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TABLE OF CONTENTS
VOLUME 3

Research Reports

RESEARCH REPORTS

PME 36

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STUDENTS' INITIAL CONCEPTS OF GEOMETRIC TRANSFORMATIONS AND UNDERLYING COGNITIVE ABILITIES

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The purpose of this study is to develop a teaching unit on geometry for secondary school mathematics. In this paper, we focus on transformation geometry and investigate secondary school students' initial concepts of geometric transformations and underlying cognitive abilities. A qualitative analysis of a series of mathematical lessons about geometric transformations suggests that students tend not only to move figures discursively but also to construct a comprehensive figure, which is underlain by the cognitive abilities to make various composite units of figures, including the pre-image and image of transformation and make use of its figurative properties for the mathematical inquiry into transformations.

INTRODUCTION AND BACKGROUND

Students need to understand many concepts in learning geometry. A deep understanding of figural concepts (Fischbein, 1993) is, for example, essential for the shift from consideration of figures to the mathematical inquiry into the system of geometry. Having a dynamic view of figures is essential for a deep understanding of figural concepts (Okazaki & Kageyama, 2010), and we need to develop learning activities that foster such a view in the school mathematics curriculum (NCTM, 2000). Transformation geometry is one approach for geometry education, in which the specific pattern is abstracted from the motion of figures and formulated as a transformation group through understanding it as a one-to-one correspondence or mapping (Mac Lane, 1986).

In Japan, moving or transforming figures is used as the method of equivalent transformation for measurement in elementary school mathematics. Abstraction of motion is used in secondary school and formulated as translation, rotation, and reflection. In high school mathematics, students are expected to understand the epistemological shift from motion to mapping as concepts of transformation by describing motion using concepts such as vector, matrix, and complex number. However, the elements for producing such a shift seem to be complicated. For example, a deep understanding of figural concepts as well as cognitive abilities related to geometrical thinking, such as spatial structuring (Clements, 2004), are needed, and it is critical to demonstrate how the students construct concepts of geometrical transformations through learning experiences using figures in mathematics education (Edwards, 2009; Hollebrands, 2003; Yanik & Flores, 2009).

For example, considering the teaching practice with high school students using technological tools, Hollebrands (2003) implies that it is important for students' concepts of transformations as functions that they understand the domain of transformations, variants, parameters, relationships, and properties. According to Yanik & Flores (2009), focus on the parameters including translation vectors, reflection lines, points for centers of rotations, and measures of angles of rotation will foster the concepts of transformations as defined motions of single objects. From the analysis of a teaching experiment for future elementary school teachers, they suggest three conditions for understanding the rigid geometric transformation: translations as undefined motions of a single object, translations as defined motions of a single object, and translations as defined motions of all points on a plane. Yanik & Flores (2009) suggest that focusing on the domain enhances the shift to the last condition, and that it seems to be necessary for comprehending this shift to understand that a plane is not an invisible background but a set of points, that geometrical figures are not on a plane but on a subset of a plane as a set of points, and that mapping is performed on all points on a plane (Edwards, 2009). These are mathematicians' understandings, and the shift from the concepts of transformations as motions to mapping cannot be completed without the conceptual shift from a figure as a single object to a plane, with awareness of discretization (Lakoff & Núñes, 2000).

In this paper, we clarify the secondary school student's initial concepts of transformations and underlying cognitive abilities. To achieve this, we analyze a series of seventh grade mathematics lessons through which the students are expected to understand transformations of figures on the basis of structuring a plane, which is assumed to be the first step to discretization in geometry. We believe that this investigation demonstrates the necessary conditions for the epistemological change from concepts of motions to mappings in transformation geometry. Furthermore, it will enable us to obtain useful resources for developing teaching units in secondary school mathematics.

METHODOLOGY

Participants and settings

We implemented mathematical lessons with 40 seventh graders in a classroom of a university attached secondary school, with the collaboration of a teacher who had 15 years' teaching experience (Table 1). The students participating in all lessons had already learned geometrical figures, their properties, and area formulas of the figures through activities with isometric changes in elementary school. However, they had not yet learned the methods to construct basic geometric concepts, such as a perpendicular bisector and bisector of an angle, and geometric transformation.

In designing lessons with the teacher prior to the implementation, our goal was to facilitate a spontaneous development of the students' understanding of the transformations of figures during these lessons. We also intended that the students should apply their learning experiences gained in elementary school. The students already had experiences in manipulating figures during the observation and construction of the tessellation pattern and during measurement. Therefore, we expected the students to have naive ideas of transformations and functional recognition of figures by considering the transformation of figures and its effects using a global, proactive image of a tessellation pattern.

Lesson	Learning contents	Main classroom activities
1–2	Understanding of tessellation task and individual solving	
3	Communication of tessellation methods	Justification of the rationality of the methods
4	Integration of tessellation methods	Labeling methods
5–6	Application of tessellation methods	Extension and integration of the methods
7–8	Understanding of the transformation task and individual solving, and communication of transformation methods	Justification of the rationality of the methods

Table 1: Mathematical lessons designed.

Tasks

During the teaching practice, the students were given two tasks: tessellation and transformation. First, the students were asked to cover an entire plane with only one of the given figures (isosceles triangle, parallelogram, and an arrow-shaped figure; Fig.1) that fit together without any overlaps or gaps (lessons 1–6).

Figure 1

The tessellation task is a method used for teaching geometry, whose aims are to let the students discover the properties of figures, recognize a two dimensional plane, and so on. In this practice we intended to focus their learning on how to construct various patterns with tools by adding the following conditions:

- The students were given figures printed on a paper with dotted points in a grid pattern, and they had to construct a tessellation with tools—ruler, compass, and protractor—instead of manipulating concrete materials.
- The students had to justify themselves by showing the tessellation and construction method as evidence.

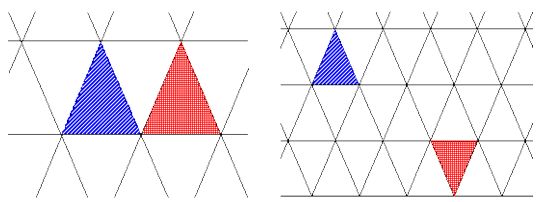


Figure 2

Second, the students were given two figures as the pre-image and image of transformation on a tessellation pattern consisting of isosceles triangles, and they were asked to explain how to transform the figure from pre-image to fit on the one as image (the transformation task; Fig. 2). A significant feature of this task was that the students were required to investigate transformations on the tessellation pattern that they had constructed in the tessellation task and not on a computer screen.

Procedure

In this study, data were collected and analyzed using a qualitative research methodology (Flick, 2009) because we were interested in how the students performed given tasks and exchanged as well as developed ideas during lessons that aimed to facilitate the development of the students' understanding of geometric transformation. We collected three types of data: video data recorded with one camera for eight hours of the seventh grade lessons described above, field notes by the author as an observer, and the students' individual notes. The students' ideas and methods of thinking identified in the classroom were interpreted with the collaboration of the teacher after each lesson by referring to written data. We then made transcripts of the video data, analyzed it in terms of the students' activities, and discussed the results with the teacher. Because this study focused on the students' mathematical thinking and cognitive abilities in performing given tasks, non-participant observation was conducted and lesson plans were not modified.

RESULTS

Results are presented for each category of tasks: tessellation and transformation. We describe in detail the methods of tessellations and cognitive abilities identified in the classroom, and then the methods of transformations.

Student's activities in the tessellation task

The students worked both individually and collectively on the tessellation task. Methods of tessellation by the students can be categorized as follows:

(a) Construction of figures one by one: The students determine a basic figure and construct others one by one around the basic figure by overlapping corresponding edges of the figures. In the case of the arrow-shaped figure, they construct figures by making polygon lines, paying attention to the length of the edges and measurement of the angles.

(b) Extension of the edges of the figures: The students make straight lines by extending the edges and construct figures by defining the position of points on the line according to the length of the edges. Then, they make lines parallel to the above lines such that the width between the lines is the height of a figure, and they define vertexes in the same way.

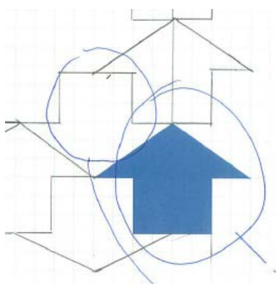
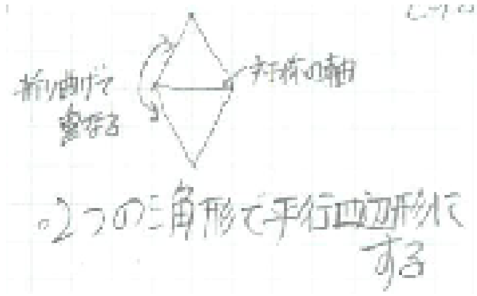


Figure 3

(c) Use of parallel straight lines: The students make parallel straight lines such that the width between the lines is the height of a figure. Congruent figures are constructed using several lines.

(d) Construction of repetition patterns: The students construct a pattern consisting of a figure by method (a), (b), or (c) and arrange the same pattern along with the original one (Fig.3). Categorization by this method is different from the above methods because of the awareness and application of the repetition of patterns.

(e) Construction of composite units: The students construct alternative figures as new composite units by composing figures and using them for tessellation (Fig. 4).

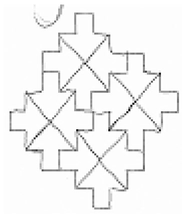


"I make a parallelogram by putting 2 triangles"

Figure 4

Another method of tessellation, in contrast to (e), was to apply (b) or (c) to the arrow-shaped figure, and thus to decompose one into pieces of triangle and rectangle.

In the 3rd–6th lessons, the students communicated with each other and discussed whether they could cover a plane entirely by using each method. In particular, the method of constructing useful units of units was accepted, as developing (e) through discussions about an arrow-shaped figure (Fig.5).



"I should make something like a gear, I thought."

Figure 5

Students' activities in the transformation task

The teacher presented the students with the tessellation pattern consisting of isosceles triangles with two colored ones and then asked with gesturing: "An original figure is the blue one (*left). We want to *transform* the blue to the red (*right)." (7th lesson, left in Fig. 2); "Now, using a ruler and compass, *move* this one (*left) to this one (*right)." (8th lesson, right in Fig. 2). Although there were slight differences in the teacher's presentations, the students explored how to move a blue triangle as the pre-image to a red one as the image by performing the task.

In performing the transformation task, a concrete method (for example, "we should fold a given paper") and a misunderstood method of constructing congruent figures one by one in the same manner as in the tessellation task were presented. Then, in exploring the transformations or moving on the tessellation pattern, the students tended to explain a type of transformations with gesturing and some spoken language representing motions (for example, "rotate it around."), rather than changing one to another by moving randomly. Four types of transformations—sliding, rotating, folding back, and point symmetrical motion—were shared through mutual communications in the classroom. The students called these types translation, rotation, symmetrical reflection, and point symmetrical reflection, respectively, instead of labeling them with mathematical terms.

The students' methods of transformations can be categorized as follows.

(A) Manipulation with gesturing: The students explain a type and the direction of transformations with gesturing and spoken language, fully explaining the motions. They are aware of one type of parameters.

(B) Motion without awareness of parameters: The students transform a figure as pre-image to one as image by composing several types of transformations. They are not necessarily aware of all parameters.

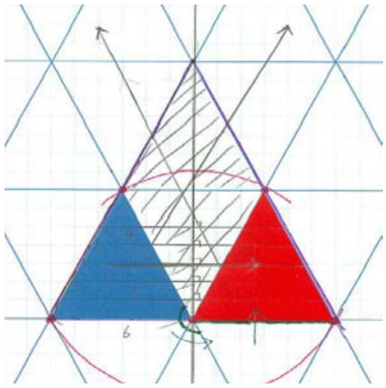


Figure 6

(C) Motion with awareness of parameters: The students transform a figure by composing several types of transformations in the same way. They are aware of all parameters.

(D) Use of corresponding points: The students focus on determining conditions for figures and attempt to match certain points of a pre-image with the corresponding points of an image (Fig.6). They try to transform the image by using the correspondence of points rather than moving figures as single objects.

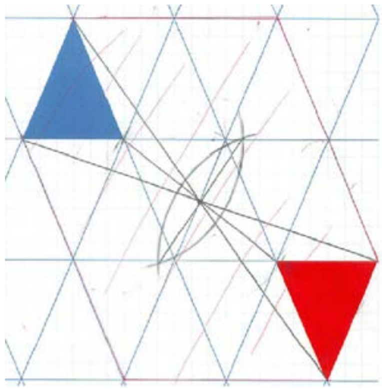


Figure 7

When the students attempt to translate and transform images symmetrically by (D), they are aware of all parameters, such as direction and distance of translation, and perform mapping on all points. However, when transforming by rotation, some students change the position of the center of rotation in relation to the vertexes of the pre-image. This error demonstrates that all students using (D) do not necessarily understand mapping mathematically.

The detailed analysis shows that there were not only students who understood the relation between the pre-image and image of transformation but also those who constructed comprehensive figures including both the pre-image and image and attempted to investigate the transformations by using their figural properties. For example, a large isosceles triangle in Fig. 6 and a hexagon in Fig. 7 are comprehensive, symmetrical figures constructed by the students. In the latter case, the students found three pairs of corresponding points between the two original figures by using the property that the center of point symmetry and corresponding points lie on the same straight line, and that the center is the midpoint of line segments.

DISCUSSION

The results of this study describe the students' initial concepts of transformations reported by Hollebrands (2003) and Yanik & Flores (2009) in greater detail. The students tended to inquire into some routes from the starting pre-image to the target image by composing the transformations that were accepted and shared spontaneously in a classroom. In the inquiry, the students either (i) thought of discursive moving and transformations of figures as single objects or (ii) interpreted the figural properties of comprehensive figures—in particular, symmetry—as transformations and applied that concept to the task. Noting that (A)–(D) described above do not necessarily indicate levels of concepts of geometric transformations, we describe below the underlying cognitive abilities for transformations on a tessellation pattern and propose the hypothetical learning trajectory from motion to mapping.

First, the students had the experiences that they could cover a plane more effectively by making use of repetition patterns (d) and composite units (Fig. 4–5). Through these activities, they seemed to comprehend a plane as a discursively structured, patterned object. We assume that this comprehension supports idea (i) because the discursive arrangement of figures corresponds with the composition of motion in sequence, although having the potential to impede a view of physical motion. The cognitive abilities that not only move figures discursively but also construct composite units of figures flexibly seem to be radical and essential abilities for constructing useful and comprehensive figures in idea (ii).

Second, Yanik & Flores (2009) found that the parameters of transformations are important and that focus on them develops the concept of mapping as a defined motion of a single object. In their teaching experiment, they used methods that allowed the students to focus on the parameters by investigating the effect of transformations through the consideration of the relationship between a pre-image and an image. In the present study, the students focused on the parameters spontaneously through activities on a structured plane, that is, the students recognized the parameters as not only necessary commands for executing transformations but also conditions for achieving the transformations. The reason is that when investigating the relationship between pre-image and image, the figural properties of comprehensive figures must define the parameters of transformations.

In summary, there seem to be mixed concepts of transformations as motion and the correspondence of individual points based on the figural properties of comprehensive figures in the final state (Fig. 6–7), which is a transitional state to shift from motion to mapping (Fig. 8).

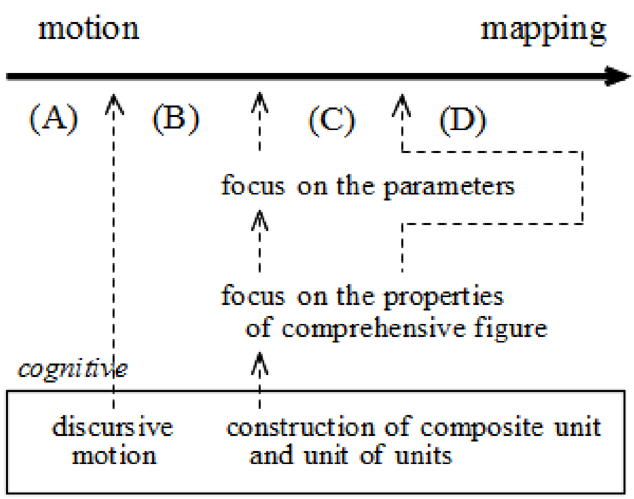


Figure 8

IMPLICATIONS

The cognitive abilities identified of this study are also the big ideas for the area of geometry and spatial sense (Clements, 2004). On the one hand, students understand various figural concepts and properties by using these abilities, but the other hand, students should have the experience of using figural properties as tools to enhance the shift from motion to mapping as concepts of transformation. For example, learning activities such as discussing whether two given figures are congruent without manipulating may develop students' concepts of congruent transformation. For future research, we should make and conduct the teaching designs to let students inquire into the relations among figures using properties and methods of transformations.

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CONSTRUCTING THE ACCUMULATION FUNCTION CONCEPT

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This report is one in a series on a research about learning integral calculus in school. In designing a curriculum that supports an improved cognitive base for a flexible proceptual understanding of the integral and integration, the notion of accumulation function is, in our opinion, one of the core concepts. In this paper, we focus on the following questions: What is the structure of the accumulation function concept in terms of elements of knowledge? What are operational definitions for these elements that allow the researcher to identify concept construction? How does the process of knowledge construction occur during an instructional intervention?

INTRODUCTION

The research study presented in this paper is situated in the wider context of a study of learning integral calculus in school. The literature (Bagni, 1999; Thomas & Hong, 1996; Thompson, 1994) shows that students have a tendency to see integral calculus as a series of procedures with associated algorithms and many do not develop a conceptual grasp giving them the desirable versatility of thought. Thus, instead of having a proceptual view of the symbols, they have, at best, a process-oriented view. This may be due in part to students having no opportunity to experience these processes directly. In order to develop an improved cognitive base for a flexible proceptual understanding (Gray & Tall, 1994) of the integral and integration, it has been proposed to construct the integral concept on the basis of the idea of accumulation (Thompson, 1994). Accumulation is at the core of understanding many ideas and applications of integration, such as curve length, volume, and work (Thompson & Silverman, 2008). There are two facets to accumulation: a) we accumulate a quantity by getting more of it; b) in case we don't have information about some whole thing, we look for and accumulate information about small parts of the whole. There is a direct connection between these two intuitive facets of accumulation, and the formal mathematical definition of the integral as accumulation function: Given a function *f*(*x*), defined and bounded on [*a*,*b*], the function $F_a(x) \approx F_{\Delta x, a}(x) = \sum_{i=0}^{\left[\frac{x-a}{\Delta x}\right]} f(a+i\Delta x)\Delta x, a \le x \le b$, also defined on [*a*,*b*], is called the accumulation function or integral of *f(x)* (equality is obtained in the limit $\Delta x \to 0$), or

$$F_a(x) = \int_a^x f(t)\,dt = \lim_{\Delta x \to 0} \sum_{i=0}^{\left[\frac{x-a}{\Delta x}\right]} f(a+i\Delta x)\Delta x, a \le x \le b.$$

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 11-18. Taipei, Taiwan: PME.

The idea of accumulation contributes to a coherent understanding of rate of change (Thompson, 1994). “When something changes, something accumulates. When something accumulates it accumulates at some rate” (Thompson & Silverman, 2008, p. 127). Hence, accumulation and its rate of change are two sides of a coin. Understanding this deep mutual relationship reveals the strong connections between derivative and integral. Moreover, we agree with Thompson and Silverman (2008) that the precise thinking and thoughtful use of the ideas required to understand the accumulation function well is, in itself, a valuable mathematical activity.

In spite of the natural support of the intuitive meaning of accumulation, the idea of accumulation function is anything but straightforward. It requires a proceptual understanding of function, a covariational understanding of the relationship between x and $f(x)$ (Thompson, 1994), as well as the ability to imagine accumulation. With respect to undergraduates, Thompson and Silverman (2008) "believe that understanding accumulation … can be part of a coherent calculus that focuses on having students see connections among rate of change of quantities, accumulation of quantities, …" (p. 13). We claim that, with minor changes this is relevant also for high school students. We designed and tested a curriculum supporting high school students in constructing integration as a conceptual aggregate of knowledge elements from approximation via accumulation to the Fundamental Theorem of Calculus (Kouropatov & Dreyfus, 2009, 2011). Here, we report empirical evidence from teaching episodes where students deal, for the first time and in an intuitive manner, with the properties of the accumulation function. Hence, this paper focuses on the following questions:

Question 1. What is the structure of the accumulation function concept in terms of elements of knowledge and what are suitable operational definitions for these elements that allow the researcher to identify concept construction?

Question 2. How does the process of knowledge construction occur during an instructional intervention?

KNOWLEDGE CONSTRUCTING AND ABSTRACTION IN CONTEXT

The main aim of this paper is to analyze the process of constructing knowledge about accumulation as a result of an appropriate instructional intervention. For this purpose, we have adopted Abstraction in Context (AiC) (Hershkowitz, Schwarz and Dreyfus, 2001) as theoretical framework. AiC takes abstraction to be an activity of vertical reorganization of previous mathematical constructs in order to arrive at a new (to the learner) construct. The activity is interpreted in terms of epistemic actions performed by the learner or by a group of learners for a specific purpose, in a particular context. The context includes the social environment as well as the learner’s personal background. The previous mathematical constructs result from previous abstractions. Reorganization includes establishing new connections between such constructs, making generalizations, and discovering new strategies for solving problems. "Vertical" implies building a new level of abstraction above a previous level. An essential component of AiC is a model of three epistemic actions for describing and analyzing at the micro-level the process by which learners construct new knowledge:

- R The learner *recognizes* a previous mathematical construct as relevant in the present situation.
- B The learner *builds-with* the recognized constructs to achieve a local goal such as solving a problem or justifying a claim.
- C The learner uses B-actions to assemble and integrate previous constructs so that a new (to the learner) *construct emerges* by vertical mathematization.

In processes of abstraction, R-actions are nested in B-actions, and R- and B-actions are nested in C-actions. These epistemic actions have been chosen because they are relevant for processes of abstraction and observable. This claim is based on studies reviewed in Schwarz, Dreyfus and Hershkowitz (2009). In most of these studies, students' constructs emerged in an instructional setting designed to offer opportunities to learn specific intended knowledge elements.

DESCRIPTION OF THE RESEARCH

We designed a ten-session integral unit based on the idea of accumulation and implemented it with five small groups of advanced-level mathematics high school student volunteers. The unit was independent of what the students were learning at school. In this paper, we report on an activity of a group of two female students from their sixth lesson. We worked with groups rather than single students in order to observe them discussing the problems together, thus making the knowledge constructing process more observable. We decided to analyze the knowledge construction of the pair rather than in each of the students separately.

THE ELEMENTS OF KNOWLEDGE AND OPERATIONAL DEFINITIONS

AiC analysis requires an a priori analysis of the content domain, and of the activities proposed to the students in terms of the intended knowledge elements, their constituents and links between the constituents. We used theoretical considerations including that accumulation function is a kind of meta-knowledge whose proceptual nature is particularly important, and didactical considerations, including accumulation function as a process of change (e.g., change of accumulating area beneath graph of function while the "right border is moving") and the objectification of this process (e.g., as a graph describing it). In light of these considerations, we focus, within the limitations imposed by this paper, on the following knowledge elements:

- CC "Complex Co-variation": When considering accumulation, the value of the accumulation depends on where we stop accumulating (upper limit) as well as on the function that accumulates.
- AF_M "Accumulation Function Meaning": For a given function *f*, defined and continuous on a closed interval [*a*,*b*], if we chose a sub-interval and change

the right end-point of this sub-interval (within [*a*,*b*]), the accumulating value on this sub-interval changes accordingly.

AF_P "Accumulation Function Property": The properties of the accumulation function (e.g., increase) follow from properties of the given function *f*.

As part of our methodology, the following operational definitions will be used to assess whether students have constructed these knowledge elements:

CC We will say that students have constructed this element if they describe and/or exemplify how the value of the accumulation depends on where we stop accumulating (upper limit) as well as on the accumulating function.

AF_M We will say that students have constructed this element if they explain the meaning of the accumulation function in terms of how the accumulating value changes according to the change of right end-point.

AF_P We will say that students have constructed this element if they explain how the properties of the accumulation function follow from the properties of the given function *f*.

Students' previous constructs, from the preceding lessons of the unit or from earlier are likely to be relevant during the activity; they include constructs for 'function' or 'increase' from earlier, as well as the following, which we assume to have been constructed during preceding lessons of the unit:

AVA "Accumulating value approximation": The accumulating value of any continuous function can be approximated in a rectilinear coordinate system, by replacing the given object with known objects (e.g., rectangles).

DIS "Definite Integral Symbol": An accumulating value (also called 'definite integral') can be symbolized for each interval as $\int_a^b f(t)\,dt$ with the components of the symbol interpreted as the given function *f*, the numerical values of the interval end-points *a* and *b*, terms of accumulation that are products of a value of *f* and the length of a corresponding sub-interval, and summing the terms of accumulation.

DIPS "Definite Integral with Parametric-Upper-Limit Symbol": As DIS but with the upper bound of integration being a letter symbol rather than a number.

SC "Simple Co-variation": A change of the value of the independent variable causes a change of the value of the dependent variable.

SC may in some cases have been constructed earlier rather than during the preceding lessons of the unit. Moreover, we note that almost all these knowledge elements are aggregates of sub-elements. For example, AVA includes such sub-elements as approximation, for example by geometrical shapes, terms of accumulation (e.g. thin rectangles, and the height of these rectangles as values of the accumulating function).

FINDINGS

As an introduction to the reported activity students used geometrical considerations to express an accumulating value (definite integral) for a linear function on an interval like $[1,\beta]$ where β denotes where we stop accumulating. The resulting term was symbolized like $A_f(\beta)$ and called "accumulation". Later, the letter β was replaced by x. This x was not considered variable, but a notation for where we stop accumulating.

At the beginning of the activity, the students, A and B, were asked to characterize the function $f(x)$ defined on $[a,b]$ as in Figure 1 and also to characterize $A_f(x)$ using such words as 'constant', 'positive', 'negative', 'increasing', and 'decreasing'.

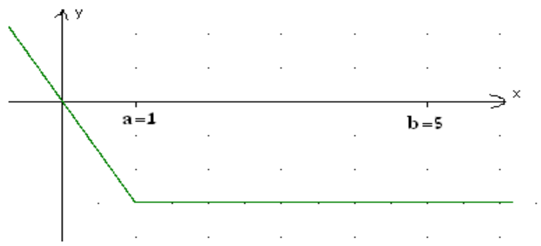

Figure 1

The students easily characterized f as constant and negative. Also, they recognized the symbol $A_f(x)$ (elements DIS and DIPS). Afterwards the researcher, R, asked them:

302 R: About the function f we said that it's constant and negative. Now, what is its accumulation?

303 A: First of all, its accumulation is negative because it's under the graph. Thirdly...secondly, I don't know if it's called increasing or decreasing but like, the accumulation grew. It grew to minus because it's under the graph

From this excerpt we infer that the students are able to refer to two different objects: the function represented by the graph and some accumulation process of this function that is not represented at all. It seems that the students' experience from preceding lessons allowed them to simultaneously construct AF_M and AF_P (line 302) in the context of given task. It seems that x for the students becomes variable and not only the place at which we stop accumulating. But the task was not very challenging; thus the construction process was fast and almost not observable. The episode is too short to point to epistemic actions and we can observe only the result of a cognitive process. The situation radically changed in the next task, where the students were asked the same questions concerning the function defined on $[a,b]$ as in Figure 2.

Here also, they easily characterized the given function as linear, decreasing, partially positive and partially negative. Next, they recognized two accumulating values (element AVA), one before the intersection point at 3 and the other after it, and concluded that their sum is zero. The researcher challenged them:

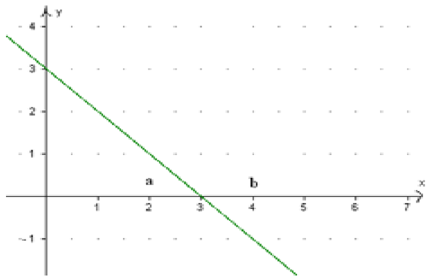

Figure 2

335 R: You need to decide something. First of all, if it's negative or if it's positive or...

336 B: Because, it depends on how you ask. If you're asking about the whole...if you're asking what's happening with the values...

337 R: I'm asking about...what am I asking about, actually? What about?

The students recognize co-variational change (element SC) concerning f but the researcher is not interested:

346 R: Why? That's the function $f(x)$. What about $A_f(x)$?

The questions helped the students to pass from thinking about fixed to thinking about changing values, and to start a "building-with" stage in their constructing process:

348 A: ... it always accumulates because the areas accumulate...

349 B: But it can be accumulating to the negative side

350 R: Well, well, then something accumulates, then try to say here...

351 B: Plugging in positive values and plugging in negative values that get smaller and cancel each other out

352 R: Okay, how does it look? What's accumulating? What's happening with the accumulation here? This point? What comes next?

353 A: Each time a smaller part...

361 A: So actually all the values keep getting smaller

In 348-361, they make claims about simultaneously changing values of $f(x)$ and $A_f(x)$ (elements CC and AF_M) but still find it difficult to relate to properties of $A_f(x)$.

362 R: Why? Try. What's here? At this point [the researcher points on the intersection point of graph of *f(x)* and x-axis where x=3] what will be the value of the accumulation? At this point it'll be greater than the last...

This intervention seems to have been helpful:

372 A: It grows

373 B: Until what point does it grow?

374 A: Until 3 then from 3...

...

395 B: It comes out like a triangle, like, like this...

396 R: It increases, right?

397 B: Gets to point 3......

398 R: To a certain point...

399 B: Decreases, starts to...

402 R: Why does it suddenly start to decrease?

403 A: Because of the negativity

406 R: Negative. And to which point do we get to in the end?

407 A: Zero

408 R: Okay. So what can we say about...?

409 B: Maximum point...

Here the students reached conclusions regarding some of the properties of $A_f(x)$. Building with AVA, its sub-elements, SC, and with the previously constructed CC and AF_M allows them to construct AF_P: They not only correctly describe a property of $A_f(x)$ (395-407) but also explain the origin of this property (403). This closely corresponds to the operational definition of AF_M. After another exchange, which confirms the construct AF_P, the students summarize their work by drawing Figure 3, which we considered as additional evidence for the construction of AF_P.

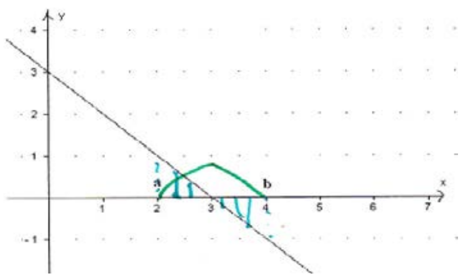

Figure 3

DISCUSSION

In the study presented here, we tried to find an answer to the question how processes of knowledge construction occur during an instructional intervention in the form of a teaching interview on the concept of accumulation function. The data gathered allows us to answer this question. From the excerpts we can see that at the current stage of learning, during a limited time (all reported episodes took place within 50 minutes) the students succeeded to construct totally new for them elements of knowledge and the constructing process is sufficiently observable using RBC methodology. The analysis of the excerpts shows that the students recognize existing constructs (AVA, SC, DIPS) and their sub-constructs, build-with them new elements (AF_M, CS) and use these new elements for then constructing AF_P with little outside intervention. We stress that in spite of the linear presentation imposed by the paper, the constructing process is not linear. A more detailed analysis reveals processes of combining and branching constructions (Dreyfus & Kidron, 2006) in the course of students' work.

The main conclusion we draw from this study is that the notion of accumulation has allowed us to design a didactical tool to support students' knowledge constructing processes on integration, and that the adopted research methodology has allowed us to observe these processes. Results similar to the ones reported here were obtained with other groups of students. Moreover, we have some data about consolidation of the constructs reported here. We hope to present these data at the conference.

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MODES OF SIGN USE IN EPISTEMIC PROCESSES

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This paper presents first results of an ongoing empirical study about the role of sign use in epistemic processes. A concept of sign use is defined integrating it into that of the semiotic bundle. According to their purposes for constructing mathematical knowledge three modes of sign use are distinguished and described in terms of their roles in epistemic processes.

INTRODUCTION

Unlike in other sciences, mathematical objects are not directly palpable and only accessible through signs. Therefore, building mathematical knowledge is deeply intertwined with sign use. However, the use of signs may cause problems, for instance difficulties may arise because students get confused by blending signs - these are signs that refer to different (mathematical) objects (Sabena, 2010). Sometimes complex signs composed of a number of partial signs cause problems for example in the use of the area dimension m^2 (Bikner-Ahsbahs, 2008). Sign use in mathematics can also be demanding because mathematical objects are mental objects that "must never be confused with the representations that are used" (Duval, 2006, 107). According to Duval the coordination of various representations of a mathematical object, the transfer of signs within the same and translating them into another sign system are essential for an adequate sign use in mathematics (Duval, 2006), hence, for acquiring mathematics.

In epistemic processes different signs provide different resources for constructing mathematical knowledge. These resources shape a semiotic bundle consisting of gestures, inscriptions and language and relations among them evolving in time (Arzarello, 2006). However, there is not yet a clear picture about how the interplay of semiotic resources may support student's constructing mathematical knowledge. This picture will be clarified in this project[1] following the question how constructing mathematical knowledge may be supported by semiotic resources. Therefore, we investigate modes of sign use of students with high proficiency within epistemic processes expecting to learn how fruitful sign use is shown. In this paper, three modes of sign use and their role in epistemic processes are illustrated by empirical data.

THEORETICAL FRAME

The construction of knowledge is described by an epistemic action model encompassing three different collective epistemic actions: gathering mathematical meanings, connecting mathematical meanings and structure-seeing (GCSt-model, Bikner-Ahsbahs, 2005). Gathering means collecting single mathematical entities and connecting means linking a finite number of mathematical. Structure-seeing takes place when a new entity is built or a known entity is re-built in a new context. In this project, constructing knowledge is focused through a semiotic lens. According to Peirce we consider as

> "sign, or *representamen*, […] something which stands to somebody for something in some respect or capacity. It addresses somebody, that is, creates in the mind of that person an equivalent, its *object*. It stands for that object, not in all respects, but in reference to a sort of idea." (Peirce CP, 2.228).

This specific object is created in a person's mind within a situation. It is called immediate object and allows observing semiosis, i.e. the sequencing of signs in social interaction in which a knowledge object is negotiated through evolving immediate objects. This is exactly what shapes the construction of knowledge according to the CGSt-model (Bikner-Ahsbahs, 2005).

Semiotic resources of the semiotic bundle may be sorted according to their *degree of self-creation* on the one hand and their *degree of fugacity* on the other. For example gestures like in figure 1 are fleeting signs (high degree of fugacity) that pass by quickly. In contrast, inscriptions (fig. 2) consist of fixed signs, they are of low fugacity. In learning processes the degree of *self-creation* of signs is also important. It has two characteristic values: signs can be *given*, for example by a task, or they can be *self-created* for example by the students (Wetzels, Kester, & van Merriënboer, 2010). Turning attention to one single sign, we obtain four possible features of its use (fig. 3): self-created/fleeting, self-created/fixed, given/fleeting and given/fixed.

A synchronic interplay of various used signs of the semiotic bundle is called a semiotic composition. Figure 1 shows two gestures with the feature self-created/fleeting building a semiotic composition. The left represents an axis of reflection and the right index finger the starting point of shaping the right part of the parabola. Every feature of a sign constitutes its function within the *semiotic composition* it belongs to. Interpreting these functions with respect to the specific task situation we are able to identify the purpose of the semiotic composition within an epistemic process. In fig. 1 both gestures together show the axis of symmetry of the given curve as part of an explanation of how the parabola can be built by its symmetric character. The purpose here is contributing to an explanation.

In this paper, sign use is described by the relation between a composition of signs and its situational purpose within a process of constructing mathematical knowledge. It shapes synchronic relationships described by the interplay of different semiotic resources and how they are used simultaneously. Observing diachronic relationships one can analyze sign development and why and how sign use changes over time (Arzarello, 2006). Some signs receive a symbolic character by being used repeatedly. Those signs are called basic signs (Arzarello & Paola, 2007). They may be used later in similar contexts without needing further explanation. Basic signs provide resources for constructing knowledge individually and socially. That is why they can be a powerful heuristic for teaching (Arzarello et al., 2007; 2009). An example for such a basic sign taken from our data is the hand in front of the chest as seen in fig. 1 representing the axis of reflection. In this case this gesture is given and fleeting.

Fig.1: Axis of reflection as a basic sign

METHODS AND METHODOLOGY

In this project[1], three tasks have been constructed. Three couples of strongly performing students of grade 10 solved these tasks in pair. Their epistemic processes were videotaped from three perspectives considering as many semiotic resources used as possible: a front-perspective for videotaping the gestures, a second one for filming the inscriptions and a third one for the monitor, if needed. The transcription (for the transcription key see fig. 5) of the video recordings captures verbal expressions as well as the non-verbal actions. In this paper we will refer to one task only: the parabola as a geometrical locus. It is chosen because various representations are introduced in this task.

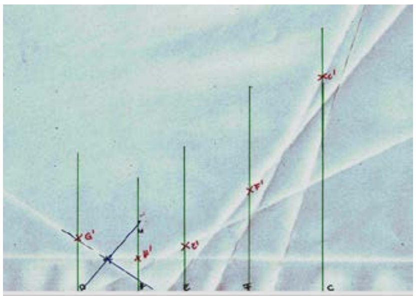

Fig. 2a

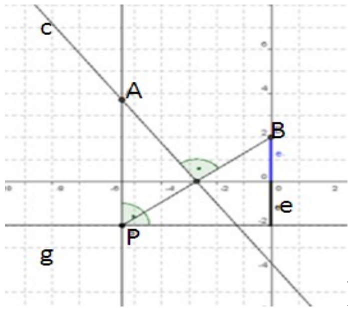

Fig.2b

Fig. 2a and 2b: Folded sheet and paper work sheet of the parabola task

The first one has to be constructed according to an instruction as follows (see fig 2a): (1) Take any point P on the bottom edge of the given sheet of paper. (2) Bend the sheet such that the chosen point touches the given point B. (3) Through point P, draw the line perpendicular to the bottom edge. (4) Mark the point of intersection with the fold (point A). Keep on doing that until you recognize a curve.

The marked points indicate a parabola and the folds represent the tangents touching the curve at the corresponding marked intersection point, as well as an axis reflecting P on B (fig. 2a).

Thereupon the students work with a worksheet of the interactive geometry software Geogebra that allows creating the trace of a curve by dragging the variable point P to and fro. In addition it is possible to vary the distance e between the fixed point B and the x-axis. One possible situation is seen in fig. 2b.

The task consists of (a) identifying the curve as a parabola, (b) justifying this by means of a function expression and (c) formulating a definition of the parabola as a point set following the exemplary definition of a circle. The students get as many printouts of the worksheet (2b) as they require.

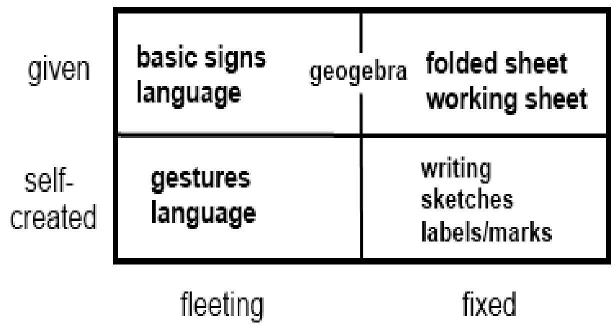

Fig. 3: Four features of the use of one single sign

All composition's signs in the data are analyzed by the four features in figure 3. Considering also the specific task situation we may extract the situational purpose of sign use by an interpretative approach analysing the social interactions with respect to the use of semiotic resources. This way we gain modes of sign use with their roles in epistemic processes. The unit of analysis is determined by the semiotic composition referring to the meaning unit of a scene that encompasses at least an utterance and its accompanying signs such as gestures and used inscriptions.

The parabola task and its setting provide encountering a wide range of different semiotic resources. Therefore they are taken as initial situations to be analyzed in order to gain first modes of sign use. Additional analyzes will be done by permanent comparisons (Krummheuer, 2000) with results already gained according to commonalities and differences. Through grouping, typical modes of sign use will be gained.

PRELIMINARY RESULTS

As seen in the diagram (fig. 3), language can be given or self-created, depending on the words or formulations used. Technical terms are considered to be given, using everyday speech to express an idea is self-created. Also, the interactive geometry software showed the characteristics of two forms, depending on whether it was used dynamically or to focus on a fixed situation.

Simplifying verbal argumentation by *connecting* a fixed and a fleeting sign

One of the findings concerns the simultaneous use of self-created/fleeting gestures and given/fixed representations. This synchronic relationship occurs often when a fixed Geogebra-situation or a printout is used to confront the captured situation (given/fixed) with a hypothetical situation (self-created/fleeting) combined with a linguistic act of argumentation. This comparison facilitates communication: Since a visual access to the argumentation is given language can be simplified, as the competing properties of the two compared situations or objects do not have to be expressed verbally. The following episode gives an example of this interplay of representations:

619 Tim: anyway the exponential function would ,wouldn't go up again' (*first indicates the course of the curve with his left finger beginning at the right side down to the left and up again the edge of the hand to the left, where the curve is not represented anymore, then goes to the deepest point of the curve*) because one but it would flatten further. (*goes with the edge of the left hand from the represented part of the curve down to the left along the x-axis and looks at M* [*or at I*]) (..) that would than haven u-h ,as its asymptote (.)

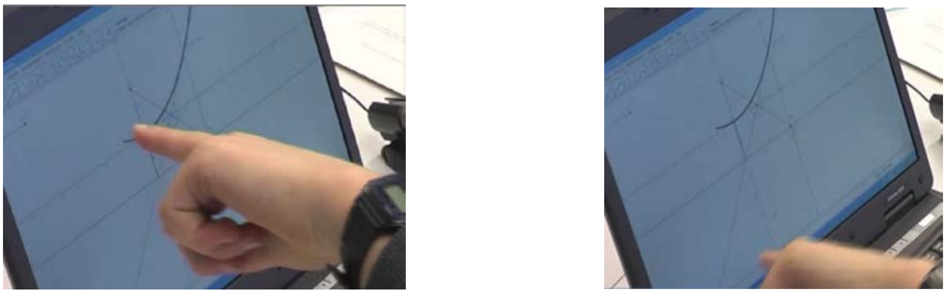

Fig. 4: Composing a fixed with a fleeting sign

exact.	dropping the voice	exact'	raising the voice
		,exact	with a new onset
EXACT	with a loud voice	(.),(..)...	1, 2 ... sec pause
<u>exact</u>	emphasized	(....)	more than 3 sec pause
e-x-a-c-t	prolonged	/S	interrupts the previous speaker

Fig. 5: Transcription key

Comparison of given/fixed and self-created/fleeting representations reduces the complexity and simplifies verbal argumentation in this situation. The gesture representation of a hypothetical curve is performed in a second layer in front of the screen in order to visualize the difference to the curve on the screen. This direct comparison concretizes the arguments mentioned in speech in matters of “how” and “where”.

Successive outsourcing of fleetingly expressed ideas and recomposing (*connecting*) the components using inscriptions

Another situation shows how a self-created inscription may emerge from the use of an inconvenient gesture. Diachronic analyzes provide insight in the additional benefit of one sign use compared to the other one.

Rosa and Lisa are searching for a definition of the parabola as a set of points having a certain property and Rosa tries to explain an observation to Lisa. She uses gestures and speech simultaneously to express her idea focusing on the distance between the points on the parabola and its axis of reflection. She represents the axis of reflection with her left hand using a basic sign and shapes the graph on the right side mentioning the increasing distance (Fig 6a, Rosa: “This <u>parabola</u>- ,where the points x-value- ,if we see it like <u>this</u>'”, 1029.). With the following gesture she visualizes the condition of regarding increasing y-values by indicating a horizontal line to the right side. The more conditions she wants to express in the fleeting representations of gesture and speech, the more fragile becomes her speech and at the same time her gestures become less precise (for example the left hand sinks). She reschedules speech and gesture, slightly modifying both, with the same result (Rosa:” Then (..) wides ,uh no uh. ,increases- , x-value' from the axis of- r r reflection- to-“, 1031.). Saying “,wait a moment” (1031) she switches sign use to producing an inscription. Sketching one mathematical idea after another (6b - 6d) while terming them verbally, Rosa switches from a complementary use of speech and gesture to a complementary use of speech and inscription: experiencing limits of the use of two fleeting signs makes her change sign use in order to benefit of their different and more convenient characteristic.

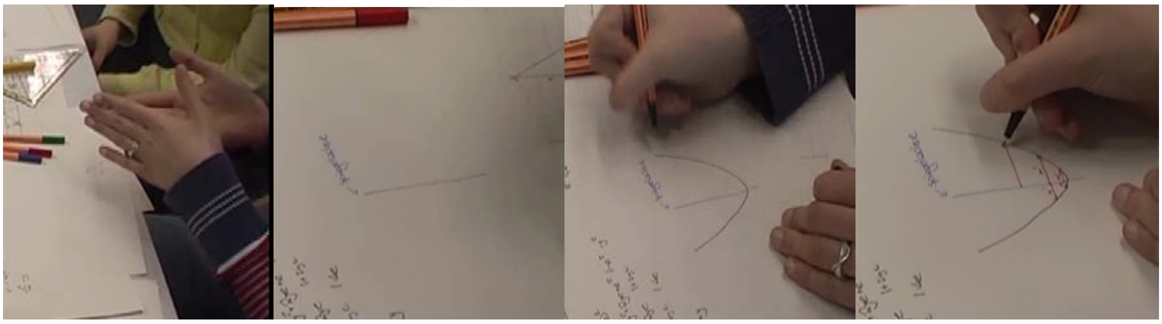

Fig 6 a, b, c, d: Successive genesis of the inscription: The axis of reflection as basic sign (6a) and as inscription (6b), the parabola (6c), arrows indicating the distance of parabola points from the y-axis (6c, 1039 “that here – is theoretically the same”)

The successive outsourcing of the components she mentions in speech allows to refer to single aspects and eases the social interaction in two ways: Lisa can pose questions to follow Rosa’s train of thought (Fig 6b, “Lisa: What is for you the axis ,axis of reflection now' “, 1034.) and Rosa can focus on what seems important to her while representing it in the diagram. In the social act of establishing a common knowledge base, composing a diagram hence goes along with arranging their thoughts.

At the beginning, the situation is too complex to describe it by gestures. Making inscriptions slows the process down because one idea after the other is gathered and worked out (Fig 6b - 6d), connecting them and building-with (Dreyfus, Hershkowitz, & Schwarz, 2001) the complex interplay of ideas step by step. Inscriptions allow unburdening explanations because of their material memory. The function of sign transfer here is to allow handling complexity and freeing from cognitive overload.

Goal-orientated exploring by comparative scanning as *gathering* action

A third example also concerns the simultaneous use of the printout as given/fixed representation and a gesture. But unlike in the first example (fig. 4), the gesture does not represent an object to compare with. It is a basic sign representing the distance. Before, the students wrote down a formula based on the Pythagorean theorem but still with the pointed distance between B to C as an unknown.

1297 Tim: then now one could theoretically (...) if we now know the (*points at the worksheet*) <u>equation</u> of <u>this</u> one here. ,then we could theoretically solve.

1298 I: then just take a look at it. ,at the B C. (15s)

[...]

1304 /Tim yes. ,but what equals this. *(points at the segment between B and C with two fingers*) (...) or' (6s) wait ,just let me

Fig. 7: Composing a temporary fixed with fleeting signs

Tim types the two fingers showing the distance-gesture repeatedly on the two points, following the prosody of his speech ("what" and "this") and stressing three words in the German version. Neither the gesture nor the speech stands for itself and each of them carries different pieces of information. The gesture locates what exactly is meant with "this" (the distance from B to C) and the language conveys what is searched for (two other points having the same distance). The synchrony of prosody and the repeated pointing let us assume a certain priority for Tim to express the distance in terms of x and y to fulfil his need for an equation expressed in 1297. The distance-gesture formed by two fingers now serves him as a temporarily fixed sign. This allows scanning (*gathering*) the diagram (in the pause of six seconds) in order to recognize another segment whose distance corresponds to that one between B and C (*connecting*).

SUMMARIZING DISCUSSION

We presented three different modes of sign use that provide insight into ways of fruitfully *connecting* signs for solving a problem: First, fleeting and fixed signs are used simultaneously in order to support argumentation and reduce its verbal complexity. Second, if signs are inconvenient to represent a complex situation *connecting* cannot be realized. It is more fruitful to change sign use; in this case the inscriptive memory is used to successively *build-with* the whole explanation step by step. Third, solving a problem sometimes demands focusing on one aspect only while ideas may flow, this is made possible by temporarily fixing an otherwise fleeting sign to compare with. Already these three modes of sign use indicate that availability to a rich semiotic bundle and flexibility in sign use may strongly foster constructing mathematical knowledge since reducing complexity, unburdening cognitive load and extending ones working memory by an inscriptive one enhance the capacity to act epistemically.

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COOPERATION TYPES IN PROBLEM SOLVING

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We present the results of a study analysing the cooperative behaviour of fifth-grade student pairs when solving problem tasks. As analysing method we used content analysis. The cooperation activities collected empirically in this study complement the cooperation types observed by other cooperation studies in the analysis of routine processing work. Comparison between those studies and our own leads us to the assumption that we are dealing with cooperation types specific to problem solving.

With regard to the poor performance of German students in the PISA studies, there seems to be a consensus "that the quality of teaching at German schools must be improved. Cognitively stimulating subject-oriented learning must be established everywhere – for instance by the use in lessons of cooperative working methods which promote independence" (Rabenstein & Reh 2007, p.23). By now there exist a lot of cooperative learning methods. For teachers, problem-solving is a mathematical topic in school which is especially appropriate for group work (Good et al. 1990). "The reasons given for the use of group work in problem solving include the opportunity for pooling of ideas, the natural need that arises to explain and express ideas clearly, and the reduction in anxiety for tackling something hard." (Stacey 1992, p.261) Not least because of the partly inconsistent quantitative results (also in problem solving contexts) researchers have called for more research focussing on the learning processes, e.g. patterns of interaction (Sahlberg & Berry 2002; Webb 1991).

THEORETICAL FRAMEWORK

Both, in mathematics and in psychology, a problem is understood as a task in which the problem solver is "not aware of any obvious solution method" (Mayer & Wittrock 2006, p.287) and has to overcome a barrier (cf. Dörner 1979, p.10). In contrast to a *routine step*, in our study a *problem-solving step* is taken to be a processing step for which the student does not have such a schematic procedure – if a task has such a problem-solving step, it is described as a problem task. For solving the problem the solver can work heuristically. Mathematicians by introspection, as well as researchers looking at processes, model this problem solving process in phases, like *reading the problem*, *analysis*, *exploration*, *planning*, *implementation* and *verification* (Schoenfeld 1985; in analogy see Polya 1973).

Researchers use different, not always synonymous concepts to define cooperation (or collaboration): In some studies cooperation (or collaboration) is seen as an ideal way of working together where students focus on their joint work (Slavin 1983). In contrast, Naujok (2000) defines cooperation as an "empirical phenomenon" and describes different ways of cooperating when solving tasks. Since various kinds of cooperation are imaginable when solving mathematical problems, we join Naujok in regarding

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 27-34. Taipei, Taiwan: PME.

cooperation from an empirically neutral viewpoint. In both cases cooperation is defined as a subset of interaction – in the first as an ideal form of interaction, in the second as "every kind of task-related interaction" (Naujok 2000, p.12). In our own study we are interested in such forms of task-related interaction.

Studies differ according to their chosen theoretical focus: In some studies cooperation is described in acts like *explaining*, *asking*, *answering*. These cooperation acts could be interpreted as one-person-actions (e.g. Webb 1991; Gooding & Stacey 1993) or as joint actions instead of "isolated activities of individual participants" (Naujok 2000, p.164). Naujok subsumes an egocentric *giving help* and *receiving help* under *helping* as practised jointly in the group. In our study cooperation is seen as a shared work, so that the students refer to each other and negotiate meanings.

There is some evidence, that the task (and the understanding of a task) can influence the way of cooperating. So, if a task is challenging, the students might cooperate in a specific way (e.g. explain) (Goos et al. 1996). While Webb's research on cooperation in the context of routine tasks has influenced other researchers who have used a similar analytic tool, only a few researchers have made use of problem solving tasks (e.g. Gooding & Stacey 1993). Therefore in this study we examine the ways of cooperating when solving mathematical problems and try to model the variety of cooperation acts. Although Naujok also selects routine tasks, her descriptions of subject-oriented cooperation acts seem to be useful: In her study with elementary students in a weekly schedule scenario, she reconstructs features to characterize the phenomenon of cooperation (e.g. *intensity*, *duration*, *thematic focus*) and describes the subject-oriented cooperation acts *explaining*, *asking*, *comparing*, *prompting* and *copying* in terms of these reconstructed features. She describes these acts as follows (Naujok 2000):

- *explaining*: The students in Naujok's work use the term explaining as synonymous to "saying-how" the task can be solved. The explaining person teaches sth. and so helps the partner.
- *prompting*: This cooperation act corresponds to a "saying-what", that means that a prompting person says the solution of the task and so helps the other person.
- *copying*: The copying person looks at the notes of another persons and copies the solution of the task. Thus copying represents a form of self-help.
- *comparing*: At the moment of comparing the comparing persons have solved the task, so they (or only one person) inform themselves about the solution of the other person.
- *asking*: This act represents a search for help. It is possible that this act initiates another.

Since we use mathematical problem tasks we expected, that we would find further cooperation acts. For characterizing these acts Naujok's features seem to be helpful.

DESIGN AND METHOD

Study: Between November 2008 and June 2010 we organized a math club at the University of Hanover (MALU), an enrichement project for fifth-grade students (age 10-12). In this math club the students had to solve one or two mathematical problem tasks in pairs at one afternoon a week. After working in pairs, we discussed possible ways to solve these tasks with the whole math club group. As the kind of cooperation may depend on the students' achievement (Webb 1991), we administered both, a general giftedness test (CFT-20R) and a mathematical giftedness test in almost all classes of fifth-graders in four different grammar schools in Hanover (Gawlick & Lange 2011). Based on these test-results we selected a group of 9 to 14 fifth-graders with different results (some had above average results in both tests, some had above average results only in one test and some had only average results in both tests) for each of the four MALU semesters (MALU 1-4). Some pairs should be homogeneous, the other heterogeneous with respect to the results in the achievement tests. Since the kind of cooperation may also depend on the way of grouping fifth-graders into pairs, in two MALU-groups we changed the pairs during the semester, in the other two MALU-groups the pairs were held constant.

Altogether each fifth-grade student in MALU solved more than 21 problem solving tasks with the help of another fifth-grader. The problem solving processes were videographed and the students' notes were collected. In addition, a log was kept of the children's main thoughts and the observers' subjective impressions.

Tasks: Cooperation may vary with different problem tasks (Cohen 1994), so we looked at tasks-collections (e.g. competition tasks) and problem solving books, analysed the tasks and chose tasks with different features. So, tasks for this study were selected from a variety of mathematical subject areas and with different heuristic potential. Fifth-graders should be able to solve the chosen problem solving tasks with their mathematical knowledge. One of the problem tasks we presented is the following (another one is the two squares-task (Schoenfeld 1985, p.77)):

The seven gates A man picks up apples. On his way into town he has to go through seven gates. There is a guardian at each gate who claims half of his apples and one apple extra. In the end the man has just one apple left. How many apples did he have first?

Fig. 1: The seven gates problem (Bruder 2003, p.12)

Evaluation Method: Because of the large amount of data, we chose a cross-section from the data (different pairs, different semesters, different tasks) – altogether ten processes (the analysing method is very time-consuming). The videographies of the ten processes were transcribed, and the transcripts were revised before the coding process started. Since fifth-graders could cooperate in different modes (verbal and non-verbal), about different themes and in different ways simultaneously, a coding in realtime seems difficult and we decided to transcribe the processes. As coding method for identifying cooperation acts we used qualitative content analysis (Mayring 2008). We were able to build upon Naujok's cooperation acts and upon her descriptions of the cooperation phenomenon (see above). Since the setting in Naujok's investigation differs in some kinds from our setting (routine tasks, tasks from different subjects in school, younger students, school setting) the cooperation acts were collected on the one hand deductively from her study and on the other hand empirically from the

MALU-transcripts.

The subject-oriented cooperation acts reconstructed by Naujok were defined and differentiated: So that the cooperation acts described by Naujok could be found in the transcript, the part acts of the cooperation partners constituting a cooperation act (e.g. *say how that works* by one person and the *understanding* of what has been explained by the partner in the cooperation act *explaining*) had to be singled out more distinctly than with Naujok. In addition, aspects which could be relevant in solving problem tasks were included in the definitions (e.g. explaining aspects of the path to a solution rather than complete paths to a solution). Besides, we had to accentuate aspects in which the cooperation acts differ: Therefore we took a look at Naujok's investigation (2000). In Naujok’s weekly schedule scenario, the children differentiate between prompting and explaining by associating *say what* with prompting, i.e. stating the solution, and explaining with *say how something works* (Naujok 2000, p.165f.). For the children, comparison is *finding out afterwards* about the work of the other or of each other, when at the time of copying information a child has *not yet solved the step* of the task him/herself (Naujok 2000, p.169). The students describe a copying phase as a *non-verbal* act and a prompting phase as a *verbal* act. When we tried to adapt these distinctions to all of Naujok's cooperation acts, we recognized that for the cooperation acts *explaining* and *prompting* the helping person doesn't have to have solved the problem yet. So, the criterion *solved the step* / *not yet solved the step* is not suitable to distinct between all cooperation acts. Instead, more suitable seems the distinction in different *cooperation intentions* (helping in the sense of assisting, and comparing in the sense of interrelating), so that we prepared the following figure:

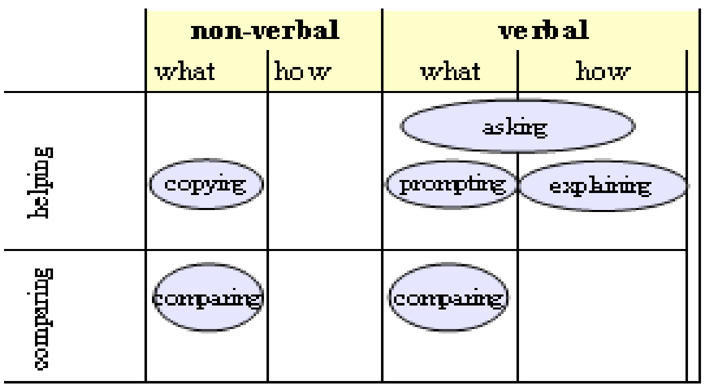


Fig. 2: Differentiation between Naujok's subject-oriented cooperation acts

In Fig.2 some cells aren't filled yet and could be filled when thinking about problem solving activities, so that our analysis had to be open to the possibility of new acts.

