Rivera, F. (2009). Visuoalphanumeric mechanisms that support pattern generalization. In I. Vale & A. Barbosa (Orgs.), *Patterns: Multiple perspectives and contexts in mathematics education (pp.123-136).* Viana do Castelo: Escola Superior de Educação.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM, 3*, 75-80.

Stein, M., & Smith, M. (2009). Tarefas Matemáticas como quadro para a reflexão. *Educação e Matemática*, 22-28.

Vale, I. & Pimentel, T. (2011). Mathematical challenging tasks in elementary grades, In M. Pytlak, T. Rowland & E. Swoboda, *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education*, pp.1154-1164. Rzeszow: ERME.

HOW STUDENTS UNDERSTAND ASPECTS OF LINEARITY: SEARCHING FOR OBSTACLES IN REPRESENTATIONAL FLEXIBILITY

Wim Van Dooren[1], Dirk De Bock[12], Lieven Verschaffel[1]

[1] Centre for Instructional Psychology and Technology, University of Leuven, Belgium

[2] Hogeschool-Universiteit Brussel, Belgium

This study investigates students' conceptual understanding of linear and "almost linear" functions, and the way in which this understanding is mediated by various external representations. 65 university students made connections between a given graph, table, or formula and several other graphs, tables, or formulas. Results indicate that students made most errors for decreasing functions, where they most often confused negative affine and inverse linear functions. Accuracy and the nature of errors also strongly depended on the nature of the representational connection that had to be made.

THEORETICAL AND EMPIRICAL BACKGROUND

Mathematics educators very often emphasize the (stimulating) role of multiple external representations in mathematics. As Matteson (2006) explains, learning mathematics is like learning a foreign language. Representations are key elements in the vocabulary of that language, and students need to become fluent in their use if they want to succeed in expressing and understanding mathematical ideas with correctness and precision. After all, external representations are inherent to the discipline of mathematics, since a characteristic of any mathematical concept is that it can be represented by a set of symbolic, linguistic or graphic symbols. External representations have also been shown to facilitate mathematical problem solving (e.g. Duval 2002; Kaput 1992). The NCTM Standards (1989) therefore hold a strong plea for establishing "mathematical connections" through the use of multiple external representations:

> Different representations of problems serve as different lenses through which students interpret the problems and the solutions. If students are to become mathematically powerful, they must be flexible enough to approach situations in a variety of ways and recognize the relationships among different points of view (p. 84).

However, research has shown that students are not always fluent and flexible in using all the external representations that are available to solve a problem, and in translating between representations (e.g. Bieda & Nathan, 2009; Yerushalmy, 1991). Recently, it was also shown that students' ability to select a mathematical model for a given real-life situation is mediated by the representational mode of that model: A particular representation may highlight aspects of a model that are easily noticed by students and therefore facilitate correct reasoning, but be misleading when the given situation is represented with another model (Van Dooren, De Bock, & Verschaffel, 2011).

In this study, we focus on students' fluency in linking multiple representations of functions. The importance of this skill in the mathematics curriculum is widely acknowledged (e.g. Elia, Panaoura, Gagatsis, Gravvani, & Spyrou, 2006). In the literature, research on representational fluency and flexibility often focuses on the concept of function (e.g. DeMarois & Tall, 1999; Leinhardt, Zaslavsky, & Stein, 1990) because functions are a typical example in the domain of mathematics where different types of representations (such as graphs, formulas, and tables) can be used (Even, 1998).

More specifically, we focus on the domain of *linear* functions. The reason is that previous research has shown that students of various ages often exhibit a very limited understanding of linear situations and tend to assume linearity in situations that are not linear at all. Students' overreliance on linear models has already been studied extensively in a variety of mathematical domains (e.g., elementary arithmetic, algebra, (pre)calculus, probability, and geometry, for an overview, see Van Dooren, De Bock, Janssens, & Verschaffel, 2008). In the domain of functions, the straight line graph prototype proved to be very appealing for many students. Leinhardt et al. (1990) showed that students of different ages have a strong tendency to produce a linear pattern through the origin when asked to graph non-linear situations, such as the growth in the height of a person from birth to the age of 30.

As becomes clear from the above overview, research on improper linear reasoning so far has mostly focused on the modelling aspect, i.e. on tasks in which real-life situations have to be expressed in mathematical terms. Much less research exists on students' tendency toward improper linear reasoning in abstract mathematical tasks where no modelling of real-life situations needs to take place. One example of such research is that of Markovits, Eylon, and Bruckheimer (1986). When they asked to 14- to 15-year-old students to draw a graph of a function that passes through two given points, they typically drew straight lines. Similarly, Karplus (1979) found that when students interpolate between two graphed data points in a science experiment, they strongly tend to connect the points using a straight line.

This paper focuses on the hypothesis that students' limited conceptual understanding of the domain of linear functions will lead to difficulties in distinguishing them from other functions that are conceptually related (such as affine and inverse linear functions - see below). We assume that this limited conceptual understanding will show up when students have to connect various external representations of these functions to each other. This paper will therefore investigate how accurate students are in connecting external representations of linear and "almost linear" functions to other external representations of these same functions. Additionally, we wanted to investigate whether accuracy in connecting representations and the confusion between linear and "almost linear" functions depend on the nature of the external representations that have to be connected to each other.

METHOD

Sixty-five students following a Bachelor programme in Educational Sciences participated. They were confronted with a written multiple-choice test consisting of twelve items. In each item, a graph, formula or table was provided, that had to be linked to one of four graphs, formulas or tables. Only one of the graphs, formulas or tables accurately represented the same function as the graph, formula or table.

The test offered graphs, formulas and tables of four kinds of functions. The first type were linear functions, i.e. functions of the form $y = ax$, graphically represented by a straight line through the origin. The other types of functions were "almost linear" ones, in the sense that they share certain characteristics with linear functions but not all. The first type of "almost linear" functions were affine functions with a positive slope ($y = ax + b$ with $a > 0$ and $b \neq 0$). The second type of "almost linear" functions were affine functions with negative slope ($y = ax + b$ with $a < 0$ and $b \neq 0$). As the third type, we used inverse linear functions ($y = a/x$). Positive affine functions for instance share with linear functions the property that their graph has the shape of a straight line, and that the same increase Δx in x always results in the same increase Δy in y. But while in linear functions, doubling x implies doubling y, this does not hold for affine functions, but students may nevertheless assume this.

Examples of items are shown below, in Figure 1. The twelve items were offered in a random order to students.

Choose among the four graphs below the one that describes the same mathematical relationship as the formula $y = 1.08\ x$

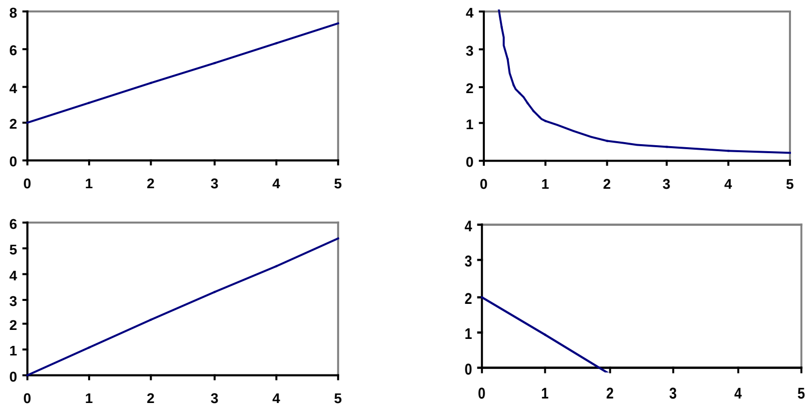


Figure 1: Example items from the multiple choice test

Data were analysed by means of a repeated measures logistic regression analysis followed by multiple pairwise comparisons. Additionally, an error analysis was conducted in order to investigate the most frequently chosen incorrect answering alternatives.

RESULTS

The logistic regression first of all showed a significant main effect of the type of function, *Chisquare* (3) = 28.322, $p < .00015$. Pairwise comparisons indicated that accuracy is considerably higher for items dealing with linear and positive affine functions (with average accuracy rates of 0.90 and 0.88, respectively) than for items dealing with negative affine and inverse linear functions (with average accuracy rates of 0. 73 and 0.77, respectively). Thus, students had more difficulties in appropriately linking representations of functions where a larger value of x implies a smaller value of y than representations of functions where a larger value of x implies a larger value of y.

Second, the logistic regression analysis indicated a main effect of the type of representational connection that students had to make, *Chisquare* (5) = 97.109, $p < .00015$. Pairwise comparisons indicated that linking a given graph to a table and linking a given table to a graph was done best by students (average accuracy of 0.95 in both cases), while linking a given formula to the correct table and a given table to the correct formula was significantly more difficult (average accuracy of 0.89 and 0.84, respectively).The worst accuracies occurred for linking a graph to a formula and a formula to a graph (accuracies of 0.76 and 0.62, respectively). This finding indicates that students can deal best with representations that involve concrete function values: Given concrete function values in a table, the correct graph and/or the correct formula can be rather easily retrieved (and reversely), while the link between graphs and formulas is more difficult, probably because there are no concrete function values given as an intermediate step containing concrete values.

Third, and most important, a significant 4 x 6 interaction effect between the type of function and the type of representational connection was found, *Chisquare* (15) = 31.314, $p = .006$. This interaction effect globally indicates that for some functions, certain representational connections are made easier than for other functions. In order to get a good understanding of this complex interaction, the accuracy rates for the various types of items and connections between representations are summarized in Table 1. To allow a proper interpretation, the date from Table 1 will be completed with data coming from an analysis of the most frequently chosen incorrect answering alternatives. As can be seen in Table 1, the items dealing with linear functions were generally solved rather good (average accuracy of 0.90), but nevertheless connecting a graph to a formula and vice versa led to errors in about 20% of the cases. A further analysis showed that when students had to select a graph given a linear formula, they often selected an affine (positive or negative) graph, probably because they were misled by the fact that these graphs are also straight lines. When selecting a formula for a given linear graph, affine formulas were hardly selected. Nearly all incorrectly selected formulas were inverse linear ones. Apparently, students were aware that they had to select a formula without an intercept, but sometimes picked a function with x in the denominator of the formula.

Also the items dealing with positive affine functions were generally solved rather good (average accuracy 0.88), but choosing the right graph for an affine formula led to a considerable number of mistakes. About half of the errors were due to the fact that students selected the graph of a negative affine function instead of the positive one, indicating that they were looking for a straight line graph that does not pass the origin, but not taking into account the positive value of the slope in the formula.

The affine items with a negative slope were the most difficult ones overall (average accuracy 0.73). When looking more closely at the types of representational connections, it becomes clear that not only linking graphs to formulas led to errors (as for the other functions), but also linking tables to the correct graph, and linking tables to the correct formula. In all these cases, about half of the errors was due to the fact that students selected the positive affine alternative. Apparently, students most often recognize the affine character of a function (be it in a graph, formula, or table), but have trouble interpreting the negative slope in the formula (a negative value of *a*). Also in the table (where the negative slope can only be noticed by seeing that larger values of *x* imply smaller values of *y*) this leads to difficulties. For the negative affine functions, a second major error was noticed: In about one third of the errors, the inverse linear alternative was chosen. These students realized that larger values of *x* implied smaller values of *y*, but then erroneously selected a hyperbola graph instead of a straight (decreasing) line, or a formula with *x* in the denominator instead of a formula with *x* in the numerator but with a negative coefficient.

Finally, the inverse linear items were generally solved slightly better than the negative affine ones (average accuracy 0.77). A closer look at the error patterns shows a quite diverse picture for the different representational connections. As for the other function types, connecting the right graph to a given formula and vice versa were the most difficult tasks for students. But it is remarkable that when students had to select a graph for a given inverse linear formula, the most frequent error (about two thirds of the errors) was the choice of the linear graph, while when students had to select a formula for a given inverse linear graph, they chose in about two thirds of the cases for the negative affine alternative. Remarkable is also that students had considerable difficulties in connecting the right table to a given inverse linear formula, while this phenomenon was not observed for the other type of functions. A closer look at the errors indicates that in almost all erroneous answers, the linear table was chosen instead of the correct one.

Table 1: Overview of accuracies for different function types and representational connections

Linear functions				
From …		To …		
		Graph	Formula	Table
	Graph		0.82	0.97
	Formula	0.80		0.94
	Table	0.97	0.92	
Affine functions positive slope				
From …		To …		
		Graph	Formula	Table
	Graph		0.88	0.97
	Formula	0.72		0.91
	Table	0.97	0.85	
Affine functions negative slope				
From …		To …		
		Graph	Formula	Table
	Graph		0.60	0.91
	Formula	0.43		0.91
	Table	0.83	0.71	
Inverse linear functions				
From …		To …		
		Graph	Formula	Table
	Graph		0.66	0.95
	Formula	0.49		0.75
	Table	0.95	0.82	

CONCLUSIONS AND DISCUSSION

The results of this study indicate that students have difficulties in distinguishing linear and various kinds of "almost linear" functions (inverse linear functions, affine functions with positive slope, and affine functions with negative slope). Particularly the decreasing functions (in which a larger value of *x* implies a smaller value of *y*) are less well understood: Affine functions with a negative slope are often linked to representations of affine items with a positive slope (which share the straight line graph that does not pass the origin) or with representations of inverse linear functions (which share their decreasing character). In the same way, inverse linear functions are often linked to representations of affine functions with a negative slope, but additionally, students often link an inverse linear function to representations of linear functions.

The representational connection that was most difficult was the one between a formula and a graph, and vice versa. All representational connections wherein a table was involved were made considerably better. The reason we suspect for this last trend is the absence of concrete function values when linking graphs and formulas. The exemplary function values that are given in a table allow a student to test very concretely which graph or formula fits (and similarly, a formula and graph can be concretely tested against a few alternative tables). For the items used in our test, an expert would be able to immediately recognize the appropriate graph for a given formula (and vice versa) without turning to concrete values, merely by comparing the formula to the global shape of the graphs. Apparently, the students involved in this study had not acquired this skill.

In further research, it would be interesting to investigate students' conceptual understanding of linear and "almost linear" functions and the way in which this understanding is mediated by the external representation by the properties of the functions. For instance, given a representation (table, graph, formula) of a linear, positive affine, negative affine or inverse linear function, one could ask students to indicate whether the statement "when *x* doubles, *y* doubles" is true. This way, a more refined understanding of the difficulties students have distinguishing linear and "almost linear" functions could be achieved.

An implication for mathematics education is the need for drawing sufficient attention to representations and for explicitly discussing differences between linear and different types of "almost linear" functions.

REFERENCES

Bieda, K. N., & Nathan, M. J. (2009). Representational disfluency in algebra: Evidence from student gestures and speech. *ZDM–The International Journal on Mathematics Education, 41*, 637–650.

DeMarois P., & Tall D. (1999). Function: Organizing principle or cognitive root? In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 257–264). Haifa, Israel: PME.

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education, 1*(2), 1–16.

Elia, I., Panaoura, A., Gagatsis, A., Gravvani, K., & Spyrou, P. (2006). An empirical four-dimensional model for the understanding of function. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International*

Group for the Psychology of Mathematics Education (Vol. 3, pp. 137–142). Prague, Czech Republic: PME.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behaviour, 17*, 105–121.

Kaput, J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515–556). NY: Macmillan.

Karplus, R. (1979). Continuous functions: Students' viewpoints. *European Journal of Science Education, 1,* 397–413.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*, 1–64.

Markovits, Z., Eylon, B.-S., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics, 6*(2), 18–24, 28.

Matteson, S. M. (2006). Mathematical literacy and standardized mathematical assessments. *Reading Psychology*, *27*, 205–233.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of students' over-use of linearity. *Journal for Research in Mathematics Education, 39*(3), 311–342.

Van Dooren, W., De Bock, & Verschaffel, L. (2011, July). How students connect descriptions of realistic situations to mathematical models in different representational modes? Paper presented at the *15th International Conference on the Teaching of Mathematical Modelling and Applications*. Melbourne, Australia: ICTMA.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning*, *7*, 42–57.

SEARCHING FOR A WHOLE NUMBER BIAS IN SECONDARY SCHOOL STUDENTS – A REACTION TIME STUDY ON FRACTION COMPARISON

Wim Van Dooren, Jo Van Hoof, Tristan Lijnen, Lieven Verschaffel

Centre for Instructional Psychology and Technology, University of Leuven, Belgium

A major source of errors in rational number tasks is the inappropriate application of whole number rules. In this study, we investigated whether this manifests itself in secondary school students when they compare fractions, and whether this can be considered as an instance of intuitive reasoning. This was tested by means of a reaction time method, relying on a dual process perspective that differentiates between intuitive and analytic reasoning. 129 first and fifth year secondary school students solved a series of fraction comparison tasks on a computer. Responding correctly to items where whole number knowledge had to be inhibited took significantly longer than to items where this knowledge was helpful.

THEORETICAL AND EMPIRICAL BACKGROUND

The whole number bias in comparing fractions

Mastery of the rational numbers is considered an important aspect of mathematical literacy. It is, however, widely documented that learning about rational numbers presents students with many difficulties. Students are found to make systematic errors when a task requires reasoning that is not in line with their prior knowledge and experience about natural numbers (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2010). On the other hand, students deal effectively with rational number tasks when these are compatible with whole number knowledge (e.g., Nunes & Bryant, 2008), one such example being the part-whole aspect of fraction (Mamede, Nunes, & Bryant, 2005; Moss, 2005). Another example is that presentation of information in terms of frequencies instead of rates facilitates probabilistic reasoning—a finding that Butterworth (2007) explains as preference for whole numbers.

Because students' reliance on whole number reasoning when dealing with rational number tasks appears to have the aforementioned characteristics—facilitation of reasoning when it is appropriate, adverse effect when it is not—the term *bias*, first introduced by Ni and Zhou (2005) in relation to this phenomenon, seems justified.

There is no consensus regarding the origins of the whole number bias. Although some researchers argue that early quantitative representation is limited to discrete quantities—and thus the whole number concept is privileged by early experiences or even biologically—this issue is still controversial (Ni & Zhou, 2005). However, it is clear that the externalization and systematization of rational number concepts is typically much less socially supported in the first years of a child's life (Greer, 2004). In addition, early instruction focuses on whole number arithmetic, thus supporting the systematization and validation of children's initial understandings of number as whole numbers.

Thus, before they are introduced to rational numbers through instruction, students have already constructed a rich understanding of number, which is tied around their knowledge of whole numbers (Gelman, 2000; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010). This explains why systematic errors occur precisely where the behavior of rational numbers deviates from that of the whole numbers. The literature reports various kinds of rational number tasks that elicit such systematic errors, but we will focus on one specific case, namely the comparison of fractions. Comparison tasks have been widely used to assess understanding of rational numbers. In the comparison of fractions, a crucial factor is students' difficulty to understand that the magnitude of a fraction depends on the relation between its terms (Moss, 2005; Ni & Zhou, 2005; Smith et al., 2005). Instead, students initially tend to interpret the symbol a/b as two independent whole numbers, separated by a bar (e.g., Stafylidou & Vosniadou, 2004). This leads them to conclude that a fraction increases when its numerator, its denominator, or both increase. Focusing on each term of the fraction separately can result in correct judgments in cases such as $2/5 < 3/5$, but primary school children have also been shown to incorrectly state that $2/5 < 2/7$ (e.g., Mamede et al., 2005). Recently, there also has been an interest in educated adults' processing of fractions. Neuropsychological studies indicate that also adults access the terms of the to-be-compared fractions separately before responding correctly (e.g., Meert, Grégoire, & Noël, 2009, 2010).

A gap in the literature – as far as we know – concerns the intermediate stage between children's first encounters with fractions wherein they commit a lot of the above-mentioned errors, and adulthood, where such errors have almost completely disappeared. Throughout upper primary and lower secondary education, students frequently encounter fractions, thus it may be assumed that gradually they become better at fraction comparison tasks. However, even when they do not longer commit errors, students' thinking may still be affected by the whole number bias, as their first thoughts – consciously or unconsciously – may still be directed by the whole numbers in the fractional expressions. A method to investigate this can be found in research traditions focusing on the distinction between intuitive and analytical processes in mathematical reasoning. This will be elaborated in the next subsection.

Intuitive and analytic processes in mathematical problem solving

In the field of mathematics education, pioneering work on the role of intuitions was done by Fischbein (1987). Fischbein described intuitions as (cognitive) beliefs characterized by – among others – self evidence, intrinsic certainty, coerciveness, and globality. Intuitions are also characterized by perseverance, implying that once established they are robust and therefore not easily eradicated by instruction. Fischbein made the rather strong claim that some intuitions are never completely abandoned, but survive—and may coexist with scientific accounts—throughout a person's life.

This analysis of mathematical intuitions is obviously relevant to the whole number bias. The question arises: Is there empirical evidence that the whole number bias is indeed an instance of intuitive reasoning? Recently, it has been argued that the dual-process theories in cognitive psychology and their accompanying methodologies could be a valuable tool in establishing the intuitive nature of erroneous reasoning in various mathematical domains (Gillard, Van Dooren, Schaeken, Verschaffel, 2009; Leron & Hazzan, 2006, 2009). Dual-process accounts of reasoning (e.g., Evans & Over, 1996; Kahneman, 2000) were originally developed to account for poor performance in reasoning and decision-making by individuals who otherwise possessed the knowledge and skills necessary to deal with the tasks at hand. In these theories, it is assumed that humans have an intuitive/heuristic (S1) and an analytic processing system (S2). S1 is deemed fast, automatic, associative and undemanding of working memory capacity, whereas S2 is deemed slow, controlled, deliberate and effortful. Fast S1-heuristics often lead to correct responses, but sometimes they do not. In these latter cases, either an incorrect response is provided, or S2 needs to intervene and override the initial response. Hence, errors may be attributed to S1's pervasiveness and S2's failure to intervene. There are at least two processing claims within the dual-process framework that may serve as a basis to empirically identify whether a response is the result of a heuristic or an analytic process. The first is the differential processing speed—S1 is faster than S2—and the second is the differential involvement of resources—S1 is less demanding in working memory resources than S2.

Recently, reaction time methods and working memory manipulation have been fruitfully employed in the case of mathematical tasks that are known to systematically elicit erroneous responses (Gillard et al., 2009). For example, Babai, Levyadun, Stavy, and Tirosh (2006) studied a widely documented error that students make when comparing two polygons with respect to their perimeter, which these researchers take to be an instance of a more general intuitive rule termed *more A – more B* (Stavy & Tirosh, 2000). They showed that incorrect responses in line with the *more A – more B* rule were provided faster than correct answers.

From a dual-process framework perspective, these findings support the hypothesis that systematic errors in a variety of mathematical tasks are the result of intuitive, heuristic reasoning. Applying this framework in the case of the whole number bias can provide a method to test for its intuitive character that go beyond the mere observation of errors. So also in case of the comparison of fractions, a reaction time method can be useful: It does not only allow to look at how accurate students are in comparing fractions for which their whole number knowledge is not helpful, but also at whether the reasoning process was affected by whole number reasoning, even if this was successfully inhibited and a correct answer was given.

Research questions and hypotheses

The central research question in this study is whether the interference of whole number reasoning in comparing fractions can be characterized as an intuitive process. This would imply that whole number-based errors are persistent and present even in older students who have in principle the knowledge and skills necessary to respond correctly. We assumed that when a student is confronted with a fraction comparison item, an intuitive response grounded in whole number reasoning comes to mind first. When an item is compatible with whole number properties—hereafter called a congruent item—(e.g., "*which is bigger: 1/3 or 2/3?*"), the intuitive response leads to a correct answer. For incongruent items (e.g., "*which is bigger: 1/5 or 1/9?*") this intuitive response needs to be inhibited for a correct answer to be given. We thus predicted that incongruent items would trigger more incorrect responses (because the intuitive response was not inhibited) than congruent ones and that correct responses for incongruent items would have a longer reaction time than correct responses for congruent items (because inhibiting the intuitive response required time).

An additional research question was how this phenomenon would evolve with age, from the first until the fifth year of secondary school. We hypothesized that the number of erroneous answers to fraction comparisons would decrease with age, but at the same time, we expected that – given its intuitive character – the whole number bias would not completely disappear, but instead that, with age, decreased error rates in incongruent items would turn into increased reaction times for correct responses.

METHOD

129 students from a secondary school in a middle-sized city in Flanders participated in this study: 57 students from the first year of secondary school (mean age 12.1 years) and 72 students from the fifth year (mean age 16.2 years). These two age groups were comparable with respect to their mathematics performance, as inferred by their mathematics exam grades during the first semester of their first year of secondary school.

All students worked individually on a computer, where they solved 190 fraction comparison items, that were offered in random order in E-Prime software. Figure 1 shows for one example item how it was shown on the computer screen. For each item, students had to press a key indicating whether the left or the right fraction was largest. Accuracy and reaction times were logged by the software.

70 items were congruent. They consisted of two fractions with an equal denominator (e.g., 10/27 and 12/27). Whole number reasoning led to a correct answer for these items. In half of these 70 items, the left fraction was larger, in the other half, the right fraction was larger. Another 70 items were incongruent, consisting of two fractions with an equal numerator (e.g., 10/27 and 10/21), so that whole number reasoning would lead to an erroneous answer. Again in half of the items, the larger fraction was on the left, and in half on the right. Finally, 50 buffer items were added, in order to

avoid that students would discover systematicity in the items set. In the buffer items, neither numerators nor the denominators of the two fractions were equal.

$$\frac{10}{27} \qquad\qquad \frac{10}{21}$$

Figure 1: Example item as shown on the computer screen

The experimental items were designed in a rigorous way, using the following principles: Numerators and denominators were between 1 and 50, and all fractions were smaller than 1. Items were moreover selected so that the average size of the numerators, denominators, and the fraction as such were not significantly different for the set of congruent and incongruent items. This was done to avoid that size comparisons would be easier (and therefore faster) for either the congruent or the incongruent items merely because of the larger distance (see the distance effect, shown by Moyer & Landauer, 1967). Fractions were always written in their simplest form, and fractions for which students can easily remember the decimal form (e.g., 1/10, 1/2, 2/3, or 3/4) were avoided.

The experiment was split up in two parts. After a first set of 95 items (a mixture of congruent, incongruent and buffer items), students took a break, after which they completed the second set of 95 items.

RESULTS

Accuracy

After removing the buffer items, accuracy results were analysed by means of a repeated measures logistic regression analysis with two factors: congruency (congruent vs. incongruent items) and age (first vs. fifth year of secondary school). Table 1 summarises the accuracies per item type and per age group. First year students performed slightly better than fifth year students, but this difference was not significant, $X^2(1, N = 116) = 2.650$, $p = .104$. There was no significant difference in accuracy between the congruent and incongruent items either, $X^2(1, N = 116) = 0.031$, $p = .861$,

Table 1: Percentages of correct responses to congruent and incongruent items per age group

	Congruent items	Incongruent items	Total
First year	92.1	90.7	91.4
Fifth year	86.0	88.6	87.3
Total	88.7	89.5	89.1

Table 2: Mean reaction times in *ms* (and standard deviations) of correct responses to congruent and incongruent items per age group

	Congruent items	Incongruent items	Total
First year	2734 (1300)	3122 (1530)	2926 (1431)
Fifth year	2316 (1260)	2649 (1394)	2484 (1340)
Total	2511 (1295)	2863 (1476)	2687 (1400)

indicating that our expectation that students would commit more errors on incongruent items was not confirmed. This was neither the case in the first year students, nor in the fifth year students, as indicated by the non-significant interaction effect, $X^2(1, N = 116) = 1.366$, $p = .242$.

Reaction times

Reaction times were analysed by means of a repeated measures ANOVA, and this was done only for correct trials. Again congruency (congruent vs. incongruent items) and age (first vs. fifth year of secondary school) were predictors.

Table 2 summarises the reaction times for the correct responses per item type and per age group. In line with our expectation, correct responses to incongruent items took significantly longer than correct responses to congruent items, $X^2(1, N = 116) = 85.261$, $p < .00015$. There also was a main effect of age, $X^2(1, N = 116) = 21.250$, $p < .00015$, indicating that first year students responded significantly slower than fifth year students. Finally, there was no significant interaction effect between item type and age, $X^2(1, N = 116) = 0.489$, $p < .484$.

CONCLUSIONS AND DISCUSSION

The central research question in this study was whether the interference of whole number reasoning in comparing fractions can be characterized as an intuitive process. A first indicator for the intuitive character would be that whole number-based errors are persistent and present even in older students who have the knowledge and skills necessary to respond correctly. This was not confirmed by our findings: Performance was very good, and more importantly, there was no significant difference between congruent items (where whole number knowledge is helpful in giving the correct response) and incongruent items (where whole number knowledge needs to be inhibited). A second indicator would be that when a student is confronted with a fraction comparison item, an intuitive response grounded in whole number reasoning comes to mind first, but is then inhibited when this intuitive response is incorrect, leading to longer reaction times to give a correct response to an incongruent item than to a congruent one. This was confirmed by our results: Correct responses for incongruent items had significantly longer reaction times than correct responses to congruent items.

An additional question was how this phenomenon would evolve with age, from the first until the fifth year of secondary school. We hypothesized that the number of erroneous answers to fraction comparisons would decrease with age, while the decrease in error rates would turn into increased reaction times for correct responses. This was not confirmed by our results. We found that already in the first year secondary school students, there were hardly any errors. Still, the reaction times clearly indicated the presence of a whole number bias in both age groups.

Remarkably, the whole number bias (as found in reaction times) was still present to the same extent in fifth year secondary school students, even though they had considerably more experience in working with rational numbers than the younger students. It was also found that first year students overall took significantly longer to respond correctly (both to congruent and incongruent items) than fifth year students. A possible explanation (which is corroborated by the observation that first year students also performed slightly better) is that the first year students have invested more effort in the fraction comparison task than the fifth year students. This would result in longer reaction times, but also in a higher accuracies. Fifth year students might have been inclined to work a bit faster, but as a consequence, they sometimes did not notice that they had to inhibit the whole number-based response to incongruent items, resulting in more errors. Of course, this interpretation is speculative, and needs to be confirmed in further research.

Taken as a whole, our study indicates that first and fifth year secondary school students are still affected by a whole number bias when they compare fractions. This does not manifest itself in students committing errors, but items in which whole number reasoning does not lead to a correct response require significantly more time to be solved correctly than items where whole number reasoning does lead to a correct response. Given the dual process theoretical framework explained earlier in this paper, one can conclude that there are clear indications that the interference of whole number reasoning in comparing fractions can be characterized as an intuitive process.

REFERENCES

Babai, R., Levyadun, T., Stavy, R., & Tirosh, D. (2006). Intuitive rules in science and mathematics: a reaction time study. *International Journal of Mathematical Education in Science and Technology, 37,* 913–924.

Butterworth, B. (2007). Why frequencies are natural. *Behavioral and Brain Sciences, 30*(3), 259-260.

Evans, J. St. B. T. & Over, D. E. (1996). *Rationality and reasoning.* Hove, UK: Psychology Press.

Fischbein, E. (1987). *Intuition in science and mathematics.* Dordrecht, The Netherlands: D. Reidel Publishing Company.

Gelman, R. (2000). The epigenesis of mathematical thinking. *Journal of Applied Developmental Psychology, 21*, 27–37.

Gillard, E., Van Dooren, W., Schaeken, W., & Verschaffel, L. (2009a). Dual processes in the psychology of mathematics education and cognitive psychology. *Human Development, 52*, 95-108.

Greer, B. (2004). The growth of mathematics through conceptual restructuring. *Learning & Instruction, 14*, 541-548.

Kahneman, D. (2000). A psychological point of view: Violations of rational rules as a diagnostic of mental processes. *Behavioral and Brain Sciences*, *23*, 681-683.

Leron, U., & Hazzan, O. (2006). The rationality debate: application of cognitive psychology to mathematics education. *Educational Studies in Mathematics*, *62,* 105–126.

Leron, U., & Hazzan, O. (2009). Intuitive vs analytical thinking: four perspectives. *Educational Studies in Mathematics*, *71*, 263–278.

Mamede, E., Nunes, T., & Bryant, P. (2005). The equivalence and ordering of fractions in part-whole and quotient situations. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education: Vol. 3.* (pp. 281-288). Melbourne: PME.

Meert, G., Grégoire, J., & Noël, M-P. (2009). Rational numbers: Componential versus holistic representation of fractions in a magnitude comparison task. *The Quarterly Journal of Experimental Psychology*, *62*, 1598 –1616.

Meert, G., Grégoire, J., & Noël, M-P. (2010). Comparing the magnitude of two fractions with common components: Which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology*, *107*, 244-259.

Moss. J. (2005). Pipes, tubes, and beakers: New approaches to teaching the rational-number system. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: Mathematics in the classroom* (pp. 121-162). Washington, DC: National Academic Press.

Moyer, R. S., & Landauer, T. K. (1967). Time required for judgements of numerical inequality. *Nature, 215,* 1519–1520.

Ni, Y., & Zhou, Y-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27–52.

Nunes, T., & Bryant, P. (2008). Rational numbers and intensive quantities: Challenges and insights to pupils' implicit knowledge. *Anales de Psicologia*, *24*(2), 262-270.

Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology, 51*, 101–140.

Stafylidou, S. & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14*, 503-518.

Stavy, R., & Tirosh, D. (2000). *How students (mis-)understand science and mathematics*. Teachers College Press, New York.

Vamvakoussi, X., & Vosniadou, S. (2010). How many *decimals* are there between two *fractions*? Aspects of secondary school students' understanding about rational numbers and their notation. *Cognition and Instruction, 28*(2), 181-209.

MATHEMATICAL COMMUNICATION IN THIRD-GRADERS' DRAWINGS IN CHILE AND FINLAND

Leonor Varas[1], Erkki Pehkonen[2], Maija Ahtee[2] & Salome Martinez[1]

[1]University of Chile & [2]University of Helsinki

Abstract: In this article we analyze the drawings made by third grade students from Chile and Finland, when asked to draw their math class. This activity was the starting test of the Chile – Finland comparative study, that was applied to pupils from 10 classes (N = 153) in Finland and 10 classes (N = 326) in Chile. Through analyzing these drawings we tried to reveal third-graders' conceptions about communication on mathematics and its teaching. The results show that pupils' conceptions about communication are very different in these countries, although there are also many similarities. In the drawings of both countries, a teacher's actions are seen similarly: The teacher teaches, observes.

This paper is linked to the background studies of the comparative research project between Finland and Chile 2010–2013, a research project funded by the Academy of Finland (project #135556) and CONICYT in Chile (project # AKA 09). The purpose of the project is to clarify the development of pupils' and teachers' mathematical understanding and performance when dealing with open-ended problems.

PUPILS' DRAWINGS AS A RESEARCH OBJECT

Kress, Jewitt, Ogborn & Tsatsarelis (2001) have studied the multiplicity of modes of communication (speech, image, gesture, action with models, writing, etc.) that are active in the classroom when the teacher presents the subject matter and seeks to shape pupils' understanding of it through the interactive processes of classroom teaching and learning. According to them communication always draws on a multiplicity of modes of representation, of which language is only one and not necessarily the dominant one. Learning is much more than a matter of speaking or writing; it is a dynamic process of transformative sign making. Drawing is an alternative form of expression for children. Barlow, Jolley and Hallam (2010) have noted that free hand drawings helped children recall and express more details about events they depicted. Drawings tend to facilitate the recalling of events that are unique, interesting or emotional, but not routine events or isolated bits of information that are not part of a narrative. Finally, we want to emphasize that pupils' drawings open a holistic way to evaluate and monitor the classroom communication.

About twenty years ago, it was observed that pupils' classroom drawings opened a non-verbal canal to children's conceptions. Many researchers (e.g. Weber & Mitchell 1996) have used pupils' classroom drawings, and realized that they form rich data to rearch children's conceptions on teaching. For example, Weber & Mitchell (ibid) state that a part of everything that we experience – what we have seen, know, think or imagine, remember or reject – will be convoyed in drawings that thus reveal sub-conscious connections or inaccuracies. According to Harrison, Clarke & Ungerer (2007) pupils' views can be captured with drawings more meaningfully than with other research data.

Ruffell, Mason & Allen (1998) question surveys as a method to investigate children, since small children do not necessary understand the words and statements used in questionnaire in the way a researcher means. According to Hannula (2007) it is not easy to get from young children verbally rich answers to questions, since young children tend to give monosyllabic answers to questions that they do not consider relevant to them. Young children may have some difficulties with reading surveys and expressing themselves clearly in writing. Within interview contexts, children of all ages may be hesitant to discuss with an often relatively unknown researcher.

On pondering the reliability and validity of mathematics drawings, Stiles, Adkisson, Sebben & Tamashiro (2008) concluded that drawings offer more opportunities to show a stronger and more personal opinion than responses to the questionnaire statement ”I like mathematics”. In pictures pupils may draw hearts when loving mathematics, or a gun in order to destroy mathematics showing that they don't like mathematics. In cases like this, the data based content analysis of drawings can concentrate on both a qualitative exploration of what is drawn, as well as quantitative inspecting how often particular themes/categories appear. Furthermore, the focus of content analysis on inter-rated reliability is the potential for researchers to check credibility of drawings. Also Dahlgren & Sumpter (2010) consider pupils' drawings as a reliable means to catch pupils' conceptions of mathematics teaching.

The focus of the study

Our aim is get answers to the research problem: ”What is communication like in drawings of mathematics classrooms?” Especially we want to clarify, what kind of differences and similarities there are between Chile and Finland. Therefore, we will restrict ourselves in those parts of the drawings where communication can be found. Thus we enter the research problem through following four research questions:

(1) How does the teacher communicate with the pupils?

(2) What is communication within the classroom like?

(3) What is the quality of communication in the classroom like?

(4) What kind of differences and similarities can be found between Chilean and Finnish pupils' drawings?

METHODS

The aim of the Finland – Chile comparison study is to clarify the development of pupils' mathematical understanding and problem solving skills, from grade 3 to grade 5 when using open problem tasks at least once a month. The data dealt with here is pupils' drawings from September 2010 (in Finland) and from March 2011 (in Chile) that belongs to initial measurements in the project.

Participants and data gathering

Finnish project participants are third-graders (N = 153, about 9– 10-year-olds) from classes taught by 10 different teachers in five elementary schools in Great-Helsinki, and their Chilean peers are third-graders (N = 326, about 8– 9-year-olds) from 10 different teachers in five elementary schools in Santiago de Chile. Pupils did the drawing task during their mathematics lessons in the beginning of the school year (September 2010 resp. March 2011). The task for the pupils was, as follows:

> *”Draw your teaching group, the teacher and the pupils, in a mathematics lesson. Use speaking and thinking bubbles to describe discussion and thinking.”*

Drawings were collected and analyzed from a total of 131 Finnish pupils and of 250 Chilean pupils; thus the non-response rate is in Finland 14.4 % and in Chile 23.3%. The contents of the speaking and thinking bubbles enabled the researchers to investigate the communication in class.

Data analysis

A drawing as an observational data can be divided into content categories. A content category means the phenomen on which data are gathered. Here we selected our content categories to be the following: *a teacher’s communication with his/her pupils (1), communication within class (2), quality of communication in class (3)*. For the analysis, the content categories were specified into following sub-categories.

- Teacher-pupil communication (1): Teacher gives instructions; keeps order; teaches; gives feedback; observes quietly while the pupils work.

- Internal classroom communication (2): Pupil remarks on matters related to teaching; asks for help; discusses with others; remark is off-topic.

- Quality of classroom communication (3): Pupil thinks ‘mathematics is fun/easy’; ‘mathematics is boring/difficult’; is competent in mathematics; helps another pupil; is tired and cannot continue.

Additionally in each content category there is also the option ‘non-recognizable’.

Teacher-pupil communication was classified as a separate category. In the beginning, we had more different sub-categories, but at the end we arrived at just five. For example, the sub-categories *Teacher gives praise* and *Teacher criticises* were combined into the sub-category *Teacher gives feedback*, since there was only one case of negative feedback. After several attempts, communication between pupils and the quality of that communication were classified as individual categories.

In both countries, two researchers classified all pupils’ drawings, and the cases with differences were looked through again. All drawings were classified carefully, and the estimation for concensus was received by calculating the differences between two classifiers. The analysis of the drawings was qualitative, and therefore, it can be classified as inductive content analysis (cf. Patton 2002). The drawings were looked through focusing on one content category at time. In each content category, we looked at each drawing, to see whether there are sub-categories belonging that content category. In every category, the last sub-category was ”non-recognizable”. The agreement between two classifiers in Finland in all sub-categories is very good, i.e. over 90% (range 91%–95%). The similar percentages from Chile are 90% to 95%. We have also coded a small sample of drawings all four together, in order to calibrate the coding within each country.

The pupils’ drawings are informative. In many drawings you can see only stick figures, in a few of them the hands start at the face, and in some of them you can only see desks with pupils’ head. However, some of the third-graders are very talented illustrators, thus there are many details found in the drawings. The pupils’ thoughts about the mathematics lesson and the classroom atmosphere are written in bubbles, although the pupils’ presentation of a turn of speech – either aloud or by whispering – or thinking in bubbles is not always logical.

RESULTS

The first result will be the comparison of the educational system in both countries. Then in the analysis of drawings, we found answers to the following question in both countries: What is teacher-pupil communication like in the drawing? And finally we will combine these results, in order to answer the question: What kind of differences and similarities can be found between Chilean and Finnish pupils’ drawings?

Comparison of primary education in Chile and Finland

In the following, the comparison on the education system in mathematics in both countries is written in a compressed style (Table 1) and is based on the following raster: description of the dominant school system, curricula development, respect for mathematics (i.a. teachers’ position), learning results, equipment in schools.

	FINLAND	CHILE
School system	Inclusive, integrated, homogeneous, free of charge	Segregated, high variabilityFree of charge in public schools (40%), some copayment in subsidized schools (50%).
Benefits for all pupils in compulsory schooling	free warm meal, textbooks, transportation, medical and dental care	Freetexbooks for subsidized schools (90%). Free meals according to vulnerability in subsidized schools.
Coverage	95% complete comprehensive school (9 grades)	99.7% complete elementary school (8 grades)
	about 50% complete high school	87.7% complete high school
Curriculum	government determines the core subjects, national objectives, the number of hours allocated for each subject	government determines the core subjects, national objectives, the number of hours allocated for each subject
Number of hours per week allocated for math	3–4 hours per week in grades 1–6; 3 hours per week in grades 7–9	6 hours per week in grades 1–8

Teachers preparation	Competitive selection of teacher students master level degree	No selection of teacher students No guarantee of the quality of the teacher preparation programs
Teachers specialization	class teachers take care of grades 1-6	class teachers take care of grades 1-8
Core ideas about teaching math and their implementation	Constructivist conceptions, traditional practice	Constructivist conceptions, traditional practice
PISA 2009 Math	ranking 4. (with 541 points)	ranking 49. (with 421 points)

Table 1: Comparison of educational systems: Main differences in Chile and Finland

Results from the Finnish drawings

Firstly, we discuss the category "A teacher's communication with his/her pupils" (cf. Table 2). Since in the drawings of many pupils, there are several indicators, the total frequency is here 145. This totality is divided rather uniformly between three factors. The mode value (36; or 25 %) in this category is "teaches" that contain both a teacher's own questions and expository teaching. But the frequencies are almost as high in the sub-categories "follows quietly pupils' working" (33; or 23 %) and "non-recognizable" (28; or 19 %). Thus most of the pupils have an impression that the teacher makes questions and delivers knowledge in mathematics lessons.

Secondly, we take the category "Communication within class" (cf. Table 2). Since in the drawings, there were several indicators, the totality is here 191. The highest frequency is in the sub-category "a pupil makes/ asks/ thinks a remark in connection to teaching" (65; or 34 %). The next highest frequency (48; or 25 %) is in the sub-category "a pupil makes/ thinks an unproper remark". The frequencies of the rest of the three sub-categories are under half of the maximum frequency. Therefore, we could say that the communication within the category is a compound of pupils' remarks where the highest share form the remarks connected to teaching/ learning of mathematics, but there is also a great many of improper remarks.