The coding procedure consists of two steps: First we had to decide in which transcript-rows a new cooperation act, a new content of cooperation or a non-cooperation phase begins. Afterwards we tried to subsume the cooperation-phases under Naujok's cooperation acts, so that for these acts, anchor examples could be attached from our own transcription material.

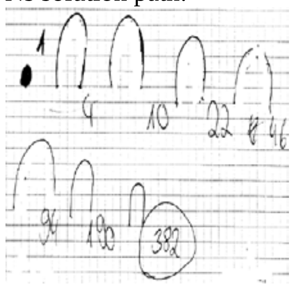
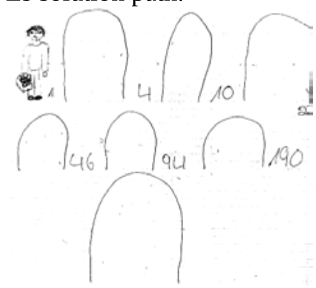
In our transcripts we found cooperation-phases, which we could not assign to Naujok's framework. They had to be defined and named. In order to keep cooperation categories of similar degrees of abstraction to Naujok's when creating categories inductively, Naujok's interpretation of cooperation *acts* and her criteria for describing cooperation formed the basis of naming and describing empirically formed cooperation acts. When Naujok uses activity verbs to describe cooperation types, which refer "usually to features of individual interaction partners", she still sees cooperation acts as "joint acting and interacting" and not as "isolated activities by individual participants" (Naujok 2000, p.164) (s.above). Activity verbs should equally be used to describe the cooperation types in the MALU processes. The descriptions of the cooperation acts chosen by the fifth-graders could be used here as descriptions of categories and the differentiations suggested by them could be considered as encoding rules.

After defining and characterizing the new cooperation acts and after revising (especially completing some features belonging to problem-solving tasks) Naujok's cooperation acts we trained two students in coding with this category system. The trainer and the students, marked independently in four transcripts the rows where new phases begin (step 1) and subsumed in other four transcripts the marked phases under the cooperation acts or under non-cooperation (step 2). The pairwise interrater agreement was as follows: In both decisions (marking the begins of new phases and subsuming the marked phases) the agreement is good (for the first decision the agreement varies between 60% and 69%, for the second decision Cohen's Kappa varies between 0.64 and 0.68). After coding independently, we discussed the decisions where we disagreed. Some of these cases remained unclear, so that we added a category "vague cooperation" to the coding system.

RESULTS AND DISCUSSION: COOPERATION ACTS

We found all cooperation acts from Naujok also in the MALU-transcripts. However, we had to adapt the definitions of the cooperation acts to our data in order to include passages, which are mostly similar, but in some ways different from the descriptions of the acts. When we look for example, at the copying-category, in Naujok's work this category includes only the copying of the solution of a task. An indicator for a copying-phase is therefore the similarity between the partner notes at the moment of copying and the copied information. The following table shows some of the original and the copied information in MALU-processes of the seven-gates-task:

1	**JP&Lu (00:55-01:15)**	After reading, JP invites L to copy information and begins to write some equations (see alongside). Lu copies these two equations.	JP writes: 1·2 = 2 2·2 = 4	Lu writes: 1·2 = 2 2·2 = 4
2	**M&HF (06:36-06:45)**	After copying the solution path with all gates from HF, HF says which answer he has written. So, M writes the firs part of HFs answer: "In the beginning the man had"	HFs answer: "In the beginning the man had 382 apples."	M completes his answer: "In the beginning the man had 382 apples."

3	**N&L (06:22-07:11)**	L paints a man in her notebook. During this time N solves the task by starting with the number of the apples, the man had in the end (1). So she goes back to the beginning (see alongside) and gets the correct solution 382. After informing L about this solution, both girls write the same answer ("In the beginning the man had 382 apples."). Then L copies the solution path by painting the gates and writing down the numbers of apples the man had before every gate (see alongside).	Ns solution path: 	Ls solution path: 

In this table you can see that the fifth-graders in MALU copy not only the solution of the task, but also the whole solution path (ex. 3), parts of this path (ex. 1) or (parts of) the answer (ex. 2). Copying passages can refer to the "how" of the solution path (s. Fig.2), too. If you interpret the "what" as the product and the "how" as the process, you can identify the answer as "what".

Furthermore, if a fifth-grader copies the whole solution path, he has to look alternately at the partners and at his own notes. The information is copied *bit by bit*. So, a copying-phase can last several seconds (or minutes) and cannot always be characterized as spotty (Naujok 2000). Beyond this, we have to revise the condition for a comparison: In Naujok's work the persons have solved the task at the moment of comparing (see above). The differentiation between parts of the answer, parts of the solution path and the solution allows the specification of this condition: At the moment of comparing both persons did not need to have solved the whole task, but the part of the solution path, that is compared.

We also had to differentiate some of Naujok's cooperation acts, so that these acts become disjoint. So, we observed that a cooperation act can be *self-initiated* or *partner-initiated* (like in the first example: JP invites Lu to copy). Since the asking act includes the initiation of other acts, we reserved the asking-act for a residual-category. In addition to Naujok's acts, further cooperation acts could be defined and labeled:

- *presenting an idea*: One person mentions an idea about a possible solution or about possible parts of the solution path (e.g. next steps, task-features, argumentation) before this person has solved the task or the parts of the solution path, which the person presents.
- *evaluating, checking, pointing out a mistake*: If the persons *check* something, they go through the solution path and reflect the steps. Instead one person can *point out a mistake* or can assess the correctness or usefulness of something (*evaluating*). These three cooperation acts occur in the MALU-processes as *saying-what* as well as as *saying-how*. In the case of these acts the person could either have the solving step already done or haven't yet.
- *informing about something*: One person says her/his (part) solution or says how she/he solved the task or parts of the task.
- *commenting on the task*: The persons say something about the task (e.g. familiar, funny) or comment on the difficulty of the task (e.g. easy, difficult).

In Naujok's investigation the students distinguish between some cooperation acts. If we take these dimensions of differentiation (s. Fig.1) as a basis for the structure of the cooperation acts observed in the fifth-grade students' processes concerning a variety of tasks and pairs (s. chapter "method"), we arrive at the following picture (in blue-grey the cooperation acts that Naujok reconstructed):

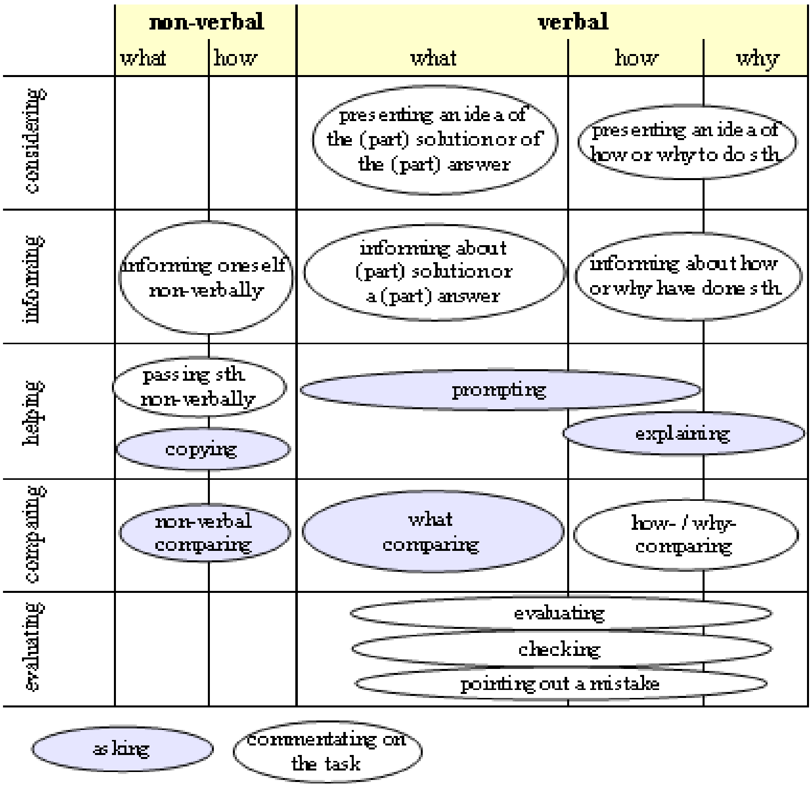


Fig. 3: Structure of subject-oriented cooperation acts observed at MALU

If we compare the cooperation acts reconstructed by Naujok (Fig.2) with those cooperation acts occurring in the MALU processes (Fig.3) the following becomes apparent: In the MALU-processes we found cooperation acts occurring in a stadium where the acting person has not done the solution step yet (*considering*-row in Fig.3). This person presents only ideas of possible solutions/answers or of possible procedures. Besides, we could assign a lot of cooperation acts we found to the *saying-how/why-to-do-sth.*-column.

The cooperation acts *checking*, *evaluating* and *pointing out a mistake* were not discussed in Naujok's investigation. Because of other studies using routine-tasks (for an overview s. Webb 1991), these acts could not be regarded as special for problem-solving steps. But it might be that these acts occur more frequently in problem-solving than in routine activities because of the uncertainty of the chosen solution path.

These results could be explained by the nature of problem-solving: If the problem solver first has to look for a solution procedure or to weigh up alternative methods, then an explorative *presenting ideas* might be more likely than if the solution procedure is clear to the problem solver and he/she "only" has to apply this. So it is possible, that these cooperation acts, assigned to the first row in Fig.3, occur above all in *exploration* and *planning* phases (Polya 1973; Schoenfeld 1985) whereas the cooperation acts assigned to the other rows in Fig.3 might occur above all in *implementation* and *verification* phases (Schoenfeld 1985).

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ENGAGING PROSPECTIVE TEACHERS IN THE ASSESSMENT OF GEOMETRICAL PROOFS

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The aim of this study is to examine the effects of engaging prospective mathematics teachers in peer assessment of geometrical proofs, both as assessors and assessees, on the developing of their own evaluation skills as part of their professional development. The research was conducted within a method course in which peer assessment activities were employed. Sixteen prospective mathematics teachers participated in the research and had to act both as assessors and assessees. Analysis of the research data reveals that during the various phases of the study the prospective teachers developed skills concerning the construction of criteria set and weights for the assessment of their peers' work and the constructed criteria set referred to meanings and roles of mathematical proof. They also realized that assigning scores without providing justification or explanation is ineffective.

Keywords: peer assessment, roles and meanings of proof, feedbacks to scoring

INTRODUCTION

Many issues relating to various aspects of mathematical proofs were investigated during the last two decades. Among these issues are the role of proof in mathematics classrooms (Hersh, 1993), students' difficulties in providing proofs (Moore, 1994), students' perceptions of proof (Harel & Sowder, 1998) and the construction of proofs (Weber & Alcock, 2004). However, the assessment of mathematical proofs received little attention especially from the students' perspectives (Alcock & Weber, 2005). Both students and teachers have difficulties in evaluating proofs for correctness (Selden & Selden, 2003; Alcock & Weber, 2005). There is no doubt that among the important skills that have to be developed during the learning of teaching practice are assessment skills, since evaluation of students' work is one of the teachers' responsibilities. Engagement in assessment activities in general and of mathematical proofs in particular of prospective teachers (PTs) may help them develop the required skills that are important to the process of their professional development. The importance of qualifying PTs to properly evaluate students' work stems from the fact that the way teachers evaluate their students can either reinforce or undermine the learning process. In the context of assessing mathematical proof, assessment ability skills include: the construction of criteria set that addresses the roles and meanings of proof (i.e. structure, verification, explanation, systematization, communication, and intellectual challenge); the assignment of relative weights to each criterion. During the evaluation of the student's proof, to be able to provide an effective justification to the scoring process – a justification that enable the assessee to correct his proof. To empower this process we decided to employ peer assessment which is an arrangement for learners to consider and specify the level, value or quality of a product or performance of other equal-status learners (Topping, 2003). Using peer assessment the PTs are acting at the same time both as assessors and assesses. We addressed the following questions: (i) how students choose, assign weights and justify categories for the assessment process; (ii) how do they react to feedbacks received from their mates to their own work; (iii) in what ways the students' exposition to their classmates' work effect their own work. In this paper we focus on the first and the second issues.

THEORETICAL BACKGROUND

In what follows we present a brief literature survey regarding roles and meanings of mathematical proof, peer assessment and assessment of mathematical proofs.

Roles and meanings of mathematical proof

Various issues relating to mathematical proofs were investigated during the last two decades. Among these issues are students' difficulties in providing proofs (Moore, 1994), the role of proof in mathematics classrooms (Hersh, 1993), the construction of proofs (Weber & Alcock, 2004) and students' perceptions of proof (Harel & Sowder, 1998). Researchers broadly discussed the meaning and roles of proof in the mathematics practice (Bell, 1976; de Villiers, 1991; Hanna, 2000). First of all the role of a proof is to verify/confirm a given statement. The proof also provides insight into why the statement is true (explanation). Proof may also provide the organization of various results into a deductive system of axioms, concepts and theorems (systematization). During the process of finding a proof new outcomes may emerge. The proof has an important role in communication by the transmission of the mathematical knowledge and the process of looking for a proof is many times an intellectual challenge. Harel and Sowder (1998) defined a student's proof scheme to be the processes she uses to become certain of the truth of a mathematical statement, and to convince others of this certain truth. Many educators and researchers have investigated students' learning and abilities with proof concepts when using a dynamic geometry drawing tool (e.g., Stevenson, 2007; Thompson, 2007). Students who are accustomed to technology in all aspects of their lives may consider proofs done via technology as providing evidence to both ascertain and persuade relative to the truth of a claim.

Peer Assessment and assessment of mathematical proofs

Peer assessment is an arrangement for learners to consider and specify the level, value or quality of a product or performance of other equal-status learners (McDowell & Mowl, 1996). The assessors and the assessed may be individuals or pairs or groups. Feedback is essential to the development and execution of self-regulatory skills (Paris & Paris, 2001). Peer assessment involves students directly in learning, and might promote a sense of ownership, personal responsibility and motivation. Giving positive feedback first might reduce anxiety among the assessees and improve acceptance of negative feedback. Peer assessment might also increase variety and interest, self-confidence, and empathy with others for both assessors and assessees. Cognitive and meta-cognitive benefits might accrue before, during or after the peer assessment. Peer assessment demands social and communication skills, negotiation and diplomacy (Riley, 1995), and can develop teamwork skills. Learning how to give and accept criticism, justify one's own position and reject suggestions are all useful transferable social and assertion skills (Marcoulides & Simkin, 1991). The goal of the peer assessment processes is to verify whether the work satisfies the accepted standards, as well as to provide constructive feedback which includes suggestions for improvements (Herndon, 2006). The use of peer assessment is based on the assumption that peers can recognize each other's errors quickly and easily, and that a larger and more diverse group of people might find more weaknesses and errors in a work. In their literature review, Falchikov and Goldfinch (2000) pointed to the increase in students' involvement in assessment across the spectrum of discipline areas. This approach, however, is rarely implemented in higher education (Zevenbergen, 2001). The issue of reading and assessing mathematical proofs received minor attention in the research literature (Alcock & Weber, 2005). Both students and teachers have difficulties in evaluating proofs for correctness (Selden & Selden, 2003; Alcock & Weber, 2005). More specifically, in-service high school teachers accepted proofs according to their format regardless their content (Knuth, 2002).

THE STUDY

In this section we present the following: data concerning the study participants; description of the Method course in which the study took place, and the course of the study. In addition, we present the methodology used for the data collection and analysis.

The study participants. A group of sixteen PTs from an academic college of education participated in the study. These PTs were studying in their third year (out of four) toward a B.A. degree in mathematics education and computer sciences or physics for middle and high school.

The Method course. The Method course in which the PTs experienced peer assessment is a two-semester course and is the second mathematics Method course the PTs were required to take. In the first course the PTs studied and discussed teaching methods and approaches for the practicing of various high school mathematics topics. In the second course, in which the research took place, they were mainly engaged in mathematical explorations and peer assessment activities.

The course of the study**.** The study participants were engaged in the following: (1) The PTs were asked to solve a given problem in which they had to construct geometrical proofs; (2) The PTs' solutions were scanned each PT had to assess two classmates' solutions. The PTs were asked to construct a criteria list, to assign each criterion a numerical weight, and to provide justification for each criterion and weight; (3) Each PT received two anonymous evaluations and scoring of his or her work and was asked to respond on them. These stages were repeated again however, in stage (1) they had to pose their own new problem based on the original one.

Data collection and analysis methods. Following analytic induction (Taylor & Bogdan, 1998), the data collected from the entire process was analyzed according to the following focal points: Evaluation criteria: the nature of the criteria and its relation to the roles and meaning of geometrical proof; the weighting scale assigned to the criteria set; types of justifications given for each criterion's weight; Numerical scores and feedbacks: an examination of the scores assigned to each criterion; types of justifications for the scores.

RESULTS AND DISCUSSION

In what follows, we present results and discussion on the construction of assessment criteria and relative weights; and discuss the PTs perceptions concerning the issue of providing and receiving feedbacks to the scoring process.

Selecting evaluation criteria and setting weight for each criterion.

A total of 64 assessments were obtained in the two assessment tasks. Analysis of these assessments revealed that the PTs used 13 different evaluation criteria in the first task and 8 criteria in the second one (Table 1).

Difficulties in constructing criteria set. Many PTs reported on encountering difficulties in generating a satisfactory set of criteria for evaluating their peers' work in the first task, ascribing their difficulties to their lack of prior experience which resulted in the following indecisions : *I had difficulties deciding whether the criteria should relate to the methods, techniques, line of thought, or perhaps the final solution…. Which is more important?* (Dan). "*My most significant dilemma related to the nature of the evaluation criteria…. I wanted to generate a set of evaluation criteria that would reflect creativity, simplicity, and original observation of the problem. I had no idea how to do that*" (Dina). Dan expressed hesitations about selecting the most relevant evaluation criteria from the ones that seemed to him appropriate to include in his list. Dina, on the other hand, was concerned with the characteristics of the evaluation criteria she had to select. Her aim was to select the criteria that will enable a valid scoring of simple solution, original ideas and creativity. Some PTs argued that they are aware to the fact that the evaluation criteria should address aspects of proof but they have difficulties in formulating them.

The assessment of the second task was more difficult than the first one since in while in the first task they had to construct criteria set for a proof they already did, in the second task they had to assess a proof to a problem which is not familiar to them. Nevertheless, none of the PTs reported on difficulties in constructing criteria set for the second task.

The criteria set. Table 1 summarizes the list of the evaluation criteria provided by the PTs in the first and the third task. The average weight assigned to each criterion, the standard deviation (SD) of the assigned weights and the number of PTs that selected the criterion (and frequency).

Aspects of proof	**Evaluation criterion**	**Task 1**		**Task 2**	
		Average weight (AW) and standard deviation (SD	**Frequency**	**Average weight (AW) and standard deviation (SD**	**Frequency**
structure	1. Customary proof structure	AW=30 SD=10.44	11 (68.75%)	AW=21.81 SD=6.81	11 (68.75%)
verification	2.Correct solution	AW=46.1 SD=21.1	8 (50%)	AW=36.53 SD=14.34	14 (87.5%)
	3.Correct final answer	AW=23.3 SD=6.87	6 (37.5%)		
	4.Calculations errors	AW =10	1 (6.25%)		
Communication	5.Correct quotation of geometrical theorems	AW=29 SD=11.93	5 (31.25%)		

	6.Easy and understandable solution	AW=12.2 SD=10.2	5 (31.25%)		
	7.Adding a clear sketch of the problem	AW=13 SD=6.7	5 (31.25%)	AW=10 SD=0	3 (18.75%)
	8.Clear use of mathematical notations	AW =10	2 (12.5%)	AW =10 SD=0	3 (18.75%)
	9.Additional explanations to claims	AW =27.5	2 (12.5%)		
Systematization	10.Reasonable organization and clarity	AW=15 SD=7.07	4(25%)		
	11.Formulation of correct conclusions	AW =25	1 (6.25%)	AW=22.27 SD=8.33	11 (68.75%)
	12.Demonstration of connectedness knowledge between mathematical areas	AW =27.5	2 (12.5%)		
Intellectual challenge	13.Providing the shortest possible proof	AW =10	1 (6.25%)		

Table 1: Evaluation criteria constructed by the PTs in the first and the third tasks

Observation of Table 1 reveals that in the first task the PTs constructed 13 criteria and in the second one they constructed only 5 criteria concerning aspects of proof. In the second task they constructed 3 additional criteria that refer to the *character* of the examined problem: well-defined problem (AW=17.7 SD=5.65, 9 (56.25%)), suggestions for elaboration/generalization (Average=20 SD=10.27, 10 (62.5%)) and interesting problem (AW=20, 1(25%)).

The first criterion refers to the structure of proof. The relatively high number of PTs who constructed this criterion, both in the first and the second task is in line with Martin & Harel (1989) and Knuth (2002) who found that many pre-service teachers accept the validity of a geometry proof mainly according to its structure. Namely, valid if the proof was in the standard two-column format and invalid if it was in paragraph form, regardless of its mathematical content. Due to space limitations we focus only on the noticeable differences between two tasks. A significant increase in the number of PTs who chose the following two criteria: 'correct solution' which refers to the verification role of proof and 'formulation of correct conclusions' which refers to the systematization role of proof, in the second task can also be observed. This may be a result of the comments they received from the assessors of their work.

The decrease in the number of the constructed criteria from the first to the second task can be explained by the difference between the first and the second task. While in the first task, all the PTs had to provide first a geometrical proof to the *same* problem and then to assess two other similar ones, in the second task, each PT had to choose his own new problem and provide its proof and then to assess two solutions to different new problems from the one he posed.

Assigning Numerical weights. The PTs expressed hesitations about the numerical weight they should assign to each evaluation criterion, realizing that the greater the criterion's numerical weight is the greater is its relative importance. The PTs argued that the numerical weight itself "*bears a certain message*"; namely, "*if a teacher wishes to educate her students to work in a certain way, she should weight specific criteria in accordance*".

After the first task, more than half of the PTs reported a process of assigning numerical weight to criteria similar to the following: *"I had many doubts in assigning weight to each criterion. After much hesitation I wrote down the criteria list and gave each one 10 points in the first round. Since I wrote down 5 criteria, I ended the first round with 50 points. Then, in the second round I asked myself which of the criteria are more important to me in relation to the others and added to these criteria 10 additional points until I reached 100"*.(Ella)

The rest of the PTs acted similar to the following process:"*The process of giving weights to the evaluation criteria was done along with the criteria selection. There were the 'must' criteria which received relatively high weight and there were the 'unique' criteria which received relatively low weight*"(Dina).

The above excerpts refer to two different numerical weight assignment techniques. Ella describes a relatively systematic method according to which she decided to assign weight to each of the evaluation criteria she selected. Ella assigns weights in an iterative way. In the first round she assigns a basic low weight (10 points) to each of her selected criteria. Then after deciding which criteria are more important to her, she raises the weight accordingly. She arrives at the last iteration when the total weight is 100. Dina, on the other hand, describes a more intuitive method of weight assignment. She divides her criteria set into two groups: the 'must' group which includes the problem-oriented criteria and the 'unique' group which includes criteria such as creativity and originality. To the first group she assigns a relatively high weight which is divided among the criteria belonging to this group and to the second group she assigns a relatively low weight. After the second task there was no reference in the PTs' reflections to the process of assigning weight to the evaluation criteria. This fact may imply that the PTs overcame the difficulties they encountered in the first assignment.

Feedbacks – assessors and assessees perspectives

An interesting phenomenon was revealed when reviewing the PTs' comments on the feedbacks they had to give as assessors and receive as assessees. When they were asked to relate to the feedbacks as assessors, most of them said the feedback has to include reference to relevant roles and meanings of proof and be phrased in a constructive rather than a negative manner. Their comments were practical and proof-oriented: "*In case quotations of geometrical theorems were missing in the student's proof, I asked him to provide justification to each claim and avoid non established conclusions*". *"One of the proofs I had to assess had many logical jumps and I could not follow the course of it. I asked the student to provide step-by-step claims and justify each of them.* In these comments the PTs refer to the lack of systematization and communication in the proofs they had to assess. However, when they were asked to relate to feedbacks they had received as assessees, their responses were mainly emotional. They were concerned with the character of the feedback and less with its relation to aspects of proof. *"Although the score was relatively high, I was not satisfied, since the feedback was general and shallow and I had the impression that my work was* *not treated seriously…"…the feedback I received left me with no clues how to improve my work". "Looking at the few explanations he gave, it occurred to me that he did not understand my solution".* The above quotations refer to various impressions the feedback left on the one who received it. In the first quotation the PT expresses feelings of dissatisfaction and frustration since he feels his work was not treated seriously. In the second quotation the PT expresses feelings of helplessness that stems from his inability to figure out how to improve his work. In the third quotation the PT expresses also frustration which stems from his feeling that his proof was not been understood.

In major part of the PTs' reflections concerning feedbacks, they compared what they did as assessors (assuming they did their best efforts) and expected to receive a similar 'treatment' from their assessors. *"In the assessments I gave I did my best to justify each score carefully and I expected to get the same attention".*

There were also others who said similar to the following: *"The score I received in one of the assessment forms was not high but since it was logically justified down the smallest details, I felt I deserved the score I received. From the explanations, it was clear that the assessor read my solution carefully and provided me with ideas on how to improve it".* To conclude, from the PTs reflections concerning feedbacks on the scoring process, we may say that as assessors they began to develop their ability to evaluate and justify (Topping, 1996) . The process enabled them to receive two opinions of their work and by reviewing the works of their peers they gained insight into their own performance (Berkencotter, 1995).

CONCLUDING REMARKS

Developing assessment skills is essential to PTs' professional development as future mathematics teachers (Lovemore & David, 2006; McGatha et al., 2009). In the PTs selected set of criteria for the peer assessment, the PTs referred to various roles and meanings of mathematical proof. We may say that the PTs engagement in peer assessment both as assessors and assessees contributed to the development of assessment skills. Comparison between the first and the second assessment tasks revealed that the PTs developed their abilities to select a proper criteria list and assigned a reasonable numerical weight to each criterion. In order to substantiate the obtained results, the present research should be repeated with a larger group of PTs.

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THE AUTONOMY TO CHOOSE: PERCEPTIONS AND ATTITUDES OF NINTH GRADE STUDENTS TOWARDS MATHEMATICS

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Abstract

The effects of choosing exam levels on the attitudes towards mathematics of ninth grade students were explored in this study. The research was carried out during the second semester of ninth grade. The students were provided with the autonomy to choose between two exam levels: regular and extended, and were asked to justify their choice. The students' justifications were classified into the following categories: goals, self-perception of mathematical ability, performance experience, motivation, and external factors. The students were classified into two groups: those who did not choose different level of exam during the three and those who did. Different focal points were observed between these groups. The class atmosphere completely changed and expressions of frustration and dissatisfaction disappeared.

INTRODUCTION

In Israeli high schools tenth grade students are usually grouped according to their past achievements in mathematics, to three learning levels: high, medium and low. Although this grouping is done very cautiously, many students are disappointed and have difficulties to accept the learning level they are assigned to. One of the reasons underlying this disappointment stems from their feeling that they had no influence in the decision-making and that the ability to improve their learning level is blocked. Moreover, since the grouping affects their future vocational opportunities, the grouping to mathematics levels becomes even more significant. This is the reason why parents feel they must also interfere in this process. These feelings of frustration are reflected in ninth grade students' behavior during the mathematics lessons. The class atmosphere becomes unpleasant and the students' tension comes to fruition through expressions of anger and misbehavior. In order to change students' attitudes, we decided to change the usual decision-making process concerning the grouping and give the students the autonomy to take part in this process. In the present study we employed the following process: during the 9th grade we allowed students to choose the level of the exams they would take during the second semester, and we then explored the various aspects of this change.

THEORETICAL BACKGROUND

In this section we present a brief literature survey concerning the following: the self-regulated theory, the causal attribution theory, self-efficacy, and motivation.

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Self-regulation theory. This theory derived from the cognitive approach in education, which was developed in the 1980s and emphasized the self-perception of the individual and his patterns of actions (Deci & Ryan, 1985). Self-regulation was defined as a process in which the learner defines his own learning goals, and executes controlled actions in his mind, his motivation and his behavior to fulfill them (Pintrich, 2000). Pintrich (2000) pointed out four premises in the theory. The first premise is derived from the constructivist theory, which argues that learners generate knowledge and meaning from an interaction between their experiences and new incoming ideas. The second premise is the control premise, in which the learner has the potential to criticize, control and supervise the characteristics of his learning environment as well as personal characteristics, like motivation. The third premise is the target premise that claims the learner can set the goals he wants to achieve in his studying. He can supervise the process toward obtaining them and regulates and adjusts his efforts, thinking, motivation and behavior to fulfill them. The last premise concerns the mediation between personal attributes and performance; that is, the personal attributes and the environmental characteristics that formulate the student's self-regulation, which affects performance and achievements. The self-regulated theory indicates three basic psychological needs: the need for autonomy, relatedness and competence (Ryan & Deci, 2002). The need for autonomy is the student's needs to feel that his actions express his values and tendencies, and correspond to his needs. Autonomy develops when support from well-meaning adults is provided. The need for relatedness concerns the individual's need to feel accepted by the people around him and the need to be loved. In addition, relatedness concerns the urge to belong to a group that one can identify with and feel secure in. The need for competence is the individual's need to perceive his capability to accomplish tasks. The fulfillment of the above needs is essential for the proper emotional and cognitive development of the individual (Ryan & Deci, 2000).

The causal attribution theory. The causal attribution theory (Weiner, 1986, 1992) postulates that the student attributes a cause to the outcome of his performance. Furthermore, the student's subsequent action, in addition to motivation, is influenced by this attributed cause more than by the outcome itself. The causal attribution can be viewed according to three dimensions: The first refers to the character of causality, which can be either internal or external; namely, the student may perceive the cause of his success or failure to be in his ability and in the extent of the efforts he invested in the activity, or he can attribute it to external factors like the difficulty of the assignment, the quality of teaching and so forth. The second dimension is control, and refers to the extent to which the student feels he can control and influence the causes of his success/failure. The last dimension is stability, in which the student either perceives the cause to be a stable factor such as personal ability or the difficulty of the task, or a changeable factor such as luck or exceptional effort or mood. Causal attribution mostly accrues when the outcomes are important, unexpected or negative to the student (Weiner, 1992). Students tend to attribute success to personal causes like ability, and failure to environmental causes. The reason for this phenomenon may stem from the assumption that in situations that are meaningful for the student, factors concerning ego and social appraisal are involved (Stephanou, 2003).

Mathematical self-efficacy. Research concerning self-efficacy has received special attention in the area of mathematics education. Mathematics is considered to be one of the difficult learning areas and it is often used for students' grouping into learning levels that affect entrance into special academic programs. For this reason it has been called a "critical filter" for students aiming to develop technical careers (Sells, 1980). Researchers have demonstrated that self-efficacy predicts mathematics performances to a greater degree than does anxiety about mathematics or previous mathematics experience (Pajares & Miller, 1994). Pajares and Kranzler (1995) found that the influence of self-efficacy on mathematical performance is the same as the influence of general mental ability. Stephanou (2004) found that students' self-efficacy beliefs in mathematics are more significant for them than in other domains. These findings are in line with other findings that performance in mathematics is strongly related to students' egos (Mason, 2003). Across ability levels, students whose self-efficacy is higher are more accurate in their mathematical calculation and show greater persistence with difficult tasks than students whose self-efficacy is low (Pajares & Graham, 1999). Hall and Ponton (2005) found that past experiences and frequent failures in mathematics usually dictate student opinions concerning perceptions of their ability in mathematics, as well as their levels of optimism about career choices. Another concept concerning self-efficacy is self-concept, which refers to a multidimensional perception of self (encompassing academic, social, and physical aspects). A review of the literature revealed that a positive association between mathematical self-concept and mathematical performance has consistently been found across different countries (Pajares & Graham, 1999). That is, students with a high level of mathematical self-concept generally show greater engagement, persistence, and effort in tasks relating to mathematics and in turn perform better than students with lower levels of mathematical self-concept.

Motivation. Motivation is the integration of the individual's behaviors, objectives and aspirations that lead him to fulfill his targets. Researchers defined three dimensions of motivational behavior: direction, intensity and quality (Maehr & Midgley, 1996). The behavior's direction refers to the choice of a specific behavior from a range of possible behaviors, and preserving it despite any obstacles encountered. The behavior's intensity refers to the extent of the effort invested in the activity, and the behavior's quality refers to the choice of a behavior that is creative and innovative rather than technical and routine.

Researchers described motivation according to the level of autonomy. The concept of autonomy support was coined to understand the social-contextual factors that affect students' learning (Deci & Ryan, 1985; Ryan & Deci, 2000). This concept describes a person in a position of authority (e.g. a teacher) providing another person (a student) with information and choice, and minimizing the use of pressure and control (Williams & Deci, 1996).

THE STUDY

The study's participants. Fifty 9th and 10th grade students from a regional high school in the northern part of Israel took part in this study. Twenty-six students were from the 9th grade and twenty-four from the 10th grade. Most of the students were from rural settlements. The data and analysis presented in this study refers only to the 9th grade students.

Method. During the whole study year of the 9th grade, the learning topics were taught in two levels starting with a basic level and then extending it to a higher one. During the second semester the students had three exams each in two versions: regular and extended. Each student had the autonomy to choose the exam's version. After each exam, the students were asked to fill a questionnaire in which they had to provide the reasons underlying their actual choice.

Data collection – the questionnaire. The research data comprises the students' responses to the questionnaire and the grades they received in the three exams. The questionnaire included general questions referring to the process of choosing, such as: "which version did you choose and why?"; statements concerning their attitudes towards their perceived level of mastery of the topics under examination, such as: "set your degree of mastery concerning the trapezoid topic" (the student had to choose from a scale of 1 to 5); and several statements to examine the students' attitudes towards mathematics. In addition, they were asked to place themselves with relation to their classmates in a scale of 4 levels.

Data analysis. Following analytic induction (Taylor and Bogdan, 1998), the data collected from the students' responses to the questionnaires were analyzed, and the following categories emerged: *Goals*- Statements in which students indicated information concerning their internal and external goals; *Self-perception of mathematical ability*- Statements referring to the students' perceived mathematical ability; *External factors* - Statements that referred to the exam's level and the extent to which students were preparing for the exam; *Performance experience* -Statements that indicated reasons for their decisions as a result of their experience in the previous exams and *Motivation* - Statements that referred to motivation.

RESULTS AND DISCUSSION

In what follows we analyzed the data across different parameters such as the students' distribution according to the above categories with special focus on students who changed the exam versions. The findings were also interpreted by self-regulation and self-efficacy theories principles (Pintrich, 2000; Pajares, 2002).

The effects of self-determination on the exam level chosen and on the students' mathematical competence.

We used external resources (providing the students the autonomy to choose their examining level) to help students develop their mathematical competence. Ryan and Deci (2002) defined the need for autonomy as one of the basic psychological needs whose fulfillment nurtures one's self-regulation. Figure 1 presents the distribution of the choice process. Observation of the data presented shows a clear tendency of shifting from the extended to the regular version of the exam. The distribution of students in the third exam converged towards a normal distribution of a homogeneous class – two thirds chose the regular level and one third chose the extended one.

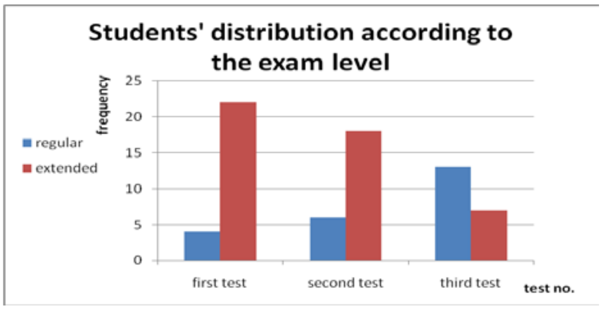

Figure 1: The students' distribution according to the exam level in the three exams

Among the 20 students who attended to all three exams, eight students (40%) did not change the exam version over the three exams, while twelve students (60%) changed versions once or twice during the three exams. In order to trace the students' reasoning in the process of choosing their exams, we differentiated between the students who changed exam versions and those who did not.

Reasons underlying the students' choices. Ninety justifications for the process of choosing the level of the exam were received from both student groups. Analysis of them revealed that these justifications fall into five main categories: factors concerning self-perception of mathematical ability, motivation, goals, performance experience, and external factors. First we refer to the categories in general and then we focus on results received from the student group who changed the exam's version.

Goals (26 responses out of 90). In this category we included statements concerning short and long term aims, such as: "*I really want to improve my mathematical knowledge*"; "*I changed the exam version because I wanted to get a better grade*"; "*I intend to learn in the high level group*"; and "*I want to be able to enter university to learn in a good quality faculty, like the computer science faculty, for example*".

Self-perception of mathematical ability (22 responses out of 90). In this category we included statements such as: "*I think I am capable of learning at a high level and I wanted to see if it would be difficult for me*"; "*the extended level is difficult for me and I feel I do not know enough mathematics*"; "*my mathematical knowledge is quite good and I think I will manage in the high level*"; and "*this* [extended level] *reflects my abilities*".

External factors (22 responses out of 90). In this category we included statements which refer to the exam's level and the extent to which students prepared for the exam, such as: "*I looked at the problems and they seemed simple*"; "*the extended exam was more difficult and I did not have enough time to study to the exam*"; and "*I did not study to the exam so I choose the regular version*".

<u>Performance experience</u> (11 responses out of 90). In this category we included statements which indicate that the reasons for their decisions were a result of their experience in the previous exams, such as: "*in the previous exam I chose the regular version and it was easy for me*"; "*I did not succeed in the extended version so I decided to do the regular one*"; "*usually I do the extended exam but this time I did not understand the questions in it so I chose to do the regular exam*"; and "*until now I chose the extended exam and I did not get good grades so this time I chose the regular exam to get a better grade*".

<u>Motivation</u> (9 responses out of 90). In this category we included statements such as: "*I want to be in the high level so that I will not be left behind*"; "*it is very important to me to succeed*"; "*I do not want to let myself down*"; "*I like challenges*"; and "*I want to do my best to get better grades*".

Concerning the students who changed versions (Table 1), most of them tried the extended version in the first exam believing they are able to cope with it. However, after receiving a low grade, a decrease in their self-efficacy may have resulted in a change of versions from the extended to the regular. These results are in line with Pajares (2002) who found that in cases where students have the freedom to choose their preference, they tend to choose tasks in which they feel confident and avoid the ones in which they do not.

category	**First exam**	**Second exam**	**Third exam**
Goals (26)	3(11.5%)	2(8%)	0
Ability (22)	7 (32%)	2 (9%)	0
External factors (22)	4(18%)	6(27%)	7(32%)
Performance experience (11)	1(9%)	2(18%)	6(55%)
Motivation (9)	2(22%)	4(44%)	0

Table 1: Students' distribution according to categories – the students who changed versions

The first column of Table 1 refers to the category list and the total number of reasons provided for each category. The following columns refer to the three exams while the first number indicates the number of provided reasons and in brackets is the relative percentage of reasons with the total number of this category. Observation of Table 1 reveals that in the first exam the most dominant reasons refer to their perceived ability. We may assume that the goals set for the first exam seemed to the students to be less achievable following the received grades. After the second exam the number of reasons relating to performance experience and external factors raised. We may say that they began to develop a sense of responsibility to their learning. Namely, they attributed their lack of success in the exam to their invested efforts. These results are in line with the casual theory according to which students tend to attribute success to personal factors and failure to external ones (Weiner, 1992). The research literature distinguishes between specialization goals, which focus on improving one's ability and developing particular skills, and operational goals, which focus on achievements and the demonstration of abilities (Ames, 1992; Maehr & Midgley, 1996). Viewing the goals as one of the constituents of self-directed process (Pintrich, 2000) present a different point of view. The student sets goals for herself and then adjusts her behavior to fulfill them. During the whole process she uses control and assessment actions to test the “quality” of her performance and products (e.g., tests' scores) that may result in the setting of new goals. This process can be observed in both the changing and the non changing groups.

The class atmosphere. The following excerpt is taken from the class teacher[1] reflection on the whole process: “For many teaching years in 9th grade most students expressed frustration and complained about not being able to take part in the decision-making process concerning the mathematics level they would be grouped in. I applied the above idea of providing the students with the autonomy to choose the exam versions for two successive years and observed a big change: the class atmosphere completely changed, and expressions of frustration become very rare. Students and parents alike more readily accepted the mathematics level they were grouped in, since they had the feeling the choice was theirs”. These results are in line with Black and Deci (2000), who found that an autonomy supportive school classroom environment has a positive effect on students’ perceived competence as well as on their interest, enjoyment, efforts, and school performance.

CONCLUDING REMARKS

Providing the students with the autonomy to choose the level of the mathematics exams encouraged the students to develop a sense of self-regulation. According to the research results, the ability to choose the exam levels nurtured the students’ need for autonomy, and provided an opportunity for each student to examine his goals and check them according to his performances. Observation of the students’ distribution according to the exam level in the three exams (Fig. 1) demonstrates a shift from a case in which most of the students chose the extended level (exam 1) towards a 'normal distribution' (exam 3) in which two thirds of the class chose the regular exam and one third the extended one. In this case we may say that the students successfully managed to monitor their mathematical self-perception, motivation and behavior. In addition, a positive change was observed concerning the class atmosphere. Expressions of dissatisfaction and frustration became very rare. To establish the above results we think this research should be repeated throughout the whole school year of the 9th grade in a larger research population.

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TALKING AND LOOKING STRUCTURALLY AND OPERATIONALLY AS WAYS OF ACTING IN A SOCIO-POLITICAL MATHEMATICAL PRACTICE1

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This paper proposes a reconceptualization of the psychological constructs of structural and operational conception. Drawing on Morgan, Moschkovich and Sfard, we present a theoretical perspective that views talking and looking operationally and structurally as ways of acting in a socio-political mathematical practice. We use transcripts of first-year undergraduate students solving a function problem to illustrate this perspective. We argue that student action is a complex interplay of the ways of talking about and looking at the mathematical objects, together with discursive, social and political ways of acting in the classroom.

INTRODUCTION

The dual ontological nature of mathematical objects as both processes and objects has been captured in psychological theories of mathematics learning such as *Action, Process, Object, Schema Theory* (Dubinsky, 1991), the theory of *reification* (Sfard, 1991), and the notion of *proceptual thinking* (Gray & Tall, 1994). It has been argued that both a *structural* and an *operational conception* of a mathematical object are necessary for mathematical understanding (Sfard, 1991), with some of these arguments focusing on the learning of functions (e.g. Breidenbach, Dubinsky, Nichols & Hawks, 1990; Moschkovich, Schoenfeld & Arcavi, 1993). These theories have dominated research on the learning of undergraduate mathematics since the early 1990s (e.g. Hazzan, 2003; Maharaj, 2010; Stewart & Thomas, 2009). This research has made a substantial contribution to our understanding of student action on mathematical objects. However, we argue that this research does not have the discursive, social and political action of students in the mathematics classroom in view and it does not take into account the macro-social issues that figure in the classroom action.

In this paper we draw on the work of Morgan (1998) as well as the later work of both Moschkovich (2004, 2007) and Sfard (2000, 2008) to present a reconceptualization, from a socio-political practice perspective, of the psychological notions of structural and operational conception. We present ways of talking and looking operationally and structurally (along with other ways of acting such as ways of endorsing arguments and of interacting socio-politically) as ways of acting in a socio-political mathematical practice. We illustrate the use of the perspective in the analysis of transcripts representing the verbal and non-verbal action of first-year undergraduate students on a function problem, and show that while the interplay between operational and structural in student action is central, it is only part of the story of the students' action.

THEORETICAL FRAMEWORK

A socio-political perspective of mathematical practice is based on the work of Fairclough (2001, 2003) in critical linguistics. Action of undergraduate students solving a function problem is located in the socio-political practice of first-year undergraduate mathematics. This practice is *socio-political* since it is a relatively stable way of doing things in which certain material and mental activities, objects, participants, socio-political relations, beliefs, and discourse are valued. Undergraduate mathematics is networked with other practices such as school mathematics and professional research mathematics. Power relations are at work both between these practices and within the action of the students in the mathematics classroom. *Mathematical discourse* is the language aspect of a socio-political mathematical practice and gives meaning to the practice; it *represents* the practice in a certain way, it is used for *interaction*, and it *identifies* participants as particular types of people. Sfard (2008), Morgan (1998) and Moschkovich (2007) agree that there is something distinctive called *mathematical discourse,* while acknowledging that this is changing, has no fixed boundaries, and is used in a variety of mathematical practices.

Using the work of Fairclough and that of Morgan, Moschkovich, and Sfard in mathematics education, in interaction with empirical data of first-year undergraduate student action, we have identified interrelated ways of acting mathematically in discourse. These ways of acting capture the mathematical, discursive, social, and political nature of this action. We focus on those ways of acting that relate specifically to the dual nature of mathematical objects (points 1 and 5).

1. Ways of talking and writing about objects and ways of representing objects
2. Ways of making links between objects, texts, events and practices
3. Ways of endorsing arguments about mathematical objects
4. Ways of evaluating the pronouncements of other participants
5. Ways of attending (ways of looking at mathematical objects and their representations, ways of listening to talk about objects)
6. Ways of operating on mathematical objects
7. Ways of identifying oneself and others, ways of interacting socio-politically.

We use *talking operationally* and *structurally* for the talk about mathematical objects identified by Morgan (1998) and Sfard (2008). Morgan (1998) argues that such objects can be talked about as processes or objects, for example, the nominalization *rotation* represents the process of rotating as an object that can act or be acted upon. A subsequent replacement of "rotation" with the reference pronoun "it" may indicate structural talk. In her recent work on mathematical discourse, Sfard moves away from seeing *reification* as a mental shift in which a "process solidifies into object, into a static structure" (1991, p.20) to viewing this notion as involving "replacement of talk about processes with talk about objects" (2008, p.171). For example process talk about the signifier $5/7$ may be "I divided the whole by 7 and took 5 of the parts", whereas object talk would be "I have $5/7$ of the whole" (Sfard, 2008, p.171).

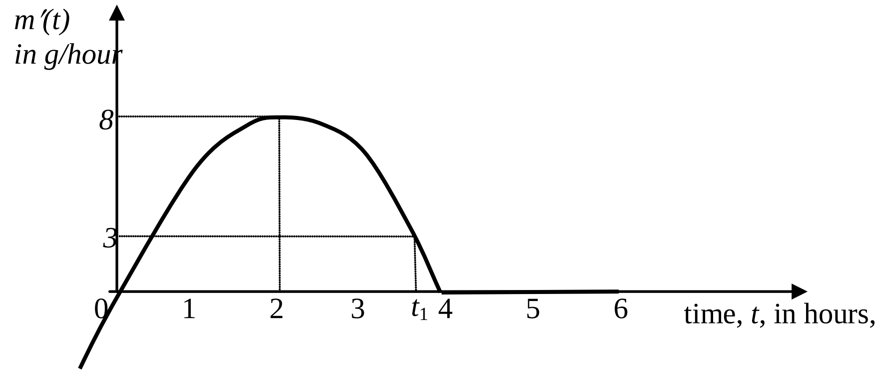
Sfard and Moschkovich propose a concept of mathematical discourse that includes "what we see" (Sfard, 2008, p.146) or *perspectives/ways of seeing* (Moschkovich, 2004, p.50). Moschkovich (2004) argues, from a socio-cultural perspective, that viewing a linear function as a process or as an object is part of expert mathematical practice for working with functions, and does not refer to "an individual competency" (p.57) as in the psychological perspective. The notions of *looking structurally* and *operationally* as ways of attending in mathematical discourse are productive in our analysis of non-verbal student action. For example, substituting the value $t = 2$ to check the accuracy of the formula $y = -2t^2 + 8t$ as a representation of a parabola graph requires an operational view of the quadratic function, while using transformations to identify the formula for a parabola graph suggests that a student is looking structurally at the quadratic function.

THE STUDY

The transcripts presented in this paper are taken from a wider study (Le Roux, 2011) that investigates innovation in the form of practical problems and a learner-centred pedagogy in a first-year undergraduate mathematics course at a South African university. The course is offered to students identified as disadvantaged by enduring inequities in the schooling system, and is designed to facilitate students' transition from school mathematics to advanced mathematics.

In this paper we present, in brief, the action of four students, Kelsa, Lwazi, Ndumiso and Thokozile (pseudonyms), as they work in a small group (with the support from a tutor) to answer question (d) of the Chemical Reaction Problem (see Figure 1). This data has been chosen as it represents the complex interplay of the different ways of acting mathematically in discourse.

Quantities of two chemicals A and B are mixed together in a reaction chamber, and they react to form a new product. The **rate** at which the produce is formed is given by $m'(t)$, where m is the mass of the product formed, in grams, and the time t from the start of the reaction is measures in hours. The graph of $m'(t)$ is a parabola graph until time $t = 4$ hours, after which it is zero. It is also given that, from the start of the reaction, some of the product X is removed from the reaction chamber at a constant rate of 3g/hour.



d) Find the equation of the parabola graph - it will express $m'(t)$ as a quadratic function of t.

Figure 1: Chemical Reaction Problem, question (d)

Answering question (d) in Figure 1 involves the student making a link to school mathematics and viewing the quadratic function structurally by identifying one of a number of possible general formulae ($y = at^2 + bt + c$ or $y = a(t - r_1)(t - r_2)$ or $y = a(t - p)^2 + q$) as representing a family of functions and hence an appropriate representation of a parabola graph. The student makes links between certain points on the given parabola graph and particular symbols in the selected formula. Then, viewing the function operationally, the student acts operationally by substituting and rearranging the subject of the formula. Acting in this way, the student arrives at the formula $y = -2t^2 + 8t$ for the function. In this discussion we do not focus on the nature of this quadratic function as the derivative function $m'(t)$; this, together with additional data from the study, is the subject of a further paper.

ANALYTIC TOOLS

The transcripts were analysed using Fairclough's three-stage method of critical discourse analysis. We focus here on the first two stages; the detailed textual analysis of the transcripts and the use of these textual clues to identify the ways of acting mathematically in discourse. Fairclough's method was supplemented with Sfard's (2000) method of focal analysis so as to bring the specificity of mathematical action into focus. We illustrate with action on question (d) in Figure 1:

After Thokozile has proposed "x^2" as a formula for the graph, Ndumiso responds:

261 Ndumiso: That's a that's a <u>general</u> parabola↑ that's the <u>easy</u> one … this has <u>moved</u> ((*Showing shift with his fingers over the graph, looking at Thokozile as he speaks*))

The pronounced focus is what Ndumiso says in line 261. The attended focus is what Ndumiso is "looking at, listening to" when speaking (Sfard, 2000, p.304); he looks at the given parabola graph and listens to Thokozile's proposed answer of "x^2". We then perform critical discourse analysis on the pronounced focus. In his use of the demonstrative pronouns "that" (for Thokozile's "x^2") and "this" (for the parabola graph), Ndumiso is talking structurally about the graphs as objects that can move. This analysis is supported by Ndumiso's description of the graph as having "moved" and his gesturing of a shift in the graph; here the material process of moving suggests that Ndumiso is viewing the function as an object that can be acted on. In addition, through the emphasis on the adjective "<u>easy</u>", Ndumiso identifies himself as attending to a more complicated graph than Thokozile.

We use this method of analysis to now describe the four students' ways of acting mathematically in discourse as they answer question (d) in Figure 1.

THE STUDENTS' WAYS OF ACTING MATHEMATICALLY

The students begin by making links to school mathematics and they recruit three possible general formulae for the quadratic function. Ndumiso pronounces "that equation" $y = a(x - x_1)(x - x_2)$, by verbally naming the symbols from left to right. Kelsa's responds negatively by verbally stating an alternative general quadratic formula from school ($y = ax^2 + bx + c$) and stating, "<u>That's</u> the equation for parabola↑". Her emphasis on the reference pronoun "<u>that's</u>" represents her formula as the only one. Ndumiso is persuaded that his formula ("this") represents something else; "What's <u>this</u> equation for?" These references to formulae suggest that the students do not identify the two general formulae as equivalent, suggesting that they are looking operationally at the formulae as representing different processes. This argument is supported by Ndumiso's discussion with Lwazi. Here Lwazi is promoting a third general formula, $y = a(x - p)^2 + q$, although this is not pronounced verbally:

248	Lwazi:	There <u>is</u> an equation like <u>that</u>
249	Ndumiso:	For what though? ((*Looking at Lwazi*))
250	Lwazi:	But that's not <u>it</u> though
251	((*Ndumiso and Kelsa laugh, Thokozile is looking at Lwazi*))	
252	Lwazi:	No really↑ the <u>*a*</u> is right at the beginning↑ but the stuff in the middle isn't ((*Pointing to Ndumiso's equation* $y = a(x - x_1)(x - x_2)$))
253	Ndumiso:	It IS
254	Lwazi:	It's not
255	Ndumiso:	I <u>know</u> it is
256	Lwazi:	It's <u>not</u> … I'm <u>telling</u> you it's not

In this action, the use of the demonstrative pronoun "that" and the reference pronoun "it" suggests that the students are looking operationally and identifying the different formulae as representing different graphs.

Other ways of acting mathematically in discourse can be identified in the transcript lines presented so far. The students' endorsements are in the form of personal opinion ("I <u>know</u>", line 255), statements of fact with no supporting evidence ("<u>That's</u> the equation for parabola↑"), and alternative forms of the general quadratic formula. There is an absence of endorsements that make use of the properties of the quadratic function and links between the formulae and the graph are not given significance. Furthermore, the socio-political interaction between the students is competitive as they claim personal ownership ("I") of the different versions of the quadratic formula, action that may prevent identification of the three formulae as equivalent.

Unable to resolve whose quadratic formula is "it" (line 250), the students then pursue a link, made initially by Thokozile, to a method for finding the equation of a parabola used elsewhere in the undergraduate mathematics course. This method, named "the whole movement" thing by Thokozile, involves finding quadratic functions of the form $y = a(x - p)^2 + q$ (where a is ±1 only) using translations and reflections. The analysis (see for example Ndumiso's focus in line 261**)** suggests that the students view the quadratic function structurally as they consider how the graphs "shift" up and down. However, the students' ways of acting mathematically constrain the productive use of this method. Firstly, in making a link to other parts of her course, Thokozile is constrained by not identifying the course method as only applicable to a certain class of functions of the form $y = a(x - p)^2 + q$ and hence assumes that $a = \pm 1$ in her formula. Secondly, the students pronounce possible quadratic formulae verbally, from left to right, as in Thokozile's "minus x squared plus 4". Such verbal pronouncements constrain the students' attention to the appropriate use of symbols; in this case the students interpret Thokozile's pronouncement as $-x^2 + 4$, when it could be written as $-(x^2 + 4)$. Thirdly, Kelsa's evaluation of Thokozile's expression "$-x^2 + 4$" involves her substituting the point $x = 2$ into the graph to test whether a y-value of 8 is obtained, as shown in line 276 (for ease of reading we use symbols in the brackets {…} to represent verbal pronouncements):

276	Kelsa:	No it doesn't↑ … minus 2 squared plus 4 gives you <u>zero</u> $\{-2^2 + 4 = 0\}$
277	Thokozile:	Minus 2 squared $\{-2^2\}$ ((*Looking across briefly at Kelsa*)) the x is ... ... this minus is <u>here</u> ((*Pointing to her equation*)) ja minus x squared $\{-x^2\}$
278	Kelsa:	So why don't you just say x squared plus ... ... plus 4 $\{x^2 + 4\}$? … cause your <u>minus</u> putting your <u>minus</u> there means negative 1 times x squared $\{-1 \times x^2\}$ … … so you must just make it x squared plus 4 $\{x^2 + 4\}$↑

By providing a way of evaluating formulae, Kelsa identifies herself and is identified by the other students as an authority in first-year undergraduate mathematics. Her way of evaluating becomes the valued way in the group, as used by Thokozile in line 277. Kelsa also uses her method of substitution to adapt the proposed formulae, suggested by her repeated use of "just" in line 278. Kelsa's use of substitution for both evaluation and construction of formulae points to an operational view of the quadratic function.

After an ongoing struggle with the question, Ndumiso and Lwazi enlist the help of the tutor, who begins by re-establishing the link to school mathematics ("something uhm like … you did at school"). Lwazi and Ndumiso pronounce their versions of the school formulae simultaneously in lines 477 and 478 (overlapping text in square brackets):

477	Lwazi:	Yes we <u>did</u> but I forgot the <u>formula</u> there… it's got an <u>*a*</u> it's got an <u>*l*</u> … [you <u>know</u> what I'm talking about it's got a <u>bracket</u> ((*Gesturing in the air, looking at the Tutor as he speaks*))]
478	Ndumiso:	[It's y equals to a x minus x_1 x minus x_2 $\{y = a(x - x_1)(x - x_2)\}$↑ ((*Nodding his head in time as he says each term*))]
479	Lwazi:	No ((*Shaking his* head)) it's not [[it's]]
480	Tutor:	[[YES ((*Looking at Ndumiso*))]]
481	Ndumiso:	<u>Exactly</u>
482	Tutor:	What is x_1? ((*Pointing at Ndumiso*))
483	Ndumiso:	x_1 is going to be your first intercept↑ ((*Pointing to something on his graph*))

The Tutor's validation of Ndumiso's formula $y = a(x - x_1)(x - x_2)$ in line 480, an evaluation that resides in his personal authority rather than a mathematical argument, together with the absence of further attention to Lwazi's formula, constrains any exploration of links between the two formulae. Having endorsed one formula from school mathematics, the tutor has a further role to play in making links between the symbols in the formula $y = a(x - x_1)(x - x_2)$, different points on the graph and different sketches of parabola graphs, as he begins to do in discussion with Ndumiso in line 482. This intervention by the tutor enables Ndumiso to produce a correct solution, who in turn assists Kelsa on her request; she only identifies Ndumiso as an authority after validation from the tutor. However, Lwazi and Thokozile do not identify with Ndumiso's formula. Lwazi starts to use it, but skips steps to the correct answer and continues debating possible formulae with Ndumiso. Thokozile does not produce any written work, but proceeds to use the correct formula in the next question.

DISCUSSION

This analysis points to the interplay between students' ways of talking about and looking operationally and structurally at the quadratic function, in particular how these ways may enable or constrain their action and how they adopt different ways of talking and looking within the same problem. This interplay talks to the psychological research on structural and operational conceptions. For example, initially, the students' identification of general formulae suggests a structural view of these formulae as representing families of functions (Sfard & Linchevski, 1994). Yet they may not be invoking a structural view (Moschkovich, Schoenfeld & Arcavi, 1993), suggested by their not identifying the three school formulae as equivalent and looking operationally at the formulae as representing different computational processes (Sfard, 1992). Yet their subsequent drawing and gesturing of transformations of the parabola graph suggests a structural view of the quadratic function (Moschkovich, Schoenfeld & Arcavi, 1993). Throughout the students' way of evaluating their proposed formulae using substitution suggests an operational view of the function (Sfard, 1991).

The analysis suggests, however, that the socio-political perspective of mathematical practice allows the researcher to view student action at the micro-level of the classroom as more than "an intricate interplay between operational and structural conceptions" (Sfard, 1991, p.1) that are located in the mind of the individual student. Rather, it is a complex interplay of mathematical, discursive, social and political ways of acting mathematically. The students' ways of talking and looking operationally and structurally interact with their ways of making links (between practices, within the course, and between mathematical representations), their ways of endorsing arguments and evaluating, their ways of talking, their ways of identifying themselves and others, and their ways of interacting socio-politically. The framework we have presented also has the potential to link the classroom action to the wider socio-political space in which it is located, but this is the topic of a further paper.

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STUDENTS' UNDERSTANDING OF GEOMETRIC PROPERTIES EXPERIENCED IN A DYNAMIC GEOMETRY ENVIRONMENT

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The study presented in this paper is intended to identify students' qualitatively different ways of seeing parallel lines in a dynamic geometry environment. The ability to discern and simultaneously focus on critical aspects of a geometric figure is considered to be essential for meaningful geometry learning. The manipulation of pre-constructed dynamic geometric figures allows more tangible yet challenging variations to be experienced. Through task-based interviews with secondary two students, we explore the nature of their focus of attention and shift in attention while varying a geometric figure with dragging to generate examples with certain properties.

INTRODUCTION

There has been much qualitative research done on dynamic geometry software investigating how this virtual environment can change the perception of mathematics and doing mathematics (in particular geometry) in hope of enriching the pedagogical practices in mathematics classroom (see for example, Jones et al. (2000);Healy et al. (2002); Kadunz (2002); Lopez-Real and Leung, 2006). A key feature of a dynamic geometry environment is its ability to visually represent geometrical invariants constructed by a learner amidst simultaneous variations induced by dragging activities. The variation visualized from the drag-mode in dynamic geometry may give students a new reasoning pattern that diverges from the traditional deductive thinking. In particular, the pseudo-accurate representation of geometrical objects and measurements under dragging offers a confluence of simultaneities that could bring about discernments which might be different from a static paper-pencil environment (Sinclair, 2004).

When the drag mode acts on a figure in a dynamic geometry environment, the figure undergoes transformations in a domain in which the dual nature of mathematical object (Sfard, 1991) can be "lived out". On the one hand, a learner operates on a dynamic geometry object via dragging and the feedback process might lead to concept formation. On the other hand, a learner constructs (or manipulates) a dynamic geometry object having prescribed geometrical structure representing a mathematical concept. The juxtaposition of these two cognitive learner activities is an epistemic essence of a dynamic geometry environment.

How one experiences the world of dynamic geometry will determine the kind of knowledge that s/he gains in it. Dynamic geometry is rooted in variation in its design. It is a natural experimental ground to experience variation since it has built-in mechanisms that enable generation (via intelligent construction and dragging by a learner) of various qualitatively different ways of literally seeing geometrical phenomena in action. This is in harmony with the phenomenographic premise that learning and awareness are interpreted under variation (Marton and Booth, 1997). Thus a phenomenographic approach to do research in dynamic geometry might open up a channel to understanding the kind of geometrical knowledge that can be acquired in such an environment.

A DYNAMIC GEOMETRY TASK ENVIRONMENT

We have been studying students' work on dynamic geometry (DG) tasks which require students to drag one or two free points in a pre-constructed geometric figure in order to produce an example with certain properties or relations. The task we chose to discuss in this paper is shown in Figure 1. A quadrilateral with a freely movable vertex D and 3 other fixed vertices (A, B and C) is given. Some measurements of angles and sides are also displayed. Students are asked to drag the point D to make a quadrilateral with a pair of parallel sides. We are interested in students' choices through their dragging to vary the figure. The key question is what geometric elements and relations do they focus on when thinking about parallel lines in this DG environment. Furthermore, how might different ways of seeing and interpreting a geometric figure co-exist or dominate in different stages of the process of varying a geometric figure?

In the early stage of our study, we have collected responses to this type of DG tasks from about 1589 students in Secondary 1 to Secondary 4 (equivalent to Grade 7 to Grade 10), chosen from 11 schools in Hong Kong. Through an online platform, students' results of dragging were submitted and recorded.

In figure 1, a scatter plot of translucent black dots is included to indicate choices of position of D collected from those students in the early stage of study. The findings suggest that students' choices were preferential to prototypical cases like a parallelogram or isosceles trapezium. It is noteworthy that choices merely associated with specific properties are also common (such as quadrilateral with a pair of equal opposite angles but no parallel sides).

THE STUDY

In the second stage of the study, which is the focus of this paper, task-based interviews were conducted in order to understand how individual students discern and relate geometric elements in this dynamic geometry task. 6 schools of varying academic standards were invited to take part in this stage of the study. In each school, 2 female and 2 male students from secondary 2 were invited to individual interviews of 30 minutes. They were students with average academic performance recommended by their teachers for this study. These schools and students had not taken part in the first stage of the study.

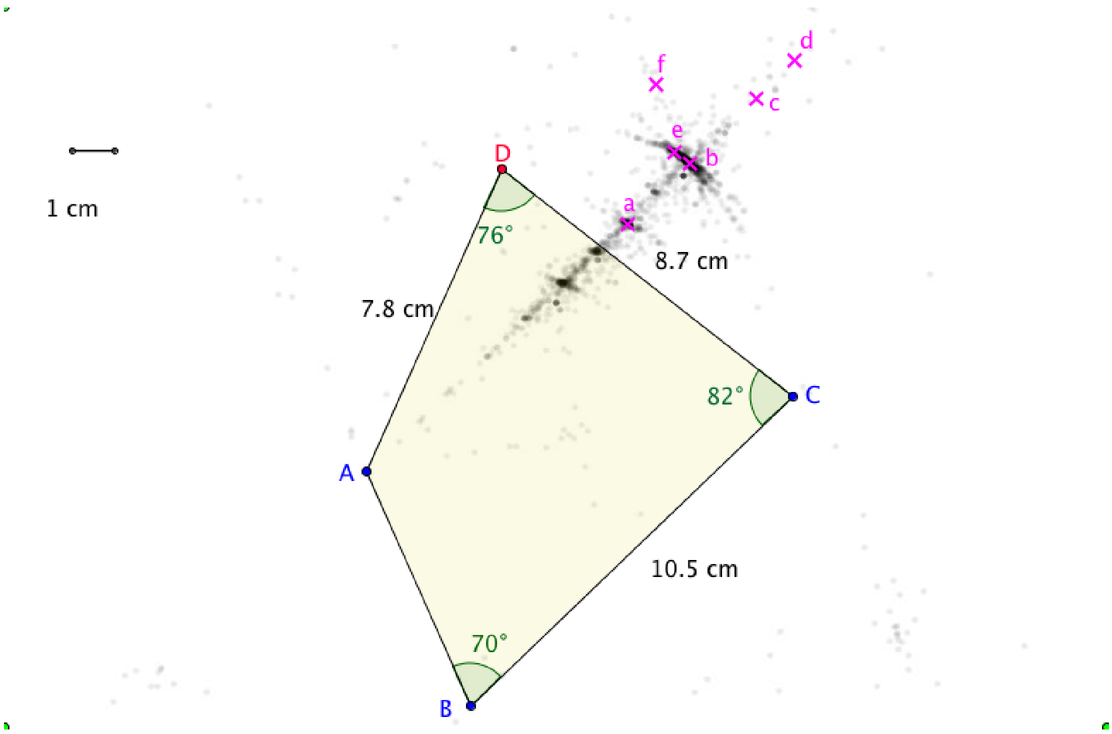

Figure 1 This figure shows (i) the selected task given a quadrilateral with a movable vertex D (ii) scatter plot with translucent black dots representing choices of D from 1589 participants in stage 1 of the study (iii) various positions (a to f) for point D chosen by Sam and discussed during an interview (refer to a latter section in the paper)

Each interview lasted about 30 minutes. The student completed the dynamic geometry task in front of a notebook computer and discussed his work with the interviewer. The interviews were recorded with video camera and screen video capturing programme.

In each interview, the following prompts and questions were set to outline the discussion.

- Student worked on the selected task. S/he had to read and interpret the question on the screen and respond directly with action through the mouse.
- Student signalled when considered the task finished. S/he then described in his/her word what s/he was required to do and in what way the resulting figure satisfied the requirements.
- The student was particularly asked to clarify which sides were parallel and how s/he knew this. S/he might be asked to tell explicitly which parts of the figure s/he was focussing on.
- Based on this first figure produced, s/he would be asked to consider other possible positions of vertex D keeping any pair of sides parallel.
- If other positions of D could be found, s/he would explain in each case how s/he knew there were parallel lines.

RESULTS

The following category of descriptions (a phenomenographic perspective) summarises those explanations given by the students in all the interviews during different stages of talking about parallel lines described above. They are considered as different qualitative ways of seeing parallel lines in the dynamic figures articulated by the students. In particular, they include ways of relating different parts of the figure to the parallel property.

- *Visual Judgement:* Parallel lines are recognised just because they looks parallel. They may also be described as 'almost parallel', 'more or less parallel', etc. In such cases, students may be aware of the need to determine more accurately by other means. They may point with fingers over the screen how the lines can be extended. Some students making merely visual judgement do not notice any relevance with the given measurements of angles and sides on the screen.
- *Prototypical Shapes:* Parallel sides are recognised mainly because the figure is considered as a special quadrilateral, particularly a parallelogram or (isosceles) trapezium. In some cases, the figure is considered as part of a rectangle or square and thus should have parallel sides. Some of these special quadrilaterals are deliberately made from the beginning. Others may be recognised in the process of free dragging.
- *Relating Parts:* Students can state clearly which parts of the figure and their relation are being considered in determining parallel lines. This may or may not be supported by knowledge about geometric theorems.
  - *Equal Angles:* Most of these considerations of angles involve the pair of opposite angles shown. A few cases focus on the pair of adjacent angles being equal. These are usually associated with recognition of prototypical shapes. The most common one is the description of equal opposite angles linking with identification of a parallelogram. The case of trapezium may be associated with a pair of equal adjacent angles.
  - *Equal Sides:* Primary focus is put on the lengths of sides and their equality. Similar to the previous case, such description is often associated with identification of parallelogram. In some cases, such concern about equal lengths is reflected in their failure in altering length of one side while keeping a parallel pair.
  - *Interior Angles:* Most students, when noticing a pair of interior angles, can state clearly the condition for being supplementary. These students are usually aware of the relevant theorem learnt in school. In a few cases, while students can notice in individual figures such condition of supplementary interior angles; they cannot predict where the vertex can probably be placed to make parallel sides.
- *Relating Changes:* Students judge the parallel lines or their formation mainly by reasoning about the 'changes' in the angles or sides. For example, one

student keeps record of the angle measurements during dragging and argues about how the lines can be kept parallel by referring to these changes of angle size. This happens only in a small number of cases but is quite a significantly different kind of argument from the rest and may have special influence from the dynamic features of the task.

The distinctions above present some major ways of identifying parallel lines by the students in different figures. These are useful for distinguishing students' seeing and reasoning in different stages of doing the tasks and responding to interviewer's questions and prompts. However, this is not the same as classifying students as having particular ways of understanding parallel properties throughout the tasks. Indeed, another major observation from these interviews is some sorts of mixing and shifts among these various ways of seeing the parallel lines. These shifts can be commonly found when the attention is moved from one figure to another while examining alternative figures created or in the process of dragging.

The shifts identified may be obvious to different extents. Some obvious changes occur when the dragging comes close to the special case of a parallelogram in this task where different ways of interpreting parallel sides may apply.

The following case of Sam (pseudo-name) is chosen to illustrate in details how this interplay among different ways of seeing is identified. His choices of placing D to give various quadrilaterals during the interview are shown in Figure 1 (marked as positions [a] to [f]).

INTERVIEW WITH SAM

Visual Judgement and Prototypical Shapes

Sam first got a pair of parallel sides by making 2 adjacent right angles (position [a]). He saw the upper part of the figure as part of a square (later considered as rectangle) that he imagined. He sketched with his finger how a rectangle could be formed by dropping a perpendicular from A to BC. He did not explain the existence of parallel lines from any properties like the right angles created. What he did was describing his imagined rectangle and referring to the fact that rectangles have parallel sides.

Relating Parts and Prototypical Shapes

He was then asked to consider other possible positions of D that could still keep the same pair of parallel sides. He could not think of any alternative. He was then suggested to have a try and started further exploration with dragging. Then he realised that a parallelogram can be made. In the second figure (position [b]), he recognised that both pairs of opposite sides as parallel. To explain how he knew there were parallel lines, he tried to formulate a proof but failed to state clearly any critical relation shown in the figure. He seemed to assume that some explanation could be made by relating the angles.

Sam: I think this is a parallelogram.

Interviewer: Is there any reason why you think it is a parallelogram? [Long pause for about 1.5 min] A brief description will be fine. Since you stop here, and you think it is a parallelogram, is there any information that makes you think so?

Sam: I think … quadrilateral's interior angles have a sum of 360 degrees. So, with the three angles adding together, this angle [A] is … 110 degrees.

Interviewer: So, you know the angle A is 110 degrees. Then, what …[Noticing Sam is still thinking; pause for about 1 min] What are you thinking now?

Sam: I'm thinking how to prove a parallelogram.

Relating Changes and Relating Parts

He was then suggested to put aside the idea of proof and predict where else the vertex D can be put. Different from his previous uncertainty in locating alternative positions of D, he could now show with his finger how to move D away from A by keeping DA in the same direction. He thus made 2 more figures (positions [c] and [d]), in which he could explain clearly with the condition of supplementary interior angles. He also admitted that the idea of interior angles was not considered in the previous cases.

Interviewer: Now which sides do you think are parallel?

Sam: AD and BC.

Interviewer: AD and BC. Looks parallel?

Sam: Yes.

Interviewer: Are you sure they are parallel?

Sam: Not really sure.

Interviewer: Do you think of any ways to be more certain? According to what you see now.

Sam: This angle [C] adding with this angle [D] …

Interviewer: What would you get?

Sam: Adding to 180 degrees.

He continued to make another pair of sides parallel. This was supposed to be done by returning to position [b], but instead, he reached position [e], which resulted in almost (yet not exactly) a parallelogram. He noticed that angle D and B were equal in this case while angle C and angle A were unequal. He explained that in this figure AB and DC were parallel because of the equal opposite angles B and D. Meanwhile, he clarified that the other pair of sides were not parallel because the other pair of opposite angles were not equal. While still focusing on the pair of parallel sides AB and DC, he was asked again to consider alternative positions of D. He considered this impossible initially. He was then reminded of the previous exploration of extending AD along the direction of BC. With such prompt, he was able to describe how DC could be extended in the same direction and reached position [f] eventually. To explain how AB and DC were parallel in this figure, his reverted to the condition of supplementary interior angles (A and D in this case). He was also aware of the difference with the previous explanation based on equal opposite angles.

Sam: AB and DC are parallel.

Interviewer: Any reason?

Sam: Angle A and angle D have a sum of 180.

…

Interviewer: Why you suddenly look at sum of these angles? It seems to be different from what you said before.

Sam: I do consider the sum of angles A and D … and if they are 180 degrees, the two lines are parallel. … What I did earlier was … if both angles were 70 degrees, the lines were parallel.

Interviewer: Now, which way of saying is more reasonable?

Sam: ‘Angle A adding angle D’ is more reasonable.

Interviewer: Why?

Sam: Because angle D is not necessarily kept at 70 degrees to have AB parallel to DC. But if angle D adding angle A gives 180 degrees, then AB and DC are always parallel.

DISCUSSION

The case of Sam is chosen for discussion because of his relatively obvious shift in attention and contrast among various ways of seeing the parallel lines.

There were moments in which justification of parallel sides relied on visual recognition of prototypical shapes. This happened at the beginning when Sam produced a figure with 2 right angles as parts of a rectangle. This reliance was also apparent in the later case of getting a parallelogram. Sam’s case was a critical example because in it we could see shifting of attention between different ways of seeing parallel sides. For example, consideration of supplementary interior angles became a dominant reasoning when Sam extended to more general cases (positions c, d, f) but it soon turned into focusing on equal opposite angles when the shape reverted to almost a parallelogram.

One important feature of the task in the interview was continuously encouraging participants to consider more and more examples satisfying the condition specified. This corresponds to a useful pedagogic approach advocated by Mason and Watson (2005) in opening up dimensions of possible variation (DofPV) and extending ranges of permissible change (RofPCh). In the case of Sam, we noticed how a learner may struggle with this process. It did not come naturally to him that the figure could be varied in some ways in order to generate more examples. This became possible for him only when he was prompted or really explored with further dragging. Moreover, even when he could produce examples in different ways, it was not immediately obvious to him how a coherent explanation could be applied to all these cases. In other words, these individual examples did not constitute DofPV yet. Though, near the end, he became aware of the difference in his explanations and could compare them.

More work need to be done to study how focus and shifting of attention open up DofPV and extend RofPCh. According to phenomenography, learning is regarded as improved discernment and simultaneous awareness of critical aspects of the object of learning (Marton and Booth 1997). This task-based dynamic geometry environment may provide a platform for us to see what students could discern and keep attention at (hence differentiating) while interacting with dynamic geometrical figures that are inherited with possibilities of undifferentiated variation.

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THE MEDIATION OF EMBODIED SYMBOL ON COMBINATORIAL THINKING

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This research investigated if the embodied symbol using a turtle metaphor in a microworld environment works as a cognitive tool to mediate the learning of Combinatorics. It was found that students were able to not only count the number of cases systematically by using the embodied symbols in a situated problem regarding Permutation and Combination, but also find the rules and infer a concept of Combination through the activities manipulating the symbols. Therefore, we concluded that the embodied symbol, as a bridge that connects learners' concrete experiences with abstract mathematical concepts, can be applied to introduction of various mathematical concepts as well as a Combinatorics concept.

INTRODUCTION

Combinatorics is an essential component of discrete mathematics and it has an important role to play in school mathematics. Kaput(1970) suggested that Combinatorics can be used to train pupils in enumeration, making conjectures, generalization and systematic thinking. Combinatorial capacity is a fundamental and very important thing to be acquired before learning Probability. However, most students regard it abstract and difficult because Permutation or Combination is approached with a formula or calculation in school education(Batanero et al, 1997). On this, while Fischbein and Gazit(1988) stated teaching a Permutation formula actually disturbs learner's intuitive experiential strategy on Combinatorics, they found out a 10-year-old child who is not capable of formal thinking may understand a combinatorial concept with a help of tree diagram.

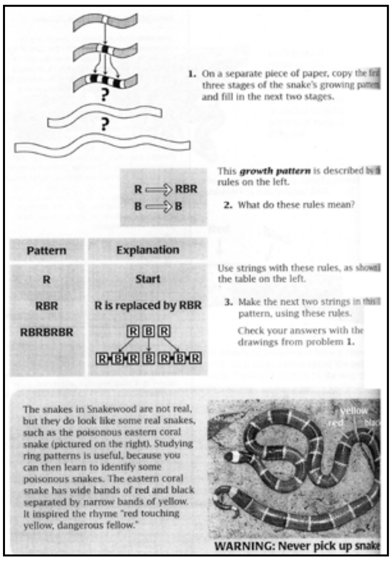

Fig 1: string of symbols

Wilensky(1991) pointed out that "formal is often abstract because we haven't yet constructed the connections that will concretize it." In addition, he explained that "concreteness is not a property of an object but rather a property of a person's relationship to an object." That is, it may not be considered abstract if a new mathematical concept is introduced by being connected to learner's concrete experience. From an educational perspective, therefore, how to provide a bridge that connects an abstract concept with a concrete experience should be considered very critical. Then, how do we provide learners with the bridge that connects their personal experiences with very dry and abstract combinatorial concepts?

Math in Context(Britanica, 2003), the textbook written by Freudenthal Research Institute, introduces the mathematical concept of recursion in the context where a string of symbols composed of R(Red) and B(Black), representing rattlesnake's colour pattern, is sequentially grown shown in Fig 1. By writing a string of symbols, learners may go through what Davis(1984) calls a 'visually moderated sequence', which means considering the symbols, manipulating, considering the new symbols, manipulating, ..., and so on, until a solution is found. Unlike a ready-made mathematical formula, a string of symbols is semi-formal. Hence, manipulating a string of symbols helps learners conjecture, observe, reflect, and explore mathematical ideas. This leads to a cognitive tool which helps learners represent and construct an object easily and effectively.

This research applies a string of symbols introduced in *Math in Context* to an introduction of Permutation and Combination concepts by merging with Logo's turtle metaphor in a technology-based microworld. By doing so, we analyse the power and educational implications of a string of symbols as a bridge that connects concreteness with abstraction.

MICROWORLD AND COMBINATORIAL THINKING

Noss & Holyes(1996) have emphasized the mediating role of computer. They propose the notion of *situated abstraction*, as a way of describing how learners can develop mathematical meanings.

> "We intend by the term *situated abstraction* to describe how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (p.122)

Microworld is a 'transitional object' which interplays between a directly manipulable concrete subject and a formal abstract mathematical concept(Hoyles, 1993). Wilensky(1991) suggested that learners engaged in microworld-based activities would be "abstracting within, not away from, the situation."

Pratt(2000) stated the microworld environment is a symbolic mediating tool, explaining interaction that occurs between learners' informal intuition and Chance-Maker microworld when understanding number of cases occurred in playing a dice game. The concrete activity experienced in the microworld environment is a useful tool to help learners make sense of the sum of two faces of dice. Unlike a ready-made formal and abstract formula, such a concrete activity makes learners approach a mathematical concept which is embodied in their body, language and cognitive nature. The activity that systematically counts the distribution which comprises sample space becomes a bridge that connects empirical probability with theoretical probability(Abrahamson, 2008). This research will use a microworld as a tool to mediate combinatorial learning which emphasizes a counting task and learners' intuition.

REPRESENTATION AND EMBODIED SYMBOLS

External representation is an important mediating tool, causes mental representation and meaning construction. Representation is an "idea" which has a certain concept in that a mathematical concept should be expressed in any ways in order to represent it in one's mind(Davis, 1984).

Godino et al(2005) stated that there are symbolic notation such as $C_{4,2}$ and visual representation such as tree diagram by analysing students' semiotics used in combinatorial problem solving. As symbols and visuals are the most representative ways to represent mathematical objects, each representation takes a mutually beneficial role to help learner understand different types of information. A string of symbols introduced in *Math in Context* is semi-symbolic, being able to be manipulated and analysed, and at the same time semi-visual, being able to be visualized. Combining these two representations is indispensable for constructing full meaning on mathematical objects(Sacristán, 1997).

A string of symbols has been used in the previous Combinatorics education. By using symbols like H and T, for example, we express it HHHT for the case that three heads and one tail come out when flipping four coins. That is, these symbols like H and T are the tool to represent the situation easily as meaning head and tail of the coin respectively. Taking a step forward from here, a string of symbols introduced in Math in Context is a tool that can represent, visualize, and explore dynamically changing recursion in sequence. We implemented a string comprised of direction indicating symbols, L(Left), R(Right) and B(Back), in the microworld environment by applying the idea of a string of symbols to Logo's turtle metaphor. This is something embodied to learners physically and mentally by projecting learners themselves to the turtle on the screen and we will call it an "embodied symbol". The embodied symbol is expected to be a cognitive tool to concrete and elaborate learners' thought and help learners communicate with others.

METHOD

To observe the process of combinatorial problem solving which does not rely on any algorithms and the role of embodied symbols, this research was conducted to 73 students aged 12 to 13, who have learned basic concept of Probability, but have not learned formulas on Permutation and Combination. The problem situations were fully explained to the students through online video lectures, texts and experimental environment and we let students manipulate and experiment combinatorial situations on their own in the microworld environment.

The experiment was proceeded with a test conducting before and after learning the embodied symbols in the online environment. The problems in both pre- and post-test were designed in an identical pattern, but we encouraged the students to use the embodied symbols that they learned when they answered the problems in the post-test. The problems asking a concept of Permutation and Combination were presented by difficulty level as shown in Fig 2 and the students were supposed to answer number of cases in the context where a turtle slides down on skis from the top of the hill to the right or left and finally arrives at a certain point and explain how they get to their answer. While level 1 and 2 problems are binominal, using L(Left) and R(Right) on two dimensions, level 3 problem is trinomial, using L(Left), R(Right) and B(Back) on three dimensions. In addition, level 1 and 2 problems include a concept of ‘Combination’ by choosing a particular number of R or L out of all possible paths or a concept of ‘Combination with Repetition’ by choosing L or R by the number of paths as allowing repetition. Adding one more dimension, on the other hand, level 3 problem includes a concept of ‘Permutation’ which enumerates a total of three paths with three symbols. We asked the students to get number of cases with their own strategy in the pre-test since the students who participated in the experiment have not yet learned a concept or a formula of Permutation and Combination while we asked them to use the embodied symbols to get the answer in the post-test after learning such symbols. In the test results, we analyzed the correct-answer rate quantitatively and analyzed students’ reasoning process on their answer qualitatively.

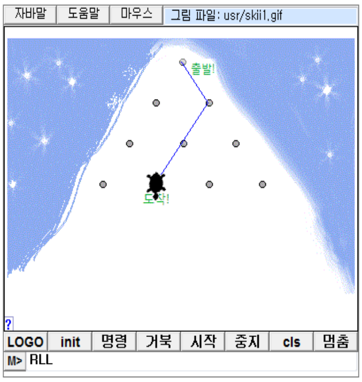

Fig 2a: level 1 problem

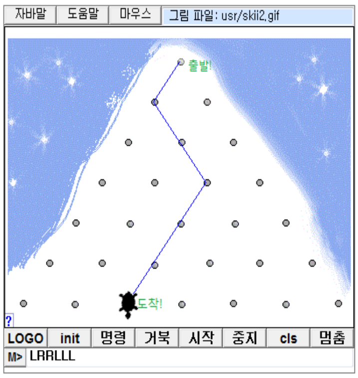

Fig 2b: level 2 problem

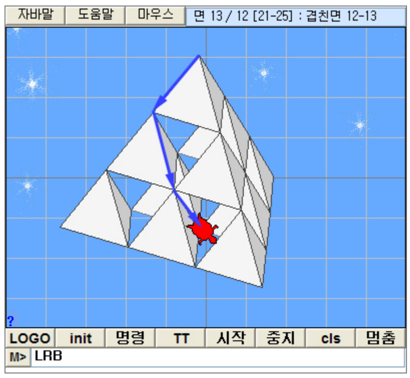

Fig 2c: level 3 problem

DATA ANALYSIS

We analyzed the pre- and post-test result to see if the embodied symbols work as a mediating tool to solve combinatorial problems. In level 1 problem, the correct-answer rates in the pre- and post-test were 46.3% and 62.9% respectively; thus the correct-answer rate in the post-test was increased compared to the one in the pre-test. We may presume the correct-answer rate was increased because students, by using a symbol of L and R, felt more concrete towards the vague problem asking number of cases.

However, level 2 problem did not show a big difference as the correct-answer rates in the pre- and post-test showed 24.1% and 26.6% respectively. On this, expanding from the problem simply asking number of cases, we asked students in the post-test if they were able to discover the rules related to a concept of Combination. Approximately 70% of the students discovered the rules, 4 Ls and 2 Rs came out in common, in their enumeration of symbols in the post-test and justified their reasoning. That is, the students were able to reach out to a concept of Combination which number of cases is the same when choosing 4 Ls or 2 Rs out of a string of 6 symbols through the concrete experience enumerating the paths which a turtle skis down the hill with the symbols.

The followings are two episodes about two students who received a help of the embodied symbols when solving a combinatorial problem. First, a student, Ye-won answered 5,040 in the pre-test in level 2 problem, but she described in the post-test "a string of 4 Ls and 2 Rs should be used. On arriving point's basis, 2 dots are on the left side and 4 dots are on the right side in the bottom line. Turning it the other way, it is correct that 4 Ls and 2 Rs are used." From this, we were able to verify that Ye-won reasoned out the number of Ls and Rs with the location of the dot in the bottom line. Another student, Han-woul, answered 57 by counting all possible paths one by one in the pre-test, but after learning, he described in the post-test, "The turtle can arrive at the bottom point with 4 Ls and 2 Rs. For example, in case of arriving at the second left point, 5 Ls and 1 R are required. Out of the 6 symbols, 1 L is removed as L moves to the left one time and 1 R is added. Therefore, the total number of paths is 15 with LLLLRR, LLLRLR, LLLRRL, LLRLLR, LLRLRL, LLRRLL, LRLLLR, LRLLRL, LRLRLL, LRRLLL, RLLLLR, RLLLRL, RLLRLL, RLRLLL, and RRLLLL." Since it was too difficult to come up with 15 cases at once, Han-woul was able to count number of cases systematically by switching L with R from the previous stage which he was able to think easily. Although many students felt difficult in counting all 15 cases accurately, we may interpret that students' concrete experience enumerating the embodied symbols actually worked as a bridge that connects to the abstract concept of combination.

Finally, level 3 problem asked number of cases that a turtle arrives at the center point of the bottom (1^{st} floor) from the peak of the regular tetrahedron shape of mountain(3^{rd} floor). The correct-answer rate in the pre-test was as low as 20.35% and the rate went up to 33.3% in the post-test after learning the embodied symbols of L, R and B (Left, Right and Back). It is assumed that the extended problem situation from binomial to trinomial and limitation of visualization led to the low correct-answer rate in both pre- and post-test. Given that the correct answer is 6, however, approximately 40% of students incorrectly answered a nonsensically large number above 20 in the pre-test, but only 5.56% of students answered such a big number above 20 in the post-test. This result confirms us that those students who did not have any clue on how to approach the problem before learning the usage of embodied symbols and therefore ended up with a nonsensically large numbers got to be more capable of presuming not exactly correct, but approximate values by enumerating number of cases with the symbols.

	Level 1	Level 2	Level 3
Pre-test	46.3%	24.1%	20.35%
Post-test	62.9%	26.6%	33.33%

Table 1: result

In the following, we looked into some of students' explanation to see what kinds of misconception appeared by analysing student's reasoning in the pre-test of level 3 problem.

Student A: There are 3 ways to go down from the 3rd floor and 9 ways to go down from the 2nd floor. To get to the arrival point, the turtle needs to go down to the 3 triangles which are connected to the arrival point and this narrows down to 6 ways. There are 3 ways to go down from the 2nd floor to the 1st floor. Therefore, the total number of paths to reach the arrival point is 54, as 3*6*3 equals to 54.

Student B: A regular tetrahedron has 4 vertices and there are 6 regular tetrahedrons; therefore 24, 6*4.

Student C: The answer is 486 because the number of all cases possible is 18 on the 1st floor, 9 on the 2nd floor, and 3 on the 3rd floor. Therefore, 18*9*3 equals to 486.

Student D: 1st floor: 3, 2nd floor: 6, 3rd floor: 3. 3*6*3=54. Likewise, I solved it the number of all cases possible on each floor.

Student E: 3 ways on the 3rd floor, 6 ways adding the number of ways on the 2nd floor, and 24 adding the number of ways on 1st floor. The sum is 24.

Student F: The answer is 21. 3 edges on the first regular tetrahedron + 9 edges on the second 3 regular tetrahedrons + 9 edges on the last 3 regular tetrahedrons .

Student A and Student D counted number of cases repetitively as counting 3 ways to go down from the 3rd floor and 6 ways to go down from the 2nd floor while the correct number of cases to go down from 3rd to 2nd floor is 6. In addition, they unnecessarily multiplied number of cases on each floor. Student C also multiplied all number of cases occurred on each floor. Student E counted correctly number of cases on the 3rd and 2nd floor, but did not count number of cases on the 1st floor systematically. Student F counted the number of cases on each floor systematically, but he made a mistake by adding all numbers on each floor. Student B made an error by multiplying the number of tetrahedron by the number of corner irrelevantly.

Based on students' explanations, we may classify these students into a certain type of error as follows:

- Error 1: Misusing of a multiplication rule ------------- Student A, B, C, D
- Error 2: Applying incorrect addition rules -------------------- Student E, F
- Error 3: Repetitively counting ------------------------------ Student A, D, F
- Error 4: Unsystematic counting in a certain part ---------------- Student E
- Error 5: Irrelevantly applying the number of solids or edges -- Student B

We observed that high percentage of students who provided a wrong answer actually applied multiplication or addition rules when they were not supposed to do so or counted some numbers repetitively. This tells us that students tend to solve the problem according to algorithm, rather than enumerating sample space systematically. We were able to conclude that enumerating number of cases with symbols can be a proper pedagogic prescription to those students.

Finally, the students responded that they used the embodied symbols easily and fount it useful to the question asking if they feel comfortable and useful in using these symbols affectively. "I had to count all number of cases one by one in my mind in the pre-test, but it felt easier when I used the symbols in the post-test. Also this looks good to me and the other people since it can be written simply. It is much better if I use symbols because I do not forget it well and also can find the rules with ease.", "I did not get confused by using symbols as solving the problem. Also this helped me find the rules in the number of cases." Such answers confirm that enumerating number of cases by using the embodied symbols works as not only a cognitive tool to solve combinatorial problems and but also a bridge that connects abstraction with concretion.

CONCLUSION & DISCUSSION

This research investigated if the embodied symbol using a turtle metaphor in a microworld environment works as a cognitive tool to mediate the learning of Combinatorics. As a result of tests, the correct-answer rate in level 1 and 3 problems increased distinctively by using the embodies symbols while no distinctive increase was shown in level 2 problem. Given that many students discovered the rules and inferred a concept of Combination in level 2 problem, however, we may say learners' concrete activity that manipulates symbols was used as a vehicle for understanding an abstract mathematical concept of Combination. Furthermore, students were able to conjecture approximate values close to the correct answer in a complicated trinomial situation of level 3 problem after introducing the embodied symbols while most of the students provide wrong answers, far away from the correct one, by using meaningless algorithms before learning such symbols. Along with it, most students evaluated the embodied symbols positively in an affective perspective.

This research confirmed that the embodied symbols, as a tool that can be easily used by learners, are useful for solving a basic combinatorial problem. We expect, furthermore, the embodied symbols can be applied to the studies of various mathematical concepts. For example, Fig 3 shows a recursion problem situation presented in *Math in Context* – growth occurs by three times as R is replaced by RWR and W is replaced by W - which is transferred to the microworld environment. In here, students accept a mathematical concept of recursion concrete through the visual images represented as a result of manipulating symbols and receiving feedback. Thus, microworld based on the embodied symbols can

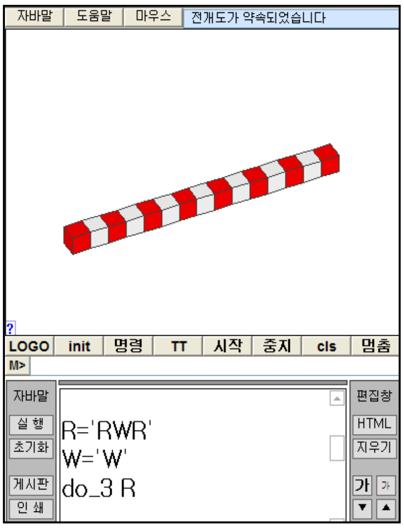

Fig 3: symbol based microworld for recursion

be a "Mathematics playground" where learners observe, experiment, conjecture various mathematical situations from learners' intuition and generate a new mathematical concept through 'what if questions'.

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THE EFFECT OF WORKED-OUT EXAMPLES WITH PRACTICE ON COMPREHENDING GEOMETRY PROOF

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This study examines how reading comprehension of geometry proof is influenced by worked-out examples. Participants are 85 Taiwanese students from grade eight who are novices at deductive proof in geometry. The result shows that average and higher level students can obtain half or more scores in reading comprehension posttest, but the average and lower level students retrogress significantly in delayed test. The main reasons for students' failures in writing proof are: (1) Students disregard the whole logical structure of proof only by repeating the steps from worked-out examples; (2) Students fail to apply proper related knowledge in proving.

BACKGROUND

Geometry proof is a complicated cognitive process of problem solving. Schoenfeld (1985) presented the view that both understanding and teaching mathematics could provide some prescriptions for students' performance in problem-solving. Therefore, we investigate into students' process of learning proofs by using this perspective. Problem solving generally means the process of finding the solutions. Abilities to deal with problems are different for diverse levels of students. For example, novices tend to focus more on the surface features of a problem (Chi, Glaser, & Rees, 1982; Atkinson & Renkl, 2007). Worked-out example is proven to be the most effective way to learn, learners to gain more knowledge during the initial stage of learning is reported to be helpful (Renkl, 2005). Hence, worked-out example is an important learning tool with widespread usage in teaching mathematics and textbooks. Atkinson, Derry, Renkl, and Wortham (2000) defined that worked-out examples typically present solutions in a step-by-step fashion including auxiliary representations of a given problem, such as diagrams. Students make fewer errors during the acquisition time and require less assistance from the teacher by using worked-out examples (Carroll, 1994); therefore, worked-out examples are more effective in learning principles and patterns on new topics.

Proving must be integrated with prerequisite knowledge, perceptive, discursive and operative apprehension (Duval, 1998) of geometrical figures. This complexity is one of the reasons for the students having difficulties when they learn the geometry proof at initial stage. Hence, the usage of assisting tools for understanding these complicated concepts must be considered. Worked-out examples are shown in different domains that they are an effective assistant tool. Deductive proof which is different from procedural knowledge may not be enough to promote students' schema construction by worked-out examples only. Gog, Kester, and Paas (2011) used three strategies - worked examples only, example-problem pairs, and problem-example pairs to

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 75-82. Taipei, Taiwan: PME.

compare with problem solving only in electrical circuits troubleshooting tasks. The result showed that worked examples only and example-problem pairs have higher learning outcomes compare to the other two strategies. Researches on worked-out examples show that worked-out examples are more effective for learning geometry proof by using different auxiliary elements (Reiss & Renkl, 2002; Reiss, Heinze, Renkl, & Groß, 2008); however, they are rarely analysed about learners' continuous effects and their proof-writing strategies. The following two questions are mainly concerned in this research:

(1) How do the different levels of students perform on reading comprehension posttest and delayed test by using worked-out examples with practice?

(2) What strategies should the students use in writing geometric proof after reading a worked-out example with similar structure?

THEORETICAL FRAMEWORK

Students spend less time and mental effort on acquisition knowledge when comparing worked-out examples with problem solving (Sweller, Van Merriënboer, & Paas, 1998). An effective design of worked-out example is presented by labeling or segmenting (Atkinson et al., 2000). Tso et al. (2011) showed that a segmented geometry proof could help either experts or novices to increase their reading willingness; however, it could decrease the task difficulty and the effort that devoted to the task by the readers. Catrambone (1996) used a label to show the mathematical meaning in probability questions, the result revealed that a label helped learners to form a sub-goal for the labeled steps.

Different strategies are suggested on how to improve novices' response to problem by the surface features when they are learning with examples. For examples, Nathan, Mertz, and Ryan (1994) showed that learners who used examples with self-explanation have greater test improvement for story-problem translation tasks. Reiss and Renkl (2002) proposed heuristic worked-out examples to develop similar ideas as mathematicians when they performed a proof. Reiss, Heinze, Renkl and Groß (2008) designed heuristic examples with self-explained that were connected with prerequisite knowledge, students with an insufficient understanding of proof were beneficial from this learning environment. Learners must spend more time to explore proving process when they use such strategies. Many researches showed good effects on procedural knowledge learning by using examples with practice in similar structure. It is worth to explore the positive effects on learning geometry proof by worked examples that require less learning time and easier to perform in classroom.

Proof is an important mathematical topic but hard to master. It requires the usage of propositions by definitions, axioms, theorems to conjecture or hypothesize, and presents in deductive form which is different from procedural knowledge. Duval (1998) defined that geometry proof had three levels of organization: micro, local, and global. They are distinct to each other by the numbers of premises and conclusions.

From this perspective, they also show different mastery levels in proving. In the intercept theorem (or Thales' theorem), it shows that when $\triangle ABC$ is a triangle, $\overline{DE}$ is a line parallel to $\overline{BC}$ that intersects $\overline{AB}$ and $\overline{AC}$ at point D and E respectively. The ratio of $\overline{AD}$ to $\overline{DB}$ and the ratio of $\overline{AE}$ to $\overline{EC}$ are equal. The theorem is generally proved by the ratios of the similar triangles. In Taiwanese mathematics textbooks, the theorem is proved by the area method. It is used to infer the related theorems of similar triangles. Hence, the proving of the theorem must be well understood before learning similar triangles. In general, the proving can be segmented in three parts by its mathematical structure. The first two parts are the same concept of areas of same height triangles which are proportional to their bases. The third part shows the concept of triangle with same base and equal height which have the equal area. These three parts are also defined in the local level of Duval's levels of organisation. They comprise the intercept theorem which is categorized in global level. Schoenfeld (1985) described the aspects of the knowledge base which was relevant to competent performance in a domain. The degree of knowledge can be used to describe the levels of students' understanding of the proof. Using the intercept theorem as an example, part of this aspect of the knowledge inventory is outlined in Table 1.

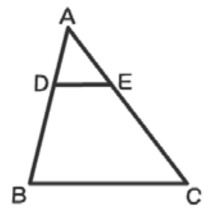
Degree of Knowledge of	Facts and	Procedures
Does the students : 1.know nothing about 2.know about the existence of, but nothing about the details 3.partially recall or suspect the details, but with little certainty 4.confidently believe	The distance between the parallel lines will be the same at all points. Triangles with same base and equal height have the equal area. Areas of same height triangles are proportional to their bases.	Drawing auxiliary line. Finding the correct triangles with proportional corresponding sides. Finding the correct triangles with equivalent area between the parallel lines.

Table 1: Part of the knowledge inventory for proving the intercept theorem

RESEARCH METHODS

Participants were 85 eighth-grade students (41 female and 44 male) who were novices of deductive proof. According to their last mathematics examination results at school, students were divided into three groups: lower level group ($0 \leq \text{score} \leq 28$; N=29), average level group ($29 \leq \text{score} \leq 58$; N=28), and upper level group ($59 \leq \text{score} \leq 100$; N=28). Students read the proofs on computer and the correspondence practices which were written on worksheet. They were able to understand the content of the proofs because they had learned the main concepts about the area of triangle and proportional relationships. The reading comprehension test was written by themselves after all learning procedures, except two students finished the entire task in 45 minutes. Before the students have learned the topic of intercept theorem, they did the delayed test which was the same as the posttest three months later.

The worked-out examples presented by multimedia are developed with Flash. Intercept theorem is segmented by the mathematical structure and Duval's deductive levels in four examples. They prove the following concepts under the condition of $\overline{DE}$ and $\overline{BC}$ are parallel:



(1) Area of $\triangle ADE$: Area of $\triangle DBE = \overline{AD} : \overline{DB}$;

(2) Area of $\triangle ADE$: Area of $\triangle DCE = \overline{AE} : \overline{EC}$;

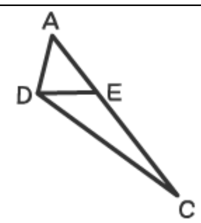
(3) Area of $\triangle DBE$ = Area of $\triangle DCE$;

(4) $\overline{AD} : \overline{DB} = \overline{AE} : \overline{EC}$ (The intercept theorem).

Students are suggested to finish a follow up practice with one dimension changed (figure showed in different position) structure of the examples before they read the next example. Four example-practice pairs are designed in learning process, one of the pairs is shown as figure 1 below:

【Example 2】

In the figure, point E lies on the segment $\overline{AC}$. Prove that the ratio of area of $\triangle ADE$ *to* area of $\triangle DCE$ and the ratio of $\overline{AE}$ to $\overline{EC}$ are equal.



【Practice 2】

In the figure, point W lies on the segment $\overline{XZ}$. Prove that the ratio of area of $\triangle XYW$ *to* area of $\triangle WYZ$ and the ratio of $\overline{XW}$ to $\overline{WZ}$ are equal.

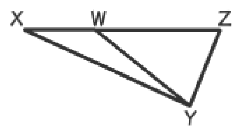


Figure 1: The problems used in example 2 and practice 2

The design of reading comprehension test is referred to the research by Tso et al. (2011). According to Duval's (1995) cognitive apprehensions (perceptual, operative, and discursive), three questions (4 sub-questions) in micro level are formed. They examine students' apprehensions in constructing figures, identifying figural elements, and recognizing the premises and conclusions. Two questions in local level which examine students' comprehension of single mathematical concept are chosen from the subconcepts of intercept theorem. A question in global level that examines the understanding of intercept theorem is asked. This question provides enough choices for novices when working on proving.

Cronbach's alpha as the measure of internal consistency reliability is 0.805. The comprehension test is 12 points in total, include micro (1 point), local (2 points) and global level questions (4 points). According to the degree of knowledge which proposed by Schoenfeld (1985), analysis of practices is divided into four levels to describe students' understanding of proof.

RESULTS

Table 2 shows the scores of students at different levels in reading comprehension posttest and delayed test.

Test	Lower third		Average third		Upper third	
	M	SD	M	SD	M	SD
Posttest	3.00	2.86	7.32	2.92	9.39	2.13
Delayed Test	2.15	2.71	5.29	3.00	8.86	2.70
p-value	0.042*		0.001**		0.298	

* $p < 0.05$, ** $p < 0.01$

Table 2: Students' performance in reading comprehension tests

From descriptive statistics, all students in delayed test have a receding trend. Instant effects have showed that average and higher level students can understand the context after they have learned. The result from paired sample t-test shows that average and lower level students regress significantly. Hence, only high level students are beneficial from learning in worked-out examples with practice for long-term learning effects. There are some evidences from analyzing the practices that students are lack of understanding of the core concepts. From different degree of knowledge described in table 1, table 3 shows the proportion of different degree of knowledge that students use for writing the intercept theorem.

Level	Description	Number (percentage)
1	No response.	6 (7.1%)
2	2a) Copied the same steps from examples.	1 (1.2%)
	2b) Tried to draw the auxiliary line but showed a great difficulty in proving.	2 (2.4%)
3	3a) Used the symbols of triangle elements which were written in the way of examples.	6 (7.1%)
	3b) At least one proportion expression that had not corresponded to the correct triangles.	22 (25.9%)
	3c) Wrote the correct proportion expression but had not showed the reason for triangles with equal area.	13 (15.3%)
	3d) Presented correct steps with minor mistake for proving.	13 (15.3%)

4	Wrote a correct solution.	22 (25.9%)

Table 3: Proportion of different degree of knowledge for intercept theorem

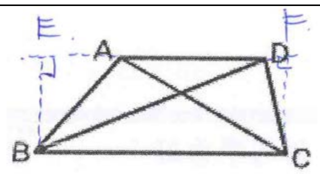
From the description in table 3, only 10% students have no comprehensive performance in the details of proof context from such learning method. There is about 25% students who are able to understand the whole proving content, and exceeded 60% students can partially recall or suspect the details with little certainty. It seems to show that students are able to understand how to write the proofs in deductive form; nevertheless, the written practices present some incomprehension that are caused by the students who are influenced by the incomplete concepts or the prototypes of worked-out examples. For example, about 7% students use the symbols of triangle elements which are written in the same notation of worked-out example in intercept theorem. Figure 2 below displays another practice content, it shows that student is influenced by the prototype of the worked-example. The heights are unusually drawn outside the triangles that are in the same position of worked- out example.

【練習3】如右圖，在梯形*ABCD*中，$\overline{AD} // \overline{BC}$，

試證明△*ABC*面積＝△*DBC*面積。

【Practice 3】In the figure, *ABCD* is a trapezoid with $\overline{AD} // \overline{BC}$,

Prove that △*ABC* and △*DBC* have the same area.



(1) 作 $\overline{BE} \perp \overline{AD}$ 且 E 在直線 $\overline{AD}$ 上.
(2) 作 $\overline{CF} \perp \overline{BC}$ 且 F 在直線 $\overline{AD}$ 上.
(3) △ABC面積 = $\frac{1}{2} \times \overline{BC} \times \overline{BE}$.
(4) △DBC面積 = $\frac{1}{2} \times \overline{BC} \times \overline{CF}$.
(5) 因為在梯形ABCD中，$\overline{AD} // \overline{BC}$.
所以 $\overline{BE} = \overline{CF}$.
(6) 因此 △ABC面積 = △DBC面積.

(1) Draw $\overline{BE} \perp \overline{AD}$ and point E is on $\overline{AD}$.

(2) Draw $\overline{CF} \perp \overline{BC}$ and point F is on $\overline{AD}$.

(3) Area of $\triangle ABC = \frac{1}{2} \times \overline{BC} \times \overline{BE}$.

(4) Area of $\triangle DBC = \frac{1}{2} \times \overline{BC} \times \overline{CF}$.

(5) $\overline{BE} = \overline{CF}$ since *ABCD* is a trapezoid with $\overline{AD} // \overline{BC}$.

(6) Hence, area of △*ABC*＝area of △*DBC*.

Figure 2: Prototype by worked-out examples showed in student's practice

This proving is written by one of the students above is correct; however, there is about 47% students in the same practice that are not only influenced by the prototype of example. Students disregard the designated triangles and notations in the problem as well. They copy the steps from the worked-out examples without reflection. Such imitation also finds commonly in other practices, this shows that students are lack of understanding of the concepts. Students usually pay less attention to context or concept comprehension that may be influenced by the explicit steps of worked-out examples.

DISCUSSION

Worked-out example is a simple learning tool. Example-practice pairs for learning geometry proof which are segmented by its mathematical structure are easy to produce instant effects. Many students, especially the average and upper levels students, have showed some positive influence of reading comprehension. From the long-term effects, the complete knowledge structure and schema of upper levels students are formed. However, average and lower levels students have significantly regressed mainly in the local deductive level. There are some immediate effects by doing practices with similar structure; however, the knowledge schema has not been constructed in long-term memory for average and lower levels students.

Learners' performance can be easily shown in level of understanding by explicit solution steps of worked examples. Nevertheless, students tend to resemble the copy-and-adapt strategy (Hilbert, Renkl, Schworm, Kessler & Riess, 2008) when they are doing tasks with similar structure. From students' written details, they present some similar but incomplete concept or prototype of worked-out examples in the solution. Thus, it shows the important factor for students' worse performance in delayed test. Based on complexity of geometry proof, especially for average or lower level students, learning from worked examples with practices is not easy to construct schema of knowledge for novices.

In practice 1 and 2 which examine the same concept, using totally different symbol for geometric elements or changing the presented format of the figure can reduce the error rate from copying the symbols in examples (from 16.5% to 1.2%). However, it is not possible to avoid students relying on the prototype of worked-out examples for solving problem. Hence, we suggest that worked-out examples can be paired with other learning strategies such as self-explanation, self-questioning or reflection which lead learners to understand the core concepts. Moreover, worked-out examples with practice are usually used for teaching instruments. The instant effects show in practice are not identical to actual comprehension. Additional practices may include both near and far transfer questions to different levels of students for deepen understanding.

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AN ERP STUDY WITH GIFTED AND EXCELLING MALE ADOLESCENTS: SOLVING SHORT INSIGHT-BASED PROBLEMS

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This paper presents a part of Multidimensional Examination of Mathematical Giftedness. This part is focuses on brain activity (using Event-Related Potentials methodology) associated with solving short choice-reaction insight-based mathematical problems. We report on the findings related to 70 right hand male-students who were chosen for comparative data analysis. We examine problem-solving performance as revealed in behavioural and neuro-cognitive dimensions. We demonstrate differences revealed in problem-solving performance of gifted as compared to non-gifted participants as well as of excelling as compared to non-excelling participants. Based on the findings of this study we argue that excellence in mathematics is not necessarily related to general giftedness.

INTRODUCTION

In resent decade a considerable body of studies has been conducted towards understanding of the neural foundation of cognition. In particular (as relevant to our research), some studies focus on neuro-cognitive bases of arithmetic operations (e.g. Santens et al., 2010) while other investigate neuro-cognitive basis of general intelligence (e.g., Deary et al., 2010). Campbell (2006) introduces implication of neuro-cognitive research to mathematics education. Though, all studies in mathematical neuro-cognition are limited to the fields of arithmetic, simple logic and mental rotations and only small number of studies focus specifically on mathematical giftedness (e.g., O'Boyle, 2008).

Our study break through the field of this neuro-cognitive research by integration of (a) mathematical performance on relatively advanced mathematical tasks, which was not touched in the field of brain research, and (b) distinctions between general giftedness and excellence in mathematics, which are usually accepted as identical constructs. Following arguments of cognitive research that insight is a specific characteristic of gifted, we implement insight-based mathematical problems to our investigation.

BACKGROUND

Insight in problem solving

Problem solving is the heart of mathematics and mathematics education (Polya, 1963). Insight problems are problems that have simple statement – that contain only a small number of objects and relations – nevertheless are very difficult to be solved (Kershaw & Ohlsson, 2004). Usually insight-based problems are not part of the curriculum, hence they are unfamiliar to students and thus has non routine nature (Davidson, 2003). As such, they require high cognitive demands even when the knowledge and the skills are developed already.

Insight problem solving and giftedness

Insight is central trait to the construct of giftedness (Davidson, 2003). Gifted children outperform their average peers in problem solving because of the increased tendency to insight (Davidson, 2003): High-ability children (in the contrast to average-ability students) are shown to understand insight-based problems immediately and to solve it quickly, whereas children with average ability are more likely to work on the sub-problems and not solve the whole problem at all (Overtoom-Corsmit et al., 1990).

Cognitive skills and brain research

Literature review demonstrates quite consistent findings that connect different mental operations associated with mathematics and the location of brain activation. For example, research shows that attention control processes and general task difficulty (Delazer et al., 2003) are associated with the prefrontal cortex, while mental rotation (Heil, 2002) and visuo-spatial strategies in mathematics (Sohn et al., 2004) with the parietal cortex. The brain of mathematically gifted show enhanced development and activation of the right hemisphere (Prescott et al., 2010)as well as enhanced brain connectivity (Jung & Haier, 2007) and an ability to activate task-appropriate regions in a well-orchestrated and coordinated manner (O'Boyle, 2008).

Neural correlates and insight problem solving

Brain imaging techniques (e.g. fMRI and ERPs) have been applied to measure brain activity during insight problem solving. fMRI studies showed that solving non-mathematical insight problems was associated with brain activation in wide cerebral areas that included anterior cingulate (ACC) and prefrontal cortex (PFC), posterior parietal cortex (Luo & Niki, 2004). There is also evidence from EEG studies that insight problem solving acquires other neural resources than the solution of conventional or non-insight problems (Qiu et al., 2008).

Our study addresses focuses on solving mathematical short insight-based problems. It uses electrophysiological measures which can illuminate neuro-cognitive characteristic of the insightful problem solving in different ability-level groups of students.

THE STUDY

The study goal

The study goal was to investigate the impact of general intelligence and that of excellence in mathematics on performance in insight-based short mathematics problems using ERP measures.

Participants

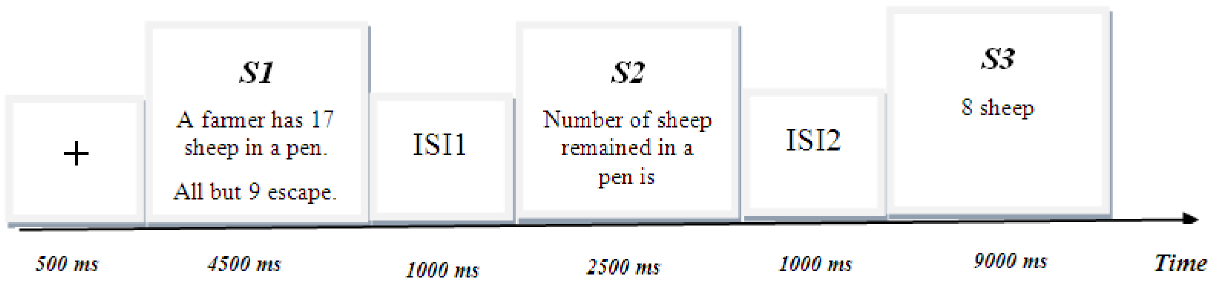
Seventy high school male students from the northern part of Israel (16-17 years old) participated in this study. The students were sampled as presented in Table 1.

Table 1: Research population

General Giftedness / Math Excellence	**Generally Gifted (G)** IQ>135 & Raven >28 of 30	**Generally Non Gifted (NG)** 100<IQ<130 & Raven < 26	Total
Excelling in math (E) SAT-M >26 of 35 (within 2%) or Math score > 92 in high level mathematics	G-E: 18	NG-E: 14	32
Non-Excelling in math (NE) SAT-M <21 of 35 or regular level of mathematics instruction	G-NE: 20	NG-NE: 18	38
Total	38	32	70
All participants were native speakers of Hebrew, right handed, without history of learning disabilities and neurological disorders and had normal to corrected vision.			

Stimuli and Procedure

A computerized insight-based test was designed using E-Prime software (Schneider, et al., 2002). The Test included 60 tasks (trials). The choice of problems in this study was based on definitions of insight-based problems borrowed from research literature. The validation of the tool (α-Cronbach=0.720) was performed with sample of subjects who did not participate in the current ERP study. Each task in each test was presented in three windows with different stimuli (Figure 1) that appeared consecutively. Time periods were determined by a pilot study.



S1 – Task condition - introduction stage, ***S2*** – Task question – question stage, ***S3*** – Suggested answer – answer verification stage

Figure 1: The sequence of events and the example of the tasks and Task Example

ERP recording and Analysis

There are 8 corners in a polygon. One corned was cut.

The number of corners in the new polygon is

Scalp EEG data was continuously recorded using a 64 channel BioSemi ActiveTwo system [we do not present here technical details of data recording due to the space constrains of this paper]. The ERP waveforms were time-locked to the onset (appearance on the screen) of *S1*, to the onset of *S2* and to the onset of *S3*. The averaged epoch for ERP, including a 200 ms pre-trigger baseline, was 2200 ms for *S1*, 2700 for *S2* and 3200 ms for *S3* (for which only the correct answers were averaged). The resulting data were baseline-corrected, and global field power (RMS) was calculated for each segment. Each condition resulted in around 40 trials.