Thirdly, the category "Quality of communication in class" (cf. Table 2) will be taken under consideration. Since in the drawings, there are several indicators, the total frequency is here 149. In this category there is no clear modal class, but three sub-categories compete with each other: "a pupil is competent in mathematics" (33; or 22 %), "a pupil says/ thinks "mathematics is fun/ easy" (32; or 21 %), "a pupil says/ thinks "mathematics is boring/ difficult" (31; or 21 %). But the share of non-recognizable drawings is remarkable (42; or 28 %). Therefore, we could state that the quality of communication in class is not clearly recognizable.

Results from the Chilean drawings

The Chilean pupils are younger, which is reflected in their drawing. The drawings were less elaborated than the ones of their Finnish peers. This can be seen in the high percentage of "Non-recognizable" category registered in each observed characteristic.

In the Chilean drawings there are not so many features to observe in the same picture like in Finland, and in more of the 60% of the drawings it is not possible to observe details corresponding to the subcategories “Communication within class” and “Quality of communication in class”. Despite of this and due to the bigger size of the Chilean sample, it is relevant to notice that in 63 drawings (25%) the teacher is shown teaching and in 62 drawings (25%) the teacher follows quietly the work of their pupils. In the other next sub-categories there are clear modal behaviors, when they are recognizable. In 48 drawings (19%) the communication within the class is connected to teaching, and the pupils express their confidence to do mathematic. We add a class in the sub-category of “Communication within class” in order to register proper thoughts that are not connected to teaching, or dialogs like greetings.

Comparison of results

In Table 2 we have combined the results discussed above from both countries.

A teacher’s communication with his/her pupils (%)	Communication within class (%)	Quality of communication in class (%)
teaches	related to teaching	is competent in mathematics
25%, 25%; --	34%, 19%; 13.33 ***	22%, 19%; 1.09 n.s.
observes quietly	is off-topic	mathematics is fun/easy
23%, 25%; -0.58, n.s.	25%, 7%; 5.62 ***	21%, 7%; 5.23 ***
keeps order	asks for help	mathematics is boring/difficult
13%, 8%; 4.35 ***	15%, 4%; 4.11 ***	21%, 5%; 4.97 ***
gives instructions	pupils discuss	helps/praises another
10%, 8%; 1.15, n.s.	12%, 1%; 3.52 ***	4%, 0%; 1.92 n.s.
gives feed-back	is off-topic but appropiate	is tired
10%, 5%; 7.12 ***	0%, 8%; -2.77 **	4%, 6%; -0.97, n.s
non-recognizable	non-recognizable	non-recognizable
19%, 29%; -2.46 *	10%, 61%; -9.51 ***	28%, 63%; -6.67 ***

Table 2. The relative frequencies in the three categories: A teacher’s communication with his/her pupils, Communication within class, Quality of communication in class (in percentages: first Finland, then Chile, last t-value).

In teacher-pupil communication (1), there are statistically very significant (p ‹ 0.1%) differences in two sub-categories: “keeps order” and “gives feedback”. In three sub-categories differences are non-significant: “teaches”, “observes quietly” and “gives instructions”. Thus pupils’ conceptions on their teachers are in both countries rather similar: They teach mathematics, follow quietly pupils’ individual workings and gives instructions during the lesson. But there are differences in teachers’ ways to keep order and to give feedback; in both cases the Finnish teachers are doing more.

The differences are the biggest in the category “Internal classroom communication” (2): altogether in five sub-categories the differences are statistically very significant ($p < 0.1\%$). These are, as follows: pupils’ remarks are ‘related to teaching’, ‘is off-topic’, ‘asks for help’, ‘pupils discuss’, ‘non-recognizable’. In no sub-category, differences are non-significant. In Finland pupils are clearly more active than in Chile: They talk on teaching and ask help from each other, but also speak more improperly or discuss with each other. One explanation for these findings might be the huge difference in the sub-category non-recognizable (10 % – 61 %). The Chilean pupils have not used much speaking bubbles in their drawings, although they were asked for.

In the category “Quality of classroom communication” (3), there are three sub-categories where the differences are statistically very significant ($p < 0.1\%$). These are, as follows: ‘mathematics is fun/easy’, ‘mathematics is boring/difficult’, ‘non-recognizable’. In three sub-categories differences are non-significant: ‘is competent in mathematics’, ‘helps/praises another’, ‘is tired’. The Finns express more their feelings (positive and negative) than their Chilean peers. But again the explanation can be seen in the sub-category non-recognizable (28% – 63%), since the Chilean pupils have not used many speaking bubbles. Pupils’ conceptions on their capabilities as learners are in both countries similar: They feel competent in mathematics.

CONCLUSION

It is evident that for pupils’ drawings there are many kinds of influences. These drawings were made in the beginning of the third grade (autumn 2010 resp. spring 2011), thus it might be that most of pupils have been thinking on their teacher and teaching in grade 1–2. Additionally, many third-graders seem to have difficulties in drawing, and therefore, they might concentrate to draw only such a situation that is easy for them.

In Finland altogether two thirds (67%) of pupils produced drawings where pupils’ thinking/ speaking and action can be seen. It is to understand that pupils draw a teacher in the first place in front of the class, although Finnish classroom observations show that the teacher walks around among the pupils. The similar is valid for Chile, too. It seems that the understanding of a teacher’s actions is very similar in both countries: The teacher teaches, observes his/her pupils’ working quietly, and gives instructions when needed. And in both countries, pupils feel themselves competent in mathematics.

But in the drawings, there seems to be more differences than similarities. In teachers’ ways to keep order and to give feedback, there are differences; in both cases the Finnish teachers are more active. Also pupils in Finland are clearly more active than in Chile: They talk on teaching and ask help from each other, but also speak more improperly or discuss with each other. Furthermore, the Finns express more their feelings (positive and negative) than their Chilean peers. One explanation for these findings might be the huge difference in the sub-category non-recognizable. In both latter categories, the Chilean relative frequency is more than 60%. The Chilean pupils have not used many speaking bubbles in their drawings.

It is interesting that we can see in drawings pupils' sayings or thoughts on their attitude toward mathematics, as mathematics is fun, easy, boring or difficult. It comes out spontaneously from pupils' drawings, without extra being asked for. Thus, we could conclude that pupils talk about mathematics, its liking and difficulty of tasks. This was very obvious in the Finnish drawings, not so clear in the Chileans.

Such a drawing task can help a teacher to understand his/her pupils' thinking, and also to hint how the teaching should be developed. Furthermore, such a ”window” to pupils' thinking is very easy to open also within a lesson, and it will not demand from a teacher so much additional work.

Acknowledgents: L. Varas was supported by grant CIE-05, CIAE-U. Chile. S. Martínez was supported by Fondo Basal CMM-U. Chile.

References

Barlow, C. M., Jolley, R. P. & Hallam, J. L. 2010. Drawings as memory aids: optimising the drawing method to facilitate young children's recall. Appl. Cognit. Psychol. DOI: 10.1002/acp 1716.

Dahlgren, A. & Sumpter, L. 2010. Children' conceptions about mathematics and mathematics education. In: *Proceedings of the MAVI-16 conference June 26-29, 2010* (ed. K. Kislenko), 77–88. Tallinn University of Applied Sciences, Estonia.

Hannula, M. S. 2007. Finnish research on affect in mathematics: Blended theories, mixed methods and some findings. *ZDM The International Journal on Mathematics Education* 39 (3), 197–203.

Harrison, L. J., Clarke, L. & Ungerer, J. A. 2007. Children's drawings provide a new perspective on teacher-child relationship quality and school adjustment. *Early Childhood Research Quarterly* 22, 55–71.

Kress, G., Jewitt, C., Ogborn, J. & Tsatsarelis, C. 2001. *Multimodal teaching and learning. The rhetorics of the science classroom.* London: Continuum.

NBE 2004. *Perusopetuksen opetussuunnitelman perusteet 2004*. [Basics for the curriculum of the comprehensive school 2004]. Opetushallitus. Vammala: Vammalan Kirjapaino Oy.

Patton, M. 2002. *Qualitative research & evaluation methods*. Thousand Oaks: Sage.

Pehkonen, E., Ahtee, M. & Lavonen, J. (eds.) 2007. *How Finns Learn Mathematics and Science?* Rotterdam: SensePublishers.

Ruffell, M., Mason, J. & Allen, B. 1998. Studying attitude to mathematics. *Educational Studies in Mathematics* 35, 1–18.

Stiles, D. A., Adkisson, J. L., Sebben, D. & Tamashiro, R. 2008. Pictures of hearts and daggers: Strong emotions are expressed in young adolescents' drawings of their attitudes towards mathematics. *World Cultures eJournal* 16 (2).

Weber, S. & Mitchell, C. 1996. Drawing ourselves into teaching: studying the images that shape and distort teacher education. *Teaching & Teacher Education* 12 (3), 303–313.

TEACHING CHILDREN TO SOLVE MATHEMATICAL WORD PROBLEMS IN AN IMPORTED LANGUAGE

Debbie Bautista Verzosa
Ateneo de Manila University
Philippines

Joanne Mulligan
Macquarie University
Sydney, Australia

Many Filipino children find mathematical word problems particularly difficult to solve because they are required to learn mathematics and solve problems in English, a language that many encounter only in school. This paper reports on one aspect of a two-year design study aimed to assist second-grade Filipino children solve additive word problems in English. With Filipino as the medium of instruction, an out-of-school pedagogical intervention providing linguistic and representational scaffolds was implemented with 17 children over 10 weeks. Pre-intervention, children experienced linguistic difficulties and were limited to conceptualising and solving simple additive structures. Post-intervention interviews revealed improved understanding of more complex structures, but only when linguistic difficulties were minimised.

In the Philippines and in many other countries, children learn mathematics in English, an imported language not widely spoken in the community. However, many Filipino children, especially those from disadvantaged backgrounds, have limited access to engaging in English outside school (Gonzales, 2006). Thus, the problem arises that many Filipino children who have completed two or three years of schooling are unable to solve even simple addition and subtraction word problems in English (Bernardo, 1999), a result replicated by our own studies (Bautista, Mitchelmore, & Mulligan, 2009; Bautista, Mulligan, & Mitchelmore, 2009). For example, of the 17 children who took part in the intervention phase of our design study, 12 could not understand the statement, “Alvin had 3 coins” (Bautista & Mulligan, 2010). It is thus not surprising that low comprehension skills is one of the reasons perennially blamed for children’s poor mathematical performance (Carteciano, 2005). However, the extent to which linguistic factors impede Filipino children’s problem-solving performance is not fully understood. This paper aims to utilise a cognitive framework to clarify the sources of the difficulties that Filipino children encounter when solving Separate and Missing Addend word problems and how these difficulties are addressed through a pedagogical intervention focused on addition and subtraction word problems. Their solutions to these problems highlight the linguistic and relational structural knowledge required for solving mathematical word problems.

THEORETICAL FRAMEWORK

Our analysis of children’s problem-solving processes was based on Kintsch’s (1994) theory of text comprehension in conjunction with Vergnaud’s (1979, 2009) mathematical theory of conceptual fields. Several mathematics education researchers have drawn on Kintsch's theory of how readers process text (e.g., English & Halford, 1995). Within Kintsch's model, word problem solving consists of three inter-related mental representations—the *textbase*, *situation model,* and *problem model.* According to Kintsch, the textbase, which by itself reflects a superficial type of understanding, depicts the meaning of the words in the text and how these relate to each other. A deeper level of understanding occurs when one has constructed a situation model, or a mental representation of the *situation* being described by the text. The problem model for additive problems is a part-whole representation of the given and unknown quantities in the problem.

Vergnaud's (1979) conceptual field theory may be used to expound on the mathematical knowledge required for the construction of an appropriate problem model representation. He notes,

> [There is a need to] portray a better cognitive status for arithmetic, instead of associating it with boring and out of date calculation…As a matter of fact arithmetic, even in its elementary aspects deals with very important mathematical concepts. (p. 263)

Thus, to solve addition and subtraction word problems, a child must grasp the concepts of cardinality and the set-subset structure. They should also be able to perform the required *relational* calculation for reasoning about set quantities. Lacking this knowledge, a child may fail to solve word problems even when these are narrated to them in their first language. Using the terminology from text comprehension research, these children may fail to construct appropriate situation models as they lack the required domain knowledge for doing so (Hirsch, 2003).

While it is acknowledged that children's solutions to word problems are influenced by a range of sociocultural factors, especially in relation to the context of a developing country where the study was conducted (Adler & Lerman, 2000; Civil, 2006; Tobias, 2005), this paper will focus on a cognitive analysis of children's solutions.

METHODOLOGY

The study employed a methodology of design research (Lesh & Sriraman, 2010), which aims to "solve complex educational problems for which no ready-made solutions are available" (Nieveen, 2007, p. 89). The study consisted of two iterations of written tests (N = 75 and N = 238) and individual interviews (N = 58) conducted with a diverse cohort of Filipino public school children in Metropolitan Manila. A 20-session pedagogical intervention was subsequently designed and implemented over 10 weeks, and was consistently attended by 17 second-grade children. The intervention aimed to assist children solve different types of additive word problems, and was piloted with a variable group of (approximately 90) children in an informal setting (Verzosa & Mulligan, 2011). The first author designed and administered all assessments and the intervention. As mentioned earlier, this article will focus on the development of children's problem-solving process for solving Separate and Missing Addend problems as they participated in an intervention (see Table 1).

Problem type	Example
Separate	Dora has 11 mangoes. Then Dora gave 6 mangoes to Kevin. How many mangoes does Dora have now?
Missing Addend	Jolina had 8 pencils. Then Alma gave her some more pencils. Now Jolina has 14 pencils. How many pencils did Alma give her?

Table 1. Two types of problems described in this article

Interview Assessments

The interview consisted of six word problem tasks, two of which (in Table 1) are the focus of this paper. Because many Filipino children still have not developed conversational fluency in English for meaningful communication to take place (Bautista & Mulligan, 2010), it was necessary to conduct all interviews in Filipino.

A scaffolded word problem interview involving linguistic and mathematical scaffolds was designed to assess children's problem-solving process that may be impeded by initial, usually linguistic, obstacles (Bautista & Mulligan, 2010; see Figure 1). Linguistic scaffolds include presenting a problem in Filipino or reading it aloud in Filipino for the child. If a child still could not reach a correct solution, a mathematical scaffold in the form of a concrete representation of the task was presented (Wright, Martland, & Stafford, 2000). In the Separate problem in Table 1, for example, the concrete task was to briefly display, then screen, 11 counters. Without allowing the child to see, six counters were then removed from this set. The interviewer said (in Filipino), *There were 11 counters, but then I took away 6 counters. How many counters are there now?*

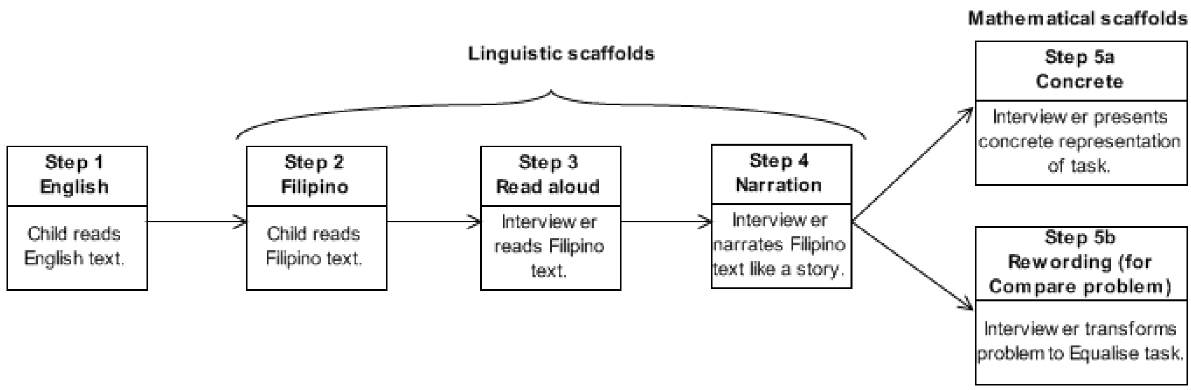


Figure 1. The interview assessment schedule

Pedagogical Intervention

The pedagogical approach was informed by Kintsch's (1994) three-component model of problem solving. The aim was to strengthen each component as well as appropriate mappings between them. The basic conjecture was that the three components of word problem solving may be developed *simultaneously*, rather than sequentially. In practical terms, this meant not having to wait for children to acquire skills necessary to construct a coherent textbase before providing them with opportunities to develop mathematical concepts required to conceptualise additive structures.

Mathematical concepts were conveyed in Filipino, using a range of representations. For example, the tasks in Table 2 show the various ways of representing the Missing Addend problem in Table 1. In this way, it became possible to communicate mathematical structures while circumventing the English language difficulty.

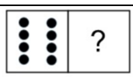
Mode of representation	Typical tasks or activities
Concrete	Screening task (Wright et al., 2000): Briefly display 8 blocks. “I will join some blocks to the 8, but I will not tell you how many.” Join 6 blocks to the original 8, without showing the child the number of additional blocks. “Now, there are 14 blocks altogether. How many blocks are in the bag?” [presented in Filipino]
Pictorial	 = 14
Verbal-pictorial	Show 8 dots. “I have 8 dots. I want 14. How many dots do I need?” [presented in Filipino]
Textual	Gina had 8 bags. Ramon gave her some more bags. Now, Gina has 14 bags. How many bags did Ramon give Gina? [presented in English or Filipino]
Symbolic	$8 + \square = 14$

Table 2. Various representations for the Missing Addend task $8 + \square = 14$

RESULTS

Children’s difficulties

Figure 2 shows the types of scaffolds necessary for success, pre-intervention. Darker areas in the graph represent instances when linguistic scaffolds were both necessary and sufficient for success. The extent of the dark regions shows that the children were dependent on linguistic scaffolds—there were very few instances where a child solved problems in English, without assistance. However, the linguistic scaffolds were primarily helpful for the Separate problem. Even when linguistic or mathematical scaffolds were available, more than half the children could not solve the Missing Addend problem. Some children were limited to conceptualising and reasoning about disjoint subsets with known quantities. For example, one child repeatedly constructed two disjoint sets, instead of adding on to one set for the Missing Addend problem, even when a corresponding concrete task involving smaller numbers ($6 + \square = 9$) had been provided.

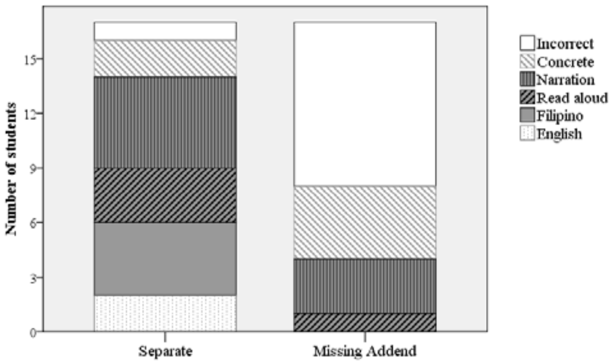


Figure 2. Pre-intervention

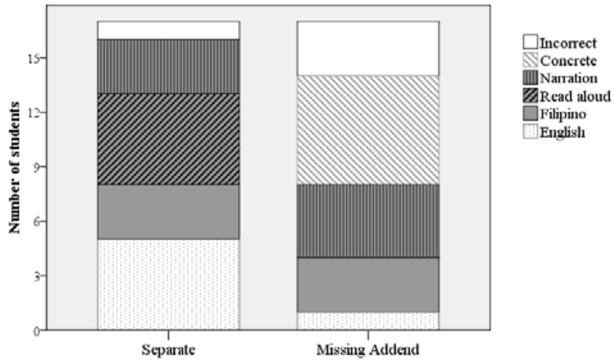


Figure 3. Post-intervention

Intervention outcomes

The development of children's problem-solving solutions is demonstrated in Figure 3, which presents the point in the post-intervention interview at which a correct solution was achieved. Although there were more children who could solve word problems in English after the intervention, reliance on linguistic scaffolds was still strong for both problems. There were also some improvements with respect to children's solutions to the Missing Addend problem as children generally made more correct solutions and required fewer scaffolds after the intervention. However, for this problem, many children were still reliant on a concrete representation of the task after the intervention. The concrete representation was thus crucial for helping children conceptualise the problem's mathematical structure.

DISCUSSION

Children's difficulties

Most of the children in this study could not solve word problems when they were presented solely in written English, pre-intervention. Indeed, their difficulties were more pronounced than those commonly reported in the literature, which tends to relate to difficulties with academic rather than conversational language (Fillmore, 2007). However, the children in this study had not even acquired the English language skills necessary for daily social interactions. Thus, their textbase representations remained impoverished and could not form the basis of an appropriate situation model.

While the children in this study had considerable linguistic difficulties, their mathematical difficulties were compatible with those of monolinguals struggling to conceptualise additive structures beyond situations involving joining sets or breaking them apart (Nunes & Bryant, 1996; Vergnaud, 2009). Thus, their fragile understanding of cardinality and the set-subset structure prevented them from conceptualising the situation being described by the text, thereby resulting in an inaccurate situation model. This idea is in common with general text comprehension theories that stress the importance of knowledge of the problem domain (in this case, of additive structures) on the construction of cohesive situation models (Hirsch, 2003).

Intervention outcomes

While many children could not conceptualise the situation being described by Missing Addend problem pre-intervention even when a corresponding concrete task was presented, the post-intervention results suggest that they had developed the mathematical knowledge required for conceptualising the set-subset structure. Thus, the intervention showed that it is possible to develop children's understanding of additive structures even before they have acquired the skills necessary to construct a coherent textbase. In alignment with Vergnaud's (1979) theory of concept development, children's progress was achieved through their engagement with situations that conveyed additive structures beyond joining sets and taking them apart. In particular, the representations of the Missing Addend structure in Table 2 were effective in the sense that these allowed children to visualise and conceptualise the quantities and relations in the problem.

Improved understanding of additive structures, however, did not necessarily translate to better performance in solving word problems in English. As children in this study had minimal exposure to English, it seems highly unreasonable to expect them to solve word problems in English unless adequate English language pedagogical support is provided. There are no short-cuts—even when children develop the necessary mathematical knowledge to conceptualise various additive structures, they still need to develop their English language skills before they can solve problems in English. As linguist Ekkehard Wolff (2011, p. 92) aptly states, "Language is not everything in education, but without language, everything is nothing."

The post-intervention results also provide some insight into the claim that many Filipino children fail to solve word problems because they have low comprehension skills (Carteciano, 2005). If one takes understanding to mean the construction of an appropriate situation model, then understanding involves not only linguistic but also mathematical knowledge. Further, this mathematical knowledge involves more than just computational skill as the interviews revealed that mathematical difficulties were primarily due to the inability to carry out a *relational* (rather than a numerical) calculation required for solving the problem.

GENERAL DISCUSSION AND IMPLICATIONS

While children were found to be more successful in solving word problems after the intervention, they continued to struggle with word problems when these were presented in English. This persistent difficulty, however, should not suggest that the intervention was a futile attempt to improve Filipino children's solutions to additive word problems in English. On the contrary, the results are promising given that the children in this study could not engage in English social conversation. Additionally, on the basis of Cummins' (2000) well-established theory of bilingualism, the children's seemingly modest progress is not to be dismissed—they do not need to relearn mathematical concepts in English, as they are expected to be capable of solving word problems in English once their fluency in the language catches up with the demands of the task.

The findings of this study have relevant implications, as the Philippines is about to introduce mother tongue-based education in the coming school year. As this study has shown, better mathematical understanding is not achieved simply by changing the language of instruction to the first language because linguistic difficulties are not the only factors that impede problem solution. The provision of a range of representations for various additive structures is also important for developing mathematical understanding. Additionally, as there is often a gap between implementation and language policy (Gonzales, 1996), the value of scaffolded interviews in assessing children's linguistic and mathematical difficulties should also be realised.

References

Adler, J., & Lerman, S. (2003). Getting the description right and making it count: Ethical practice in mathematics education research. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 441-470). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Bautista, D., Mitchelmore, M., & Mulligan, J. T. (2009). Factors influencing Filipino children's solutions to addition and subtraction word problems. *Educational Psychology, 29*, 729-745.

Bautista, D., & Mulligan, J. T. (2010). Why do disadvantaged Filipino children find word problems in English difficult? In L. Sparrow, B. Kissane & C. Hurst (Eds.), *Shaping the future of mathematics education* (Proc. of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia, pp. 69-76). Fremantle, WA: MERGA.

Bautista, D., Mulligan, J. T., & Mitchelmore, M. (2009). Young Filipino students making sense of arithmetic word problems in English. *Journal of Science and Mathematics Education in Southeast Asia, 32*, 131-160.

Bernardo, A. B. I. (1999). Overcoming obstacles to understanding and solving word problems in mathematics. *Educational Psychology, 19*, 149-163.

Carteciano, J. A. (2005). *NRCP/DOST addresses dire state of S&T education in the regions*. Retrieved December 23, 2008, from http://nrcp.dost.gov.ph/index.php?option =com_content&task=view&id=31&Itemid=106

Civil, M. (2006). Working towards equity in mathematics education: A focus on learners, teachers, and parents. In S. Alatorre, J. L. Cortina, M. Saiz & A. Mendez (Eds.), *Proceedings of the 28th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 30-50). Merida, Mexico: Universidad Pedagogica Nacional.

Cummins, J. (2000). *Language, power, and pedagogy: Bilingual children in the crossfire*. Clevedon, UK: Multilingual Matters.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum.

Fillmore, L. W. (2007). English learners and mathematics learning: Language issues to consider. In A. H. Schoenfeld (Ed.), *Assessing mathematical proficiency* (pp. 333-344). Cambridge, UK: Cambridge University Press.

Gonzales, A. (1996). Using two/three languages in Philippine classrooms: Implications for policies, strategies, and practices. *Journal of Multilingual and Multicultural Development, 17*, 210-219.

Gonzales, A. (2006). The language planning situation in the Philippines. In R. B. Baldauf, Jr. & R. B. Kaplan (Eds.), *Language planning and policy in the Pacific: Fiji, the Philippines, and Vanuatu* (Vol. 1, pp. 114-153). Clevedon, UK: Multilingual Matters.

Hirsch, E. D., Jr. (2003). Reading comprehension requires knowledge—of words and the world. *American Educator, 27*, 10-13, 16-22, 28-29, 48.

Kintsch, W. (1994). Text comprehension, memory, and learning. *American Psychologist, 49*, 294-303.

Lesh, R., & Sriraman, B. (2010). Re-conceptualizing mathematics education as a design science. In L. D. English (Ed.), *Theories in mathematics education: Seeking new frontiers* (pp. 123-146). Heidelberg, Germany: Springer.

Nieveen, N. (2007). Formative evaluation in educational design research. In T. Plomp & N. Nieveen (Eds.), *An introduction to educational design research* (pp. 89-101). Proc. of a seminar conducted at East China Normal University. Enschede, the Netherlands: SLO.

Nunes, T., & Bryant, P. (1996). *Children doing mathematics*. Cambridge, MA: Blackwell.

Tobias, B. (2005). Mathematical word problems: The student's voice. *Perspectives in Education*, *23*, 61-75.

Vergnaud, G. (1979). The acquisition of arithmetical concepts. *Educational Studies in Mathematics, 10*, 263-274.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*, 83-94.

Verzosa, D. B., & Mulligan, J. T. (2011). Overcoming language barriers in word problem solving: A framework for intervention. In J. Clark, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds.), *Mathematics: Traditions and [new] practices* (Proc. of the conference of the Australian Association of Mathematics Teachers and the Mathematics Education Research Group of Australasia, pp. 892-299). Alice Springs, NT: AAMT-MERGA.

Wolff, E. (2011). Background and history: Language politics and planning in Africa. In A. Ouane & C. Glanz (Eds.), *Optimising learning, education and publishing in Africa: The language factor*. Hamburg, Germany: UNESCO Institute for Lifelong Learning and Tunis, Tunisia: Association for the Development of Education in Africa.

Wright, R. J., Martland, J., & Stafford, A. K. (2000). *Early numeracy: Assessment for teaching and intervention*. London: Paul Chapman Publishing.

COMPARISON OF MATHEMATICAL LANGUAGE-RELATED TEACHING COMPETENCE OF FUTURE TEACHERS FROM TAIWAN AND THE UNITED STATES

Ting-Ying Wang, Feng-Jui Hsieh, William H. Schmidt*

National Taiwan Normal University, Michigan State University*

The purpose of this study is to explore and compare the uniqueness of future lower secondary mathematics teachers from Taiwan and the US in mathematical language-related teaching competences (MTC-ML). A total of 161 and 171 future teachers from Taiwan and the US, respectively, participated in the study. The participants were drawn from samples of the TEDS-M, an international comparison study sponsored by IEA. Tests including items testing MTC-ML, regarding "thinking and reasoning" (TR) and "conceptually executing" (CE), were given immediately after the participants finished the TEDS-M questionnaires. The findings include that Taiwan performed better than the US in TR, but in CE, Taiwan and US performed the same. Compared to the US, Taiwanese future teachers were relatively stronger in TR than in CE.

INTRODUCTION

What teachers know or should be able to do is regarded as factors influencing students' mathematical achievements (eg. Capraro, Capraro, Parker, Kulm, & Raulerson, 2005). After Shulman (1986) raised the idea of pedagogical content knowledge, there was a widespread interest in studying teachers' pedagogical content knowledge (PCK), skills, and abilities (eg. Borko et al., 1992; Hiebert, Morris, Berk, & Jansen, 2007). Though lacking details of what constitutes PCK, this concept has been applied to research in a variety of areas, including mathematics. Different frameworks and corresponding measures have been developed to figure out what teachers possess (eg. Ball, Thames, & Phelps, 2008; Capraro, Capraro, Parker, Kulm, & Raulerson, 2005).

Hsieh (2009) has developed a framework to analyse teachers' mathematics teaching competences (MTCs), which has been used to analyse international comparative data about MTCs (Hsieh, Lin, & Wang, 2011). This structure contains twenty MTC elements, such as "mathematics thinking" and "mathematics language", which are engaged with three operations: recognition and understanding (RU), thinking and reasoning (TR), and conceptually executing (CE). Using "mathematics thinking" as an example, under the operation TR, one may generate a MTC "judging the types of students' mathematics thinking".

"Mathematical language" (ML) is an element in Hsieh's framework that plays an important role in teaching and learning (Laborde, 1990). During the past two decades, scholarly interest in international comparison of teachers' mathematics PCK (MPCK) has increased, such as the IEA sponsored Teacher Education and Development Study in Mathematics (TEDS-M). As a type of MPCK, MTC-ML should be studied at any chance in international or multinational comparisons. The authors of this paper have cooperated in studying future teachers' MPCK for a long time (eg. TEDS-M and see Schmidt, 2011). Therefore, the authors extended TEDS-M by including a comparative study on MTC-ML in Taiwan and the US. The main research questions are:

1. How do future teachers from Taiwan and the US perform in MTC-ML?
2. What uniqueness or patterns of MTC-ML performance do these two countries' future teachers possess?

RESEARCH METHOD

Conceptual framework

The issue of mathematical language is mentioned frequently in the concerned literature and is considered important.

<u>ML should be taught</u>: As Chapman stated, "learning mathematics requires transformational shifts between 'less mathematical' and 'more mathematical' language" (Herbel-Eisenmann, 2002). Students have to learn more formal language for mathematical enculturation, and therefore, teachers should spend time instructing students to tackle the language. (Barton & Heidema, 2000).

<u>ML students can understand</u>: Researchers suggest several ways for teachers to simplify their mathematical language in order to help students' comprehension. For example, teachers can make their mathematical language more informal or dialogic, or slow down the speed of information progression of their language (Herbel-Eisenmann, 2002; Street, 2005).

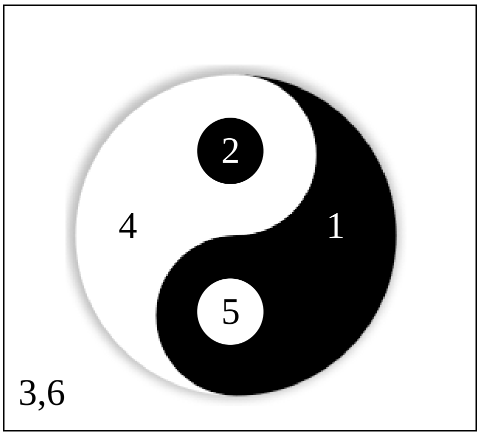
<u>Activities for teaching ML</u>: Researchers have suggested several methods to teach mathematical language, and have developed various teaching activities to improve students' comprehension (eg. Laborde, 1990; Rubenstein & Thompson, 2001).

<u>Influential Factors</u>: Researches mentioned several factors relating to mathematical language that affect students' understanding the language, such as the terms used in the sentences (Carter & Dearn, 2006), the amount of nominalizations of verbs, and the complexity degree of relations among mathematical objects (Laborde, 1990).

Based on these literatures and Hsieh's MTC framework, this study includes six MTC-ML issues that are structured in the Taijitu (太極圖) as shown in Figure 1.

Taijitu (太極圖) is a Chinese symbol which displays the concepts of yin (陰) and yang (陽). An S-shaped line divides a circle into two areas. One side is colored black and represents the yin element, the other side white, representing the yang element. Each side contains a dot (small circle) of the opposite color. This image represents yin and yang as two complementary and interacting parts. Our conceptual framework utilized a similar structure.

In Figure 1, areas 1 and 2 relate to *teaching*, 4 and 5 relate to *means*, and 3 and 6 relate to *factor (reason)* for teaching and means as described at the right part of the figure.



1: TR - Judging the types of ML which should be taught to students. (teaching)
2: TR - Judging the types of ML which are suitable to teach students with less advanced mathematics level. (teaching)
3: TR - Judging factors that influence students' understanding of ML. (factor)
4: CE - Using ML that students can understand. (means)
5: CE - Using appropriate activities to teach ML to students. (means)
6: CE - With appropriate reasons for using activities to teach ML. (factor)

Figure 1: Conceptual framework of MTC-ML in this study

Participants

This study used a randomly drawn sub-sample of the TEDS-M samples of lower secondary future mathematics teachers in Taiwan and the US. TEDS-M utilized a stratified multistage probability sampling design and drew the samples reflecting the distribution of future teachers at the end of their training in each country [1] (Tatto et al., 2009). The samples of this study include 161 Taiwanese samples out of the 365 Taiwanese TEDS-M samples and 172 US samples out of the 607 US TEDS-M samples. In Taiwan, for lower secondary level, there is only one program type, preparing to teach from grades 7 to 12, while in the US, one program type prepares to teach from grades 4 to 9 and the other prepares to teach from grade 6 to 12 (Tatto et al., 2011). The samples of this study included all these three types.

Design and instrument

Two different forms of test booklets, each was to be completed in 30-mins, were developed by the Taiwanese TEDS-M team and reviewed by the US team to measure MTC. For each country, all samples were randomly divided into two equal groups, and each group was administrated with one form of the tests. The test was administered as an extension of the TED-M survey in the way that each participant was asked to complete the test right after they finished the TEDS-M survey.

The test measured a variety of elements of MTC, including the MTC-ML which was measured through two questions with three items each. One question measured MTC-ML regarding the TR operation, the other measured CE operation. Each question's items used complex multiple-choice or constructed-response formats.

[1] The US limited its participation to public institutions.

Data processing

The complex multiple-choice items were scored as 1 point for correct and 0.5 for partially correct. For the constructed-response items, coding rubrics were developed according to the literature (described in the conceptual framework section) and reviewed by the TEDS-M team in Taiwan and the US. Each response was coded for its score and types according to the rubrics by at least two raters independently to ensure reliability and avoid bias. A third person gave final codes if the two raters' codes did not match. All constructed-response items were issued with partial credits. Some items scored as 2, 1, and 0.5 points, others scored as 1 and 0.5 points.

For each country, this study calculated the percentage of correct answers of each item as *item percent correct*. For the items with 1 and 0.5 points, the item percent correct was computed by the percentage of answers receiving 1 point plus half of the percentage of answers receiving 0.5 point. For items with 2, 1, and 0.5 points, the item percent correct was the sum of the percentage of 2-point answers, half of the percentage of 1-point answers, and quarter of the percentage of 0.5-point answers.

This study also calculated the average of all percent corrects of the TR items as *percent correct of TR*, similarly for CE. The items with maximum 2 points were weighted as 2 when calculating the average.

RESEARCH FINDINGS

Percent corrects of TR and CE

With regard to TR, Taiwan's percent correct was significantly higher than that of the US, while in terms of CE, Taiwan did not surpass the US. In regard to the comparison of TR and CE, the percent corrects of the US were the same. Compared to the US, Taiwanese future teachers were relatively stronger in TR than in CE (See Table 2). This finding demonstrates that Taiwan performed better than the US in the competences relating to thinking and reasoning about mathematical language and to making judgments and decisions about it in a teaching context. But when it comes to the competences of actually using appropriate mathematical language for students and developing or executing teaching activities to teach mathematical language, Taiwan and the US performances were the same.

	TR	CE	MTC-ML Difference
Taiwan	71.3(1.8)	54.1(2.6)	17.2^{**}
US-Public	49.9(2.3)	51.1(2.8)	-1.2
Country Difference	21.4^{**}	3.1	

Note. MTC-ML Difference=TR-CE. The numbers in the parentheses indicate *SE*.

$^{**}p$<.01.

Table 2: Percent corrects of TR and CE of Taiwan and the US

In-depth analysis of TR and CE

The responses patterns of Taiwan and the US deviate from each other drastically, regardless of items with similar or dissimilar percent corrects. Items #706D and #806B were chosen as examples to express the uniqueness and commonality of future teachers' responses in these two countries.

<u>MTC-ML regarding TR</u>

#706D was the last item of a series of items that were constructed in one question to gauge future teachers' competence relating to the use of different representations to introduce the concept of function. This item asked future teachers to determine what kind of student is most suitable for the Words representation—"If for a given value *x*, there is one and only one value *y* corresponding to it, then we say *y* is a function of *x*"—and why. Future teachers were expected to capture two types of reasons or factors that influenced students' understanding of mathematical language, one related to the characteristics of mathematical language (see Table 3, also for partial rubric) and the other related to the characteristics of students, such as logical thinking abilities, mathematics level, or affection towards mathematics or learning.

The results showed that in the item measuring the competence of judging reasons or factors influencing students' understanding of mathematical language, though Taiwan's percent correct (49.3%) of this item was significantly higher than that of the US (21.9%), a less than half correct still showed a lack of this competence.

In Taiwan, more than half (53.7%) of future teachers regarded the characteristics of mathematical language as factors influencing students' understanding of it, but only 11.5% of US future teachers could come up with any characteristic of mathematical language. The literature (Barton & Heidema, 2000) has documented that the characteristics of mathematical language would influence students' comprehension of mathematical language, but future teachers from the US seemed lacking the competence of seeing this. Almost all Taiwanese future teachers considered the characteristics of students as factors influencing students' understanding of mathematical language (97.3%). The percentage was significantly higher than that of the US (68.0%).

Characteristics of mathematical language	
Code 20:	When encountering Words representation, students need to transform it into a more concrete representation, such as pictures, expressions, etc; or construct corresponding concrete mental images of it in order to understand. (2 points, full credits)
Code 10:	Properties of mathematical language such as structural, formal, precise, or abstract influence student understanding. (1 point)

Note. Future teachers received codes as many as possible, and gained the points of the best code.

Table 3: Focuses of the characteristics of ML in the rubric of #706D

Code 20 is the code with full credits. In order to proffer this type of answer, responders must have a sense of the characteristics of mathematical language such as formal and abstract as described in code 10 (see Table 3), an idea of other forms of mathematics representations, and most important, an awareness of how mathematical language is mentally processed during the learning process. A total of 17.5% Taiwanese and 2.7% US future teachers were considered as having sound competence in this issue by proving this kind of answer. The following example 1 was a response from Taiwan and example 2 from the US that received code 20:

Example 1: 能由文字系統直接在腦海中反應出圖形或意象、…的學生。(Translation: Students who can directly produce figures or images in their minds from Words representation…)

Example 2: …The student must be able to process the actions of two variables in their heads for this to make any sense to them. They would also most likely have to visualize some sort of example in their head or make one up on paper for this work as well….

<u>MTC-ML regarding CE</u>

#806B uses a structural and formal sentence that appeared often in Taiwanese textbooks: “The product of the positive square roots of two positive numbers is equal to the positive square root of the product of the two positive numbers”. Its corresponding expression is $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. Our scenario follows: students understand the expression, but they did not understand the sentence. Their teacher desired to teach them to understand the sentence. #806B composed of two items. The first one (B1) asked future teachers to choose from five options two appropriate activities for the teaching of this sentence. The second item (B2) asked future teachers to state the reasons for their choices.

Many future teachers from Taiwan and the US (Taiwan : 62.1%; USA: 64.8%) chose appropriate activities. However, when it comes to B2, a significant portion of the reasons future teachers provided did not relate to mathematical language. Both percent corrects of the US (38.7%) and of Taiwan (26.5%) were low. Activity B, D and E were considered correct choices while the latter two were regarded as best choices. In these two activities, students would learn how to deconstruct mathematical sentences into its components, and learn to judge the relationships between the components. These skills may be used in future studies of mathematical language if the students have gained them. This idea can be exemplified best by one of the US responses,” …It is important for students to be able to break apart statements and make sense of it so they can do the same with other algorithms of statements they may come across in the future….” There were fewer Taiwanese future teachers choosing activities D or E than future teachers from the US, which is a warning to Taiwan.

The activity most future teachers chose was activity B. This option was not initially regarded as the correct answer by this study. In this activity, one had to understand the given sentence before he/she could write down the corresponding mathematical expression. Therefore, a reverse situation could exit in this activity. However, by interviewing, this study discovered that future teachers would explain the sentence in more detail in order to write down the corresponding expression to their students, thus complementing activity B and becoming the acceptable answer.

	Description of teaching activities	Taiwan	US-Public	Difference
A	Re-teach the related concept.	8.7(2.8)	15.6(7.5)	-6.9
B	Write down the corresponding mathematical expression step by step.	84.0(3.8)	63.7(5.7)	20.3**
C	Provide numerical expressions and ask students to describe it with words.	55.6(5.4)	38.1(10.9)	17.5
D	Decompose the sentence into parts and explain the meaning part by part.	23.2(4.4)	39.5(6.6)	-16.3*
E	Ask students to start with a part of the sentence and gradually expand it, and judge the relations among the mathematical objects.	17.0(3.3)	26.4(4.9)	-9.5

Note. $^{*}p$<.05. $^{**}p$<.01.

Table 4: Percentages of choosing each teaching activity of B1

Some future teachers chose teaching activities B to E, of which learning mathematical language was the focus, but the reasons they provided for their choices revealed that they indeed focused on the learning of mathematical concepts rather than language (Taiwan: 59.2%; the US: 23.9%). Taiwan's percentage was significantly higher than that of the US by 35.3%. This is not a pleasant result for Taiwan.

CONCLUSION

Compared to the US, Taiwanese future teachers showed strength in the competences related to thinking and reasoning about mathematical language. However, both countries shared some weaknesses. Very few future teachers were able to analyse why students could not understand mathematical language through analysing students' minds or cognitions based on the features of mathematical language. There were also few future teachers who were able to choose the teaching activities that could cultivate students' mathematical language related competences. This revealed that future teachers in both countries lack some important MTC-ML, which were important when teaching students to learn mathematical language. More opportunities for future teachers to develop these competences are needed.