Based on inspection of the ERPs grand mean waveforms and topographic maps the mean amplitudes of 10 intervals of 100 ms were measured and use later as measures. Repeated measures MANOVA was performed on the ERP mean amplitude at seventeen electrodes (AF3, AFz, AF4, F3, Fz, F4, FC3, FCz, FC4, P3, Pz, P4, PO3, POz, PO4, O1, O2) taking E (excellence) and G (giftedness) as between - subject factors and Caudality (anterior, posterior) and Laterality (left, middle, right) as within - subject factors. The examination of the time course was done for each of the 3 stages of a task (*S1, S2, S3*).The electrodes for statistical analysis of the peaks in each stage were chosen based on the preliminary examination of global field power (RMS) on each electrode and on the observation of ERP topographical map. Table 2 depicts the chosen electrodes.

Table 2: Electrodes chosen for statistical analysis

Introduction stage(S1)			**Question stage (S2)**			**Answer verification stage (S3)**		
Peak	time epoch	chosen electrodes	Peak	time epoch	chosen electrodes	Peak	time epoch	chosen electrodes
P1	100-175 ms	PO3, POz, PO4, O1, O2	P1	90-200 ms	PO3, POz, PO4, O1, O2	P1	80-200 ms	PO3, POz, PO4, O1, O2
N2	180-280 ms	P3, Pz, P2, PO3, POz, PO4	P2	200-380 ms	P3, Pz, P2, PO3, POz, PO4	P3	300-460 ms	P3, Pz, P2, PO3, POz, PO4
N4	380-550 ms	F1, Fz, F2, FC1, FCz, FC2	N2	200-380 ms	F1, Fz, F2, FC1, FCz, FC2	P6	500-700 ms	P5, P3, Pz, P2, P4, PO3, POz, PO4
			N4	390-550 ms	F1, Fz, F2, FC1, FCz, FC2			

MANOVAs were used for latencies and mean amplitudes on the chosen electrodes with G and E as between-subjects factors. This was done for each identified peak and for each trial stage (*S1, S2, S3*). For all consequent ANOVAs results were corrected for deviations according to Greenhouse-Geisser.

RESULTS

Behavioural data

We first performed analysis of the behavioural measures of the participants' performance on the Test: Reaction Time (RT), Reaction time for correct responses only (RTc) and Accuracy (Acc). RT and RTc were measure in ms; Acc was measured in percentage of correct answers. ANOVA examination for the effects of G and E factors on Acc, RT and RTc demonstrated the following significant effects:

1. *E and G factors have significant effect on accuracy of the responses*:

1.a Acc(E) = 57.9(9.5)% vs. Acc(NE)= 51.8(8.5)%: $F(1,66)=8.127$, $p<0.01$; (Values depicted: Mean(SD); Acc(E) denotes accuracy found for E participants)

1.b Acc(G)= 57.7(8.0)% vs. Acc(NG)=50.8(9.7)% : $F(1,66)=10.861$, $p<0.01$.

2. *E factor has significant effect on RT and RTc* (in ms):

2.a RT(E)= 1989.0(490.8) vs. RT(NE)= 2327.6(626.1): $F(1,66)=5.398$, $p<0.05$

2.b RTc(E)= 1881.1(482.30) vs. RTc(NE)= 2353.1(644.8): $F(1,66)=5.398$, $p<0.01$

G factor did not have significant effect on RT and RTc. E and G factors did not interact on Acc, RT and RTc. These findings lead to the hypothesis that excellence in mathematics is not necessarily a function of general giftedness.

Electrophysiological scalp data

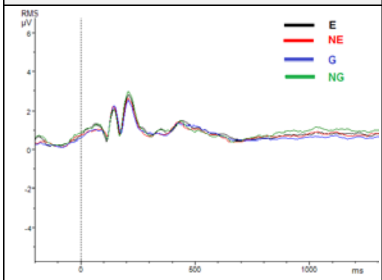
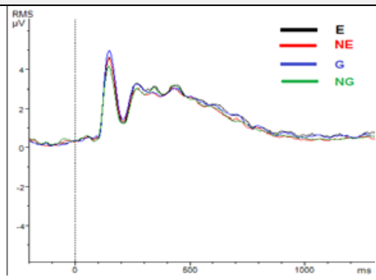
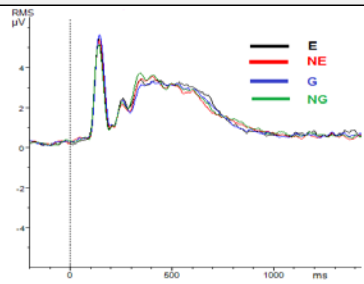
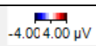
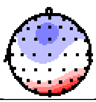
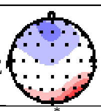
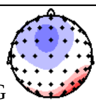
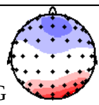
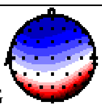
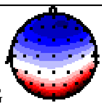
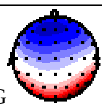
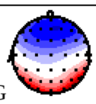
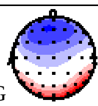
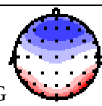
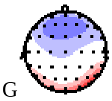
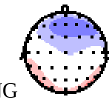
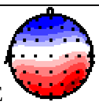
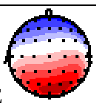
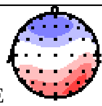
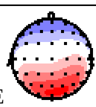
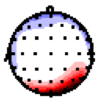
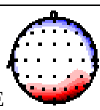
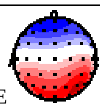
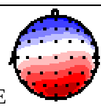
Introduction stage (S1)	Question stage (S2)	Answer verification stage for correct answers (S3)
		

Figure 2: ERP waveforms of RMS for three stages of the task (S1, S2 and S3) for G and E factors

Table 4: Scalp topography: Significant main effects and interactions

Stage	Main effect or significant interactions 							
S1	Excellency				Giftedness			
	Time Epoch (TE): 300-400				TE: 300-400			
	E 	NE 			G 	NG 		
	E vs. NE: F(1, 66) = 3.871*				*G vs. NG: F*(1, 66) = 4.511*			
S2	Laterality × Giftedness							
	TE: 500-600		TE: 600-700		TE: 700-800		TE: 800-900	
	G 	NG 	G 	NG 	G 	NG 	G 	NG 
	F(1.864, 123.004) = 4.397*		*F*(1.898, 125.235) = 3.706*		*F*(1.837, 121.214) = 2.936*		*F*(1.868, 123.314) = 3.347*	
S3	Excellency				Laterality × Excellency			
	TE: 700-800		TE: 800-900		TE: 200-300		TE: 700-800	
	E 	NE 	E 	NE 	E 	NE 	E 	NE 
	F(1, 66) = 5.597*		*F*(1, 66) = 4.419*		*F*(1.931, 127.432) = 3.326*		*F*(1.690, 111.520) = 4.181*	

[1] E- Excelling, G – Gifted, NE – non Excelling, NG – non Gifted [2] * p <.05, ** p <.01, *** p <.001

Figure 2 demonstrates the ERP waveforms of RMS for three stages of the task (*S1*, *S2* and *S3*) for G, NG, E and NE participants. Table 4 depicts significant differences in Brain topography revealed in the study. MANOVAs on the mean ERP amplitude for each stage demonstrated no significant factors' effects and no significant interactions between different factors. Consequent ANOVAs revealed significant effects of E and G factors as well as their interactions with laterality at particular time frames. We found interactions between laterality and G factor at *S2* while laterality interacts with E factor at *S3*. These finding demonstrate different brain topography for G vs. NG participants with higher activation of right hemisphere for G participants at S2. These finding support our hypothesis raised in behavioural dimension that excellence in mathematics is not necessary depended on general giftedness.

At the next stage of analysis we examined effects and interactions of E and G factors for each identified peak at each of the task's stages (*S1, S2, and S3*). Space constrains of the paper do not allow us presenting all the finding. Table 5 exemplifies findings for *S2* (at the conference we will demonstrate also findings for *S1* and *S3*).

Table 5: Significant main effects and interactions at *S2*

Stage	Peak			MANOVA	ANOVA	Mean (SD)
S2	P1 F (5, 62)	Amplitude	G	NS	PO3: $F(1,66)=4.518^{*}$ PO4: $F(1,66)=4.608^{*}$ POz: $F(1,66)=4.726^{*}$	G vs. NG: 8.91(5.2) vs. 6.6(3.4) 10.1(5.5) vs. 7.5(3.6) 8.4(5.4) vs. 6 (2.8)
	N2 F (6, 61)	Latency	G	NS	F1: $F(1,66)=4.934^{*}$ Fz: $F(1,66)=4.083^{*}$ F2: $F(1,66)=4.331^{*}$ FC2: $F(1,66)=6.883^{*}$	G vs. NG: 308.3(35.6) vs. 326.8(34.6) 310.7(35.3) vs. 328.5(37.2) 309.1(40.4) vs. 328.3(39.1) 301.6 (30.8) vs. 314.3(42.9)
	N4 F (6, 61)	Latency	E × G	NS	FC2: $F(1,66)=8.701^{**}$ FCz: $F(1,66)=5.010^{*}$	on FCz G-E vs. NG-E: 474.3(43.8) vs.440.8(23) G-NE vs. NG-NE: 451.8(34.2) vs. 476.2 (53)
		Amplitude	E	NS	FCz: $F(1,66)=3.886^{*}$	E vs.NE: -4.24(1.65) vs. -3.46 (1.64)

[1] E- Excelling, G – Gifted, NE – non Excelling, NG – non Gifted [2] $^{*}p<.05$, $^{**}p<.01$, $^{***}p<.001$

Table 5 demonstrates the following findings:

(a) Main effect of G factor on amplitude of P1 (electrodes PO3, PO4, POz): This finding refers to perception of the question by the participants that lead to stronger brain activity in G participants at the parieto-occipital areas.

(b) Main effect of G factor on latency of N2 (electrodes F1, Fz, F2, FC2): This finding demonstrates that brain activation related to attention allocation associated with understanding the problem question appears earlier in G than in NG participants.

(c) Main effect of E factor on amplitude of N4 (electrodes FCz): This finding demonstrates that brain activation related to semantic processing is stronger for E participants.

(d) Interaction between E and G factors on latency of N4 (electrodes FC2, FCz): This finding demonstrates that brain activation related to semantic processing appears later in G-E participants than NG-E participants and earlier in G-NE participants than in NG-NE participants.

CONCLUSIONS

Present study examined differences in brain activity of Gifted vs. Non Gifted and Excelling vs. Non Excelling male adolescents while performing short insight-based mathematical problems.

Our finding that gifted participants who excel in mathematics were found more accurate and faster is consistent with the claim that people who are more intelligent tend to be more insightful (Davidson, 2003). Absence of significant *interactions between G and E factors* suggests that *G* and *E* are not necessarily interrelated traits.

Differently from previous ERP studies we use relatively complex problems and introduce ERP examination in which we divide the time course of problem solving into three stages: introduction stage, question stage and answer verification stage (contrary to the 2 stage-examination that usually is used).

The waveforms showed at the selected sites had similar components in all experimental groups (G-E, G-NE, NG-E and NG-NE). However the electrophysiological data have revealed the differences in mean amplitude and time course for each of three stages for G and E factors and G and E factors had different effects on the three stages of examination. This finding supports the hypothesis about differences in nature of G and E factors as well as leads to the hypothesis that this nature is associated with differenced in effect of E and G factors at each stage of problem solving: introducing situation, question presentation, answer verification.

The posterior P1 is related to a visual processing and is sensitive to physical stimulus characteristics (e.g. Di Russo et al., 2002). The present study found G-related increase in amplitude as well as latencies of the P1. The larger P1 with delayed peak latency for G participants could be interpreted as less automatic processing of visual information. N2 was found at parietal sites for S1 stage and at fronto-central sites for S2 stage and involved with attention allocation (Fabiani, et al., 2000). Shorter latencies' of N2 at S2 stage for gifted individuals might mean more efficient allocation of attention.

The significant difference in latency of N4 (stage S1), N4 (stage S2) and P3 (stage S3) suggested that longer latencies of gifted at stage S1 and long latencies of G-E at stages S2 and S3 are due to stimulus analysis and planning (note - we did not present all the data in this short paper). However that in spite of long latencies that are similar to those of NG-NE participants, G-E participants finished the tasks more accurately and quickly. The longer latency for G-E participants may be due to the adoption of a strategy of devoting more time to stimulus encoding and planning for future events.

The significant effect of G factor and its interaction with Laterality is found at stage S2 while significant effect of E factor and its interaction with laterality is found at stage S3. This observation leads to suggestion that G participants start finding answers to posed question immediately after question presentation, while E participants put more emphasis on answer verification stage.

AKNOWLEDGMENT

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THE MISINTERPRETATION OF HISTOGRAMS

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Histograms are frequently used in statistics and mathematics education and in reporting statistical data. Recent studies have however shown that students often have difficulty interpreting these representations of data distributions. In this study we analyzed reaction times and accuracy rates in order to investigate whether intuitive reasoning based on salient features of the histogram might underlie one specific difficulty in students who compare the means of two histograms. We found evidence for the occurrence of intuitive reasoning, both in the patterns of accuracy rates and in reaction times. We conclude that the salient height of the modal bar in histograms leads students to the incorrect interpretation of these representations.

THEORETICAL AND EMPIRICAL BACKGROUND

Histograms are omnipresent, both in research and education, but also in printed press. Despite their omnipresence, various misinterpretations concerning histograms have been reported. First, students mistakenly use height differences between the individual bars of histograms as an indicator of variation, instead of using the range and overall shape of the figure (Baker, Corbett, & Koedinger, 2002; Cooper & Shore, 2008). Second, students confuse histograms with bar graphs, thinking that the height of a bar in a histogram represents the value of that bar instead of the frequency or proportion (Baker et al., 2002; delMas, Garfield, & Ooms, 2005). Third, students interpret the horizontal axis as a time scale (delMas et al., 2005). Fourth, students misinterpret classes of histograms, creating difficulties with reading off frequencies of groups or values (delMas et al., 2005). Finally, students think that when one of two ungrouped histograms has a mode with a higher frequency, it automatically also has a higher mean (Lem, Onghena, Verschaffel, & Van Dooren, 2011; Watson & Moritz, 1998).

With respect to this last misinterpretation, Lem et al. (2011) proposed that histograms are not optimally designed, taking into account how people reason about graphs. This claim was based on the graph design principles proposed by Tversky (1997). According to Tversky, the way we interact with the world influences the way we interpret graphs. One of these principles is that the vertical dimension is very important in daily life (e.g., gravity works vertically), making us focus more on the vertical than on the horizontal dimension of a graph. However, when we need to compare the mean of two histograms, the *horizontal* dimension is very important, while we intuitively tend to pay more attention to the vertical dimension. This can explain why students misinterpret the difference in height of the modal bars of two histograms as representing the difference in means in both histograms (Lem et al., 2011). In this

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study we wanted to investigate whether there indeed is an intuition that makes us focus more on the vertical dimension than on the horizontal dimension when comparing the means of two histograms. In order to do this, we used a methodology often applied in research using the dual process theoretical framework.

Dual process theories have been proposed to explain why people often fail to give the correct answer to a task for which they do have the required knowledge to solve it (e.g., Kahneman & Tversky, 1972). Not only in cognitive psychology, but also in the psychology of mathematics education the dual processes framework has been used (Leron & Hazzan, 2006), among others to study phenomena, like logical reasoning in mathematics (Inglis & Simpson, 2004). Various dual process theories have been described, but all of them make the distinction between intuitive and analytic reasoning processes (e.g., Kahneman, 2000). While intuitive processes are described as unconscious, automatic, fast, and undemanding of working memory capacity, analytic processes are described as conscious, slow, deliberative, and effortful (Evans, 2008). Intuitive processes are often used in problem solving situations, frequently leading to the correct solution. However, in some tasks intuitive reasoning leads to an incorrect solution and analytic reasoning is necessary to detect and inhibit the intuitive error.

According to the revised and extended heuristic-analytic model of Evans (2006), intuitive and analytic processes work sequentially but in constant competition and interaction. When confronted with a task, one will – in the intuitive mode – immediately construct the most plausible or relevant default model, based on salient task features, the goal of the task, and background knowledge. Only after this initial intuitive processing, analytic reasoning may occur, depending on various factors such as general intelligence, time available, and the task instructions. In cases where analytic reasoning does take place, the default model's validity is evaluated before a final response is given. The most important consequence of Evan's model is that even when analytic thinking takes place after the intuitive processing of salient task features, reasoning is still biased by that default model and therefore by (irrelevant) salient task features, possibly causing interference in the analytic stage of reasoning. Applied to histograms, this means that when confronted with a mean comparison task, one is likely to focus on the height of the (modal) bars, which is assumed to be the most salient feature of histograms. Even when one then starts reasoning analytically, for instance by taking into account the range and overall shape of the histograms, the first idea based on the comparison of the height of the modal bars may still influence the final reasoning outcome.

METHOD

Participants

Participants were 116 first year university students at the University of Leuven. All students had completed the same introductory statistics course several weeks before participation, covering the mean as well as histograms among various other topics.

Materials

Participants were presented with 40 comparison items in random order. Each item consisted of two histograms representing fictitious exam results of two groups of students. Their task was to determine in which of the two groups the average exam result was highest. In all items, the only two things being varied between both histograms was the height and/or the location of the histogram, in order to specifically address Tversky's (1997) design principles. Five item types were constructed: two congruent and three incongruent item types (see Figure 1). In congruent items, the intuitive response is the correct response, while in incongruent items the intuitive response is incorrect, making analytic reasoning necessary to obtain the correct response. *Congruent equal items* presented two identical histograms: Both the correct and the intuitive response were that both histograms had the same mean. Also in *congruent unequal items*, the intuitive response was the correct one, as the histogram with the higher average also had higher bars. In *incongruent equal items*, the correct response was that the average was the same in both histograms, while the intuitive response would be that the histogram with higher bars had a higher mean. In *incongruent inverse items*, the histogram that was placed more to the left had higher bars, leading to the intuitive response that this histogram had a higher mean. The correct response, however, was that the histogram that was placed more to the right had the highest mean, independent of its lower bars. Finally, *incongruent unequal items* showed two equally shaped histograms of which one was placed more to the right. The intuitive response would be that both histograms had the same mean as the height of both histograms was the same, while the correct response was that the histogram that was placed more to the right had a higher mean.

Procedure

The 40 items were administered in groups of approximately 20 students in a computer class where each student worked individually on a computer. After the general introduction of the task, two sample items were provided. Students were randomly assigned to one of three conditions. First, the *analytic condition* started with a warning for the fact that graphs can be misleading, and students were told that they were provided as much time per item as needed. Also, students were told they should try to answer correctly and to check their response before giving it. Second, in the *intuitive condition* students were told they would only get four seconds per item to provide their answer, and that they hence should work fast, but also try to give the correct responses. Third, the *control condition* involved no time constraint and no warning; The students were told to work at their own pace and to try to provide the correct responses.

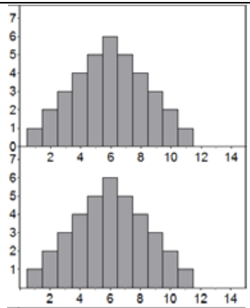
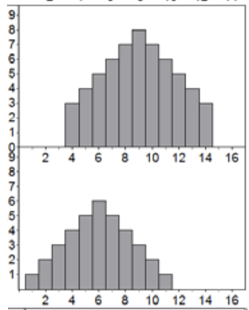
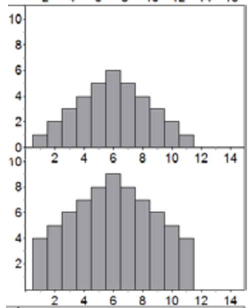
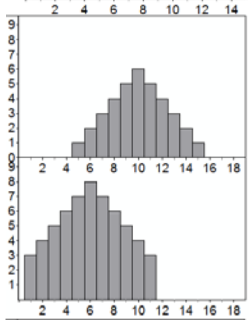
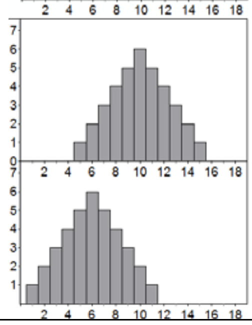
	Correct response	Intuitive response	Example
Congruent equal	Same position and shape, so same mean	Same height, so same mean	
Congruent unequal	Histogram positioned more to the right has the highest mean	Histogram with highest bars has the highest mean	
Incongruent equal	Same position and shape, so same mean	Histogram with highest bars has the highest mean	
Incongruent inverse	Histogram positioned more to the right has the highest mean	Histogram with highest bars has the highest mean	
Incongruent unequal	Histogram positioned more to the right has the highest mean	Same height, so same mean	

Figure 1: Overview of item types with example items

Predictions

First, we expected congruent items to be solved better than the incongruent items, as for congruent items the intuition leads to the correct response, while to solve incongruent items correctly analytical reasoning is necessary. Second, we expected correct responses to congruent items to be given faster than correct responses to incongruent items: When the correct response to an incongruent items is given, slow analytic reasoning has occurred, while for the correct response to congruent items only fast intuitive reasoning was required. Third, we expected that accuracy would be highest in the analytic condition and lowest in the intuitive condition, as analytic reasoning would be activated by the warning and, also, because there was enough time for analytic reasoning to occur while in the intuitive condition there would be insufficient time to reason analytically.

RESULTS

In the intuitive condition, 25 trials in which no response was provided in time (i.e. 1.60% of all trials in the intuitive condition) were removed.

Accuracy

As expected, accuracy for the congruent items, 94%, was higher than that of incongruent items, 62% (see Table 1 for an overview of the accuracy rates). A generalized linear mixed model with correctness as the dependent variable and condition and congruency as independent variables only showed a main effect of congruency, $F(1,193) = 421.26$, $p < .001$, but not of condition, $F(2,193) = 2.47$, $p = .087$, nor an interaction effect, $F(1,193) = 0.96$, $p = .386$. So, in contrast with our third prediction, our time and warning manipulations did not have a significant effect on accuracy. This suggests the occurrence of a strong intuition that is difficult to overcome.

	Congruent items		Incongruent items		
	Equal	Unequal	Equal	Inverse	Unequal
Intuitive	91.7	98.1	40.1	67.5	82.7
Control	90.6	95.0	41.9	61.9	77.5
Analytic	95.4	95.7	40.1	60.6	85.7

Table 1: Accuracy rates (in %) per item type and condition.

A closer look at the incongruent items, using a generalized linear mixed model with correctness as dependent variable and item type as independent variable, showed significant differences in accuracy for the three different incongruent item types, $F(2,2533) = 349.17$, $p < .001$. More specifically, the average accuracy for incongruent equal items was only half (41%) of the accuracy for incongruent unequal items (82%), while the accuracy for incongruent inverse items was in between (63%). These differences can be explained by studying the items more closely. The incongruent unequal items elicited a lot of correct responses, since there was no height difference that could distract them, while there was a difference in location that could lead to the correct conclusion. In the incongruent equal item, which elicited much fewer correct responses, the only difference between the two histograms was the height of the bars, easily leading students to the erroneous intuitive response. In the incongruent inverse item, finally, which had a medium accuracy rate, both the position and the height differed, making it attractive to focus on the height difference as, according to Tversky's (1997) design principles, one focuses more on the vertical dimension.

Reaction times

We analyzed the log transformed reaction times. Overall, there was a main effect of congruency on reaction time, with faster reactions to congruent than to incongruent items, $F(1,4410) = 20.74$, $p < .001$, and a main effect of condition, $F(2,4410) = 427.88$, $p < .001$, with the longest reaction times in the analytic condition and the shortest reaction times in the intuitive condition. When only including the correct responses in the model, however, the effect of congruency surprisingly disappeared, rejecting our second prediction, $F(1,3320) = 1.65$, $p = .199$. A closer look at the reaction times per condition and incongruent item type, revealed that correct responses were given particularly fast to "incongruent unequal" items (see Figure 2). This is in line with the observed higher accuracy rates for incongruent unequal items.

DISCUSSION

In this study we confirmed the hypothesis that intuitive reasoning processes lead

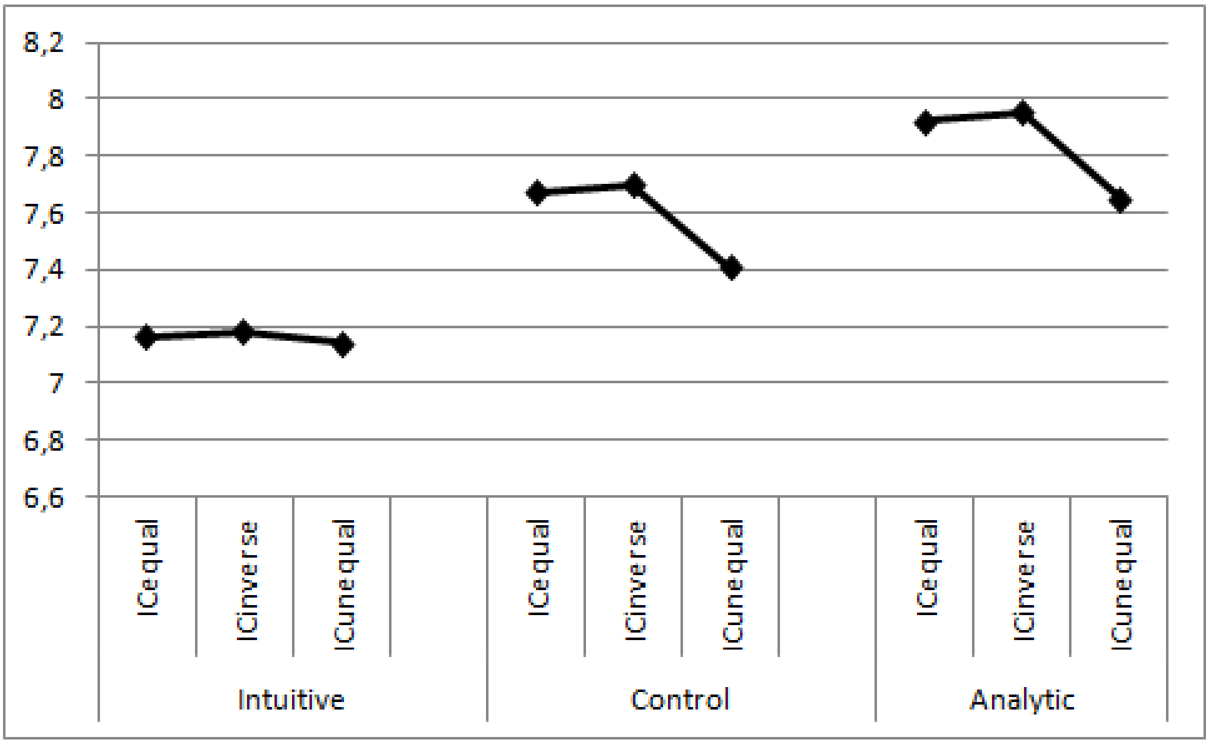


Figure 2: log reaction times for the incongruent item types and per condition, for the correct responses only.

students to use the height differences of histograms to compare the mean of two histograms. Both accuracy and reaction time data convincingly confirmed this hypothesis. Remarkable is that our analytic and intuitive condition did not differ in terms of accuracy, even though significant differences in reaction times between the three conditions were found. So, explicitly warning students and providing ample reasoning time did not result in higher accuracy than solving the same items within four seconds and without warning. This suggests that students' intuitive reasoning was strong, and hard to overcome, not only when reasoning time was restricted, but also when ample processing time was provided and even when also a warning was given to students. This is in accordance with the revised and extended heuristic-analytic model of Evans (2006), in which is assumed that the first impression of a task or representation influences analytic processes if they occur. The differences we found in accuracy rates and reaction times between the different incongruent item types further confirmed the second part of our hypothesis that students focused more on the vertical than on the horizontal dimension, as most errors were made when height was the only difference between two histograms and fewest errors were made when the heights were the same, making students focus on the horizontal dimension. These findings have several implications, which we will discuss below.

First, we have shown that Tversky's (1997) graph design principles can be used to predict the occurrence of a common error students make when comparing the means of two histograms. This suggests that the design of histograms is not ideal for comparing the mean of two distributions. Similar graph design problems have been reported with respect to box plots (Lem, Onghena, Verschaffel, & Van Dooren, 2012).

Second, with respect to mathematics and statistics education, we have shown that even histograms, which are generally assumed to be very easy representations to interpret, can make students reason incorrectly. Special attention should hence be paid to this specific misinterpretation, in order to help students interpret histograms better.

Third, with respect to future studies, it would be interesting to also test expert users of histograms. This would inform us about the strength of the intuition, as these experts have much more experience with and knowledge about histograms. Furthermore, previous studies (e.g., Lem et al., 2011) have shown that even when students do provide the correct answer, they are not always able to correctly explain why this is the correct answer. In our study, we did not ask students to explain their responses, so we do not know whether correct responses reflected a correct interpretation of the histograms. Future studies could address this. Also, it would be interesting to study whether other misinterpretations of histograms or other external representations can be explained by graph design principles and/or the dual process framework.

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AGENCY AND IDENTITY: MATHEMATICS TEACHERS' STORIES OF OVERCOMING DISADVANTAGE

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Trajectories through schooling in general and mathematics in particular seem to be framed by social class and other factors at a very early age, leading some to succeed and many to fail. Sociologists such as Bernstein and Bourdieu give accounts of how education reproduces society. However people are not totally determined by their social circumstances. They are able to exhibit agency and change the direction of their lives. In this paper I report on a study in which individuals, who have taken up mathematics as their careers in spite of what might be described as disadvantaged backgrounds, tell stories of their trajectories. I analyse one story through a particular construction of notions of identity in the hope of raising an image of possibilities and potentialities.

STRUCTURE AND AGENCY

In mathematics education we have traditionally taken cognitive perspectives in research, developing notions of 'normal' development of mathematical concepts and as a consequence pathologising those children who do not develop those concepts at the appropriate age and stage. More recently we have turned to studying teaching, recognising its essential mediating role in the learning process. Texts and tasks have also come under scrutiny but the approach has remained predominantly a cognitive one. Many of us have realised over recent decades, particularly perhaps through the resilience of working class and other disadvantaged students' failure in mathematics despite changes to teaching, textbooks and styles of assessment, that sociological frameworks address these issues in ways that psychology is not able to, though Vygotsky's insights into the social origins of consciousness (Daniels) take the same perspective, being a Marxist orientation, as many of the sociological theories, such as those of Bernstein, Apple, Bourdieu, Zeichner and others whose work informs education.

I have been engaged in research drawing on Bernstein's work (Lerman & Tsatsaroni, 1998; Morgan, Tsatsaroni & Lerman, 2002; Lerman, Xu & Tsatsaroni, 2003) for some years. Bernstein shows (e.g. 2000) that disadvantage, in relation to children from working class backgrounds, has its origins in the home, in restricted as compared to elaborated language, and is reproduced at an early stage of primary schooling. Thus the option to choose to study mathematics at the upper levels of schooling appears to have been taken away from working class students at a very early age. However, some students can and do find ways to resist and make choices that may appear surprising to teachers and researchers, and indeed the students' families. The structuralist account is perhaps not the whole story.

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Identity is multiple and a person apparently powerless in one discourse may feel themselves as powerful in another. As researchers we need rich descriptive tools for analysing identity and the resources potentially available through shifts in discourses.

In this paper I will tell the story of Denise[1], the child of an immigrant family who came from the Caribbean to the UK in the 1950s. Her background was working class and expectations on her future life path from her parents, on the rare occasions that career/job was mentioned, were very limited. In spite of this she went to university and became, eventually, a University lecturer. Subsequently I will discuss notions of identity that might account for measures of agency which, I argue, enable apparent resistance to structures that continue to play powerful roles in people's lives.

Retrospective accounts such as these two stories, and the others I am collecting, clearly do not enable the listener/reader/observer to have anything to say about the formation of the person's trajectory. But in not enabling those insights it does enable other possibilities; in providing her view of her life story the narrator presents herself as she chooses. She makes her selection from the twists and turns in her life path, giving her considerable control in an interview situation in which she might otherwise feel vulnerable. The narrative emerges somehow whole, but potentially, subject to the researchers' motives and interpretations, revealing of other identities along the way.

INTERVIEWING

Interviewing is generally thought of as a neutral method for gathering data, "a conversation with a purpose" (Lincoln & Guba, 1985, p. 268), 'neutral' in the sense that any sensitive interviewer asking the same question of interviewee X will receive the same response. Similarly, the analysis of interview data is, perhaps rather less often, thought to be straightforward in that meanings can be gleaned from interviewees' responses and categorised in a manner that can be replicated by another.

Mishler (1991), amongst many others, questions the neutrality of the interview by indicating that what a question means to the interviewer may well not be the same as for the interviewee. Furthermore, "Changing the interviewer changes the interview results, even if the new interviewer asks the same set of questions" (Scheurich, 1997, p. 62; see also Labov, 1992). Consider for example a female child being interviewed, with the same set of questions, by her teacher, or an unknown male adult researcher, or a female adult researcher, or a female relative, or another child. We would hardly be surprised if the responses were to differ in each case.

In order to analyse interview transcripts the nuances, ambiguities and uncertainties need to be simplified to enable data reduction and categorisation. Of course there is a range of methods of discourse analysis on which to draw but the variety of underlying assumptions of these different methods emphasises the range of meanings that can be

[1] All names in this paper are pseud

offered for the same data. This is not to say that interviews are un-analysable and unreliable. Researchers have the audiotapes or the videotapes, as well as field notes, and impressions. It is, instead, to recognise the contingent nature of the whole interview process and hence the kinds of responses elicited and the theory-driven nature of ascribing meaning to the interviewee's utterances.

The researcher's account, therefore, will be a narrative of its own and calls for reflexivity and openness by the writer and multiple elements to be presented by the researcher (Lather, 2007; Scheurich, 1997). In this respect, then, I must note that Denise was conscious of my status as a full professor in the same field as herself and once again this calls for awareness on my part of the power relations and their potential effects. I will write my account of my writing of Denise's story later in this paper.

DENISE'S STORY

Denise: I didn't have a very privileged background and I think I struggled.

Interviewer: What do you mean by…

Denise: Because I was born in Jamaica and I came here in 1962 at the age of 3, my father was already over here, my mother brought 3 girls, there were five of us, three girls. My parents weren't educated, my dad was, he responded to the 50s and 60s drive to get people from the Caribbean to work in the jobs that people over here didn't want to do, so he worked in the railways. He did that for forty years. My mother worked in factories, in clothes factories … kind of unskilled. My parents didn't take any notice of my education they never went to parents' evenings… If my mother was given a chance she would have done more than she did. If anyone influenced me it was my elder sister and my elder brother who was in Jamaica but came over later. So the most my mother wanted for us was to be able to leave school and be able to get a good job like working in a bank. By working in a bank I'm talking about as a cashier. I was the fourth out of five… I was the first one that was allowed to stay on and go to University and it was – and I remember at the time I should have been told that I was lucky that I didn't have to leave school and go and work in Woolworths. Any help I ever got with my work was … from my elder brother and sister. I muddled through O levels… and muddled through A levels… Even then I never got good grades. But I enjoyed chemistry, I enjoyed practical part of chemistry and we had a good teacher, Mr J_ you know? Probably we all had these teachers that made a – Mr J_ made an impact on me and I went on to do a chemistry degree at Q_... When I arrived at University that was just a total culture shock. Going back a bit, I remember… I was given career advice to do food technology at S_... I got into Q_ through clearing. Working class black children from T_ didn't go to University… I just remember, I hated it, I just didn't fit in… They used to take the mickey out of the way I talked. They were talking posh and I had a London accent. I did the first two years and I didn't pass my second year exams so I took a year out and I worked as a science technician at a school. I retook my exams and passed them, but I still wasn't ready to go back, so I took another year out… After two years I was ready to come back. I actually did better in my third year than I did in my first two years. When I did maths and chemistry (at school) I remember I was the only girl. I got a job at a training centre and it was teaching technician skills at a women's training centre. It was a women's collective, I was one of the founder members. I worked there for five years, but I didn't have a teaching qualification… so it wasn't secure. But then I went for a job in P_. My job was science and technician skills but I made more of the numeracy, developing the numeracy provision; that was a two-year project. As we were approaching the end of the two years… I ended up having to write the report and then they created a full time post as numeracy lecturer/organiser… I worked there for 15 years, within that time I was promoted to team leader for numeracy. While I was there I got bogged down in the every day teaching. During this time I started to do my teacher training qualification… and that's when I started to get interested in teacher training so in the last two years I started to teach on teacher training, and I got a job on a short term project training numeracy teachers.

Interviewer: Back at the school stage, I was wondering whether you ever had any self-doubts or whether because if (brother and sister) said you could do it you just went ahead and did it.

Denise: Oh no, no, no, I always had self-doubts, I was always kind of average, I was never at the bottom, I was never at the top, I worked hard, but I had no direction. One of the things I remember, in terms of maths, we had four streams for maths and I was put in the third stream, then I got moved up the second stream and then I got moved up to the first stream, so I was obviously progressing, but I never felt clever.

Interviewer: So you had those doubts but nevertheless you worked hard and kept going.

Denise: When we were talking in the summer I think it was where I started, the starting point was really the issue. Now I am working with people who do have much more privileged backgrounds. I was always kind of average.

Interviewer: In terms of, you know you were saying working class, there were two elements to your background, there was the working class thing but also the immigrant thing. Do you think that that made the difference (from your peers)?

Denise: The problem is, where I grew up most of the people were, I am thinking of secondary school now, I probably don't know of those people, I probably dealt with more people like that in my primary school, in my secondary school most of us were from immigrant backgrounds, if they weren't from the Caribbean where I came from there were lots of Asian but also Turkish and Greek… I can remember there's a picture of my maths A level group and there's one white British person, and all the rest were… When I reflect back I kind of think, it's a shame that my mother didn't want more, because her goal for me was to work in a bank, so I was never pushed, I was never – it was never have you done your homework or anything.

NOTIONS OF IDENTITY

Accounts of identity are ubiquitous in the social sciences and increasingly in mathematics education too (Lerman, 2006; Sfard, 2008). It is used in many different ways and it is therefore vital that one sets out how the notion is understood, read, and operationalised in one's work. One needs also to be aware of the dangers of collecting ever more fragmented stories without having any way of talking about the production of identity (Arnot & Reay, 2006). In education we cannot be content with the description that sociologists can provide. Our task is to assist young people to develop themselves in useful and productive ways and hence we need to be able to offer them possibilities, at least, for their developing sense(s) of self.

In this paper I work with the framework provided by Holland and Lachicotte in their 2007 chapter in which they contrast Erikson's and Mead's notions of self/identity. They describe Erikson's interpretation (he is credited by Gleason (1983) as having put the term 'identity' into common use in the 1950s), drawing on Penuel and Wertsch (1995), as "a sense, felt by individuals within themselves, and as an experience of continuity, oriented toward a self-chosen and positively anticipated future" (p. 83). On the other hand Mead took identity to be a sense of oneself as a participant in the social roles and positions defined by a specific, historically constituted set of social activities – multiple, perhaps even contradictory, and performative. Holland and Lachicotte suggest that there is a strong complementarity with Vygotsky's elaboration of the sociogenetic formation of self, as "ways in which social interaction, mediated by symbolic forms, provide crucial resources … for self-making" (p. 105). They add: "In Vygotskian terminology, an *identity* is a higher-order psychological function that organises sentiments, understandings, and embodied knowledge relevant to a culturally imagined, personally valued social position." (p. 113, emphasis in original)

Drawing on Burke and Reitzes' (1991) Meadian account, they suggest that:

> "Identities are simultaneously (1) *social products*, that is, collectively developed and imagined social categories; (2) *self-meanings*, developed through a sociogenetic process that entails active internalization; (3) *symbolic*, when performed they call up the same responses in one person as they do in others; (4) *reflexive*, providing a vantage point from which persons can assess the "implications of their own behavior as well as of other people's behaviors"; and (5) *a source of motivation for action*, particularly actions that result in the social confirmation of the identity." (Holland & Lachicotte, p. 109, emphasis in original)

I will draw these elements into the narrative I will construct of Denise's story as an analytical framework.

What was striking in Denise's account was the sense of a trajectory, a coherence of life path, leading to who she is today, a gradual unfolding of who she could be. To an extent this progressive narrative will have been a product of the interview situation, having been invited to look back and tell her story to me to account for how she arrived at her current situation from a background that could be seen to threaten a dead end. She was eager to tell her stories and needed very little prompting from me.

Denise's account of her life trajectory has a powerful sense of coherence and direction in it, fitting well with Erikson's notion of identity. Denise says that she was able to focus on her University Chemistry studies because she didn't go to the pub like her fellow students. At the same time there are multiple elements in Denise's background: low parental expectations but encouragement from her elder siblings *(social products* and *motivation for action)*; an awareness of the privilege of being the first, because of family financial constraints, to be given the opportunity to go to University *(symbolic)*; a strong feeling of not fitting in because of social class background *(symbolic* and *self-meaning)*; taking time out for two years and yet returning to studies, which Denise said was not expected by her tutors *(motivation for action)*; applying for positions that might have been seen as beyond her qualifications/experience but succeeding at them and gaining responsibility and promotion *(self-meaning* and *motivation for action)*. I asked her questions concerning the immigrant issue though they seemed not to find a connection with her experience[2]. In response to my prompt, Denise said that she always had self-doubts, and a sense of herself as average *(reflexive)* but these doubts did not lead to her giving up. The support and encouragement of her siblings, and perhaps a consequent family pressure of expectation, provided a more influential discourse/identity for Denise *(self-meaning)*.

DISCUSSION

Although it might appear that Erikson's and Mead's (also Vygotsky's) notions of identity are dichotomous paths for the reader's analysis of Denise's account, it seems to me that both provide theories of identity that are revealing in the two stories. In retrospective accounts at least, people have a need to present themselves externally and perhaps internally too, as a coherent identity/self, constructed possibly from pushes and pulls in different directions through our lives, but purposefully, in pursuit of the person we wish to be. At the same time we reveal elements of that construction that would have produced and been produced from a range of different situations and hence identities, some in which one feels powerful and others in which one feels powerless. Both notions are helpful in writing an account of a person's retrospective story.

I referred above to Arnot and Reay's (2006) warning concerning the danger of being unable to use ever more fragmented accounts in ways that might be helpful to learners of mathematics. What Denise's story confirms is that although structures of social class and others represent still present trends in society, no individual's potential can be reduced to those trends, and teachers (also parents and others) need to beware of the damage that can be done by the self-fulfilling effects of expectations of students' abilities and potential. Second, teachers might be significant others for students,

[2] I recognise the possible influence of being born from recent immigrants as my own story rather than Denise's.

offering them visions of themselves, projected identities (Sfard) that can be empowering identities for those students. What a teacher might do, consciously, about that well-known information is rather more problematic! Third, I would argue that teachers, as well as students, need to be offered a range of alternative ways of being teachers or students. From this account, and so much other research too, it is clear that there is no single way of being a student (or teacher) that will or will not lead to a rich life. Being offered a range of alternative images of how to become can be supportive of alternative identities/selves.

In terms of theory, it is tempting to use the term 'identity' when discussing learning in general or learning mathematics in particular and an increasing number of researchers in the field employ the signifier. Wenger's (1998) book has given great impetus to the idea of seeing learning as developing identity in a community of practice. We have argued (Kanes & Lerman, 2007) that Lave and Wenger's (1991) book gives a more nuanced and conflicting, and therefore more productive, sense of participation and trajectories in communities of practice. In this analysis, which is still a work in progress, I have sought to present how I am employing the signifier and have made it clear that two apparently dichotomous uses have both been productive through, I propose, being both explanatory and useful in the sense of our need, in education, to find ways of assisting in particular those learners who generally fail in mathematics.

The researcher's narrative

With a principled commitment, any commitment, in this case my desire to reduce the reproductive effect of education in general and mathematics education in particular as the leading exchange token for future choices, comes the reading of data for what one wishes to read. I am looking for clues, leads, possibilities, for identity development that can enable students who might otherwise fail in school mathematics to succeed. I am not so naïve as to believe that research ever produces knowledge, given that we are working in the field of social science. I look to my own story for accounts that may give a sense of vision of the consequences of one's actions and thus taking/creating power for oneself. I recall an African Caribbean primary school head teacher telling how he realised as a teenager that if he continued with the rebellious and aggressive group of boys from the same background he would never live a life that was to any extent within his control. Separating from his group was painful in all sorts of ways but for him it was the only possibility. Another retrospective account.

Part of my story is the inspiration I have found in many aspects of Vygotsky's theories. Holland and Lachicotte suggest:

> "Vygotsky's ideas about semiotic mediation clarify the role of culture in the formation of identities and envision how actively internalized identities enable one to control one's behavior and, thus, have agency. (p. 109, emphasis in original)

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TOWARDS A MODEL FOR PARENTAL INVOLVEMENT IN ENHANCING CHILDREN'S MATHEMATICS LEARNING

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I am a mathematics teacher educator who worked closely with an elementary school teacher and conducted a case study on parental involvement; the common goal is to enhance children's learning in mathematics. In an elementary school near my university, I offered bi-weekly Friday Math Camp for parent-child year round. Data sources are questionnaire, diaries, parents' focus group interviews, worksheets and videos and they were analysed qualitatively. Results indicated that it is feasible to use research-based activities to engage group of parents in assuming various roles over time: parents as learners; parents as teachers; parents as facilitators, parents as observers; parents as partner/opponent of child. The various roles also enabled parents to take charge of children's math learning in school and at home.

INTRODUCTION

Educating the next generation is a responsibility for all. In large scale comparative studies Asian students, including that of Taiwan, scored exceedingly well in mathematics among the list of participating countries (e.g. PISA 2009). In Taiwan, whenever there is a call for mathematics curriculum reforms, there is also an increase in parents' attention. Research on large scale comparative studies already confirmed the importance of parents' involvement and students' performance (e.g. Chiu & Zeng, 2008). Parents who are involved in their children's education contribute not only to higher academic achievement, but also to positive behaviours and emotional development (e.g., Stevenson & Lee, 1990; Weston, 1989).

In view of the importance in parental involvement there are large scale studies on partnership of home, school and community worldwide. In US, the project NNPS guides school teams of educators, parents, and community partners to use a research-based framework of six types of involvement: parenting, communicating, volunteering, learning at home, decision making, and collaborating with the community (Epstein, 1995). Studies also attend to families of different socioeconomic status: high socioeconomic status (Maher, 2007); or low socioeconomic groups such as immigrants (e.g. CEMELA, Latinos families, Civil & Bernier, 2006). Cai (2003) examined parental support and found that among five parental roles in middle school students' learning of mathematics: motivator, monitor, resource provider, mathematics content adviser, motivator and monitor are predictors of children's success in problem solving. However, it is unknown how parents specifically motivate and monitor their children. In this report, I myself as a teacher educator describe my journey as I worked with elementary school teacher and parents. In an elementary school near my university, I offered bi-weekly Friday afternoon Math Camp for parent-child year round. The research questions are: What roles do parents assume as they attend this Math Camp over time? What exemplars can be collected as evidence of roles that carry potentials in enhancing children's math learning?

LITERATURE REVIEWTHEORETICAL FRAMEWORK

Studies on enhancing children's math learning

The content areas as in math learning as revealed from curriculum standards. For schools in Taiwan, the mathematics curriculum standards include four content strands (Number, Space, Algebra, Data-handling) (MOE, 2003) and the first two are mostly emphasized in elementary schools. In alignment to content, the *Standard* suggests student-cantered approach, real life applications, and include reminder for parents.

Research studies on enhancing math learning of elementary school children in number sense and in spatial sense addressed to curriculum materials and teaching strategies. In review, the activity can be in form of a game (van den Heuvel Panhuizen & Buys, 2008; Leung & Lo, 2010), reading a picture book (van den Heuvel Panhuizen, Boogard, & Doig, 2009), posing problems and diary writing (Leung & Wu, 2000), or completing a math trail (English, Humble & Burmes, 2010). However, few studies referred to how parent devoting time and engaging in such activities in order to enhance children's math learning. In Taiwan, Leung (2011) reported her action research on parental involvement as a teacher educator working in an intervention project (http://www2.nsysu.edu.tw/leung/home.html). Leung went to her sons' school and played math games during home room period and these math games used by myself as a parent to motivate children's learning and activities were shared in a book (Leung, 2008). For the above two major content strands being emphasized, parents (also adults who take care of children such as grandma) can use math games or real life applications to enhance children's learning to address motivation and monitoring.

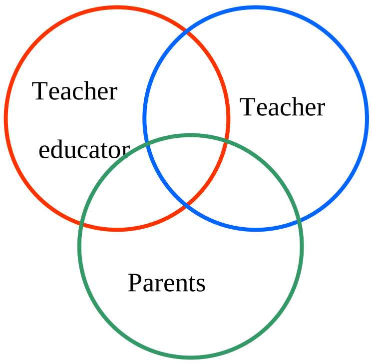
Intervention projects for parental involvement in math learning

In both US and in Taiwan, parental concern for children's learning of mathematics is emphasized in curriculum standards documents. According to the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000), when parents understand and support the schools' mathematics program, they can be invaluable in convincing their daughters and sons of the need to learn mathematics and to take schooling seriously. Families become advocates for education standards when they understand the importance of high-quality mathematics education for their children. (p. 378). In project CEMELA (Civil & Bernier, 2006), the investigators included parents as learners, teachers and facilitators. As learners, parents experienced learning math again but according to the *Standard*s (e.g. children can solve a math problem in a variety of ways).

In Taiwan math curriculum standards documents (MOE, 2003), there is a part on reminding parents; "learning math should be a happy experience to students." When parents find children having hard time with learning math or having low scores there is no need to be anxious. In this document, parents are urged to ensure students to complete homework with full concentration. If not, students make mistakes, get frustrations and finally give up learning math.

Theoretical Framework

The model I used is an extension to the sharing knowledge model by Jaworski (2008), when a Venn diagram represents how a mathematics teacher educator (MTE) and teachers share knowledge. One circle carries MTE's knowledge of research and theory while the other circle holds teachers' knowledge of students and schools. The sharing of knowledge, given in the intersection of the two circles, means that passing of knowledge is bi-directional: teachers also pass knowledge to MTE. In this study, a third circle is used to include parents: parents share knowledge of children's behaviour at home.



METHOD

The case and parent-child activities.

This is a case study and I used the method as in Yin (1994). The teacher educator is I who had been out to elementary schools for 4 years teaching my two sons' classes. The elementary school teacher used to be my undergrad students and also my graduate student and her master degree thesis is on parents' study group. Participants came to meet in the elementary school that was only 5 minutes walk from subway station and from me university. By referring to Civil and Bernier (2006), I started with Parents as learners and have parent-child pairs learn math together. I referred to curriculum standards and 4 textbooks series (children and parents are from 4 schools) to develop research-based activities suitable for parent-child Math Camp during Friday afternoons. Two characteristics of each activity are: Learn Math together, Promote Parent-Child interaction. To explain how to motivate parents and child in learning math concepts and as well enhance parent-child interaction, I will include one example in Space Strand (S). The activity can be in form of a game, reading a picture book, posing problems, paper folding, writing a diary, or, completing a math trail.

<u>Write a story using plane figures (Grade 3 to 6; Space Strand)</u>. For plane figures, the child and the parent each do paper folding that results into paper Tan grams. Each will use all pieces and create an object (e.g. a tree; a kite). Then parent and child will make

a story using a tree and a kite and record the story in a diary. Sharing and communication is emphasized and spatial sense is enhanced, as they can remove the pieces and make other shapes and write different stories.

Although the activities were developed by me, I sent to three elementary school teacher to check if they were appropriate. Revisions were made until all agree on the timing, math materials, and interaction with parents.

A typical Friday Math Camp (Time: 13:40 to 15:20)

During the first session parents and children attended class in two separate classrooms. I shared with parents how children learn math. This session is also used for focus group interviews and filling in questionnaire. One elementary school teacher and the children went to another classroom. The lesson is mainly a hands-on activity. For example, asking children to make up teaching aids required for playing math game with parent in lesson two; folding paper to form a card for Mothers Day or, shading colours on fraction cards and cut them into a deck of poker cards.

Lesson 1: Parents Lesson 1: Children Lesson 2: Parent-Child 1, Parent-Child 2…n

During the second session parents and children attended class together. I introduced math games (so that learning took place in families). The seating arrangement is: one table, one family. I made sure each family could play on their own before all families played together. Finally, there was whole class discussion and I sent out a follow-up activity and a diary sheet to do at home for next week.

Data source and analyses. Over one school year, there is total of 18 bi-weekly Friday Math Camps. Thirty one families attended (6 fathers, 25 mothers; some families bring 2 children). Data source are: questionnaire, diaries, parents' focus group interviews, worksheets and videos; analysed qualitatively as in Creswell (2009). Data were coded by two independent raters and reliability was checked. Disagreements were discussed and resolved through e-mails or telephone discussions.

RESULTS AND DISCUSSIONS

In this project data set, I witnessed multiple roles identified by Civil and Bernier (2006). In the following I report on my findings on roles and clip in episodes on how these parental roles changed parents' interaction with children. I also discuss how the change might cause an effect on math learning, though the data analysed on children's math learning is reported elsewhere.

<u>Parents as Learners</u>: In the activity (Tan grams, Space Strand) parent-child pairs folded papers and made shapes the created a story to share with other families. All parents (n=31; 25 mothers, 6 fathers) expressed that this experience was new to them and they were working with their children as peers. "*My husband and I never know Tan grams can be so challenging. During the Moon Festival break, we fold papers when we bring the boys to see grandma and granddad. Dad pretended not to be interested (in this simple task), then joined in after we five started. The 6 of us had a wonderful time written a story by using all 6 figures*." (Focus group interview, 2011/10/14).

(Diaries, 11/05/13.)

<u>Parents as teachers</u>: In the activity (Fair Trade) the math materials is on conversion of units. A card is picked. On one side of the card wrote a smaller unit (e.g. ___days) and one the other side wrote one larger and one smaller unit (e.g. ___weeks ___ days). No matter which side faced the parent, he or she acted as the teacher and posed a problem (e.g. 36 days) the child then solved problem (5 weeks 1 day). The teacher told the child if the answer was correct. If correct, they shook hands and said, "This is a Fair Trade". If the answer was incorrect, the parent did not shake hand with child and said, "This is not a Fair Trade, please try again". "*This activity is so exciting and it made me feel that I am my child's teacher…*" (Parent focus group Interview, 11/11/11). Another evidence is from the Math Trail, all parents made up problems by reading photo and campus map. The final version of the activity included at least one problem from each parent and next to the problem I write the parent's name as author of problem. They acted as teachers for all those who signed up for this math trail. Finally, four parents were taking charge in the Math Fun Fair by the end of the school year. Parents expressed that they learned a lot, as they compared children's responses. "*When I watched how other children of the same age as my child attempted the problems and used various approaches, I came to understand why my child failed and I began to realize why…This Fun Fair enabled me not to worry too much, as I see that making mistakes is no big deal*". (100/06/10)

<u>Parents as facilitators</u>: A parent was doing story using one picture book by Marilyn Burns (Meat Ball Spaghetti) to teach perimeter/area. After doing so other parents would join in. A questionnaire will be used to ask which parents will be willing to teach children. The parent asked her girl to make 32 children's faces and 8 dining

tables. She linked the activity to math teaching successfully. When 8 tables are all separate then 32 guests can sit and 32 is the maximum number.. "*It was not the first time telling a story but telling a story to link to perimeter and area is a brand new experience. My girl returned home and moved the tables to check again…*" (Video, 11/12/09)

In addition to the above mentioned roles, I added two roles in math games. <u>Parents as partners to child</u>. In solving puzzles in Soma cubes (Leung & Lo, 2010), parent-child worked towards a common goal—to solve the puzzle as given in the figure. "*My child rarely learns to do things together with a peer. When two persons stood at different positions and stared at the cubes from different angles, the views helped to solve the puzzle. We two changed positions whenever we disagreed with each other. Taking other's perspectives is what we both learn.* " (Diary, 11/05/27). In the fraction card game adding to "1", each of them picked up a fraction card and see if they added to '1". After playing this game, parent and child were told to use the same deck of card and make up rules for new games. The ways were reported in diaries. <u>Parents as opponents to child</u>. When they played the game called Millionaire arithmetic (without banknotes and coins), each child and parents are competing. They threw a die, walked through the streets that were named by them and after each transaction each of them wrote expression to record the amount of money they had. Each one challenged the other if the record was incorrect. In this case, the two were opponents. "*I now realize the secret in successful parenting. It is to lose! If it is always the parent who wins the child will lose interest.* ^0^" (10/06/10) "*If the teacher gave a ditto with 20 horizontal addition and subtraction facts as homework my child will not complete it unless I remind her. Now with this game, she wrote 3 A4 pages of record and still interested to continue...*" (10/06/10)

Conclusion and Recommendations

Attendance is a often a challenge to parents project but when parents' needs or voices were attended and when parents' learning math with child is arranged over the Fridays, eager parents would choose to attend. In families with two jobs, father and mother took turns to attend. In this study, we collected evidence of parental interactions during Math Camps and also examples of parent-child interaction extension for carrying the math learning at home. The development of research-based tasks align to math curriculum cannot be done by MTE without consulting the elementary school teacher. The tasks designed by MTE and teacher, cannot be embedded in intimate parent-child activity nor be appropriate without referring to parents' completed questionnaires during the beginning of the term. In the model I propose, the intersection region of three overlapping circles is the key to success. The findings indicated that working with parents over a school year, it is feasible engage parents to take actions to motivate and monitor children (Cai, 2003). All in all, the model of three intersecting circles, together with identification of parents' roles, as given in Civil & Bernier (2006)can be used as guidelines for future intervention study on parental involvement in math learning of children. I close with anticipation to my next step: to empower parents, to include them as exemplars; and, to assist them to lead other parents in the community. Parents, as Civil and Bernier (2006) reminded, are resources; the resources are not limited to helping out in cafeteria or doing notice boards; the resources can be an enhancement in children's math learning.

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A PORTRAIT OF DISAFFECTION WITH SCHOOL MATHEMATICS: THE CASE OF HELEN

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Disaffection with mathematics is a complex problem, and one that is not easy to characterise in research terms. It is argued in this paper that it is necessary to look beyond the quantitative study of attitudes and beliefs in order to investigate the phenomenon more fully. It is proposed that a focus on the motivational and emotional aspects of disaffection from a phenomenological perspective might offer new insights. The paper describes a study of disaffected students studying mathematics at a Further Education College in the UK. The novel theoretical framework of Reversal Theory and the novel methods used in the study are described. An Interpretative Phenomenological Analysis is applied to the data concerning one student, and her intense subjective experience of disaffection is reported.