References

Ball, D. L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Barton, M. L., & Heidema, C. (2000). *Teaching reading in mathematics: a* supplement *to "Teaching Reading in the Content Areas Teacher's Manual (2nd Ed.)"*. Washington, DC: Office of Educational Research and Improvement. (ERIC Document Reproduction Service No. ED469111)

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194-222.

Capraro, R. M., Capraro, M. M., Parker, D., Kulm, G., & Raulerson, T. (2005). The mathematics content knowledge role in developing preservice teachers' pedagogical content knowledge. *Journal of Research in Childhood Education*, *20*(2), 102-118.

Carter, T. A., & Dean, E. O. (2006). Mathematics intervention for grades 5-11: teaching mathematics, reading, or both? *Reading Psychology, 27*(2-3), 127-146.

Herbel-Eisenmann, B. A. (2002). Using student contributions and multiple representations to develop mathematical language. *Mathematics Teaching in the Middle School, 8*(2), 100-105.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education*, *58*(1), 47-61.

Hsieh, F.-J. (2009). *Research on the development of the professional ability for teaching mathematics in the secondary school level (3/3)* (Rep. No. NSC 96-2522-S-003-008). Taiwan: National Science Council. (In Chinese)

Hsieh, F.-J., Lin, P.-J., & Wang, T.-Y. (2011). Citing advance online publication: Mathematics-related teaching competence of Taiwanese primary future teachers: Evidence from TEDS-M. *International Journal on Mathematics Education*. Advance online publication. doi: 10.1007/s11858-011-0377-7

Laborde, C. (1990). Language and mathematcis. In Nesher, P., & Kilpatrick, J. (Eds.), *Language and comprehension* (pp. 53-69). New York: Cambridge University Press.

Rubenstein, R. N., & Thompson, D. R. (2001). Learning mathematical symbolism: challenges and instructional strategies. *Mathematics Teacher, 94*(4), 265-271.

Schmidt W. H., Blömeke, S., Tatto, M. T., Hsieh, F.-J., Cogan, L., Houang, R. T. et al. (2011). Teacher Education Matters: A Study of Middle School Mathematics Teacher Preparation in Six Countries. NY: Teacher College Press.Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Street, B. (2005). The hidden dimensions of mathematical language and literacy. *Language and Education, 19*(2), 135-140.

Tatto, M. T., Schwille, J., Senk, S., Rodriguez, M., Bankov, K., Reckase, M, D., Ingvarson, L., Peck, R., & Rowley, G., Dumais, J., Carstens, R., Brese, F., Meinck, S. (2009). *Teacher education study in mathematics (TEDS-M): Technical summary.* East Lansing, MI: Teacher Education International Study Center, College of Education, Michigan State University.

Tatto, M. T., Schwille, J., Senk, S., Ingvarson, L., Rowley, G., Peck, R., Bankov, K., Rodriguez, M., & Reckase, M. (2011). *The teacher education study in mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics.* New York: Springer.

AN INVESTIGATION ON THE PROJECT-BASED LEARNING OF MATHEMATICALLY GIFTED ELEMENTARY STUDENTS

Wen, Shih-Chan — An-ho Elementary School, New Taipei City, Taiwan.

Leu, Yuh-Chyn — National Taipei University of Education Taipei, Taiwan.

The paper is to investigate the project-based learning of elementary mathematically gifted fourth graders. It is a case-study. There are 21 students in the class and the teacher has 18 years of teaching experiences. The research period is 11 weeks. We prove that, through a comprehensive course structure, students can experience the praxis of project-based learning. The course significantly facilitates students' cooperative learning ability and communication/sharing ability. There is also an improvement in their critical thinking ability, even though less than two above abilities. However, students do not make much progress in their problem-solving ability. We propose three teaching strategies in amending the course to assist students to develop their ability on solving the "driving question".

INTRODUCTION

One of the focuses on gifted education is to promote gifted students from knowledge receivers to knowledge producers so that they have greater contribution to the society (Tsai, 1998). Much research (such as VanTassel-Baska & Stambaugh, 2006) indicates that the learning needs of the gifted students include: providing critical thinking activities, engaging in challenging learning activities, participating in cooperative learning (with peers of similar mental and intellectual development), providing opportunity to investigate the inter-relationship among each knowledge domain, utilizing the knowledge into the real world, and emphasizing on creativity and problem-solving ability.

Project-based learning is one of main teaching strategies in gifted curriculum (Davis & Rimm, 2005). It helps students to develop much different ability: high-order thinking, self-learning, data collection and analysis, peer learning and communication skills (Thomas, 2000). Baş & Beyhan (2010) comments that the investigation on project-based learning is still very recent and immature and there is not substantial data on empirical research. The goal of this research is to investigate the learning process of elementary mathematically gifted fourth graders during a project-based learning and their performance on high-level thinking ability, cooperative learning ability and communication/sharing ability.

LITERATURE REVIEW

According to the literature, there are three categories of the characteristics of project-based learning.

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 219-226. Taipei, Taiwan: PME.

The first category relates to the characteristics of “driving question” . Some characteristics of driving question include: realistic, endurance and relating to student’s daily life. It should be able to challenge students and sustain their motivation in further investigation. It should be open-ended, multiple and it’s not a single-answer question. It should offer students a realistic and meaningful context that they can integrate and apply their knowledge from different subjects (Thomas, 2000).

The second category relates to the teaching methods of project-based learning. In project-based learning, it emphasizes that the role of the teacher is to guide and facilitate students to investigate and construct new knowledge on their own. Through group discussion, public presentation and peer review, students can amend their results for better final products (Krajcik & Mamlok-Naaman, 2005).

The third category focuses on the abilities students develop through project-based learning. Project-based learning can facilitate students’ high-order thinking skills (Gültekin, 2005), cooperative learning abilities (ChanLin, 2008), communication and sharing abilities (Grant & Branch, 2005). Most of the researches mentioned above did not provide explicit explanation on the teaching strategies employed, in other words, how teachers guide students through the process of project-based learning. The high-order thinking ability mentioned in this paper includes critical thinking ability and problem-solving ability (Miri, David, & Uri, 2007).

Bell(2010) argued that standardized tests cannot evaluate students’ improvement on critical thinking and problem-solving skills through the process of project-based learning. Hence the present research conducts qualitative analysis and focuses on the development of three abilities on these elementary mathematically gifted students during the process of project-based learning: high-order thinking ability, cooperative ability and communication/sharing ability.

RESEARCH METHODS

Research Subjects

This is a case-study research. The research subjects are twenty-one elementary mathematically gifted fourth graders. All of them had taken a mathematics achievement test and had scores higher than two standard deviations. The coefficient of Cronbach α of this mathematics achievement test is 0.90 and this test demonstrates satisfactory validity test. The teacher is an in-service elementary school teacher who has eighteen years of teaching experiences. The main teaching method is group discussion.

The Course Structure and Its Implementation

The researchers and the teacher drew up two principles in current curriculum design, according to the characteristics of project-based learning. The first is to design questions on the worksheets according to the characteristics of driving question. The second is to emphasize that the teacher needs to guide students to do the following: engaging in group discussion that centers on the driving question, accomplishing their products on their own and presenting their results publicly and accepting peer review. The goals of the design principles are to facilitate students' development on skills of high-order thinking, cooperative learning, communication and sharing.

The teaching procedures are as followed. The teacher took students to the "Aiming high for a Low-Carbon Taiwan" Exhibition to guide students to realize the importance of "Energy Conservation and Carbon Reduction" from their life experiences and promote students' learning interests in this project. The teacher divided 21 students into four groups and provided each group different worksheets. Due to the limitation of paper length, here we only present a group's learning portfolio on this project-based learning. We chose Group Three and their driving question on the worksheet states: "The conditions for wind power are: long wind period, high wind speed, steady wind force and free of terrain and other surface obstacles. So, to set up wind turbines, we need to consider the geographical location and its wind quality. The cost for wind power is now comparable to the cost of generating power from gas in Taiwan. . . . . . If the cost for wind power and gas power is comparable, then why don't we change all power supply to wind power?" This question composes of the characteristics of open-ended, covering different perspectives, challenging and endurance. Students need to investigate actively and associate much knowledge from various resources. They need to carry out interdisciplinary learning and it would be hard for them to solve this question individually. They need to depend on the cooperative learning methods of group discussion and communication/sharing to solve the question.

Each group needs to do two project presentations (with an interval of 6 weeks). The reason for two presentations (instead of one) is to provide students an opportunity, through the process of teacher and peer review and questioning, to modify and revise their projects. The purpose is to sustain students' interests in investigating the topic and to cultivate students how to self-correct and refine their results, through public presentation and peer review. The research lasts for eleven weeks. Each student needs to fill out a form to evaluate the performance of themselves throughout this project-based learning as a practice of reflection and feedbacks.

Data Collection

The procedures of data collection are as followed:

1. Collecting worksheets on the day of field-trip: All worksheets were collected to serve as a baseline for their further performance.
2. Video-taping and transcribing: Each group had to present their results twice. Both presentations were video-taped and transcribed.
3. Collecting self-evaluation sheets: After the second presentation, each student filled out an evaluation sheet. It includes four rating questions (scales from one to ten, please refer to Table One in Results and Discussion) and one open-ended question. The open-ended question is "What do you gain most in this project-based learning?"

RESULTS & DISCUSSION

The results prove that there is significant progress on Group Three students' cooperative learning ability and communication/sharing ability and an enhancement in their critical thinking ability. However, students do not have much improvement on their problem-solving ability. We include Group Three's worksheets, two presentations and evaluation sheets to demonstrate their learning process and performance on this project-based learning. We also include one paragraph on the learning performance on all students on this project-based learning. In the results, the denotation of S42 means the number two student in group four and T represents the teacher.

On the day of students went to see an exhibition, Group Three's answer for the driving question, after group discussion, is that "The noise produced by the wind turbines affects the habitat of sea birds". They didn't consider other factors listed on the worksheet, such as wind period, wind speed and wind stability. It signifies that the students in Group Three did not yet develop diverse thinking at that time.

One the day of first presentation, we excerpt some of the transcripts (April 23rd, 2011)

S32: The first reason is the loud noise would affect the sea birds. And not every place in Taiwan has big wind. So the power supply would not enough.
S42: We are surrounded by seas here in Taiwan. There is always strong wind from the sea. Why don't we just build all wind turbines by the sea?
S32: There is a distance required between each wind turbine. So it may be because there is not enough area to build wind turbines.

Compared to the worksheet, S32 pointed out that "the wind force at site" and "insufficient power supply" might be the possible reasons. This first presentation by Group Three indicates that, even though they were still immature on the comprehension, analysis and perspective of the driving question, they did collect data from different resources and they provided much more knowledge to other groups of students.

S12: If there is only a limited amount of available space, can we make the fans smaller and set up more turbines?
S32: If there is a large surface area of the fans, then that should make the fans to turn faster.
S42: For larger fans, it means they are heavier. And more wind is used to turn the fans. It seems that larger fans do not generate more power.
S32: Smaller fans mean smaller surface areas.
S33: The fans need to be large enough to produce power.
T: ……There are many factors to be considered. . . . . The structure of the wind turbine is important.

Though that the students in Group Three could communicate and share from different perspectives with other students, they did not consider the aspects provided in the driving question. It indicates there was still room for improvement for their critical thinking ability and problem-solving ability.

According to the researchers' and the teacher's understanding, the biggest problem for wind power in Taiwan is that the naturally varied weather patterns in Taiwan lead to the instability of wind power. Students in Group Three only focused on the problems of wind turbines (the size and weight of fans) and neglected "wind period", "wind speed" and "wind stability" as possible reasons, as mentioned in the driving question.

We propose three reasons for this outcome. The first one is that since the students in this project-based learning are mathematically gifted, they are better at mathematics knowledge. Therefore they tend to consider quantitative numbers, such as "size" and "weight".

The second reason is that each group's students only had their worksheet at hand, but not other groups' worksheets. As we (the researchers and the teacher) were setting up the driving question for each group, we designed them from different topics in different subjects so that the students would have an opportunity to engage in interdisciplinary learning. Since each group is on a different topic, the teacher did not provide other groups' worksheets. However, from the abovementioned teaching experience, the researchers assumed that this condition can be alleviated if students could have other groups' worksheets, they could remind the presented students the information they overlooked in the worksheet.

The third reason is that when students were not on the right track, the teacher could provide some guiding question. When S32 reported that the sole reason for insufficient supply of wind power in Taiwan to be that "*not every place in Taiwan has big wind*", the teacher can ask some probing questions, such as "What places have strong wind and what places have weak wind in Taiwan?", "For the same place, is the wind strong throughout the whole year?" and "What seasons have strong wind and what seasons have weak wind?". By doing so, students may start to think about some aspects mentioned in the driving question, such as wind period, wind speed and wind stability. The teacher's guiding questions may improve the quality of the presentation.

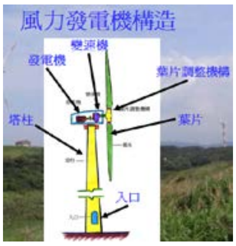


Figure 1

After six weeks, students did the second presentation. The following is the excerpt of some of transcripts for the second presentation of Group Three (June 4th, 2011). They presented the structure of a wind turbine (as in Figure 1), as well as provided some more information about wind turbine.

S33: The material of the wind turbine will influence the price. The wind turbine with higher power capacity requires higher start-up wind speed. The larger the wind turbine, the harder it can get started.

S32: We can make the fans larger but they need to be light. So the larger wind turbine would have higher cost.
S33: We can make the fans lighter to lower the cost.

We can see that, compared to the first presentation, Group Three had much more complete content in their second presentation. Unlike the first presentation in which they focused only on “not every place has strong wind” and “insufficient power supply”, they took account of the questions mentioned by their peers such as the size and weight of the fans and collected much more data/information. They digested, analyzed and interpreted the data/information and extended the factors such as the material of the fans and the start-up speed. They also proposed some solutions for the driving question, such as “making the fans light” and “increasing the size of the fans but reducing their weight”. It demonstrates that there is an improvement in students’ critical thinking ability.

However, we found that they still did not answer the question put forth by S42 (“We are surrounded by seas here in Taiwan. There is always strong wind from the sea. Why don’t we just build all wind turbines by the sea?”). They still did not take account of some aspects mentioned in the driving question, such as wind period, wind speed and wind stability. This indicates that there is still room for improvement on students’ problem-solving ability. The researchers recommend that students may be able to notice other factors mentioned in the driving question if the teacher could guide students to organize and evaluate questions put forth by their peers after their first presentation.

We present the scores of the self-evaluation sheets of all students from Group Three in Table 1.

Questions \ Student No.	S31	S32	S33	S34	S35	AVE
Provide unique point of view	7	8	6	7	5	6.6
Do well on the job assigned	8	7	7	8	8	7.6
Communicate and share positively with others	8	9	9	8	7	8.2
Participate actively in the discussion	9	9	9	9	8	8.8
Total Scores	32	33	31	32	28	31.2

Table 1: Students’ Self Evaluation from Group Three

According to Table 1, the total score of each student in Group Three is around 30. This demonstrates that all students in Group Three regard themselves to have significant improvement in those four items. However, the lowest average score in these four questions is on “providing unique point of view”. This indicates that students’ improvement on high-order thinking ability is far less than their improvement on cooperative learning ability and communication/sharing ability.

There is also one open-ended question in the self-evaluation sheet, “what do you gain most in this project-based learning?” The following is each student’s response: S31 answered “how to accept other people’s opinions”; S32, S33 and S34 all answered “the ability to cooperate and communicate/share with classmates”; S35 answered “the ability to cooperate with classmates”. Three out five students all had the same feedbacks that the biggest gain from this project-based learning is “the ability to cooperate and communicate/share with others”. In Group Three, four students (S31, S32, S33 and S34) had reported their results in either or both of the two presentations but one student (S35) did not report his results publicly during the two presentations. The data indicate that there is a consistency between the learning performance of the two presentations and the evaluation sheets.

Generally speaking, out of the 19 students (2 students were absent on the day of self-evaluation) in the class (including Group Three), 14 students (74%) have the total score higher than 30 in their self-evaluation sheet. They all regard themselves to have significant progress in three areas: high-order thinking ability, cooperative learning ability and communication/sharing ability. And all students all scored the lowest in the question of “providing unique point of view”, indicating that all students’ improvement in high-order thinking is far less than the improvement in cooperative learning and communication/sharing ability.

CONCLUSION & SUGGESTION

For the last few years, project-based learning has been applied in different subjects domains, such as technology education (Gültekin, 2005), information education (ChanLin, 2008), and science education (Grant & Branch, 2005). However, there is not much research of project-based learning centered on mathematically gifted elementary students. We provide an empirical research to prove that, through a good design of driving questions and appropriate teaching methods, students can demonstrate significant improvement on cooperative learning ability and communication/sharing ability. They also have improvement on their critical thinking ability, even though less than the two above abilities. Unfortunately, due to a shortcoming in teacher’s teaching strategy and skill, students’ improvement on “problem-solving skill” is not satisfactory.

We propose that for a project-based learning to be successful, the factors in curriculum design should include more than “the characteristics in driving question” and “teaching method”, as reported in previous literature. The teaching strategy involved in conducting a project-based learning is also a critical factor. We propose three relevant teaching strategies. First, the teacher needs to have the awareness and skills on how to use his/her interdisciplinary knowledge to guide students' discussion and make students to think and research in the right track according to the aspects mentioned in the driving question. Second, the teacher needs to guide students on how to organize questions put forth by their peers and how to assess whether these questions fit into the aspects mentioned in the driving question. Third, other than providing students with different research topics, the teacher should make each group's worksheet available to other groups as well so that there can be more valuable feedbacks from other groups during peer review. We recommend that the teacher should utilize these three teaching strategies while conducting project-based learning. The results of the present paper can be the guideline for designing curriculum on project-based learning in teacher training program.

REFERENCE

Baş, G. and Beyhan, Ö. (2010). Effects of multiple intelligences supported project-based learning on students' achievement levels and attitudes towards English lesson. *International Electronic Journal of Elementary Education*, *2*(3), 365-386.

Bell, S. (2010). Project-based learning for the 21st century: Skills for the future. *The Clearing House*, *83*, 39-43.

ChanLin, L. (2008). Technology integration applied to project-based learning in Science. *Innovations in Education and Teaching International*, *45*(1), 55-65.

Davis, G. A., & Rimm, S. B. (2005). *Education of the gifted and talented* (5th ed.). New York: Allyn and Bacon.

Grant, M. M., & Branch, R. M. (2005). Project-based learning in a middle school: Tracing abilities through the artifacts of learning. J*ournal of Research on Technology in Education*, *38*(1), 65-98.

Gültekin, M. (2005). The effect of project based learning on learning outcomes in the 5th grade social studies course in primary education. *Educational Sciences: Theory and Practice 5*(2): 548–56.

Krajcik, J. S. & Mamlok-Naaman, R. (2005). Using driving questions to motivate and sustain student interest in learning science. In Tobin, K (Ed.), *Teaching and Learning Science: A Handbook*, vol.*2*, pp.527-543. Westport, CT: Greenwood Publishing Group.

Miri, B., David, B.-C., & Uri, Z. (2007). Purposely teaching for the promotion of higher-order thinking skills: A case of critical thinking. *Research in Science Education*, *37*(4), 353-369. Springer. Retrieved from http://www.springerlink.com/index/10.1007/s11165-006-9029-2

Thomas, J. W., (2000). *A review of research of project-based learning*. Retrieved online 02/17/ 2010 from http://www.bobpearlman.org/BestPractices/PBL_Research.pdf.

Tsai, T.M. (1998). Training Gifted Students to Be Knowledge Producers. *Gifted Education Quarterly*, *69*, 1-5.

VanTassel-Baska, J., & Stambaugh, T. (2006). *Comprehensive curriculum for gifted learners* (3rd ed.). Needham Heights, MA: Allyn & Bacon.

DESIGNING PROFESSIONAL LEARNING TASKS FOR MATHEMATICS LEARNING TRAJECTORIES

P. Holt Wilson[1], Paola Sztajn[2], Cyndi Edgington[2]

The University of North Carolina at Greensboro[1], North Carolina State University[2]

Our paper presents an emerging set of learning conjectures and design principles to be used in the development of professional learning tasks that support elementary teachers' learning of learning trajectories. We outline our theoretical perspective on teacher knowledge of learning trajectories, review the literature concerning mathematics professional learning tasks, offer a set of initial conjectures about teacher learning of learning trajectories, and articulate a set of principles to guide the design of tasks. We conclude with an example of one learning trajectory professional learning task taken from our current research project.

INTRODUCTION

In recent years, researchers have extended Simon's (1995) hypothetical learning trajectory construct to include empirically-supported descriptions of the ways in which student thinking evolves over time. Based on syntheses of the research literature, clinical interviews, teaching experiments, and large-scale assessment data, learning trajectories (LTs) have come to represent empirically defined descriptions that trace the ways in which students' informal ideas mature through appropriate instructional opportunities into sophisticated mathematical understandings (Confrey, Maloney, Nguyen, Myers, & Mojica, 2009). In the United States, LTs are purported to be a "tool for reform" (Corcoran, Mosher, & Rogat, 2009) and are growing in their influence over national standards development, assessment systems design, and mathematics curricula development.

Early accounts suggest that teachers' knowledge of an LT improved their own mathematics content knowledge (Mojica, 2010), guided their instructional decisions (Wilson, 2009), and enhanced their abilities to use student thinking (Clements, Sarama, Spitler, Lange, & Wolfe, 2011). Though there is a call to the research community to "translate the available LTs into tools for teachers" (Daro, Mosher, & Corcoran, 2011), research on teacher learning of LTs is only beginning to emerge. Empirical work is needed to examine not only the ways in which teachers come to learn about these trajectories but also to define the ways in which teacher educators can design professional learning tasks (PLTs) that support such learning.

The goal of this paper is to present an emerging set of learning conjectures and design principles to be used in the development of PLTs that support elementary teachers' learning of LTs. As part of a larger design experiment to examine a professional development setting in which elementary teachers learn about one particular learning trajectory, this paper highlights the design aspect of the empirical work under way in

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4, pp. 227-234. Taipei, Taiwan: PME.

the project. We begin the paper with background information about LTs and outline our theoretical perspective on teacher knowledge of LTs. Next, we review the existing literature concerning the mathematics PLTs. In the tradition of design research, we offer a set of initial conjectures about teacher learning of LTs and articulate a set of principles to guide the design of PLTs for mathematics LTs. We conclude with an example of one LT PLT taken from our current research project to illustrate the ways that our conjectures and principles may be instantiated.

BACKGROUND

LTs have been defined as "descriptions of children's thinking and learning in a specific mathematical domain, and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking" (Clements & Sarama, 2004, p. 83). More recently, Confrey and her colleagues defined an LT as "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey et al., 2009, p. 347). This definition establishes the impossibility of separating student learning from instruction in school settings; it centralizes the fundamental role that mathematics teachers play in the growth of students' understanding of mathematics.

Much of the research in developing LTs makes a distinction between the logic of the discipline and the logic of the learner or the learners' cognitive development (Corcoran et al., 2009). This distinction indicates that, rather than organizing mathematical topics and learning experiences for children based on logical analysis of disciplinary knowledge, LTs allow mathematics instruction to be based on "research about how students' learning actually progresses" (p. 8). This distinction shifts the organizing focus of mathematical instruction from the discipline to the students. Thus, a fundamental characteristic of LTs is attention to the ways a learner's logic matures into the logic of the discipline.

THEORETICAL PERSPECTIVES

In our work, we consider Ball, Thames and Phelps' (2008) notion of *mathematical knowledge for teaching* (MKT) in light of the distinction between the logic of the discipline and the logic of the learner. Ball and colleagues built on Shulman's (1986) work on understanding teacher knowledge to conceptualize MKT grounded in an examination of the mathematical knowledge teachers need for teaching. At the heart of their MKT framework was a careful analysis of the mathematical demands teachers face in practice. Their work resulted in "refinements to the popular concept of pedagogical content knowledge and to the broader concept of content knowledge for teaching" (p. 390).

MKT is organized as two large domains: pedagogical content knowledge (PCK) and subject matter knowledge (SMK). Each of these domains is further divided into three categories of teacher knowledge. Within the PCK domain of the MKT framework, Knowledge of Content and Students is defined as the "knowledge that combines knowing about students and knowing about mathematics" (p. 401) so that teachers may anticipate what students are likely to think as well as what they find confusing, interesting, or motivating. Knowledge of Content and Teaching refers to knowledge about the design of instruction for a particular content, including choosing examples, sequencing tasks, and evaluating advantages and disadvantages of various representations, in ways that bring together mathematical understanding and an understanding of the pedagogical choices that affect student learning. Finally, Knowledge of Content and Curriculum is placed as part of PCK.

Within SMK, Ball and colleagues explained that Common Content Knowledge is the knowledge of mathematics not specific to teaching whereas Specialized Content Knowledge is the mathematical knowledge not typically needed for purposes other than teaching. This specialized knowledge is exemplified as the knowledge teachers need to explain patterns in student errors or decide whether a nonstandard approach would work in general. A third category, Horizon Content Knowledge, represents "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403).

Although we recognize the importance of examining the knowledge demands of teaching and the contribution Ball and colleagues made in defining MKT, our work on LTs focuses on the careful attention to the logic of the learner and the analysis of the relation between this logic and the mathematical disciplinary knowledge. This attention necessitates an interpretation of the MKT categories in light of the relation between the learner and the discipline, particularly when trying to understand teacher learning about LTs. From an LT perspective, we consider the PCK domain as related to the knowledge that emerges from a focus on the learner's cognitive development and is based on teachers' understandings of the learner's logic and how it progresses over time. The SMK domain represents aspects of teacher knowledge that are centred on the logic of the discipline and allow teachers to situate the logic of the learner within the larger framework of shared mathematical knowledge. Together, teachers' PCK and SMK guide the instructional work needed to support students' movement along various levels of an LT as informal ideas develop into sophisticated mathematical knowledge. Therefore, we contend that teachers' knowledge of LTs spans both the PCK and the SMK domains of MKT, and teachers' learning of LTs impact both these domains.

TEACHER LEARNING AND PROFESSIONAL LEARNING TASKS

Silverman and Thompson (2007) propose a framework for developing MKT. In their model, MKT develops when a teacher uses a key developmental understanding (Simon, 2006) of a mathematical idea to consider what students might understand about the idea, how they might come to a deeper understanding including types of learning activities to support that deepening of understanding, and the ways that new understanding positions students to learn other mathematical ideas. Implicit in this framework is the notion that teachers need to examine their own mathematics in relation to the logic of the discipline prior to considering students' mathematics. Thus, an understanding of students' mathematics follows teachers' development of their own mathematics. Further, the work of developing teachers' MKT starts with attention to SMK.

Similarly, Silver and colleagues (2007) outline a cycle of PLTs to develop teachers' MKT that use practice-based materials and always begins with an activity in which teachers solve a mathematics problem themselves. Next, teachers individually read and analyse a narrative case followed by a whole group discussion and concluding with collaborative work where teachers consider implications for their own practice. The authors note that the use of practice-based materials in this PLT cycle "integrates and interweaves several domains of knowledge germane to teaching: mathematics, pedagogy, and student thinking" (p. 266).

Contrary to this notion of first developing teachers' own understanding of mathematics, Phillip, Thanheiser, and Clement (2002) and Phillip (2008) suggest that in the case of *elementary* teachers, it is important to attend to the student prior to the mathematics. These authors propose that children are at the center of what elementary teachers care for, and therefore elementary teachers see the mathematics through the child. For Phillip and colleagues, the work of developing elementary teachers' MKT starts with a focus on students, which we interpret as a focus on the logic of the learner and, therefore, a focus on PCK.

In reporting findings from a mathematics professional development program with experienced teachers, Swan (2007) lists a set of general principles taken from a review of the literature for the design of tasks—whether they focus on PCK or SMK. These principles indicate that tasks need to include a focus on significant cognitive obstacles, understand and build from students' prior knowledge, and create "surprise, tension, and cognitive conflict" (p. 219). Similarly, Smith and Boston (2009) suggest that mathematics professional development that embraces a social constructivist perspective are built around PLTs that take into account teachers' prior knowledge and beliefs and purposefully create cognitive conflicts between teachers' prior views and new conceptions.

LEARNING TRAJECTORY PROFESSIONAL LEARNING TASKS

In her review of a special issue of the *Journal of Mathematics Teacher Education* entitled 'The Role and Nature of Mathematics-Related Tasks for Teacher Education', Zaslavsky (2007) compared the creation of mathematics PLTs to design experiment research. Initial selection or creation of a PLT is informed by professional literature, theories of learning, and personal experiences. Iteratively, PLTs are implemented and refined. Our current research project involves partnering with elementary grades teachers in a professional development setting designed to support their learning of the equipartitioning learning trajectory (Confrey et al., 2009) and therefore the development of their MKT around LTs. In the context of this work, we conduct our design experiment. The overall research question guiding our work focuses on understanding the ways in which teachers use their existing MKT to engage with the PLTs designed to support their learning of the LT. The purposeful use of a design experiment methodology within this professional development setting is meant to provide “systematic and warranted knowledge about learning and to produce theories to guide instructional decision making” (Confrey, 2006, p. 136). Design experiments “entail both ‘engineering’ particular forms of learning and systematically studying those forms of learning” (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003, p. 9). In the case of this study, we are “engineering” LT PLTs.

In creating our PLTs, we begin with a two-part learning conjecture to guide our design work: (1) PLTs that focus on the logic of the learner engage teachers in using MKT to learn about LTs; and (2) Despite the tasks’ focus on the learner, teachers use both pedagogical and subject matter knowledge domains of MKT to learn about LTs. The first part of our conjecture builds on the work of Phillip, Thanheiser, and Clement (2002) and Philipp (2008) and stands in contrast to Silverman and Thompson’s (2008). It indicates that our design of PLTs begin with attention to elementary teachers’ PCK. The second part of our conjecture highlights our claim that teachers’ MKT about LT spans both the PCK and SMK domains, combining the findings from Mojica (2010) and Wilson (2009). Further, it recognizes that elementary teachers’ own content understanding is often underdeveloped, making it important for PLTs to provide teachers with opportunities to learn about SMK despite the an initial focus on PCK.

Together, our conjectures indicate that PLTs focused on the logic of the learner engage teachers in using all domains of MKT as they learn about an LT. Guided by this conjecture, we propose and use in our own research work a set of design principles to develop our PLTs for LTs. They state that LT PLTs: a) attend mostly to the PCK aspect of the LT; b) embed opportunities for teachers to examine all aspects of their MKT; c) employ instructional sequences that start with practice-based activities that challenge elementary teachers’ views of students’ mathematics and mathematics learning; and d) use artifacts similar to the ones researchers used in developing the LT to highlight the logic of the learner.

The first two principles are informed by our theoretical perspective on teacher knowledge and follow from our learning conjectures. PLTs that are closely aligned with teachers’ daily practices (Smith, 2001) frequently draw upon their PCK domains. Nonetheless, purposeful design may provide opportunities for teachers to engage both their PCK and SMK domains when learning about LTs. The third principle draws on literature on mathematics PLTs. LT PLTs should be grounded in teacher practice (Smith, 2001), create surprise or cognitive conflict (Swan, 2007), and allow for discussion and opportunities to consider learning in relation to their own practice (Silver et al., 2007). The final principle is based the work of researchers using various materials for professional learning including clinical interviews, video recordings, and analysis of student work.

AN EXAMPLE

In what follows, we provide a brief example of how we used our design principles from above to create LT PLTs as sequences of activities to engage and support teachers in developing their understandings of single or coupled levels of the LT. These PLTs and the ways in which teachers learn from engaging with them are the focus of our design research.

Each sequence begins with a *challenge* where we pose a question related to students' mathematics and present teachers with artifacts from practice (Smith, 2001) including videos of clinical interviews with children or student written work on diagnostic assessment tasks. The goal of these challenges is to problematize teachers' current views of the logic of the learner, focusing on teachers' PCK. Following the challenge, teachers engage in an *exploration* activity that requires them to consider all aspects of their MKT to examine and resolve the challenge at hand. Resolutions from these explorations are *formalized* in whole-group discussion and are used in an *application* closely related to instruction, such as work with curriculum materials or examination of videos from whole-class instruction.

One of our LT PLT sequences aims for teachers to learn three cognitive processes children must coordinate when equipartitioning as described in two levels of the LT. These three equipartioning criteria (Confrey, Rupp, Maloney, & Nguyen, in review) include creating the correct number of groups or parts, creating equal-sized groups or parts, and exhausting the original collection or whole. We begin the sequence by challenging teachers with the question, “Based on their written work, what do these students know about equipartitioning?” and providing them with carefully selected written responses to diagnostic assessment items which exhibit different partial understandings of the three criteria. In whole group discussion, teachers explore the similarities and differences among the work samples. Guided by the facilitator, the teachers formalize their observations as the three equipartitioning criteria and then apply this learning to an analysis of a new set of written work samples.

The goal of this LT PLT is to problematize teachers' current PCK about equipartitioning (principle a) by challenging teachers' views of students' mathematics and learning in the practice-based activity of analysing student written work (principle c). It engages teachers in analysing responses to diagnostic assessment items used by researchers when developing the LT (principle d). Finally, it provided an opportunity for teachers to engage with their SMK as well as their PCK, specifically the part-whole relationship that exists only when the three criteria are met (principle b).

CONCLUSION

Research on LTs is quickly moving from an agenda for examining student learning to an agenda for promoting teacher learning and researchers need to attend to the ways in which teachers come to learn about the recently developed, empirically-tested descriptions of how students learn about specific mathematics content topics over time. The focus on teacher learning highlights the need to attend to teachers MKT in relation to LTs. In our design experiment research, we claim that teacher knowledge of LTs expand all domains of MKT, and we have proposed a set of learning conjectures and design principles to guide our work in developing and empirically testing a set of LT PLTs. We contend that our learning conjectures and design principles are an initial attempt at explicitly articulating required features of PLTs that aim at supporting teacher learning of LTs.

Note: This report is based upon work supported by the National Science Foundation under grant number DRL-1008364. Any opinions, findings, and conclusions or recommendations expressed in this report are those of the authors and do not necessarily reflect the views of the National Science Foundation.

REFERENCES

Ball, D., Thames, M. & Phelps. G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59(5),* 389-407.

Clements, D. & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6(2),* 81-89.

Clements, D. H., Sarama, J., Spitler, M. E., Lange, A. A., and Wolfe, C. B., (2011). Mathematics learned by young children in an intervention based on learning trajectories. *Journal for Research in Mathematics Education, 42*(2), 127-166.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*, 9-13.

Confrey, J. (2006). The evolution of design studies as methodology. In Sawyer, K. (editor) *Cambridge Handbook of the Learning Sciences*. Cambridge Press, Cambridge, England.

Confrey, J., Maloney, A. P., Nguyen, K. H., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd conference of the psychology of mathematics education: In search for theories in mathematics education* (Vol. 2, pp. 345-352). Thessaloniki, Greece: PME.

Confrey, J., Rupp, A., Maloney, A., & Nguyen, K. H. (in review). Toward developing and representing a novel learning trajectory for equipartitioning. *Journal for Research in Mathematics Education.*

Corcoran, T., Mosher, F. A., & Rogat, A. (2009). Learning progressions in science: An evidence based approach to reform. Retrieved July 2, 2009, from http://www.cpre.org/images/stories/cpre_pdfs/lp_science_rr63.pdf

Daro, P., Mosher, F., & Corcoran, T. (2011). Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction. Retrieved May 2, 2011, http://www.cpre.org/images/stories/cpre_pdfs/learning%20trajectories%20in%20math_ccii%20report.pdf

Mojica, G. (2010). *Preparing Pre-Service Elementary Teachers to Teach Mathematics with Learning Trajectories*. Unpublished dissertation. North Carolina State University.

Philipp, R. (2008). Motivating prospective elementary school teachers to learn mathematics by focusing upon children's mathematical thinking. *Issues in Teacher Education, 17(2),* 7 – 26.

Phillip, R., Thanheiser, E., & Clement, L. (2002). The role of a children's mathematical thinking experience in the preparation of prospective elementary school teachers. *International Journal of Educational Reform, 27(2)*, 195-213

Schulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15(2),* 4–14.

Silver, E., Clark, L., Ghousseini, H., Charalambous, C., & Sealy, J. (2007). Where is the mathematics? Examining teachers' mathematical learning opportunities in practice-based professional learning tasks. *Journal of Mathematics Teacher Education, 11*, 261-277.

Silverman, J. & Thompson, P. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499 – 511.

Simon, M. (2006). Key developmental understandings in mathematics: A direction for investigating learning goals. *Mathematical Thinking & Learning*, 8(4), 359–371.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26(2)*, 114-145.

Smith, M. S. (2001). Practice-*Based Professional Development for Teachers of Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Smith, M. & Boston, M. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education, 40(2),* 119 – 156.

Swan, M. (2007). The impact of task-based professional development on teachers' practices and beliefs. *Journal of Mathematics Teacher Education, 11*, 217-237.

Wilson, P.H. (2009). *Teachers' Uses of a Learning Trajectory for Equipartitioning*. Unpublished dissertation. North Carolina State University.

Zaslavsky, O. (2007). Mathematics-related tasks, teacher education, and teacher educators. *Journal of Mathematics Teacher Education, 11*, 433-440.

UNIVERSAL AND EXISTENTIAL QUANTIFIERS REVISITED

Ruili Ye

Hostos Community College/ City University of New York

Bronislaw Czarnocha

Hostos Community College/ City University of New York

Investigation into students learning of logical quantifiers is extended to include both universal and existential quantifiers. The work confirms, in a spectacular fashion, the impact of natural language on the mathematical understanding of negation by identifying, during the student interview, a source of misconception initiated from incorrect French/English translation. The paper extends the analysis of the data into root causes of student difficulties and with the help of innovative teaching-research interviews, which identifies possible routes for the improvement of learning, when needed. This way, every research clinical interview of a student about mathematics can be enhanced, without loosing its scientific character, as it also becomes a carefully crafted, research-based tutoring session, exemplifying the reduction of time lag between research and practice to zero.

INTRODUCTION

This report continues the discussion of the development of mathematical reasoning, and in particular understanding and negating quantifiers "all" and "some" (Bardelle, 2011; Ferrari, P.L. 2004; Dubinsky, 1997). There are interesting reasons why the negation of quantifiers is full of challenges for students, to a larger extent than for example the negation of the conjunction and the disjunction or even the conditional. Transformation of the conditional to the equivalent disjunction reduces it to the negation of that disjunction. While (Bardelle, 2011) and (Ferrari, P.L. 2004) point to the impact of the natural language register upon understanding the meaning of mathematical statements, following the line of reasoning of, e.g. (Cornu, 1981, Mason and Pimm, 1984), Dubinsky and his co-workers initiated a large investigation program to understand learning of quantification assisted by the general Piaget-based theory of learning APOS (Action, Process, Object, Schema) with a moderate success (Dubinsky et al. 1988; Dubinsky, 1997; Dubinsky and Yiparaki, 1997). They point to student cognitive difficulties in constructing the meaning of the quantifiers and their negations. Meanwhile several empirical reports analyzed the difficulty of negating single quantifier statements (Lin et al, 2003, Zepp et al, 1987). The present work continues the path of empirical investigations motivated by the excellent (Bardelle, 2011) discussion, which analysed the difficulties of the general quantifier without including the existential one. We extend the analysis of difficulties to include the existential quantifier and have designed the set of questions (Appendix) similar to those used by (Bardelle, 2011).

2012. In Tso,T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 235-242. Taipei, Taiwan: PME.

The report presents the results and analysis of the pilot teaching experiment investigating understanding of the single quantification by students in three sections of the first college level course Introduction to College Mathematics for Liberal Arts majors at the Hostos Community College of the City of New York in the Bronx. The college is the only bilingual (English/Spanish) institution of higher education in the CUNY public system, and the majority of students (65%) are ethnically Spanish speaking, while around additional 15 % are speakers of languages other than English and Spanish. Thus it was expected that we might observe not only impact of the English natural register upon the meaning of mathematical statements but the impact Spanish or French languages as well.

The course introduces students to elementary notions of Set Theory, Logic, Probability and Numerical Patterns. Each instructor has the freedom to design their own course syllabus, relating to the four specialized domains, as well as the content of all tests and the final exam.

The aim of this presentation is twofold: on one hand, it is to complete the assessment of the learning process for both quantifiers, and on the other, to demonstrate a new approach to a clinical student interview, called the teaching-research student interview, which of course, has a twofold nature as well: to inquire and to teach. The interviews, on the basis of this investigation, are teaching-research interviews because they fulfil two roles, that of investigation and that of teaching. They are also named in the sequel as “investigative teaching”.

THE THEORETICAL FRAMEWORK/METHODOLGY

The study has been conducted within the theoretical framework of Teaching-Research NYCity model (Czarnocha, B. 2002, Czarnocha and Prabhu, 2006) utilizing cyclical investigations for the design of classroom intervention, its implementation, collection and analysis of data and the refinement for the intervention of the next cycle. The research questions of the study were:

What are students’ difficulties in negation and understanding of the quantified propositions in the bilingual context?

What are the routes of improvement for students’ understanding and mastery of the negation of quantified propositions?

Each cycle of the teaching experiment in TR NYCity methodology thus plays a dual role, that of an inquiry and that of inquiry-based improvement of learning. TR NYCity model develops adaptive methods of instruction (CPRE Report, 2011).

TEACHING EXPERIMENT

The teaching experiment was conducted by two collaborating instructors whose teaching techniques were similar, except for the degree of symbolic notation employed. Instructor T1 used a minimal degree of symbolism, relying on verbal explanations. Instructor T2 taught the quantifiers topic with intensive use of symbolical notation. In the second test, after the students were introduced to using universal quantifier $\forall$ and existential quantifier $\exists$, instructor T2 asked them to first symbolize the original statements, then write the negation of the symbolized version, and ultimately translate the symbolized negations back to English. As an example:

Original sentence: "All men are mortal."

Symbolize: $(\forall x)M(x)$

Negation: $\neg(\forall x)M(x)$

Move the negation to inside the scope of the quantifier: $(\exists x)\neg M(x)$

Translate the negation into English: Some men are not mortal.

Following this procedure, there was an immense improvement of the results.

This difference in teaching was used to investigate the differences in effectiveness of each technique on student learning of quantifiers. Whereas together both classes had N=112 students, only 43 completed the pretest and posttests components of the data. The pretest contained 4 propositions to negate (Propositions 1,3,5,7) to assess student initial knowledge of the subject. Instruction of the quantifiers and their negations followed the assessment and, after this instructional intervention, a posttest was administered (Appendix). Each instructor taught the topic according to individual preference. The essential difference was the degree of introduced abstract logical notation.

	Propositions to transform	T1, Pr/Po N=32, + %	T 2 Pr/Po N=17, + %
1	Negation: <u>All men are mortal.</u>	14/75	37/100
2	Equivalence: <u>It is not the case that all men are mortal.</u>	34	18
3	Negation: <u>All integers are whole numbers.</u>	0/84	44/88
4	Equivalence: <u>It is not the case that all integers are whole numbers.</u>	28	36
5	Negation: <u>Some athletes win gold medals.</u>	11/16	44/88
6	Equivalence: <u>It is not the case that some athletes win Gold medal.</u>	19	36
7	Negation: <u>Some numbers are not even</u>.	14/63	30/88
8	Equivalence: <u>It is not the case that some numbers are even.</u>	75	27

Fig. 1 The table of pre/post tests results (Pr/Po).