BACKGROUND

Disaffection with school mathematics is a widely acknowledged problem, and one that impacts at national level, but above all on practitioners and students themselves. It has received attention within the mathematics education research community. In the UK, survey data has established a quantitative picture of negative attitudes, and some qualitative studies have been reported. Nardi & Steward (2003) have described the notion of 'quiet disaffection', and Brown, Brown, & Bibby (2008) noted student perceptions of mathematics as boring and difficult. But disaffection is clearly a complex phenomenon, and as such, it is necessary to move beyond quantitative studies, and beyond the simple characterisation of disaffection as negative attitude. Outside of the UK, Schorr & Goldin have also studied negative affect, and they argue for "the need to study affect more deeply than the study of attitude permits." (2008, p. 132) And further "It is increasingly clear that the functioning of affect is far more complex than is suggested by considerations of positive versus negative emotions and attitudes." (2008, p. 133).

The study reported here is an attempt to widen the study of disaffection to include a more detailed investigation of the motivational and emotional aspects of the problem, and to widen the methodologies used to include more innovative approaches to understanding the complexity of the subjective realities of the perspectives of students. Thus the study was undertaken from a phenomenological and constructivist perspective, and Reversal Theory was used as a theoretical framework since it provides a coherent account of motivation and emotion from a phenomenological perspective (Apter, 2001). The theory proposes that there are eight motivational states that are ordered in binary oppositional pairs. We are in one state from each pair at any one time, although only one of the four states may be focal – that is, central to our experience of the world at that moment. We switch from a state to its opposite at frequent intervals

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(called a reversal). All states are seen as necessary for psychological health and well being. However, since each motivational state is associated with particular needs and values, we may be satisfied or not in our moment-by-moment experiences of the world. The combination of states and our experience of the world in this way gives rise to our emotions, whether positive or negative. A more detailed description of Reversal Theory and its application in this context, together with evidence of motivational states operating in a mathematics education context can be found in Lewis (2011b).

THEORY AND METHODS

The study involved students in two further education (post-16) colleges in the UK. On leaving compulsory education and entering college, all students who have 'failed' mathematics in national school examinations are required to study a Use of Mathematics course. It is thus likely that many of these students will be disaffected with mathematics. A mixed method approach was adopted, but in this paper we will focus primarily on the qualitative data acquired through interviews. Within the context of the interviews a range of methods was used. Prior to interview, students completed a survey instrument which sought data on their perception of stress and their experience of the frequency of negative emotion in relation to mathematics. The survey was adapted for this context from the Tension and Effort Stress Inventory (TESI-ME) (Svebak, 1993). The results were available at interview. Within the context of the interview projective psychological and graphical techniques were used to gain insights into the 'lifeworld' (Smith & Osborn, 2003) of the students. The graphical technique involved plotting the highs and lows of their relationship to mathematics on a timeline. The projective technique involved sorting cards showing motivationally and emotionally valent terms. These methods, and their justifications are described more fully in Lewis (2011a).

The interviews were recorded and transcribed. The method for analysis chosen for this study is Interpretative Phenomenological Analysis (IPA). The aim of IPA is to explore in detail how participants are making sense of their personal and social world, and the main currency for an IPA study is the meanings that particular experiences, events, and states hold for participants (Smith & Osborn, 2003). It is phenomenological in that it is based on the lifeworld, personal experiences and perceptions of individuals, and it is hermeneutic in that it is concerned with the meanings and sense-making of those individuals. It is seen as a counter to the prevalence of quantitative and experimental approaches of much cognitive psychology, with their focus on nomothetic or aggregate picture of the world. It thus favours the qualitative and the idiographic perspective. This approach lends itself to the study of individuals in complex realities, and is entirely consistent with the theoretical position adopted for the study as a whole. It is seen as a way to getting to the 'insider perspective' – the reality behind the subjective experience of young students.

The process of analysis followed the approach described by Gee (2011) which involves a series of steps and iterative passes through the transcript examining and re-examining the data from multiple perspectives.

THE CASE OF HELEN

Helen scored quite highly for overall stress on the TESI-ME (4 out of 7) with high scores for boredom (7), anger, humiliation and shame (6). These scores suggest a high degree of negative affect even compared to some of her peers in this sample. Her life history scores are medium with increases in year 5 and 7 (at aged 10 and 12 years), and then a significant decline in year 10 (aged 15 years). Helen's story can be told in relation to four key themes.

Decline into disaffection

Although Helen reports a positive relationship to mathematics over many years, her attitude to the subject is ambivalent throughout as is evidenced by the tentative way she uses language ("It wasn't that bad…"). Her decline into serious disaffection starts in year 9 when the maths gets harder and (as she sees it) the teaching is poor.

> mmm...basically we'd just go over the same stuff and we'd never learn...like he never explained it…and he left half way thro' the year so we always got supply teachers.

Her description of the confusion and procession of supply teachers being constantly swopped around throughout year 10 is heart-rending. As she says: "…we had no one to build us up…and then I didn't get entered into the year ten exam 'cos everything got messed around." By this time she was lost. "I just hated it." and "I'd just switched off by then."

One of Helen's key narrative themes is confidence. Her relationship with maths ebbs and flows in tandem with her confidence. And as well as being volatile, there is no sense of agency or internal regulation for her confidence or competence – it won't sustain on its own without the external push from the teacher. This is evident in many examples of the forms of language like:

> ok ...(pause)…mmm I know I got more confidence in year 5…'cos all my reports in year 5 say I was really good in maths

> we had no one to build us up...

It seems that as much as they can confer confidence, they can also take it away.

The experience of disaffection

In this section we look at Helen's description of the motivational and emotional landscape of the disaffection itself. In her early description she says "I hated it…I lost confidence." Later on she elaborates some of the detail. When asked about her high score on anger she provides a vivid description of how it comes about:

> It's like…I can't explain it… like nervous…like I can't do it…like you know when you can't do it...on the edge...'cos everyone else can do it…you're looking around...everyone's doing it…and I'm sitting there…

And then on the anger itself:

> Well...like sometimes when we're doing like decimals...I can't do 'em…nothing…and actually anger…and even how much she explains it to me I can't do 'em because it just doesn't register…then I get frustrated and then I'm not doing the work...because obviously I can't do it (laughs).

In motivational terms this is an interesting sequence. We can infer from her description that, when faced with a difficult task, she is in a serious-conforming state – i.e. wanting to complete the task as set. The nervousness or anxiety she describes is the emotion that arises in this state combination when arousal is high, and it is high because of her realisation that she can't do the task. She thus feels under threat. Her self-comparison to others suggests a feeling of mastery (losing), which in turn feels unfair and this trips her into anger (associated with a reversal to the rebellious or negativistic state). Helen's behavioural response to the anger is:" I can't do it so I don't do it…I'll sit there drawing." Later on she describes separately the sense of humiliation, which is self-mastery (losing):

> Like...humiliation sometimes…when you sit there like I say...you look back...and you think everyone else…and I can't do this and I should be able to do it.

And later on she amplifies the description:

> I was just staring at it…I don't know none of this…..and I used to sit there and look at it and think…that's jibber jabber to me…like I don't know nothing..

This sense of helplessness and hopelessness when faced with a task that she can't fulfil, and the motivational, emotional and behavioural sequence that follows is a recurrent pattern for Helen, and an influential part of her experiences with mathematics. One of the problems for Helen is that it takes time and repetition for her to achieve competence. Listening to explanations often doesn't provide learning for her.

> I don't know what it is…she explains it to me…sometimes I get it straight away…she keeps saying it to me...but I don't seem to register a lot…it don't seem...

Helen also seems sensitive to the nature of the delivery by the teacher. Here she describes how boredom comes about:

> When you're sitting there and someone's got such a dull voice and you're listening to them over and over again...it's just...like…you switch off…because…erm…you don't feel like it's something to listen to…does that make sense?

I **can't believe I'm any good**

This theme gives a brief account of the more positive experiences and satisfactions in Helen's relationship with mathematics. We have already noted the important and periodic boosts to her confidence, provided by transfer to a new school (and a new teacher) and transfer to a higher set.

yes...because I actually went into a higher class...and obviously that gives you a boost

But we have also noted the tentative nature of that confidence. So, when describing positive aspects of her experience of maths, her language also reflects the tentative hold she has on competence. Thus we have: "I ***actually*** (my emphasis) went into a higher class."; "I actually learnt it."; "I actually thought I was a bit smart ...to be in one of those top classes."

Notice the surprise implied by the 'actually', that she was only a 'bit' smart, and 'those top classes' also implies a non-personalisation – not my class, but their class. Even when she is temporarily successful, she appears to feel like mathematics is a club to which she does not fully belong. She expresses this in the third person, through the experience of her mother, when she says "It's become immune to her." The notion of immunity is an interesting metaphor (as is the fact that it is immune to her, and not the other way round). It suggests the hostile nature of maths to her, and the fact that she needs protecting in order to engage with it. The evidence in the current study suggests that this is a common phenomenon amongst these students.

Despite that, she does report some positive experiences. Helen chose the cards *feeling part of the group* and also *helping others*. Both of these involve socialising, and in Helen's mind, both involve the notion of help: "Yeh...I like doing groups 'cos I like everyone supporting each other more." And further on:

'cos...obviously you know if everyone's working in that group and if everyone can't do it...so if you're working in a small group it works out better because everyone can help each other... that's why I like helping others.

And helping others also helps her to learn:

Yeh...because when you're helping 'em…you're saying what you've just been learning...so it's helping because you're re-saying it like over and over again.

And the net result is "in the end I actually learnt it." – but again, note the use of actually, as if she is surprised by this. Help can also come from unlikely sources, such as being allowed to use a calculator (even though it feels a bit like cheating):

But were allowed to use a calculator…so I feel like I've got more help if that makes sense… so it feels like it's a backup to help.

Other positive aspects of her experience that she describes include having fun, maths that is 'real', and a teacher who is energetic and enthusiastic. She clearly responds to an element of intimacy in her relationships with teachers:

mmm...there's one teacher at my secondary school…I liked him quite a lot…and...in some ways he always used to boost my confidence..

You get on with them (teachers) when they teach you more.

The utilitarian contract – "I don't need you…well, I do need you"

Helen's story has shown that her fragile confidence and hold on mathematics was broken by her adverse experiences in year 9 and 10 at school, which led to her failure in national examinations. At the point of interview she was doing a course of mathematics that she was required to do. It would be easy to think that being a conscript rather than a volunteer would negatively affect her attitude and motivation, but the reality is more complex than that. Her ambivalent attitude, that she would like to say "goodbye for ever", but knows she can't, is reflected in the quote in the section title. In this she has been influenced by the experience and advice of her mother, who also struggled with maths to qualify as a nurse, but who stuck at it to achieve competence and success. So, despite the odds, she retains a sense of purpose and a sense of the importance of maths in her future life. When asked what she would do if she had to learn some (for her) complex maths in order to fulfil a future job role and gain promotion, she says:

> I would try it...I **would** (my emphasis) try it...I would take it...if it was an upgrade…you're obviously gonna have to take it… you're gonna have to learn it...there is gonna be people in that job that will know it and will be able to help you.

This desire to do what she sees as necessary frames much of her current experience.

> I'm sitting there and I'm bored but I know I've got to go there...I've got to pass my maths.

She is aware that she can take a long time to grasp an idea or achieve competence in a topic, but, however, difficult, she somehow maintains her effort:

> and sometimes I sit on my own and do it…I have to keep ...to read it over and over again. And further: (I get there) in the end…but it takes me a long time…see that decimals took me about two years.

However, this doesn't mean that all of her motivational effort is focussed entirely on the end goal. That would be difficult to sustain for a young person whose recent experience of mathematics is one mainly of failure. At college she encounters a regime different to school, and one that has elements of motivational satisfaction that sustains her. And her experience at college is different to school:

> It's not just M (teacher)...working in groups...we all laugh about things…she's quite laid back...she lets us just talk if were all doing the work...most teachers don't let you do that…they like you in silence.

An important point here is that there is sometimes an implication in research literature that low-achieving and disaffected students lack metacognitive skills such as effort and persistence. On the contrary, Helen is typical of many students in this sample who have to demonstrate a range of motivational and behavioural resources in the face of quite severe difficulty.

> like I'd sit there and if I'm in my own little world…I'll sit there and do it…I came out 100%...so….I think…I've got to concentrate loads as well.

She is also quite self-aware of how she learns: "I have to keep ...to read it over and over again."; and later: "I think I have to learn it for ages."

Her relationship to mathematics represents what we might call a utilitarian contract. From the evidence here, we can interpret the 'terms' of this contract from Helen's point of view as : it's important to get maths; it will be useful to me; if you give me confidence I can do it and if it's not too painful; then I will try and try. Many of the students in this sample have settled for a similar version of such a contract.

DISCUSSION

It is not appropriate to generalise from a case of one, and although there is a universality in our experience of motivational states, they are experienced and expressed in highly individual ways. The case of Helen demonstrates a number of points. Her up and down relationship with mathematics shows that we should not rely on statistics to tempt us to think that students are fixed on some positive-negative scale of attitude. Like Helen, the data in the current study shows that for many, their relationship to mathematics is volatile and fragile. Among other things, this study shows some of the ways in which students become disaffected, and this is vital if we are ever to gain traction on the problem at the level of policy, curriculum or practice.

Another thing that the case study illustrates is the motivational and emotional complexity of students' relationship to mathematics. The whole range of motivations, together with positive and negative emotions weaves in and out of their experience. Perhaps surprisingly, many disaffected students like Helen are highly motivated (but often unsatisfied), and apply a range of skills related to effort and persistence.

Finally, the theoretical framework, the methods of data collection and analysis, although perhaps new to the field seem to warrant further investigation.

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TWO CASES OF RAPID AND PROFOUND CHANGE IN MATHEMATICS TEACHERS' PRACTICE

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Change in mathematics teachers' practice is often characterized as something that takes time and sustained intervention. In this article, I present the results of research that highlights a different kind of change – a profound change that takes place very quickly. Based on the analysis of 42 cases of such rapid and profound change, I have produced a disaggregation of this phenomenon into five distinct mechanisms of change, each one rapid and profound – two of which are presented here. This disaggregation shows that not all changes, even when outwardly similar, are the same.

INTRODUCTION

Teacher education literature is full of examples of teachers' changing their practice. Usually, these examples are found in research examining specific professional development models such as action research (Jasper & Taube, 2004), lesson study (Stigler & Hiebert, 1999), communities of practice (McClain & Cobb, 2004; Wenger, 1998), or more generally, collegial discourse about teaching (Lord, 1994). Such research has very effectively delineated different mechanisms by which teachers change while participating in a variety of professional development setting. Conclusions show that with time and continued intervention, support, and collaboration teachers can make significant and robust changes to their practice.

As a mathematics inservice teacher educator working in a variety of professional development settings I have witnessed teacher change of the form exemplified in the aforementioned research. But I have also witnessed change of a different kind – rapid and profound change in practice – examples of which are not to be found in the literature. In what follows, I first introduce the unobtrusive research methodology that allowed me to identify examples of rapid and profound changes in practice. I then present the results of a deeper analysis of this phenomenon, which led to its stratification into five distinct mechanisms of change.

METHODOLOGY

Working as both a mathematics inservice educator and a researcher interested in the contextual and situational dynamics of the inservice setting, I find myself too embroiled in the professional development activities to adopt the removed stance of observer. At the same time, my specific role as facilitator prevents me from adopting a stance of participant observer. As such, I have chosen to adopt a stance of *noticing* (Mason, 2002). This stance allows me to work within the inservice setting to achieve the professional development goals called for in the different settings, while simultaneously being attuned to the experiences of the teachers involved. At the same time, this stance allows me to engage in these experiences as a researcher without the requirement of an a priori research question. In so doing, from time to time I notice phenomena that strike me as interesting [1]. Occasionally, these are phenomena that occur in more than one setting and speak to invariance between individuals, settings, contexts, or behaviours.

Using this methodology of noticing within a diverse number of mathematics professional development settings – from workshops to learning teams to graduate programs – I have, from time to time, noticed teachers undergoing *rapid and profound change*[2] in their beliefs and practices. This phenomenon is rare. Most teachers engaged in inservice work follow a trajectory of change that is much more pedestrian. At first such changes surprised me and I was immediately suspicious of the self-reported accounts of the almost instantaneous revision of practice that some teachers spoke about. But as more and more of these accounts accumulated I decided to investigate further. I stepped out of my role as facilitator and assumed a stance of researcher. I began to interview teachers, to visit their classrooms and to observe their teaching. I spoke with their colleagues, their administrators, and their students. I took field notes and I wrote narratives (Clandinin, 1992; Clandinin & Connelly, 1996). In the end, I compiled data on 42 cases of mathematics teachers' rapid and profound changes in practice, which were analysed using a grounded theory (Charmaz, 2006) approach in general and a constant comparative method in particular.

Results

From this analysis, the phenomenon of rapid and profound change fragmented and converged into five different types which I have come to call: *(1) conceptual change; (2) accommodating outliers; (3) reification; (4) leading belief change;* and *(5) push-pull rhythm of change.* Each of these types embodies a different mechanism for transformation of teaching practice that needs unique analysis in order to bring it into sharper focus. For purposes of brevity, in what follows I focus only on two mechanisms: *accommodating outliers* and the *push-pull rhythm of change*[3]. I chose these categories because they exemplify nicely the nuanced differences inherent in the five mechanisms. I present each first with an abbreviated case, chosen for its ability to

[1] Of course, what I notice is first and foremost predicated on what I find interesting. As a person who works in professional development settings I find all things associated with teacher change interesting.

[2] In the settings in which I work my contact with teachers is discrete; constituted of a series of meetings at regular intervals. Teacher change in this setting is overwhelmingly observed to be incremental, gradual, and tentative, stretching out over several meetings and involving encouragement, planning, experimentation, and refinement. My use of the term *rapid and profound* is meant to describe changes that stand in stark contrast to this more usual form of change. So, for example, a teacher who is observed to change from no (or little) use of group work to ubiquitous use of group work wholly between two consecutive meetings without apparent trepidation is seen to have made a change that is rapid and profound.

[3] All five cases are presented in Liljedahl (2010).

succinctly exemplify the category of change, and then with a brief analysis of the individual case as well as the category as a whole.

Accommodating Outliers

Mitchell is a middle school teacher with eight years of teaching experience. He has always taught mathematics and he has a very clear sense of what is important for students to learn in mathematics and what his role as a teacher is in this context. For Mitchell, mathematics is really just a game – a game with set rules and very clear outcomes. Mathematics is a collection of skills and facts that need to be mastered before going on to the next level. As a teacher, he sees his job as assuring that each student learns these skills and facts – and to not let anyone advance to the next grade until they have done so. He also has a very clear idea of what the students' role is. Their job is to learn the material that is being taught and to be able to demonstrate mastery at the end of a unit … and at the end of the year. Mitchell is a traditional mathematics teacher in every sense of the word and he has no issues about stating so.

Mitchell's mathematics classroom is a pillar of traditional teaching. He adheres to a standard lesson of review-demonstrate-mimic-practice and students are expected to seek his help if they are stuck or do not understand. He only uses questions that are unambiguous and lead to closed-form single solutions. He feels that mathematics needs to be taught (and learned) in this fashion and that all of the problems that face mathematics education are due to deviation from this tradition.

Ironically, Mitchell is not this traditional in teaching his other subjects so there are some aspects of more reform oriented teaching that have seeped into his mathematics lessons. For example, he does allow his students to sit in pairs and to work together on in-class assignments. He also, from time to time, gives a problem solving activity to students, but he sees this as extracurricular and does not allow it to figure into his assessment and evaluation schemes.

Mitchell does not shun professional development opportunities, engaging in them with the expectation that he will "get something out of them". However, he openly admits that many of these opportunities turn out to be things that he already does, which further reinforces his conviction that his method of teaching is "on the right track". Occasionally he learns something that is interesting, which he then implements in his teaching. This is how he came to start doing some problem solving activities in his classroom, and as he states, "there are some really fun activities that I now do with my kids". Of course, there are things he sees in workshops that he also dismisses outright as being "completely pointless", such as a session on performance based assessment that he once attended.

Although I had met Mitchell as a participant in a number of single session workshops, I did not begin to interact with Mitchell until he became a member of a district-based learning team that I was facilitating. This team was formed for the purpose of creating numeracy tasks for district wide assessment. Mitchell came to this learning team with the expressed purpose of offering some of his expertise in creating "really comprehensive final exams".

The first task of this team was to co-construct a definition of numeracy. Initial attempts to do this resulted in definitions that were more closely associated with fluency of arithmetic. In order to get past this initial definition I suggested that they think of students that they had taught in the past who were very good in mathematics, and to further think what qualities they possessed that allowed them to be good in mathematics. This dramatically changed the discourse about numeracy and rather quickly a more sophisticated definition emerged – "*Numeracy is not only an awareness that mathematical knowledge and understandings can be used to interpret, communicate, analyze, and solve a variety of novel problem solving situations, but also a willingness and ability to do so*."

> The team then set out to design a task that would measure some of the capacities embodied within this definition. Over the course of four additional meetings stretched out over approximately eight weeks the team went through three iterations of a design-test-refine process before they arrived at the final task. During this process I saw Mitchell undergo tremendous changes in his teaching. After pilot testing the initial version of the task he was talking about things that needed to change in his classroom in order for his student to be successful. He was restructuring the way he thought about and facilitated group work; he was redefining his own notion of what constituted a good mathematics question and a good mathematics answer; and he was trying to find ways to change the dispositions of the students in his classroom. In a very short period of time Mitchell came to change most of what he held to be true about mathematics and mathematics teaching and learning.
>
> This is not to say that Mitchell completely reconstructed his teaching practice overnight. He spent the remainder of that school year struggling to actualize some of his ideas as he swam against the current of already entrenched student expectations and dispositions. At the beginning of the following school year, however, his classroom was truly transformed. Lessons were now modeled on explorations initiated by interesting (often open ended) tasks which were worked on in groups and concluded with whole class discussion. His assessment practices were also completely redesigned, although still very much a work in progress.

Mitchell did not come to the learning team looking for answers. In fact, he came for quite the opposite reasons – to provide answers. In a way Mitchell's teaching can be seen as impenetrable. He participated in a wide variety of professional development opportunities, but nothing had any effect on his practice. Invoking the discourse of adaptation á la Piaget we might say that Mitchell was not accommodating new ideas into his existing schema of teaching mathematics (Piaget, 1968). He tended to deal with new ideas about the teaching of mathematics in one of three ways. First, and most common, Mitchell would find within his professional development experience something specific that resonates with his current teaching. This point of commonality was then used to support his basic assumption that "*I already do that*". He was able to make this claim no matter how minute the point of commonality was. A nice example of this was Mitchell's insistence that he used effective questioning in his teaching because, when teaching from the front of the class, he asked a lot of questions and when he attended a workshop on effective questioning he learned that sometimes an effective question can be short and very directed, as his always were. Aside from ensuring that very little of substance penetrated his practice, this strategy of assumed commonality also served to entrench Mitchell's teaching practice as he was constantly reassured that he was "*going in the right direction*".

If Mitchell did not find any points of commonality, but the new ideas that presented to him caught his interest he tended to incorporate them into his practice. But he would do so without letting it impact on his conception of himself as a mathematics teacher, or on his general notions about mathematics and the teaching and learning of mathematics. As an example, Mitchell very easily introduced a program of problem solving into his practice. He had a small, but good, collection of problem solving tasks that he gave to his students from time to time. There was no attempt to assess their performance on these tasks or to pull some of the affordances such tasks could offer into the rest of his teaching. As such, he kept it very much as an extracurricular activity not allowing it to redefine his teaching or his conception of himself as a teacher. Mitchell was not only assimilating these problem solving experiences – he was actively not accommodating them.

Finally, if Mitchell found no points of commonality and no points of interest he would simply dismiss the new ideas presented to him as "*pointless*". As mentioned, this very effectively allowed him to deal with the complex nature of alternative assessment in general and the specific ideas around performance based assessment in particular.

Ironically, as effective as Mitchell was at avoiding accommodation, it was accommodation that eventually led to the revision of his practice. From interviews with Mitchell it became clear that a real turning point for Mitchell was the construction of the definition of numeracy. The definition itself meant very little to him, it was more the consideration of the good students Mitchell had had in the past that lead to deeper changes. Mitchell had always been aware of these students, but he had effectively not allowed their existence to impact on his aggregated vision of a mathematics student. To him they were outliers, as were the capacities that they possessed. For Mitchell, a student was seen from a deficit perspective. They were learners that lacked specific knowledge – knowledge that he possessed and would apportion out to them over the course of the school year. When he started to think of these outliers he not only saw them as capable, but he also saw a whole spectrum of skills that he had never really considered before. Problem solving abilities, divergent thinking, awareness of the mathematics inherent in a specific context, the ability to use mathematical concepts broadly in different contexts, etc. were suddenly seen as capacities that all students needed in order to be successful in mathematics.

As Mitchell struggled to reform his teaching for the remainder of that first year my work with him continued. In subsequent interviews Mitchell revealed that he was now beginning to make sense of why the problem solving tasks that he had previously been using as extracurricular were so effective at developing some of these aforementioned capacities. At the same time, Mitchell was beginning to see a new set of capacities requisite for students to be successful at the numeracy tasks that he had participated in designing. Group work, ability to articulate thinking, persistence, tolerance of ambiguity, and comfort with being stuck were now the deficiencies that he wanted to address.

In the consideration of both talented students and good problem solving tasks Mitchell was finally accommodating information into his practice. But this was not new information. Rather, it was information that he had previously incorporated into his schema by keeping it as outliers. That is, he had kept it compartmentalized and away from his normal constructs of what constituted a mathematics student and a mathematical activity respectively. In the end, the reform of his teaching happened when he began to accommodate these outliers.

Mitchell is not the only teacher that I have encountered who was so effective at not accommodating new information. There have been many others. It is easy for a teacher to become entrenched in their practice, and it is easy for them to stay entrenched by using the "*I already do that*" strategy that Mitchell did. In my data there are five additional cases in which such teachers reformed their practice. In each case their change was initiated by an eventual accommodation of outlying information – information that was acquired through their own teaching.

Push-Pull Rhythm of Change

> Karen is a high-school teacher with 16 years of experience. She loves teaching and prides herself on the level of care she brings to her practice – especially the care she shows for her students and their success in the exam at the end of the year.
>
> Karen's practice can best be described as meticulous. She is very careful to make sure that she covers every prescribed learning outcome in a timely manner so that there is plenty of time left over at the end of the year for review. This is not to say that she rushes through the curriculum – she doesn't. Her pace is deliberate and methodical with not a minute wasted. Karen builds her instruction around homework, which is carefully selected from the textbook as well as a databank of old exams. Each lesson begins with a review of any problematic questions from the previous homework. She then moves quickly into a lesson carefully designed to prepare the students for the forthcoming homework assignment. This usually involves taking the class through a carefully designed sequence of examples each one incrementally different from its predecessor.
>
> Karen makes herself available to her students before school, after school, and at lunch and actively encourages students to come and work on their homework under her supervision and tutelage during these times. Karen's greatest pride is the level of success some 'challenged' students are able to achieve through diligent use of this service.
>
> I first encountered Karen in a numeracy task-design team that I was facilitating. This team was similar in focus to the one Mitchell partook in although slightly different in terms of end-users. In this particular learning team, we were working on designing tasks for the team's use rather than for the whole district. For the first two sessions of the learning team, Karen worked very happily on the activity of creating a numeracy definition and a numeracy task. Then, suddenly in the third session, she revolted against the entire enterprise espousing beliefs that the educational system as a whole did not allow for students to succeed in such activities. Surprisingly, she came to the fourth session quite excited about the shortcomings of her students in their abilities to complete the task and committed to making changes in her teaching to ensure that they would be successful. She talked about implementing group work, developing thinking and communication skills, as well as the need to engage the students in more open-ended activities as a way to nurture these capacities. Karen continued talking about changes to her teaching over the next few sessions. Then in the seventh session her discourse suddenly shifted from teaching to learning as she began sharing with us her efforts around particular students.
>
> Karen's practice continued to change for the remainder of the year. The following fall, with the start of a new school year, her teaching practice was unrecognizable in comparison to the type of teacher she had been a short 10 months previously.

Outwardly, Karen's case can be seen as being similar to Mitchell's. In fact, their discourse on the changes they needed to make in their teaching in order to allow their students to succeed at the numeracy tasks was almost identical. Deeper analysis, however, revealed that Karen's transformation was more steeped in a natural rhythm of change than in accommodation.

The *push-pull rhythm of change*, as I have come to call it, is a phenomenon comprised of a series of two, three, or four distinct phases, always in the same sequence, each having its own associated name. The names are an amalgamation – the prefixes *exo-* and *endo-* come from Greek meaning outer, outside, external and inner, inside, internal respectively; while *-spection* comes from the Latin *specere* which means 'to look at'.

Phase 1: exo-spection (x)

The teachers work on an activity, which, at the time, occupies their focus. This could be a problem solving exercise or the designing of a lesson, task, or assessment rubric. Whether or not the activity is relevant to their own teaching practice is immaterial as the teachers' focus is on the completion of the task, rather than on the potential for the task to inform their own practice. That is, the teachers are looking at the activity as lying outside of themselves. Although not a push, this phase can be characterized as a holding of the activity away from themselves.

Phase 2: eXo-spection (X)

The teachers realize that the problem they have solved, or the lesson or task they have built, is not commensurate with their own classroom context. They see this as a large- scale problem bemoaning the poor state of affairs of all students and the educational system in general. They look at the source of the problem as laying far outside of themselves, for example, societal expectations, the curriculum, the evils of external examinations or deterioration of standards, and speak of systemic reform as the only solution. As such, they are not only pushing the problem further outside of themselves, but also broadening its scope.

Phase 3: eNdo-spection (N)

Suddenly there is a change in the teachers' disposition – the problem, regardless of where it lies, must be solved within their own practice in the scope of the classroom. Now the conversations are about what they can do within their teaching in order to enable their students to be successful in solving a specific problem, completing a specified task, or performing well on a given assessment. The teachers are no longer pushing the problem, and any subsequent solutions, away from themselves, but are rather *pulling* it back to their locus of control.

Phase 4: endo-spection (n)

Finally, there is a shift of attention to the plight of individual students. The conversations shift from the classroom to a particular student or subset of students, and with this shift comes a narrowing of focus on their influence as teachers. This final shift is also marked by a subtle shift in discourse from *teaching* to *learning* as they *pull* the problem closer to themselves.

It should be noted that I have deliberately avoided using the term *introspection*, which means to examine one's own thoughts and feelings. This is not what I am trying to capture here. *Endo-spection* is not about looking inside oneself, but about looking at something as lying inside of oneself or one's locus of control. Conversely, *exo-spection* is about looking at something as lying outside of oneself or one's locus of control.

Of the 16 teachers who exhibited this rhythm of change in their practice three went through the XN phases of change only, while ten went through the xXN sequence. The remaining three teachers, Karen included, went through the full xXNn sequence. This is not to say that I haven't also encountered teachers who have exhibited just one of these phases – I have. However, one phase does not a change make.

CONCLUSIONS

Not all change is the same. I began this article by drawing the distinction between the more common forms of teacher change that teachers experience as participants in the popular sustained and collaborative intervention settings and the phenomenon of rapid and profound change in practice that I had witnessed. But even instances of this phenomenon, as rare as it is, are not to be seen as being the same. Within this phenomenon I identified five distinct mechanisms of change which I came to call: *(1) conceptual change; (2) accommodating outliers; (3) reification; (4) leading belief change;* and *(5) push-pull rhythm of change*. Awareness of rapid and profound change in general and the nuanced characteristics of these five distinct mechanisms in particular have alerted me, as a person who works within the field of mathematics professional development, to new possibilities within both my practice and my research. Personally, I now question the assumptions upon which much of the professional growth literature has been predicated. Deeper analysis of each one of these richly diverse and complex mechanisms is certainly warranted.

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ASSESSING IMPULSIVE-ANALYTIC DISPOSITION: THE LIKELIHOOD-TO-ACT SURVEY AND OTHER INSTRUMENTS

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The likelihood-to-act (LtA) survey is a 32-item instrument that measures impulsive and analytic dispositions in solving math problems. In this research report, we compare it to other instruments related to the impulsive-analytic construct such as Frederick's Cognitive Reflection Test (CRT) and the Barratt Impulsive Scale in terms of mean scores, Cronbach alpha values, and correlation values. Both LtA-Impulsive and LtA-Analytic subscales have acceptable reliabilities of 0.79 and 0.83 respectively. The LtA-Analytic and LtA-Difference (analytic-impulsive difference) correlated well with other the Need for Cognition subscale and CRT scores. The correlations involving LtA-Impulsive subscale were unexpected and call for further investigation.

INTRODUCTION

The behaviour of "doing whatever first comes to mind … or diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 307) is quite common among students while solving problems in mathematics. The term *impulsive disposition* refers to a tendency to proceed with an action that comes to mind without analysing the problem situation and without considering the relevance of the anticipated action to the problem situation (Lim, Tchoshanov, & Morera, 2009). In contrast, the term analytic disposition refers to a tendency to study the problem situation prior to taking actions. The premise underlying our work is the view that learning opportunities should be provided to help students progress from impulsive disposition to analytic disposition.

Related Theoretical Constructs

Various researchers in cognitive psychology have posited two distinct cognitive systems of reasoning: implicit-explicit (Reber, 1993), associative and rule-based (Sloman, 1996), and System 1 and System 2 (Stanovich & West, 2000). According to Evans (2006), "System 1 processes are rapid, parallel and automatic in nature: only their final product is posted in consciousness" whereas "System 2 thinking is slow and sequential in nature and makes use of the central working memory system" (p. 454). Sloman (1996) points out that the two systems often work cooperatively despite having different goals and specializing at different kinds of tasks. At times, they may each try to generate a response. A response is considered impulsive when System 1 hijacks one's attention, and reflective when System 2 overrides System's 1 response.

Whereas the dual system model can explain the impulsive-reflective distinction in terms of general functioning of cognitive processes, cognitive style can account for

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individual variability in impulsiveness. The *Matching Familiar Figures Test* was developed to assess children's *cognitive tempo* (Kagan et al., 1964). An impulsive is one whose response time is faster than the median and whose accuracy rate is below the median, whereas a reflective is one whose response time is slower than the median and whose accuracy rate is above the median.

In mathematics education characterizing students' impulsivity-reflectivity in terms of problem-solving disposition is arguably more useful than in terms of cognitive style. A disposition is context-dependent whereas a cognitive style is a personality trait that is stable across situation and across time. In addition, viewing impulsivity-reflectivity as a continuum, rather than as a dichotomy, is more likely to influence educators to help learners progress from impulsive disposition to analytic disposition.

Instruments for Assessing Impulsive-Analytic Disposition or Related Constructs

A reliable way to investigate students' problem-solving disposition is through task-based interviews (Clement, 2000; Goldin, 1998) and think-aloud protocols (Ericsson & Simon, 1993). Impulsive anticipation and analytic anticipation can be identified from the careful analysis of students' responses to interview tasks (see Lim, 2008). Although well-suited for uncovering problem-solving disposition in individual students, this mode of data collection is not practical for large-scale assessment.

Well-designed mathematical problems can be an effective and efficient means to assessing impulsive-analytic disposition. Frederick (2005) designed a three-item test for assessing *cognitive reflection*—" the ability or disposition to resist reporting the response that first comes to mind" (p. 35). For the bat-and-ball problem in Figure 1, the wrong answer of 10 cents is considered *impulsive*. A person with only a moment of reflection would realize that the difference between $1.00 for bat and 10 cents for ball is not $1.00.

1. A bat and a ball cost $1.10. The bat costs $1.00 more than the ball. How much does the ball cost? Answer: ________ cents

2. If it takes 5 machines 5 minutes to make 5 widgets, how long would it take 100 machines to make 100 widgets? Answer: ________ minutes

3. In a lake, there is a patch of lily pads. Every day, the patch doubles in size. If it takes 48 days for the patch to cover the entire lake, how long would it take for the patch to cover half of the lake? Answer: ________ days

Figure 1: The three items in Frederick's Cognitive Reflection Test

Another way to measure is to use questionnaire. The *Barratt Impulsiveness Scale Version 11* (BIS-11) is a 30-item self-report instrument for assessing one's impulsivity. Patton, Stanford, ad Barrrett (1995) performed a principal components analysis and confirmed three subtraits of impulsiveness: attentional, motor, and non-planning. A representative item for each subtrait is listed accordingly in Figure 2.

6. I have racing thoughts.
17. I act on "impulse".
27. I am more interested in the present than the future.

Figure 2: The three items in BIS-11

A questionnaire that assess a different but related construct is the *Need for Cognition* (NfC) scale. This 18-item self-report instrument measures one's "tendency to engage in and enjoy thinking" (Cacioppo & Petty, 1982, p. 116). An example of NfC item: "I would prefer complex to simple problem".

In the context of mathematics problem solving, Lim et al. (2009) developed the Likelihood-to-Act (LtA) survey as a means to measure impulsive-analytic disposition. For each LtA item (see Figure 3) respondents indicate on a scale of 1 to 6 how likely they are to respond to a given mathematical problem in the described manner.

1. $78 + 987x + 654 + 321x = x + 987x + 654 + 321x$ When asked to solve for x, how likely are you to begin by studying the equation and noticing the solution?	**1 2 3 4 5 6**
4. When solving a problem in mathematics, how likely are you to read and understand the problem thoroughly before deciding what to do?	**1 2 3 4 5 6**
17. $90 + 1234n + 567 + 89n = n + 1234n + 567 + 89n$ When asked to solve for n, how likely are you to begin by combining like terms?	**1 2 3 4 5 6**
20. When solving a problem in mathematics, how likely are you to use the first idea that comes to mind?	**1 2 3 4 5 6**

Figure 3: Four items in the current of the LtA instrument

The first version with nine pairs of items (one impulsive and one analytic per pair) was administered to 318 undergraduates, mostly pre-service teachers; the reliabilities of the impulsive and analytic subscales are found to be 0.64 and 0.63 respectively (Lim et al. 2009). The second version with 16 item pairs was administered to 119 pre-service and in-service teachers; the reliabilities for the two subscales are 0.74 and 0.81 (Lim & Morera, 2011). The written work of 92 participants for 6 open-ended math problems were analysed and coded; the coded scores for written responses were found to significantly correlated to both the LtA subscales (Lim & Mendoza, 2010). Based on the findings, one pair of items was replaced and seven items in the second version were modified to produce the current version of the LtA.

The 32-item version has a 2×4×4 structure: 2 types (impulsive and analytic), 4 categories (algebra, fraction, word problem, and non-mathematically-specific), and 4 items per type per category. Item 1 is analytic-algebra, Item 8 is analytic-nonspecific, Item 17 is impulsive-algebra, and Item 24 is impulsive-nonspecific.

The purpose of this research report is to (a) present students' responses to the LtA items, (b) compare the reliabilities of the LtA subscales to those of other instruments, and (b) present the correlations among measures obtained using these instruments.

METHOD

A total of 495 undergraduates, mostly pre-service teachers, participated in this study. A convenience sample involving 17 classes was used because some instructors chose not to give up their class time for us to collect data. Out of 470 participants who specified their program, 29 majored in either math or engineering, 80 are pre-service 4-8 teachers specializing in either math or science, 54 are pre-service 4-8 generalists, and the remaining 307 are pre-service elementary or bilingual or special education teachers. Out of 466 who specified their gender, 72 are males.

Within a 100-minute class period, participants took a set of three surveys (NfC, BIS-11, and LtA), took a version of a math test consisting of eight multiple-choice items (see Figure 4 for examples) and three cognitive-reflection items (see Figure 1), received warning that some of the items in the math test are "tricky", and took another version with structurally-equivalent items. Because of page limit constraint this research report does not focus on the effect-of-warning part of the study. Instead, it focuses on comparing students' scores as measured using the various instruments.

4. Benito needs to increase the amount of money he has now by 20% so that he can buy a $540 laptop. How much more money does he need?
   (a) $90
   (b) $108
   (c) $432
   (d) $2700

5. A 20-member choir takes 15 minutes to sing a song. How long does it take a 100-member choir to sing the same song?
   (a) 3 minutes
   (b) 15 minutes
   (c) 75 minutes
   (d) 95 minutes

Figure 4: Two multiple-choice items in the math test

The data analysis is based on a sample of 460 participants, with the exclusion of 23 students who had taken the LtA survey before and 12 students who had more than 2 missing entries in the LtA survey. These 460 students took an average of 8.8 minutes to complete the LtA survey, ranging from 3 to 19 minutes (based on students' self-record of start time and end time).

RESULTS AND DISCUSSION

Table 1 shows the percentage of responses for the four items in Figure 3 across the six likelihood numbers, the mean likelihood score (1 = extremely unlikely; 6 = extremely likely), and the standard deviation. For the pair of algebra items, students tended to choose higher likelihood for the impulsive Item 17 than the analytic Item 1. When comparing the mean response for Item 17 to Item 1, the estimated difference is -0.75 ($p < 0.001$). The analytic-impulsive difference is also negative for ten other math-specific pairs. As a group, the mean of the 12 math-specific analytic items is 3.61 and the mean of the 12 math-specific impulsive items is 4.64, the analytic-impulsive difference is -1.0 ($p < 0.001$). For the pair of non-math-specific items (#4 and #20), the analytic-impulsive estimated difference is 0.72 ($p < 0.001$). As a group, the mean of the four non-math-specific analytic items is 4.52 and the mean of the four corresponding impulsive items is 4.29; the analytic-impulsive difference is 0.23 ($p < 0.001$). These results suggest that students tended to think that they are analytic but when specific mathematical situations are used they tended to respond in an impulsive manner.

	Percentage of Responses						Mean Score	Std. Dev.
	1	2	3	4	5	6		
Item 1	3	6	8	22	33	28	4.59	1.32
Item 4	2	2	8	21	26	41	4.91	1.19
Item 17	2	2	3	7	25	61	5.34	1.08
Item 20	4	9	15	22	31	17	4.19	1.38

Table 1: Results for the two pairs of LtA items in Figure 3

Table 2 presents the statistics for the various instruments. LtA-Difference refers to the difference between the analytic item and the impulsive item in each pair (a negative value means more impulsive than analytic). The reliability of all the measures, except LtA-Difference, are greater than 0.7, above which is considered acceptable. The LtA-Difference has a lower reliability than the individual subscale reliabilities (0.79 and 0.83) because of the combined variability in two subscales.

Reliabilities	Number of Items	Number of Subjects	Mean	Std. Dev.	Cronbach's Alpha
LtA-Impulsive	16	460	4.55	0.69	0.790
LtA-Analytic	16	460	3.83	0.76	0.826
LtA-Difference	16	460	-0.72	0.81	0.681
BIS-11	30	458	2.02	0.33	0.804
Need for Cognition	18	459	3.30	0.57	0.817
Math-MCQ	16	426	6.83	3.20	0.738
Math-CRT	6	426	0.67	1.37	0.818
Math-Confidence	22	426	4.03	0.55	0.900

Table 2: Statistics for the measures obtained in the various instruments

Math-CRT is computed by summing the scores for the cognitive-reflection items; students averaged 0.67 corrects out of six items. When restricted to the three original CRT items (Figure 1), our students, mostly prospective teachers, averaged 0.32 out of 3 items, which is lower than those found by Fredericks (2005), ranging from 0.57 in the University of Toledo to 2.18 in MIT. In fact, 342 out of 433 students who attempted the CRT items got all three items wrong. The skewness of CRT towards zero raises the issue of its discriminatory power, especially for the population of pre-service teachers.

Math-MCQ is computed by summing the scores for the multiple-choice items in the math test; students averaged 6.8 corrects out of 16 items with a standard deviation of 3.2. Unlike Math-CRT, Math-MCQ is less skewed and more normally distributed.

For each math item, students indicated their level of confidence on a 5-point scale (1="I'm certain I'm wrong, 2="I think I'm wrong" … 5="I'm certain I'm right"). Math-Confidence is computed by averaging their confidence scores across the 22 items. Interestingly, students generally thought that they are correct (mean of 4.03) yet got only got 43% and 11% correct for the MCQ items and CRT items respectively. More specifically, 12% and 57% of students got Item 4 and Item 5 in Figure 4 correct, and only 10%, 12%, and 10% for the three CRT items in Figure 1. These results are consistent with Frederick's (2005) findings that respondents who missed the problems thought they were easier than the respondents who got them right.

Table 3 presents the Pearson correlations and associated false discovery rate adjusted *p*-values among the mean scores (Benjamini & Yekutieli, 2001). The highlighted correlations indicate that the LtA-Impulsive did not behave as expected. LtA-Impulsive items should be in oppositon to LtA-Analytic items, yet the mean scores exhibit significant correlations with the LtA-Analytic, with the NfC, and negatively with the BIS-11 mean scores. On the other hand, the significant negative correlation between LtA-Impulsive and Math-CRT mean scores suggests that LtA-Impulsive items might be valid in assessing non-cognitive reflection.

	LtA-I	LtA-A	LtA-D	BIS	NfC	MCQ	CRT	Conf
LtA-Impulsive	1							
LtA-Analytic	0.38**	1						
LtA-Difference	-0.49**	0.62**	1					
BIS-11	-0.21**	-0.16**	0.03	1				
NfC	0.21**	0.37**	0.17**	-0.45**	1			
Math-MCQ	-0.04	0.09	0.12*	-0.12*	0.25**	1		
Math-CRT	-0.16**	0.20**	0.34**	-0.07	0.24**	0.51**	1	
Math-Confidence	0.18**	0.28**	0.12*	-0.22**	0.28**	0.28**	0.31**	1

Table 3: Correlations among various scores (** *p*<0.01; * *p*<0.05)

The LtA-Difference scores are behaving as expected since they are significantly correlated with the other measures except BIS-11, which measures impulsivity in everyday actions rather than in mathematical situations. This may explain why the negative correlation between BIS-11 and Math-CRT is not significant.

Interestingly, BIS-11 and Math-Confidence means scores are significantly negatively correlated, which suggests students who are impulsive in life tend to be not confident in math. The correlation of 0.24 between NfC and Math-CRT is consistent with the correlation of 0.22 found in Frederick's (2005) study.

The high correlation between Math-CRT and Math-MCQ can be taken to mean that the two set of math items are related by a common factor—possibly the impulsive-analytic disposition. Both Math-CRT and Math-MCQ correlated equally well with NfC. The LtA scales correlate better with the Math-CRT than with the Math-MCQ.

CONCLUSION

The LtA instrument is still in its early phases of development. By having four categories of items in the LtA, we learn that non-math specific items seem to behave differently from the other three math-specific categories. This finding is consistent with the results we found in our principal factor analysis, which will be presented elsewhere. In this paper, we investigate the criterion-related validity for the LtA survey by examining the Pearson correlations between the LtA-Impulsive, LtA-Analytic, LtA-Difference, BIS-11, NfC, Math-MCQ, Math-CRT, and Math-Confidence mean scores. Results support the criterion-related validity of the LtA-analytic subscale and LtA-Difference. The unexpected correlations involving the LtA impulsive subscale call for further investigation. We are currently analysing interview data to study 15 respondents' problem-solving dispositions and their interpretations of the LtA items.

In addition, our study confirms reliability of three established instruments. Our results involving pre-service teacher corroborate with those involving undergraduates reported by Frederick (2005); our study adds credence to his three-item CRT. The strong correlations between NfC score and other scores, except LtA-Impulsive, suggest students' high need for cognition is related to analytic disposition and cognitive reflection. We also found that the type of impulsiveness (attentional, motor, and non-planning), as measured by BIS-11, does not seem to be tightly coupled to mathematical impulsiveness, as measured by CRT and LtA-Difference.

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FIFTH GRADERS' MATHEMATICS PROOFS IN CLASSROOM CONTEXTS

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The study intends to identify what mathematical proof looks like in primary classroom contexts. It describes the three components of mathematical proof observed in the episode of fraction comparisons in a grade 5 class. Students' written work collected from classroom, videos and transcriptions of classroom observations were the main data gathered in the study. Empirical induction from finite number of discrete cases for verifying a conjecture to be true and offering more than one counterexamples rather than only one counterexample for refuting a false proposition were two common forms of reasoning used by the fifth graders. Teacher's authority was another way of proof in the fifth grade classroom. The fifth graders accepted $Q \rightarrow \sim P$ as the negative of a proposition $\sim(P \rightarrow Q)$.

INTRODUCATION

Mathematical reasoning is a focus of mathematical instruction that involves students in exploring, investigating, conjecturing, explaining, and justifying mathematics (NCTM, 2000). The document states that teachers should provide students from K to12 grade level with the opportunities "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematics arguments and proofs, and select and use various methods of proof."

Proof is a vehicle to enhance students' understanding of mathematics concepts and promote mathematical proficiency and reasoning (Hanna, 2000). Proving is an important means of exploring in mathematics. Research shows that engagement in proving can support students to explore why things work in mathematics and explain their disagreements in meaningful ways, thus providing them with a solid basis for conceptual understanding (Stylianides, 2007).

Researchers draw more attentions on mathematics reasoning, proof and proving, and conjecturing. For example, international conference ICMI 19 study has a special issue of proof and proving in mathematics education (Lin, Hsieh, Hanna, & de Villiers, 2009). Some studies focus on students' verification with mathematics (Reid, 2002), and emphasize on the formal and informal aspects of proof (Canadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007). Some focus on the roles that proof might take in the mathematics curriculum if it is to be taught effectively. Teaching that supports proof (Stylianides & Ball, 2008) and teachers' conceptions of proof that influence the opportunities they create for students to engage in proving are also addressed (Ko, 2010). These increasing studies of proofs and proving has been limited to high school geometry. Additionally, existing empirical studies suggest that some high school students or undergraduates have difficulty with mathematical proof in school mathematics (Ko, 2010). The above discussion indicates that students should have early and appropriate opportunities to incorporate mathematical concepts and proof into their mathematical learning. Taking this consideration, this study encouraged elementary school teachers to provide students the opportunity to engage in proving activities. The proofs and proving were a new experience for primary school students and teachers involving in the study. Thus, this study is intended to identify what mathematical proof looks like exploring in a primary classroom context.

CONCEPTUAL FRAMEWORK

Stylianides and Ball (2008) distinguish the definition of proof from two considerations: mathematics as a discipline and students as mathematical learners. From mathematics as a discipline, a proof is a mathematics argument that requires three components: a set of true statements, valid modes of argumentation, and appropriate modes of argument representation (Stylianides, 2007). According to the criteria, empirical arguments cannot count as proof at any level of schooling, because empirical arguments utilize invalid modes of argument by promoting acceptance of mathematical claims based on incomplete evidence (Stylianides and Ball, 2008).

Regarding the consideration of students as mathematical learners, proof is defined as a set of accepted statements, known modes of argumentation, and accessible modes of argument representation to a classroom community. According to the criteria, students' proofs can be distinguished into four levels (Balacheff, 1988). *Naïve empiricism* at Level 1 refers that a conjecture is true after verifying some cases. *The crucial experiment* at Level 2 involves testing a conjecture by using a special or extreme case. The distinction from Level 1 is in that the students are aware of the generality. *The generic example* at Level 3 refers to showing the truth by manipulating a representative objective. The proof is indicated by the effect of operations. Students' poof at Level 4 *thought experiment* is indicated by looking at the properties of the objectives, not at the effects of operation on the objects. The conceptual framework of this study takes the consideration of students as mathematical learners. Thus, students' proofs consist of the three components: the set of accepted statements, understandable modes of argumentation, and accessible modes of argument representations, which do not focus on correctness or success that students produced at primary mathematics classroom will be identified.

RESEARCH METHOD

The proving activity described occurred in the grade 5 classroom of Lu-Lu, a teacher who participated in the first year of a three-year project that is designed to help teachers to create conjecturing tasks for engaging students in the activity of proving. Hence, the design of conjecturing tasks is new learning for Lu-Lu. Help students approaching to do valid proofs is a new experience for Lu-Lu, too.

The conjecturing task being explored was that "*2/4 is greater than 1/3, 4/5 is greater than 3/4. The teacher, Lu-Lu, finds out that both denominator and numerator of the 2/4 are greater than those of 4/5, respectively. Thus, Lu-Lu concludes: To compare any* *two fractions, if both denominator and numerator of the one fraction are greater than those of the other fraction, then the one fraction is greater than the other fraction. Do you agree? Why? Write down your reasons.*"

The conjecturing task being explored is to ask students to judge and verify. Lu-Lu stated that she would rather give fifth grade students two cases in advance to assist them in understanding the verbal statement than give them the proposition directly.

29 students in the class were grouped heterogeneously in groups of 4 or 5. When given the problem, students first worked independently and jotted down their solutions; then they came together in groups to compare their solutions, and last they reported to the whole class. The students were videotaped throughout their group and whole class discussion. Each student's written work was collected.

The first part in the result session was aligned to the Stylianides and Ball's (2008) three components of proof by using students' written work, videotapes of classroom observation, and transcriptions of discussion. In accordance with the three components of the proofs, the analysis of students' written solutions were first split into two piles and then a pile is assigned to each group of the two groups consisting of six school teachers studying in graduate program. Afterwards, they took turns to review the other pile for increasing the validity and reliability of analysis. The second part of the result session was from a small sampling from the whole class in order to illustrate a particular type of proving.

RESULTS

The Set of Statements Accepted by the Fifth Graders

After the task being explored, 23 (79%) students made incorrect judgement by accepting the teacher's conclusion, while only 6 (26%) students were correct by rejecting the conjecture; out of the 6 students, 3 (13%) students verified successfully.

The set of statements referred to the statements accepted by the classroom community. Replicating the teacher's statement was the most common statements accepted by those who were in favour of the conjecture. 20 students' set of arguments were "*if both denominator and numerator of the one fraction are greater than those of the other fraction respectively, then the one fraction is greater than the other fraction.*".

Once they figured out the new cases, the equivalence of fractions regularly became as the part of their arguments consisting of a set of statements. For instance, Janice and Krin, two of the students, successfully showed their accepted statements by the classroom community as follows.

Janice's statements:

$\frac{2}{100} < \frac{1}{2}$, *since* $\frac{1}{2} = \frac{50}{100}$, *but here 1 is not greater than 2 and 2 is not greater than 100.*

$\frac{2}{1000} < \frac{1}{2}$, since $\frac{1}{2} = \frac{500}{1000}$, but here 1 is not greater than 2 and 2 is not greater than 1000. $\frac{12}{1000} < \frac{1}{2}$, since $\frac{1}{2} = \frac{500}{1000}$, but here 1 is not greater than 12 and 2 is not greater than 1000. So that, I did not agree with the teacher's saying.

Krin's statements:

$\frac{3}{6} = \frac{1}{2}$, here 3 is greater than 1 and 6 is greater than 2, but 3/6 is not greater than 1/2

$\frac{2}{2} = \frac{3}{3}$, here 2 is smaller than 3 and 2 is smaller than 3, but 2/2 is not smaller than 3/3. Thus, I did not agree with what the teacher said.

The Janice's and Krin's accepted statements further indicated that the equivalence of 1/2 and 1 became mathematical cognitive tools as part of the counterexamples.

Modes of Argument Known by the Fifth Graders

Overall, the modes of argument the fifth graders regularly utilized were empirical induction from finite number of discrete cases with one, two, or three for accepting a proposition and offering one more counterexamples for refuting a false proposition.

Furthermore, one or two cases for empirical induction was from their teacher offering in the task. For instance, of the 23 students' incorrect conjecture, 20 students' verifications were based on the two cases (2/4 > 1/3 and 4/5 > 3/4) given by the teacher, as shown in Table 1. In addition, 3 students argued that the conjecture is true after verifying three cases (e.g., 2/4>1/3, 4/5>3/4, 7/8> 6/7).

For the 6 students' correct judgement, they used counterexamples to refute a false proposition. Only 3 students verified successfully by utilizing a single counterexample to refute it, while 3 students did not agree that a single counterexample is sufficient to refute a false proposition. Thus, they offered more than one counterexamples (e.g., 2/4 is not greater than 1/2; 2/6 is not greater than1/2) to refute the proposition. Moreover, 1/2 and 1 as the referent points while comparing with the other fraction were more likely successful when finding out a counterexample.

Argument Representations Accessible by the Fifth Graders

Modes of argument representations were the forms of expression for communicating with the classroom community. For this conjecture task involving fractions, word description was the most popular form of argument used by the fifth graders. As shown in Table 1. 22 (85%) students convinced others by using word descriptions. These students utilized the two cases ($\frac{2}{4}>\frac{1}{3}$, $\frac{4}{5} > \frac{3}{4}$) given by the teacher to look for a pattern as follows. (T: the teacher, Yi: a student)

16 T: Why do you agree with the statements I proposed?

17 Yi: Because for $\frac{2}{4}>\frac{1}{3}$, the denominator 4 of $\frac{2}{4}$ is greater than 3 of $\frac{1}{3}$, the numerator 2 of $\frac{2}{4}$ is greater than 1 of $\frac{1}{3}$. Likewise, for $\frac{4}{5} > \frac{3}{4}$, the denominator 5 of $\frac{4}{5}$ is greater than 4 of $\frac{3}{4}$, the numerator 4 of $\frac{4}{5}$ is greater than 4 of $\frac{3}{4}$. Thus, I believe it is true.

Within the word descriptions for convincing others, 7 students combined the written words with pictorial representation and 2 students combined the words with symbols. In other words, 13 students used word descriptions only to explain the conjecture to be convinced. Additionally, 13 students also frequently used pictorial representation combining with either word descriptions or symbolic representations to explain the comparison of the two pairs of fraction given by the teacher.