Data is from two instructors (N_1 =32,N_2=17); …/… indicates the percentage of correct responses on the pre test and on the post test. It is clear from the pre-test scores that the level of initial preparation for the concept was quite different in both cohorts. This difference in preparation does not allow any conclusions to be drawn concerning the appropriate level of symbolic notation. The excerpt below shows the need for the intermediate step between the verbal and symbolic descriptions. The pre test/post test comparison is given only for the task to negate solely, as the task to find the equivalent statement was judged to be too difficult for the pre-test. Small positive results in the task equivalence confirm the difficulty of the transition. This, as the interviews indicate, is rooted in the absence of clarity as to the meaning of negation of the proposition. The collected data demonstrates 76%-86% of incorrect student responses, on the pre test, indicate the process of negating the verb in the proposition as mathematical negation. Confirming, yet again, the impact of natural language on mathematical understanding. Two broad tendencies are revealed by the data in the above table:

1. A decisive weakness in the notion of equivalence of negated propositions. (Statements 2, 4, 6 (with one out layer: statement 8), as compared with the strength of responses in statements 1, 3, 5, 7).

2. An unclear evidence concerning the existential quantifier: weakness in 5, 6 and strength in 7 with 8 inconclusive.

To obtain better insight into students' problems, we turn to analysis of the interviews conducted from a sampling of students. Students were asked to come to the interview on the basis of the types of errors they made on the written test. The interviews were semi-structured. Each student was asked to provide the answer with explanation to at least one universal and one existential quantifier question. The interviews were conducted before a Smart Board, which videotaped the writings on the board together with the accompanying audio, without the images the student or instructor. This set up satisfies the requirements of IRB. The recorded interviews are available to the public on request. The interviews were divided into two sections: first, a diagnostic section, which ascertained the knowledge of the student, then a didactic section, addressing the diagnosed weaknesses of the student. The teaching strategies discovered during the interviews become part of the instruction for the next iteration of this teaching experiment next semester.

ANALYSIS OF THE INTERVIEWS

Missing Integration

From this example of investigative teaching, a missing aspect to student understanding was discovered during the interview. Once identified, the incorporation of this missing aspect immediately helped the student to understand different cases.

1 Instructor: What does this proposition mean for you, "It is not the case that all men are mortal"

2 Student: "Not all men are mortal."

3 Instructor: Makes sense. What does it mean for you: "Not all men are mortal"?

4 Student: It means that "some men are mortal and some men are not mortal."

5 Instructor: And which of those two is the negation of "All men are mortal"?

6 Student: "Some men are mortal."

At the same time student starts introducing the variable P symbolizing a proposition and writes ~(~P) saying that therefore it's P.

7 Instructor: OK. What is P? Is it "All men are mortal?"

8 Student: Yes.

9 Instructor: So what we have in front of us is we have this (he circles "It is not the case..") which is a negation ~, and this (he circles "All men are mortal") is our P, proposition. So we have ~(all men are mortal).

10 Student: (writes under the text) Some men are not mortal.

11 Instructor: What made you so certain suddenly?

12 Student: Once you put a symbol like there, then it's become clear. Before I tried to figure it from the words, but here, once you have a formula…

Two issues are revealed in the above fragment:

Issue 1: Lines (5, 6) reveal a frequently encountered conceptual misunderstanding in perceiving "Some men are mortal" as the negation of "All men are mortal", as if "Some men who are mortal" is not a subset of "All men who are mortal" and therefore cannot be its negation. In the fragment above, the instructor attempted to direct student attention towards that contradiction (line 5), but the student took another route.

Issue 2: Lines (9 - 12) reveal that student understanding lacked the intermediate step between the verbal and symbolic levels of mathematical expression, which requires the symbol of negation of the proposition "All men are mortal" in front of it. This hypothesis was rapidly confirmed by the student's response to the next question:

13 Instructor: Find the equivalent to "It is not the case that some athletes win Gold medals."

14 Student: writes the following 3 lines:

It is not the case (that some athletes win Gold medals).

~ (that some athletes win Gold medal).

All athletes do not win Gold medal.

Coordination between verbal and symbolic levels will become part of next semester's iterated instruction.

Impact of the natural language on student understanding

The claim of (Ferrari, 2004) and (Bardelli, 2011) that natural language has an impact on student understanding of logical statements has been confirmed in our investigations. The negation of "All men are mortal" as "Some men are mortal" (line 5, 6) is common among a large number of students. From the linguistic perspective, it is connected to negating the subject of the sentence. A second error encountered during the negation of "All men are mortal" is "All men are not mortal", where the negation of the verb occurs. The impact of natural language discussed in (Bardelle, 2011) generates both errors. Our hypothesis for the organization of the pedagogy in the next cycle iteration of the teaching experiment is: The <u>negation of the subject</u> AND <u>the verb</u> is the content of logical negation of the proposition, which of course distinguishes it from the negations encountered in a natural language.

An excellent example of the investigative teaching (teaching-research) in a bilingual (English/French) mathematical environment provides a very strong evidence for such an impact. The following series of instructor's questions and student responses led them to identify the source of difficulty in the incorrect translation of a corresponding phrase from French by the interviewed student:

1 Instructor: Find the equivalent statement to "It is not the case that some athletes win Gold medals."

2 Student: "All athletes win Gold medals."

3 Instructor (perplexed): Find the equivalent statement to "It is not the case that some numbers are even."

4 Student: "All numbers are not even."

5 Instructor (intrigued): Find the equivalent statement to "It is not the case that some athletes are winners."

6 Student: "All athletes are not winners"

7 Instructor (certain of formed hypothesis): Find the equivalent statement to "It is not the case that some athletes are winners."

8 Student: "All athletes win Gold medals".

After the last exchange it became clear that the issue is in the grammar, and on closer inspection it turned out that the issue was caused by inappropriate translation from French. The full transcript of the discovery will be presented at the conference.

CONCLUSIONS

Our central aim was to investigate two research questions: the first inquires into the state of student knowledge vis-à-vis negation of both quantifiers, and the second inquires into the routes of improvement of that knowledge, once the need for improvement had been identified.

State of student knowledge

The pretest demonstrates that students enter the subject strongly impacted by their intuitive knowledge of the logical negation of the proposition as "verb negation". [Some integers <u>are</u> whole numbers]→ [Some integers <u>are not</u> whole numbers]. That bias changes a significant degree upon the instruction. The interviews reveal that student understanding has misconceptions of which careful elimination is necessary to reach complete understanding.

Discovery of the routes for improvement takes place primarily during the individual interviews with students. Two strategies had been discovered: creating the verbal symbolic notation through which the transition to the symbolic one can be made, and using one particular structure "It is not the case that …" as the probe in the clarification of the meaning of negation.

Appendix: The instrument of assessment

Choose correct answer to each question.

NOTE: There are TWO different INSTRUCTIONS among the problems:

Find the <u>negation</u> or find the <u>equivalent</u>:

1A) Which of the following sentences <u>is the negation</u> of **"All men are mortal**"?

a. No man is mortal.

b. Some men are mortal.

c. Some men are not mortal.

1B) Which of the following sentences <u>is equivalent</u> to **"It is not the case that all men are mortal**"?

a. Some men are mortal.

b. Some men are not mortal.

c. No men are mortal.

2A) Which of the following sentences <u>is the negation</u> of the sentence: **"All integers are whole numbers"**?

a. Some integers are not whole numbers.

b. Some integers are whole numbers.

c. No integers are whole numbers.

2B) Which of the following sentences is <u>equivalent</u> to **"It is not the case that all integers are whole numbers"**?

a. Some integers are not whole numbers.

b. Some integers are whole numbers.

c. No integers are whole numbers.

3A) Which of the following sentences <u>is the negation</u> of the sentence: **"Some athletes win Gold medals"**?

a. No athlete wins Gold medal.

b. Some athletes do not win Gold medal

c. All athletes do win Gold medals.

3B) Which of the following sentences <u>is equivalent</u> to the sentence: **"It is not the case that some athletes win Gold medal"**?

a. No athletes win Gold medals

b. Some athletes do not win Gold medals.

c. All athletes win Gold medals

4) Which of the following sentences <u>is the negation</u> of the sentence: **"Some numbers are not even"**?

a. All numbers are not even

b. All numbers are even

c. Some numbers are even.

5) Which of the following sentences <u>are equivalent</u> to the sentence:" **It is not the case that some numbers are even"?**

a. Some numbers are even

b. All numbers are even.

c. All numbers are not even

References

Bardelle, C. (2011) Student Understanding of the negation of statements with universal quantifier. *Proceedings of the 35th Conference of the International group for the Psychology of Mathematics Education*, V.2, pp.97-104.

Daro Phil, Frederic Mosher, Corcoran Tom (2011) *Learning Trajectories in Mathematics,* Consortium of Policy Research in Education.

Dubinsky, E., Etlerman, F., Gong, K (1988) The student construction of quantification. For the Learning of Mathematics.

Dubinsky, E. (1997) On Learning Quantification. *Journal of Computers in Mathematics and Science Teaching* 16(263), pp.335-362.

Dubinsky, E. Yiparaki, (1997) O. Predicate Calculus and Learning of Students. http://www.math.kent.edu/~edd/publications.html#C.)

Ferrari P.L., (2004) Mathematical Language and Advanced Mathematical Learning. In Johhnsen Heinees & A. Berrit Fuglestead (Eds) *Proceedings of the 28th conference of the International group for the Psychology of Mathematics Education* V.2, pp. 383-390

Lin, F.L., Wu Yu,, Lee, Y.S. Student understanding the proof by contradiction. *Proceedings of the 27th Conference of the International group for the Psychology of Mathematics Education*, V.4, 443-449, Honolulu, Hawai, USA.

Mason, J., Pimm,D., (1984) Generic examples: Seeing the General in the Particular. *Educational Studies in Mathematics,* 15, pp.227-289

Zepp,R., Monin, J., Lei, C.L. (1987) Common Logical Errors in English and Chinese. *Educational Studies in Mathematics,* V.18, N1, pp 1-17

SHORT ORAL COMMUNICATIONS

RECOGNIZING TEXTS IN UNDERGRADUATE MATHEMATICS

Hoda Ashjari, Eva Jablonka, Christer Bergsten

Linköping University, Luleå University of Technology, Linköping University

Findings from interviews investigating how Swedish first year engineering students recognize undergraduate mathematics texts as being more or less "mathematical". The results indicate a relation between the students' understanding of the principles for knowledge classification and their success in their mathematics studies

Introduction We aim to get insight into the students' awareness of the type of mathematics that is institutionalised in the beginning undergraduate mathematics courses – in comparison to upper secondary school mathematics. Attempting to clarify some of the issues related to the meaning of "higher level" of mathematics, in particular the extent to which the students are aware of changes in knowledge criteria.

Theoretical background For success, students need to understand the principles for distinguishing between contexts and recognize the speciality of the discourse, thus acquire the *recognition rules*. (Bernstein, 1981) We are interested in the students' recognition of what counts as legitimate mathematics texts, a necessary condition for their own capacity of producing such texts. The knowledge classification of undergraduate university mathematics creates specific subject-related recognition rules that differ from university mathematics. We investigate whether and on what grounds the students were able to distinguish between different mathematical texts that resemble weaker or stronger principles of knowledge classification and whether there are differences in relation to achievement.

Methodology The students' were confronted with four different mathematical texts and asked which of those appear "more mathematical" to them. For the selection and description of the different texts, as well as for the analysis of the students' responses, we employed analytical tools developed in the context of systemic-functional linguistics. (cf. Halliday & Hasan, 1989)

Findings & Discussion Whether recognition of differences between texts would be necessary for success, can be answered positively. The outcomes reflected that recognition of the knowledge classification is necessary but not sufficient for success. We observed a relation between the students' understanding of the principles for knowledge classification (*recognition rule*) and their success in their examinations.

References

Bernstein, B. (1981). Codes, modalities, and process of cultural reproduction: A model. *Language in Society*, *10*(3), 327-363.

Halliday, M.A.K., & Hasan, R. (1989). *Language, context, and text: aspects of language in a social-semiotic perspective* (Second Edition.), Oxford: Oxford University Press.

VIDEO AS RESOURCE FOR MATHEMATICAL VISUALIZATION

Rúbia Barcelos Amaral

UNICAMP - Brasil

In this paper I analyze the use of video in classrooms as well as the process of mathematical visualization. From an example regarding the concept of scale factor between similar objects, I argue that the use of video can contribute to the construction of external visual images, which often support the construction of mental images, which some students have difficulty to conceive. Therein, video can be an educational resource to enlarge mathematical visualization, enhancing learning.

The production of video for educational purposes is not a novelty. In the Mathematics Education environment, in particular, it is also possible to refer to investigations regarding the use of video. Some example is the work of Wood and Petocz (1999) that say it "is clear from students' comments that video should be used more frequently in Mathematics teaching and learning" (p.227).

Regarding visualization, it's clear that it's part of the mathematical "doing". Further studies stress the role of visual thinking and visual representations as mediating artifacts for the teaching and learning of mathematics and the relevance of spatial thinking in early mathematical development (Góes & David, 2010). When studying Mathematics, the visualization is associated to the ability of interpreting and understanding pictorial information. For that, two processes can occur: to interpret a visual image, or create a visual image from a non-figural information. Visualization is also considered as a "process of forming images (mentally or with paper and pencil or other technology), used in order to get a better mathematical understanding and stimulate the process of mathematical discovery" (Borba & Villarreal, 2005, p. 80).

Based on the analysis of an example, the video "A certain scale factor", from a Brazilian collection of multimedia materials in the area of Mathematics – the Project M^3, I believe it is possible to discuss how some perspectives of mathematics visualization and use of video could be related, in the sense that the use of videos could help in the processes of teaching and learning mathematics. For students who have difficulty in constructing mental images, this may be a "barrier" to learn mathematics concepts. Therefore, the use of other visual resources, not only the textbook or other printed media, can be a way to expand the formation of images by students.

References

Borba, M.C. & Villarreal, M.E. (2005). Humans-with-Media and the Reorganization of Mathematical Thinking: Information and communication Technologies, Modeling, Visualization and Experimentation. New York: Kluwer.

Góes, M & M. David. (2010). Visualization: a mathematical mediating artefact for the architectural students. In Pinto, M.M.F. & Kawasaki, T. (Eds.), *Proc. 34th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 37-37). Belo Horizonte, Brazil: PME.

Wood, L.N. & Petocz, P. (1999). Video in Mathematics learning at the secondary-tertiary interface. The challenge of diversity: *Proceedings of the Δ'99 Symposium on Undergraduate Mathematics*, Rockhampton, Queensland, 223-228.

ASSESSING YOUNG CHILDREN'S UNDERSTANDING OF MULTIPLICATION

Patrick Barmby, David Bolden, Stephanie Raine & Lynn Thompson
Durham University, UK

Research (for example Anghileri 2000; Greer 1992) has suggested that multiplication is significantly more difficult for young children than the addition and subtraction operations. Nunes and Bryant (1996) suggest that "multiplication and division represent a significant qualitative change in children's thinking" (p.144). One source of difficulty is the range of contexts in which the concept of multiplication can arise. Greer (1992) highlights a range of different 'classes of situations' for multiplication, including equal groups, equal measures, rate, multiplicative comparison, multiplicative change and Cartesian product situations. This study is part of a funded project aimed at developing primary children's understanding of mathematics. As part of this project, a test of primary children's understanding of multiplication was developed. Although there has been research on developing tests of understanding in other mathematical topics (e.g. fractions) and in particular for older (e.g. late primary or secondary) children, there has been little reported work on tests for multiplication for younger primary children. In this study therefore, a test of multiplication was constructed based on the range of contexts associated with multiplication. The 19-item test was administered to a sample of mainly Year 3 (7-8 years) pupils (*n*=272) with a small sample of Year 4 (8-9 years) pupils (*n*=18) in 10 primary schools. The age of the pupils meant that test questions were read out by the teacher, and most were multiple choice items. The data obtained from the test was analysed using Rasch analysis in order to examine the reliability of the overall measure, and the validity of individual items. Overall, the measure was show to have a good reliability indicated by a Cronbach α value of 0.79. The test results were also validated against the results of task-based interviews carried out with a small sample of 12 pupils, with pupils' responses (both verbal and written) recorded on a tablet computer. The study highlighted the relationship between the test scores of pupils and the quality of pupil responses in the task-based interviews. Recommendations for further improvements in the test are put forward.

References

Anghileri, J. (2000). *Teaching number sense*. London: Continuum.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Nunes T., & Bryant. P. (1996). *Children doing Mathematics*. Oxford: Blackwell Publishers.

THREE QUALITY COMPONENTS OF EPISTEMIC PROCESSES

Angelika Bikner-Ahsbahs, Julia Cramer, Thomas Janßen
University of Bremen (Germany)

The German-Israeli project "Effective knowledge construction in interest dense situations" (GIF grant: 946-357 4-2006) has the aim to network two epistemic action models, to further develop the two models and their combination, and to further understand what drives an epistemic process socially and individually. So far, the concept of general epistemic need (GEN) has been worked out (Kidron, Bikner-Ahsbahs, Dreyfus, 2011). The GEN is an unspecific need capturing what drives epistemic processes and leading to the development of a new construct rooted in seeds for constructing processes. Sometimes while trying deeply to solve a problem the GEN is transferred to a very specific need that arises when the students search for an adequate mathematical idea that might solve the problem at stake. This need is called the need for a new construct (NNC) (Kidron et al. 2011). It is expressed on the meta-cognitive level and drives the epistemic process in a strongly directed manner. Besides actions on the meta-level epistemic processes comprise two different qualities of epistemic actions: producing knowledge directly and supporting to produce knowledge in an indirect manner. In this short oral, the last two qualities of actions and their creation are described integrating them into a 3-component-model. They comprise types of epistemic actions that are illustrated by data from a process of solving a parabola task. These types are reconstructed by analyses using two epistemic action models, the RBC-model (Schwarz, Dreyfus, & Hershkowitz, 2009) and the GCSt-model (Bikner-Ahsbahs, 2005). The analyses were conducted in three steps: Reconstructing single epistemic actions and their roles within the epistemic process, comparison of these actions in order to aim at building ideal type actions that differentiate the two different roles (direct knowledge production, indirect knowledge supporting) and the meta-level of acting, and finally structuring all the ideal type actions into three components.

Reference

Bikner-Ahsbahs, A. (2005). Interesse zwischen Subjekt und Situation. Empirisch begründete Konstruktion einer Theorie interessendichter Situationen. Hildesheim, Berlin: Verlag Franzbecker.

Kidron, I., Bikner-Ahsbahs, A., & Dreyfus, T. (2011). How a general epistemic need leads to a need for a new construct: A case of networking two theoretical approaches. In: Pytlak, M.; Rowland, T. & Swoboda, E. (Eds.), Proceedings of the 7th Congress of the European Society for Research in Mathematics Education (pp.2475-2485). Rzeszów: University of Rzeszów, Poland.

Schwarz, B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In Baruch Schwarz, Tommy Dreyfus & Rina Hershkowitz (2009): Transformation of Knowledge through Classroom Interaction (pp. 11-41). New York: Routledge

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 248. Taipei, Taiwan: PME.

USE OF DIFFERENT SYMBOLIC REPRESENTATIONS IN THE TEACHING OF COMBINATORICS

Rute Borba, Fernanda Barreto and Juliana Azevedo

Universidade Federal de Pernambuco, Brazil

Combinatorial problems are enriching situations in terms of mathematical learning and development because they are characterized by a great variety of contexts and problem types that require careful analysis in order to consider how elements must be chosen and combined. A previous study (Borba, Pessoa, Barreto and Lima, 2011) had shown that children, young people and adults present limited understanding of combinatorial problems before formal instruction. The main difficulty shown was in listing all possibilities of a given combinatorial situation through systematic thinking and organization of given information. Fischbein, Pampu and Minzat (1970) had observed that the use of trees of possibilities allowed 10-year-old students to progress in their combinatorial reasoning by helping them to systematically enumerate combinations. Two studies that aimed to help students – adults in initial process of schooling and Primary School children – in the development of their combinatorial reasoning are here presented. After only one session of intervention, there was a decrease of incorrect answers; there were more partially correct answers and some completely correct answers – not present in the pre-test. The interventions proposed – by use of systematic lists and trees of possibilities – did not teach how to reproduce procedures but offered participants the opportunity to think about combinatorial relations and to improve their initial understanding of Combinatorics. Thus, the understanding of combinatorial situations may be increased by the use of powerful symbolic representations offering opportunities to think about combinatorial relations and properties.

Acknowledgements: This research was partially financed by the Fundação de Amparo à Ciência e Tecnologia do Estado de Pernambuco (Facepe – APQ 1095-7.08/08) and by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (MCT/CNPq – 476665/2009-4).

References

Borba, R., Pessoa, C., Barreto, F.& Lima, R. (2011). Children's, young people´s and adults' combinatorial reasoning. In: B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 2, pp. 169-176). Ankara, Turkey: PME.

Fischbein, E., Pampu, I., & Minzat, I. (1970). Effects of age and instruction on combinatory ability in children. The British Journal of Educational Psychology, nº 40.

DEVELOPING STATISTICAL LITERACY: A DESIGN EXPERIMENT APPROACH TO PROBABILITIES APPLIED IN NATURAL SCIENCES[1]

Sandra Campelos & Darlinda Moreira

Universidade Aberta / UIDEF, Portugal

Statistics plays a key role preparing young people to interpret and make critical evaluations about media information as well as to develop students' capacities for reasoning and communication, providing them with mathematical techniques and procedures (Batanero, 2002). Different views and goals for Probabilities and Statistics teaching, the latest ideas about statistical literacy, and the new Portuguese Mathematics Curriculum (Ponte et al., 2007) were taken in consideration to develop a teaching experiment within a three academic years intervention project implemented in two classes of a Portuguese school. The follow up was done by the teacher/researcher.

In the first phase (7th grade) tasks were implemented to detect potential problems in the organization and interpretation of dummy data and analyze the ability of application of statistics to everyday situations. The second phase (8th grade) included tasks aiming at studying the advancements made and examining how the statistical expertise is mobilized to other areas of knowledge and situations. In 9th grade tasks implemented on the subject of Probability and Statistics, aimed to explore communication skills in a probabilistic context and to characterize how students apply Stochastic reasoning to other school subjects, namely to Natural Sciences (genetics) and to gambling situations. This paper describes one of these tasks involving peer work and collective discussion.

A research methodology of design experiments was followed based on data that was collected and videotaped in its natural environment and the teacher/researcher was a primary mean of data gathering. Some details to future memory, allowing and supporting reflection after the classes, were recorded by the teacher in a field diary.

The results suggest that although students have not shown significant difficulties in the understanding, resolution and application of probability to the context of Natural Sciences, they show difficulties in articulating the justification in writing as well as in the forms of mathematical representation, such as tables and diagrams. Although students haven't yet reached a statistical literacy to a desired level, this is emerging from their continuous work with statistics.

References

Batanero, C. (2002). Los Retos de la Cultura Estadística. Jornadas Interamericanas de Enseñanza de la Estadística. Buenos Aires. Conferencia inaugural.

Ponte, J. P. et al. (2007). *Programa de Matemática do Ensino Básico*. Lisboa. ME, DGIDC.

(1) Paper produced within Project P3M supported by FCT, Portugal (**PTDC/CPE-CED/098931/2008**).

THE MATHEMATICS INQUIRY-BASED CLASSROOM PRACTICE OF CELIA[1]

Ana Paula Canavarro
University of Évora and Research Unity of Institute of Education of University of Lisbon, Portugal

Hélia Oliveira
Institute of Education of University of Lisbon, Portugal

Luís Menezes
Higher School of Education of Viseu and CI&DETS, Portugal

This study has been developed in the context of the research project P3M Professional Practices of Mathematics Teachers. One of its main aims is to propose a framework for mathematics inquiry-based classroom practice, combining theoretical perspectives and the transversal analysis of cases of experienced teachers of different school levels (one per level) that regularly conduct inquiry-based teaching of mathematics at different school levels. Developing an inquiry-based approach is a complex practice for most of the teachers (Stein, Engle, Smith, & Hughes, 2008) and we hope that our intended framework could contribute as a tool for reflection about this practice, namely in the context of teacher education.

We adopt an interpretative approach for the investigation, considering the importance of capturing teachers' perspectives to understand the actions they performed while teaching (Sowder, 2007). Data were collected by observing 2 lessons of each teacher and by pre and post interviews, the latter supported by videoepisodes of the lessons.

Here we focus on the case of the primary teacher, Célia, teaching her 4th grade class. To describe her practices, we adopted a four phases model for the lesson structure: 1) Introduction of the task; 2) Development of the task; 3) Discussion of the task; and 4) Systematization of the mathematical learning. For each phase, we describe the actions that Célia intentionally performs with two main interrelated purposes: to promote the mathematical learning of the students and to manage the students/class.

References

Sowder, Judith T. (2007). The Mathematical Education and Development of Teachers. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 157-223). Charlotte, NC: Information Age Publishing.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Helping teachers learn to better incorporate student thinking. *Mathematical Thinking and Learning*, 10(4), 313-340.

[1] This paper is supported by national funds through FCT – Fundação para a Ciência e Tecnologia in the frame of the Project P3M - *Professional Practices of Mathematics Teachers* (contrato PTDC/CPE-CED/098931/2008).

MENTAL COMPUTATION WITH RATIONAL NUMBERS: AN EXPERIENCE WITH GRADE 5 STUDENTS

Renata Carvalho Carrapiço
Unidade de Investigação do Instituto de Educação
Universidade de Lisboa

João Pedro da Ponte
Instituto de Educação
Universidade de Lisboa

Mental computation with rational numbers is a challenge both for students and teachers and tends to be undervalued by school curricula and teaching practice. Teachers need to know how to propose and carry out mental computation activities in the classroom and also to understand students' strategies, to be able to help them to develop more efficient and faster strategies. The literature on students' mental computation strategies show that they need to understand (i) numeration, that is the size and the value of numbers, (ii) the effect of an operation on a number, (iii) number facts, and (iv) how to make estimates to check reasonableness of a solution (Heirdsfield, 2011). Empson, Levi and Carpenter (2010) suggest that children use a set of strategies when they work with fractions, based on mathematical relationships, that are essential to the understanding of algebra, namely relational thinking. Such strategies arise from the children's understanding about numbers and operations and familiar numerical relationships familiar, that they may use to establish relationships and do computations.

This study, designed as a teaching experiment, involved 35 grade 5 students and the first author as the teacher. Data were collected using video recording of classroom episodes and data analysis focuses on episodes of collective discussions. The teaching experiment consists of 6 mental computation tasks with rational numbers carried out weekly for about 15-20 minutes at the beginning of the class.The study shows that students have more difficulty in mental computation in questions with percent than in questions involving addition/subtraction with decimals and fractions. Their strategies vary with the representation. With decimals and percent or when there are two representations (such as decimal and fraction), students often use change of representation. With fractions, they use equivalence and mental forms of written algorithms. In questions with percent, the strongest strategy is to repeat the operation. In solving problems students' preferred strategy is using bridging. Throughout the teaching experiment, they developed a diverse set of strategies, and at the end they showed improved ability to compute with less errors and using more complex strategies.

References

Empson, S. B., Levi, L., & Carpenter, T. P. (2010). The algebraic nature of fraction: Developing relational thinking in elementary school. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 409-428). Heidelberg: Springer.

Heirdsfield, A. (2011). Teaching mental computation strategies in early mathematics. *Young Children*, *66*(2) 96-102.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 252. Taipei, Taiwan: PME.

MEANING OF THE PART-WHOLE RELATION AND THE CONCEPT OF FRACTION FOR PRIMARY TEACHERS

Elena Castro-Rodríguez, Luis Rico, Pedro Gómez

Universidad de Granada

The part-whole relation is complex and raises questions that affect different disciplines. Researchers have proposed different interpretations of the notions of fraction and rational number (e.g., Behr, Lesh, Post & Silver, 1983; Kieren, 1976). We highlight three kinds of relations in the study of rational numbers—the part whole-relation, the part-part relation, and the functional relation—through which we organize the different subconstructs of rational number. We claim that the meaning of fractions should be understood through three components: their mathematical structure, their representations and their senses.

We performed an empirical study that focused on the meanings that primary future teachers in their initial stage of training have of the multiplicative part-whole relation. We designed a questionnaire for this purpose. The answers were organized according to the presence or absence of the aforementioned components of the part-whole relation. The analysis revealed that the future teachers in early training who participated in this study considered a significant plurality of meanings for the concept of fraction based on the multiplicative part-whole relation and showed different levels of mastery in using this relation. The participants in the study gave priority to the action of dividing, followed by actions of distributing and dividing into parts. In representing this action, the students gave priority to regular figures divided into equal parts. Finally, for the senses of fractions, family situations took priority over mathematical ones in discrete contexts and contexts with continuous surfaces. In continuous linear contexts, however, personal and mathematical situations occurred almost in equal proportion.

Acknowledgement

This study was performed with support from project EDU2009-10454 of the National Plan for R&D&r, Subprogram EDUC, and group FQM-193 of the 3rd Andalusian Plan for Research, Development and Innovation (PAIDI).

References

Behr, M. J., Lesh, R. Post, T. R., & Silver, E. A. (1983). Rational-number concept. In R. Lesh & M. Landau (Eds.), *Acquisitions of mathematics concepts and processes* (pp. 91-126). New York, NY: Academy Press.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement* (pp. 101-144). Columbus, OH: ERIC-SMEAC.

DO TEACHERS WITH DIFFERENT RELIGIOUS BELIEFS HOLD DIFFERENT VALUES IN MATH AND MATH TEACHING?

Yip-Cheung CHAN[1], Ngai-Ying WONG[1], Yuh-Chyn LEU[2]

[1]Chinese University of Hong Kong, [2]National Taipei University of Education, Taiwan

It is common knowledge that math teaching is not value-free. Religious beliefs, as an important source of personal values, might have impact on math teaching. Despite the fact that there are some empirical studies which support this (Leu, 2005), there is no systematic research focusing on the relationship between religious beliefs and values in math teaching. There are two major research questions in this paper. Besides whether teachers with different religious beliefs hold different values in math and math teaching, we would also like to know if religiosity has an influence. A questionnaire was administered among 249, 257 and 107 math teachers in Taiwan, Hong Kong, and the Chinese Mainland respectively. Besides demographic data, the questionnaire comprises three parts: religiosity, beliefs about math, and beliefs about math teaching. They are adapted respectively from the *Spiritual Involvement and Beliefs Scale* (Chui, Cheng, & Wong, 2011), our previous works (Wong, Lam, & Wong, 1998 and Leu & Wen, 2001) and Fennema et al's *Mathematics Beliefs Scale*. Results reveal that respondents holding Christian beliefs see more math is precise and is a subject of *calculables* than their counterparts, whereas respondents holding Chinese religious beliefs view math involves thinking more than others. At the same time, Christians possess relatively weaker constructivist view on math teaching than others. Furthermore, the degree of religiosity is mildly correlated to some subscales of beliefs about math in the positive direction and mildly correlated to the constructivist view about math teaching in the **negative** sense. More in depth studies are needed, yet we believe that we have opened up a door of a fruitful research area.

References

Chui, W. H., Cheng, K. K., Wong, L. P. (2011). *Spirituality and punitiveness: An exploration of Christian, Buddhist, and non-religious attitudes towards crime*. Unpublished manuscript. Hong Kong: Department of Social Work and Social Administration, The University of Hong Kong.

Leu, Y. C. (2005). The enactment and perception of mathematics pedagogical values in an elementary classroom: Buddhism, Confucianism, and curriculum reform. *International Journal of Science and Mathematics Education*, *3*(2), 175-212

Leu, Y. C., & Wen, S. C. (2001). A study on the related beliefs of mathematical teaching among mathematics teachers in elementary schools, junior high schools and senior high schools. *Journal of National Taipei Teachers College*, Vol *XIV*, 459-490.

Wong, N. Y., Lam, C. C., & Wong, K. M. (1998). Students' and teachers' conception of mathematics learning: A Hong Kong study. Paper presented at the 1st *ICMI-East Asia Regional Conference on Mathematical Education*. Seoul, Korea.

ENUMERATION AND MAGNITUDE COMPARISON AS INDICATORS OF MATHEMATICS DIFFICULTIES

Fiona Cheam[1], Tony Cline[2]

[1]Psychological Services Branch, Ministry of Education, Singapore

[2] Educational Psychology Group, University College London

Introduction

Number sense (NS) is an innate ability to perceive quantity (Dehaene, 1997). It has been suggested that poor number sense underlies some difficulties in learning mathematics. In particular, Butterworth (1999) hypothesised that a specific aspect of NS, subitisation (automatic recognition of quantities less than 4), is the core deficit in mathematical disabilities.

Enumeration and magnitude comparison are measures of NS that could be used to screen for individuals with mathematics difficulties. The present study is aimed at comparing the efficacy of these two measures in distinguishing between children of different mathematical abilities.

Method

Measures of NS (Enumeration and Magnitude Comparison), mathematics attainment (fact retrieval fluency and general mathematics attainment), non-verbal ability and receptive vocabulary were administered to 219 children (aged 6 – 11) from Singapore.

Results and Discussion

ANCOVAs revealed that, after controlling for differences in reaction time, age, non-verbal ability and language proficiency, both enumeration and magnitude comparison differentiated children of different mathematical abilities. Differences in NS were observed across all mathematics ability groups (above average, average, below average) on the measure of fact retrieval fluency. However, on a measure of general mathematics attainment, differences in number sense were observed only between children of below average performance and those of average and above average performance. Number sense did not distinguish between those of average and above average ability. It may be that number sense is necessary for an adequate performance in computation, but additional competencies are required for high performance in mathematics. The results suggest that both measures of NS are viable tasks to screen for children who may have mathematics difficulties.

References

Butterworth, B. (1999). *The mathematical brain*. London : Macmillan.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York ; Oxford : Oxford University Press.

COLOR EFFECTS IN READING GEOMETRY PROOFS: EVIDENCE FROM EYE MOVEMENTS AND RECALL TESTS

Chi-Yao Chen, Chao Jung Wu

National Taiwan Normal University

Reading comprehension of geometry proofs needs to build a coherent mental representation of all information conveyed in figures and text. Duval (1998) claimed that “geometry involves three kinds of cognitive processes: visualization, construction and reasoning” (p. 38). In most mathematic textbooks, colored sides and angles have been proposed to promote more effective learning. This study investigated how colored figures affect the integration of text and figures while reading geometry proofs. We used four junior high school level geometry proofs with colored/uncolored figures to examine 31 Taiwanese undergraduate students who were not mathematics majors. We collected both of their recall performance and eye movement data for analysis.

Sample screenshots for the initial reading comprehension time are depicted in Fig. 1. The circles and the numbers indicate the fixation location and the order in the sequence. The initial reading comprehension of $\overline{AB} = \overline{AD}$ includes fixations 1 to 7 which imply encoding and visualization of a geometric proposition. The regression is the time after the fixation locations come back from another proposition to the original one. Colored figures caused the time of initial reading comprehension in one proposition to decline, but did not affect the regression time and the comprehension reflected in the recall test.

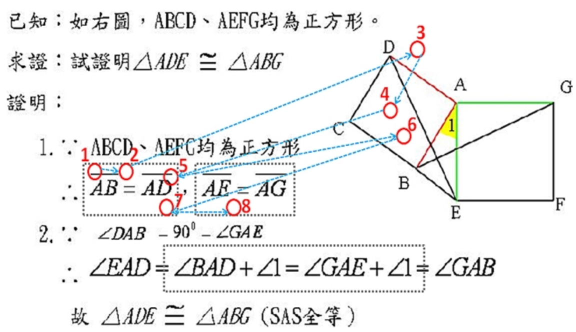


Figure 1 Example of the initial reading comprehension time

The results indicate that color cues in reading geometry proofs promote visualization which relates to the initial reading comprehension. Color cues can’t improve logical inferences so that there is no effect on regression which relates to the reasoning and also on the recall test. It shows the eye-tracking technique can be used to provide real-time cognitive processing measures, but the valid analysis of color effects of geometry proof reading of junior high school students is another challenging question for future investigation.

References

Duval, R. (1998). Geometry from a Cognitive Point of View. In C. Mammana & V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century*. An International Commission on Mathematical Instruction (ICM) Study, 37-52. Boston, MA: Kluwer Academic Publisher.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 256. Taipei, Taiwan: PME.

ON STUDENTS' COGNITION IN PROPORTIONAL COMPUTATION

Hui-Ju Chen[1] Shian Leou[2] Haw-Yaw Shy[3]

[1]Tainan Municipal Min-De Junior High School, Taiwan

[2]National Kaohsiung Normal University, Taiwan

[3]National Changhua University of Education, Taiwan

Proportional computation is not just a technique for students to learn. The notation of proportion equation ($a:b=c:d$) includes an equivalence relation between two ratios. In transforming the proportion equation"$a:b=c:d$" into the fractional equation "$a/b=c/d$", if students consider *a/b* as a value, they might ignore the covariant relation within the ratio and the invariance relation between ratios. In other words, the cognition on the equal symbol in the equation significantly affects students' construction of the proportion concept. This study is aimed to explore students' proportion learning by examining students' cognition and the use of symbols in computing the proportional equation. This is a case study of four objects selected from 88 participants to represent different performance levels and representation styles. Inspired by Noelting (1980) and Grade 1-9 Curriculum Guidelines released by Taiwan Ministry of Education (2003), two Proportional arithmetic examinations Test A and Test B are designed for selecting desired objects of the study and for data collection. Problems in Test A are parallel to those in Test B and the coefficient correlation between these two tests is 0.90.

Two major findings are: First, students with high mathematics achievements are stable in using symbolic representation in solving proportional-computation problems due to their fluency in and inertia on symbols operation; the student with medium mathematics achievement, who has good arithmetic ability, manipulates algebraic representations merely by "imitation"; the student who has low mathematics achievement can only solve simple proportional-computation problems with between-multiple relation of integral ratios in a proportional expression. Second, students' use of symbols and operation coincide with students' operation on equal fractions during computing proportion problems, and the character of equal sign in a proportional equation is understood quite differently among students.

References

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part II - Problem structure at successive stages; problem solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics*, *11*, 331-363.

Taiwan Ministry of Education. (2003). The outline for the nine-year integrated curricula for the elementary and secondary education. Taipei: Author.

WHAT AFFECT THE CHANGE FROM ADDITIVE TO MULTIPLICATIVE REASONING

Hui-Ju Chen[1] Shian Leou[1] Haw-Yaw Shy[2]

[1]National Kaohsiung Normal University, Taiwan

[2]National Changhua University of Education, Taiwan

Proportional reasoning is pivotal in mathematics curriculum as it is a culmination of arithmetic and also a corner stone of algebra (Lesh, Post & Behr, 1991). One of the fundamental questions in proportional reasoning learning is the mechanism of the change from additive to multiplicative reasoning (Lamon, 2007). This study is aimed to explore the factors affect the change from additive to multiplicative reasoning. Researchers choose the case student, who misuses additive and multiplicative reasoning, from a group of 88 seventh to eighth graders according to their performance on a standardized examination. Inspired by Noelting (1980), we designed a series of juice making activities as proportion problem situations to inquiry the case student's reasoning. Besides to Noelting's experiments about lemon juice activities, we added problems involving three quantities and problems of value finding. Questions in juice making activities are grouped into four sets according to five dimensions: problem category, problem condition, juice content, unit quantity and the operation of quantities.

Two major results are found. First, three phases about the case student's understanding of multiplicative reasoning structures are classified as: 2-multiple relationship reasoning, integral multiple relationship reasoning, and fractional multiple relationship. The 2-multiple relationship reasoning plays a role as the bridge between the additive and the multiplicative reasoning. Although the competency of arithmetic computation is influential in students' understanding of proportional reasoning, the authors find that the development of these two capabilities can be individually and interlaced. Second, three processes are found in the course of students' proportional reasoning, they are: unit-presuppose, unit-conversion, and unit-combination. These three processes are similar, in part, to *unitlizing* and *norming* proposed by Lamon (1994). We also find students' unit-presuppose is a critical factor that hinder their re-unitizing of different units.

References

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part I - Differentiation of stages. *Educational studies in Mathematics*, 11, 217-253.

Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds), *The development of multiplicative reasoning in the learning of Mathematics*, 89-121. Albany, NY : SUNY Press.

Lamon,S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In K. Frank & J. Lester, (Eds), *Second handbook of research on Mathematics teaching and learning* (pp. 629-667). NCTM.

Lesh,R., Post,T. R., & Berhr,M. (1991). Proportional Reasoning. In J. Hiebert & M. Berh(Eds.) *Number concepts and operations in the middle grades*,93-118. Reston, VA: National Council of Teachers of Mathematics.

THE DIFFERENCES OF 2ND TO 6TH GRADERS' ABILITY OF FINDING HIDDEN PROBLEM FROM TWO-STEP PROBLEMS

Tzu-Liang Chen	Der-bang Wu	Hsiu-Lan Ma	Jing-Wey Chen
National Taichung University of Education, Taiwan	National Taichung University of Education, Taiwan	Ling Tung University, Taiwan	Overseas Chinese University, Taiwan

This study was undertaken to explore the differences of finding hidden problem from two-step problems among 2nd through 6th graders based on Krulik and Rudnick theory. Krulik and Rudnick's (1995) mathematical problem-solving process referred to as a heuristics plan. It is divided into five stages: (a) Read and Think, (b) Explore and Plan, (c) Select a Strategy, (d) Find an Answer, and (e) Reflect and Extend.

The participants were 140 elementary school students, randomly selected from 3 counties/cities in Taiwan. The research tool used the idea in Wu and Ma's book "Elementary school math reasoning and problem-solving exercise books" (2009), which is based on Krulik and Rudnick's theory, from the page 56-57. In the test, the respondents were presented a series of two-step problems. They identified the hidden question for a set of given problems in multiple choices. There are four questions (Q1 demo, Q2-Q5), each of which had five answers. The respondent got a point when they chose the correct answer.

The conclusions drawn from this study, for elementary school students, were: (a) Most students of 6th grade were able to correctly identify the problem of hidden questions and the passing rate to each question (Q1 demo, Q2-Q5) were higher than 90%. Except the question 3, the average passing rate of 2nd, 3rd, 4th, and 5th graders were less than 60%. (b) Gender affected the test results, and females performed better than males in finding the hidden questions. (c) Grade affected the test results, and 6th grader performed better than other graders in finding the hidden questions.

KEYWORDS: prime numbers, independent-sample t-test, three-way ANOVA

Acknowledgements

The research reported in this paper was supported by the National Science Council of Taiwan under Grant No. NSC 100-2511-S-142 -005. Any opinions, viewpoints, findings, conclusions, suggests, or recommendations expressed are the authors and do not necessarily reflect the views of the National Science Council of Taiwan.

References

Krulik, S., & Rudnick, J. (1995). *The new sourcebook for teaching reasoning and problem solving in elementary school*. Boston: Allyn and Bason.

Wu, D. B., & Ma, H. L. (2009). *Elementary school math reasoning and problem-solving exercise books*. Taipei: Psychology Press (in Chinese).

THE PROCESS OF CONJECTURING A CONDITIONAL PROPOSITION IN DYNAMICS GEOMETRY SETTING

Ying-Hao Cheng | Hui-Yu Hsu | Jian-Cheng Chen

Taipei Municipal University of Education | National Taiwan Normal University | MingChi University of Techonology

Many recent national-wide survey results show that many students had heavy difficulties in applying learnt theorems in a geometry argumentation task(Healy and Hoyles, 1998; Reiss, Hellmich and Reiss, 2002; Lin, Cheng and linfl team, 2003). In order to retrieve a theorem for inferring, the students have to distinguish whether a geometry setting satisfied the condition of a theorem or not. However, the school math often simplified a theorem to be just a ‘fact’ under the given setting. It may cause the difficulty of retrieving the theorem in geometry task.

We established a conjecturing lesson for conditional proposition in dynamics geometry setting. The results show that there are three types of conjecturing process. That is *finite induction* on random numerical data, *systematic induction* by generating examples systematically , and *dynamic induction* by grasping corresponding-changing relation on dynamic examples. The finite induction approach seems not effective. All students in this type failed. The systematic induction and dynamic induction approached seems be successful. Students in these two approaches grape the corresponding-changing relations, produce and validate good conjectures. Another interesting finding is that students in dynamic induction approach will set a mediated observing object for “connecting” the condition and result. This mediated object may transfer the noticing information to be *visible* and enhance the quality of conjecturing.