	Set of statements		Modes of argument		Argument representations	
Correct judgement	Offering new cases	#	Counterexamples	#		#
	◦ $\frac{1}{2}>\frac{2}{6}$	1	◦ One case	3	◦ word descript.	1
	◦ $\frac{2}{2} > \frac{3}{4}$	1			◦ Pictorial	0
	◦ $\frac{1}{2} = \frac{2}{4}$	1			◦ Symbolic	0
	◦ $\frac{3}{6} = \frac{1}{2}, \frac{2}{2} = \frac{3}{3}$ ◦ $\frac{2}{4} = \frac{1}{2}, \frac{2}{6} < \frac{1}{2}$	2	◦ Two cases	2	◦ Word+Picto. ◦ Picto.+Symb	3 1
	$\frac{2}{1000} < \frac{1}{2}, \frac{12}{1000} < \frac{1}{2}, \frac{2}{100} < \frac{1}{2}$	1	◦ Three cases	1	◦ Word+Symb.	1
Incorrect judgement	Using teacher's giving cases	#	Empirical induction from	#	Word descriptions	12
	--	--	◦ One case	0	Pictorial	0
	$\frac{2}{4} > \frac{1}{3}, \frac{4}{5} > \frac{3}{4}$	20	◦ Two cases	20	Symbolic	1
	Offering new cases ◦ $\frac{2}{4} > \frac{1}{3}, \frac{4}{5} > \frac{3}{4}, \frac{7}{8} > \frac{6}{7}$ ◦ $\frac{4}{6} > \frac{1}{2}, \frac{6}{8} > \frac{2}{4}, \frac{4}{8} > \frac{1}{4}$ ◦ $\frac{8}{10} > \frac{5}{5}, \frac{4}{5} > \frac{9}{10}, \frac{9900}{10000} > \frac{9800}{9900}$	 1 1 1	◦ More than two cases	3	Word+Picto. Picto.+Symb. Word+Symb.	4 5 1

#: the number of students

Table 1: Fifth graders' proofs of the given conjecturing task.

Many Fifth Graders Accepted the "~(P→Q) is (Q→~P)

During engaging the proving activity, Wei, one of the students who did not agree with the statements the teacher proposed, revised the original proposition to "*if the differences of denominators and numerators of two fractions are equal to 1,* (P)*, then the one fraction with greater denominator or numerator is greater than the other fraction.*"(Q), represented as P→Q. The teacher stated that she was not sure at that moment whether the proposition is true when we discussed right after the classroom observation. Immediately, the teacher invited all of the students to judge and justify if Wei's statements is true or false.

24 of 29 students engaged in previous activity of proving realized that to verify the false proposition is to find counterexamples. Hence, they tried hard to find counterexamples to verify the false proposition. 12 of them did not recognize that a counterexample is sufficient to verify a false proposition. Most of the students attempting to figure out the counterexamples accepted that the negative of the false proposition (~(P→Q)) is Q →~P instead of P∧~Q. That is, most of the students accepted that the negative of the proposition "*if the differences of denominators and numerators of two fractions are equal to 1,* (P)*, then the one fraction with greater denominator or numerator is greater than the other fraction.*"(Q)," became as "*if one fraction is greater than the other (Q), then the differences of the denominators and numerators of two fractions respectively are not always equal to 1 (~P).*" Four students, S3, S6, S17, and S29, displayed their statements on their worksheets as follows.

S3: The counterexamples are 8/10>3/5, 6/9> 1/3, but the differences of denominator and numerator 8, 3 and 10, 5 of one pair (8/10, 3/5) are not equal to 1. Likewise, 6-1=5, 9-3=6 in the other pair, both 5 and 6 are not equal to 1, either.

S6: 4/6>1/2, 6/8>2/4, 4/8>1/4, but the difference of denominator and numerator in each pair of fractions is not equal to 1. 4-1=3 , 6-2=4; 6-2=4 , 8-4=4; and 4-1=3 , 8-4=4.

S17: 2/2=3/3. The difference of denominators of the two fractions is equal to 1, and the difference of numerators of the two fractions is equal to 1, too. But 3/3 is not greater than 2/2.

S29: 99/100<98/99. The difference of denominators of two fractions, 100-99=1, is equal to 1. The difference of numerators of two fractions, 99-98=1, is equal to 1. But 99/100 is not greater than 98/99.

The common error of the students S3, S6, and S29, for verifying the false proposition displayed that the negative of "if P, then Q", i.e., ~(if P, then Q) is "if Q, then ~P". That is, given two pairs of fraction comparisons, such as S3's two pairs (8/10>3/5, 6/9> 1/3), they do not satisfy the condition: the differences between two denominators and two numerators of the pair of fractions are equal to 1. The episode further showed that fifth graders' failed in refuting a false proposition in that they accepted "$\sim(P \rightarrow Q) \equiv Q \rightarrow \sim P$".

Furthermore, S29's statements revealed that she did not have a correct knowledge of comparing two fractions with unlike denominators. The result also indicted that without solid mathematical concepts, it is impossible for students to engage a valid mathematical proof.

For S17's arguments, he perceived with finding a counterexample to refute the conjecture. That is, an existed a case, 3/3 is not greater than 2/2, but it does satisfy the condition: the difference (3-2=1) of two denominators and the difference (3-2=1) of two numerators in these two fractions are equal to 1. Thus, this counterexample resulted in some of the students in the class to revise the proposition proposed by Wei. They further stated that they must be proper fractions.

CONCLUSION AND DISCUSSION

Relying on the three components of mathematical arguments, the first component of proof suggested by Stylianides and Ball's (2008), the result of the study suggested that only 26 % of the fifth graders were readily to offer successful new cases as the set of mathematics arguments. It seems that the successful new cases they were more likely to provide were the equivalent fractions of 1/2 and 1.

Regarding the modes of argument as the second component of proof, the result of the study indicated that empirical induction from finite number of discrete cases for verifying the task and offering more than one counterexamples for refuting a false proposition were the most two popular forms of reasoning used by fifth graders. Only about 10% of the fifth graders accepted that a single counterexample is sufficient to refute a false statement. This finding of the study becomes a literature of Reid and Knipping's (2010) reviewing that many secondary students do not accept counterexamples as refutation. The level of fifth graders' proof was characterised at Balacheff's (1988) first level, naïve empiricism.

In addition, making reference to an authority (the teacher) was also popular method of verification of the fifth graders. The result is consistent with Reid's study (1995). The result also showed that almost fifth graders' accepted the truth of "$\sim(P \rightarrow Q) \equiv Q \rightarrow \sim P$".

With respect to the third component of proof, the result of the study displayed that word description and it combines with pictorial representation were regular two representations of modes of argument used by the fifth graders.

The study suggested that knowledge of students' mathematics concepts and teachers' knowledge of mathematics embedded in the conjecturing tasks were two essential factors of affecting fifth graders' proofs and proving. Without solid mathematical concepts underpinning the arguments, it is impossible for students to produce logical proofs. The episode reported in the study showed that the teacher did not recognize whether Wei's statements were true or false. It leaded to the teacher's uncertainly to when and where to terminate students' discussions. The finding of the study supported

Stylianides and Ball’s (2008) claim. Unless the teacher had a good understanding of general rules of the comparison of fractions and sound knowledge of proof, we cannot expect that she was able to effectively promote proving. Helping teachers to assist students in engaging activities of valid proofs and proving is the purpose of the further study.

ACKNOWLEDGEMENT

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THE INFLUENCE OF READING FIGURES IN GEOMETRY PROOF ON EYE MOVEMENT

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The goal of this study is to explore the role of cognitive processes in the reading of geometry proof as measured by the ratio of total fixation time and the number of saccades. The eye movements of 50 undergraduate students were analyzed during the reading of 2 geometry proof of different difficulty levels. Significant results include a greater average ratio of total fixation time on figures than other kinds of text with similar layout. Goal-driven reading was also found. Finally, the implications of the number of saccades occurring during visualization while reading geometry proof are discussed.

INTRODUCTION

Empirical research has increasingly focused on the reading of geometry proof. Much of this work has involved manipulation of the layout or content of geometry proof to determine the factors that influence reading comprehension and geometry problem solving. (Gal & Linchevski, 2010; Yang, Lin, & Wang, 2008). One of the most significant aspects of this research is the failure of students to integrate information between figures and test. Cheng and Lin (2005) studied students' answers to geometry tests and instructional interviews and proposing that students failed to integrate information between figures and proof. However, the cognitive processes of student failure are not well understood. Yang, Lin and Wang (2008) examined how geometry proof with different layouts affected students' reading comprehension. The results showed no interaction and that the generalizability of results is limited by the design of tests. In general, the limitation of paper-and-pencil tests and interviews makes it difficult to infer cognitive process. Therefore, one of the purposes of this study is to solicit the specific reading patterns of reading geometry text for eye movement analysis.

Most research application of eye movement to cognitive processes and reading comprehension has been proposed to explain the process of the coding, integration, and construction of mental representations. Experimental materials typically consist of text, graphs and the integration of texts with graphs, as well as scientific text, comics, biology text and advertisements. Number of saccades has been the main index of analysis of cognitive processes (Hegarty & Just, 1993; Hannus & Hyönä, 1999; Rayner, Caren, Rotello, & Stewart, 2001). Despite this, little research has examined eye movements during the reading of geometry proofs. Our study aims to expand understanding of the importance of the number of saccades as a measure of cognitive processes during the reading of geometry text.

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THEORETICAL FRAMEWORK AND LITERATURE REVIEW

Duval (1999) contends that representation and visualization are essential in the learning of mathematics. His research emphasizes the importance of the semiotic system in representation, which he believes is necessary to coordinate and translate components during the learning of mathematics. At the same time, visualization plays an important role in the beginning phase of the comprehension, coordination and organization of elements within figures and texts. Visualization is one of the three kinds of cognitive processes suggested by Duval which each fulfill specific functions. Its processes primarily address space representation which comprises the heuristic exploration of a complex situation, the synoptic glance (visual reception), and a subjective verification. Visualization, composed of visual reception, permits individuals to coordinate and integrate information between texts and figures during the beginning stage of reading geometry proof.

Research examining eye movement during the cognitive processes of reading text with both text and figures, whose layouts is similar to those of geometry text, has found the ratio of total fixation time on figures to be from 6% to 29% less than that for geometry text (Hannus & Hyönä, 1999; Rayner, Caren, Rotello, & Stewart, 2001). Examination of the number of saccades and the role of eye movement in the comprehension of layouts containing integrated text and figures has found the combination of figures and text can be used to infer cognitive processes. (Schmidt-Weigand, Kohnert, & Glowalla, 2010).

Epelboim and Suppes (2001) studied model eye movement and visual working memory during the solving of geometry problems. One result was that participants performed eye movements between figures and proof in order to maintain the mental images of figures in their working memory. In conclusion, the reading comprehension of geometry proof counts for beginning learners. This study investigated the average ratio of the total fixation time on figures to confirm the significance of these figures playing a significant part in the reading of geometry proof for the construction of mental models. The number of saccades during reading was analyzed to infer cognitive process while reading the contents of geometry proof.

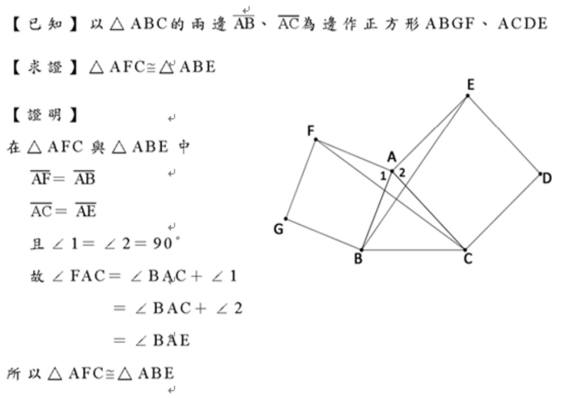
METHOD

Participants

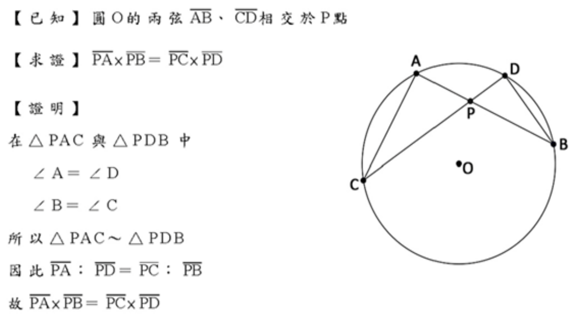
Sixty-five non-expert undergraduate students were selected from universities in Taipei. After exclusion of participants majoring in Mathematics, passing of a calibration, and validation procedures, the valid sample was fifty.

Materials

Geometry proof written in Chinese was adapted from junior-high-school mathematical textbooks. Their layout was arranged in a formal way (shown as Fig 1). Mathematics specialists estimated the difficulty of the problems based on their mathematical structure and phase of learning. Square item is simpler than a circular item. In order to motivate participants to read with attention, a true-or-false item appeared after reading each geometry proof.



Square item



Circle item

Fig 1 geometry text

Apparatus

Eye movements were recorded by an Eyelink 1000 with a sampling rate of 1000Hz. A chin rest was used to minimize head movement. Proof was displayed on a 19-inch LCD monitor. The screen resolution was set to 1024*768 pixels. Participants sat approximately 65 cm from the monitor.

Procedures

The formal experiment started after instructions were delivered and calibrations were completed. Participants read two geometry proofs that were presented randomly. Reading time was not limited. Participants then pressed the button onto the next page with a geometry proof without the example of proof. Finally, they pressed the specific keyboard unto the other proof.

Data selection and analysis

Gazes, apart from the calibration points, were removed by checking each eye tracker so that the final valid sample was fifty. Contents of proof was divided into a variety of “Areas of Interests (AOIs)” based on the CKIP system developed by Taiwan’s Academic Sinica (shown in Fig. 2). Additionally, the number of saccades appearing when participants read **mathematical symbols** located in the AOIs within “given” and “worked proof” (inside the bolded squares) were analyzed.

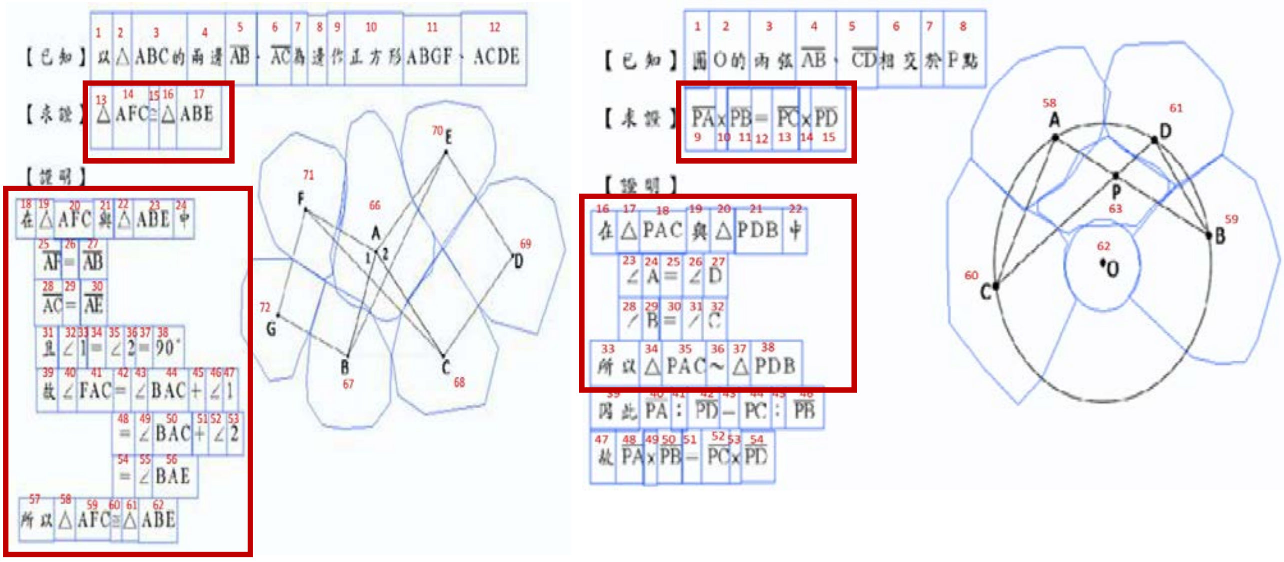


Fig 2 AOIs of two items

RESULTS

Eye movement indices are shown in Table 1. The average ratio of total fixation duration in reading figures was 50% higher than that found in work cited above. The total number of saccades for the two items was similar, but the average number of saccades for the two items and that of the worked proof for the Square item showed less difference, except for that of the worked stage of the Circle item (shown in Table 2). The results of the within-subject analysis of variance revealed that the interaction was significant for stages and items, $F(1, 49) = 17.87$, $p < .001$. Simple main effects occurred in the difference between the given proof and the worked proof of the Circle item. This meant that the number of saccades of the worked proof of the Circle item were significantly larger than that for its given stage.

	Square M(SD)	Circle M(SD)
figure total fixation time(%)	54	47
number of saccades	20.74 (8.65)	20.67 (7.80)

Table 1 analysis of eye movement indices

proof / items	given M(SD)	worked proof M(SD)
Square	.51(.37)	.51(.25)
Circle	.45 (.36)	.72 (.32)

Table 2 descriptive data of the number of saccades for proof of items

Additionally, thirty-eight eye trackings were recorded representing the goal-driven pattern, the first gaze fixation on "proof" and the refocus on "proof" after reading the conclusion (shown as Fig3).

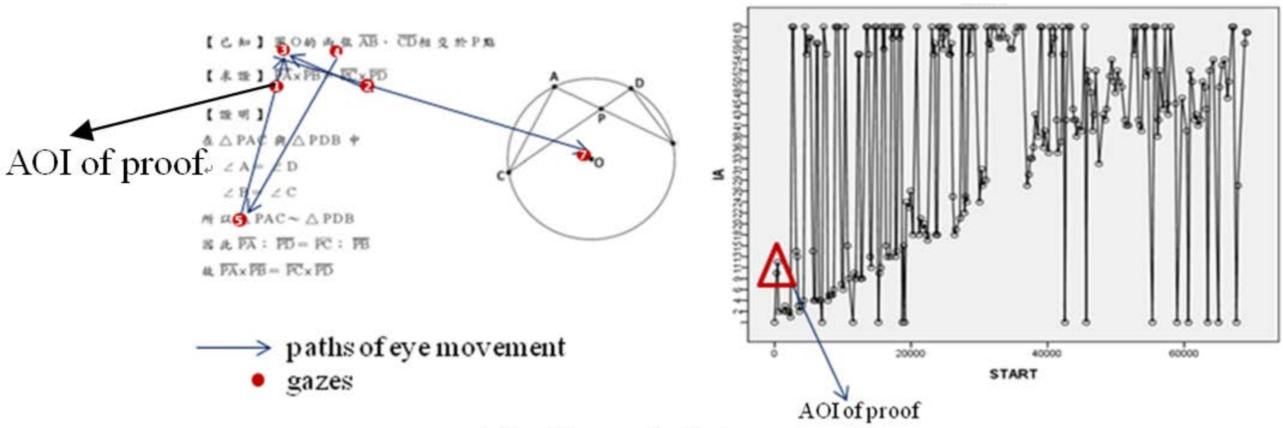


Fig 3 goal-driven pattern

DISCUSSION AND CONCULSION

Four main points that are relevant to the research question can be concluded from these results. First, the average total fixation time on figures of geometry proof is higher than that of proof with similar layouts, including scientific text, biology text, and advertisements. This study further confirmed that figures function as the concrete referent for readers to construct a mental model. Another main finding was a goal-driven reading pattern, in accordance with aspects of schema theory. Because confirming the goal of answering geometry items is a necessary step and strategy in solving geometric problems, a problem-solving schema was formed, and gradually activated while reading geometry proof. Finally, the difference between the number of saccades for proof with different levels of difficulty was significant. "Given" consists of known properties for readers to match and establish spatial representations, corresponding to **visual reception**. When reading "worked proof", readers need to keep spatial representations in their working memory and to look between proof and figures, which corresponds to **visualization**. More complicated cognitive processes would be required in reading "worked proof"; and as such, the number of saccades reading "worked proof" would be significantly greater than that of "given". The result corresponds with the reference, indicating that participants tried to search for functional information and match mental models from the figures.

The number of saccades between "given" and "worked proof" for the Circle item indicates that a greater number of saccades are essential for readers to integrate the more complex information of the more difficult item. Conversely, that of "given" and "worked proof" involved in the Square item shows that it took readers a similar amount of time to read the two portions of the Square item. Due to its simpler properties, it took less time to construct mental models, so initial reasoning occurred during visualization.

In other words, visual reception and visualization occur simultaneously. These results point to the differing effects on eye movement of proof with different levels of difficulty. We have proposed the model in Figure 4 to explain these results.

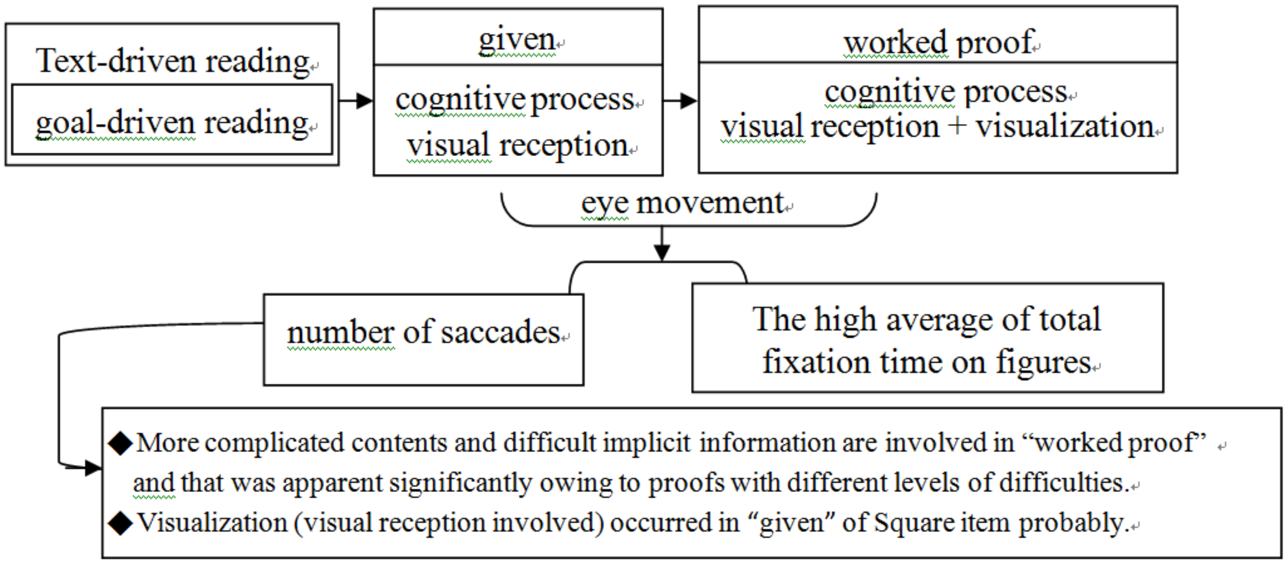


Fig 4 cognitive process of reading geometry text

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USING NARRATIVE APPROACH CASE STUDY FOR INVESTIGATING TWO TEACHERS' KNOWLEDGE, BELIEFS AND PRACTICE

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The purpose of this study was to use narrative approach case study to investigate two mathematics teachers' development of inquiry-based mathematics teaching during their participation in a professional development program. By means of narrative analysis based on the data collected from interviews, classroom observations, concept maps, researchers' field notes and post-observation interviews, the study construct a retrospective explanation of how the two teachers' own experiences were reflected in the change of their beliefs, knowledge and practices about mathematics inquiry. Findings indicate that (1) a high level of knowledge regarding mathematics inquiry might not the guarantee that the teacher will employ better inquiry-based mathematics teaching; (2) the teacher who has changed her beliefs might perform better inquiry-based teaching than the one who always holds positive beliefs toward inquiry.

INTRODUCTION

Inquiry-based mathematics teaching is considered to be an effective means to facilitate students' development of mathematics understanding and mathematics thinking (Chapman, 2007). It is characterized by students' active engagement in meaningful mathematical problems and activities that involve conjecturing, investigating, collecting and analyzing data, reasoning, making conclusions, and communicating (National Council of Teachers of Mathematics, 1991; Wilkins, 2008).

In the literature of teaching, research provides evidence to support the concern about the integrated view of teachers' knowledge, beliefs and practice (Ernest, 1989). For example, the work of Chapman (2007), provided a model which connects the preservice teachers' beliefs and conceptions of mathematics, mathematics concepts and procedures, students and instructional practices. In this model, the investigation of teachers' inquiry-based teaching should focus on the four factors: *beliefs about mathematics, beliefs about students' learning, how to hold knowledge of mathematics and pedagogical knowledge of engaging students in inquiry*. In particular, the *relationships* among these factors seemed to be the key to explaining teachers' use of inquiry-based teaching. Under the same integrated view about teachers' inquiry-based teaching, the research of Wilkins (2008) indicated that beliefs were found to partially mediate the effects of content knowledge on practice and content knowledge was found to be negatively related to beliefs in the effectiveness of inquiry-based instruction and teachers' use of inquiry-based instruction in their classrooms instructional practice.

There were some studies considered how teachers understand and implement inquiry-based mathematics teaching, but researchers typically find it still challenging to facilitate teachers to use inquiry-based teaching. Research suggests that the researchers often do not hold a sound understanding of the teachers' knowledge, beliefs and practices (Oslund, 2012). Therefore, the researchers need to advance their understanding about the factors that influence teachers' inquiry-based teaching. The research method of narrative inquiry might be a useful means to achieve such an understanding since experience plays an important role in influencing teachers' knowledge, beliefs and practice. Clandinin and Connelly (2000) claimed that teachers' knowledge and beliefs can only be thought of in narrative terms. The stories teachers tell can facilitate the interpretation and understanding of teachers' experiences and offer a way of recovering, articulating and understanding the meanings and intentions embodied in teachers' behaviors (Chapman, 2008). Thus, using the narrative as a research method seems to allow us to understand teachers' knowledge, beliefs and experiences as interconnected and interrelated systems.

The purpose of this study was the use of narrative inquiry as a research method in understanding teachers' knowledge, beliefs and practice with regard to inquiry-based mathematics teaching. Unlike an ordinary case study, this investigation employed a mathematical biography of two participants (Kaasila, 2007). Through this type of retrospective explanation the researchers could better understand the source of the participants' knowledge, beliefs and practice as well as why they changed and how they progressed. Understanding one's personal history may be of more use in explaining their behaviors.

METHODOLOGY

A narrative inquiry approach design (Chapman, 2008, Kaasila, 2007, Oslund, 2012) using a holistic perspective was undertaken to gain an in-depth understanding of teachers' knowledge, beliefs and classroom implementation of inquiry teaching. To that end, data were gathered on multiple aspects of a professional development project to assemble a comprehensive picture of the participants.

Participants

From a total of 15 participants (7 males and 8 females), two participants (Table 1) were selected from the professional development project focusing on learning and implementing inquiry-based mathematics teaching. This cohort was composed of three high school mathematics teachers, eleven junior high school mathematics teachers and one elementary school teacher.

Selection of the focal teachers for this study was based on their different characteristics in their academic background and early performance in the research project. Wendy did not major in mathematics at the university; rather she became a mathematics teacher by taking postgraduate mathematics education courses. This is not a typical way to become a mathematics teacher in Taiwan. The teachers who take these courses are given quick training by encapsulating the needed courses into one year while the regular training at the university is four years. Therefore, the postgraduate-courses mathematics teachers are usually exposed to less university mathematics than the regular-courses mathematics teachers. In contrast to Wendy, Sara graduated with a mathematics major from a university specializing in education. She had regular training in becoming a mathematics teacher. Based on her high scores in the university, she seemed quite confident of her teaching in comparison with Wendy. Further, she was a very mathematics-oriented teacher. That is, she paid much attention to mathematics content but less to pedagogy.

Name	Gender	Teaching Experience	Grade Level	Undergraduate Degree in
Wendy	Female	8 years	7th	International Business
Sara	Female	10 years	7th-9th	Mathematics

Table 1: Backgrounds of two focal teachers

Data collections

This study employed five different data sources: **(1) Interviews**. The teachers were interviewed individually once a month during the year-long experience with the interviews lasting from 50 to 90 minutes. These interviews were unstructured and open-ended. They did not involve a detailed series of questions but instead used an open-ended prompt such as “tell me…”, which was recommended by Gellert (2001). The purpose of these interviews was to get the interviewees to recall their previous experiences in both teaching and learning mathematics, and their current experiences in learning and implementing inquiry-based mathematics teaching. They were asked to think of these experiences as a story and then were questioned about important or meaningful things and events within these experiences. For example, interviewees were asked to describe positive or negative experiences with mathematics in their childhood. **(2) Concept Maps.** At the beginning and again at the end of the professional development project, the participants were asked to make a concept map (flow diagram) that reflected their understanding of mathematics inquiry. The two focal participants were both in the midst of pursuing master’s degrees in mathematics teaching. In their graduate courses, they had been shown an example of a concept map and how it was structured to reflect a particular topic. In addition, a follow-up interview was conducted in which participants explained their ideas on the construction of the concept maps. **(3) Classroom observations.** In addition to participating in the interviews, the two focal participants were observed teaching mathematics through inquiry during their participation in the professional development project. For each 10 class sessions were observed and videotaped. **(4) Researchers’ field notes** and **(5) Post-observation interviews.** Researchers’ field notes for all observations and post-observation interviews were collected during and after the classroom observations. These notes recorded the teachers’ goals for the observed lessons, any questions the two observers had about the lessons and anything that surprised the observers during the lessons. In post-observation interviews (after the observed lessons subsequently), the observers would ask the questions recorded in their notes. For example, in one observed lesson, Wendy asserted that she would use open-ended mathematics problems; however, it seemed to observers that these problems were not so open-ended. One of the two observers then recorded the question and asked her after the observation.

Data analysis (narrative analysis)

Narrative analysis is the procedure through which the researcher organizes the data elements into a coherent developmental account (Polkinghorne, 1995). The researchers used narrative analysis to deal with all kinds of data.

The interviews were analyzed structurally and thematically. The initial analysis of the interviews involved using their structural elements to focus on how the teachers represented and interpreted their experiences. Each interview was coded according to Drake's (2006) categories of *tone* (statements containing expressions of positive effect or negative effect) and *specificity* (descriptions of the event: i.e., ''What happened?'', the timing: i.e., "When did it happen?'', and the mathematical content: i.e., ''What mathematics were involved?''). All interviews were coded by three researchers. Any differences between the three coders were resolved through consensus. The thematic analysis of the content of the stories involved identifying and organizing information about the roles of teachers, students and mathematics in the plots.

The concept maps were analyzed and coded according to: (a) map structure and (b) essential features of mathematics inquiry. Based on Kinchin and Hay (2000), concept map structures can be classified as *spokes, chain I, chain II* or *nets* (from level 1 to 4 respectively). Each of these reflects a different level of conceptual understanding, complexity, degree of relatedness between concepts including feedback levels, and ability to accommodate the addition of new material. The second analysis tool, essential features of mathematics inquiry, ranked participants' concept maps according to a system adapted from Borasi et al.'s (1999) specific elements of an inquiry approach: (1) *students are actively engaged in the construction of mathematical knowledge*; (2) *students develop ownership of mathematics inquiry*; (3) *mathematics is portrayed as the product of human activity*; (4) *priority is given to developing problem solving, understanding and confidence*; and, (5) *rich mathematical situations provide opportunities for learning*. Each concept map was qualitatively ranked based on the presence or absence of an essential feature, with a minimum rank of 0 and a maximum rank of 5.

The videotaped classroom observations of the teachers' practices were coded and analyzed using the class discussion structure (Warfield, Wood & Lehman, 2005) – a framework that focuses on the forms of the teacher-student interaction and discourse. These practices were categorized as: (1) *Conventional* The interaction mainly focuses on Initiate Respond Evaluate (Hoetker & Ahlbrandt, 1986) and giving lots of hints to solution. (2) *Strategy Reporting* The main focus in a strategy reporting classroom is on children's presentation of different strategies of the problems. The focus of teacher questioning is on "how" and "what" to prompt students to describe what they did to solve the problem. (3) *Inquiry* Children offer different solution methods in the classroom, as in strategy reporting classes, but they provide reasons for their thinking in order to make sense to others.

FINDINGS

The different measures and analyses of these two participants' knowledge, beliefs and practice are summarized in Table 2 (Phase 1=2007/01-07; Phase 2=2007/08-2008/02):

	Wendy		Sara	
Phase*	1	2	1	2
Knowledge	Spoke / Rank 1	Chain II / Rank 4	Spoke / Rank 1	Chain I / Rank 3
Beliefs	Toward Inquiry	Toward Inquiry	Backward Inquiry	Toward Inquiry
Practice	Conventional	Strategy Reporting	Conventional	Inquiry

Table 2: Wendy's and Sara's knowledge, beliefs and practice about inquiry-based teaching

Table 2 shows Sara taught in a conventional way based on her negative beliefs about inquiry (Phase 1).Then she learned some new information, accordingly revised her belief, and changed her practice to inquiry (phase 2). On the contrary, Wendy had more positive beliefs to inquiry in phase 1 but struggled to implement it. Although she had improved her inquiry teaching in phase 2, the improvement was not as much as Sara.

How we plotted Wendy's and Sara's stories?

In the initial construction of the plot, the researchers specified the outcomes of Wendy's and Sara's stories. Wendy had more inquiry-oriented beliefs and better inquiry knowledge than Sara at the start of their participation in the professional development project, but she did not develop better teaching practice at the end of participation. The researchers then asked: "*Why did this happen?*" and began seeking clues to explain the phenomenon. The researchers found that Wendy had more positive beliefs regarding inquiry-based teaching and consequently had a stronger will to understand what mathematics inquiry is, which might led to her better beliefs and knowledge about mathematics inquiry. However, her weakness in mathematics knowledge (content knowledge) might restrict her development in inquiry-based mathematics teaching. In contrast, Sara had a more conceptual understanding of mathematics knowledge (evidence from classroom observations). That is, she understood various mathematical topics and was able to connect these topics. In the second stage of the plot, researchers focused on "*Why did they have these beliefs and knowledge about mathematics inquiry to start with?*" and "*What could cause them to change?*" The researchers found that Wendy's and Sara's early beliefs and later knowledge development might be influenced by their early memories of learning mathematics, their perceptions of themselves as mathematics learners, and their interactions with family members. Then their private reflections and group investigations into their teaching changed the beliefs and knowledge. When constructing the final versions of Wendy's and Sara's stories, the researchers arranged the data elements chronologically.

Wendy: a romantic philosopher

> Motto: The more learning autonomy a teacher gives to students, the better those students will performance.

Early memories of learning mathematics Wendy had mainly positive memories of learning mathematics during her time in school. She reported having the most positive experiences in high school and university: "I've liked mathematics since I was in elementary school. I got high mathematics scores in high school and even in the university I also got high grades in calculus. I was confident with mathematics". A positive role model was her high school mathematics teacher: "He was a good talker. He had a sense of humor. He made me feel that it's possible to both talk with students and teach mathematics. He just focused on the contents of textbook without putting too much emphasis on problems outside the textbook". From this experience, Wendy's beliefs of good mathematics teaching also focused on the best way to teach the basic content of a textbook (basic content knowledge than on challenging content knowledge). This is unusual in Taiwan because many of teachers use teaching handouts that contain separate content and challenging problems. Moreover, her belief of what makes a good mathematics teacher involved being able to relax students' anxiety over learning mathematics.

Interactions with family members Wendy grew up in a democratic family. Her parents allowed her autonomy in learning. They did not force her to study hard. This influenced her beliefs regarding teaching. She did not put pressure on students to learn mathematics. She believed that "students will explore for themselves in an open-ended atmosphere and in doing so will perform better".

Beginning inquiry-based mathematics teaching The first time Wendy heard about mathematics inquiry was during the graduate school course in which constructivism was discussed. This led her to connect the term constructivism with mathematics inquiry in the beginning of her teaching practice. During the phase 1, her inquiry-based teaching focused on guiding students to have discussion among students. "I think student discussion is one important element in the inquiry teaching; I try to do this". Although Wendy talked much about student discussion but she did not actually allow much time for them from our classroom observations. Thus, the researchers labelled her teaching as "Conventional" since the class remained teacher-centered.

Further inquiry-based mathematics teaching In the phase 2, Wendy's teaching still mainly focused on student discussion, but she allowed more opportunities for students to explain their ideas. Although she often took over the students' explanations to add detail or provide a rationale for what the students said, she would continue by asking if anyone used a different strategy. In this way, Wendy's teaching practice seemed to move into the category of "Strategy Reporting".

Sara: an artist of teaching

> Motto: It's easy to get high scores in mathematics as long as you are able to comprehend the basic concepts and problem solving principles.

Early memories of learning mathematics Sara also had positive experience in learning mathematics. Sara was found to have performed better in school compared to Wendy (according to their university entrance examination scores). She said she was a mathematics gifted student: "I thought learning mathematics was quite easy. When you understand fundamental mathematical concepts, you don't even need to do many exercises". As a mathematics high-achiever, she did not have a mathematics teacher role model. She believed that anyone could learn mathematics well if one categorized the problems first and then learned the corresponding skills needed to solve those problems. Her beliefs about teaching involved encouraging students to get high scores.

Interactions with family members Similar to Wendy, Sara's parents did not put stress on her with regard to learning. They gave her the chance to choose the things she wanted to do. Thus, when Sara said that she wanted to be a mathematics teacher, her parents did not oppose it. Parents in Taiwan often hope their child will be a doctor or engineer if the child is intelligent and good at learning. This type of parental influences seemed to allow her to change beliefs later when she began inquiry-based teaching.

Beginning inquiry-based mathematics teaching Sara did not agree with the use of the inquiry-based teaching in her beginning phase for, in her opinion, it only complicated the teaching by allowing students the opportunities to learn on their own using more naïve problem solving strategies. Thus, she did not change her teaching method during this time. In her teaching, she usually introduced mathematical ideas in a conceptual way and then discussed the concepts with students. Her students did understand what she was saying in this type of teaching, but Sara still controlled most of the teaching time. Hence, the researchers labeled her teaching also as "Conventional" during the phase1.

Further inquiry-based mathematics teaching In the subsequent period (phase 2), Sara developed more complex teaching practice which could be termed as "Inquiry." Sara used different ideas of a mathematical concept to elicit students' debate. In this circumstance, her class context included "Strategy Reporting" characteristics, but was further distinguished by the posing of clarifying questions from the listeners and Sara, and the provision of reasons by the explainer. The impetus leading to this change seemed to be her participation in the professional development project. In this project, she experienced and came to understand more about inquiry-based mathematics teaching. “I now know the meaning of student-centered teaching. It does not mean that the problems should only be solved by the students’ in their unsophisticated or simplistic way, but that teachers and other student listeners can elaborate on their thinking”. While Sara changed her beliefs and developed her knowledge about inquiry, she performed better inquiry-based teaching than Wendy.

DISCUSSION

In terms of teaching, Sara, equipped with stronger mathematical knowledge, seemed more knowledge-based (to teach by utilizing mathematical concepts themselves to stir students’ discussion and learning) while Wendy seemed more pedagogy-based (to teach by adapting different approaches). The evidence to support this finding came from our observations and interviews. We found that Sara could make many connections among mathematical contents and also mathematical arguments were involved in her explanations about mathematics concepts. That is, Sara used mathematics knowledge as a source to elicit students’ inquiry but Wendy relied much on pedagogy approaches (e.g., cooperative groups).

The findings also seemed to indicate teachers’ knowledge of inquiry-based teaching does not decide the quality of inquiry-based teaching more than teachers’ mathematics knowledge, which is not consistent with previous studies (Wilkins, 2008). As shown in the findings, Wendy’s knowledge of inquiry-based teaching in phase 2 was better than Sara’s, but her result of practice did not surpass Sara’s. This might also result from the substantial change of Sara’s beliefs of inquiry teaching from negative essence (backward inquiry) to positive support (toward inquiry), while Wendy remained positively supportive throughout the whole study. Nevertheless, this should be further investigated with more research. Finally, the findings also suggest that it is important to further investigate the parental influences on teachers' change of beliefs.

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MAKE YOUR CHOICE - STUDENTS' EARLY ABILITIES TO COMPARE PROBABILITIES OF EVENTS IN AN URN-CONTEXT

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As data analysis and probability gained interest in primary education, the need for a thorough knowledge base on students' early capabilities increased. Children in grade 4 already master basic concepts of probability and distinguish different degrees thereof. However, the abilities of students in the domain of probabilistic reasoning are affected by misconceptions as research could show. Especially, problems of representativeness, availability, and problems with the adjustment of probabilities of events are identified. Our research addressed whether students of grades 4 and 6 were able to compare probabilities of events. In an urn context, children were asked to solve problems and compare events in respect to their probability. We found problems resulting from misconceptions. However, students also used facilitating features of tasks like sure-to-win draws. The results rest on standardized interviews.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Data analysis and probability is a topic, which is not easy to understand for young children. However, on the one hand, it found its way in primary school curricula. The standards of NCTM (2000) and the German standards (KMK, 2004) suggest e.g. an early introduction of these topics. On the other hand, in traditional curricula as well as in the common core standards initiative (CCSSI, 2010), education in the area of data analysis and probability will not start before grade 6. The right point in time for teaching data and probability can hardly be deducted from theoretical considerations. We argue that in order to better understand the development of probabilistic knowledge, it is important to learn more about early ideas of students. Empirical results can be a basis for defining a curriculum based on adequate presumptions about students' capabilities.

In prior research, we found that students have a basic understanding of concepts of probability already in grade 2 (Lindmeier et al., 2011). Most students in grades 2, 4 and 6 were able to distinguish certain, possible and impossible events in an abstract-formal urn context although difficulties were found when students had to discriminate between improbable and impossible events. These findings can be seen in analogy to findings of Shtulman and Carey (2007) for everyday problems. So, a basic understanding of probability and chance can be assumed and lays ground for further investigations into probabilistic reasoning in primary and early secondary age.

Research in the area of probabilistic reasoning describes relatively stable difficulties of students and adults that can be explained with the help of misconceptions. These

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misconceptions lead to a biased estimation of the probability of events. In consequence, these misconceptions influence the ability of successfully comparing two events with respect to their probability.

Three problem fields are described as resulting from an understanding of probability based on ideas of representativeness of events, availability of events and problems resulting from adjustment and anchoring processes that are used to estimate probabilities (Tversky & Kahnemann, 1982). For the following, three phenomena are relevant: As a consequence of the misconception of representativeness, the probability of events that seem to represent the sample space is overestimated. For example, Fischbein and Schnarch (1997) present in a lotto context the two set of numbers 1-2-3-4-5-6 and 39-1-17-33-8-27 to students in grades 5, 7, 9, 11 and college students. Although both sets have the same chance of winning, students up to grade 7 attribute a higher probability to the second set that looks more representative for the parent population.

Other difficulties are described for the comparison of two events that satisfy a sub-event/event relation. The probability of a sub-event is always lower than the probability of the event. This can also be described in terms of conjunction. For example, in Fischbein and Schnarch's (1997) study, students had to compare the probability of a person to become a student and to become a student at a medical school. The sub-event "to be a student at the medical school" can be described as the conjunct event "to be a student" and "to be at the medical school". Students up to grade 9 failed in identifying the more probable event when primed with a description of the person that emphasized a medical interest of the person. These difficulties are usually described as the conjunction fallacy. In Tversky and Kahnemann's (1982) description of difficulties, the conjunction fallacy is seen as a result of inadequate adjustment of probabilities. In this understanding, the probability of conjunct (and in analogy disjunct) events is estimated as an adjustment of the probability of related elementary events. The adjustment of the probability – anchored by the probability of the elementary events – should result in a reduction of the probability for the conjunction of events (and an increase for the disjunction). However, under certain circumstances, this adjustment fails. In the example above, the priming description of the person leads to wrong estimations, although a sub-event/event relation usually facilitates the comparison of events. Fiedler (1988) showed, that prior experiences and expectations of students influenced their abilities to deal with conjunct events as they facilitated the imagination of events. The availability of events biased in turn the probability of events and lead to an overestimation.

The third phenomenon relevant for our work is described as a difficulty to compare compound and simple events. For example, imagine the simultaneous rolling of two dice. Comparing the events "5,6" "6,6", Fischbein and Schnarch (1997) found that even college students failed to identify the event with higher probability. With Tversky and Kahnemann (1982) such difficulties may also result from failed adjustment processes for disjunctive events, as described in the analogue case of conjunctive events above.

The classical studies usually used text-based assessment formats and embedded the problems in everyday contexts. As a consequence, performance was influenced by the individual availability of the particular events. It is an open question, whether students could compare the probability of events in a more formal-abstract context adequate for their age. In the following study we aimed at investigating abilities of probabilistic reasoning of students in an urn-context. The following questions were addressed:

- Description: Which abilities have students of grades 4 and 6 when comparing probabilities of events? Is there evidence for (well-known) misconceptions?
- Development: Is there evidence for a development of abilities between grades 4 and 6?

DESIGN OF THE STUDY

Methods

Children's ability to compare probabilities was evaluated in a series of tasks presented in an interview situation. Their basic understanding of concepts of probability was additionally assessed in order to control for essential prerequisites.

Assessment of basic understanding of probability

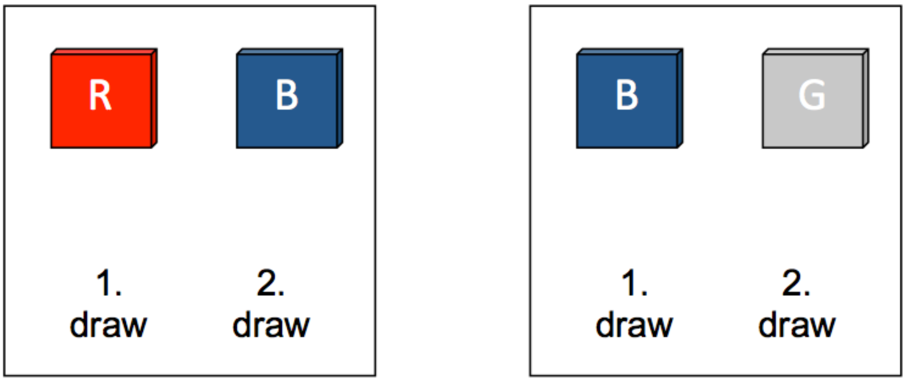
First, the students were presented bags with cubes. The cubes were of two different colors and the students knew the distribution of the colors. They were asked to evaluate, if a blue cube could be drawn in one blind try. The students answered the question by pointing on a scale of probability that consisted of three segments. The segments stood for certain, possible, and impossible events and were labelled in a way that was adequate for their age. In order to avoid problems of understanding, no technical language was used and certain events were described as events that would happen in either case, whereas possible events were described as events that might occur and impossible events as events that would happen in no way. Six items of that type were presented, whereof three exemplary items are given in Figure 2.

One tasks represented an impossible event (GK4). For two tasks, the probability exceeded 50% (e.g. GK1). The probability of drawing a cube of the target color ranged from 1% to 10% for the remaining three tasks (e.g. GK6). In other studies it could be seen, that the differentiation between impossible and improbable events was difficult for children at primary and beginning secondary age. Thus, these tasks seemed to be apt to control for a basic understanding of probability and chance at this age.

Assessment of abilities to compare probabilities of events

In order to get to know students' abilities to compare probabilities, students were presented two events in an urn context and they were asked to evaluate them with respect to their probability. The urn was composed of 10 red and 15 blue cubes and did not change during the series of task. The students knew the distribution of colors. A stack of event-cards was presented. The events referred to results from the subsequent blind drawing of cubes, with replacement. For example, in Figure 1, two cards are presented. The card on the left stands for the event "draw first a red cube, lay it back, mix the urn and then draw a blue cube" whereas the right card presents the event "draw first a blue cube, lay it back, mix the urn and then draw either a red or a blue cube". Hence, the depicted gray cube stands for the (certain) draw of either a blue or a red cube. A player wins with a card, if the represented event is fulfilled. The students were laid open certain pairs of events. They had to make a choice and to decide which event had a higher probability to occur. Therefore, the questions were: "Make your choice – what is the better card? With which card will you win more likely?" Technical terms, e.g. the notion of probability, were not used in order to avoid problems of understanding resulting from technical language. Children were in addition asked to give reasons for their choice.

Eight pairs of cards were presented to the students as given in Figure 2. The tasks encompassed events with clearly different probability as well as nearly equiprobable events. Moreover, some pairs consisted of events that had a sub-event/event relation (e.g. item WV2). Moreover, conjunct and elementary events had to be compared for some items (e.g. item WV7). The event "first red, than blue" can be seen as representative for the bag's distribution of 10 red and 15 blue cubes. Thus, in some tasks, students had to compare these representative events with other events (e.g. WV8). The last column in Figure 2 describes the event-pairs in respect to misconception-relevant characteristics.



Card for Event „First red, then blue"

Card for Event „First blue, then red or blue"

Figure 1: Assessment of abilities to compare probabilities: Example cards for events

Both experiments were implemented as imagined activities. However, red and blue cubes as well as the bags and urns were available when we presented the tasks. Moreover, the game contexts were explained at the beginning of the interview and control items ensured that students understood the context.

Sample

The sample comprised 40 primary school children (21 male, 19 female; 17 from grade 4, 23 children from grade 6). The children were asked to solve the tasks as described above in individual interview situations. Each interview encompassed 17 problems. Children were interviewed in a school setting outside their classrooms by trained interviewers. Videotapes of the interviews allowed for differentiated coding of the students' argumentations after transcription.

RESULTS

Understanding of basic probability concepts

The students' understanding of basic concepts of probability was assessed with 6 items. Students of grade 4 solved in mean 4.12 items (SD = 1.41), whereas students of grade 6 estimated in mean 4.52 item correct (SD = 1.16). The difference between the performance in grade 4 and grade 6 is statistically not significant ($t(38) = 0.99$, $p > .05$).

Item	Number of cubes		Probability of blue	Solution rate (SD)	
	Total number	Blue		Grade 4	Grade 6
GK 4	10	0	Impossible (p=0)	1.00 (.00)	1.00 (.00)
GK 1	10	6	Possible (p=0.6)	.94 (.24)	.96 (.21)
GK 6	100	1	Possible (p=0.01)	.41 (.51)	.39 (.50)

Figure 2: Assessment of basic understanding of probability concepts: Sample items and solution rates

Figure 2 displays the solution rate for the three sample items. All students evaluated the impossible event correctly. As for the example item GK1, students of both grades showed high solution rates if the probability was mid-range. Items that represented possible, but improbable events ($p<0.1$), were often evaluated as impossible events by children of this age. These results are in line with prior findings.

Abilities to compare probabilities of events

The items used to assess students' abilities in comparing probabilities of events demanded them to select the better event out of a pair of events. This task design results in a probability of .5 by guessing the right answer. Therefore, only solution rates that significantly deviate in a positive direction from .5 can be interpreted as ability. The solution rates are given in Figure 3 and items are ordered according to the solution rates for grade 6. Actually, two tasks were not solved significantly better than it would have been expected if students guessed.

One of these two tasks, the comparison of "red cube" against "two blue cubes" (WV7) cannot be solved with elementary means. Thus, the result lies in line with our expectations. The other case will be reviewed below.

	Probabilities		Card 1		Card 2		Solution rate (SD)		
Item	Card 1	Card 2	1. draw	2. draw	1. draw	2. draw	Grade 4	Grade 6	Characteristics
WV5	.24	.60	R	B	B	G	1.00* (.00)	1.00* (.00)	sure-to-win draw representativeness (aggravating)
WV8	.40	.24	R	G	R	B	.94* (.24)	1,00* (.00)	sure-to-win draw sub-event representativeness (facilitating)
WV6	.24	.16	R	B	R	R	.88* (.33)	.96* (.21)	representativeness (facilitating)
WV1	.60	.40	B		R		.82* (.39)	.96* (.21)	elementary events
WV4	.60	.36	B		B	B	.88* (.33)	.91* (.29)	sub-event conjunct-elementary
WV3	.36	.24	B	B	R	B	.76* (.44)	.87* (.34)	representativeness (aggravating)
WV2	.40	.24	R		R	B	.47 (.51)	.52 (.51)	sub-event representativeness (facilitating) conjunct-elementary
WV7	.40	.36	R		B	B	.47 (.51)	.48 (.51)	conjunct-elementary

* Differs significantly from the probability of guessing the right answer, t-Test, p<.05

Figure 3: Assessment of abilities to compare probabilities: Items, solution rates, and characteristics of items in respect to common misconceptions

As the comparison of the elementary events “blue cube” against “red cube“ (WV1) has been solved very well, the item format seems to be understood by the children. If students hold a misconception of representativeness, they should have difficulties in solving items WV5 and WV3, as the representative events have lower probability than the contrasting events. Indeed, item WV3 has low solution rates. However, all students solve item WV5, so that a misconception of representativeness – if existent – does not impact the solution in this case. An explanation may be the special structure of the event displayed on card 2 for item WV5. Here, a gray cube is used and stands for either a red or blue stone. As this is a certain event, the gray cube actually can be interpreted as a joker (although the instruction was designed not to use this extremely positive connoted term). They gray cube was also used in item WV8 that was solved almost perfectly. Indeed, in both cases, the card with the gray cube represented the event with higher probability. This indicates, that the gray cube worked as a highly facilitating factor for the children, such that an aggravating effect of a misconception of representativeness could not show up for WV5.

If students hold a misconception of representativeness, it would in addition facilitate items WV8, WV6 and WV2 as events that look more representative for the parent population ("first red, then blue") are indeed the events with a higher probability in these tasks. WV6 showed accordingly high solution rates, whereas WV2 was an item where the solution rate did not significantly differ from 50% (for WV8 see argumentation above). This is an interesting finding, as the item is mathematically equivalent to item WV8, which was solved by almost all students. In order to solve WV2, students had to compare the conjunct event "first red then blue" with the elementary sub-event "one red cube". For WV8, in contrast, a gray stone was used to represent the same elementary sub-event "one red cube" as a conjunct event "one red cube, then a blue or red cube". Thus, students seemed to have difficulties in comparing conjunct and elementary events, even if a subevent-event structure could be used as an elementary mean to compare the two events. This is also backed by the findings for item WV4, where the event "two blue cubes" must be evaluated against the elementary sub-event "one blue cube". These findings are in line with the prior findings and indicate problems of students with anchoring and adjustment.

Development of abilities

We did not observe performance differences between grade 4 and 6 so that in particular students' abilities in comparing probabilities of events did not increase. The mean score differences for the 8 items are not statistically significant (grade 4: $M = 6.24$ ($SD = 1.03$), grade 6: $M = 6.70$ ($SD = 1.15$), $t(38) = 1.31$, $p > .05$).

In our study, students had to give reasons for their choices. The in-depth analysis of these argumentations might shed light on small developments of abilities, even if no development can be seen on the performance-level.

The arguments could be classified as arguments that rely on probabilistic reasoning and other arguments. For example, a simple probabilistic (although not universally valid) argument would be "there is only a blue cube (points on event) and this blue cube is easier to get". Non-probabilistic arguments were mostly based on what the student wished ("I want to get a red and a blue one"), on animistic beliefs, or on physical explanations ("the blue ones build a layer above the red ones").

In grade 5, already 51.5% of the argumentations could be coded as probabilistic. In grade 6, the percentage of probabilistic argumentations was 63.6% and thus even higher, although this did not yield a statistically significant difference ($t(38) = -1.85$, $p > .05$).

Discussion

In this study we could show, that students of grade 4 already have basic abilities to compare probability of events in an abstract-formal urn context. However, there are indications that the comparison of events is influenced by certain misconceptions already at this age. In line with prior findings, the students tended to assign a high probability to events that looked representative. The comparison of elementary events with conjunct events proved to be difficult. This can be explained with the help of anchoring and adjustment effects. Sure-to-win draws were identified and facilitated the evaluation of events. In tendency, students of grade 6 showed higher abilities than students in grade 4. This was reflected in the probabilistic level of their argumentations.

The design of the study as highly structured interviews proved to be suited to elicit the abilities of students. Although the tasks seemed to be complex, our material-based implementation as a game of chance made the comparison of probabilities accessible even for school children of grade 4. Further research is necessary, to explore whether even younger children are able to compare the probabilities of events in an abstract-formal context. Moreover, the results from this study may help to understand difficulties of students. The study may contribute to improve instruction in the area of probability and chance by reducing the disequilibrium between students' cognition and teacher knowledge about students' cognition.

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PROSPECTIVE ELEMENTARY TEACHERS' CONCEPTIONS OF FRACTIONAL UNITS

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In order to design effective materials for use with prospective elementary teachers we need to a more fine-grained analysis of their conceptual understandings. The research study reported here is aimed at doing this in the area of number and operation. Sixteen prospective teachers (PTs) of varied mathematics backgrounds participated in four one-hour interviews. The subset of tasks discussed here focus on PTs' understanding of fractions. The data reveals not only varying levels of understanding of basic fraction concepts, but also highlights the impact of contextual features and general approaches to solving problems on PTs' performance, and help identify fraction tasks that are more powerful in illuminating conceptual weaknesses.

PURPOSE OF THE STUDY

Prospective elementary teachers must understand fraction division deeply in order to meaningfully teach this topic to their future students. The National Mathematics Advisory Panel in the United States (2009) has identified proficiency with fraction operations as a major goal for K-8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). Unfortunately, many PTs enter and leave their teacher education program with insufficient understanding of fractions. For example, Newton (2008) reported that, on a post-test given at the end of a required mathematics course focusing on numbers and operations at a major research university in the Northeast U.S., 85% of the participating PTs changed both 2/4 and 3/6 to the equivalent fractions of the same denominator when solving 2/4 - 3/6=? rather than recognizing both to be equivalent to ½. Toluk-Uar (2009) reported that most of the Turkish PTs in the control group still did not fully understand the meaning of finding a common denominator when adding or subtracting fractions on a post-test given at the end of a method course.

The main reason behind this less than satisfactory result is the lack of fine-grained empirical data of what mathematics knowledge PTs bring to the college classroom that can be used as the basis for designing effective curricular and instructional materials. In this paper, we report the results of a research study with 16 PTs of a wide range of mathematics performances to capture the breadth and depth of the mathematical knowledge of numbers and arithmetic operation that needs for teaching. The following research questions guided the design of this study:

- What conceptual structures do PTs have for making sense of whole numbers and fractions?

- What conceptual structures do PTs have for making sense of arithmetic operations?
- How do PTs' concepts of numbers and operations influence their performance on solving arithmetic word problems?
- How do PTs' concepts of numbers and operations influence their performance on evaluating solutions of arithmetic word problems?

In this paper, we focus on the results of the first research question within the context of fractions. Results of the full study will be discussed during the presentation.

THEORETICAL FRAMEWORK

One complexity of understanding fractions lies in the multiple sub-constructs of rational numbers: part-whole (3 out of 4 equal parts), measure (three (1/4) units from 0 on the ruler), quotient (the result of sharing 3 pizzas with 4 people), operator (shrink down to ¾ of original length), and ratio (3 boys vs. 4 girls) as discussed by Lamon (2007). Thompson & Saldanha (2003) argue that the traditional approach of interpreting fraction ¾ as three parts out of four parts implies an inclusive relationship that is suggestive of an additive relationship between the numerator and denominator; therefore the authors suggest two alternative conceptualizations of ¾ as three one-fourths or three times as big as 1/4 that are multiplicative in nature.