In conclusion, the crucial process of conjecturing a condition-result proposition in a complex setting is to recognize and formulate the “*corresponding-changing*” relation between the figural elements. Making examples systematically or dynamically seems to be an effective way to find out the corresponding-changing relation.

Reference

Healy, L. & Hoyles, C.(1998):Justifying and proving in school mathematics. Summary of the results from a survey of the proof conceptions of students in the UK. Research Report Mathematical Sciences, Institute of Education, University of London.

Lin, F. L., Cheng, Y. H. and linfl team(2003):The Competence of Geometric Argument in Taiwan Adolescents. International Conference on Science & Mathematics Learning. Department of Mathematics. National Taiwan Normal University. 2003.12.16-18

Reiss, K.; Hellmich, F.; & Reiss, M. (2002): Reasoning and proof in geometry: Prerequisites of knowledge acquisition in secondary school students. Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education. Norwich (Great Britain) University.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 260. Taipei, Taiwan: PME.

STUDENTS' MATHEMATICAL INTERACTIONS IN WHOLE GROUP WORK

Judit Chico and Núria Planas

Universitat Autònoma de Barcelona, Spain

Whole group discussion has become a common practice with a modest amount of research compared to pair and small group work. In our study, we examine the emergence of relationships between the students' learning of mathematics and their interaction in pair (Chico & Planas, 2011) and whole group. The motivation for the study is the increasing use of collaborative classroom settings, along with the need to refine scientific arguments around them. We aim to examine the potential of groups from the perspective of creating learning opportunities. We interpret the notions of learning and learning opportunities as conceptually and empirically similar. Both refer to the favorable negotiation of circumstances toward the construction of knowledge. Cobb, Yackel and Wood (1991) already suggested the value of thinking of learning at an operational level in terms of identifiable learning opportunity environments.

We designed and analyzed six lessons based on problem solving in a secondary classroom with a group of students aged 15 to 16, and the teacher. Our expertise in and the curricular relevance of the transition from arithmetic to algebraic thinking made us choose this topic for the sequence of problems. We identified pair and whole group moments in which mathematical practices were at the core of the discussion due to the existence of diverse meanings. Diversity of meanings was thought of as a learning opportunity. Qualitative comparative methods were applied to develop narrative themes that are central to what takes place in the interaction, marking in particular the positive effect of collaboration on the students' mathematical activity. Our results show progress in algebraic thinking related to generalization processes that were fostered at different points of the interaction. However, we have found a few moments that become exceptions in that they represent a non-positive influence on learning.

Acknowledgments

Project EDU2009-07113, 'Estudio sobre el desarrollo de competencias discursivas en el aula de matemáticas', Grant BES-2010-030877, Ministerio de Economía y Competitividad, Spain.

References

Chico, J., & Planas, N. (2011). Question-answer, validation, and follow-up in lessons of mathematics. In B. Ubuz (Eds.), *Proceedings of the 35th International Conference of the Psychology of Mathematics Education Group* (vol. 1, p. 279). Ankara, Turquey: PME.

Yackel, E., Cobb, P., & Wood, T. (1991) Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22*(5), 390-408.

EXPLORING STUDENTS' LEARNING MOTIVATION TOWARD MATHEMATICS UNDER A CONJECTURING-BASED TEACHING CONTEXT

Erh-Tsung Chin, Da-Wei Chien, Huei-Yin Lin

National Changhua University of Education, TAIWAN

Introduction- Motivation is a crucial factor of learning. For the students in Taiwan, both TIMSS and PISA reports point out that our students' math achievements are highly graded in the world, but their learning motivation toward math is lower than the global average. Chen and Lin (1998) indicate that mathematical conjecturing is closely related to problem solving, argumentation and reasoning. They also suggest that conjecturing-based activities should be beneficial to promote students' learning motivation toward math. Therefore, we are interested in examining the influence of conjecturing based teaching on vocational high school students' learning motivation, who are low achievers in math in general.

Methods- Quasi-experimental design with pretest and post-tests is adopted in the study. The research subjects are 43 tenth graders in a vocational high school in central Taiwan. The teaching is based on the framework of conjecturing (Chen & Lin, 1998): propose a conjecture, verify or refute, and confirm, and lasts a whole semester. The main research instrument is a questionnaire with six sub-domains including *self-efficacy*, *active learning strategy*, *mathematics learning value*, *non-performance goal*, *achievement goal*, *learning environment stimulation*, modified from the SMTSL questionnaire ("student motivation toward science learning" questionnaire by Tuan, Chin & Shieh, 2005), and is analysed by ANOVA and multivariate regression analysis.

Result- The results show that students' learning motivation toward math has improved after having the conjecturing-based activities. It reaches statistically significant difference ($p < .05$) in the whole questionnaire as well as five of the sub-domains, excluding in the aspect of "non-performance goal" ($p = 0\ .08$). The multivariate regression analysis reveals that the main sub-domains influencing students' learning motivation toward math in turn are active learning strategy, self-efficacy, learning environment stimulation, and achievement goal with the explanatory power R^2s = 56%、15%、13% and 6% respectively, while the total explanatory power R^2 reaches 90%.

Reference

Chen, I. E. & Lin, F. L. (1998). A thinking model of mathematics conjecturing. *Chinese journal of science education, 6*(2), 191-218.

Tuan, H. L., Chin, C. C., & Shieh, S. H. (2005). The development of a questionnaire to measure students' motivation towards science learning. *International Journal of Science Education, 27,* 639-654.

THE INFLUENCE OF CONJECTURING-CENTRED INQUIRY TEACHING ON VOCATIONAL HIGH SCHOOL STUDENTS' MATHEMATICAL LEARNING ACHIEVEMENTS

Erh-Tsung, Chin [1], Ruei-Chang, Lo[1], Ke-Hsu Chang[2]

[1]National Changhua University of Education, TAIWAN, R.O.C.

[2]Taichung Home Economics and Commercial High School, TAIWAN, R.O

This study is aimed at examining the influence of conjecturing-centred inquiry teaching on vocational high school students' mathematical learning achievements. The research design adopted in this study is the quasi-experimental, non-equivalent pretest-posttest design. The experimental group is one of the classes with thirty-nine girls of the third author which receives the conjecturing-centred inquiry teaching, whilst the control group is another class with thirty-nine girls of the same teacher receiving the traditional expository teaching. The conjecturing-centred inquiry teaching integrates the conjecturing process (Chen & Lin, 1998, which contains posing a conjecture, examining/rebuttal, confirming the conjecturing), into the 5E inquiring learning cycle (Bybee & Landes, 1998), which consists of five stages including engagement, exploration, explanation, elaboration and evaluation. The research period is a whole academic year. The instruments of assessing students' learning achievements are the six period examinations in the academic year which are transformed to T scores and compared with all the thirteen classes of the same grade, and four self-designed tests for pretest, posttest and delayed test which are analysed and compared within the experimental and control groups by means of ANCOVA.

The research results reveal that, compared with the traditional expository teaching, conjecturing-centred inquiry teaching has significantly improved students' mathematical learning after a whole academic year. Moreover, from the results of the six period examinations, the average scores of the group of conjecturing-centred inquiry teaching show a trend of going backward first, then upward. A probable reason should be that both the teacher and students need time to adapt themselves to the new pedagogy. Once they get used to it, not only the students' performance towards mathematics gets improved, but also they could actively construct more meaningful mathematical knowledge through the process of inquiry.

Keywords: conjecturing, inquiry, vocational high school students

References

Bybee, J. W. & Landes, N. M. (1988). The biological sciences curriculum study (BSCS). *Science and Children, 25*(8), 36-37.

Chen, Y. E. & Lin, F. L. (1998). A thinking model of mathematics conjecturing. *Chinese Journal of Science Education*, *6*, 191-218.

MAKING SENSE OF MATHEMATICS THROUGH PERCEPTION, OPERATION & REASON: THE CASE OF TRIGONOMETRIC FUNCTIONS

Kin Eng Chin David Tall

Universiti Malaysia Sabah University of Warwick

This paper builds on the three-world framework presented in PME30 by Tall (2004) in which mathematical thinking in the individual evolves through embodiment, symbolism and formalism. It is here refined to consider how ideas become sophisticated through *perception*, *operation* and *reason*. It is based on foundational ideas developed by Bruner, van Hiele, Fischbein, Skemp, Collis and others relating to the full development of mathematical thinking from child to all forms of adult mathematics. In particular, it considers the mismatch between the longer-term ideas developed in succeeding levels of sophistication by graduate students destined to become teachers and the ways of thinking of the learners that they will teach in school. It is a significant evolution of ideas developed throughout the history of PME.

The empirical data focuses on three successive levels of learning in trigonometry, namely *triangle trigonometry* involving lengths that are magnitudes without sign, angles in right-angled triangles strictly between 0 and 90°, *circle trigonometry* involving angles of any size, sides that have signs and functions that vary dynamically, and *analytic trigonometry* that gives new insights unavailable to school children involving infinite power series and complex numbers.

It builds theoretically on the notion of *supportive* and *problematic met-befores* where some experiences of mathematics that the individual has encountered before either support or impede learning in new situations. This applies both to the learner and to the teacher. It leads to a completely new way of planning the education of teachers to enable them to teach in a way that can make sense to learners. We consider this to be an essential long-term evolution of ideas highly relevant to the teaching and learning of mathematics fully consistent with the long-term aims of PME. The submitted paper is a blend of highly relevant theory and specific empirical detail, considered ‘excellent’ by one reviewer and criticised on methodological grounds by others. You may your own mind up by reading the full version from

http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2012c-Chin-making-sense.pdf

The ideas arise in the PhD of Kin Eng Chin (in preparation) and are included in *How Humans Learn to Think Mathematically* by David Tall, (forthcoming from CUP: NY).

Reference

Tall, D. O. (2004). Thinking through three world of mathematics. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education,* Bergen, Norway, 4, 281–288.

DEVELOMENT OF COVARIATIONAL REASONING ABILITY IN A LOGO-BASED JAVAMAL MICROWORLD

Han-Hyuk Cho, Dong-Jo Shin, Ahn-Sung Woo

Seoul National University

This paper explores gifted students' covariational reasoning abilities. Three tests were developed in order to assess and analyze their reasoning abilities building on previous research about covariational reasoning. Giving consideration to the arising problems in the tests, we designed a LOGO-based microworld environment which engages students in an active learning environment. This environment was designed by applying several theoretical background, which are Papert's constructionism and Vygotskian perspective of semiotic mediation.(Papert, 1980; Vygotsky, 1978)

In the learning environment, students used "move △x, △y" to construct given graphs. The amount of change within graph expression can be represented by the command "move △x, △y" in this environment.(see table 1)

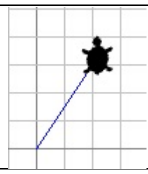
move △x, △y :moves turtle horizontally △x steps and vertically △y steps	

Table 1. move command and move 2,3

From this activity, we led students to internalize the fact that graphs can be constructed by using contiguous secant lines for the domain. And then, students discussed about factors which determine the shape and concavity of graphs in small group. In this discussion, the role of the teacher was crucial in leading student to face the move from individual ideas to collective common idea. Following this discussion, we gave last written tasks which student had to construct graphs while modelling dynamic events.

We analyze students' verbal discourse and graph drawings in discussions and written tasks. This analysis show that the expression "move △x, △y" and computerized graphs serve as means of semiotic mediation and led students to grasp the cognitive strategies that support the covariational reasoning.

Acknowledgement

This research is supported by the Gwacheon National Science Museum for developing Informal Mathematics Education Program.

References

Papert, S.(1980). *Mindstorms:children, computers, and powerful ideas*. Cambridge, Massachusetts: Perseus Publishing.

Vygosky, L. S.(1978). *Mind and Society: The Development of Higher Psychological Processes.* Cambridge: Harvard University Press.

FORMAL AND INFORMAL REASONING PROCESSES IN COMPARING AND ORDERING RATIONAL NUMBERS[1]

João Pedro da Ponte
Instituto de Educação,
Universidade de Lisboa

Marisa Quaresma
Unidade de Investigação do Instituto de Educação, Universidade de Lisboa

This study aims is to know to what extent a teaching unit in which students work with different representations and different meanings of rational numbers, mainly in exploratory tasks, provides an effective learning environment to develop informal into formal reasoning processes. We take reasoning as the process of making inferences from given information. Informal reasoning includes ways of thinking from everyday life and formal reasoning involves attending to the definition of mathematical objects, their properties and formal procedures. A critical problem of mathematics teaching is to promote the progressive articulation between these two kinds of reasoning processes. For Gravemeijer (2005), emphasizing informal reasoning processes helps to bridge the gap between students' personal informal and formal knowledge.

This study is a teaching experiment, including an in-depth case study of a pupil (Leonor). Qualitative data was collected by video record of classroom discourse, interviews with the pupil and analysis of her written work. The teaching unit had 12 classes (90 minutes each), with pupils' autonomous work (mostly in pairs) and collective discussions for sharing results and strategies. The study provides a possible way to build on students' intuitive knowledge of rational numbers. Throughout the teaching unit, the students could choose to work with formal representations, or, if they experienced difficulty, to revert to the use of informal representations. They gradually developed their mastery of formal representations in two ways (i) by reference to informal representations, as suggested by Webb, Boswinkel and Dekker (2008), relating fractions with pictorial representations; or (ii) using formal representations (seeking to extend the properties of fractions or exploring the conventions of the decimal representation system) and converting between formal representations.

References

Gravemeijer, K. (2005). What makes mathematics so difficult, and what can we do about it? In L. Santos, A. P. Canavarro & J. Brocardo (Eds.), *Educação matemática: Caminhos e encruzilhadas* (pp. 83-101). Lisboa: APM.

Webb, D. C., Boswinkel, N., & Dekker, T. (2008). Beneath the tip of the iceberg. *Mathematics Teaching in the Middle School, 14*(2), 110-113.

[1] This study is supported by national funds by FCT – Fundação para a Ciência e Tecnologia through the Project *Professional Practices of Mathematics Teachers* (contract PTDC/CPE-CED/098931/2008).

UNIVERSITY STUDENTS' INCOHERENT CONCEPT DEFINITION IMAGES OF CONTINUITY AND ASYMPTOTES

Bettina Dahl (Søndergaard)

CSE, Faculty of Science and Technology, Aarhus University, Denmark

The term *concept image* denotes all the mental pictures, properties, and processes that an individual has related to a concept while the term *concept definition* is the words used to specify it (Tall & Vinner, 1981). Students often do not rely on the concept definition but on the concept image when solving problems (Rösken & Rolka, 2007) and the concept definition might not be coherent (Jukić & Dahl, 2011). Thus, the aim is to further study why students are reluctant to use the concept definitions. The research questions are: How do the students in this study refer to the concept definitions when solving the given tasks in continuity and asymptotes?

This explorative case study took place in 2011 at a US university in an introductory single variable calculus course which most students take to satisfy the university's disciplinary breadth requirement. 9 females and 5 males were interviewed in pairs to create a process of sharing to help the explorations. Due to practicalities, some were interviewed alone. There were 12 tasks. Some tasks asked for definitions, others to e.g. explain if, and where, a particular function is continuous, or if a function can cross its vertical and horizontal asymptote. The students had just had a mid-term test on these topics. Most tasks were revised tasks from the textbook or the mid-term test.

11 students wrote at least a partially correct definition of continuity. Some students referred to the definition when solving the tasks but some tasks exposed the definition as incoherent. A few students correctly reproduced the concept definition but did not understand it, nor use it. Others knew the formal definition and were able to refer to it but preferred the concept image as it appeared to be the easier way. 10 wrote at least one correct definition of asymptotes. The students were all combinations of giving a definition (coherent or not) and using it, or not, in deciding if a function could cross its asymptote; including referring to incoherent concept definitions when solving tasks.

References

Jukić, Lj., & Dahl, B. (2011). University students' concept image and retention of the definite integral. In B. Ubuz (Ed.), *Proc. 35th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 73-80). Ankara, Turkey: PME.

Rösken, B., & Rolka, K. (2007). Integrating intuition: The role of concept image and concept definition for students' learning of integral calculus. *The Montana Mathematics Enthusiast, 3,* 181-204.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics, 12,* 151-169.

HOW DO TEACHER EDUCATORS AND INDIGENOUS TEACHERS, JOINTLY, CONTRIBUTE FOR THE REVITALIZATION OF A NATIVE LANGUAGE?

Maria do Carmo S. Domite

Faculty of Education, University of São Paulo, Brazil

We present an action research led among and with Terena indigenous teachers and non-indigenous teachers in which we gathered information about how these indigenous teachers would be more enthusiastic and encouraged using their native language and acknowledging that there is a world-wide cultural movement in the sense of the revival of the native tongues as well as means, methods and working styles towards this revitalization.

The data/facts came from three days of work in one Terena village - starting from the necessities, desires and conditions of native language fluency of this indigenous group - producing along with them materials for learning language and mathematics (mental calculation, measurements, spatial orientation etc.) focusing on five different procedures: a) special indigenous language classes; b) activities of immersion of the pupils in the communal cultural practices; c) bilingual schooling; d) involvement of native language speaking elders in early childhood education; and e) the indigenous people as researchers of the reasons just few people speaking the native language.

The issues of this language revitalization pedagogic context led us to focus three emergence categories of analysis: a) what the Terena indigenous teacher requires to be able to conveniently exercise his/her tasks regarding the usage of their native language and the process by which they succeed; b) the manner in which Terena people negotiate their relationship with the bilingual perspective for the Indigenous Education Program and, c) difference of expectancy of improving Terena language fluency.

Findings indicate that along with indigenous people a movement trying to reverse a native language demands an effort in both domains: context and of the organized tasks. One of the main struggles for the Terena people was the identification of what would contribute to the diminishment of the level of degradation of their language. This represents a powerful tool that unfortunately has become forgotten (i.e. evening talking group at House of Prayer) in light of unbalanced competition with modern media, primarily television. In terms of what it takes teachers to perform this reversal, the elders would be fully available to the inquiries of the children and youngsters in order to share old ways and general ancient knowledge.

References

Freire, P. & Macedo, D. (1987). Literacy, Reading the Word and the World. South Hadley: Bergin and Garvey.

D´Ambrosio, U. & Domite, M.C.S. (2007). The potentialities of (ethno) mathematics education: Ubiratan D Ambrosio an interview with Maria do Carmo Domite. In: Internationalisation and Globalisation in Mathematics and Science Education. Dordrecht, the Netherlands: Springer Science + Business Media B.V., 2007, v. 1.

McIvor, O. (2009). Strategies for Indigenous language revitalization and Maintenance, *Encyclopedia of Language and Literacy Development*, 1-12. London: Canadian Language and Literacy Research Network. Retrieved October, 10th, 2010, from http://www.literacyencyclopedia.ca/pdfs/topic.php?topId=265

WHICH QUALITIES DID ASPIRING TEACHERS VALUE IN THEIR 'BEST' MATHEMATICS TEACHERS?

Susan K Forsythe and Gareth Lewis

University of Leicester, UK

Candidates for a post graduate teacher training course specialising in mathematics at our own institution were asked to describe a mathematics teacher who had an impact on them. When adults are asked to look back to their school days and identify their 'best' teacher they often refer to the personal attributes of their teachers rather than describe the way they taught their subject (Gossman, 2011). That these attributes are an important influence on the motivation of pupils has also been noted by other researchers, for example Ryan and Patrick (2001).

We analysed seventy five written responses. When analysing the transcripts, statements in the narratives were marked and transcribed. These were then organised into categories which emerged iteratively and inductively (Thomas, 2009). The initial readings indicated that teachers' personal attributes appeared more important than their style of teaching or even their competence as a teacher. Given that the candidates writing these descriptions have been reasonably successful at school mathematics, a surprising finding was evidence of disaffection in some of the accounts. The data suggests that, over a school career, many pupils will experience disaffection to some degree. It also suggests that teachers here are chosen as role models precisely because they make a substantial impact in terms of providing positive experiences, changing attitudes and beliefs.

In addition to evidence of disaffection, a number of clear themes were identified, and these fell into two broad categories which are broadly consistent with Gossman's teacher-as-person and teacher-as-teacher classification. The first of these categories encompasses aspects of personality such as enthusiasm/passion for the subject and human/caring. Teacher-as-teacher themes included the use of real world interest; clarity of explanation; sense of fun; use of variety of methods; use of discipline and challenge. The data supports prior research that suggests that core beliefs and personal attributes are important qualities in teachers, and evidences this specifically in relation to the environment for learning mathematics.

References

Gossman, P. (2011). 'My best teacher' *TEAN journal 3:1*

Ryan, A.M. and H. Patrick, (2001), The classroom environment and adolescents' motivation and engagement during middle school. *American Educational Research journal, 38:2 437-446*

Thomas, G. (2009). How to do your research project. *Sage publications*.

USING DYNAMIC GEOMETRY TO GAIN FRESH INSIGHT INTO HOW STUDENTS VISUALISE 2-DIMENSIONAL SHAPES

Susan K. Forsythe

School of Education, University of Leicester, UK

Basic elements of Euclidean geometry are generally taught from a static perspective but students gain new insights into these fundamental principles when Dynamic Geometry Software (DGS) is used. An important affordance of DGS is the drag mode which allows users to drag objects in a geometrical figure in order to investigate which aspects change and which remain invariant. A number of different dragging strategies have been described by Arzarello et al (2002) who noted that students' use of dragging can provide an insight into how they conceptualise figures.

This study explored the reasoning of 13 year old students whilst generating geometric shapes from a generic figure by dragging two internal rigid perpendicular 'bars' (AC and BD in figure 1). Pairs of students working together were asked to describe the position of the bars as they generated specific shapes such as the kite. The aim was for them to investigate the properties of diagonals within quadrilaterals and the base and height within triangles. In conducting this task the dragging strategies proved to be inherently interesting.

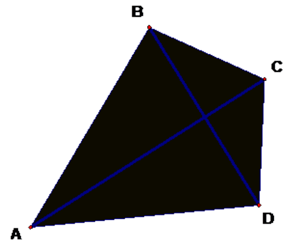


Figure 1

Recordings of the dialogue and 'on screen' activity indicated that the students used an innate sense of symmetry to guide them when positioning the bars. This was most obvious when moving between shapes, for example, from the isosceles triangle through the kites to a rhombus. The students typically dragged one bar along the path of the perpendicular bisector of the other bar, maintaining symmetry while dragging. This was observed when the bars were orientated vertically and horizontally and also when the bars were orientated at an angle to the vertical as shown in figure 1. The students also used symmetry to help them identify equal sides and angles. In conclusion; it may be more intuitive for children to focus on symmetry when they first learn about 2 dimensional shapes and to use this to develop an understanding of other shape properties.

References

Arzarello, F., F. Olivero, D. Paola and O. Robutti (2002). A cognitive analysis of dragging practices in Cabri environments, *ZDM*, *Vol. 34(3), pp. 66-72.*

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 270. Taipei, Taiwan: PME.

AN ANALYSIS OF MATHEMATICAL MODELLING IN SWEDISH TEXTBOOKS IN UPPER SECONDARY SCHOOL

Peter Frejd

Linköping University

The Swedish government has introduced and implemented a new curriculum for upper secondary school valid from the autumn 2011, where the section on mathematics emphasizes mathematical modelling as one out of seven teaching goals, i.e. to develop students' ability to "interpret a realistic situation and design a mathematical model and to use and validate a model's properties and limitations" (Skolverket, 2010, p. 1, my translation). When teaching of mathematics in Sweden to a large extent depends on textbooks, both as a guide for teachers on what to teach and for students to work individually with exercises (Jablonka & Johansson, 2010), it is critical how the mathematical content and notions are described, presented and interpreted by the textbook authors, in order for students to have the opportunity to develop the abilities described in the seven goals.

This study is a content analysis of 14 'new' mathematical textbooks, with the aim to investigate how the notion of mathematical modelling is described. The analysis, focusing on *implicit* and *explicit* descriptions of modelling, indicates that there is a large variety of descriptions, both in terms of how frequently the words models, modelling etc have been used and how the notions have been described. The range spans from more *explicit* descriptions of modelling such as a cyclic problem solving method and an activity to solve Fermi problems to more *implicit* descriptions in the form of tasks for the students to solve as well as tasks that include the word model without further explanations. There may be many reasons for why the descriptions of modelling found in the textbooks varied so much. Some suggestions are that the modelling ability in the curriculum is not interpreted as a holistic ability by the textbook authors, the authors emphasise other aspects of mathematics as more important, or the authors did not have enough time to reflect about the new curriculum and relied on older textbooks. These results also indicate that, with a strongly textbook based teaching, not all students get the same opportunities to develop a modelling ability as stated in the curriculum. This implies that teachers may need to complement their teaching of mathematical modelling with other teaching material in order to fulfil this curriculum goal.

Reference

Skolverket. (2010). Förordning om ämnesplaner för de gymnasiegemensamma ämnena. Retrieved 1/1, 2012, from http://www.skolverket.se/forskola_och_skola/

Jablonka, E., Johanssson, M. (2010). Using texts and tasks: Swedish studies on mathematic textbooks. In B. Sriraman et al. (Eds.), *The first sourcebook of Nordic research in mathematics education* (pp. 269-282). Charlotte, NC: Information Age Publishing.

CONTENT ANALYSIS OF THE NEW FIRST YEAR SECONDARY MATHEMATICS TEXTBOOK FROM TEACHERS' PERSPECTIVE

Zahra Gooya Morteza Hasanpour

Shahid Beheshti University Uromieh School Board

The strong justification for the grade 9 mathematics textbook (entry year for the secondary school in Iran) is to enhance mathematics literacy among students (De Lange, 2003). This is especially important in Iran since there is only one national textbook for all at this grade. In addition, the drop-out rate of entry year of secondary school is extremely high and its main cause is mathematics subject. Consequently, two major reasons for changing the previous textbook and launching the new textbook in 2008 as declared by the officials were enhancing the mathematics literacy of all 9^{th} graders and lowering the drop-out rate.

However, many teachers who taught the previous textbook and are already teaching the new one have complained that the new effort does not serve the above purpose. We therefore believed that many studies are needed to shed more light into this issue and help the decision- makers for further revision of the new book (Pingel, 2009). Along this line, the present study was conducted using content analysis and semi- structured interviews with eight mathematics teachers who taught the previous and the new textbook as well as the second author's experiences of teaching both- the old and the new- textbooks. For the data analysis, we designed a framework based on the components that have been used for the organisation of the content of the new book.

The results of this study showed that the strength of the new book is the effort to take the challenge of confronting difficulties that were caused by the previous textbook. To give an example, teachers had complain about the separation of "factorization" and "identity" sections in the previous textbook, and in the new one, they are integrated as one section. However, this integration is not coherent enough and there is an excessive use of geometric representations to introduce the identities.

However, the major difficulties of the new textbook include the vagueness of its aims considering its vast and diverse audiences, the lack of coherency of the learning theories and curriculum theories to support the changes, ambiguous strategies used for the content organization and last not the least, the unclear expectancy level for the learners.

References

De Lange, J. (2003). Mathematics for Literacy. In B. L. Madison & L. A. Steen (Eds.), Quantitative Literacy. Why Numeracy Matters for Schools and Colleges (pp. 75- 89). Prinston, NJ: The National Council on Education and the Disciplines.

Pingel, F. (2009). Guidebook on Textbook Research and Textbook Revision. Unesco.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 272. Taipei, Taiwan: PME.

CAN STUDENTS LEARN PROBLEM SOLVING WITH A DYNAMIC GEOMETRY ENVIRONMENT (DGE)?

Reinhold Haug, University of Education Freiburg

It appears that most empirical research looking at the efficiency of Dynamic Geometry Environment (DGE) was carried out qualitatively (Furinghetti & Paola, 2003 & Hölzl, 1999). Accordingly, most of the authors agree that a successful implementation of a DGE is highly reliant on the students' previous knowledge and requires a differentiated methodological as well as didactical and structural consideration (Olivero, 2001).

Based on this situation, this paper reports the outcome of quantitative and qualitative empirical research activities with 120 students (intervention group n=59, control group n=61) from grade 7 (13-years) who worked with DGE to learn problem solving techniques. The introductory phase lasted for three weeks. The students were also introduced to recording their learning activities in a learning protocol as well as working in pairs with a computer. A pre-test took place between the introduction and the intervention phase. This test documented the pre-knowledge of different problem solving techniques such as "making conjectures", "identifying invariants" and "using auxiliary lines". In the following four lessons (intervention) the students were working on the subject "line reflection" to increase their mathematical knowledge without the help of a teacher. The research was completed by a post-test which was conducted after the intervention, and a follow-up test which was carried out six months later. All three tests each contained eleven test items related to problem solving techniques and regarding the "use of auxiliary lines", "making conjectures" and "identifying invariants".

The results of this experimental research show that students have the ability to acquire basic problem solving techniques with the help of DGEs. Furthermore, the students of the intervention group were also able to document more their own problem solving processes in the learning protocol. These problem solving processes included making design descriptions, making conjectures and reflecting on their own learning processes.

References

Furinghetti, F. & Paola, D. (2003). To produce conjectures and to prove them within a dynamic geometry environment: A case study. In Dougherty, B. J., Pateman, N. A. & Zilliox, J. T. (Hrsg.) *Proceeding of Psychology of Mathematics 27th international Conference, 2003*, pp. 397-404.

Hölzl, R. (1999). *Qualitative Unterrichtsstudien zur Verwendung dynamischer Geometrie-Software*. Augsburg: Wißner-Verlag.

Olivero, F. (2001). Conjecturing in open geometric situations in a dynamic geometry environment: an exploratory classroom experiment. In Morgan, C. & Jones, K. (Eds.), *Research in Mathematics Education, vol.3*, pp.229-246.

ASSESSING LEARNING AND SPECIFIC MATHEMATICAL TEACHING PRACTICES IN ADVANCED ALGEBRA CLASSROOMS

Stephen Hegedus and Sara Dalton

University of Massachusetts Dartmouth

We report on data collected during one study of a five study program of longitudinal research across four years which included four cluster randomized trials and one quasi-experimental study implementing SimCalc MathWorlds® (hereon called SimCalc) in Algebra 1 and Algebra 2 classrooms across seven districts in Massachusetts, US. We have discovered that we can relate how students perform at different levels to specific mathematical teaching practices as measured by teachers' self-reports on daily logs.

SimCalc combines two innovative technological ingredients to address core mathematical ideas in deep and sustainable ways for mathematics learners. First, the software addresses content issues through dynamic linked representations and, second, the use of wireless networks can enhance student participation and mathematical communication in the classroom. The materials developed at the Kaput Center fuse these two important ingredients through new curriculum materials that replace core mathematical units in high school Algebra 1 & 2 courses. We have measured the impact of implementing these materials on student learning, and high-stakes State examinations in a mix of urban and smaller local districts in Massachusetts.

An important component of the SimCalc theory of change, a model for analyzing a teacher's "faithfulness" to SimCalc, is reasoning across mathematical representations and communication of ideas and concepts through classroom discussions. This is stressed in the curricular materials, as well as in the teacher training that occurs prior to beginning the intervention and we found that student performance is highly correlated to teachers focusing on this in their teaching.

The research questions for the overall study focus around the intersection of learning and motivation following an implementation of SimCalc materials in Algebra 1 and Algebra 2. We report on a finding within one study focused on student learning from the Algebra 2 materials (quadratic and exponential functions) and teachers self-reported focus on instructional strategies during the teaching of their Algebra 2 lessons. We use statistical methods such as ANCOVA and HLM to analyze student performance and compare those findings with graphical analysis of teacher log data (by experimental group and individual teacher).

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 274. Taipei, Taiwan: PME.

YOUNG STUDENTS LEARNING ORDER WITHIN FORMAL ALGEBRAIC NOTATION

Dave Hewitt

School of Education, University of Birmingham

Over the last twelve years there have also been reports of students of primary school age successfully using letters in a meaningful way, although this has been with relatively simple expressions. This study looks at how 9-10 year old students learnt order of operations within complex formal algebraic expressions, on their way to learning to solve linear equations. This was a mixed ability class of 21 students who had never met letters in a mathematical context, nor formal algebraic notation. They were taught over just three lessons using the computer software *Grid Algebra*. The lessons were taught mainly with an interactive whiteboard, although there were also some pencil and paper activities and two sessions in a computer suite. The software is based upon making journeys around a multiplication number grid. Any number can be picked up and dragged horizontally (resulting in addition or subtraction within a multiplication table) or vertically (multiplication or division between multiplication tables) on the grid with the process, rather than the resulting answer, shown in formal notation. For example the number 3 sitting in the one times table can be dragged four spaces to the right, resulting in 3+4. This expression can then be dragged down to the two times table row, resulting in 2(3+4) being shown. This meant that students had at least two potential meanings for an expression, an arithmetic meaning where 2(3+4) is seen as a mix of addition and multiplication; and a visual meaning, there 2(3+4) is seen as a mix of moving across and moving down. It is the dynamic of these two spaces which is the focus of this report. Fauconnier and Turner (2002) talk about a blended space where two or more spaces are brought together which can help develop a new structure. Here the blended space of arithmetic operations and physical movements operated together. In the early stages of meeting formal notation through movements on the grid there was evidence of students becoming confused between the arithmetic and the visual meanings. However, later on, a number of students were able to focus purely on the visual dynamics of creating expressions through movement which helped them successfully learn order of operations, which was then transferred into arithmetic meanings when they later began solving linear equations.

References

Fauconnier, G. and Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities.* New York: Basic Books.

KNOWLEDGE OF ASSESSMENT AMONG NOVICE ELEMENTARY MATHEMATICS, THEIR BEHAVIOR AND THE LINKAGE

Liora Hoch & Miriam Amit

Ben-Gurion University and Orot Israel College, Israel; Ben-Gurion University, Israel

Theoretical Background

Assessment is an integral part of the teaching and learning process. Teachers are responsible for conducting assessment of students' knowledge and achievement on a routine basis and in a reliable manner (Birenbaum et al., 2006; NCTM, 2000).

Methodology and Some Research Findings

This study is intended to examine pedagogical knowledge in the area of assessment of students' achievements among preservice and novice elementary mathematics teachers (EMT), and to what extent novice EMT make use of it in their work. In order to check knowledge, two questionnaires were given to 42 preservice and 25 novices ENT: Declarative Knowledge questionnaire, which includes 18 terms in assessment, and Actual Knowledge questionnaire, which include 11 multiple choice questions. Novice EMT had also to complete Declarative Behaviour questionnaire, that include the same terms appeared in Declarative Knowledge questionnaire, and examine their deeds in the classroom.

On the average, the participants think they have a moderate degree of familiarity with assessment-related concepts: 2.7 (out of 4). But the Actual Knowledge questionnaire reveals lack of knowledge in some terms and confusion, mix-up and fuzziness in others, for example between the terms norm test and criterion test. The average score for this questionnaire was 40%. For both questionnaires no significant differences were found between the two groups.

On the average, the application of knowledge concepts in the classroom is low: 2.18 (out of 4). A significant difference was found between the participants' declarative knowledge and its application for most of the terms.

The correlation between the knowledge terms and their applications was found extremely high ($r=.855$) and significant ($p<0.01$). Due to the high and significant correlations, it was examined which assessment terms of knowledge predict behaviour. It was found that five terms predict use of knowledge. In the presentation, we shall show these terms and try to explain the meaning of this result.

References

Birenbaum, M., Breuer, K., Cascallar, E., Dochy, F., Dori, Y., Ridgway, J. & Wiesemes, R. (2006). A learning integrated assessment system (Position paper). *Educational Research Review*, *1*(1), 61 – 67.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics.* Reston, VA: Author.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 276. Taipei, Taiwan: PME.

SUBSTANTIAL LEARNING ENVIRONMENTS IN PRE-SERVICE TEACHER TRAINING

Alena Hošpesová

University of South Bohemia in České Budějovice, Faculty of Education

The paper describes the results of a qualitative interpretative study, whose aim was an analysis of achievements of pre-service primary school teachers who had attended a learning course focusing on work with pupils in substantial learning environments (SLEs) (Wittmann, 2005). Our effort was (a) to equip pre-service teachers with a set of intellectual tools and dispositions that enable them to engage in mathematical inquiry and (b) to change their beliefs about mathematical teaching.

The students worked in groups and pairs in different arithmetical and geometrical SLEs, e.g. number pyramids, number houses, various types of arithmetical patterns, mosaics. The teaching was organized in several steps: (a) content-related characteristics of the SLE (relation to the curriculum, to the area of mathematics it focuses on), (b) students' own activity when solving problems, (c) discovery of properties and relations of the SLE, (d) posing problems for primary school pupils, also suggestions on how the SLE can be further developed, (e) practical testing of a selected problem in their teaching practice (focus on pupils' learning and understanding) followed by self-reflection.

The results show a gradual increase in students' motivation and a change in their conception of the objectives and sense of mathematical education. The qualitative analysis of the students' written reflection showed that they appreciated motivational and cognitive merit of their seminar work. Surprisingly they often mentioned also the organizational aspect of teaching. But even though we tried to show more profound links to previously studied areas of mathematics (especially algebra) in the summary which followed every students' activity in SLE, when working on their own the students used the same solving procedures they could expect from their prospective pupils. We did not manage to extent their horizon knowledge of mathematics (Ball, Thames, Pelps, 2008).

References

Wittmann, E. (2005). Mathematics as the Science of Patterns - A Guideline for Developing Mathematics Education from Early Childhood to Adulthood. Plenary Lect. presented at the International Colloquium "Mathematical Learning from Early Childhood to Adulthood". Retrieved from http://irem.u-strasbg.fr/php/publi/annales /sommaires/11/WittmannA.pdf

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education,* 59(5), 389-407.

EQUAL SIGNS IN TAIWANESE ELEMENTARY MATHEMATICS TEXTBOOKS

Kai-ju Hsieh and Li-Chuen Wang

National Taichung University of Education, Taiwan, Republic of China

Research results have shown that most students view equal signs as “figure out the result”, instead of “two equal amount”. McNeil et al. (2006) argue that the problems might be in the textbooks. The purpose of this study was to explore the meanings of equal signs were included in the textbooks. Content analysis was performed on four most popular series of elementary mathematics textbooks (1st ~ 6th grades) in Taiwan. The content presented were partial results of the project funded by the National Science Council of the Executive Yuan (Project No.: NSC 99-2511-S-142-007-)

The coding system included two main categories, “operation = answer” (e.g., 3 + 4 = 7 or 3 + 4 = x), and “others” (not “operation = answer”) (e.g., 5 + 2 = 3 + 4 or A = l × w). All equal signs appeared in these four series of textbooks were labelled and calculated. Chi-square tests were performed in order to analyse the frequency differences among these series.

The result indicated about 60% of equal sign were found in “operation equal answer” situations (Table 1), which was inconsistent with the research results conducted by McNeil, et al. (2006). There were also significant frequency differences of equal signs appeared in these four series ($\chi 2$ values were between 523.323 and 1014.983, p<.001). It might be interesting to explore the impact of textbooks on students’ understandings of the meanings of equal sign.

	Frequency (Percentage)				χ^2	p
Series	A	B	C	D		
Overall	2821	4235	1827	3492	1014.983	.000
Operation = answer	1870 (66.29%)	2508 (59.22%)	1171 (64.09%)	2262 (64.78%)	523.323	.000
Others	951 (33.71%)	1727 (40.78%)	656 (35.91%)	1230 (35.22%)	545.699	.000

Table 1: Chi-square results: comparison of frequencies of equal signs by series

References

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., et al. (2006). Middle-school students’ understanding of the equal sign: The books they read can’t help. *Cognition & Instruction, 24,* 367-385.

THE TYPES AND IMPLEMENTATION OF MATHEMATICS TASKS IN THREE TEACHERS' MATHEMATICS TEACHING

Wei-Min Hsu

National Pingtung University of Education, Taiwan

The key that directly influences the student's learning performance is teacher's mathematics teaching in the classroom (Stein, Remillard & Smith, 2007). However, How to understand and analyse teacher's mathematics teaching? Hiebert and Grouws (2007) argue that teaching is composed of the teacher-pupils interaction toward the course content for achieving the learning goals. From this point of view, there are two main focuses in mathematics teaching: the course content and the teacher-pupils interaction toward the course content. Usually, the course content appears in the textbook in the forms of mathematic tasks. And some researchers began to analyse teacher's teaching that focusing on mathematic tasks (Henningsen & Stein, 1997).

Therefore, this study used mathematics tasks as units to exam the mathematics teaching of elementary teachers according to types and implementation of tasks. Case study was used as method, and three sixth-grade teachers were chosen as subjects. The main data sources were classroom observations and solving records of students that collected during the school year of 2009.

The results revealed that three teachers presented different 'pictures' in mathematics teaching. One teacher used more high-level mathematics tasks and open ended questions to interact with students. She provided many opportunities for students to explain their thinking processes and results. The other two teachers adapted the close ended dialogues when they implemented the tasks even though many of the tasks were considered as high-level ones. What we found in this study in terms of the relationship between the task types and the implementation was that if the teacher used group collaboration to implement the tasks, mostly high-level, then the teacher usually asked open questions to students and provided the opportunities to them to present their solving processes and results. However, when the teacher used closed ended dialogues or decreased students' cognitive demand of task solving to implement the tasks, students would usually learn the high-level tasks in a low cognitive way.

Reference

Henningsen, M. & Stein, M. K. (1997). Mathematic tasks and student cognition: Classroom-based factors that support and inhabit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, *28*(5), 524-549.

Hiebert, J. & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In Frank K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp.371-404). Gweenwich, CT: Information Age.

OPPORTUNITIES TO BECOME AWARE OF VALUES AND THE SIGNIFICANCE OF LEARNING MATHEMATICS

Kazuhito Imai, Daijirou Kusunoki

Fukuoka University of Education, Togiri Elementary School, Japan

Raising children's awareness of values and the significance of learning mathematics is one of the aims of mathematics education in Japan. The theory of situated learning (Lave & Wenger, 1991), a new viewpoint on learning which emphasizes paying attention to 'activity', is considered to have potential to achieve this aim. Transforming mathematical content into a form of activity should make it easier for students to become aware of the values and significance of learning mathematics. The purpose of this research is to design a mathematical learning environment (lesson) based on the theory and to form specific principles of the introductory setting design with the purpose of creating a mathematical activity for students in the classroom. In this paper, the creation of mathematical activities for students is interpreted as evidence of their awareness of values and significance of learning mathematics. This research has formed general principles of lesson design as follows (Imai, 2010): (DMLE1) Choose a mathematical activity from daily life in order to stage it in the classroom. (DMLE2) Make students involved in the actual mathematical activity staged by the teacher. Based on these principles, a case-study lesson on the concept of per unit quantity for a class of 14 fifth graders who hadn't studied the concept before was conducted by one of the authors of this paper in 2011. The following conditions provided platform for the lesson: the number of students in the class and the setting of crowdedness which is generally used in Japanese mathematics textbooks to introduce the concept. At the introductory stage of the lesson the students were told they were going on a school trip, and three pictures of *tatami* rooms (1 room of the size of 12 mats and 2 rooms of the size of 10 mats) in a hotel were shown. The realism of the situation was compared to that from a daily life, and it was the students who tried to decide the number of students to stay in each room. Examination of the impulse which caused the students to get involved in the mathematical activity of thinking about the number underlined the importance of: (1) Making students recognize that they have some connection with the setting. (2) Preparing a setting which sets a clear goal and provides a means (in our case, the principle of crowdedness) to achieve it.

References

Imai, K. (2010). Forming the principles of the introductory setting design of creating students' mathematical activity in the classroom. In Y. Shimizu, Y. Sekiguchi, & K. Hino (Eds.), *Proc.5th East Asia Regional Conference on Mathematics Education* (Vol. 2, pp. 63-70). Tokyo, Japan: Japan Society of Mathematics Education.

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*, Cambridge: Cambridge University Press.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 280. Taipei, Taiwan: PME.

LESSON STUDY BASED PROFESSIONAL DEVELOPMENT : A CASE STUDY OF SCHOOL PROJECT IN THAILAND

Maitree Inprasitha Narumon Changsri Kasem Premprayoon

Center for Research in Mathematics Education, Khon Kaen University, Thailand

Japanese lesson study is attracting attention around the world (Isoda & Nakamura, 2010; Murata, 2011). Effective lesson study can have an impact on teachers' daily practice (Yoshida, 2005) and influence teachers' attitudes about being professional (Takahashi & Yoshida, 2004). In Thai context, the CRME has been implementing lesson study in the Professional Development Project since 2002. This project modified the Japanese lesson study by incorporating Open Approach and emphasizing on a ***"unique collaboration"*** in each phase of lesson study cycle.

This study aimed to describe activities for mathematics teachers' professional development based on lesson study approach. The target group was fourteen teachers from a school project since 2007 academic year. Data collection involved participatory observation on lesson study cycle at a school project 4 days per a week through one academic year, interviewing and questionnaire distribution.

The result revealed that activities for mathematics teachers' professional development based on lesson study in the school project were as followings; 1) A workshop for teachers to be conscious of their worldview about teacher profession 2) Teaching practice was set under the cycle of lesson study as following; collaboratively designing research lesson at least once a week, collaboratively observing their friend teaching the research lesson 3-4 hours per a week and collaboratively doing post-discussion or reflection on teaching practice once a week. 3) Extra activities for support self confident in teaching practice under the cycle of Lesson Study.

References

Isoda, M. & Nakamura, T. (2010). The Theory of Problem Solving Approach. *Journal of Japan Society of Mathematical Education, Special issue (EARCOME5)*. Tokyo; Bunshoudo Insatusho.

Murata, A. (2011). Introduction: Conceptual Overview of Lesson Study. *Lesson Study Research and Practice in Mathematics Education*. In L.C. Hart, A.S. Alice, A. Murata. New York: Springer.

Takahashi, Y., & Yoshida, M. (2004). Idea for establishing lesson study communities. *Teaching children mathematics 10(9).*

Yoshida, M. (2005). An Introduction to Lesson Study. *Building Our Understanding of Lesson Study*. In Patsy,W-I & Makoto, Y. (eds.). Philadelphia: Research for Better Schools.

Acknowledgement :This research was supported by the CRME and Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission, through the Cluster of Research to Enhance the Quality of Basic Education.

TEACHERS' REFLECTIONS ON A MATHEMATICAL TASK OF TEACHING: TEACHERS' KNOWLEDGE WHEN FIGURING OUT NON-STANDARD STUDENTS' WORK

Arne Jakobsen[1], C. Miguel Ribeiro[2]

[1]Department of Education, University of Stavanger (Norway); [2]Research Centre for Spatial and Organizational Dynamics (CIEO), University of Algarve, Lisbon School of Education (Portugal)

Teachers' mathematical knowledge involves much more than *knowing how to do and perform*. There is a direct relationship with the nature of tasks teachers prepare in and for teaching; thus their knowledge influences the way tasks are carried out in the classroom, in particular with respect to regulating the mathematical demands involved. Such knowledge is here conceived as mathematical knowledge for teaching — MKT (Ball, Thames and Phelps, 2008). In research conducted in Norway and Portugal, we focused on trainee and qualified teachers' knowledge in mathematical critical situations. Because such episodes have not received prior consideration, the teachers' knowledge comes much more to the fore because their response to such situations is so much more intuitive. To get a better understanding on the factors leading to teachers' difficulties in such situations, we combine a qualitative approach and an instrumental case study. Data used is sourced from teachers' reflections gathered in focus group interviews in Norway, and prospective teachers' answers to a questionnaire focusing on critical situations in Portugal, — both related to MKT and students reasoning.

We will present results focusing on the knowledge associated with making sense of non-standard students' work in relation to the use of alternative approaches to subtraction algorithms, and non-usual definitions in geometry. We will report from observations made independently, which, combined have developed a momentum on their own. By discussing and reflecting on what we have learnt about teachers' MKT, we hope to contribute to the discussion on how to improve and develop MKT, both in prospective and/or qualified teachers.

Acknowledgements: The first author would like to thank the Norwegian Oil Industry Association for its supports for the Norwegian part of the research project. The work for this short oral presentation has been partially supported by the Portuguese Foundation for Science and Technology (FCT). It forms part of the research project "Mathematical knowledge for teaching with respect to problem solving and reasoning" (EDU2009-09789), General Directorate for Research and Management of National Plan I + D + i, Ministry of Science and Innovation of Spain.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

GIFTED STUDENTS' CONSTRUCTING PRE-FORMAL PROBABILISTIC LANGUAGE

Irma Jan and Miriam Amit

Ben-Gurion University, Israel

The aim of this study was to investigate the influence of a "Probability Active Learning Environment" on the understanding of pre-formal probability. The study was conducted with mathematically gifted middle-school students who participate in "Kidumatica"- an after- school math club in Israel's southern region. The study is an extension of our preceding studies (Jan & Amit, 2010) in which we used an interactive study unit, based on the constructivist approach (Carpenter & Lehrer, 1999).

We followed the development of students' probabilistic language concerning central probabilistic key concepts: independence, theoretical and empirical probability of an event, probability comparison, and sample space. The study consisted of 13 experiment - oriented lessons carried out in a dynamic environment that serves as a 'nursery' for internalizing and utilizing these concepts. Pre-post tests were designed to assess students' understanding and perception of these concepts. For example, following is one of the items that addressed to the concept of independence:"In a TV game, there is a secret three - digit number. The first digit can be 1 or 2; the second digit can be 3 or 4; the third digit can be 5 or 6. Suppose you correctly pick the first digit. Has your chance of picking the second digit changed? Explain". Since the nature of the research was an interactive study unit, students were constantly required to justify their claims in order to communicate with their fellow peers. Therefore in the process they developed an informal probabilistic language based on everyday language. Evidence of this language can be found in the posttest. After the intervention (posttest) we found in students' justifications an intensive use of expressions like: It doesn't change the chance; it has no influence; they are unlinked; they are independent. These phrases are to students' ability to distinguish the relevant information from the irrelevant one which was presented in the question. The use of these phrases is evidence to students understanding the concept of independence which was presented in the questions. More evidence of students' extensive use of pre-formal probabilistic language after the intervention, concerning other probability key concepts, will be presented in the session.

References

Carpenter, T.P. & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19–32). Mahwah, NJ: Erlbaum.

Jan, I., & Amit, M. (2010). Students' multifaced strategies on predicting outcomes of random phenomena. Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 129-136). UFMG, Belo Horizonte, Brazil: PME.

DEVELOPING ALGEBRAIC STRUCTURE SENSE: A STUDY TO SUPPORT INSTRUCTION AND INFORM THEORY

Thomas Janßen, Angelika Bikner-Ahsbahs

University of Bremen (Germany)

Motivated by the work of Hoch (2007) on algebraic structure sense, its development in everyday classroom interaction is investigated by means of a design study. As Hoch's notion is referring to a hypothetical final product of algebra instruction, a redefinition of structure sense as a more open and dynamic entity is also a goal of the study.

In this endeavor the cultural-historical activity theory as made useful for the research in mathematics education by Roth and Radford (2011) serves as the theoretical background. The hypothesis is that algebraic structure sense develops through processes of objectification (development of knowledge) and subjectification (development of personality), and that structure-seeing as described by Bikner-Ahsbahs (2005) supports these processes.

The first cycle of the design study consisted of a unit on linear equations. In the preliminary evaluation of the data, algebraic structure sense was seen as developing in two stages. At the first stage the students show a *general structure awareness*. The students' decisions for and against certain structural elements from the set they are initially aware of can then be described as one of objectification as proposed by Roth and Radford. By different means—e.g. a metaphor as a support in the practical activity in one case and a high attentiveness to notational features in another one—the students discover the motive of the activity: to reduce the equation in order to solve it. As subjectification the very same process forms the personality of the students, for this instance in that they are able to fluently work with equations. Further research is expected to deepen insight in how such processes of objectification and subjectification evolve in the long run, how structure sense can be characterized in these terms, and how this description relates to Hoch's notion of structure sense. A focus will be on the importance of metaphors and other elements of spoken language, the development of written records as a means and document of objectification, and the role of the teacher acting in the *zone of proximal development* (see Roth & Radford 2011, 91-109).

References

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation. Theorie interessendichter Situationen – Baustein für eine mathematikdidaktische Interessentheorie*. Hildesheim, Berlin: Franzbecker.

Hoch, M. (2007). *Structure Sense in High School Algebra*. Unpublished doctoral thesis. Tel Aviv: Tel Aviv University.

Roth, W.-M. and Radford, L. (2011). *A Cultural-historical perspective on mathematics teaching and learning*. Rotterdam: Sense.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 284. Taipei, Taiwan: PME.

SECONDARY INSERVICE TEACHERS' KNOWLEDGE OF STUDENTS' THINKING IN PATTERN GENERALIZATION

Murad Jurdak and Rabih EL Mouhayar

American University of Beirut

The purpose of this paper is to explore secondary school in-service mathematics teachers' knowledge as reflected in their ability: (1) to explain students' abductive-inductive actions in near generalization tasks which involve students in employing different ways of counting and structuring parts of figural-steps in a pattern in an algebraically useful way, (2) to explain students' symbolic actions in far generalization tasks which involve students in translating the abductive-inductive action into algebraic generalization, and (3) to account for the variation in teachers' explanations of students' symbolic actions in terms of task and teachers' factors. A questionnaire was developed to measure teachers 'ability to explain students' abductive-inductive actions in near generalization tasks and their symbolic actions in far generalization tasks. The questionnaire was given to a sample of 83 secondary school in-service mathematics teachers from 22 schools in Lebanon.

Analysis of data shows that teachers' explanations of both students' abductive-inductive actions in near generalization tasks and of students' symbolic actions in far generalization tasks is lacking in terms of their ability to identify variable-related counting elements that involve counting which is dependent on the step number. The results of our study parallel the findings from other studies that report that students (Stacey, 1989) and pre-service teachers (Rivera & Becker, 2007) have difficulties in establishing and justifying a rule for the far generalization tasks. However, this result does not parallel findings from previous research which indicate that near generalization tasks were accessible to the majority of students (Stacey, 1989).

The results of step-wise multiple regression show that teachers' ability to explain students' symbolic actions in far generalization tasks mainly depends on their ability to explain students' abductive-inductive actions in near generalization tasks. This assertion supports and extends similar research findings on pre-service teachers' pattern generalization as an outcome of combined abductive-inductive and symbolic actions and that these actions play an essential role in generalizing patterns from a finite, incomplete class of particular instances (Rivera & Becker, 2007).

References

Rivera, F., & Becker, J. R. (2007). Abduction-induction (generalization) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behavior*, 26, 140-155.

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematic*, 20, 147-164.

CHANGING THAI TEACHERS' VALUES ABOUT TEACHING MATHEMATICS IN PILOT SCHOOLS IMPLEMENTING LESSON STUDY AND OPEN APPROACH

Thanya Kadroon Maitree Inprasitha

Center for Research in Mathematics Education, Khon Kaen University, Thailand

Values which teachers of mathematics brought to various aspects of their work profoundly affect what and how they teach, and therefore what and how their students learn (Bishop, et al., 2003). Values are expressed through views and behaviour (Veugelers & Kat, 2000). Changing behaviour of others can ultimately lead to changes value (Listenberger, 2004). An individual's values change over time and that their development is a lifelong process (Cairns, 2000 cited in Cole, 2002).Value change as either derivative or direct, value change caused by changes in the society's operating environment (Rescher, 1969 cited in Seah, 2004). Any significant development in mathematics education probably implies a change in values (Bishop, et al., 2003). The aim of this study was to analyse and explanatory values change about teaching mathematics of Thai teachers in pilot schools implementing Lesson Study and Open Approach. The research design is mainly qualitative. The study was structured through questionnaire survey of 83 teachers in 4 pilot schools. Case studies were then conducted with 3 of teachers in Kookham Pittayasan School, involving participatory observations and video recorded in 3 phase of Lesson Study, interviews, and document analyses. Theoretically, the conceptualization of professional development with Lesson Study and Open Approach, values and values change in this study helps to explain changing Thai teachers' values about teaching mathematics. The study shows that changing values about teaching mathematics of Thai teachers be changes in a gradual manner and increasingly the level of recognition and attention. The changes that occur as a result of the teachers involved in the process and the implementation of project activities continued from Year 1 to Year 6. To see the evolution and changing of the students in the class from their own view and feedback from colleague administrators and expert also participated in raising awareness for teachers and lead to changing values about teaching mathematics and found changing values about teaching mathematics of Thai teachers in 3 aspects: 1) Values in designing research lesson 2) Values in teaching practice and 3) Values in assessment.

References

Bishop, A. J.; Seah, W. T. & Chin, C. (2003). Values in Mathematics Teaching – The Hidden Persuaders? In: A. J. Bishop et al, (Eds.), *Second International Handbook of Mathematics Education. Pt. 1and 2* (pp. 717–765). Dordrecht. Netherlands: Kluwer Academic publishers.

Acknowledgement: This research is supported by Center for Research in Mathematics Education, Khon Kaen University, Thailand.

LESSONS ON ANGLES IN KOREAN AND U.S. ELEMENTARY MATHEMATICS CURRICULA

Ok-Kyeong Kim Jee Hyon Lee

Western Michigan University Seoul Women's University

This study compares lessons on angles in two elementary mathematics curriculum programs: *Mathematics* (Korea) and *Math Trailblazers* (U.S.). In this study, curriculum programs refer to written curriculum materials for day-to-day teaching and learning. *Math Trailblazers* is one of the reform curricula developed with funding by the National Science Foundation. *Mathematics* is based on the National Curriculum of Korea, where students have outperformed those in other countries in a number of international studies in mathematics. The comparison examines the characteristics used in each program to represent and develop the concept of angle. Specific guiding questions are: What are similarities and differences in the two programs in terms of key mathematical ideas and their development and organization in the lessons? What characteristics does each program exhibit in those lessons?

Angles are introduced and extensively explored in the lessons for grade 4 in both curriculum programs. The key content covered in those lessons is nearly the same, including identifying right, acute, and obtuse angles, representing angle measures in degrees, and using protractors to measure and draw angles. They, however, exhibit different approaches to teaching such content and distinct organization of the lessons *Mathematics* uses the static meaning of angles and operates on angles as objects from the beginning, using protractors early on and finding the sum of and difference in angle measures. In contrast, *Trailblazers* introduces angle as the amount of turning and asks students to do motions to represent angles, such as making a complete turn, and then describes angles as the amount of opening. In fact, *Trailblazers* promotes dynamic as well as static meanings of angle, with much emphasis on dynamic interpretation of angle throughout the lessons.

Precision and approximation are appropriate words to describe overall characteristics of *Mathematics* and *Trailblazers*, respectively. *Mathematics* emphasizes and promotes accuracy and exactness throughout the lessons. Estimating, thinking intuitively and manipulating angles are often required in order to emphasize the necessity of using protractors. *Trailblazers* uses approximation extensively throughout the lessons. Using benchmarks, such as 90°, to estimate, show and sketch angles is a core activity in the lessons. Also, expressions such as "little more/less than 90°" are commonly used to describe angles.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 287. Taipei, Taiwan: PME.

MEASURING PROFESSIONAL COMPETENCIES OF PRIMARY SCHOOL MATHEMATICS TEACHERS

Imke Knievel, Aiso Heinze

Leibniz Institute for Science and Mathematics Education Kiel, Germany

In the last years, several studies examined knowledge of mathematics teachers using Shulman's (1986) taxonomy of teacher knowledge. However, according to Weinert (2001), teachers' professional competencies comprise more than only *knowledge*: they are considerd as teachers' learnable context-specific cognitive resources (knowledge, strategies, skills, abilites) to master the core teaching tasks (e.g., lesson planning, lesson evaluation, teaching). Based on Weinert's concept, Lindmeier (2011) developed an extended structure model to describe the competencies for teaching mathematics. The model consists of three components: (1) underlying basic knowledge, (2) reflective competencies, and (3) action-related competencies. The basic knowledge includes the pedagogical content knowledge and the content knowledge. The reflective competencies subsume the abilities and skills required for pre- and post-instructional processes. Finally, the action-related competencies comprise the abilities needed to fulfil the spontaneous demands in classroom situation.

This project aims at (a) developing a valid and reliable instrument assessing all components of primary mathematics teachers' competencies, in particular the action-related competencies, and (b) analyzing the relation of the competency components. We developed a computer-based test containing items with short video clips of typical classroom situations. The teachers are asked to spontaneously react to a problem shown in the clip. To ensure the quality of the instrument, firstly, the content validity was approved by an expert review of 3 mathematics education professors. After that, a small pilot study with 8 in-service teachers was conducted to validate the authenticity of the presented classroom situations and to check the feasibility of the test administration. The feedback of the teachers strengthened the assumption that this new kind of items enables the measurement of action-related competencies. In March 2012, we started to collect data for the main study with 100 in-service teachers. The data will give information about the instrument's reliability. Moreover, by a principal component analysis and a correlation analysis, we will examine if the structure of the three components model can be confirmed and how the components are interrelated.

References

Lindmeier, A. (2011). Modeling and Measuring Knowledge and Competencies of Teachers: A threefold Domain-Specific Structure Model for Mathematics. Empirische Studien zur Didaktik der Mathematik: Vol. 7. Münster: Waxmann.

Shulman, L. S. (1986). Those Who Understand: A Conception of Teacher Knowledge. *American Educator*, *10*(1), 9-15 and 43-44.

Weinert, F. E. (2001). Concept of competence: A conceptual clarification. In D. S. Rychen & L. H. Salganik (Eds.), *Defining and Selecting Key Competencies* (pp. 45–65). Seattle.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 288. Taipei, Taiwan: PME.

INTUITIVE RULES IN DESCRIPTIVE STATISTICS REASONING

Ida Kukliansky Haim Eshach

Ruppin Academic Center Ben Gurion University of the Negev

Data analysis is an important step in science. However, students have significant difficulties using statistical tools. Intuitive rules as can be seen as a framework for understanding some of the errors made in descriptive statistics. According to Stavy and Tirosh (2000) novice learners reacted in the same manner to a number of tasks having a similar outer appearance but with no conceptual connection. Many alternative conceptions can be a result of using intuitive rules. Two such intuitive rules, for making comparisons are: 1) *Same A-Same B*: If ($A_1 = A_2$), thus it is intuited that ($B_1 = B_2$) 2) *More A-More B*: If $A_1 > A_2$, thus it is intuited that $B_1 > B_2$. In cases where the relation is a different one, using these rule will lead to error. This study aims to explain some of the misconceptions in descriptive statistics by using these rules. The subjects in our study were 77 engineering and physics university students. The research tools included a questionnaire and personal interviews. The results that can be explained by using these rules are exemplarily showed here. In one of the questions two data sets having the same number of observations and the same averages were provided. About a fifth of the participants wrongly chose the proposition that the standard error in both cases was the same because the averages are the same, and about a fifth chose the proposition that the standard error was the same because the number of measurements was the same. In another question we added to the data set a new measurement which was equal to the average of the previous ones. 14% of the subjects concluded that the standard error would be unchanged.10% of participants wrongly stated that raising the number of observations would always result in a bigger standard error. Further evidence for using these rules can be found in the participants' justifications of their answers, for example: "The standard error rises as we take more observations." The plausibility of intuitive rules can be a cause for data analysis errors. Educators may thus face students with situations were these rules manifest and expose their fallacy.

References

Stavy, R., & Tirosh, D. (2000). *How students (mis)understand science and mathematics: Intuitive rules.* New York: Teachers College Press.

ONE TEACHER'S STRUGGLE TO TEACH EQUIVALENT FRACTIONS WITH MEANING MAKING

Ruchi S. Kumar K. Subramaniam

Homi Bhabha Centre for Science Education, Mumbai, India

Change in teachers' beliefs and practices is a "process rather than an event, it must be considered in terms of continuous growth over time" (Sowder, 2007, p.97). In this light it is important to understand the challenges that arise for teachers as they attempt to change their teaching. Adopting a case study approach, we report one teacher's struggle (Mrs. A) to teach equivalent fractions to fifth grade students after expressing an intent to teach conceptually following a professional development workshop. The findings being reported here are part of a larger study (2009-11) on collaborating with teachers to develop classroom practices aimed at teaching mathematics for understanding having the components of professional development workshops and collaborative follow-up of classroom teaching by the first author. Audio records and researcher notes of the 14 classes of 35 minutes each (10 in 2009 and 4 in 2010) and teacher-researcher post teaching discussions focusing on the topic of equivalent fractions were analysed.

The initial challenge was for evaluating student understanding since Mrs. A tended to ask leading questions in response to wrong answers and then accepted correct answers without probing into students' thinking. Eventually she realised that student responses to fraction representations in terms of whole numbers ("Two" instead of "Two-fifth") as a conceptual problem and tried to address it. Secondly she focused on perceptual similarity to establish equivalence among different representations which she later overcame by using a linear model like fraction strips that allowed student to make conjectures after overlapping strips of different sizes. Her initial choice of representations were based on what she thought would be interesting to students like a colored wheel but later she used representations that allowed students to make meaning. Since some students were already aware of procedures for operating fraction through coaching classes, she made attempts to connect procedures with representations but was not able to do justice because of her lack of adequate pedagogical content knowledge. We argue that as a result of engaging with these challenges the teacher developed sensitivity towards mathematics as well as sensitivity towards student understanding.

References

Sowder, J. (2007). The mathematical education and development of teachers. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 157-224). Charlotte NC: Information Age Publishing.

APPLYING THEORETICAL KNOWLEDGE TO REFLECT PROFESSIONAL DEVELOPMENT IN ELEMENTARY SCHOOL MATHEMATICS INSTRUCTION USING A LEARNER-CENTERED APPROACH

Lee, Y.S.(1); Guo, C.J.(2), and Wang, M.C.(1)

(1)Department of Mathematics and Graduate School of Mathematics Education, Taipei Municipal University of Education

(2)Graduate Institute of Science Education, National Changhua University of Education

Based on the learner-centered concept, social constructivism, and the interaction of teacher educators and teachers, this study examines how elementary teachers apply theoretical knowledge in teaching mathematics. This article reports the professional development process of a lower grade elementary school teacher T1. This research used case methodology, and the study was conducted for three years. The researcher (the first author, and a teachers' educator) pursued teachers in a school who were willing to participate in a group for teacher instructional development. Teachers observed each other's teaching every two weeks and participated in a course on mathematics teaching. They used their time in afternoon advanced courses to discuss teaching issues together. The qualitative data include teacher instruction videos, teacher interviews, discussions between the teacher and teacher educators, and research journals. The triangulation method was used to test the reliability and validity of this quantitative research.

The findings show that the teacher was improved his professional development. For example, T1 said, "*These three years I have learned many things. …this afternoon's discussion has clarified mathematics knowledge and key concepts, misconceptions, and how to teach*" (20110630T1, Interview).

This study found that the participating teacher was able to use theoretical knowledge to reflect her own teaching. For example, T1 said, "*Today I gave students some exercises which were the same types of questions about two-place subtraction without carrying a number. All of these exercises were about the take-away type and the comparison type of word problems. These two types of word problems had already been practiced in the first grade, so I gave them these exercises to practice simultaneously*" (20101013 Teaching and Discussion). T1 also reflected the four mathematical operations, including the situation structure (discrete quantity, one-dimensional or two-dimensional continual quantity), the meaning structure (change, merge, comparison, and equivalence), the operation structure (augment, addend, or unknowns), and the representation theory (manipulative, pictures, or written symbolic), the key concepts of each unit, and possible misconceptions students may have, to enhance mathematics teaching (20101223T1, Teaching and Discussion, 20110302, Teaching and Discussion, and 20110630T1, Interview).

STUDENT GENERATED EXAMPLES AS A PEDAOGOICAL TOOL: THE CASE OF ALGEBRAIC EXPRESSION

Allen Leung — Hong Kong Baptist University, Hong Kong SAR, China

Doris Ming Yuen Ong — St. Edward Catholic Primary School, Hong Kong SAR, China

Using learner generated examples as a pedagogical approach has been studied with respect to students' mathematical learning experience and conceptual development (Watson & Shipman, 2008; Goldenberg and Mason, 2008). This paper presents a classroom case on introducing algebraic expression to Hong Kong Primary 5 students where student generated examples were integrated into pedagogy. A sequence of lessons was designed to guide students to experience how symbols could be used to express geometric patterns. Student generated examples during class constituted an example space that is a contingent driving force behind the design of the lesson sequence. An example space is "an experience of having come to mind one or more classes of mathematical objects together with construction methods and associations." (Goldenberg and Mason, 2008) A characteristic of an example space is the bringing about of awareness of the mathematical theme 'invariance in the midst of change' (Watson and Mason, 2005). The class teacher in this study made use of the variation in students' ways of understanding figure patterns as a pedagogical tool to design teaching/learning activities in the context of using an algebraic expression as a generalized description of many possible patterns. Students constructed different figure patterns to represent the same algebraic expression by a generalization that based on variation of constant position in the figures. These student generated examples were "possible and permitted variation that maintains examplehood". (Goldenberg and Mason, 2008) Thus an example space may emerge from a mutual interaction between the intended lessons (what the teacher planned), the enacted lessons (what actually happened in the classroom) and the lived lessons (how students responded to the teaching from what they have learnt in the lesson). These three pedagogic modes are contingent on student generated examples and are consequently growing out of each other.

References

Goldenberg, P. & Mason, J. (2008). Shedding light on and with example spaces. Educational Studies in Mathematics, 69:183–194

Watson, A., & Mason, J. (2005). Mathematics as a constructive activity: Learners generating examples. Mahwah, NJ: Erlbaum.

Watson, A. & Shipman, S. (2008). Using learner generated examples to introduce new concepts. Educational Studies in Mathematics, 69:97-109

A STUDY ON PARENTS ATTENDING MATH STUDY GROUP DESIGNED BY GRADE ONE ELEMENTARY SCHOOL CLASS TEACHER

Shuk-Kwan S. Leung[1], Pei-Jou Wu[2], Chia-Huang Chen[3]

National Sun Yat-sen University[1],

Wun-Fu elementary school Kaohsiung[2],

Kun Shan University[3]

The aim of this research is to study changes resulting from parents attending a study group designed by class teacher that include reading a research-based activity book for parents (Leung, 2008), engaging in teacher's designated activities relating to grade 1 elementary school mathematics contents; and trying ideas at home with school children. By referring to literature on numbers (Fuson, 1992) and Geometry (Clements & Battista, 1992) she decided on a study group format and adopted equal interaction and co-operative discussion setting. The math contents are: Number (1-10); Knowing about Geometric Shapes, Classifications (Color, Shapes). Data collection included questionnaire about study group meetings, children's interviews, parents' interviews, teacher's diaries, video tapes of in-class activities. Analyzes of interviews is by Goldin (2000). The findings are four: through the above mentioned study group (1) parents upgraded themselves in math ability and in confidence and set goals relating to different age of children; (2) children were given the opportunities to learn in a variety of ways; (3) promoted parent-child interaction, especially at home; and, (4) enhanced parent-teacher relationships and stimulated creativity in teacher's instruction.

Keyword:

Grade One Elementary Level, Parent-Child Math, Math Study Group

References

Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In Grouw, D. A.(Ed.). Handbook of research on mathematics teaching and learning, (pp.420-464). NCTM, New York: Macmillan publishing company.

Fuson, K. C. (1992). Research on whole number addition and Subtraction. In D. A. Grouws(Ed.), Handbook of research on mathematics teaching learning(pp.243-275). New York: National Council of Teachers of Mathematics.

Leung, S. S. (2008). The mathematics teaching survival kit. Kaohsiung, Taiwan: OutSide The Box (OSB) Publishing Co, Ltd.

A SURVEY STUDY ON PARENTAL INVOLVEMENT IN MATHEMATICS LEARNING FOR ELEMENTARY SCHOOL CHILDREN

Shuk-kwan S. Leung

Hui-Chi Chou

National Sun Yat-sen University

Recent cross comparative studies increased attention from children's performance to family factors (Chiu & Zeng, 2008). In this survey study, the investigator studied parental involvement in mathematics learning; parents' needs; and, relationship of involvements to needs. In order to conduct the survey, the investigator referred to Epstein (1995) and developed a questionnaire with items specific to mathematics learning of young children. There are 3 parts in the questionnaire: background information (6 parent variables and 8 children variables), parental involvement in mathematics learning and parents' needs. A total of 958 parents from 40 classes completed the questionnaires; they came from ten elementary schools (2 classes in each grade) in south Taiwan. Data analyzes were completed by descriptive statistics, t tests, ANOVA-one way, and, Pearson product-moment correlation coefficient. First, results on parental involvement are two: among the items in Epstein, the involvement is highest in *Learning at Home* category and lowest in *Decision Making* category; and, three among the six parent variables and seven among the eight children variables were related to parental involvements. Second, results on parents' needs indicated that the overall parents' need is high. Third, there is positive correlation between parental involvements and parents' needs. The investigator closed with recommendations for future research and practice.

Keywords: Lower grade, parental involvement in children's learning, parents' needs, survey study.

References

Chiu, M. M. & Zeng, X. (2008). Family and motivation effects on mathematics achievement. *Learning and Instruction*, 18 ,321–336.

Epstein, J. L. (1995). School / family / community partnerships: Caring for the children we share. *Phi Delta Kappan*, *77*(9), 701-712.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 294. Taipei, Taiwan: PME.

FACILITATING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' LEARNING THROUGH IMOVIES AND GAMES

Qing Li Olive Chapman

Towson University University of Calgary

Studies involving technology in mathematics teacher education have focused mainly on its use technology by prospective teachers to explore mathematics tasks or problem solving. Limited research, however, has focused on employing new technologies in ways that encourage teachers taking active part in the process of knowledge generation and co-authoring, therefore preparing them for the emergent culture of participation. In this study, we explored prospective secondary mathematics teachers' learning by immersing them in a process of designing, creating, sharing and using of media products with specific focus on iMovie and digital games for instructional purposes. We report on: the nature of the iMovie tasks and games they engaged in; the mathematical pedagogical knowledge they constructed through the iMovies and digital games; and their experience and challenges in the creation and use of iMovies and games in their learning, with implications for mathematics teacher education.

METHODS AND RESULTS

The study was framed in a qualitative, naturalistic research perspective that focused on capturing and interpreting the participants' experiences and knowledge. The participants were 34 preservice secondary mathematics teachers. Data sources included their iMovies and games for an algebra concept, all of their written assignments associated with the iMovie/game processes (e.g., their storyboarding), and reflective journals on assigned topics, including, their rationale, choice of topic, meaning of mathematics concept involved, assumptions, and changes made.

The findings provide insights into how prospective secondary mathematics teachers are likely to use iMovie and digital games without any explicit intervention to influence their thinking. The participants used them to establish connections, present problems, motivate, entertain, and be innovative in their teaching. Mathematic topics were linked to real world contexts, to students' interests, to self-interest/experiences, and to mathematics history. Mathematics problems were presented either as action scenarios or through pictorial modes. Humour, music and cartoon animations were integrated to motivate and entertain students. Creative thinking, problem solving, logical and planning skills, and analytical skills also emerged from the study. The participants were pushed to think creatively about how to teach the mathematics concept as they tried to take full advantage of the tools. Problem solving became the a key theme as the participants planned, tested, reflected on, and revised their production and made decisions about how to represent the mathematics content meaningfully. Analytical skills were constantly exercised in the unpacking of the mathematics concepts to develop their storyboards and working with these tools.

THE DIFFERENCES OF MENTAL NUMBER LINE REPRESENTATION BETWEEN STUDENTS WITH AND WITHOUT MATH LEARNING DISABILITIES

Wen-Hung Lien [a] Li-Yu Hung [a] Hui-Ting Chuang [a]

[a] National Taiwan Normal University, Taiwan

Numbers are represented along a so called mental number line, oriented spatially with increasing values from left to right (Dehaene, Bossini, & Giraux, 1993; Fischer, Castel, Dodd, & Pratt, 2003). Furthermore, the number line is represented with multiple scales, compressive/logarithmic and linear, which are used flexibly depending on the demands of the task (Dehaene, Izard, Spelke, & Pica, 2008). This view is supported by the finding showing a developmental trend from compressive to linear scaling (Siegler & Booth, 2004). This study aims to investigate the differences of the scaling associated with this representation between the students with math learning disabilities (MD group) and with normal math achievement (NA group) in second (G2) and forth grade (G4). All participants pointed at the line shown in notebook screen to tell the position of the number using 0-to-100 randomly. Two major results are found: (1) Both MD and NA groups' estimates for each number show linear mental number line function, no matter what grade he/she is. The improvements in the accuracy of estimates from G2 to G4 are both found in the MD and NA groups. Nevertheless, the accuracy of estimates in the G4 MD group is the same as the G2 NA group. (2) However, variation distinguishes MD from NA group. In G2, 26% of the MD students' mental number line representation is logarithmic, while only 14% of the NA group's is. In G4, the MD group's percentage of logarithmic representation is 8%, while the NA group's is 0%.

The findings illustrated that both MD and NA groups' mental representations for the 0-100 number line fit linear scaling in G2 and G4. Generally speaking, the accuracy of estimation increased with grade on both groups. However, the improvement of the accuracy with grade in the NA group was greater than that in the MD group. With further analysis, more individuals using logarithmic representation were found in the MD group than in the NA group both in G2 and G4.

References

Dehaene S, Bossini S, Giraux P (1993) The mental representation of parity and number magnitude. *Journal of Experimental Psychology: General*, *122*, 371–396.

Fischer MH, Castel AD, Dodd MD, Pratt J (2003) Perceiving numbers causes spatial shifts of attention. *Nature Neuroscience, 6*, 555–556.

Dehaene S, Izard V, Spelke E, Pica P (2008) Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science, 320*, 1217–1220.

Siegler RS, Booth JL (2004) Developmental of numerical estimation in young children. *Child Development*, *75*, 428–444.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 296. Taipei, Taiwan: PME.

USING NON-PROTOTYPE EXAMPLES TO HELP STUDENTS EXPLORE QUADRILATERALS: AN ACTION RESEARCH

Boga Lin

Beitou Junior High School, Taiwan

The classification of quadrilaterals is a difficult task to students (de Villiers, 1994). The ability to recognize squares, rectangles, rhombuses, parallelograms and trapeziums influences the learning of the inclusion relations among quadrilaterals. There were strong prototype phenomena related to the shapes of the square and rectangle in Japan, and related to angles in Scotland (Fujita and Jones, 2007).

This research contained two stages: (I) to understand students' ability to recognize quadrilaterals and to identify the difficulties and misconceptions they might have (II) to design a teaching activity to help students face and deal with their difficulties and misconceptions. 32 students of 7th grade participated in this action research for two months. Data were collected through questionnaires, interviews, learning worksheets, and videotaping.

The results of the first stage were that students recognized quadrilaterals better through definitions than simply through names. For example, the definition of a rectangle is any quadrilateral with four right angles. In the research, 20 students could recognize all quadrilaterals with four right angles through the definition, whereas only 4 students could recognize correctly through the name of "rectangle" from the 16 graphs. Similarly, the numbers of students recognizing correctly squares, rhombuses, parallelograms and trapeziums through definitions were 22, 14, 10, and 17, while the numbers of those who could recognize correctly through names were 23, 8, 11, and 7. In the second stage, focusing on prototype phenomena and the results of the first stage, a teaching activity was designed with many non-phenomenon examples. The core idea was to make students observe common features from those graphs, write down their observations, and then take turns delivering their observations in class. When one student presented an observation, the others had to determine whether the features observed were correct. Next, correct observations were recorded on the blackboard. Those features recorded should be enough for students to figure out correct definitions of quadrilaterals so they could connect definitions with names in their own terms and ways of reasoning. In the end, there were 12, 24, 14, 26, and 14 students who could recognize correctly rectangles, squares, rhombuses, parallelograms and trapeziums.

References

de Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *For the learning of mathematics*, 14, 11-18.

Fujita, T. and Jones, K. (2007). Prototype phenomena and common cognitive paths in the understanding of the inclusion relations between quadrilaterals in Japan and Scotland. *Proceedings of the 31st PME Conference*, 4, 41-48.

THE RELATIONSHIP OF MATHEMATICS ABILITY AND ACHIEVEMENT EMOTIONS

Shin-Huei Lin

National University of Tainan, Taiwan

Su-Wei Lin

National University of Tainan, Taiwan

Recently, emotional experiences related to learning and achievement have been noticed in educational research. (Goetz, Frenzel, Hall & Pekrun, 2008). The purpose of this research is to explore the relationship between the academic emotions and ability. The academic emotions in this study are defined as the domain-specific academic emotion, i.e. mathematics, including the positive and negative emotions. The total sample included 2986 10th graders from 43 different schools in this study. To assess students' performance in mathematics, the mathematics achievement test items based on the Taiwan Assessment of Student Achievement in Mathematics. This study adopted ConQuest software to estimate the students' mathematic ability. Based on the mathematic ability, the students were further classified as the highest-ability group, the higher-ability groups, the lower-ability groups, and lowest-ability group by their ability parameter, a total of four groups. There were ten items in mathematics achievement emotions, of which four items measuring anxiety, two items measuring helplessness, and four items measuring confidence. This study performed a correspondence analysis by SPSS statistics software to compare the emotional profiles of the four ability groups. The results revealed that the low-ability group students present the emotional traits of the high confidence, high helplessness, and high anxiety at the same time, and the highest group possesses the emotional traits of "low confidence, low helplessness, and low anxiety. The findings underlie the importance of mathematical emotional traits for students and how these constructs interact to facilitate the learning in mathematics domain. However, different methodologies in the study of emotions have multiple, complementary advantages and disadvantages. In research on achievement emotions, a multi-perspective methodological paradigm is required that integrates qualitative vs. quantitative approaches, experimental vs. non-experimental designs, as well as laboratory vs. field studies. In addition, there is a need to develop better measures and dynamic modeling methods allowing for an analysis of emotional processes over time.

Reference

Goetz, T., Frenzel, C A., Hall, N. C., & Pekrun, R. (2008). Antecedents of academic emotions: Testing the internal/external frame of reference model for academic enjoyment. Contemporary Educational Psychology, 33, 9-33.

PROFESSIONAL LEARNING OF PEDEGOGICAL LEADERS

Terry Wan Jung Lin

McGill University

It is generally agreed that effective professional learning communities can promote and sustain the learning of the participating professionals in order to enhance students' learning (Louis & Kruse, 1995; Vescio, Ross & Adams, 2008). As part of a Professional Development and Innovation Grant project, I have worked with a group of secondary cycle 2 (sec. 3, 4 & 5) mathematics teachers from an inner city school to develop a professional learning community within their department. For one school year, the group met regularly, once every other cycle of 6 days, to plan lessons, to discuss and reflect on their teaching practices and to evaluate their implementation of strategies. Data is drawn mainly from my field notes and reflective journal, and partly from the teachers' reflections.

For a professional learning community to develop into an effective one, many challenges must be surmounted (Bolam *et al.* 2005). This presentation highlights several obstacles that I faced as a leader in the implementation and development of a professional learning community. Firstly, the nature of the secondary school structure is challenging as these mathematics teachers are specialists and teach multiple grade levels and courses. The context of this inner city school adds another layer of complexity with the characteristics of its student population and its frequent turnover of teaching staff. Secondly, the composition of the members participating in building this professional learning community posed a challenge at the beginning stages of the process. Finally, the most difficult challenges in this process were linked to my professional learning and development. I questioned my role as the pedagogical consultant who was there to support the teachers. As well, I pondered the question, "What is the nature of the professional learning that is happening in this context?" The reported reflection will be extended to a further study on professional learning of pedagogical leaders in school districts when these are working with teachers.

References

Bolam, R., McMahon, A., Stoll, L., Thomas, S., & Wallace, M. (2005). *Creating and Sustaining Effective Professional Learning Communities.* University of Bristol.

Louis, K. S., & Kruse, S. D. (1995). *Professionalism and community: Perspectives on reforming urban schools*: Thousand Oaks, CA: Corwin Press.

Vescio, V., Ross, D., & Adams, A. (2008). A review of research on the impact of professional learning communities on teaching practice and student learning. *Teaching and Teacher Education, 24*(1), 80-91.

'NOT NORMAL' CLASSROOM NORMS

Minnie Liu and Peter Liljedahl

Simon Fraser University

Classroom norms (Yackel and Cobb, 1996) are both the *cause* and *effect* of stable classrooms. But what happens to these norms in a situation where a classroom is destabilized—a classroom in flux, where a teacher is making change? In a climate of teaching reform it is quite common to find such classrooms. Surprisingly, as much research as there is on mathematics teachers and how they negotiate changes in these settings there is very little research on how the students—the primary stakeholders in educational change—experience them. In this paper we present the results of a study designed to examine the way a group of 8^{th} grade students (n = 30) respond to a teaching style predicated on the pervasive use of numeracy tasks.

For the purposes of this study, a numeracy task is defined as a novel problem solving task contextualized in a real world setting. As such, these tasks are inherently ambiguous, open ended, and rely on the sophisticated use of rudimentary mathematics. Working in groups of two to four, participants were assigned one of these tasks each week from the beginning of the school year. The teacher (lead author) gave little to no direction as to how to solve them and did nothing to resolve any inherent ambiguity. On the other hand, she continually provided encouragement, facilitated discussions, and mobilized knowledge in the room. Data include field notes, survey responses, and transcriptions of informal and spontaneous interviews.

Results indicate that classroom norms, even if very different from the norms the students are used to, are quickly established. This is not surprising. What was surprising, however, were the results pertaining to what we have come to call *institutional norms*. As much as the students were quick to adapt to their new environment and to adopt the normative behaviours of the (new to them) expectations they were always aware that these new norms were 'not normal'. Their awareness went beyond how things were last year, may be next year, or currently are in other classrooms. The students were keenly aware that there existed, for them, a set of institutional norms—meta-patterns of behaviours that all classrooms (should) adhere to. The fact that their current classroom did not conform meant that they saw their experience in that class as temporary and restricted to that specific context. This is problematic for a number of reasons, not the least of which is the way it prevents the expansion of their understanding of what mathematics is and can be.

Reference

Yackel, E., & Cobb, P. (1996, July). Norms, Argumentation, and Autonomy in Mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 300. Taipei, Taiwan: PME.

EMERGING DEVELOPMENT OF ELEMENTARY SCHOOL MATHEMATICS TEACHER LEADERS

Yu-Jen Lu*   Jing Chung**   Hak-Ping Tam*

*National Taiwan Normal University **National Taipei University of Education

In Taiwan, each school must appoint teacher leaders for each subject. From 2008 to 2011, the Math Buddy program was instituted in Yilan County as a means to equip primary school math teacher leaders in their roles. The theoretical framework behind the program was based on the personal, professional, and social development perspectives that described the development of math teacher leaders as discussed in Lu & Chung (2010). The purpose of this paper is to report the framework and the implementation of the program followed by an analysis of the patterns of emerging leadership development among the participating math teacher leaders. The research subjects for this study were 41 mathematics teacher leaders who participated in the program. As a means to assess the effect of the program on the participants, a survey was administered that comprised of sixteen Likert five-point rating items and six open-ended questions pertaining to the three aspects of leadership development. After data preprocessing, 32 valid questionnaires were retained for detailed analysis, which was conducted both qualitatively and quantitatively.