Steffe (2003) and Olive (1999) have identified several main types of fraction operations summarized in the following table.

Operation	Description
Unitizing	Treating an object or collection of objects as a unit, or a whole
Partitioning	Separating the unit, or the whole, into equal parts
Disembedding	Imaginatively pulling out a fraction from the whole, while keeping the whole intact
Iterating	Repeating a part to produce identical copies

These operations can be used to understand the mental processes one has to use in order to solve a variety of fraction tasks. For example, partitioning can be used to figure out the shaded part of the square as given in the diagrams below.

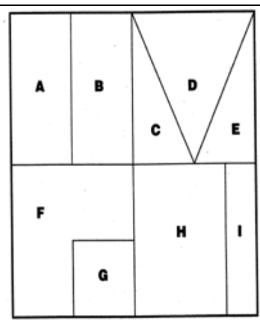
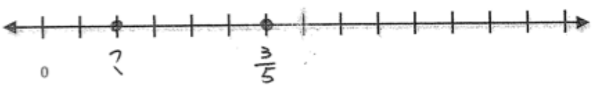
New operations can be developed from applying the same operation twice or from combining two or more operations. For example, recursive partitioning is taking a partition of a partition in the service of a non-partitioning goal and splitting is a composition of partitioning and iterating.

METHODOLOGY

The participants of this study were 16 PTs from a university in the Midwestern region of the United States: 2 white male, 2 African American female and 12 white female. Three PTs had completed the first required mathematics course for prospective elementary teachers, which is focused on number and operation, and earned either a B or BA in the course. The other thirteen PTs had not yet taken this class and had ACT math scores in three main groups: 18 or below (4), 19-23 (4), and 24 or higher (5). During a three-month period, each PT participated in four one-hour interviews.

Data Collection

All interviews were videotaped and written work was scanned for analysis. During the interviews, PTs were encouraged to solve the same tasks in multiple ways. They were also encouraged to justify their solution methods in a manner that an elementary student could understand. The goal was to uncover the processes and conceptions used by PTs to arrive at a particular conclusion, rather than leading them through, or assuming, predetermined processes. Each interview included of a set of core tasks that were administrated to all participating PTs to establish a common baseline data for comparison across individuals. The interviews also contained items that could be scaled up or down to further probe PTs' conceptions when needed. In this paper, we discuss the results of four main sets of tasks that were used to help us answer the first research question with respect to fractions. Each set of tasks contained 4-6 subtasks arranged from easy to hard. Table 1 includes example of these tasks.

Tasks	**Examples of subtasks**
Pieces	The largest outer square represents one unit. a) What fraction is represented by D? b) What fraction is represent by E and F combined? c) If H represents one unit, then what fraction is related to piece D? 
Number Line	What fraction is represented by the "?" a)  b) 
Boxes	Joe's **box** contains **60 oz.** of chocolate. a) How many oz. are in **2/3 of the box**? b) Joe's box of chocolate is **3/4 the weight of Dana's box**.

	How much does Dana's box weigh?
Bars	a) Show **3/4** of the following bar. Can you find another way to do it? b) The length of the bar shown below is **6/5 of the length of a red bar**. *Show me the length of the red bar*. 

These tasks are aimed at investigating a variety of unit-related concepts when numbers are conceptualized either as quantities (amount of space taken by piece D) or operators (2/3 of a box). To solve these tasks successfully, PTs must to be able to work with multiple units, move freely among quantities and the numerical relationship among them. No complex computation is required to solve any of these tasks. All tasks could be solved in multiple ways either through common arithmetic procedures, algorithms or alternative reasoning-based strategies.

Data Analysis

The overarching technique for on-going and retrospective analyses throughout the entire project was comparative analysis (Strauss, 1987). Following each interview, notes were created for the PT's responses to each interview task. Based on these notes and the PT's written work, each response was first coded as correct (1) or incorrect (0). To be coded a 1, the response needed to contain both the correct answer and sound reasoning. The responses were then further coded for the types of reasoning and the nature of errors that may have occurred. We looked specifically for patterns in the types of problems PTs got correct versus incorrect. Based on emerging themes and evidence, hypotheses were formulated about the main characteristics of PTs' existing fraction knowledge and the challenges they face when they reason and justify with fractions. Interview episodes were used to provide further evidence to confirm or disconfirm hypotheses. Our overarching goal is to generate models that are "useful to us, which have support in the data, that fit or interact productively with our larger theoretical framework, and that give us a sense of understanding by providing satisfying explanation about hidden processes underlying the phenomena in an area" (Clement, 2000; p. 559). In this paper we share initial analyses and hypotheses of PTs' conceptions of fractions based on these 4 sets of tasks.

RESULTS

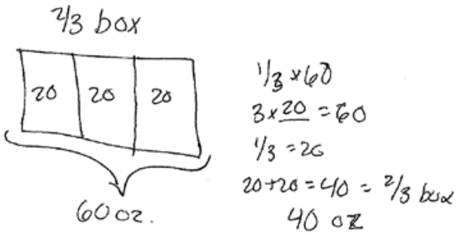
Though no complex computations are required for any of the tasks included in this analysis, only 9 out of 16 PTs were able to provide correct answers and reasoning for more than 80% of these tasks. These 9 PTs include all three PTs who had completed the required number and operation course, and 46% of those who had not taken this course. Of those PTs who had not yet taken this course, approximately 30% answered less than 60% of the questions correctly. A closer look at how PTs approached these tasks provides more insight into the strategies they used, and the struggles they faced.

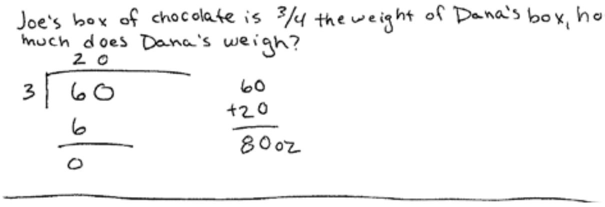
PTs' Strategies and Overall Performance

Recall that the Pieces tasks involve determining the size of different shaped portions of a whole. Depending on the shape of the given piece, PTs might solve these tasks either by directly partitioning the whole into congruent pieces of the given pieces (e.g. Piece I), or identifying a subunit that could co-partition both the given unit and the whole. For example, to name Piece D, PTs need to reason that D is made up of two shapes like Piece C, and the whole unit could be divided into 16 pieces of C. Thus D is 2/16 or 1/8. On each of the first three subtasks, 75% of PTs were able to provide reasonable solutions. The final and hardest subtask asked: if Piece H is 1, then what is piece D? Out of 11 PTs who were able to correctly identify Piece D as 1/8 and Piece H as 3/16, only 6 PTs were able to answer this question correctly by reasoning directly with the quantities (1/8 is 2/3 of 3/16) or finding an intermediate sub-units (for example, Piece C) and compare both H and D to that unit visually.

For the Number Line tasks, the most common strategy was to first determine the fraction corresponding to one (or a group of units) on the number line, then determine the value of the smallest sub-unit, and iterate it until landing on the "question" mark. For example, to solve the number line problem in Table 1, many PTs would iterate the "1/3" unit three times to land at the spot right before the question mark. They then conceptualized the "1" as 12 line segments; thus, 1 segment is 1/12, and the ? in this case is $1\frac{1}{12} or \frac{13}{12}$. A few PTs utilized equivalent fraction idea by asking themselves, *if 4 lines segments = 1/3, then I need to find ? in the following equivalence* $\frac{1}{3}=\frac{4}{?}$, which leads to the same conceptualizing about the "1" being 12 line segments. Overall, PTs performed better on the Number Line tasks than they did on the Pieces tasks. On each of the Number Line tasks, at least 75% of PTs were able to correctly determine the position in question. We attribute this, in part, to the fact that the Number Line tasks provided students with equal partitions, whereas the Pieces tasks did not. However as the Number Line tasks became more difficult (typically when it was not possible to iterate the given fraction to determine 1) an increasing number of PTs used guess-and-check rather than direct reasoning. Interestingly, those PTs who were not able to solve all the Pieces tasks, but did solve all the Number Line tasks were more likely to have used guess-and-check on these more difficult Number Line tasks.

The Boxes of Chocolate tasks yielded two general types of strategies. The first type made use of the concept that "*a*/*b* is *a* parts of 1/*b*, where 1/*b* is obtained by dividing the whole into *b* equal parts." These could be done for task a) either with a picture (below left) or direct computation (below right):





The second strategy was to use fraction multiplication (e.g., $\frac{2}{3}\times 60$). Only two students attempted this method: one was successful; the other was not because she chose to convert the fractions into decimals and made errors in the conversion. These basic strategies could be used to solve Box task b) if PTs could unitize the weight of Joe's box as a fraction other than 1. Only 9 out of 13 PTs could solve both a) and b) correctly. It's worth noting that all 4 PTs who were able to solve a) but not b) were able to correctly identify whether the answer should be smaller or larger than 60 ounces; furthermore none of these PTs attempted to create any kind of picture, and went directly to some kind of computation to solve the task.

The Bar tasks have essentially the same mathematical structures as Box tasks, except the former are set in a real life context with specific quantities, and the later are set in a pictorial format. Because there were no numbers, PTs were forced to focus solely on the picture to solve these tasks. Generally speaking, PTs' performances on Bar and Box tasks were almost all identical. However some PTs were able to solve Box problem with improper fractions but not the corresponding Bar task. In this instance it seems that the context may have made all the difference to these PTs.

PTs' Conceptual Weaknesses

The majority of the students were able to solve the earlier tasks in each category, and thus the progression of tasks helps to highlight where their thinking breaks down. The fraction conceptions possessed by the lowest performing PTs we interviewed were extremely limited. They conceptualized a/b as "a parts out of b parts" and were able to iterate unit fractions. These conceptions were sufficient to solve simple tasks, like Number Line a) and Box a), but not tasks that required partitioning or unitizing like Piece a). Middle-performing PTs faced two distinct challenges: conceptualizing *a*/*b* as "*a* parts of 1/*b*, where 1/*b* is obtained by dividing the whole into *b* equal parts" and the being able to use partitioning and disembedding to create new unit and to iterate with both unit and non-unit fraction in numerical and geometrical settings. Their inability to answer the question "If Piece H is 1, then what is piece D?" despite knowing that H is 3/16 and D is 2/16 was an indication of the need for more experience with unitizing tasks.

Not all PTs in the top-performing group possessed solid understanding of the fraction conceptions described above. Some of them relied on their good number sense and trial-and-error approaches to get correct answers. For example, to solve Box b) some would first determine that the weight for Dana's box should be heavier, and then experimented with different computations, checking to see which ones fit this criteria.

Thus they might add ¼ of 60 to 60 to get 75, and then check if Dana's box being 75 oz would fit their assumption about which box should be bigger. If it didn't, they might add 1/3 of 60 to 60 to get 80 and check the answer again. These students did not do as well on the Bar tasks in which a way of checking was not as obvious.

Similarly, a low performance score did not necessarily mean a complete lack of conceptual understanding. Consider Liz, whose overall performance on these four tasks was not stellar: she got 8 out of 20 correct. However a closer look at her approaches reveals a better understanding of many concepts than her numerical score suggests. In general Liz prefers to do problems quickly, looks for patterns between tasks, and rarely changes her mind once she decides on a course of action. Thus in several cases Liz failed to notice the change of patterns between sub-tasks. In addition, Liz prefers to use algorithms over non-algorithmic reasoning, but often performs those algorithms erroneously: for example, in solving Box a) Liz first converted ¾ to 0.75, and could not carry out the standard algorithm for 0.75 x 60 correctly. Liz demonstrates reasonable number sense, and the ability to think through problems in non-algorithmic ways, but only does so when she has no other choice. For example, unlike 9 other PTs who did not know how to solve the problem "If H represents one unit, then what fraction is related to piece D?" because they could not fit D inside H, Liz were able to re-unitize Pieces D and H in terms of Piece C, and recognize that Piece D was equivalent to two Piece Cs and H was equivalent to three Piece Cs. Thus D is 2/3 of H. In fact Liz was the only PT who was able to solve this problem without correctly solving all of the Pieces tasks. The analysis of Liz's approaches highlight some of the difficulties PTs fact that are distinctly different from those facing elementary students learning fraction concepts for the first time.

CONCLUSION

The findings provide a diverse set of fraction conceptualizations possessed by PTs before their first specialized mathematics course. They also help identify fraction tasks that are more powerful in illuminating previous conceptual weaknesses. As we work to meet the challenge of developing courses that help prospective elementary teachers understand fractions deeply, we need a clear sense of their conceptions and the issues that inhibit them from either fully developing those conceptions and/or drawing upon those conceptions in the context of problem solving (as students themselves, and as future teachers). Although existing frameworks for children's conceptual understanding of fractions are helpful, it is clear that general approaches PTs use to solve problems also play a critical role in the ways in which they are, and are not, able to utilize their knowledge to successfully solve problems. Thus in our work with PTs, not only do we need to help them re-learn and deepen their understanding of key concepts, but we must simultaneously help them to adopt a reasoning approach to mathematics more generally. Doing so may prove most difficult in cases where PTs have a false sense of security, as they often do with standard algorithms.

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GENDER FACTORS IN PRIMARY-AGED SINGAPOREAN STUDENTS' PERFORMANCE ON MATHEMATICS TASKS

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This study investigated the performance of 607 Singaporean students who completed mathematics tasks sourced from national assessments (tests) in Singapore and Australia. Specifically, this study examined student performance on graphic and non-graphic tasks from the respective tests. The results of the study revealed significant performance differences in the favour of males on graphics items sourced from both countries. There were no gender differences across non-graphic tasks.

INTRODUCTION

For more than thirty years, there has been an extensive body of literature which has examined performance differences between school-aged males and females on mathematics tasks. Although performance differences are widely acknowledged (Linn & Petersen, 1985), the extent of these differences, the age when these differences occur (and/or diminish), and the nature of the tasks have raised considerable debate. Some studies have indicated that gender differences have diminished in the past ten years (e.g., Spelke, 2005) while others have reported emerging differences in favour of males (e.g., Hill, 2011; Thomson, De Bortoli, Nicholas, Hillman & Buckley, 2010) or females (e.g., Dindyal, 2008). In terms of specific mathematics knowledge, males tend to perform better on tasks that require high levels of spatial reasoning (Spelke, 2005), and tasks that involve number lines and maps (Lowrie & Diezmann, 2007). This study investigated whether there were gender differences in the performance of Singaporean students on a mathematics sense making test which comprised graphic and non-graphic items from both Singapore and Australia.

BACGROUND

Singapore students and mathematics assessments: Gender perspectives

Singapore has a proud history of excellence when it comes to performance on international comparative assessments such as the Trends in International Mathematics and Science Study (TIMSS) and more recently, the Programme for International Student Assessment (PISA). Indeed, there has been considerable interest in the performance of Singaporean students on mathematics tasks for more than 30 years (Ministry of Education, 1988). Some of the analysis undertaken with respect to student performance has been associated with international comparisons, specific performance within content areas, and gender differences. Dindyal (2008) noted gender differences in favour of females in the number and data content strands from the Grade 4 2003 TIMSS data. By Grade 8, these differences expanded to include number, algebra and geometry content strands. By contrast, Kaur (1995) commented that as students continued through the schooling years, performance differences in favour of males on tasks which required spatial reasoning and cognitive complexity emerged. Such contradictions may be due to: the mathematics tasks within the respective instruments; the way in which students interpret the context of tasks; or cultural attitudes or beliefs with respect to pre-conceived ideas about the mathematics in the tasks.

To date, most studies that compare students' mathematics performance are derived from mathematics items which have either been produced within country (e.g., a national assessment) or from hybrid items developed for international tests (e.g., TIMSS). By contrast, this study assesses students' performance on items sourced within a country and from outside that country. Within both Singapore and Australia, national assessment of primary-aged students' mathematics knowledge through high-stakes testing has become common place. Singapore has implemented the Primary School Leaving Examination (PSLE) since 1960, with the main purpose to allocate student placement into secondary schools (Tan, Chow, & Goh, 2008). The PSLE is administered to Primary 6 students (aged 11-12 years). In 1988, the Singapore Ministry of Education undertook a study to examine gender differences in the PSLE. In contrast to the TIMSS data, males outperformed females on the mathematics test within the PSLE. No other studies were found to have examined gender differences of Singaporean students on this examination.

In Australia, the national testing regime is much more recent, having only commenced in 2008. The National Assessment Program for Literacy and Numeracy (NAPLAN) is administered annually to students in Grade 3 (aged 8-9); Grade 5 (aged 10-11); Grade 7 (aged 12-13); and Grade 9 (aged 14-15). The purpose of this assessment is to measure Australian students against national standards at the various grade levels and provide comparable data about student performance over time (Australian Curriculum Assessment and Reporting Authority [ACARA], n.d.). Both the PSLE and the NAPLAN are designed and developed within their respective countries. Since it is the case that assessment items are culturally based (see Cooper & Dunn, 2000) and typically reflect the teaching practices and curricula of a particular cohort (Lowrie & Diezmann, 2009), a comparison of student performance across *differently constructed and represented* mathematics tasks is undertaken to better understand how Singaporean students make sense of contextually-based mathematics tasks.

Graphic and non-graphic mathematics tasks

The entire representation of mathematics tasks has changed dramatically in recent years as graphical and visual representations become increasingly embedded (contained) within tasks (Lowrie & Diezmann, 2009). Graphic-based tasks have begun to replace more traditional word problems in assessment—including tests in countries like Singapore and Australia. As a consequence, mathematics tasks are more likely to include graphics-based information, with these graphics represented with more detail and increased richness. With respect to the mathematics PSLE and NAPLAN (Grades 3 & 5) instruments, graphic tasks constituted 41% and 70% of the total item banks respectively (Lowrie, in press). Thus, increased attention is being placed on primary-aged students' capacity to decode and interpret the various elements that comprise a task.

This paper is derived from a larger study investigating Singaporean students' performance on mathematics assessment tasks sourced from both the PSLE and the NAPLAN. It focuses on whether there are gender differences in the performance of Singaporean students on these mathematics tasks. Furthermore, this study looks specifically at performance difference with respect to both graphic and non-graphic items within these country-based instruments. The study contributes to the research base of the field since it moves beyond measuring performance *only*, to considering approaches and strategies employed by students.

DESIGN AND METHODS

The aims of the study were to:

- Determine whether there were gender differences in Singaporean students' performance across Singaporean and Australian tasks with respect to task type; and
- Analyse student responses (visual and analytic) on tasks where differences occurred.

Participants

The participants comprised 607 Primary 6 students (aged 11-12 years; 320 males and 287 females) from five Singapore schools (three government and two government-aided).

The instruments and administration

The participants in the study undertook two activities, namely: a mathematics sense making test; and a Mathematics Processing Instrument (MPI). These activities were designed and developed by the research team using items from the Australian NAPLAN tests from Grades 5 and 7 and the Singapore PSLE tests from Grade 6. The Mathematics sense making test was a 24-item instrument used to determine students' performance across mathematics tasks. It consisted of six graphic and six non-graphic tasks from the NAPLAN and six graphic and six non-graphic tasks from the PSLE. A graphic task is an item that has a graphic (e.g., picture, diagram, table, chart, graph or map) embedded within the task, where the graphic contains information essential for task solution. A non-graphic task is an item where there is only text (similar to a traditional word problem). The MPI provided students with the opportunity to describe how they solved the 24 tasks in the sense making test by choosing a strategy which best represented how they worked out their answer.

Two members of the research team attended the participating schools in Singapore during their morning classes. Both the test and the MPI were administered to whole (intact) classes to minimise disruption to both the school and the students' daily classroom routine. The classroom teachers and the research staff administered the activities in two parts. Firstly, students answered the 24 mathematics tasks as a pencil and paper test over the course of one hour. After a short break, students were then given one hour to complete the MPI. A common set of instructions (for both parts) were read to the students in each school.

Data analysis

The data in the study were coded according to two criteria: (a) whether the answer was correct or incorrect in the mathematics test; and (b) the solution strategy used to solve the task as indicated in the MPI. The solution strategies were coded as either visual or analytic. A visual response was one where the student used either a picture or diagram (on paper or in the mind's eye) to generate a solution. An analytic response was recorded as a solution where no picture or diagram was used in the solution process. Typically, a student's analytic solution involved the use of a computation or algorithm.

RESULTS AND DISCUSSION

The two research questions were investigated through an analysis of variance (ANOVA) to determine whether there were statistically significant differences between the mean scores of males and females (Gender) on four independent variables, namely: Australian graphic items; Australian non-graphic items; Singapore graphic items; and Singapore non-graphic items. The ANOVAs revealed statistically significant differences between the mean scores of males and females across the two graphic variables; Australian graphic [$F(1,606=10.06$, $p=.002$]; and Singapore graphic [$F(1,606=6.05$, $p=.01$]. By contrast, there was no statistically significant differences on the non-graphic variables: Australian non-graphic [$F(1,606=3.45$, $p>.05$]; and Singapore non-graphic [$F(1,606=.42$, $p>.05$]. Table 1 presents the means (and standard deviations) for gender across the four variables.

Item type	Total /6	Males N=320	Females N=287
Australian non graphic	3.60	3.72	3.48
	(1.60)	*(1.60)*	*(1.59)*
Singapore non graphic	2.73	2.77	2.68
	(1.70)	*(1.74)*	*(1.67)*
Australian graphic	3.78	3.96	3.58
	(1.49)	*(1.43)*	*(1.52)*
Singapore graphic	4.19	4.34	4.03
	(1.56)	*(1.48)*	*(1.62)*

Table 1: Means (*Standard Deviations*) of males and females on the four categories

Subsequent univariate analysis was conducted on the two graphic variables in order to determine which items within the respective categories had statistically significant differences. The Bonferroni correction method (alpha levels were adjusted to p=.008) was used to avoid Type II error. Two of the Australian graphic items revealed statistically significant differences in favour of males: namely, the *Spinner Item* [F, (1,605)=15.39, p=.001]; and the *Tree Design Item* [F, (1,605)=8.3, p=.004]. The Spinner item (see Figure 1) required the students to interpret a graphic based around notions of probability. The Tree Design Item (see Figure 2) required the students to interpret a pattern that increased geometrically and encouraged students to use pre-algebraic thinking.

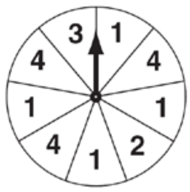
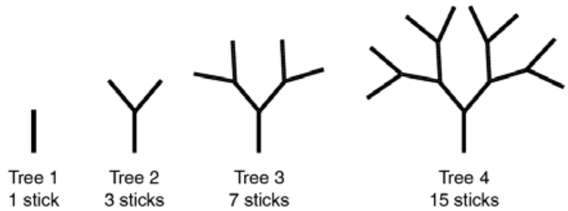
The spinner is used in a board game.  Sanjay spins the arrow. On which number is the arrow **most** likely to stop?	Lucy made 4 tree designs using sticks. There is a pattern in the way the trees grow.  Lucy continues the pattern in the same way. How many sticks will Tree 5 have?

Figure 1. The Spinner Item (ACARA, 2010a)

Figure 2. The Tree Design Item (ACARA, 2010b)

For the Spinner Item, 73% of males correctly solved the task compared to 59% of females. Both males and females tended to use a similar proportion of visual and analytic methods to solve the task (approximately 80% of males and females used an analytic approach). We observed that many of the females' incorrect solutions were a result of choosing the numeral "4" as the most likely occurrence for the spinner to land on. This error is a result of selecting the second most common number occurrence (rather than the most common occurrence "1"). Thus, an ability to accurately decode the graphic resulted in an incorrect solution. We argue that this visual recognition error is a result of females' performing significantly lower than males on *memory for location* tasks (Lowe, Mayfield, & Reynolds, 2005) which require the recall of the location of patterns or objects. It is also the case that notions of probability are not explicitly taught in Singaporean primary schools. This is noteworthy, since the participants had to rely more on the graphic to interpret the question than they may if they were familiar with such tasks. Consequently, the visual demands of the task were relatively high.

For the Tree Design Item, 64% of males and 52% of females correctly solved the task. The solution strategies for males and females on this item were different, with 54% of males employing an analytic response, while only 41% of females used an analytic response. Generally, the male students tended to make connections between the graphic features of the task and the corresponding information presented below the respective tree patterns. By contrast, the females (59% used a visual response) tended to either rely on the graphic itself or the text stimulus below to interpret the pattern. Since females were more likely to draw the 5th tree, there was an increased chance of making a calculation error. In fact, females were almost twice as likely to produce an incorrect response if they employed a visual method, rather than an analytic method, to solve the task (15% incorrect using analytic and 28% incorrect using visual). By contrast, males tended to be relatively successful regardless of the method employed (see Table 2). This finding is supported by a recent analysis of U.S. students' performance on international tests. The American Institutes for Research (2005) noted that "girls use different strategies for solving mathematics problems than do boys, and those strategies do not work as well on more complex problems" (p. 21).

	Analytic		Visual	
	% Incorrect	% Correct	% Incorrect	% Correct
Males	15	39	15	30
Females	15	26	28	31

Table 2. Analytic and visual responses for the Tree Design Item by gender

With respect to the Singapore graphic items, there was no statistical difference between males and females on any items at the conservative p=.008 level required under the Bonferroni principle. This was despite the overall significant value for this cluster of six items at the first level of analysis. It is noted that males outperformed females on each of the six items within this category. Nevertheless, these differences occurred due to an overall trend rather than large differences between the means of males and females on a particular item. By contrast to the Australian items, the Singapore graphic items required less decoding proficiency—in the sense that information contained in the graphic was not as critical for solution. In addition, the Australian graphic items were more difficult (see Table 1) which also may explain trends towards difference rather than specific item differences.

CONCLUSION

The major finding of this study revealed a difference in the performance of males and females on particular types of mathematics representations. Of particular relevance was the fact that males were more proficient at solving graphics-based mathematics items. This finding is in contrast to TIMSS data which indicated that Singapore females outperformed their male counterparts at similar age levels. Since the mathematics tasks in this study (especially the graphic items) could be indicative of *new* assessment tasks, the current study may be a guide to future developments in the area.

The finding of gender differences in favour of males on graphic items has three educational implications. First, everyday instruction in mathematics needs to provide opportunities for females to become proficient at interpreting graphics tasks and especially those tasks where the information presented in the graphic is critical for tasks solution. Explicit guidance and support should be provided for developing visual recognition skills associated with locating patterns and objects through activities such as subitising and memory games. Second, females should be encouraged to develop a wider repertoire of strategies for decoding graphics tasks. In this study, we noted that females tended to rely on visual encoding approaches rather than using the graphic as a scaffold to utilise more efficient analytic strategies. Thus, it would be helpful to encourage these students to find the mathematical patterns within the particular graphic. Finally, studies that analyse the performance of students on mathematics achievement tests should go beyond the description of mathematics content (e.g., algebra or measurement) and also consider the way in which particular tasks are represented. To solve today's mathematics tasks, students need to acquire different spatial-reasoning skills which allow them to consider all the elements of a task, including specific features of a graphic and the surrounding text.

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MATHEMATICS EXPERIENCES WITH DIGITAL GAMES: GENDER, GEOGRAPHIC LOCATION AND PREFERENCE

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Numerous digital games contain mathematics ideas and representations. This study investigated the game playing preference of 410 Australian students who classified the types of mathematics content and ideas present in digital games they played in out-of-school contexts. The results of the study revealed significant gender differences in the type of games primary-aged students played. Females tended to prefer playing games that required logic and problem solving while males preferred games that contained maps. Results also revealed an interaction effect between gender and geographic location. In each case, gender differences were more pronounced in non-metropolitan locations than metropolitan locations.

INTRODUCTION

The broad focus of this study was to determine the type of mathematical experiences students engage with in out-of-school settings. Of particular interest was participants' engagement with digital games that contained specific mathematics content or ideas. Our definition of mathematics content includes not only quantitative reasoning and literacies but also measurement ideas, graphical data and spatial reasoning. Elsewhere (e.g., Lowrie & Jorgenson, 2011), we argue that the mathematics concepts are contained within most digital games—with representations that include large numbers; algorithms; graphs which may show energy levels or similar measures; or games that display maps so that the gamer can locate objects and navigate through spaces. Increasingly these games have dual screens and multiple representations of maps and perspectives. As Wheatley and Brown (1997) posited, many aspects of mathematics are image-based and as a result, it is important to understand the effectiveness of new technologies on students' mathematical development as they spend increasing amounts of time with digital technologies. This study considered the mathematics content of the digital games students' play and the extent to which such engagement differed in metropolitan and non-metropolitan areas. Furthermore, the study considered potential differences between males and females' engagement *within* these contexts.

Diverse geographical contexts

Australia is a country which is both diverse culturally and geographically. The population of Australia is clustered around relatively large coastal cities, with very few large metropolitan cities located away from the coastal areas. The OECD Programme for International Student Assessment (PISA) results indicate that Australia scores in the top ten countries in numeracy. However, there is cause for concern as the

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Australian results demonstrate greater variability between the best- and worst-performing students than other high-performing countries. Some of the most at-risk students are those who live in rural/remote areas (37% below those students living in metropolitan areas) (MCEETYA, 2005). It is not that these learners are cognitively inferior but due to their life circumstances, the opportunities to immerse themselves in a numerate culture are very limited. Moreover, for these students, issues of culture and language compound their access to the discourses of school mathematics (Sullivan, Zevenbergen, & Mousley, 2005). Unlike their urban counterparts whose worlds are saturated with mathematical constructs—transport timetables, street directories, signage, advertising and so on, students who live in geographically remote areas have less exposure to these signifiers and hence many of the taken-for-granted aspects of school mathematics are very restricted in their lifeworlds (Lowrie, 2007).

Gender and digital game play

Since the investigation involved students playing with digital technologies it was also timely to consider the role gender played in such engagement—given the extensive literature base which points to differences in how males and females engage with games (Bananno & Kommers, 2005; Brumbaugh, 2009) and the types of games they prefer to play (Cassell & Jenkins, 1998; Lowrie & Jorgensen, 2011). With respect to the type of games student play, studies have focussed on game genre and male preference for particular types of games—for example, games that simulate violent acts (Jansz, 2005) or games which require a competitive mindset (Hartmann & Klimmit, 2006). On the other hand, there is evidence to suggest that females prefer playing games that involve logic (Quaiser-Pohl, Geiser, & Lehmann, 2006) and games which promote skills such as perceptual speed and time skills (Ziemek, 2006).

Of particular interest to mathematics educators, is the fact that males prefer games which are graphically sophisticated and require good spatial awareness and visualisation skills (Lowrie & Jorgensen, 2011). This point is noteworthy since males tend to outperform females on mathematics tasks which require high spatial reasoning (e.g., tasks which require the interpretation of number lines and maps, see Lowrie, Diezmann, & Logan, 2009).

The present study goes beyond previous research by investigating the actual mathematics content contained within digital games rather than the types of games student play. Moreover, we take note of Fennema and Leder's (1993) challenge to ensure that studies that consider gender differences in mathematics are focused and strategic. This is achieved by specifically asking the students about the mathematics in the games they play—and thus focuses on the students' perceptions—rather than what the games designers stipulate the game constitutes.

METHOD

The project focused on the development and subsequent implementation of a Digital Landscape survey with primary school students. Data from the survey were analysed to address the three research questions of the study using the Statistical Package for the Social Sciences (SPSS for Windows 17.1). The broad aim of the study was to determine the nature of student engagement with entertainment-based digital games with specific reference to mathematics content. Two research questions were formulated:

1. *Are there gender differences in students' engagement with digital games?*
2. *Are there geographic-location differences in students' engagement with digital games?*

Participants

The participants comprised 410 students (M=195; F=215) from two geographically-distinct locations (Metropolitan=171 and Non-Metropolitan=239) in Australia. The metropolitan location was one of Australia's largest coastal cities while the non-metropolitan location was a regional inland city. The populations of the two sites were distinctly different. The sample was purposively selected from a general expression of "participation interest" from schools in both catholic education and state systems. The participants were aged 10–12 years from Grades 5 and 6. The questionnaire was administered to all Grade 5 and 6 students at the respective schools, with students completing the questionnaire at home once parental consent had been received.

Survey instrument

The survey was designed to describe the nature of student use of a range of entertainment-based digital games with specific reference to mathematics content. The survey sought information that would provide patterns of student behaviour in relation to preference for the content and processes involved in playing the games. The survey used in this study was based on the British Educational Technologies and Communications Agency (BECTA) (2002) questionnaire on Young People's Use of ICT and adapted specifically for this study. Further information regarding survey validity is described elsewhere (see Lowrie & Jorgensen, 2011). The present study focuses on the items which asked students to provide information about the presence of mathematics content contained within their game-playing. These questions were in a 5-point Likert scale (in the form *frequently* through to *never*) format. Descriptive data were also collated and included students' gender, grade level and favourite game. The four survey questions appear in the Appendix.

RESULTS

Comparisons of digital game play

The two research questions were investigated through an analysis of the participants' responses to four items from the survey. These items (the independent variables) included questions regarding algorithms, maps, graphs and problem solving (see Appendix). A multivariate analysis of variance (MANOVA) was used to analyse mean scores across Gender and Geographic location dependent variables. The MANOVA revealed statistically significant differences between the mean scores of students across the *Gender* [F(4,403=13.2, p<.01] variable. There was no statistically significant difference on the *Geographic location* [F(4,403=2.19, p>.05] variable. Noteworthy, there was statistically significant interaction (*Gender* x *Geographic location)* [F(4,403=6.59, p<.01]. Table 1 presents the means (and standard deviations) for gender and geographic location across the four questions.

Location	Gender	Algorithms	Maps	Graphs	P. solving
Metro	M (n=84)	2.42	2.81	2.27	2.36
		(1.2)	*1.3*	*1.2*	*1.4*
	F (n=87)	2.45	2.21	2.15	2.45
		1.1	*1.2*	*1.0*	*1.2*
Non-Met	M (n=111)	2.18	3.12	1.79	1.69
		1.1	*1.4*	*1.1*	*0.9*
	F (n=128)	2.68	2.20	2.18	2.64
		1.1	*1.1*	*1.0*	*1.4*

Table 1: Means (*Standard Deviations*) of Student Scores by Grade and Gender

Subsequent post-hoc analysis was conducted to determine where differences across the Gender variable lie. ANOVA's revealed statistically significant differences in game playing preference on *map content* [F(1, 410=24.13, p<.001] and *problem solving processes* [F(1, 410=17.78, p<.001] variables. Males ($\bar{x}$=2.97) were more likely to play games containing map content than females ($\bar{x}$=2.21). By contrast, females ($\bar{x}$=2.55) were more likely to play games involving problem solving than males ($\bar{x}$=2.03).

With respect to interaction differences, there were three statistically significant differences including *map content* [F(1, 410=6.14, p<.01], *graph content* [F(1, 410=5.65, p<.01] and *problem solving processes* [F(1, 410=12.08, p<.001]. Figure 1 displays the interaction between males and females across geographic locations for these three questions. In each of the three instances, the interaction was due to there being smaller gender differences at the metropolitan site than was the case at the non-metropolitan site.

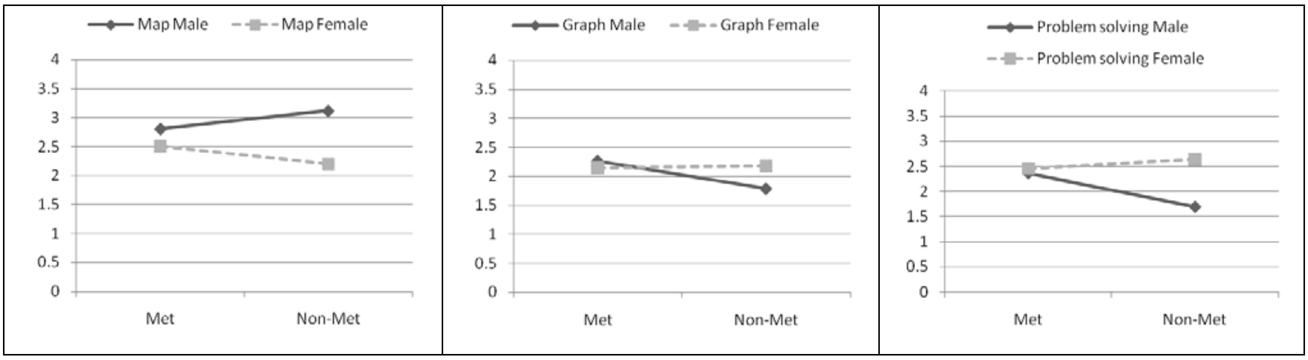

Figure 1. Interaction between gender and geographic location for *map content*, *graph content* and *problem solving processes*.

Game playing preferences

The questionnaire also provided opportunities for the students to describe the types of games that they preferred to play. These data were collated and categorised by gender and location. The analysis below provides a description of examples of games identified through the descriptive data as being played by non-metropolitan males and females in order to illustrate the interaction effects described above. That is, the identification of the different types of games non-metropolitan males and females played which elicited responses indicating the mathematics content they perceived to be accessing whilst engaged in gameplay. The intent of this analysis was to describe features contained within these games and the explicit mathematics concepts identified. Figure 2 features a game (*Runescape*) that non-metropolitan males frequently played involving map content and concepts. Figures 3 and 4 represent games commonly played by non-metropolitan females which contained graph concepts (*Nintendogs*) and problem solving (*Club penguin*).

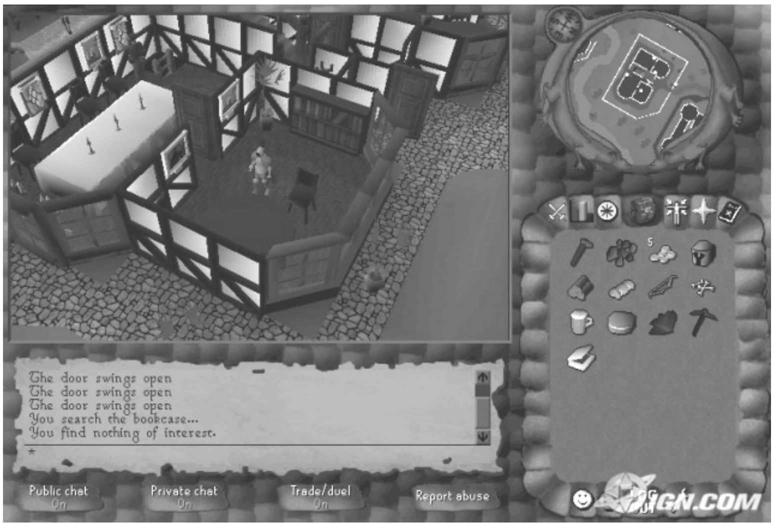

Figure 2. A preferred game of non-metropolitan males with mapping content

The game Runescape was a favourite among males. This game, and others of this genre, typically provides mapping content with multiple map perspectives. The perspective on the left side of the screen is approximately at 45^{o}. This screen is utilised to move between rooms and navigate the immediate environment. The text below the representation provides information concerning the location and movement of objects within the immediate space. The map on the top right is a birds-eye-view perspective of a region much larger (and less detailed) than that of the left side. This map is used as a location device and provides positioning information in relation to the larger 'world'.

In Figure 3, the Nintendogs game, generally played by females, displays two of the four screens with graph information. The top left frame contains a form of picture graph represented with images and numeric symbols. The top right frame contains a visually complex bar graph which represents information on a percentage scale. The data in these frames are related and provide information pertinent to the game.

Figure 3. A preferred game of non-metropolitan females using graphs

Figure 4. A non-metropolitan female preferred game with problem solving.

Figure 4 (Club penguin) is an example of a game involving problem solving in which there is a scenario that needs to be solved. The textual information requires the gamer to investigate an incident at the Stadium by searching the entire location with a clue “if you run into any locks, your Spy Phones have a new tool”. These new tools are utilised to solve several tasks in order to achieve an overall objective. These scenarios are often described as challenges or components of a ‘staged’ and sequenced set of problems that need to be completed to achieve a main goal. Females were much more inclined to play this game, and others within this genre, than the males.

DISCUSSION AND CONCLUSIONS

Our study examined the nature of student engagement with entertainment-based digital games with specific reference to mathematics content. There were gender differences for two of the four variables; namely, *map content* (in favour of males) and *problem solving processes* (in favour of females). There were no statistically significant differences in students’ digital game engagement across geographic location. There was, however, an interaction effect between gender and geographic location across the *map content*, *graph content* and *problem solving processes* variables.

The gender differences identify the type of mathematics content that males and females prefer to engage with. One explanation for males’ preference for digital games that contain maps could be associated with the general assumption that they prefer games which are graphically sophisticated and often involve competitive traits (Hartmann & Klimmit, 2006; Zevenbergen, 2007). Other studies (e.g., Lowrie & Diezmann, 2011; Lowrie & Diezmann, 2007) have shown that boys outperform girls on mathematics items that require the decoding of spatial information—and particularly tasks that required the interpretation of map-based graphics. By contrast, females prefer playing games that required problem solving and scaffolded challenges that involve logic (Lucas & Sherry, 2004; Quaiser-Pohl et al, 2006).

Although there is a generally accepted view that non-metropolitan students are disadvantaged in terms technology engagement and opportunity—due in part to isolation and a range of factors associated with the digital divide (Gee, 2007)—the results of this study did not find differences in student engagement across location. This finding is noteworthy considering previous research on geographic location identified differences in the performance of primary aged students from metropolitan and non-metropolitan locations on map items that required interpretation of coordinate maps and landmark maps in favour of metropolitan students (Lowrie, Diezmann & Logan, 2011).

The results of the study reveal three consistent interaction patterns between gender and geographic location of the students. Males at the non-metropolitan location engaged with a higher number of map content games than males in the metropolitan area. By contrast, females in the metropolitan area were more likely to engage with these games. This interaction pattern was reversed for students' engagement with problem solving games. With respect to the graph content, males at the non-metropolitan location were less likely to engage with these games compared with that of the metropolitan males—while females reported that they engaged with similar numbers of graph content games irrespective of location. For the problem solving, there was a slight increase in the number of non-metropolitan females playing such games, whereas there was a dramatic reduction in non-metropolitan males playing such games.

Importantly, the majority of the games *all* of the students played—irrespective of gender or location—contained rich visual environments. These visual environments, due in part to advances in technology, where often dynamic and required high levels of visual reasoning in order to process varied forms of information. Most of the games played by these students contained maps (generally games they preferred to play) or graphs (especially non-metropolitan girls) that were more diverse and sophisticated than those they would typically encounter in school curricula, and these digital representations were also quite different to the static forms displayed in text books and work samples. We argue that is it time to utilise such digitally-rich (and dynamic) representations of maps and graphs in classrooms.

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DEVELOPMENT OF REFLECTION ABILITY IN PFCM

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In this research report we present some results pertaining to the reflection upon practice of three teachers in the context of the program for Continuous Training in Mathematics (PFCM). From this study it is possible to determine that the participation of teachers Aida, Dora e Sara in PFCM has contributed to the development of reflection. However, although they all attribute it a degree of importance, they show diverse preferences for forms of reflection undertaken and divergences in the depth and content of the written reflections achieved.

INTRODUCTION

Professional development is considered to be a permanent, continuous and intentional process aiming at improving professional knowledge, teaching practice and reflection thereupon and, in consequence, contribute to student's improvement in learning Mathematics (Guskey, 2002; Sowder, 2007). Participation in training programs is pointed as a tool for professional development (Borko & Putnam, 1995; Guskey, 2002; Santos & Ponte; 1998; Zaslavsky, Chapman & Leikin, 2003; Wu, 1999). In particular, the Program in Continuous Training in Mathematics (PFCM) presets specific and innovative traits such as the kind of sessions considered, including group training sessions and classroom supervision sessions with the presence of the supervisor, group planning and subsequent task implementation, emphasis on practice, the importance of collaborative work and the elaboration of a portfolio by the trainee, comprising three written reflections (Serrazina *et al.*, 2006).

In this context we wish to present some results pertaining to the question of evaluation: In what way does the teacher's ability to reflect evolve throughout the PFCM training program?

BRIEF THEORETICAL OVERVIEW

Reflection presents itself as a connecting link between knowledge and practice (Krainer, 1996; Schön, 1983). It is one of the most quoted activities aiming at the teacher´s professional development, since it can be presented as an appropriate tool to face new situations and to improve classroom activities (Lieberman, 1994; Schön, 1983; Zeichner, 1993). Reflection could be considered as a mental process to structure or restructure an experience, a problem, existing knowledge or *insights* (Korthagen, 2001), leading to its understanding (Hatton & Smith, 1995). It is constituted as a continuous process of analysis and perfecting of the practice (Cole & Knowles, 2000).

It is fundamental for reflection to be envisaged as a deliberate, systematic and structured process, its beginning and end being located in action (Dewey, 1933; Rodgers, 2002). The recursive character and cyclical nature (Lee, 2005) briefly define the way it is processed.

"Although all teachers think informally about their experiences within the classroom, to foster habitudes of considered and systematic reflection may be the key both to improve their teaching as to scaffold their professional development throughout life" (Stein & Smith, 2009, p. 22). It is not only important to state that the teacher reflects, it matters that the teacher is aware that he is so doing, what must be considered within the process and of the intention underneath it. To that purpose several strategies may be adopted such as the intervention of a person stimulating it (mentor, tutor, supervisor, critical friend), reflections sharing within collaboration works or the use of written reflection (Sowder, 2007).

The existence of levels of reflection, which can range from the description of one aspect of a class to the consideration of ethical, social and political implications of the teaching practice (van Manen, 1977), leads to the consideration, apart from its content, of the depth it should reach.

INVESTIGATION METHODOLOGY

This investigation has followed an interpretative paradigm (Stake, 2009), with recourse to case study (Yin, 2009). Participants were three 1st cycle teachers (students aged 6 to 9), Aida, Dora and Sara (fictional names), who voluntarily enrolled in PFCM. Selection criteria used were number of teaching years and academic training. To conduct data gathering semi-structured interviews to the three teachers were used, as well as observant participation of work sessions and documental gathering. The following categories for the analysis of information relating to reflection were used: (i) importance given to reflection; (ii) forms of reflection used in PFCM; (iii) content of written reflection; and (iv) depth of written reflection.

Aida has always related well to all areas of knowledge and has always felt a taste for learning. To Dora, Mathematics has always been felt as a problem area. She was not successful as a student and claims not to like mathematics. Sara has consistently shown an interest for scientific areas mathematics having been as a student, her favourite subject.

The teaching experience of these teachers is likewise distinct. Aida and Sara have over twenty years teaching experience the former having meanwhile completed a master's degree and the latter an educational complement. Dora has less than ten years teaching experience. A taste for collaborative work has developed quite clearly in Aida and Sara counter to the idea stated by themselves at the beginning of the study.

PRESENTATION OF RESULTS

Importance attributed to reflection within PFCM

Aida acknowledges that "reflection is one of the most important parts" of PFCM, "perhaps the one we undertook the less in our daily practices" [intermediate interview]. She feels the "obligation" of, within PFCM undertaking a "profound reflection", written, on the experienced tasks is a positive aspect. She justifies it stating: "I think that the most remarkable aspect this training gave me was that realisation. I can reflect, draw conclusions, and afterwards improve" [final interview].

Dora emphasizes that her participation in PFCM has allowed her to learn how to reflect, having been able to write on the tasks experienced:

> I don't like writing. However, in this training I don't know where I get so many words from (…) I do it with pleasure, gladly, because I feel that all I write is not enough. I have a feeling to write more, (…) there is always an idea. I have to say what I felt. [final interview]

For Sara, during her participation in PFCM, reflection was the aspect which pleased her the most and she started integrating points that she had not considered to be important in a reflection, namely students' leanings:

> I didn't even know how to undertake a reflection. (…) But I did not know what to record concerning students' acquisitions. Have they learned operations?! Of course, they already know that! There, I had some initial doubts, because that would not have been something to reflect upon. But their own work reflects what we did. [final interview]

Forms of reflection within PFCM

All three teacher recognized salience to the forms of reflection undertaken within PFCM – written, undertaken individually, post-observation with the supervisor and joint reflection (undertaken in the training group) – albeit presenting different preferences.

Aida has shown a preference for written reflection undertaken individually, on the basis of her own personal characteristics, namely her fondness for writing:

> I would rather have the opportunity to think, I enjoy having that space to think about what has happened. Reflecting a bit on my own (…) I thing I reflect better when I'm writing, but this is a personal issue. I just think better about things when I'm writing. So I value written reflection the most. [final interview]

Dora has set her preference on post-observation reflection with the supervisor, fundamentally for the chance to have someone correcting her, thus constituting an improvement to her teaching practice:

> You helped me … you were careful to tell me when I was not going that well (…) you guided me, gave me your opinion, which was for me, as a matter of fact, one of the fundamental things, for me to improve (…) If I don't have someone to judge my action how can I know sere I went wrong? It is possible that they all [the different kinds of

reflection] are important, but the one done with you helped me a lot because I felt protected and corrected. [final interview]

Sara has pointed joint reflection as her favorite for the opportunity provided to change views with her colleagues and the implications of this to her teaching practice:

I think group sharing was important. I had more interest in listening to other people, to see their strong and weak points, the strategies they used. This is a personal view, I paid more attention to others in order to understand what they did, what strategies they used. [final interview]

Content of the written reflection

The reflection script given within PFCM considered the following points: (i) planning and evaluating the task; (ii) evaluating what the students might have learned with the task undertaken; (iii) importance the task had for the teacher, and (iv) the teacher's future perspectives concerning mathematics.

In Aida's case the script was followed to the letter and presents the same subcategories in all the points it comprises. For instance, in the item *Planning and evaluating the task*, for every task she justifies the kind of task to undertake, as can be seen in the second reflection submitted:

As in the course of the group training session we dealt with the subject Mathematical Investigation and their application in the classroom, and this idea excited me a lot. I decided to choose this theme as my second supervised class. From that point on I decide to create a plan to an *activity of mathematical investigation with the multiplying tables*. Between the possibility of working just one table or all of them, I opted for the second one, which seemed to me to open more possibilities for the students to make discoveries. [portfolio 2nd reflection]

In Dora's case the use of the script depended on the task under experimentation. Only in the second reflection does she follow all the points considered in the script. It is in this reflection that Dora highlights the importance that the completion of the task has had for her professionally: "It was a class that made me grow up a lot and which constituted a landmark in my professional and personal career. I had never had such an experience, in which at the beginning of the task I had gotten so few answers from the students" [portfolio, 2nd reflection] and she manages to put forward some perspectives about her future practices, showing the will to accept challenges again, valuing preparation of tasks, namely mathematical investigations:

Anyway, next time, I will again accept a new challenge, but never without having considered several possibilities of solving the questions, having always tried to know more about the issue at hand (…) It is difficult to plan investigation activities. I needed a lot of coaching from the supervisor. Preparation is an important phase. [portfolio, 2nd reflection]

One can also find in Dora some differences from one reflection to the other concerning the sub contents of each item. For instances, in the Topic *Planning and Evaluating the Task,* the fulfillment of the task involves the first and third reflections. Thus in the second reflection she states:

> The fact that I havent though all I had planned was my option, because, as I already stated, the students had perfectly understood the solution mechanism. Thus, to continue would only be good for them to aplly the already mechanized process. [portfolio 1st reflection]

In the third reflection she evaluates the fulfillment of the planning: "Planning was accomplished with most of the class, however slower-paced students finished the self-Evaluation sheet after the break" [portfolio, 2nd reflection].

Sara, although following the script, has balanced in a more definite way the sub-contents presented in each reflection, as a result, mostly, of the specificity of the tasks upon which she reflects. For instances, in the reflection on the second task, involving the organization of students in work-groups, she makes clear her opinion about advantages and disadvantages of group work:

> In my opinion group work has advantages and disadvantages. The disadvantage is that it makes them lazy, even lazier, because we have those who commit themselves and work, and those we just sit back and copy the results. The advantage is that it advances collaborative work, thus contributing, no matter how little, for the social and personal development and making learning a moment of sharing. [portfolio, 2nd reflection]

Depth of written reflection

For the analysis of the depth of the written reflection the following were considered as categories: (i) Confrontation with one's own practice (identification and description of what one considers important or problematic); (ii) Interpretation (why does one perform the way one does?); (iii) Putting into perspective (confrontation of action with what one thinks and feels about it) and (iv) Reconstruction (what ought to be kept? What can be different? What can be changed, why?) (Lee, 2005).

In the three reflections undertaken, Aida e Sara considers all levels contemplated (description, interpretation, problematizing, and reconstruction). In Aida's Case, apart from the first two levels, reflection goes through the level of problematizing when she questions her actions, and simultaneously proposes alternatives to situations arising in the course of the class: "However, I should have insisted a little more with student Raul who presented a lot of difficulties in the task" [portfolio, 1st reflection]. The level of reconstruction is attained when she rethinks the organization of classroom work: "I think that a different way of conducting this type of class, in a group with this characteristics, and aiming at potentiating communication, could be work in groups" [portfolio, 1st reflection].

In Sara's Case, problematizing is undertaken when, for instances, she question the amount of time allotted to the individual solving phase, and consequent lack of time to the phase of communicating results:

> I think I allotted too much time to individual problem solving, which didn't allow more children to go to the board and show their reasonings and to ascertain whether there were different results. [portfolio, 1st reflection]

Reconstruction is reached when she rethinks classroom practices when considering that particular class:

> I think that in class one should pay more importance to problem solving, because it will help them develop reasoning and prepare them to use personal strategies more easily and progressively assume a critical attitude face the results. [portfolio, 1st reflection]

To Dora, the depth of reflection depends on the task undertaken. Only in the second task she problematized and rebuilt her practice. The level of problematizing is visible when she shows an awareness that the first part of the task has not developed according to her wishes and expectations: “Although I still think about the first part of the ask as something which has not taken place as I intended, From then on, yes, (…) the class took on another spirit” [portfolio, 2nd reflection]. The level of reconstruction is verified when she puts forward some perspectives about her future practices, showing her determination to face challenges, and showing a conscience of the need for an adequate preparation of the tasks:

> Anyway, next time, I will again accept a new challenge, but never without having first considered several possibilities of solving the questions and having tried to know more about the issues dealt with. [portfolio, 2nd reflection]

FINAL CONSIDERATIONS

It is clear that reflection is a central issue in teacher training (Ponte, 1994; Schön, 1983, Zeichner, 1993), the development of the reflection of teachers Aida, Dora and Sara on the tasks experimented having been evident, within PFCM, as an outgrowth of the context in which they participated. Being a compulsory component of this training, reflecting activity was defined by an intentional and systematic nature (Dewey, 1933). It is however to be remarked that the starting points of the teachers considered was clearly different, which may account for differences in the results attained. Aida upholds the idea of *reflecting to improve her practices*, fundamentally when mediated by writhing, as, when writing, the teacher gains awareness of his own leaning process (Passos *et al*, 2006). On the other hand, as for Dora, who had a conflictive relationship with mathematics and showed a great deal of insecurity in her teaching, it is possible to detect the need to have the acknowledgement of the supervisor about her practices to improve her practices. She *thus favors reflection after action*. The reflection with the supervisor, intersecting with the concept of reflexive teaching put forward by Schön (1983), is essential to help her analyze planning, concretization and evaluation of teaching, in the sense of developing her ability of reflection upon her practice (e.g. Day, 2001; Hatton & Smith, 1995). To Sara, reflection needs to happen in community, in interaction with others (Rodgers, 2002). It is a *reflection involving collaboration*. In training programs, involvement in cooperation with other teacher can bestow (more) sense and meaning on formative or classroom experiences which, otherwise, would not have been so evident (Day, 2001; Hargreaves, 1994).

The script for reflection has served as a baseline for all reflections undertaken by the teachers, allowing them to attain a structured orientation. In connection with this aspect is the depth of the written reflection. Reflections can cover from simple descriptions of thought about one only aspect of a class, to consideration of ethical, social and political implications of teaching practice (van Manen, 1977). Aida and Sara followed all the levels considered while, in Dora's case, the levels of reflection attained varied according to the tasks experimented. On this point we consider that in teacher training programs attention must be given not only to importance of "having" teachers reflect but also to the aspect such a reflection should contemplate and to it is depth.

We consider, thus, that training programs, when valuing the teacher's reflection emphasize the importance of the teacher's practical theories (Zeichner, 1993). The teacher is no longer seen as someone who applies in his classroom theories generated elsewhere, he is envisaged as someone who produces his own theories. Likewise, the recognizance of the active role of the teacher in his own professional development is emphasized. This idea is rooted in the fact that professional knowledge cannot be transferred, as it actively constituted, individual and socially, through personal experiences with the environment and in interaction with others, involving reflection and adaptation (Zaslavsky, Chapman & Leikin, 2003).

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INSIGHTS INTO CHILDREN'S UNDERSTANDINGS OF MASS MEASUREMENT

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While measurement may look simple, insights gained from research into young children's concepts of mass measurement lead us to believe that the learning of measurement can be complex for children. A teaching experiment, lasting one week, delivered five lessons that offered rich learning experiences regarding concepts of mass to 119 children of 6-8 years of age. Observational data collected during these lessons and clinical interview data collected before the lessons gave insights into the diversity of children's perspectives. The focus of this paper is children's understandings of the attribute of mass and comparison and ordering of masses.

Clements and Sarama (2007) drew on their own work and that of other eminent researchers of early childhood mathematics to conclude that "young children possess an informal knowledge of mathematics that is surprisingly broad, complex, and sophisticated" (p. 462). While research elaborates early informal mathematics for a range of concepts, we have limited knowledge of children's understandings of mass measurement at the start of school or during the elementary years.

Brainerd (1974) conducted research on mass measurement examining the transitive nature of young children's ordinal ability and found that 5 year olds could, given feedback, be trained to arrange three balls of clay according to their mass. Brown, Blondel, Simon, and Black (1995), with an interest in progression in measuring, interviewed 48 Grade 2, 4, 6, and 8 children on their understandings of weight measurement. Their work focused on the underlying general concepts of Units, Number, Scale and Continuity. Results suggested that "some aspects of competence seemed to progress more smoothly by age than did others" (p. 167) and showed variation in individual performances. More recently, Spinillo and Batista (2009) studied 40 children of 6 and 8 years on their understandings of measurement, and found that children of both ages had an understanding of the relationship between the size of a unit and the number of units needed to measure an object, including for measurement of mass. They found also that mass was not difficult for children to understand in terms of the relation between units of measure and objects being measured. The researchers posited that this outcome was linked to children's experiences of weighing objects at home from an early age.