Results of the study showed that the program was most effective with respect to the professional aspect of development for the math teacher leaders followed by social development whilst the personal aspect rounded up as the weakest. So far as the aspect of personal development is concerned, the participants first showed signs of willingness to take up the roles as resources provider and helper after the training. In contrast, their willingness to self-initiation and managing professional community were weaker. As regards professional development, the participants were confident their knowledge towards curriculum, instructional practices, assessment and learning. However, they were less confident in their knowledge regarding how to motivate the members of the professional community into action. As for social development, the participants could develop connection outside of their schools by way of the program. Yet the connection within their schools was relatively weaker. In sum, the participants could pursue their role as teacher leaders under external assistance. Nevertheless, they were rather limited to the role as resources provider. With the management of external social capitals, a chance was open up for the participants to develop their personal and professional development that facilitated the actualization of their leadership role.

References

Lu, Y. J. & Chung, J. (2010). Essentials of Developing a Mathematics Teacher Leader Project. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proc. 34th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol.3, pp.233-240), Belo Horizonte, Brazil: PME.

"WE PERCIEVE TWO MINUTES TO BE A FAST ACHIEVEMENT WHYLE FOR ROBOTS THIS PRESENTS A LIFE TIME": ANALYSING MATHEMATICS LEARNING FROM A SITUATED PERSPECTIVE

Sónia Martins | Elsa Fernandes
Louros Middle School | University of Madeira

In this communication we will discuss and analyze the transparency process of the artifacts, robots and mathematical concepts, as elements that shape, in a particular way, participation and, consequently, the production of knowledge, through engagement in practice.The main purpose of this study[i] is to understand how the use of robots contributes to the development of mathematics competences and of mathematics concepts. The nature of the research related in this article is qualitative due it aims to develop an understanding of human systems, such as a technology-using teacher and his or her students and classroom (Savenye and Robinson, 2004). Thus, the methodological choices were made taking into account the nature of the phenomenon under study and the theoretical framework that was chosen. Data had been collected from May to June 2011, over 8 sessions, with two primary school classes – 2nd and 3rd grade – from a school in Portugal, working together (24 and 16 students, respectively). The students have built and program LEGO robots that will became characters in a play-story. The robots physical and emotional features were defined by the students and the main storyline was negotiated by all of them. After the story was written, students programmed their robots, regarding the characters and their roles in the story.

The transparency of the practice organization, of its contents and artifacts, is a crucial resource to increase participation and therefore learning. Lave and Wenger (1991) elaborate transparency as involving two characteristics: invisibility and visibility. Having access to mathematics practice requires that mathematics concepts and procedures are, at the same time, visible and invisibles. To occur mathematics learning, students should be capable of use instruments (robots), focusing on them when necessary, but must also be able to make them invisibles when use them as forms to construct and broaden their knowledge.

References

Lave, J., & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation.* New York: Cambridge University Press.

Savenye, W. C., & Robinson, R. S. (2004). Qualitative research issues and methods: An introduction for educational technologists. In D. H. Jonassen (Ed.), *Handbook of research on educational communications and technology*. (2nd ed., pp. 1045-1071). Mahwah, New Jersey: Lawrence Erlbaum.

[i]This work is part of a larger project DROIDE II- Robots in Mathematics and Informatics Education funded by Fundação para a Ciência Tecnologia under contract PTDC/CPE-CED/099850/2008.

DEVELOPING MATHEMATICAL REASONING WITH REAL NUMBERS: A STUDY WITH GRADE 9 PUPILS

Joana Mata-Pereira
EB 2,3 Francisco de Arruda, Lisboa, Portugal
UIDEF, Universidade de Lisboa, Portugal

João Pedro da Ponte
Instituto de Educação,
Universidade de Lisboa, Portugal

Developing mathematical reasoning is an essential condition to understand mathematics and also to use mathematics in a proficient way. Reasoning ability is crucial to students, but to develop it is challenging to teachers, particularly in everyday classrooms, in all mathematical topics. Students cannot develop this ability just by memorizing concepts and routine procedures that only suggest that mathematics, far from being a logic and coherent subject, is just a set of more or less disconnected rules. Understanding a concept does not follow in an automatic way from knowing its definition, requiring students to know how this concept connects with others and how it may be used. Knowing mathematical procedures also requires understanding how they work, how they may be used, and how to analyse their results (NCTM, 2009). In order to understand how to promote the development of students' mathematical reasoning, a fruitful step is to regard students' mathematical reasoning processes as conjecturing, testing, and justifying.

This study aims to analyse grade 9 students' mathematical reasoning while working on tasks involving real numbers and to know how those tasks contribute to develop students' mathematical reasoning in an algebraic context. Based on a collaboration between the first author and a grade 9 mathematics teacher, the study is developed within a teaching unit supported by exploratory tasks involving real numbers. Following an interpretative and qualitative methodology, data collection includes videorecording of the lessons and students' written tasks. This presentation concerns two students working on a particularly significant task.

The results suggest that, working with real numbers, exploratory tasks promote the development of students' conjecturing processes, but are not enough to make them use testing or justifying processes. However, students' development of valid mathematical knowledge should rely on justification. Therefore, we conclude that, besides exploratory tasks, teaching practices such as questioning need to play a key role in promoting the use of reasoning processes and, as a consequence, in developing students' mathematical reasoning. Exploratory tasks allied with teacher questioning practices, emerged as key elements to develop students' mathematical reasoning as well as to promote their deeper understanding of real numbers properties.

References

NCTM (2009). *Focus in high school mathematics: Reasoning and sense making.* Reston, VA: NCTM.

FRAMING CHILDREN PRACTICES IN SCIENCE AND MATHEMATICS IN A FREE OPEN LEARNING SITE

João Filipe Matos, Ana Pedro, Pedro Patrocínio

Instituto de Educação, Universidade de Lisboa

Aiming to study how teachers and students appropriate and use artefacts (e.g. activity guidelines, technological tools) as structuring resources for learning science and mathematics within a interactive science museum, we take learning as participation (Lave & Wenger, 1991) and as an integral form of development which is materialized in qualitative transformations of the activity system (in a macro-level of analysis) (e.g. within the social world where the students' practice takes place) or/and of the subject (from a micro-analytical perspective) (e.g. assuming the perspective of the student). This movement is mainly related to the progression towards a wider and expansive field, for both the subject and the context (Engeström, 2001). The need to bring into dialogue the analysis of collective activity systems and the point of view of individual subjects is addressed through the exploration of the idea of learning as participatory transformation.

Within this rationale we developed a conceptual framework and guidelines that teachers use to prepare and run activities with their pupils in a special kind of school settled in premises of the Knowledge Pavilion in Lisbon. Children have free access to all the permanent and temporary interactive exhibitions and to all the modules where experiments in science and mathematics can be done. Teachers use the framework to prepare the children to interact to existing modules in the exhibition, making experiments, collect data, analyse it and share the results with their peers (e.g. children make real experiments with objects in water, simulations, collect data formulate conjectures and align their results and theories about floating, using modules in the exhibition prepared for general visitors to understand the mathematics and physics of floating). A Design-Based Research approach is adopted.

Data analysis of focus group interview with 7 teachers documents the relevance of the guidelines as structuring resources and indicates that the appropriation of the guidelines by teachers has a fundamental role in the quality and results of children engagement and learning.

References

Engeström, Y. (2001). Expansive learning at work: Toward an activity theoretical reconceptualization. *Journal of Education and Work, 14*(1), 133-156.

Lave, J. & Wenger, E. (1991). *Situated Learning: legitimate peripheral participation*. Cambridge: Cambridge University Press.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 304. Taipei, Taiwan: PME.

ISOMETRIES AND PATTERNS – A CREATIVE APPROACH

Lúcia Matos[1], Isabel Cabrita[1], Isabel Vale[2]

[1]University of Aveiro, [2]Polytechnic Institute of Viana do Castelo

Today's mathematics curricula for basic education confer an increasing emphasis on the topic of isometric geometric transformations on the Euclidean plane (Usiskin, Andersen & Zotto, 2010). Curricula also defend that recognizing patterns and regularities underlying some structures is considered the essence of Mathematics itself (Vale & Barbosa, 2010). That process will be favoured if students are comprehensively engaged in creative tasks that involve reproducing, continuing and completing patterns and identifying repeating sets. In fact, creative tasks, implemented in a creative way, demanding creative solutions and allowing student's imagination and actions grow free can be crucial to develop mathematical creativity, strongly underlined lately (Robinson, 2011). Dynamic software can be an innovative way to achieve those goals and to promote autonomy development and communication skills (Kasten & Sinclair, 2009) as well as to contribute to build a more positive view towards geometry.

Acknowledging that research encompassing these various topics is scarce, we conducted a qualitative case study (Huberman & Miles, 2002) with 9th grade students (14/15 years old), aiming at evaluating the impact of a creative approach to Isometries, centered on Patterns and using Geometer's Sketchpad (GSP), on the knowledge acquisition on isometries as well as on the development of mathematical communication and autonomy and of a positive relationship with Geometry. The main data collection techniques where enquiry, direct observation and documentary analysis supported, mainly, by questionnaires, field notes, logbook, tests, other students' assignments, including those computer related.

The research led to the conclusion that Isometries approach, centred on Patterns and using GSP, has contributed not only to deepen students' knowledge and skills on geometry, mathematical communication and autonomy but also to develop a closer relationship with the field of geometry itself.

References

French, D. (2005). Teaching and Learning Geometry. London: Continuum.

Huberman, M. & Miles, M. (2002). *The Qualitative Researcher's Companion*. London: SAGE Publications Ltd.

Robinson, K. (2011). *Out of Our Minds: Learning to be Creative*. UK: Capstone/Wiley.

Usiskin, Z., Andersen, K. & Zotto, N. (Eds.) (2010). Future Curricular Trends in School Algebra and Geometry. *Proceedings of the Second International Curriculum Conference*, The University of Chicago and The Field Museum, Chicago, Illinois, U.S.A.

Vale, I e Barbosa, A. (Org.) (2009). *Patterns – Multiple Perspectives and Contexts in Mathematics Education*. ESEVC. ISBN: 978-989-95980-4-1

INSTRUCTORS' PRACTICAL RATIONALITY IN COMMUNITY COLLEGE TRIGONOMETRY

Vilma Mesa

University of Michigan

The theory of practical rationality (Herbst & Chazan, 2011) states, in it its most general terms, that instructional situations are governed by *norms* that are imposed on a teacher teaching a particular course and *obligations* that exist towards the profession in which teachers are attached. Together with teachers' personal resources, norms and obligations can be used to understand the rationality of teachers' actions as they teach. We used this theory to guide the analysis of interviews with two community college instructors who were teaching trigonometry, and who were involved in a year-long research project that centred on collecting information about their students' understanding of inverse trigonometric functions. We wanted to investigate whether and how tuning into students' thinking, as promoted in K-12 research (e.g., Hiebert & Wearne, 1993), would reveal teachers' dispositions towards possible instructional changes. By creating a dissonance between what community college instructors thought their students understood about trigonometry and what their students revealed through questionnaires and in-depth interviews, we created an opportunity to understand the instructors' obligations in teaching trigonometry. The instructors selected the topic for investigation and proposed the design (pre- and post-tests, student interviews). When confronted with the information about students' understanding they discussed mostly institutional obligations (e.g., curricular organization) that were mostly out of their control. Obligations towards the students, the mathematics, or the class were tangentially addressed or not at all. The discussions revealed the strength with which the context in which instructors work influences decisions they make and also help us understand how these perceptions may run counter efforts to reform the teaching of mathematics in this setting.

References

Herbst, P., & Chazan, D. (2011). Research on practical rationality: Studying the justifications of actions in mathematics teaching. *The Mathematics Enthusiast, 8*(3), 405-462.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*, 393-425.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 306. Taipei, Taiwan: PME.

DIALOGUE IN LANDSCAPE OF INVESTIGATION: ACCEPTANCE AND RESISTANCE

Raquel Milani

State University of Sao Paulo – Unesp, Rio Claro – SP, Brazil

Denival Biotto Filho

State University of Sao Paulo – Unesp, Rio Claro – SP, Brazil

Skovsmose (2001) defines landscape of investigation as a learning environment characterized by invitation from the teacher to develop an investigative activity and its acceptance by the students. The type of communication between the teacher and the students and among the students that takes place in this scenario is the dialogue (Alrø & Skovsmose, 2004). In empirical terms, the dialogue is characterized by eight dialogic acts: getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging, evaluating. The directions of a dialogue and a landscape of investigation are unpredictable. There is no guarantee that students will accept the invitation of the teacher, there is no way of knowing whether the students will be resistant to the investigative activity and, once involved in the process, there is no way of knowing what knowledge they will produce. Entering a landscape of investigation can be a hard task to the teacher and the students. This scenario of unpredictability can be characterized as a risk zone (Penteado, 2001). However there can be different opportunities for learning. In a landscape of investigation the students can be able to question what is given and what is new, conduct the activity, argue for an idea, and create new perspectives. The dialogic acts help the teacher to renew the invitation for investigative activity. Biotto Filho (2008) proposes that the teacher could approach gradually a risk zone. We suggest that the teacher can start proposing inquiry activities related to mathematics itself. We believe that moving to a landscape of investigation is a possibility for the teacher to deal with the students' different intentions.

References

Alro, H., & Skovsmose, O. (2004). *Dialogue and learning in mathematics education - intention, reflection, critique*. Dordrecht: Kluwer Academic Publishers.

Biotto Filho, D. (2008). *O desenvolvimento da matemacia no trabalho com projetos*. Master Theses, IGCE, Universidade Estadual Paulista, Rio Claro, São Paulo, Brasil.

Skovsmose, O. (2001). Landscapes of investigation. *ZDM The International Journal on Mathematics Education*, *33*(4), 123-132.

Penteado, M. G. (2001). Computer-Based Learning Environments: Risks And Uncertainties For Teachers. *Ways of Knowing Journal*, *1*(2), 23–35.

CONCEPTUALIZATIONS OF SLOPE AS OUTLINED IN THE U.S. COMMON CORE STATE STANDARDS INITIATIVE

Deborah Moore-Russo

University at Buffalo, The State University of New York, USA

The mathematics curriculum in the United States has suffered from a lack of focus and coherence. In response to research and based on studies of the practices and curricula of high-performing countries, the Common Core State Standards Initiative (CCSSI, 2010) was recently introduced. The CCSSI outlines the expectations for students' conceptual understanding and mastery of procedures in a grade-by-grade sequence for grades K-8 and across six mathematical subject areas for high school.

Slope is typically introduced as a way to introduce covariation and describe the most basic behavior of functions. Slope also plays an essential role in the development of the derivative concept in calculus. Since the CCSSI aims to provide a more focused, coherent structure for the mathematics curriculum by stressing key ideas, this study considers how slope, arguably one of the most important concepts addressed in secondary mathematics, is presented and developed in the CCSSI document.

The eleven conceptualizations of slope offered by Moore-Russo and colleagues (Moore-Russo, Conner & Rugg, 2011; Mudaly & Moore-Russo, 2011) will be presented and used as the framework for the reporting of results. Contrast between the CCSSI and the National Council of Teachers of Mathematics' (2000) curriculum standards development of slope will be discussed. Since there is no mandated national curriculum in the U.S., both of these documents will be compared to a recent report on how slope is conceptualized in the standards documents of each of the 50 states (Stanton & Moore-Russo, 2012), which more directly impact the expectations for and enacted curriculum of mathematics teachers in the United States.

References

Common Core State Standards Initiative. (2010). *Common Core Standards for Mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Moore-Russo, D., Conner, A., & Rugg, K. I. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics*, *76*(1), 3-21.

Mudaly, V., & Moore-Russo, D. (2011). South African teachers' conceptualisations of gradient: A study of historically disadvantaged teachers in an Advanced Certificate in Education programme. *Pythagoras*. *32*(1), 27-33.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Stanton, M., & Moore-Russo, D., (2012). Conceptualizations of slope: A look at state standards. *School Science and Mathematics*, *112*(4), 241-248.

A MODELLING OF MISCONCEPTION IN MATHEMATICS LEARNING: THE CASE OF THE LAW OF SMALL NUMBERS

Koji Otaki and Hideki Iwasaki

Graduate School of Education, Hiroshima University

The purpose of this paper is to clarify the concept of *misconception*. For this purpose, a *tetrahedral model of conception* (TMC; *Fig.*), which is based on the *epistemological triangle* (e.g., Steinbring, 2006) and on the *C(C,N,E) model* (e.g., Mizoguchi, 1993), is constructed as interpretative framework. TMC consists of four components, which are 'object/reference context' (O), 'sign/symbol' (S), 'notion' (N) and 'conviction' (C). N means learner's concept, ambiguous idea, knowledge, or mental model. C means learner's attitude towards mathematics or mathematical knowledge. O means learner's practical experience on object or reference context, which are laden with N and C. S then means learner's practical experience on sign or symbol, which are laden with N and C. One solid line and three broken lines show whether four components connect directly or indirectly with each other. TMC as a whole means that O, S and N are nothing but connected indirectly by C. Misconceptions are characterized as "non-shift of only C" by using TMC.

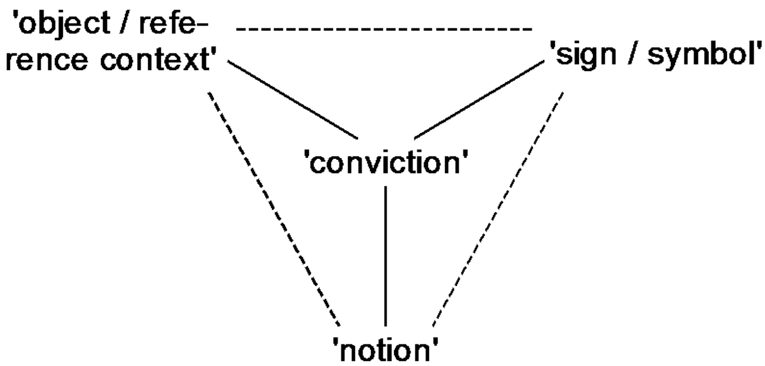


Fig. A tetrahedral model of conception

In domain of probability and statistics, there is the *misconception of the law of small numbers* (MLSN; Tversky & Kahneman, 1971). This misconception makes learners believe that a small sample will be representative of a population. MLSN can be identified by means of TMC as follow: [O] small sample; [S] population; [N] the law of large numbers; [C] determinism. C in learning of probability and statistics must be more or less indeterministic.

References

Mizoguchi, T. (1993). On shifting conviction in conceptual evolution. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. L. Lin (eds.), *Proceedings of the 17th Annual Conference of the International Group for the Psychology of Mathematics Education* (I, pp. 260-267). Tsukuba, Japan.

Steinbring, H. (2006). What makes a sign a *mathematical sign*? - an epistemological perspective on mathematical interaction. *Educational Studies in Mathematics, 61*(1-2), 133-162.

Tversky, A. & Kahneman, D. (1971). The belief in the law of small numbers. *Psychological Bulletin*, *76*, 105-110.

AN INSTRUMENTAL APPROACH TO PATTERN GENERALISATION IN SPREADSHEET ENVIRONMENT

Emel OZDEMIR ERDOGAN, Pelin TURAN

Anadolu University

It is stated that subject of patterns has contributed to skills of establishing mathematical relations; especially establishing functional relations, introduction to algebra and improving problem solving strategies (Mor et al., 2006). Additionally, the potential of spreadsheets aiming teaching and learning of algebra is researched by many researchers (Ainley, Bills and Wilson, 2005). In this study, the use of this potential of spreadsheets, which aims to find the rule of the patterns by students, is handled in terms of instrumental approach. Rabardel (1995) distinguishes artefact and instrument from each other. The instrument emerges from the encounter of an individual, who has knowledge and studying customs, and a tool, which has potential and constraints. This composition, which is called as instrumental genesis, includes two processes: *instrumentation* (the individual adapts the tool) and *instrumentalisation* (the individual makes the tool to adapt himself).

By using the instrumental approach theoretical framework, we tried to examine the usage of spreadsheets by 6th grade (11-12 ages) students, who have never experienced the spreadsheet environment before, in the questions that the generalisation of the patterns is asked. The study contains a five-week teaching process with the sixth grade (12 ages) students in a primary school. Clinical interviews have been started after the teaching programme has been finished. In this study, the data that has been collected during those clinical interviews have been considered.

The spreadsheet is seen as an instrument in each phase (introduction- research-generalisation) of exercises, and thanks to this instrument, it is found that different strategies that aim to find the pattern rule are being used. Also, while generalising the rule of the pattern, the students, who have met with algebra recently and never made algebraic expressions, can easily determine formulas for n^{th} term and take the formulas on the spreadsheet as their references.

References

Ainley, J., Bills, L. and Wilson, K. (2005). Designing spreadsheet- based tasks for purposeful algebra. *International Journal of Computers for Mathematical Learning, 10*(3), 191- 215.

Mor, Y., Noss, R., Hoyles, C., Kahn, K. and Simpson, G. (2006). Designing to see and share structure in number sequences. *International Journal for Technology in Mathematics Education, 13*(2), 65-78.

Rabardel, P. (1995). Les hommes et les technologies, approches cognitive des instruments contemporains, Armand Colin, Paris.

AN ANALYSIS OF THE CONTINGENCY DIMENSION OF THE KNOWLEDGE QUARTET IN KOREAN ELEMENTARY MATHEMATICS INSTRUCTION

JeongSuk Pang — Korea National University of Education

YooKyung Jung — Dangdong Elementary School

Many studies have been conducted to specify what constitutes mathematical knowledge in teaching. Rowland and his colleagues (2009) developed the Knowledge Quartet (KQ) while observing UK prospective primary school teachers' mathematics lessons. We use the KQ because it has been reported in several countries as an effective analytic tool to examine teacher knowledge in teaching. On one hand, we, like other researchers, explore adaptabilities of the KQ in Korea beyond the UK context. On the other hand, we look for further refinement in order to reflect on any significant context-specific phenomena.

In order to analyse Korean teachers' knowledge in mathematics teaching, we used on-line videotaped mathematics lessons which have been identified as effective instruction from the contests of instruction-research among practicing teachers. For this paper, ten lessons were analysed, specifically with regard to the contingency dimension of the KQ as an attempt to specify what expert teachers' knowledge entails. Concise descriptive synopsis about each teacher's main actions and episodes was written and then assigned to the three codes of the contingency dimension: 'responding to children's ideas', 'use of opportunities', and 'deviation from agenda'.

The results showed that the prevalent codes were 'responding to children's ideas' and 'use of opportunities'. A noteworthy aspect was that the teachers tended to adjust students' unexpected contributions to their original agenda rather than deviating from the agenda. This tendency has not been explicitly addressed in the KQ. Deviating from the agenda in order to respect students' ideas and to provide them with a new learning opportunity is an aspect of teacher expertise. However, tailoring students' ideas skilfully to her original agenda is also another desirable element of teacher expertise, which may be underestimated in the KQ. By presenting representative episodes with this matter, this paper is expected to provide an opportunity to probe various aspects of contingency and to seek for further elaboration to be used across countries.

Reference

Rowland, T., Turner, F., Thwaites, A., & Huckstep, P. (2009). *Developing primary mathematics teaching*. London: SAGE.

A COMPARISON OF TREATMENT OF PROBABILITY IN HIGH SCHOOL MATHEMATICS TEXTBOOKS BETWEEN CHINA AND SWEDEN

Aihui Peng[1,2]

[1]Linnaeus University, Sweden; [2]Southwest University, China

International studies of mathematics achievement show that in the past decades, East Asian students have consistently outperformed their counterparts in Western countries (*cf*. OECD, 2010). There are lots of factors which impact students' learning of mathematics. It is commonly assumed that textbooks are one of the most important sources for the content covered and the pedagogical styles used in classrooms and, therefore, they dominate what students learn (Haggarty & Pepin, 2001). To explain the underlying reasons which lead to the disparity of students' learning of mathematics, researchers show growing interests in cross-national comparison of textbook analysis.

Probability has been a quiet new topic included in school curriculum in many countries. This is the case for both China and Sweden, where probability is one of the central content areas in their newly developed mathematics curriculums in both countries. The current study aims to compare the treatment of content and problems of probability in Chinese and Swedish high school mathematics textbooks.

Methods employed in the study include both problem and content analysis (Son & Senk, 2010).The selected textbooks are popular used in each country. The selected Chinese mathematics textbook is published by the People's Educational Press, 2009, and the selected Swedish mathematics textbook is published by Bonniers, 2011. Preliminary analysis shows that Chinese and Swedish mathematics textbooks have similar content covered, but they differ in the presentation of concepts and cognitive expectations. Implications of findings for both textbook writing and mathematics teaching are discussed.

Acknowledgements. The work was supported by the Swedish Research Council (No. 20117349).

References

OECD. (2010). PISA 2009 results: What students know and can do – Student performance in reading, mathematics and science (Volume I). *Retrieved on January, 2012 from http://dx.doi.org/10.1787/9789264091450-en*.

Son, J., & Senk, S. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics*, 74 (2), 117-142.

Haggarty, L., & Pepin, B. (2001) An investigation of mathematics textbooks and their use in English, French and German classrooms: who gets an opportunity to learn what? *British Educational Research Journal*, 28(4), 301-332.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 312. Taipei, Taiwan: PME.

A COMPARISON OF VALUES IN EFFECTIVE MATHEMATICS LESSONS IN MIDDLE SCHOOLS BETWEEN CHINA AND SWEDEN

Aihui Peng[1,2] Kongxiu Kuang[2] Yueqiang Shang[3]

[1]Linnaeus University; [2]Southwest University; [3]Guizhou Normal University

Values are the principles, standards and qualities explicitly or implicitly considered to be worthwhile or desirable by the participants of a distinct social practice (Jablonka, 2005). Mathematics education is designated as social practices where the teaching and learning of mathematics actually occur, and it is deeply rooted in its particular culture; therefore, what is valued in one culture will not necessarily, in general, be valued in another one (Peng & Nyroos, 2010). In their detailed literature review on mathematics teaching in countries with high performance, Askew, Hodgen, Hossain, and Bretscher found that high attainment in school mathematics may be much more closely linked to cultural values than to specific mathematics teaching practices (2010, p. 12).

This study aims to investigate the differences and similarities on what are valued in effective mathematics lessons in middle schools from students' and teachers' perspective between China and Sweden, two countries where there are vast cultural differences. Two groups of students and their mathematics teachers from each country participate in this study. Data collection includes lesson observations and interviews. Structured interview questions are based on students' recall of the moments of effectiveness and the photos selected by students when they feel that they are learning mathematics particularly well and draw upon teachers' reflective thoughts relating to the lessons observed.

Research results show that there are both differences and commonalities in values in effective mathematics lessons between China and Sweden. Teaching explanation is commonly valued by both students and teachers from both countries. Values differences are related to issues such as collective and individual teaching. Reasons underlying those differences and commonalities are discussed from cross-cultural perspective.

References

Askew, Hodgen, Hossain & Bretscher. (2010). *Values and variables: Mathematics education in high-performing countries*, Nuffield Foundation.

Jablonka, E., & Keitel, C. (2006). Values and classroom interaction: students' struggle for sense making. In F. K. S. Leung, K. Graf & F. J. Lopez-Real (Eds.), *Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West* (pp. 495-521). New York: Springer.

Peng, A., & Nyroos, M. (2010). Differences and similarities of different group students' values in effective mathematics lessons. In B. Yoshi (Eds.), *Proceedings of the 5th East Asian Conference on Mathematics Education*. Tokyo, Japan.

PRE-SERVICE TEACHERS' FINANCIAL LITERACY

Vladimíra Petrášková

University of South Bohemia in České Budějovice, Faculty of Education

Nowadays everybody must face the ever increasing offer of financial products and services that are often untransparent and confusing. Only a person with a satisfactory level of financial literacy is able to make the right decisions in this situation. Furthermore the young generation, in contrast to the generation of their parents, must be more responsible in their decisions about investments in order to secure their pension and necessary healthcare.

It is evident, that pre-service teachers' knowledge in the field of finances is crucially important as it is a prerequisite to the development of students' financial literacy. We developed a quantitative study focusing on testing pre-service teacher financial literacy at the start of their studies: 47 pre-service teachers of mathematics at the lower secondary level (for pupils aged 11 – 15*)* were assigned a test on financial literacy within the frame of this study. The questions of the test were adopted from the national survey of financial literacy in the USA (FINRA 2011).

We asked the following questions at the beginning of the survey: 1.What is the level of financial literacy of pre-service teachers at the start of their studies? 2. Which components of financial literacy are developed sufficiently? 3. Which components of financial literacy should be further developed in pre-service teacher training?

The test showed the following:

1. The level of financial literacy of students at the beginning of their university studies is average.

2. The following components are sufficiently developed: (a) knowledge of the principle of simple and compound interest; (b) knowledge of the impact of inflation on savings; (c) principle of paying off a debt.

3. The results of the survey clearly show that it is necessary to focus pre-service teacher training on stock markets. This means: (i) Solve model situations from the area of investments into stock and shares. (ii) Solve model situations showing the factors influencing the price of a bond.

References

Hašek, R., & Petrášková, V. (2010). Issue of Financial Capability. *The International Journal for Technology in Mathematics Education, 17*(4), 183-190.

Hašek, R., & Petrášková, V. (2010). A Way to Improve Financial Literacy of Future teachers. In T. Bianco, & V. Ulm (Eds.), *Mathematics Education with Technology - Experiences in Europe.* (pp.199 – 218). Univ. Augsburg.

FINRA. (2011). *Financial capability study* [online]. [cit. 2012-07-01]. Available at http://www.usfinancialcapability.org.

SUMMATIVE AND FORMATIVE ASSESSMENT: A DIFFICULT DIALOGUE

Jorge Pinto and Leonor Santos

Polytechnic Institute of Setúbal, Institute of Education of Lisbon University

The relationship between assessment for formative and summative purposes is still a problematic issue in assessment practice (Harlen, 2006); although there are several ways to answer this question, they all carry risks (Black & Wiliam, 2003). In Portugal, there is a general assumption that written tests are assessment instruments with a high level of reliability. Recently, primary schools (students with ages between 6 to 10 years old), have been using mathematics written test, each trimester, applied to all the students from the same grade of each school or group of schools. This practice intend to assess the students' achievement in mathematics and to contribute to develop plans to support their learning. In other words, the main purpose of using this method is to articulate the formative and summative assessment. Considering this context we have formulated the following research questions concerning one group of schools: (i) what kind of information is collected by the teachers, from the test, about students' difficulties; (ii) how they use this information and (iii) if there are any differences by years of schooling. Using an interpretative methodology, we collected data through observation of the councils assessment by grade, conducted with the purpose of analyzing and making decisions based on the results obtained on the written test, and the collection of the minutes of these meetings. The data were analysed according to a discourse analysis. A total of thirteen teachers was involved. The results of our research evidence that, in most cases, the information that teachers get from students' tests is marked by a high level of generality, making it more difficult to build supportive plans directed to the specifications of each student. As the students progress in their education, the difficulties described move from certain mathematical topics (numbers, operations) to mathematics capacity (problem solving and mathematical reasoning and communication). These results indicate that these tests reproduce more a logic of summative purposes, than configure a moment of formative assessment.

References

Black, P., & Wiliam, D. (2003). In praise of educational research’ formative assessment. *British Educational Research Journal, 29(5),* 624-637.

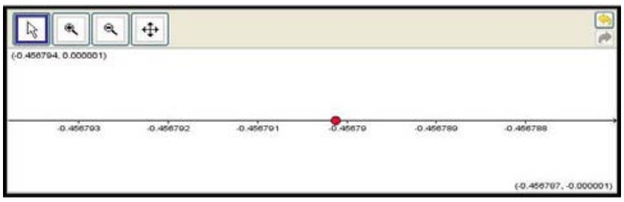
Harlen, W. (2006). On the relationship between assessment for formative and summative purposes. In J. Gardner (Ed.), *Assessment and learning* (pp. 61-80). London: Sage Publications Ltd.

LEARNING OBJECTS, REAL NUMBERS AND THE NUMBER LINE MODEL

Márcia Maria Fusaro Pinto, Felipe Pereira Heitmann

Universidade Federal do Rio de Janeiro, Universidade Estadual Paulista

Here we report on the conception, theoretical foundation and development of a learning object to study the concept of real numbers. The aim is to explore its decimal representation in a number line, and also the mathematical line model having the notion of completeness as a starting point. The methodology integrates the constitution of learning environments as a component of the object development.



The object conceived of is a very simple *applet*, which allows the movement of a point on the line, zooming in and zooming out on a selected region, and a scaling.

The theoretical foundation recalls Duval (1993), who argues that becoming able to deal with representations and to perform *conversions* allows us to get acquainted with different aspects of an object, and to perform *treatments* more efficiently. *Conversions* between representations represent a cognitive demand throughout the learning process, and must be made in both directions. The idealized learning environment using the object invites to use successive zooms: to determine the coordinates of points on the axis, rounding them up to six decimal places – a limitation of the software, in this case.

We conclude reflecting on the entire process, highlighting possibilities opened by the *software*. Firstly, we were able to propose unusual *conversions* for the study of numbers: from the graphical to the numeric representations. Secondly, limitations of the *software*, once perceived, may be used constructively (Giraldo et al, 2003): they may be a starting point for a discussion on the theoretical mathematical model for the number set and the real line. Finally, we mention the convenience of using a friendly dynamic geometry *software* for the development of our prototype, both for the possibility of using multiple representations of concepts, including those in dynamic stages (Moreno-Armella et al, 2008), and also for the possibility of the development of such objects by a teacher who is not a computer specialist.

References

Duval, R. (1993) Registres de représentation sémiotique et fonctionnement cognitif de la pensée, *Annales de Didactique et de Sciences Cognitives*, 5, IREM Strasbourg, 37-65.

Giraldo, V., Carvalho, L.M. & Tall, D. (2003). Using theoretical-computational conflicts to enrich the concept image of derivative. *Research Into Mathematics Learning*. , 5, 63 - 78.

Moreno-Armella, L; Hegedus, S. J.; Kaput, J. J. (2008) From static to dynamic mathematics: historical and representational perspectives. *Educational Studies in Mathematics*, v.68, 99-111.

PROFESSIONAL REFLECTION AND DEVELOPMENT: BECOMING A TEACHER

Anne Prescott[1], Michael Cavanagh[2], Tania Kennedy[3], Frederic Jaccard[4]

[1]University of Technology, Sydney, [2]Macquarie University, [3]Meriden School, [4]Xavier College, Llandilo

The term ‘reflective practice’ is commonly used to describe a disposition to reflect critically on one’s classroom practice and is regarded as an important part of teacher development. Wenger, et al. (2002) believe a community of practice results from a group learning together with a common focus. The relationships between the participants allow a coherent knowledge community to develop. This paper describes the early stages of the development of a community of practice for beginning secondary mathematics teachers.

Michael and Anne are mathematics methodology lecturers and Frederic and Tania are former students who have been teaching for three years. Each academic made regular visits to their classrooms, observing and discussing lessons, looking for patterns and differences and generally reflecting on the issues involved in teaching.

Three themes emerged from their discussions: improvements in self-confidence and self-efficacy, the community of practice, and reflective practice.

In the early days both teachers were looking for reassurance – as beginning teachers, self-confidence can easily take a battering with minor incidences becoming disproportionately large in their minds (Prescott, & Cavanagh, 2008). In-school mentoring has limitations that working with the academics did not have – they could be a critical friend who was not associated with the school so was objective.

As solo academics at their respective universities, Michael and Anne have developed their own community of practice but working with the teachers has allowed them to broaden this community of practice, allowing them to have a unique opportunity to reflect on their own practice.

In discussions following their lessons, Tania and Frederic have demonstrated a remarkable ability not only to identify critical learning and teaching incidents but also to analyse some likely causes and suggest remedies for dealing with them. The nature of the comments and questions posed by Tania and Frederic are an indication of a developing ability to truly reflect on their classroom practice.

References

Prescott, A., & Cavanagh, M., (2008). A situated perspective on learning to teach secondary mathematics. *(Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Brisbane*, pp 407-413). Brisbane: MERGA.

Wenger, E., McDermott, R., & Snyder, W.M. (2002). *Cultivating communities of practice.* Boston, MA: Harvard Business School Press.

COMPARING CONTAINERS AND THE UNDERSTANDING OF VOLUME IN 3RD GRADE

Silke Ruwisch

Leuphana University Lueneburg, Germany

As Reece & Kamii (2001) showed, a majority of 51% of the third-graders already were able to reason by the transitivity in the measurement of volume, whereas unit iteration only was used by 47%. Asked to compare containers regarding their volume, children have to have experience with both concepts to give judgements, which are based on mental comparison. Solving volume tasks in written tests, Hart (1980, 1984) found only little differences with regard to the form of presentation: real blocks or drawings. But she noticed that the majority of the 12-14 year olds tried to count whenever they could, although it might not have been successful.

So, the question came up, if third-graders before any formal instruction in volume are capable of comparing different containers with regard to their volume. In a case-study with 35 students we presented different containers in different representations: real objects or pictures. The children were allowed to look at the objects or to use them to solve special tasks. The pictures of the containers also were given in different ways: real size pictures, pictures on the same scale or on different scales.

The results of Reece & Kamii (2001) were confirmed by our small data: Nearly 60% were able to reason by transitivity, most of them (50%) gave transitive reasons on their own. Unit iteration was more difficult and was only shown in a second step. All successful children (51%) asked for as much cups as they needed to distribute the water, before they were able to transfer this procedure to the measurement with only one cup. 85% of the children could take into consideration the different dimensions in comparing differently shaped and filled cylinders. As Hart (1984), we found no differences in comparing real objects or drawings in this task. Given pictures of containers known from everyday life on different scales was not as difficult as we expected. Nearly 60% could compare two given containers correctly; about 50 % were able to put ten containers in a correct order (one mistake taken into account).

Apparently, the children have a lot of experience with volume in their everyday life, so the concepts of transitivity and unit iteration seem to be the most fostering task.

References

Hart, K. (1980). Secondary School Children's Understanding of Mathematics. London: Chelsea College.

Hart, K. (1984). Which Comes First – Length, Area, or Volume? *Arithmetic Teacher*, *31*(9), 16-18, 26-27.

Reece, Ch. S. & Kamii, C. (2001). The Measurement of Volume: Why Do Young Children Measure Inaccurately? *School Science and Mathematics, 101* (7), 356-361.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 318. Taipei, Taiwan: PME.

A COLLABORATIVE RESEARCH PROJECT ON PROBABILITY

Annie Savard and Terry Wan Jung Lin

McGill University

The process of collaborative research is a shared construction between the teacher and the researcher that provides an opportunity to finding answers to important questions for them (DeBlois, 2009). A collaborative research has three components: 1) cosituation or the setting of the project, where the teacher and the researcher defines their role and their aims; 2) cooperation between them through reflections; and 3) coproduction where a new knowledge is constructed. In this project, the participating teacher is Inuk and she taught a grade three class in Inuktitut. The teacher's goal for this project was to learn more about mathematics. She felt the need to know more mathematics, in general. The principal researcher was not Inuk. Her initial aim was to conduct a study on the development of a probabilistic reasoning on grade three Inuit students and on the development of critical thinking. The collaborators agreed to construct learning situation on probability using an ethnomathematics framework (Savard, 2011), and using some traditional games. During the project, the researcher's aims shifted. It became necessary to inquire into the teacher's practice because it appears that the teacher's learning intentions were implicit in some aspects. These implicit learning intentions affected the orientation of the research project. But, how to interpret the influence of the teacher' learning intention on a collaborative research project?

The preliminary results show that this collaborative research project between the teacher and the researcher presented some breaches in their aims. At the beginning of the cosituation, the collaborators' aims were defined, but during the cooperation phase, the teacher's aims changed. She said that her goal was to learn more mathematics, but her actions were shifted to focus on the sociocultural and languages arts goals of the curriculum. The change could be due to the reason that the teacher was not familiar with probability. The researcher tried to support the teacher, but it takes time for this concept to be properly understood. The concept of probability was introduced early in the phase of the co-situation, but because the teacher did not explicitly express her needs, and because the researcher did not want to offense the teacher by a detailed explanation, the concept of probability remained unclear for the teacher. It seems to us that in a collaborative research project, the mathematics involved played an important role on the teacher's representations and actions.

References

DeBlois, L. (2009). Les contextes et les besoins à l'origine de la recherche collaborative. Annales de Didactique et de Sciences Cognitives, 14, 219-229.

Savard, A. (2011). Elementary Teachers Do Not Teach Through Mathematics.In J. N. H. Moraova, Proceedings of the International Symposium Elementary Maths Teaching (pp. 297-305). Prague, CZ.

THE ROAD NOT TAKEN – EMPIRICAL RESEARCH AT THE BEGINNING OF NEW MATH

Gert Schubring

Bielefeld University/UFRJ

Had there been a chance to found the new math movement from the beginning on research into the psychology of mathematics learning and not just on contents derived from the structure of the discipline and focussed on secondary school math? This is what one infers from a revealing research report, elaborated in preparing the famous Royaumont Seminar of 1959, which launched the new math movement. The report, now found in yet unexplored archives of the OECD, constitutes the first genuinely international and well informed analysis of research into the psychology of learning and teaching mathematics. The report evidences the bold self-confident attitude of certainty to have achieved a higher level of empirical research. Its authors were both from the National Foundation for Educational Research (NFER), founded in 1946, a pioneering institution: William Douglas Wall, its director and specialised in educational psychology, and John B. Biggs, working for mathematics education (Howson 2009). The report with its 47 pages comprised an extensive bibliography, of 163 items. It was structured by 3 sections: relating research up to 1950, a second presenting the period from 1950 on, featuring in particular Piaget, and finally giving an outlook to needed future research. The bulk of research up to 1950 was characterised as "short-term, scattered and piecemeal" and "by the scarcity of attempts to develop a comprehensive theory of the psychology and methodology of mathematics" (p. 1). The authors attributed this to the isolation of its "pioneers devoting their spare time […], unsupported by any research organisation" – another part of work being done as doctoral theses, which "inevitably suffers from the limitations imposed by this (p. 2). As an example, studies on students' errors were shown as poorly designed. Most studies pretending to address "psychology of arithmetic" would "cause raised eyebrows today" (p. 8). Research from 1950 on is credited to embrace already theoretical foundations of child psychology.

One can assume that this report was instigated by Marshall Stone, the seminar's president. His introduction surprises as a fervent plea for focusing the reform on primary grades, on promoting psychological research and on founding research institutes. Given the seminar's inherent contradiction that the other participants were university mathematicians or secondary school teachers, his approach to reform became not implemented. The "remarks", which had been allotted to Wall and where he sketched an ambitious long-range assessment project, remained isolated and without the duly documented discussion. A few statements on research did not become part of the seminar's Final Conclusions.

Reference

Howson, G. (2009). The origins of mathematics education research in the UK: a tribute to Brian Griffiths. *Research in Mathematics Education*, *11*(2), 97-114.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 320. Taipei, Taiwan: PME.

PISA AS AN EYE OPNER FOR TEACHERS: A CASE WITH APPLES TASK

Heejoo Suh — Michigan State University
Rachel Snider — University of Michigan
Edward Silver — University of Michigan

The PISA mathematics assessment has received almost no attention within the U.S. mathematics education community. A consequence of the lack of attention to PISA in the U.S. is that we have underutilized a potentially valuable source of information for improvement of mathematics education.

This paper is a part of larger study that sought to explore ways to transform PISA as an accessible resource for mathematics educators. We claim that PISA items can stimulate teachers to see the unseen, because of their authenticity. To support this claim, we provide a case study of professional development program that used a PISA task as a main resource.

The task M136: Apples is one of the items that OECD released to the public. We used a modified version of the original Apples task, to address collaborating professional developers' request. The original Apples task is constituted with three subtasks. The second subtask was modified to be more open-ended. In all we collected 917 student responses to the Apples task from a convenient sample, exit surveys of a professional development program that used the Apples task for participating teachers, and personal interaction with the facilitators' of the professional development.

Our finding suggests that the Apples task initiated interesting conversation among participating teachers. First, the teachers started to see the grade level they are teaching as one point of a continuum. Using the coding result of 917 responses, teachers engaged in a lively discussion across grade levels. Next, the teachers attended to mathematical communication. Focusing on the coding result of a subtask that asked students to explain their reasoning, participating teachers discussed the importance of providing opportunity to students to explain and talk to each other about solution strategies. Lastly, the Apples task allowed the teachers to think about connections in mathematics in two different levels. One level came naturally from the different types of expressions appeared in the task (diagram, tabular, numeric, symbolic, and verbal). Another level was the connection between different types of solution paths. In particular, the need of explicit connection between recursive reasoning and explicit reasoning was addressed.

EFFECTS OF ONLINE TRAINING COURSES FROM THE PARTICIPANTS´ PERSPECTIVE

Roman Sebastian Szymanski & Regina Bruder

Technical University of Darmstadt, Germany

The research group "Didactics and Pedagogy Mathematics" of the Technical University Darmstadt, Germany, has provided online training courses for secondary math teachers since 2005. These online training courses lasting six months contain cooperative and problem-oriented learning situations (Mandl, Gräsel & Fischer, 2000) implemented by a Web Based Training (WBT). The courses are based on long-term research and development projects which aim to enhance the quality of teaching (Collet & Bruder, 2006). The main research questions are: How do participants assess the concept of the online training courses? Which aspects of the online training courses are especially beneficial for teachers and the application of the learned aspects in math classes?

Since 2011 the ongoing evaluation concept of the courses was expanded. Up to now, 48 math teachers (31 female, 17 male) took part in this new evaluation concept where we developed the online questionnaire "Questionnaire for the Evaluation of online training courses for math teachers" (QEOM). At the end of each course, the QEOM evaluates with several scales the acceptability and the subjective learning success. Also, the QEOM asked to what extend the learned content is applied in the teachers' own math class and whether there is an intention to continue applying this knowledge and skills in the future.

The results led to some relevant conclusions which have to be substantiated by further research: One of the main findings was the participants' consideration of the structure and arrangement of the online training courses. These aspects were found to be very positive. The results also showed that the professionalization of math teachers can be improved with a WBT. However, a very important factor for the improvement is that teachers use the online course contents in their own math class during the training phase.

References

Collet, C.& Bruder, R. (2006).Evaluation of a teaching concept for the development of problem solving competences in connection with self-regulation. In: *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME-NA 30).* Vol. 2, 345-352.

Mandl, H. Gräsel, C. & Fischer, F. (2000). Problem-Oriented Learning: Faciliting the Use of Domain-Specific and Control Strategies through Modeling by an Expert. In W. Perrig & Grob (Eds.), *Control of human behaviour, mental processes, and awareness* (pp. 165-182). Hillsdale: Erlbaum.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 322. Taipei, Taiwan: PME.

TEACHING AND LEARNING FOR CONSTRUCTION AND SHARE OF METACOGNITION IN MATHEMATICS EDUCATION: ADVANTAGE TO EXTENSION OF METACOGNITION

Goro Takai

Aichi University of Education, Japan

Metacognition is the concept taken in mathematics education as analysis concept of problem solving. And it is also used as "driving force" (Silver, 1985) of problem solving. In Japanese lesson of mathematical problem solving, pupils often work on "Neriage" after the step of individual solving. Neriage "describes the dynamic and collaborative nature of the whole-class discussion during the lesson. In Japanese, the term Neriage means *kneading up* or *polishing up*" (Shimizu, 1999, p.110). Although the metacognition is the analysis concept and the driving force for individual activity, what kind of function is achieved in the collective solving (Neriage)?

In the step of individual solving, pupil's metacognitive knowledge is based own experience. And the metacognitive knowledge may be polished up thorough discussion between pupils. From a point of view of "viability" (Glasersfeld, 1995), this could be regarded as the change to intersubjectivity from subjectivity. On the other hand, the pupil compares own thinking to the others thinking in Neriage, and he/she determines better solution. In the thinking process, the object of monitoring (a type of metacognitive skill) may aims at not only self cognitions but the other's cognition and whole-class. Thus, by extending metacognition, we can consider to develop metacognition in whole lesson.

The post of teacher about development of metacognition was supporter to individual solving. However, when he/she works with pupils in Neriage, he/she should change the post to work up the others monitoring and to polish up metacognitive knowledge of pupils. In particular, they are "participant" as model of various monitoring, and "umpire" of shared metacognition by view point of mathematical world of pupils.

References

Glasersfeld, E. von. (1995). *Radical Constructivism: A Way of Knowing and Learning*. London ; Washington, D.C.: Falmer Press.

Shimizu, Y. (1999). Aspects of Mathematics Teacher Education in Japan: Focusing on Teachers' Role. *Journal of Mathematics Teacher Education, 2*, 107-116.

Silver, E. A. (1985). Research on Teaching Mathematical Problem Solving: Some Underrepresented Themes and Needed Directions. In Silver, E. A (Ed.), *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* (pp.247-266). Lawrence Erlbaum Associates.

STUDENTS' LOGICAL STRUCTURE FROM EXPLORING TO PROVING

Shu-Jyh Tang and Chia-Jui Hsieh and Feng-Jui Hsieh

National Taiwan Normal University, Taiwan

This study focuses on surveying secondary school students' *logical structure* from exploring to finishing a proof. Mathematical proof usually consists of explicit chains of inference (Hanna et al., 2009). The students' *logic* in proving can be viewed as the connection among objects including propositions, concepts, and external actions. One of ways to improve the effect of proof teaching is bringing in manipulation or exploration. In Taiwan, hands-on exploration often plays a significant role in teaching mathematical concepts or properties. For example, paper-folding is common to help Taiwanese secondary school students understand and construct an auxiliary line.

This research viewed the process of proving in classroom as a process from exploration to argument and then proof production (Hsieh, Horng, & Shy, 2012). The researchers adopted the exploration approach to teach proving in a regular class with 35 ninth graders. Although those students had not yet learned a proof formally, their task was to learn and finish the proof of "the four midpoints of a quadrilateral constitute a parallelogram" (with a figure). Four explorative or argumentative activities were designed for this teaching. Two of the findings were as follows.

Firstly, 17 persons could conceive an idea to justify their previous claims by paper folding, i.e., almost one half could build a meaningful causal or justificative structure to connect two far objects throughout external actions like hands-on manipulation. Secondly, after students underwent a series of activities, 9 persons successfully construct the formal proof. 4, 10, and 2 of the other persons respectively benefited from 1-3 hints given accumulatively later to recall their activity experiences, and finally completed the task. However, 6 students still failed. Compared to the Hint 1 related to their visual experience, the Hint 2 in which more connection provided seemed the most influential key for constructing an auxiliary line. But whether the difference came from the accumulation or nature of the Hint 1 and the Hint 2 still need further analysis.

References

Hanna, G., de Villiers, M., Arzarello, F., Dreyfus, T., Durand-Guerrier, V., Jahnke, H. et al. (2009). ICMI study 19: proof and proving in mathematics education: discussion document. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *ICMI study 19: Proof and Proving in Mathematics Education* (Vol. 1, pp. xix-xxx). Taipei, Taiwan: The Department of Mathematics, National Taiwan Normal University.

Hsieh, F.-J., Horng, W.-S. & Shy. H.-Y. (2012). From Exploration to Proof Production. In G. Hanna and M. de Villiers (Eds.), *Proof and Proving in Mathematics Education: The 19th ICMI Study*. Springer.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 324. Taipei, Taiwan: PME.

INTERWEAVE OF COGNITION AND EMOTION IN SMALL-GROUP MATHEMATICAL COMMUNICATION

Sampan Thinwiangthong
Faculty of Education
Khon Kaen University
Sampan@kku.ac.th

Suladda Loipha
Centre of Excellence in
Mathematics, CHE.
Suladda@kku.ac.th

Maitree Inprasitha
Center for Research in
Mathematics Education
Inprasitha_crme@kku.ac.th

Since about 1975, research in cognition and emotion has accelerated at a rapid rate. However, few cognitive psychologists evidenced much interest in affect until the mid-1970s and cognitive psychologists preoccupied cognitive processes in the absence of affect (Ellis, Varner & Becker, 1993). One important problem in the recent research on affect is the understanding of the interaction between affect and cognition (Hannula, et. al., 2004; Comiti, et. al., 1993). Students' emotional experience is more important aspect in scene of mathematical communication (Emori, 2005). Since 2002, Lesson Study and Open Approach were implemented in Thai mathematics classroom for stimulating the students' learning processes (Inprasitha, 2010), especially, Small-group Mathematical Communication (SMC). It is one of important mathematical learning processes that occur in such classroom. From the literatures, the relation between affect and cognition still very ambiguous. This paper aimed to clarify interweave of cognition and emotion in SMC.

Ethnographic method, video-stimulated interview, and video analysis were used to study interweave of cognition and emotion in SMC in classroom innovated by Lesson Study and Open Approach. 7-grade students' SMC was analyzed in order to reveal the interweave of cognition and emotion, and the mechanism of schema change, and the generation of emotional experience in SMC. Triad feedback as unit of analysis of SMC was lens for diagnosis interweave of cognition and emotion in SMC.

The research results showed that mechanism of interweave between cognition and emotion generated in SMC. Cognitive evaluation is the process to generate the emotional experience, occurred in SMC. We can understand the interweave of cognition and emotion by analyzing of SMC through Triad Feedback.

Keywords: Cognition, Emotion, Small-group Mathematical Communication, Triad feedback, Cognitive evaluation

Acknowledgment: This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand and Center for Research in Mathematics Education, Khon Kaen University, Thailand.

References

Comiti, C., Nohda, N., Schroeder, L., and Tirosh, D. (1993). How to Link Affective and Cognitive Aspects in Mathematics Education. In Hirabayashi et al. (Eds.). **Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education**. (pp.I3-I13). Tsukuba, Japan: University of Tsukuba.

Hannula, M., Evans, J., Phillippou, G., & Zan, R. (2004). Affect in Mathematics Education-Exploring Theoretical Frameworks. In M. Johnsen Hines & A. B. Fuglestad (Eds.), **Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education**. (pp. 107-136). Bergen, Norway: Bergen University College.

ANALYSING MATHEMATICAL PRACTICES OF A SIX-YEAR LOGITUDE STUDY FOR DEVELOPING REASONING NORMS

Wen-Huan Tsai, Pi-Jen Lin

National Hsinchu University of Education, Taiwan

Developing students' mathematical reasoning is central to many mathematical curriculums. It is also one of the five process standards in the Principles and Standards for School Mathematics (NCTM, 2000). Traditionally, students encountered the concept of proof and the activities of proving in middle school or high school courses. Many researchers and curriculums recommend that proof and proving will be integrated to all grade levels (NCTM, 2000; Ball & Bass, 2003). Ball and Bass (2003) also treated reasoning as comprising a set of practices and norms that are collective and rooted in discipline. Few researchers explore deeply the developing of reasoning norms in elementary classroom discourse and how the reasoning norms affect students' reasoning processes and mathematical learning. This study will cooperate with primary school teachers to develop normative aspects of acceptable and appropriate reasoning processes in their classroom communities. This study was based on Cobb & Yackel's (1996) theoretical perspectives of the relations between the psychological constructivist, sociocultural, and emergent perspectives in order to examine both individual learning and classroom communities.

This is a sixth-year longitude study from first grade to sixth grade. There are many norms of learning mathematics were already developed from first grade to sixth grade, but we try to cooperate with teachers to develop especially the acceptable reasoning processes from fourth grade to sixth grade. Several reasoning norms were developed in those two years. For example, students need to explain their solution based on what they already learned; the methods they used need to be convinced every one first; mathematical truth or method were developed on some domain, but when domain was extended or changed you need to prove it again…etc. In this paper, author will describe how those reasoning norms affect classroom mathematical practice and students learning mathematics.

References

National Council of Teachers Mathematics. (2000). Principle and Standards for School Mathematics. Reston, VA: Author

Ball, D. L. & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, and D. Schifter (Eds.), A Research Companion to Principals and Standards for School Mathematics (pp. 27-44). Reston, VA: National Council of Teachers Mathematics

Cobb, P. & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of development research. Educational psychologist. 31(3/4), 175-190.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 326. Taipei, Taiwan: PME.

SHAPE COMPOSITION IN EARLY CHILDHOOD

Marianna Tzekaki and Ioanna Kaplani

Aristotle University of Thessaloniki

Activity related to composition and decomposition of geometric shapes is recognized as particularly important to the development of geometric thinking, as it supports children to approach geometric properties and to visualize and transform geometric relations (Callingham, 2004). Early research has depicted children's difficulties in composing 2-D shapes and especially in tilling activity and underlined structural features like figures orientation, alignment and size in matching parts (Owens & Outhred, 1998). Clements, Samara and Wilson (2004) studied a developmental progression of composing geometric shapes in 3-7 years old children and mapped details of learning trajectories related to particular actions that the children apply when trying to combine geometric elements.

In this study, we present the outcomes of a teaching intervention aiming at improving composing and decomposing abilities of young children. Twenty two children of 5-6 years old were examined in tasks of composing geometric figures of varied difficulty in terms of the number of shapes, orientation, symmetries and rotation. The same children participated for two weeks in a teaching program during which they dealt with special designed activities related to compositions, tangrams, patterns and tilling of shapes of varied difficulty and were then examined again.

Comparing pre- and post-tests, the results indicate that the children, after working in a rich environment with challenging activities, improved their skills of shape composition and developed their performance in tangrams and tesselation. Particularly, they rapidly became skilled at selecting appropriate figures to create a new form, at putting them in the right direction and matching sizes, as well as at developing a wide range of structuring strategies with increasing sophistication. The study suggests that young pupils respond successfully to this kind of activity that is important to be introduced into early mathematics programs.

References

Clements, D. H., Wilcon, D.C., & Samara, J. (2004). Young Children's' Composition of Geometric Figures. *Mathematical Thinking and Learning*, 6 (2), 163-184. Copyright© 2004, Lawrence Erlbaum.

Callingham, R (2004). Primary students' understanding of tessellation: an initial exploration. In M, Johsen Hoines & A.B. Fuglestad (eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education,* Vol 2, 183–190. Bergen University College.

Owens, K., & Outhred, L. (1997). Covering Shapes- with Tiles: Primary Students' Visualisation and Drawing. *Mathematics Education Research Journal*, Vol. 10/ No. 3/28-41.

THE BEST CLASS I HAVE EVER GIVEN, WAS WHEN I DID NOT GIVE CLASS

Lucas Vanini, Maurício Rosa

Lutheran University of Brazil, ULBRA, Canoas, RS, Brazil

This paper analyses the influences that the conceptions of the use of Information and Communication Technologies (ICT) may exert on the work of a mathematics teacher. In this context, we discuss the structure of the praxis of a teacher, suggesting that a continuous education program, based on a specific theoretical framework, may open a different and "new" path for this mathematics teacher. Our results show that continuous education may trigger transformations in the classroom, which make possible to create situations that afford the production of mathematical knowledge of pupils, as long as pre-existing educational paradigms are broken. We notice that the conceptions of the use of TCI in Mathematics Education presentify themselves in the praxis of the maths teacher (subject teacher of our research) in the break of an educational paradigm. In other words, when Constructionism (Papert, 1994) replaces a traditional practice of solving finished exercises/problems in the maths classroom, it now becomes possible for theory to come in support of the teaching practice of this teacher. Praxis takes place in a constructionism way, and this can be seen when he states that "[…] instead of entering the classroom and writing definitions, contents and examples on the blackboard and then demanding that pupils solved exercises according to a predefined model, […] I simply proposed a challenge, a project the pupils would have to carry out alone. I would be a mere moderator". Additionally, we perceive the incorporation of the set of Constructionism ideas to the teaching practice. Therefore, we understand that the praxis led to a transformation of the teaching role of teacher, triggered by the continuous education program, which included readings in education theory.

References

Papert, S. (1994). Instrucionismo versus Construcionismo. In Papert, S., *A Máquina das Crianças: repensando a escola na era da Informática* (pp. 123-139). Porto Alegre, RS: Artes Médicas.

THE ANSWER PATTERN OF JAPANESE STUDENTS IN PISA2003 MATHEMATICAL LITERACY

Koji Watanabe

Graduate School for International Development and Cooperation,

Hiroshima University

Japan attained the highest level on PISA Mathematical Literacy Test in 2003. However, Japan could not reach the highest level in the area of the probability and statistics, called “Uncertainty”. At that time, the Japanese course of study did not cover the contents of this area totally. That is why we should accept the fail in that area. The course of study, however, is revised with probability and statistics in 2008. Then, what kind of answer pattern did the Japanese students have on PISA2003 Mathematical Literacy Test, especially in the area of “Uncertainty”? If we get the picture of their answer pattern, it would be possible to make a deep analysis of secular change between the result in PISA2003 and the future result in PISA2012.

The aim of this study is to reveal the answer pattern of Japanese students in PISA2003 Mathematical Literacy Test, especially by focusing on “Uncertainty” domain and comparing 13 countries and areas (namely, Australia, Canada, Finland, France, Germany, Hong Kong, Ireland, Italy, Japan, Korea, Netherland, New Zealand and the United States), through the use of multiple group item response theory (IRT), in order to detect differential item functioning (DIF). In this study, two types of DIF will be detected, concretely speaking uniform DIF and non-uniform DIF. This kind of analysis way is beginning to attract attention in the study of cross-cultural comparison (Tasaki 2007). On the other hand, Suzukawa et al. (2008) made a mathematics education research that Japan has a most unusual pattern of item difficulties by detecting only uniform DIF. In this study, Japanese students’ answer pattern will be more deeply discussed from the viewpoint of two types of DIF.

Analysis revealed that Japan has unique answer pattern from the average answer pattern among 13 countries in the area of probability and statistics. Especially, the item regarding outlier and sample selection is different from the other items.

References

Suzukawa, Y., Toyota, H., & Kawahashi, I. (2008). Do Students in Japan Learn How to Use Mathematics in Real-Life Situation? An Analysis of Items in the 2003 Programme for International Student Assessment (PISA2003). *The Japanese Journal of Educational Psychology, 56(2)*, 206-217.

Tasaki, K. (2007). Cross-cultural Validity of Self-construal Scales. An investigation of differential item functioning using multigroup mean and covariance structure. *The Japanese Journal of Behaviormetrics, 34(1),* 79-89.

EXAMINING THE EFFECTS OF THREE INSTRUCTIONAL STRATEGIES FOR LEARNING AREA CONCEPT IN A DIGITAL LEARNING ENVIRONMENT

Huei-min Wu , Ying-Hsiu Chung, Hui-chuan Huang, Ning-Chun Tan*,

Shyh-Chii Tzeng

Fo Guang University National Taipei University of Education*

Learning by studying examples and practicing solving problems are the two most common ways to learn. The common approaches in classrooms are watching teachers' demonstration (examples) first and then practicing solving similar problems or having students attempt to construct their own solutions before seeing the solutions or demonstration. The interest of this study was to investigate three strategies of organizing the worked-out examples (W) and the practice problems (P) (W-P, P-W, WW-PP) in a digital learning environment.

The participants were 84 third graders in an elementary school in Taiwan. The content was counting unit squares which were structured into four levels of strategies. The learning outcome was measured in terms of the time-on-task, their performance during the learning session, posttest, delayed test and levels of strategies as demonstrated in the open question of drawing shapes of an equivalent area. In addition, a cognitive load and affective perception rating scale was used to take the affective domain into account when evaluating the learning and instructional efficiency as proposed by van Gog and Paas (2008).

The results showed that (1) there was no significant differences among the three instructional strategies in the learning outcome as measured in the post- and delayed post-test, but the P-W group required significantly more time-on-task. The WW-PP group was more productive in the open question and showed a higher level of applying the strategies in the open question than the P-W and W-P groups. (2) There was no significant correlation between learning outcome and the perception of cognitive load (task difficulty, and effort demanded) and motivational factors (confidence and willingness to learn), but the perception of cognitive load was negatively correlated to the motivational factors. (3) There was no significant differences in learning efficiency and instructional efficiency among the three instructional strategies.

References

van Gog, T. & Paas, F. (2008): Instructional Efficiency: Revisiting the Original Construct in Educational Research, *Educational Psychologist*, *43*(1), 16-26.

LEARNING THE DIFFERENTIABILITY OF MULTIVARIABLE FUNCTIONS THROUGH THREE WORLDS OF MATHEMATICAL THINKING

Szu-Hui Wu Chi-Jer Yu

National Chiao Tung University, Taiwan

The differentiability of multivariable functions is fundamental in analysis but difficult for learning. It is our belief that multiple representations provided by computer could offer a good bridge between the abstract formalistic argument and the tangible intuition.

This article is to report our exploratory project looking into how 2nd-year calculus students reconstruct their concept in problem-solving with Java Modules. The teaching experiment methodology was adopted for qualitative analysis. Based on the theory of three worlds of mathematical thinking(Tall, 2008), we use a special 3D modules (Wu & Yu, 2008) developed to design worksheets as learning scaffoldings to encourage students explore and connect their spontaneous concept with formal definition of differentiability and tangent plane.

Problem solving behaviour of four mathematics sophomores as well as their cooperative efforts were assessed dynamically. Data, including pretest, the protocols of peer learning and interactions, students' answer sheets, feedback questionnaires and interviews, were collected and processed with respect to information processing model to analyse students' reasoning and identify their misconceptions and difficulties.

The result asserts that a visualizabe and manipulable environment with worksheets, facilitates students' experience in the 'conceptual-embodied' world and 'procepual-symbolic' world and enhances students' capability to recognise the meaningfulness of subconceptions and their representations of the differentiability, such as tangent planes, tangent lines, partial derivatives, directional derivatives, gradient, which lends the construction of whole pictures possible. It is encouraging that some students could aware and clarify their misconceptions, some even could connect their knowledge with other domains, such as linear algebra, thus reformulate their concept images and formal definition of differentiability.

References

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5-24.

Wu, S. H., & Yu, C. J. (2008). Visually assisted learning in multivariable calculus: the cases of continuity and differentiability. *Proceedings of the 16th International Conference on Computers in Education*. Taipei, Taiwan, October27-31.

THE ROLE OF SPECIFIC TRANSFORMATION PROCESS OF SOLVER'S PROBLEM REPRESENTATION DURING PROBLEM SOLVING: A CASE OF ABSTRACTION/CONCRETIZATION

Atsushi Yamada

Aichi University of Education, Japan

From a cognitive and representational perspective, problem-solving process is often considered as constructional/transformational process of solver's mental and/or external problem representation. The purpose of this study based on the above perspective is to relate the progress of problem-solving process with a specific type of transformation pattern of solver's problem representation, which is called as Abstraction/Concretization in this article, and to analyse more broad roles of the transformational process (Abstraction/Concretization) during problem solving.

In order to describe and interpret solver's problem representation, this study use a model for a structure of problem-solving competence based on five types of mature internal representational systems (Goldin,1998). Using those five categories of representational systems in this model, we could interpret solver's internal problem representation in detail and find its transformation patterns that characterize the progress of problem solving.

In the imagistic representational systems in the Goldin's model, however, we can suppose numerous types of representations and various levels of abstraction among them. Even in the visual/spatial (sub)system, we can find out various mental representation, for example, from concrete operative imagery to abstract pattern imagery(e.g.,Presmeg,1986). But, the variety of representations can be considered as a resource of transformation process of solver's problem representation leading to the progress of problem solving. If a solver reflects her/his operative activity with concrete materials and transforms concrete operative imagery accompanying the activity to more abstract pattern imagery constructed by the reflection, she/he will, in parallel, change her/his activity to systematic pattern exploration.

Such a transformation process of solver's problem representation (Abstraction) will help the solver to increase the information about the problem, change her/his understanding of it, and lead the problem solving to another phase. This is the important role of Abstraction process in imagistic system. On the other hand, one of the roles of another transformation process in the opposite direction, Concretization, is to help the solver get a sense of confirmation or conviction about her/his solution.

References

Goldin, G. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior*, ***17***(2), 137-165.

Presmeg,N.C.(1986). Visualisation in high school mathematics. *For the Learning of Mathematics*, ***6***(3), 42-46.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 332. Taipei, Taiwan: PME.

THE RELATIONSHIP BETWEEN PRICE AND MENTAL NUMBER LINE

Shohei Yamamoto

Aoyama Gakuin University

A numerical representation, called a "mental number line," plays a critical role when we make a magnitude judgement (e.g., comparison between two numbers). This representation has a spatial property which yields a distance effect (Moyer & Landauer, 1967): Reaction times (RTs) depend on the distance between two numbers. This study uses this property to examine the human understanding of prices. Nine-ending prices is common in the U.S. When consumers compare prices between two products, they tend to ignore the right-most digits if the left-most digits are different (Stiving & Winer, 1997). Hence, when firms reduce a price from $20, they choose $19.99 rather than $19.80 to make a better profit. Japanese firms use similar pricing rules. When reducing from 13,000 yen, they prefer 12,800 yen to 12,300 yen. In the present research, we showed our participants several numbers and asked them to choose the nearest number from benchmark figures. If the RTs to a set of numbers differed (i.e., a distance effect), this indicated that they were able to distinguish the numbers from each other. Conversely, if the RTs did not differ, the numbers appeared to blur into each other.

STUDY 1

Thirty participants were required to choose the nearest number among the three round figures (10,000, 15,000, and 20,000) when presented with nineteen 5-digit numbers. The last three digits and distances from the rounded numbers were manipulated. The results showed that the RTs were different only when the thousand digits were different, indicating that they tended to ignore the three digits on the right. It also turned out that numbers in the 12,000 range tended to be associated to 10,000 rather than 15,000.

STUDY 2

We investigated the actual prices of a web market in Japan. The results revealed that among the price category of 12,000 yen the number of products priced at 12,800 yen was greater than at other 12,000 yen range prices. The results of study 1 and study 2 imply that when consumers see such prices they misconstrue these prices as being closer to 10,000 yen than to 15,000 yen.

References

Moyer, R. S. & Landauer, T. K. (1967). Time required for judgment of numerical inequality. *Nature*, *215*, 1519–1520.

Stiving, M. & Winer, R.S. (1997). An empirical analysis of price endings with scanner data. *Journal of Consumer Research, 24,* 57-68.

DEVELOPMENT OF MATHEMATICAL LITERACY TEST FOR SECONDARY SCHOOL STUDENTS IN TAIWAN

Chih-Chiang Yang, Ph.D

National Taipei University of Education, Taipei, Taiwan

Yu-Rong Chen

Jin-Cheng Junior High School, Kinmen, Taiwan

Introduction

The purpose of this study was to develop a **M**athematical **L**iteracy **T**est (MLT) to realize young students' capacities to extrapolate from what they have learned in school and to apply mathematical knowledge to authentic problems situated in a variety of contexts. The theoretical framework of MLT was based on the PISA 2009 assessment framework. The MLT contained 3 cognitive and 2 content domains. The cognitive domains were abilities of reproduction, connection and reflection. The content domains were change and relationships, and quantity.

Methodology

The MLT was comprised of 20 test items in 15 themes. The test subjects were 7^{th} grade students in the secondary schools. Samples were randomly drawn from a secondary school at Kinmen, Taiwan by using a cluster sampling method. Several techniques of item analysis, such as item difficulty index, item discrimination index and option characteristic curve (OCC) by using kernel smoothing approaches to nonparametric item characteristic curve estimation were demonstrated in the study.

Conclusions

The reliability and validity of MLT have been confirmed and the quality of test items has been also proved in the study. The item difficulty index of 20 test items ranged 0.10 from 0.86. The average was 0.54. The item discrimination index of 20 test items ranged 0.14 from 0.82. The average was 0.57. Most of OCCs behaved well and increased along the slope of 45 degree. The MLT was concluded as a reliable and appropriate assessment to measure the mathematical literacy for secondary school students.

Main References

OECD (2009). PISA 2009 assessment framework: Key competencies in reading, mathematics and science. Paris: OECD Publishing.

OECD (2010). PISA 2009 results: What students know and can do: Student performance in reading, mathematics and science (Volume I). Paris: OECD Publishing.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 334. Taipei, Taiwan: PME.

LEARNING QUANTITATIVE SYNTAX IN READING CHINESE TEXTS AND GRAPHICAL ILLUSTRATIONS

Chih-Chien Yang[1], Hsin-Ju Tsai[1,2], Yih-Shan Shih[1], Hui-Yi Liang[1]

[1]Cognitive NeuroMetrics Laboratory

Graduate School of Educational Measurement and Statistics

National Taichung University

[2]Department of Language and Literacy Education,

National Taichung University

The study is aimed to examine early developments of learning quantitative syntax via reading Chinese texts and graphical illustrations. The study inspects and broadens Cutting and Scarborough (2006) findings that varied reading comprehensions can be caused by diverse measurements. A longitudinal sample of elementary students from beginning to senior grades in central Taiwan regions was recruited for several assessments, including Quantitative Syntax Test for Arithmetic, Basic Chinese Reading Comprehension Test and Graphical Illustration Test. Developmental trajectories are revealed by latent growth models and respective predictors for the trajectories are analyzed by finite mixture structural equation models (Yang & Yang, 2007). Assorted developmental trajectories are established for linguistic, quantitative, and graphical elements of participants' cognitive abilities. Participants have a general tendency in possessing of consistent performance in recognizing Chinese texts, quantitative syntax, and graphical illustrations. However, statistical results also show that subgroups of students can own unique cognitive developments in comprehending quantitative syntax in solving arithmetic word problems if linguistic components were controlled by graphical illustrations. In addition to Cutting and Scarborough (2006) measurement differences, varied longitudinal cognitive developments in corresponding to linguistic, quantitative, and graphical elements are to be outlined and discussed. The study is concluded by discussing potentials of fundamental cognitive developmental differences among comprehending quantitative syntax, reading Chinese Texts and recognizing graphical illustrations.

References

Cutting, L. E., & Scarborough, H. S. (2006). Prediction of reading comprehension: relative contributions of word recognition, language proficiency, and other cognitive skills can depend on how comprehension is measured. *Scientific Studies of Reading*, *10*, 277-299.

Yang, C. C., & Yang, C. C. (2007). Separating latent classes by information criteria. *Journal of Classification*, *24*, 183-203.

THE DEVELOPMENT OF TEACHERS' THINKING ABOUT "TEACHING AREA"

Ru-Fen Yao

National Chia-Yi University

During **case discussions**, a mutual educational process takes place. When teachers engage in group deliberation, they can construct ideas that might not have occurred to them through personal reflection about their own teaching (Barnett, 1991). **Case discussions** serve as catalysts for changes in thinking even when no disagreement or conflict exists. The main purpose of this one-year case study was to investigate participants' thinking about area instruction in the discussion-based context. Nine in-service school teachers joined in this study voluntarily. Through observation of meetings, participants' journals, and related documents, data was collected during the period of this research. According to data analysis, there were four stages of participants' development of thinking about teaching area: "exploration", "argumentation", "modelling", and "generalization", these labels were resulted from the crucial characters of each stage. For considering the area instructional sequence and the changeable/unchangeable parts in the sequence of area instruction, participants finally revised the framework of area teaching and built a new one (showed as Figure1). They named it as the "model of Area instruction". Based upon the findings, the researcher holds opinions as conclusion and suggestion for area instruction and mathematics teacher education.

Keyword: area instruction, case discussion, elementary school, teachers' thinking

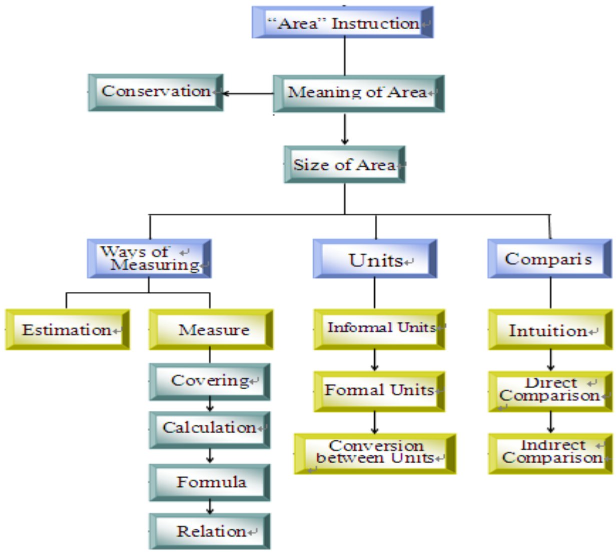


Figure1. The "model of Area instruction"

Reference

Barnett, C. (1991) Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education*, vol.2, no.4, pp.263-272.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 336. Taipei, Taiwan: PME.

TEACHERS' BELIEFS AND PROBLEM SOLVING PEGAGOGIES FOR STUDENTS WITH LEARNING DIFFICULTIES

Shirley M. Yates[1] & Anne E. Bayetto[2]

[1]Christian Brothers College, Adelaide, South Australia. [2]Flinders University

Solving problems is a process of thinking mathematically that is both a means and goal of learning mathematics (National Council of Teachers of Mathematics, 2000). Since the 1990s a trend towards student-centred, problem-based, constructivist mathematics pedagogy has been evident in many countries. Constructivist based pedagogies place emphasis on students' acquisition of conceptual knowledge rather than mathematical computational skills and procedural knowledge (Westwood, 2011). Studies show teachers endorse a focus on problem solving in the mathematics curriculum (Anderson, 2000) but little is known about their beliefs and pedagogies for students with academic learning difficulties (LD) who lack prerequisite skills and strategies (Westwood,2011).

This study investigated the beliefs and pedagogical practices of sixty-five experienced teachers, working predominantly with elementary students with LD within a constructivist mathematics curriculum framework. The teachers were surveyed about their beliefs and mathematics pedagogies for students with LD before voluntarily participating in a professional development (PD) program about mathematics problem solving for students with LD. Statistically significant relationships were evident between teachers espousing strong beliefs in constructivism and their endorsement of indirect pedagogical approaches while those with strong beliefs in basic skills favoured directive practices. The essential differences between the reported pedagogical practices of the constructivist and basic skills oriented teachers were whether or not they provided opportunities for basic skill practise and direct instruction in problem solving for students with LD. These findings have implications for the provision of PD opportunities for teachers to deepen their mathematics pedagogical content knowledge so as to be able to use problem-based approaches systematically to differentiate and scaffold problem solving instruction in mathematics for students with LD.

References

Anderson, J. (2000), *Teachers' problem-solving beliefs and practices*. Unpublished PhD Thesis. Australian Catholic University.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Westwood, P. (2011). The problem with problems: Potential difficulties in implementing problem-based learning as the core method in primary school mathematics. *Australian Journal of Learning Difficulties 16*(1), 5-18.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 4, pp. 337. Taipei, Taiwan: PME.

YOUNG CHILDREN'S APPROACH TO SOLVE OPEN-ENDED PROBLEMS

Kai Kow Joseph YEO

National Institute of Education, Nanyang Technological University

Mathematics educators including school teachers are beginning to pay attention to the kind of problems that they give to students. In working with primary school mathematics teachers in Singapore, the author frequently hear lower primary mathematics school teachers felt inadequate about their own teaching approaches to problem solving, especially with open-ended problems. There is a need to equip teachers with a bank of greater variety of mathematical problems for problem solving that can enhance their teaching methods. Students must experience intriguing mathematical problems where they can reason and offer evidence for their thinking. This study intends to examine the approach of Singapore young children on open-ended mathematics problems. Open-ended problems are one form of non-routine problems, which allow a variety of correct responses and at the same time help to elicit the thinking processes involved in solving the problems. According to Takahashi (2000), there are two types of open-ended problems: problems with only one solution but diverse approaches and problems with multiple correct answers. In this study, two contextual open-ended problems were crafted by the author with the belief that they were able to promote thinking, metacognition and application of problem-solving strategies. It examines twenty-six primary one (7-years old) students' performance in solving two open-ended problems in Singapore context. The openness of the problems includes two aspects – multiple approaches, which allow students to begin working on the problems using different approaches, and multiple acceptable answers (i.e., more than one correct answer). In this presentation, the roles of open-ended problems are presented, along with a sharing of research evidence on how young children perform in solving open-ended problems. Samples of students' solutions were evaluated under three categories, namely, "Approach", "Explanation" and "Overall Quality". The study also intends to explore the implications and offer suggestions for the improvement of young children's ability in problem solving. What was significant to the teacher then was that the students would also be able to make a decision with respect to their solution choice after deliberating on the solution. The young children also experienced much difficulty in organizing their solutions. The results suggest that more variety of problem solving strategies need to be taught and documentation is another important skill that needs to be paid attention to in problem solving.

References

Takahashi, A. (2000). Open-ended problem solving enriched by the internet. Paper presented at the NCTM annual meeting, Chicago, IL. Retrieved February 29, 2012, from http://www.mste.uiuc.edu/users/aki/open_ended/NCTM_Presentation/sld006.htm .

STUDENTS' UNDERSTANDING OF INVERSE RELATION BETWEEN ADDITION AND SUBTRACTION AT PRIMARY LEVELS

Sze Man YEUNG and Ka Lok WONG

The University of Hong Kong

This article presents the findings of a research concerning primary students' understanding of inverse relation between addition and subtraction. Analysis is based on students' responses to a written test which includes transparent inverse problems (i.e. $a + b - b = a$, $a - b + b = a$, or $a + b - a = b$), and non-transparent inverse problems in four terms (i.e. $a + b - (b +/-1)$) and five terms (i.e. the second term can only be negated by the total of the third and the fourth terms). The different types of inverse problems serve to reveal students' ability of comprehending a computational problem situation in order to take advantage of their knowledge of inversion. In the test, students were allowed to jot down in the question paper, if they wanted, their calculation work (e.g. the column arithmetic) which would provide evidence of their problem-solving and computational strategies. Apart from the written responses from altogether 152 students sampled from Grades 2, 3 and 4, six students from each grade level were also sampled after the written test for face-to-face interviews.

The results show that the students understand the concept of inverse principle, but they do not use it as frequently and properly as desired, even in the cases of transparent inverse problems. While most of the students prefer using column arithmetic (as their routine expertise) in handling computational problems, they are weak in identifying the conceptual relation embedded in a situation and thus, notwithstanding the conceptual knowledge about the inverse principle, they cannot apply it to figuring out a possible shortcut for more effective calculation. Although the inverse principle was underused in favour of routine computation, students did better in transparent inverse problems than in non-transparent ones. Moreover, regarding the order of operations, there is also an observation that, when tackling a computational problem that involved a negation between the second and the third terms, i.e. $a + b - b$ or $a - b + b$, students always identified the inverse relation better in the case where the addition came first.

We, from the perspective of a classroom teacher, believe that when students are gaining computational competence, they should also develop their conceptual sophistication. This will, in the long run, enable students to analyse a problem in terms of structures and relations, and to look for more suitable arithmetic principles to solve the problem with more effective strategies. We also consider that the order of operations is important for at least two reasons. One is concerned with the decision on setting teaching examples and test or exercise items, whereas the other is with the interpretation of students' weaknesses in handling subtraction or difference as compared with addition or sum.

AN ANALYSIS OF STUDENT'S COGNITION TO THE MEANING OF MULTIPLICATION BY A DECIMAL

Yohei Watarai

Graduate School of Comprehensive Human Sciences, University of Tsukuba

The meaning of multiplication by a decimal is known to be difficult for students to understand. In addition, many of the students don't realize that the meaning of multiplication by a decimal cannot be explained by the "cumulative addition of the same number", in the elementary school mathematics (Asada, 2006). Therefore, the purpose of this study is to clarify cognition of students who don't realize inconvenience of the meaning of multiplication "cumulative addition of the same number" when the multiplier is a decimal, and to discuss improvement of teaching about the meaning of multiplication by a decimal.

In order to accomplish the above purpose, this study analyzes student's cognition to the meaning of multiplication by a decimal in the problem of Asada's survey (Asada, 2006) by applying the framework "The Theory of Conceptual Fields" (Vergnaud, 1988), as follows. First, this study identified a theorem-in-action for solving a problem, which is a mathematical property used by the student, in the assumed student's thought process for the problem of Asada's survey. Second, this study construed a mathematical meaning of the identified theorem-in-action. Third, this study clarified a relationship between the mathematical meaning of the identified theorem-in-action and the "multiplication of a reference quantity by a ratio" which is taught as the meaning of multiplication when the multiplier is a decimal.

The results of analysis reveal that (i) students grasp the meaning of multiplication by a decimal as the "cumulative addition of the same number and remainder": the "cumulative addition of the same number and remainder" is a extended meaning of the "cumulative addition of the same number" in additive structures. Next, (ii) the "cumulative addition of the same number and remainder" is equivalent to the "multiplication of a reference quantity by a ratio" in additive structures. Finally, (iii) teaching the "cumulative addition of the same number and remainder" enables students to understand the "multiplication of a reference quantity by a ratio" as coherent extension of the "cumulative addition of the same number".

References

Asada, S (2006). The survey of students' understanding about the meaning of multiplication. *Journal of Japan Society of Mathematical Education*, 88(12), 2-10 (In Japanese).

Vergnaud, G. (1988). Multiplicative structures, In Hiebert, J. & Behr, M. (eds.), *Number concepts and operations in the middle grades*, New York: Lawrence Erlbaun Associates, National Council of Teachers of Mathematics, 141-161.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 340. Taipei, Taiwan: PME.

TEACHER'S GENDER RELATED BELIEFS ABOUT MATHEMATICS

ZHANG Qiao-Ping WONG Ngai-Ying LAM Chi-Chung

The Chinese University of Hong Kong

Mathematics is always believed as a subject in the male domain (Forgasz, Leder & Kloosterman, 2004). It is found that teachers do possess different beliefs about male and female students, and these beliefs influence their teaching which in turn affect students' learning (Leder, Pehkonen, & Törner, 2002; Wong, Marton, Wong, & Lam, 2002). In particular, boys often receive more attentions from their teachers, in terms of either praise or criticisms (Kelly, 1988). It is interesting to know whether such a phenomenon persists today.

Math teachers from Hong Kong and Wuhan (China), 187 in total, were asked to jot down the names of 3 students who immediately came to their minds, then they were asked to indicate the gender of the 3 students as well their academic standards. The teachers were also asked if they agree/disagree or neutral to 4 simple statements: Boys are much better in math than girls; Boys are more interested in math than girls; Boys are more confident in math than girls; Math is male subject. Finally, the respondents were asked to indicate their own gender.

Results reveal that more boys were recalled than girls by the teachers, regardless of their genders. More *boys came to the teachers' minds even if these boys possess an average academic standard. Most of the teachers did think that boys perform better and have more confidence. Yet they denied that math is a male subject. In a sense, such a bias, if it really is, is not realised by the teachers.*

As an exploratory study, we hope to open a box to investigate subtler gender related beliefs among math teachers. Teachers' attention and interaction worth further investigation with qualitative methods like classroom observation and interviews.

References

Forgasz, H.J., Leder, G.C., & Kloosterman, P. (2004). New perspectives on the gender stereotyping of mathematics. *Mathematical Thinking and Learning, 6,* 389-420.

Leder, G. C., Pehkonen, E., & Törner, G. (Eds.) (2002). *Beliefs: A hidden variable in mathematics education*? Dordrecht, the Netherlands: Kluwer Academic Publishers.

Kelly, A. (1988). Gender differences in teacher-pupil interactions: A meta-analytic review. *Research in Education. 39,* 1-23.

Wong, N. Y., Marton, F., Wong, K. M., & Lam, C. C. (2002). The lived space of mathematics learning. *Journal of Mathematical Behavior, 21,* 25–47.

AUTHOR INDEX

RESEARCH REPORTS

VOLUME 4

PME 36

TAIWAN

2012

AUTHOR INDEX
SHORT ORAL COMMUNICATIONS
VOLUME 4

R

S

T

V

W

Y

Z