In this paper we build on previous research (Cheeseman, McDonough & Clarke, 2011) where data from 1806 children in the first three years of school indicated that rich experiences involving measuring mass are needed. In the study reported in this paper

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we provided such experiences, listened to and observed the children, and gained insights into their understandings about the attribute of mass and what it is to compare and order masses.

THEORETICAL FRAMEWORK

The teaching experiment reported in this paper, was informed by a social constructivist perspective that was summarised by Ernest (1994) as recognising that knowing is active, "individual and personal, and that it is based on previously constructed knowledge" (p. 2). Social constructivism also holds that knowledge is not fixed; rather it is socially negotiated, and is sought and expressed through language. This theory contends that teachers need to be sensitive to learners' previous constructions, seek to identify errors and misconceptions, foster metacognitive techniques, and acknowledge social contexts of learners and content (Ernest, 1994). We believe that, through the provision of rich tasks, children have the opportunity to develop understandings of mass measurement and its related language. We believe also that it is important for teachers and researchers to come to know students' current and developing understandings. It is therefore of value to listen, to respond, and to challenge children's mathematical thinking as we have done in this study.

THE LEARNING OF MEASUREMENT

Young children encounter mathematical ideas from the time they are born and come to play with and use those ideas. They may make statements related to measurement such as "That box is big", sometimes with unconventional use of language such as "We are going to the playground today tomorrow". Children's learning continues through formal schooling, where they are taught about attributes of measure including length, mass, time, area, angle, and volume. They develop understandings related to foundational or key ideas of measure such as comparison, unit iteration, number assignment, and proportionality (Lehrer, Jaslow, & Curtis, 2003; Wilson & Osborne, 1992). But while "the basic idea of direct measurement is quite simple" (Wilson & Osborne, 1992, p. 91), there is complexity within measuring that may not be taken into account in typical measurement teaching (Stephan & Clements, 2003).

A framework of six growth points for mass measurement, developed within the Early Numeracy Research Project, describes children's developing understandings of mass measurement concepts (Clarke et al., 2002). The framework begins with *Child has awareness of the attribute of mass and its descriptive language* and *Child can compare, order, and match objects by mass*. In the paper we focus on these two aspects which are foundational to children's conceptual development.

Awareness of the attribute to be measured is important as, in order to make comparisons and to measure, children need to know what attribute is in focus. McDonald (2010) found that 5-6 year old children have an awareness of the attribute of mass, as revealed in drawings of measurement situations.

Kamii (2006) argued that it is essentially for the purpose of making comparisons that one measures. Battista (2006) described *non-measurement reasoning* which is visual or appearance-based and holistic comparison as a step to *measurement reasoning* which involves units and numbers. Charles (2005) discussed *Big Ideas*, that is "statement[s] central to the learning of mathematics … [that link] numerous mathematical understandings into a coherent whole" (p. 10). One *Big Idea* of mathematics is that "Numbers, expressions, and measures can be compared by their relative values" (p. 14). An example of a related understanding is that "mass/weights can be compared using ideas such as heavier, lighter and equal" (pp. 14-15).

METHODOLOGY

Research Participants

119 children of 6 to 8 years of age and their class teachers participated in the study. Each class was taught five lessons over one week, with three of these by the third researcher who is a practising primary school teacher, and two by the classroom teacher (who had observed the researcher teach the lesson earlier that day).

Data collection and analysis

The study can be considered design research as we engineered particular forms of learning and systematically studied the learning (Cobb, Confrey, DiSessa, Lehrer & Schauble, 2003). We assessed children's initial concepts, developed and offered rich experiences, and re-assessed the children's understanding of mass. In this paper we report data from the first assessment interview and the first lesson due to our focus on children's awareness of the attribute of mass and their comparison of masses.

A task-based interview was used to assess children's learning three weeks before and three weeks after the teaching phase of the study. Each child was interviewed by a teacher trained in the use of the protocol. Records were made of each child's responses for later examination and analysis. A detailed discussion of these data can be found in Cheeseman, McDonough and Ferguson (2012). As described in more detail below, the first two interview tasks gave insights into children's awareness of the attribute of mass and ability to compare by hefting (judging relative masses by hand) and use of a balance scale. The interview data were coded by two of the authors independently, resulting in the assignment of a growth point (Clarke et al., 2002) to each child.

A unit of work on the topic of mass measurement was developed by the authors for the Year 1 and 2 children. Five one-hour lessons were devised to replace the existing school planning for mass. The first lesson, *Party Bag Surprises* focused on the attribute then on comparison and ordering of mass. The lesson began with a discussion of heavy and light, with children identifying items in the classroom to demonstrate what they meant by these terms. They then worked in pairs, choosing and placing objects from the classroom in small opaque bags to create three sealed bags of different masses to be ordered by hefting. The children were challenged to make their bags "tricky". This meant that they deliberately made them quite similar in mass which led to careful comparisons by other children. Once each pair had filled three bags, another pair of children was challenged to order by hefting (without being able to see what was inside each bag, that is, without visual cues). Where agreement could not be reached, balance scales were made available for further comparison of mass.

Observational data, in the form of notes, photographs and audiotapes, were collected by the researchers to document children's actions and comments during the lessons.

A daily reflective summary was made by each researcher, and the teachers participated in an audio taped focus group interview at the end of the day.

RESULTS AND DISCUSSION

Interview data

Children were asked to handle and describe objects - a piece of foam, a rock, two plastic containers [short & fat and tall & thin], a ball of string, a 1 kg mass [labelled 1 kg], and a tin of tomatoes. They were then asked, "Which things are heavy and which things are light?" Based on these tasks we found from the initial interviews that 98% of the children in each year level had an awareness of the attribute of mass.

The children were then asked to estimate the relative masses of two plastic containers by hefting, to check with balance scales, and describe and justify the result. We found that 68% of Year 1s and 83% of Year 2s could compare two masses by hefting and with scales. These data suggest that, even before the classroom experiences, many of the 6-8 year-olds knew about mass and could compare masses. However, the data from the classroom reveal another perspective, that is, diversity in children's thinking.

Lesson data

The first lesson (Party Bag Surprises) focused on the attribute of mass, language of mass, and the use of balance scales for comparing the masses of *three* items. As such, it provided for children in these mixed ability Year 1/2 classes an opportunity for confirmation, consolidation or extension of understandings. Four themes that emerged from the lesson observational data are discussed below, with a selection of data included to illustrate the reasoning and complexity in the children's thinking.

Judging by sight whether something is heavy or light

At the beginning of the lesson the teacher asked questions including "What is something in the room that you think will be heavy?" Responses included "The cupboards because they are strong and can hold lots of things" and "The speaker [on the wall] may be heavy because it has metal in it". The teacher later reflected: "I was surprised at how many children chose things as heavy that they would not be able to lift. I think that's what they were using as their criteria". The responses suggested also that children may have been paying attention to materials, or strength, such as for the cupboard and in reference to metal in the speaker (see quotes above) or for a chair: "The chair … when someone sits on it, it is pretty heavy. [It feels heavy because of] the materials it is made out of". Also with a focus on materials as justification, when asked, "What would be really light?" responses included, "your hat because it is made out of light material".

The children appeared to have developed informal knowledge of mathematics and science from earlier experiences, perhaps from handling or weighing things at home (Spinillo & Batista, 2009). As we can see from their comments about the materials, children aged from 6-8 years also have emerging concepts of matter. Authors in the cognitive sciences have researched weight and found that it is a concept that is tied to children's growing understanding of the nature of matter. Carey (1999) found that "roughly half [of the children] had differentiated weight from density by the age of 9, before they encountered the topic in the school curriculum" (p. 484).

Some of the children's suggestions for heavy things focused on size including the smart-board on the wall and the cupboard, described as heavy "because they are 'big'". In giving such responses, children appeared to be using appearance-based reasoning (Battista, 2006). Although the children would not have been able to test their perceptions of the mass of all the objects by hefting or use of a set of scales, they did appear to use reasoning in making their generalisations and were confident in their responses. Perhaps they believed that things that were too big to hold would be heavy. This finding is consistent with that of McDonald (2010).

Judging by feel whether something is heavy or light

In the initial discussion the teacher also asked three children to each choose and bring to the front of the class an object that they thought would be "a bit heavy" (phrased so it would be heavy but something they could hold). She then challenged them to think about how she could know which object was heaviest and which was lightest.

One child judged an object as heaviest because of its impact on the child who held it: "[He] looks like he is struggling to hold it". Others described knowing something is heavy because "my arms are flat [that is, stretched]" and "his arms are right down".

Some other children gave responses that suggested a more specific reference to movement such as would occur with a balance scale:

> You put them in your hand. You just relax your hand and see if one is going low and one is going high. … If one is heavier it is putting more pressure on my hand [and makes it go down]

Another child added, "the other [hand] would kind of stay in the same spot because it is light".

These responses show some appreciation of the attribute of mass and of the effect of the mass upon the hand holding it. They suggest an understanding that may inform the use of the balance scale for comparing masses. These data suggest that an understanding of the workings of a balance scale may be informed by experiences, observations and generalisations children make from everyday encounters with objects.

Judging whether heavier or lighter using a balance scale

While most of these 6-8 year old children had an understanding of the workings of a set of balance scales, it became apparent that some children's understandings were still emerging.

One teacher was surprised that at the end of the lesson when a child she described as "confident and pretty cluey" shared with the class that he believed that the first bag put in the scale would always be the heavier bag. Another child offered an explanation: "no, it was just because it went in first, it has to go down". The teacher intentionally held back from telling and the class discussed and tested the two children's theories.

In another class, two children used a set of balance scales to check the relative masses of two bags and correctly stated which bag was heavier. However, when asked how they knew, one child referred to the number of items in the bag and the metal in the items but did not refer to the balance scale. The other child said the particular bag was heavier because it was in the red bucket (on one side of the scale). The teacher moved the bag to the other bucket, thus challenging the child to reflect on her theory.

These data show that children have emerging understandings of the use of balance scales where factors such as the order in which objects are placed in the scale, the materials in the items, or some other what may appear arbitrary factor such as the colour of the bucket, may be powerful forces in decision making. The teachers held back from telling and challenged the children to test their theories.

Difficulties with comparing three items

When asked to heft and order three bags, children needed to reason about the relationships between the masses of the bags in a transitive manner (e.g., if A<B, and B<C, then A<C). This was a more complex task than the comparison task in the assessment interview (two plastic containers). The teachers commented that some children appeared to have difficulty comparing the three bags methodically. To illustrate, one child held two bags in one hand and one in the other and unsuccessfully tried to order the three bags. Another child suggested he should "find out which one is lighter then hold the other one and see what that is like with the first one". Letting the children grapple with this task allowed the difficulty with transitive reasoning to become apparent and showed that children of 6-8 years will work together to solve such problems.

CONCLUDING THOUGHTS

Being the amount of matter in an object, mass is invisible. It is not surprising then that learning about mass measurement may pose challenges for children. However, we admired the development in understandings the 6-8 year old children in this teaching experiment achieved (see Cheeseman et al., 2012) from the rich learning experiences they were given over a one-week period. But we were also fascinated by the diversity and complexity in children's thoughts and theories that emerged during the five lessons. As illustrated above in relation to comparing and ordering masses, we saw children reason, theorise, experiment, observe, and challenge each other in their learning. Teacher actions appeared to facilitate these processes. In thinking about mass measurement children seemed to draw on prior experiences and sometimes on visual cues, but with appearance-based comparison for mass not as likely a reliable strategy as it might be, say, for length. Our data suggest that 6-8 year old children are capable of thinking constructively about such intricacies of mass measurement if they are given suitable learning opportunities.

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PRE-SERVICE TEACHERS' REPRESENTATIONAL PREFERENCES

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In this article we report on an exploratory study on development of pre-service teachers' competences in using different representations of multiplication. Grounded in the literature on pedagogical and psychological value of knowing and using representations we investigate student teachers preferences of representations of multiplication in the case where they are given free choice. A survey was distributed to 121 prospective elementary school teachers at University. The analyses of their answers highlighted differentiated choice of representations made by students. The choice of preferred representation was linked to problem abstractness.

INTRODUCTION

How do we see abstract mathematical concepts? How do we visually present arithmetic procedures? "The ways of representing the subject" is one of seven categories of pedagogical content knowledge identified by Shulman (1986). Knowing different representations is considered as element of subject matter knowledge of teachers as well as of students (Ball *et al*., 2008). But when one knows how to transform knowledge into forms comprehensible to learners it becomes element of pedagogical content knowledge (Livingston and Borko, 1990). This is why currently, the issue of representations is one of key areas in research in mathematics learning and teaching. The idea that representations are "tools in thinking" is well documented (Janvier, Kaput, Cuoaco & Curcio). Bruner (1960) underlined importance of representations in learning. The ability to use action representations is viewed in contrast to the usage of iconic or symbolic representations. Goldin and Shteingold explain that

> ”Effective mathematical thinking involves understanding the relationships among different representations of "the same" concept as well as the structural /similarities (and differences) among representational systems" (2001, p.9).

Contrary to educators' belief about importance of representations, Ball (1990) reported on unexpected research finding that pre-service teachers, although successful in calculation, were not able to produce good pictorial representations for a problem involving division of fraction by fraction. Looking into preparatory programs for elementary school teachers we can see a string of reasons for such weakness. Student-teachers only occasionally see professors using visual representations in their math classes. Therefore the idea of using visual aids does not become appealing for them. Even more, unfamiliarity with different representations means that they are not readily available for novice teachers. Consequently, they do not use them when teaching new concepts and procedures, when solving problems or when posing

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problems. So, if novices do not regard the issue of representations as important at all, we may expect the same for young learners of those teachers.

On the other hand, researchers highlighted socio-cultural value of visual representations (Arcavi, 2003, Sfard, 2003). Arcavi highlighted three functions of visual representations 1) as a support and illustration of symbolic representations; 2) as a tool for resolving conflict between intuition and symbolic solution and 3) as a tool to reengage and recover conceptual understanding. It was suggested that "seeing things" sharpens our understanding and serves as a springboard for questions which we would not pose otherwise. Extending an idea of representations as medium for cognition Arcavi identified visual representations as "cognitive technology aid" for thinking, learning and problem solving activities in technology driven communication (Arcavi, 2003, p. 216). On the other hand, we need to recognize certain difficulties in engaging in visualization of mathematics concepts from cultural, to cognitive to sociological. First, it is culturally unappreciated method and therefore there is little space in school curriculum for presenting such approach (Sfard, 1991). Second, it may bring some challenge to classroom whereas before gain of using multiple representations in solving problems, children have to pass a phase of learning those representations. This requires ability to translate between symbolic and visual representation. Finally, given their previous experience teachers may be prone to use only symbolic (analytical) representation as more pedagogically appropriate.

We distinguish several levels of pedagogical knowledge of representations:(1) knowing different representations of concepts and procedures, (2) knowing how to use different representations in problem solving (3) knowing how to use different representations in problem posing (4) understanding representations as a tool in teaching. Rowland, Huckstep and Thwites (2005) reported on a case study of student teacher within broader research on the ways student teachers could be instructed on mathematics content knowledge within practical training. They distinguished "knowledge quartet": foundation, transformation, connection and contingences, with special emphasis on foundation, coinciding with Shulman's "comprehension". The second category "transformation" relates directly to the issue of representations because it refers to the ability of teacher to transform the content knowledge into different forms. We found the following list of potential use of representations in classroom: distinguishing elements of process, understanding relations, discovering, grouping, reorganizing and memorizing information. The list is certainly not exhaustive enough but points to importance of good pedagogical knowledge of representations for teachers. Yet so far representations did not take prominent part in methods courses. Fenemma and Franke noticed that elementary pre-service methods courses in USA emphasize the use of manipulatives, such as Cuisinaire's rods, for representing whole number operations. Remarkable, after initial considerations of representations of numbers, the interest in representations fades away. Kaput stressed that knowing abstract mathematics ideas implies recognition of their salient features regardless of representation or notation (Kaput, 1991).

Knowledge of representations as it was noted earlier is particularly important in problem solving (Polya, 1957, Goldin, 1987). One way to help students to become confident in using different representations in problem solving is to confront them with different forms of problems. Friedlander & Tabach (2001) argue that teacher's presenting problem situation in different representations encourages flexibility in students' choice of representations. They state that "The presentation of a problem in several representations gives legitimatization to their use in the solution process" (p.176). Similarly, Singer (2011) emphasizes that "the tasks format underline cycles of transfer in order to internalize..." (p. 1-147). But in practice, teachers rarely consider different representations in problem posing as an important issue. In infrequent occasions teachers will switch from one to another representation. Some researchers believe that we should not make speculations about ways students understand or interpret representations. (Cobb *et al*., von Glasersfeld). In order not to speculate we attempted to contribute to unveiling this issue, by conducting this research.

METHODOLOGY

Our study examined pre-service teachers' preferences in using representations of multiplication. The idea is developed from our previous research on pre-service teachers' use of visual representations of multiplication (Barmby and Milinkovic, 2011). The results highlighted a number of issues, such as the influence by teachers' subject knowledge and limitations in teachers' use of representations. Earlier research identified the following types of representations of multiplication: sets representation, equal group representations, arrays, Cartesian product, number line, rectangular representations(or area model) (Skemp, 1971, Greer, 1992, Battista et al.,1986, Anghileri, 2000). Array model was pointed as a possibly useful for studying properties of multiplication (Skemp, Anghileri, Barmby). Our previous finding was that teachers recognize grouping representations (sets, equal groups, number line) as a visually preferable model for introduction of concept of multiplication as repeated addition. Also, we found that teachers did not pick 2D representations (arrays or area models) for exploring properties of multiplication as was expected (Barmby and Milinkovic, 2011).

In attempt to clarify choices teachers made in the previous study we surveyed new generation of student teachers at University.This time, during the semester, we made an overview of representations most frequently used. Also, throughout the semester, prior to survey, students were focused on problem posing activities. The questionnaire was distributed during the last class. Five questions were examining students' 1) knowledge of representations of multiplication and of commutative low and 2) competence in using different representations in problem posing. In the questionnaire students were asked (Q1) to draw representation of "$3 \cdot 4$"; (Q2) to draw representation of "$a \cdot b$"; (Q3) to draw representation of "$3 \cdot 4=4 \cdot 3$"; (Q4) to draw representation of "$a \cdot b=b \cdot a$"; and (Q5) to pose problem visually and in a textual form based on sentence $(3+4) \cdot 2$. As a reminder, the following set of representations of "number 3" were shown during the survey: a group of three coins, set with three stars, array, number line and rectangular grid.

We surveyed 121 prospective elementary school teachers during their 4th year of studying at University of Belgrade. In the questionnaire students had freedom to use whatever representation they found suitable for a given question. Students' choices of representations were marked from R1 to R9 based on nine types of representations identified in the data as it was shown in Figure 1. Unanswered or unidentifiable answers were marked with 0.

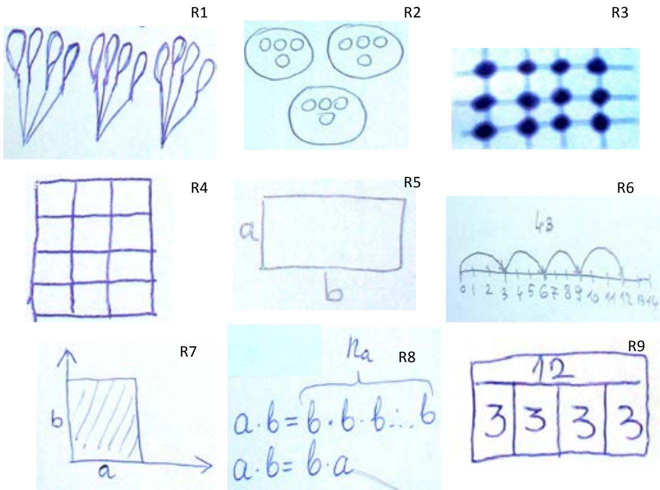

Figure 1 Representations R1-R9 produced by students

The obtained data were inputted into a spreadsheet. After initial statistical analysis the initial scores were grouped. Bar graphs are used to make clear findings.

RESULTS

The initial analysis of the pre-service teachers' responses shown in the Table 1.

Item	R1	R2	R3	R4	R5	R6	R7	R8	R9	0
Q1	20	38	56	5	0	0	0	1	1	0
Q2	1	2	4	10	69	0	4	4	1	26
Q3	13	4	84	18	0	1	1	0	0	0
Q4	2	1	7	14	67	0	5	0	0	25
Q5	44	27	38	2	0	3	1	0	0	6

Table 1 Initial marking of items

Strikingly we could see that students have preferred some representations over the others. But their choice was obviously not based on age level. Although indeed they preferred equal grouping representations such as sets or arrays for Q1. They considered notation with variables in Q3 as something that requires area model. The same model was preferred for dealing with commutative low if introduced with variables instead of particular numbers. R1, R2 and R3 considered suitable for Q1, Q3 and Q5. R3 was the most popular choice for Q3; 0.69% students considered array as the best choice for commutative law expressed on particular numbers. But, we could observe that R5 (area) was preferred in cases of symbolically expressed multiplication with variables and of commutative law with variables. Similarly, representations which were preferred for multiplication of particular pair of numbers were also preferred for expressing commutative law on an exemplary pair of numbers. How popular is a particular representation may be seen in Figure 2. Pre-service teachers found array representation to be the best choice 189 times (31%). Area model (R5) proved to be popular choice 22% times. In contrast, number line was chosen only few times regardless of the fact that school curriculum explicitly stresses its value.

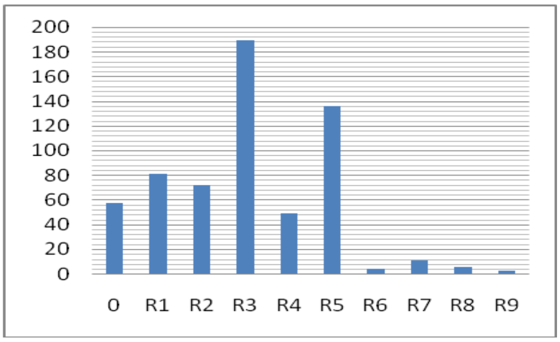

Figure 2 Total score of representations

We grouped the answers respectively for Q1 & Q3 and for Q2 & Q4 in Figure 3. As we could see regardless of the fact that Q1 and Q3 were not pointing to the same idea (one is the concept of multiplication and the other commutative law) students were prompted to use the same representation. The distinction between them is only in the kind of expressions addressed. On one side are involved concrete numbers (3 and 4) while on the other side students are dealing with general symbols (a,b).

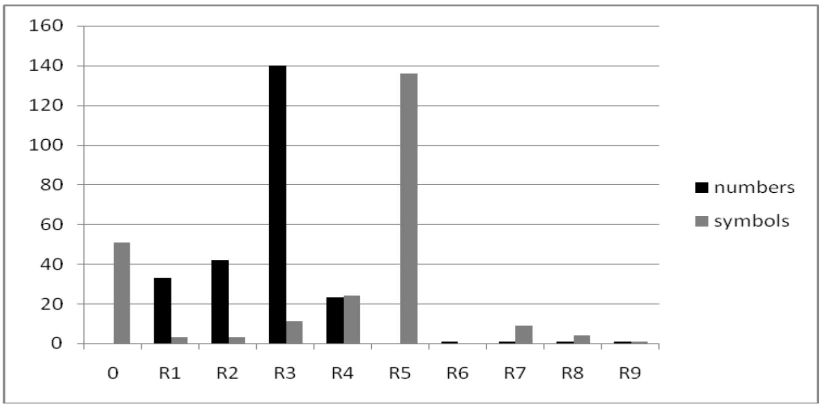

Figure 3 Marking of number items vs. symbolic items

The request of using visual representations in problem posing proved to be a challenge for students. Pattern for Q5 was less determined but we could conclude that teachers prefer concrete models which support the idea of multiplication as repeated addition (sets and equal group representation). Students opted for grouping representations R1, R2 and R3 and simplified contexts (For example, distributing flowers in vases or having two dark and tree white balloons in two hands, etc).

CONCLUSION

The analysis of student teachers questionnaire revealed interesting details. First, we found that teachers do have preferences in using representations. Second, those preferences are not necessary based on a kind of concept or procedure they are dealing with. The choice is also not based on frequency of use particular representation in schools. Third, their choice seems to be related to the level of abstractness they are dealing with. This assumption needs to be checked in future.

Somewhat limited findings from a question on problem posing point to possible weaknesses in student teacher readiness to pose problems. Consequently students are compelled to use simpler context and unsophisticated choices of representations in problem posing. We need to explore this assumption further. In any case, we need to consider ways for improving student teachers competences regarding representations. It seems that a critical issue is development of ability to use different representations while posing problems.

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LEARNING ABOUT FUNCTIONS WITH THE HELP OF TECHNOLOGY: STUDENTS' INSTRUMENTAL GENESIS OF A GEOMETRICAL AND SYMBOLIC ENVIRONMENT

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Software learning environments, especially those offering extended multi representational capabilities, are more and more complex. That is why researchers are now sensitive to the process of instrumental genesis that transforms this kind of artefact into an instrument for students' mathematical work. The study reported here deals with Casyopée, a geometrical and symbolic learning environment dedicated to functions at upper secondary level. Consistent with the curriculum, these situations aimed at approaching functions by modelling geometrical dependencies, a task for which Casyopée offers special capabilities. The study suggests that such an instrumental genesis can be a real attainment, but needs to be achieved as a two year long process. This genesis consists in a joint development of knowledge about the artefact's capabilities together with mathematical knowledge about functions during the instrumental genesis.

THE INSTRUMENTAL GENESIS OF GEOMETRICAL AND SYMBOLIC ENVIRONMENT

Many studies on the use of technology in mathematics education refer to an instrumental approach (Artigue, 2002; Drijvers, Kieran, & Mariotti, 2010). This approach derives from the analysis by psychologists of new uses of a tool by an individual, and his/her associated cognitive changes: in a 'study of thought in relation to instrumented activity', Vérillon & Rabardel (1995) stress that a human creation, an 'artefact', is not immediately an instrument. A human being who wants to use an artefact builds up his/her relation with the artefact in two directions: externally s/he develops uses of the artefact and internally, s/he builds cognitive structures to control these uses. Vérillon and Rabardel's approach helps to see that instruments are not neutral, because they have an effect on the cognitive functioning of the user. More precisely, in the case of instruments used for the mathematical activity the cognitive structure is made of knowledge about the artefact itself and mathematical knowledge related to the domain of use. For instance, Lagrange (1999) described various schemes, calculator oriented or not, algebraic, graphic or symbolic that a user of a CAS calculator (TI-92) can use to search for the properties of a rational function. Each of them mixes the awareness of affordances and constraints of the calculator for a given task and knowledge about the function itself. The necessity of considering students' and teachers' instrumental genesis when introducing new tools in mathematics teaching and learning is now widely recognized (Drijvers, Kieran, & Mariotti, 2010). It is also recognised that when a tool offers a wealth of capabilities deeply connected to mathematical knowledge, the instrumental genesis is likely to be complex and cannot be achieved on a short term. It is especially the case of tools offering means to work both on geometrical and algebraic situations (Weigand & Bichler, 2010).

However no research study yet provided precise examples of a genesis of one of these complex tools and no data exists about the period of time needed in order that this type of genesis can develop. The aim of this paper is then analyse an example of instrumental genesis of an algebraic and geometrical environment devoted to functions, Casyopée, in the process of learning about functions. It is especially expected to know the period of time that students need in order to consider Casyopée really as an instrument of their mathematical activity about functions, looking at key capabilities of this environment as constituent of their mathematical knowledge about functions.

APPROACHING FUNCTIONS BY MODELLING GEOMETRICAL DEPENDENCIES: THE CASE OF CASYOPÉE

In many countries, the choice generally made by curricula is to privilege functions at upper secondary level, in order that students consolidate their algebraic proficiencies in order to prepare for calculus. For instance, the French new 2009 curriculum pays particular attention in considering functions in application domains and recommends the use of digital technologies for approaching functions:

> The goal is to make students able to study an optimization problem or a problem of the type $f(x) > k$ and solve it by exploiting the potential of digital technologies… The situations proposed in this context come from very diverse fields: geometry, biology, economics, physics… The digital technologies available to students can be usefully exploited.

Our approach considers functions as models of dependencies in an application domain, especially of dependencies between geometrical magnitudes in a digital environment. The paper draws on a research work on approaching functions within the European ReMath project. We pay particular attention to the activities at the level of dependencies between magnitudes in geometry that allow highlighting the functional modelling and are fruitful for conceptualizing functions.

The theoretical framework underlying Casyopée' design has been exposed by Lagrange & Artigue (2009). We focus on the two main windows. The first one, (called the symbolic window) provides students with symbolic computing and representation capabilities as well as facilities for proving. The second offers Dynamic Geometry and the facility for exporting dependencies between magnitudes into functions in the symbolic window by computing a domain and a formula. This "export" capability is intended to provide help for students in problems involving modelling of geometrical dependencies.

A LONG-TERM INSTRUMENTAL GENESIS: QUESTIONS AND METHODS

The questions addressed in this paper derive from the aim expressed at the end of the first section: to study a genesis of Casyopée.

- It is expected that this genesis will articulate notions about functions and knowledge about Casyopée's functionalities. Then what are Casyopée's key functionalities that students progressively understand along this genesis, in parallel with the development of their mathematical knowledge about functions?
- Since notions related to functions are understood only in a multi-year process, it is expected that a genesis of Casyopée will be more than a year long. Then, what is the state of the process after one year? What can be achieved after two years?

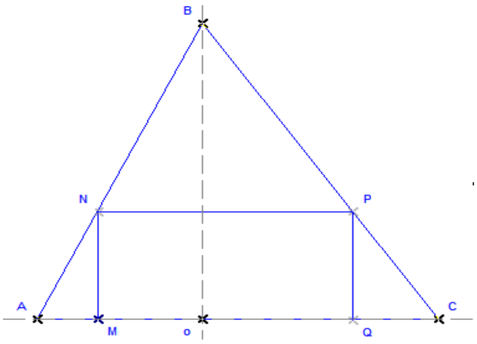
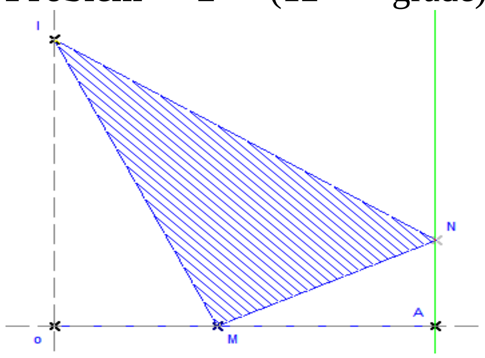
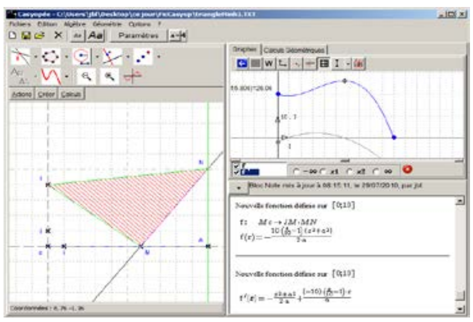
	Text	**Hint of a solution**
Problem 1 (11th grade) 	o being the origin, consider a triangle *ABC*. *A(-a;0)*, *B(0;b)* and *C(c;0)*, *a*, *b* and *c* being three parameters. Find a rectangle *MNPQ* with *M* on [*oA*], *N* on [*AB*], *P* on [*BC*], *Q* on [*oC*] and with the maximum area.	*For all values of the parameters, the maximum area is for M at the middle of segment* [oA]. See Lagrange & Artigue (2009) for a more in depth presentation of this problem.
Problem 2 (12th grade)  	Consider the point *I(0; a)*, *a* being a parameter and the point *A* of coordinates *(10; 0)*. *M* belongs to the segment *[oA]*, *N* is on the parallel to the *y*-axis passing by *A* and the triangle *IMN* is rectangle in *M*. When *M* is in *A*, then *N* is also in *A*. The problem is to find the position of *M* to maximize the triangle's area	For values of *a* greater than $\frac{10}{\sqrt{3}}$ the function is decreasing and then the maximum is for $M = o$. For other values, there is a local maximum for a position of *M* inside the segment. This maximum is the absolute maximum for *a* lower than *5* (left), otherwise the maximum is for $M = o$. For $a = 5$ there are two positions for a maximum, one for $M = o$ and the other at the middle of *[oA]*.

Figure 1: Tasks proposed in two milestone observations

Our method was to observe the same class of scientific students at 11th and 12th grades and to focus particularly on two students Elina and Chloé working as a team in this class. Observation at 10th grade took place within experiment. This experiment consisted of six sessions in three parts. Consistent with the ReMath team sensitivity to students' instrumental genesis, each part was designed in order that students learn about mathematical notions while getting acquainted with Casyopée's associated capabilities. Observation at 10th grade took place as a series of activities designed in my doctoral work together with the teacher. The three steps were: (1) a session aiming at the consolidation of Casyopée's use some months after the ReMath experimentation: the goal is to model a variable area in a square; the function at stake is quadratic, (2) a session where students have to use more completely Casyopée's functionalities, especially for the management of parameters and for symbolic calculation, again in a modelling activity, the function at stake being a third degree parametric polynomial (3) a session involving the study of a family of logarithm functions, a more classical task with regard to the curriculum as compared with the geometrical modelling in the two other sessions, the goal being that students become aware of how they can use Casyopée to prepare for the baccalaureate, an exam they have to pass at the end of the second year. A semi-directed interview was passed by the end of the second year, before the baccalaureate in order to understand the evolution of students' relationship with mathematics and with Casyopée. We consider here two milestones in this study. One took place in January of the first year (11th grade) at the concluding session of the ReMath experiment. The second milestone was in January of the second year (12th grade), as a second step of the series of activities. The interview at the end of the second year was another data for analysis.

TASKS PROPOSED IN TWO MILESTONE OBSERVATIONS

In both problems, a solution with Casyopée involves mathematical subtasks in relationship with the corresponding functionalities of the software:

Mathematical subtasks	Casyopée's functionalities
Building a geometrical figure	Creating objects in dynamic geometry
Exploring and conjecturing	Creating a geometric calculation, dragging free points, observing numeric values
Modelling a dependency	Choosing an independent variable, exporting a function
Using an algebraic procedure	Using Casyopée's algebraic transformations, and justifications
Generalising	Animating parameters

Table 1: Mathematical subtasks and Casyopée's functionalities

OBSERVATIONS

The first year

We report on the work of Chloé and Elina for each subtask in the session that we presented above (figure 1, top) as a milestone in the first year.

<u>Building a geometrical figure</u>: Students took much time constructing the variable rectangle. Modelling the variable rectangle implied to build a proper rectangle based on a free point on a segment, but students first built a "soft" rectangle, that is to say that

the quadrilateral they built was perceptively a rectangle, but did not resist to a variation of the figure by animation of a free point or a parameter. Thanks to the feedback of the software and to the help of the observer, they recognised that they were wrong, but were slow to correct.

Exploring and conjecturing: Students mistook the creation of a dependant variable representing the area for the choice of an independent variable, two actions accessible in the same toolbar of Casyopée. They did few explorations.
Modelling a dependency: Students did not understand by themselves the need to choose an independent variable and to export the dependency as an algebraic function. They hesitated on the choice of an adequate variable. They understood the exportation as a way to have a graph, rather than the creation of an algebraic model.
Using an algebraic procedure: After recognizing a parabola, students did not know how to use their previous knowledge about quadratic functions. With the help of the observer they remembered a formula for the abscissa of the vertex. The resulting expression is complex, but can be easily simplified by Casyopée. In spite of this, students could not apply this formula to the generic parametric quadratic expression and they treated only a numeric case, considering the current values of the parameters.
Generalising: As said above, the students conjectured the optimal position for a generic triangle in the geometrical window, but could not prove it in the symbolic window, because they could not take advantage of Casyopée as an algebraic tool.

This observation was disappointing because we expected that students had been carefully prepared in the five preceding sessions. The outcome was that the instrumental genesis was to be considered on a longer period. We did also observation of other students and found that Chloé and Elina were considered representative of a majority of the class. That is why we centre the subsequent observations on this team.

The second year

We report now on the work of Chloé and Elina in the session that we presented above as a milestone in the second year, again considering the subtasks described in table 1.

Building a geometrical figure: Difficulties in using dynamic geometry remained, but students corrected easily their mistakes.
Exploring and conjecturing: As said above, the exploration is more complex in this problem, and students did a lot of exploration, corresponding to different cases and values of the parameter. They commented, using the relevant functional language: growing", "decreasing"…
Modelling a dependency: There was a much better understanding of the process of modelling and of the Casyopée's functionalities, as shown by the following extract:

1	Chloé:	Choosing the independent variable? Last time we did it with the altitude?
2	Elina:	No, the distance OM, I think that it will be a good variable.
3	Chloé:	Yes (*She chooses this variable and then exports the function*).
4	Elina:	Its domain?
5	Chloé:	It is the set of real numbers. Oh, no it is [0;10].
6	Elina:	Look (*points to the screen*)! It is here.

<u>Using an algebraic procedure:</u> The students proposed a procedure using the derivative. They easily used Casyopée's functionalities like "expanding", "factoring" and also the justifications for the sign of the derivative.
<u>Generalising:</u> The students animated the parameter in order to study different cases.

We see a clear improvement, both in the use of Casyopée's functionalities and in the mathematical abilities.

The interview at the end of the second year

We report here some of the two students' answers. The first outcome is that after two years of use, the students saw Casyopée as a tool whose appropriation had not been easy:

> We did not know all functionalities, tools… in Casyopée. We obtained expressions, but we did not know how to manage them. We did not know which functionalities to use.

They recognized that these difficulties are linked to the understanding of the mathematical content:

> The most difficulty is to choose an independent variable. It is important to choose an appropriate variable.

They also indicate that these difficulties have been overcome thanks to a continuous use of Casyopée and with the help of the teacher:

> I downloaded Casyopée from Google and I use it sometimes (at home) for training… At the beginning it was hard to find functionalities to use it, but now it works… thanks to the help of the teacher. He explains us how to solve problems.

In spite of the difficulties observed by the first uses, the students also expressed positive feelings relatively to specificities of Casyopée, especially the help for modelling and the link between algebraic and geometrical windows:

> Choosing variables is the interesting part. To perform all the process is great: constructing the figure, table of variation, calculation of the derivative… We have the algebraic and geometrical sides together. We see better how a function "reacts", it is convenient and interesting.

Students identified clearly different functionalities and how they could help exploring and proving freely:

> We can try different variables, animate the figure, and visualize functions (several at the same time), draw a table of signs, find the derivative…

Other Students

During the first year, most of students had difficulty to build a dynamic figure and took a long time to complete this task. Some of them first built a "soft" rectangle MNPQ where M, N, P, Q are free points in the plan. They recognized this error by moving the points. Some students needed help from the teacher or the observer to finish the construction of the figure. The creation of a geometric calculation of the area has been carried out easily for all of them. The choice of variables is varied but the exportation of function has often required interventions of the teacher. For the algebraic proof, the students worked mainly in paper/pencil on a particular case of the parameter and didn't use much algebraic tools available in Casyopée to calculate the maximum value.

In general, the results in the second year showed a progress in realizing the given tasks. There are seven pairs of students (among eleven) who conjectured exactly the positions of the point M corresponding to the maximum area. The choice of variables and the exportation of function are more spontaneous. The choice of the variable OM was dominant. For the algebraic proof, seven pairs have used the derivative to find an answer. Students were comfortable with the Casyopée's features such as "Expanding", "Factoring" or "Derivative".

SYNTHESIS AND DISCUSSION

We recapitulate this genesis in the table 2 by identifying links between students' progress in the use of Casyopée's functionalities and the development of their mathematical knowledge.

Mathematical subtasks	Progress in the use of Casyopée	Development of mathematical knowledge
Building a geometrical figure	Quicker correction of mistakes after observing unexpected behaviour of the figure.	Better understanding of the functional dependencies in the figure.
Exploring and conjecturing	Correct definition and use of a geometrical calculation for the area	Understanding of the triangle area formula as expressing a co-variation between the mobile point M and the value of the area.
Modelling a dependency	Spontaneous and easy use of the buttons for choosing a variable and exporting a function.	Understanding a functional dependency. Distinguishing a functional dependency between co-variations.
Using an algebraic procedure and generalising	Easy use of Casyopée's algebraic transformations, and justifications. Animation of parameters	Understanding the different behaviours of the function depending on the parameter. Understanding a parametric function as a family of functions.

Table 2: Joint development of math knowledge and knowledge about Casyopée

The table 2 shows how an instrumental genesis deeply links knowledge about an artefact (here Casyopée) and mathematical knowledge (here the idea of function as a model of a functional dependency). In the first year, the use of the artefact was seen as complicated and the help of the teacher or of the observer was needed at crucial steps. In the second year, the students better identified the underlying mathematical notions and the corresponding functionalities of Casyopée. In this year the students had to prepare for the baccalaureate and the activities with Casyopée could have been seen as far from standard exercises proposed in this important exam. Nevertheless, students recognized the contribution of these activities to their learning as well as the help that the artefact could bring even for standard tasks.

Before this study, we were aware of environments like Casyopée as complex tools that students cannot be really acquainted with in a short time. Actually, as said in the paper, a mid-term experiment in the first year was disappointing and the real outcome of this study is then that the instrumentation of Casyopée can only be achieved on a long-term basis in parallel with the process of learning about functions. The analysis of observations over two years shows the progress of students observed. This progress can be interpreted as an appropriate instrumental genesis, particularly as regards the Casyopée's modeling features and the mathematical knowledge associated with them. Further results have been developed in the author's doctoral thesis (Minh, 2011) in which we indicated that the students achieved a flexible understanding of functions at the end of the second year, which means a fluency in applying procedural and structural features of functions in solving problems.

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PROOF IN GEOMETRY: A COMPARATIVE ANALYSIS OF FRENCH AND JAPANESE TEXTBOOKS

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This paper reports a part of results of the international comparative study on the nature of proof in lower secondary school geometry. In order to clarify the different natures of proof in different countries, textbooks of two countries, France and Japan, are analysed, from the ecological perspective of the Anthropological theory of the didactic. The results of the analysis show several differences in what is called "proof", in the form of proof, in the interrelations between mathematical objects that the proof creates, and in the functions of proof.

INTRODUCTION AND THEORETICAL FRAMEWORK

Learning and teaching of proof has been a topic of research in mathematics education for a long time, and a lot of research has been conducted so far. Recently Balacheff (2008) points out that the meaning of "mathematical proof" is not necessarily shared today among researchers in mathematics education, albeit prior research. He reports the different researcher's epistemologies in mathematics education. Reid & Knipping (2010, Part 1) also extensively describe the different usages of the terms "proof" and "proving" and different perspectives on their learning and teaching. Besides this diversity, taking a glance at the proof in school mathematics of different countries, one may also find diversity of proof: form of proof (cf. two-column proof in US), properties used in proof (cf. triangle congruency in US and in Japan, but not in France), and functions of proof, etc.

However, from the perspective of the Anthropological Theory of the Didactic (ATD), in particular the ecological perspective (Chevallard, 1994, Artaud, 1998), this diversity of proof in school mathematics could be, in a sense, a natural consequence. According to ATD, in different educational systems or institutions (e.g., French secondary school and Japanese one), the body of mathematical knowledge taught or to be taught would be different. A mathematical object exists there not in isolation, but in relation to other objects, with particular functions. It is like in ecology that a species lives in some places (called *habitats*) of an ecosystem with some functions (called *niches*) in relation to other species. And what allows the object or species to live in such particular places in a particular form is a system of *conditions* and *constraints* involved in an environment and imposed to that object. In the case of proof, while the proof is not really a mathematical object but a *paramathematical object* (Chevallard, 1991, Ch.4) which is an auxiliary object in mathematical practice, I consider that the nature of proof in school mathematics is also formed in the same mechanism: the proof in secondary school mathematics has interrelations with some particular objects in a body of

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 225-232. Taipei, Taiwan: PME.

mathematical knowledge formed in a given educational system; some functions are attributed to the proof for the sake of mathematical practice and also for the sake of didactical practice in an educational system; internal and external constraints of an educational system affects the nature of proof; etc. In consequence, the proof taught in different countries may have different nature, and what should be learnt, what is really learnt, the difficulties students hold, teacher's supports, etc. may also differ from country to country.

In the research on mathematics education, one could find some literature that report differences on the nature of proof taught in school mathematics. For example, Knipping (2003) reports that in the case of Pythagorean Theorem, the processes of proving play different functions in German class and in French class of 8th grade. Proving is to make clear the meaning of a theorem in German Class, while it is to explain why in French class. Cabassut (2005) reports that the proof is explicitly an object of teaching in French secondary schools and in German gymnasium, while it is not the case in German Realschule and Hauptschule, and shows that there is a mixture of different types of arguments and of different functions of validation from the perspective of didactic transposition.

The aim of this study is to further develop these prior research and examine, from the ecological perspective of ATD, the different natures of proof that may exist in secondary school mathematics of different countries. The principle research questions are: *What is the thing called 'proof' in school mathematics in a specific country? Why and how is it formed as it is?* Contributions I expect to obtain in this research are twofold. On the one hand, it would propose a coherent view of proof albeit differences. This would maximise the results obtained in prior research on proof. On the other hand, it would propose alternative ecologies of proof along with the conditions to be satisfied for their realisation, which would serve for future curriculum development. In this paper, I report a primary result, in particular a response obtained for the first research question, by means of the analyses of textbooks of two countries, France and Japan. The second question related to the system of conditions and constraints that forms the nature of proof will be studies the next time.

METHODOLOGY

In order to clarify different possibilities of the nature of proof, a comparative study will be carried out in the cases of France and Japan. It is expected from a cross cultural comparative study to make explicit what is implicit or taken for granted in other country. For the sake of a comparative analysis of proof, mathematics textbooks of each country will be analysed as data. From the perspective of ATD, especially in the process of didactic transposition, different mathematics are taken into consideration: scholarly mathematics, mathematics to be taught, and taught mathematics (Chevallard, 1991). In the case of Knipping's study (2003), the proof really taught in the classroom was an object of study. On the other hand, in this paper, the object of study is the proof to be taught that could be identified in French and Japanese textbooks.

Mathematics textbooks to be analysed

France and Japan both adopt a single-track educational system for the lower secondary level, that is, all students go to the same kind of school: four years of *collège* in France and three years of *middle school* in Japan. And in both countries, teaching contents are determined in the national curriculum written by the Ministry of Education. Japanese textbook should be approved by the ministry, while no approval is required for French textbook. For the analysis, a mathematics textbook which is relatively known and shared in each country was chosen: for Japanese textbook, *Atarashii Suugaku [New Mathematics]* series published by Tokyo-Shoseki (I call "*Tokyo-shoseki* series"), and for French textbook, *Triangle* series published by Hatier. Due to a recent change of national curriculum in Japan, the textbook will be replaced in the school year 2012. In the analysis, the new textbook obtained as a sample is used. Usually, there is no change after publishing a sample version of textbook.

Four steps of the analysis

The analysis is carried out in the domain of geometry where the proof is introduced in both countries. It consists of four steps. The first step is to identify and clarify what is called 'proof'. In the textbooks, especially in the process of learning geometry, the term 'proof' might not be necessarily used from the beginning, while the other term such as 'justify' or 'explain' could be used. Therefore, I try to identify in this step not only the object called 'proof' but also the objects that are related to the justification, by taking the meaning of proof in a broader sense.

The second step is to identify the main characteristics of the form of proof. What is called 'proof' might not have the same form. A proof given as an exemplary in the textbook will be picked up, and its main characteristics will be discussed. It would allow us to understand what aspect of proof is taken care of as a proof in each country.

The third step is to identify the interrelations of geometrical objects created by means of the proof. One of functions of proof is the systematisation (cf. De Villiers, 1991), that is to say, the creations of interrelations between mathematical objects. I consider that the nature of proof is also characterised by these objects. There would be theorems or properties often used in the proof, and those might be specific to the proof in school mathematics. In this step, I identify the geometrical properties that are proven and the properties or theorems that are employed in a deductive step of proof. In terms of the *praxeology* of ATD, this is to identify *the types of task* appeared in *the genre of task* "prove", and the properties used in *the techniques* to accomplish these types of task (see Chevallard, 1999 for the notion of praxeology). The analysis of this step is conducted mainly on the chapters of textbook where the proof is introduced.

The forth step is to identify the functions of proof in the textbook. Several functions of proof are today well known in the research on proof (cf. De Villiers, 1991). However, in school mathematics, usually, not all of functions, but just some of them could be found. In this paper, I discuss the functions attributed to the proof in particular in the chapters analysed in the prior steps.

RESULTS

In what follows, I report a part of results due to the restriction of pages. The results of the first and second steps of analysis are reported together.

Different proofs and their forms in textbooks

French textbook: In *Triangle* series, one could identify at least two terms for the justification of a statement: "preuve" and "démonstration" (I call "proof" and "mathematical proof" respectively in this paper). These terms can be found from the textbook of the first year of *collège*, *Triangle 6e* (6th grade). However, it is 7th and 8th grades where these terms are explicitly introduced. In Chapter 9 "Initiation to deductive reasoning" of *Triangle 5e* (7th grade), the term "proof" is introduced. This chapter also introduces four rules of 'mathematical debate' with which the truth or false of mathematical statement can be determined. Those are: "(1) A mathematical statement is either true or false. (2) Findings or measures on the drawing do not allow proving that a geometrical statement is true. (3) Some examples that verify a statement is not enough to prove that that statement is true. (4) An example that does not verify a statement is enough to prove that that statement is false. This example is called 'counter-example'" (p. 144). Proof is then a product of this mathematical debate. Fig. 1 shows a proof given as an exemplar in the textbook. As it is for a 'simple' statement which has only a single deductive step, the structure of proof is simple. The conclusion is followed by a colon and the property used to deduce it. The property is stated in the form of "si [if]... alors [then] ...". Most of other proofs that can be found in *Triangle 5e*, in particular in the solutions of exercises at the end of textbook, have the similar form.

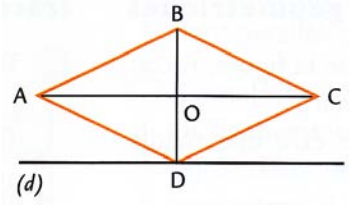
Exercise: (d) is the perpendicular bisector of [EF]. (d) cuts (EF) at I. Prove that the point I is the midpoint of [EF].

Solution: The point I is the midpoint of [EF] after the property: "if a line is the perpendicular bisector of a segment, then that line is perpendicular to this segment and passes through its midpoint".

Fig. 1 *Triangle 5e* (2010, p. 145). Only the translation is given due to the restriction of spaces.

In Chapter 8 "Geometry and initiation to the mathematical proof" of *Triangle 4e* (8th grade), the term "mathematical proof" is explicitly introduced. Its definition is: "A mathematical proof in geometry is a succession of the deductive chains which start from the givens and reach at the conclusion" (*Triangle 4e*, p. 147). The deductive chain consists of three elements: given, property and conclusion of chain. Writing of these three elements is emphasised in the textbook, and its instruction is given. The first sentence starts with the phrase "On sait que [We know that]" followed by a given. The second is a conditional statement of the form "Si [If] ... alors [then] ..." as it was in the proof of 7th grade (cf. Fig. 1). And the third starts with a conjunction "Donc [So] …" followed by a conclusion of this deductive chain. The textbook also mentions that the properties could be sometimes left out according to the level of familiarity with them and the teacher's demands (p. 147; p. 149). Fig. 2 shows a mathematical proof given as an exemplar in the textbook, where three elements of a deductive chain can be easily found. One may also notice that the mathematical proof is written as a paragraph, without many mathematical symbols. Most of mathematical proofs given as examples in the textbook or as solutions of exercises at the end of the textbook adopt this form.

Exercise: ABCD is a rhombus with the centre O. Let (d) the line parallel to (AC) which passes through D. Prove [démontrer] that (d) and (BD) are perpendicular.

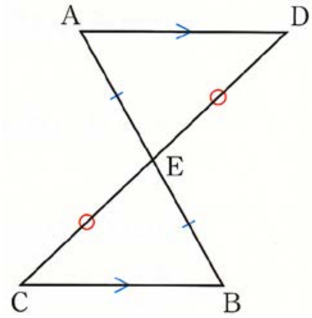


Solution: We know that ABCD is a rhombus.
If a quadrilateral is a rhombus then its diagonals are perpendicular and cut each other at their midpoints.
So (AC) and (BD) are perpendicular.
We know that (AC) and (BD) are perpendicular and that (d) and (AC) are parallel. If two lines are parallel and a third line is perpendicular to the one then it is perpendicular to the other.
So (d) and (BD) are perpendicular.

Fig. 2 *Triangle 4e* (2011, p. 149). Only the translation is given due to the restriction of spaces.

Japanese textbook: In *Tokyo-shoseki* series, the term "proof" can be found in 8th and 9th grades. A part from this term, some exercises require "explanation". In the 7th grade textbook, the exercise of explanation ask either a description of the procedure of geometrical construction or a kind of justification. In 8th and 9th grades textbooks, some exercises, not many, also require some explanations of justification: "tell/explain the reason ...", "explain why ...", etc. The explanation appears before introducing the proof and also after that. However, the method of explanation/justification is implicit in the textbook. Neither instruction nor example is given. One cannot clearly know from the textbook what is really required in explanation. It seems that it is at times the property used in a deductive step, and at other times the given or data used.

On the other hand, the term "proof" is explicitly introduced in Chapter 4 "Parallelism and congruency" of the 8th grade textbook, and the instruction how to prove is given in a sub-section "Method of proving". The definition of proof given in the textbook is: "showing the reason why a fact is true by means of the properties already known as true is called *proof*" (p. 98). Fig. 3 shows an exemplary proof given in the textbook. The proof is well-organised. Some statements are numbered for the sake of the economy of not restating them later. Mathematical expressions with symbols such as "EA = EB", "∠AED = ∠BEC" are often used and written separately from Japanese phrases. In this example, the properties used in a deductive step such as "vertical angles are equal" is always written, while the hypotheses or givens are not necessarily stated (e.g., for the statement 3, the hypothesis "AD // CB" is not stated). In the instruction of method of proving, the term "thing that could be grounds" is stressed to be written in a proof (pp. 109-112). It is either a status of statement "hypothesis" or a geometrical property. The form "if … then …" is not used to write a property, while this form is particularly used when claiming a proposition to be proven.



Right diagram is drawn so that the intersection of the segments AB and CD is E, EA = EB, and AD // CB. Let's prove ED = EC.

Proof: In ΔAED and ΔBEC
From hypotheses EA = EB ... (1)
Since vertical angles are equal
∠AED = ∠BEC ... (2)
Since alternate angles of parallel lines are equal
∠EAD = ∠EBC ... (3)
From (1), (2), and (3), since a pair of sides and their extreme angles are equal
ΔAED ≡ ΔBEC
Since corresponding sides of congruent figures are equal
ED = EC

Fig. 3 *Tokyo Shoseki 8th grade* (2012, pp. 109-110). Note: in Japanese secondary school mathematics, the congruencies of segments and of angles are not in the national curriculum. In most of textbooks, the expression "two segments are equal" are used, while it means "two segments have the same length".

Overall, in both countries, proof is an object of teaching, and two kinds of justification appear. However, explanation/justification is not an explicit object of teaching in Japanese textbook, while the proof is. As for the form of proof, the proof of French textbook uses more paragraphs than the proof of Japanese textbook.

Interrelations between geometrical objects

French textbook: The main geometrical properties that are proven in 8th grade where the proof is introduced are parallelism, perpendicularity and midpoint. The former two properties are intensively proven both in the activities for a class and the exercises of the chapter. In addition, at the end of the 8th grade textbook, there are the pages called "Sheet of methods" where the important methods are summarised. Three methods out of five are "Prove that two lines are parallel", "Prove that two lines are perpendicular", and "Prove that a point is the midpoint of a segment". These are main types of task associated with the genre of task "prove" in French textbook. And a list of several properties to be used is given for each method. For example, for the parallelism, six properties are given: "the property of the parallel lines to a same third line", "the property of the perpendicular lines to a same third one", "the property of the opposite sides of a parallelogram, a rectangle, a rhombus, a square", "the property of the line that passes through the midpoint of two sides of a triangle", "the property of the central symmetry", and "the property of alternate-interior or corresponding angles". These are properties that allow accomplishing the three types of task mentioned above.

Japanese textbook: In 8th grade textbook where the proof is introduced, several properties are proven. Among others, the congruency of triangles is quite often proven as a step to prove other properties and plays an important role in the textbook. In Chapter 5 "Triangles and parallelograms" which is a chapter right after introducing the proof, all the theorems introduced in this chapter are proven with a step of proving congruent triangles. Therefore, the congruency of triangles has many interrelations with geometrical theorems or properties, while these interrelations do not exist in French textbook, because it is not an object of teaching.

Besides congruency of triangles, the parallelism which is often proven in French textbook is also proven in Japanese textbook. But it is always by the alternate-interior or corresponding angles. Even in Chapter 7 Section 2 “Parallel lines and ratio” of the 9^{th} grade textbook where Triangle proportionality theorem (Thales’ theorem) is an object of teaching, this theorem is merely used for proving a parallelism (only a single exercise out of 22 in this section). That is to say, parallelism is tightly connected to the alternate-interior or corresponding angles in Japanese textbook, while several interrelations are made between parallelism and other properties in French textbook.

Functions of proof

The principle function of proof in French textbook is to justify a mathematical statement as it is written in the textbook: “In order to prove that some geometrical statements are true, one must carry out some mathematical proofs” (*Triangle 4e*, p. 147). The similar remark is given in the descriptions of “mathematical debate”. Because the mathematical proof is introduced as an extension of proof or mathematical debate, one can find that the function of communication is one of principal functions of the mathematical proof. This finding conforms to the result of the analysis of proof in French mathematics classroom (Knipping, 2003).

The mathematical statement to be proven in French textbook is either a theorem that can be used in other proofs or a statement only appeared in a particular exercise. In general, it is the proof that allows using the theorem in other proofs. However, this function of proof is less clear in French textbook. A theorem is sometimes admitted first without proof and then its proof is an exercise at the end of chapter (cf. the Midpoint Theorem in *Triangle 4e* Ch. 12, p. 219 and p. 233). Theorems or properties either proven or not proven are summarised in the section of lesson in each chapter. On the other hand, this function can be clearly identified in the Japanese textbook. All the theorems appeared after introducing proof in 8^{th} grade textbook are proven, and their proving is given for the activity in the classroom.

The principle function of proof in Japanese textbook is to justify a mathematical statement, not a particular statement but a general statement. The generality is emphasised. For example, there is a comment with an example of the sum of interior angles in a triangle: “One cannot check out all triangles by means of experiments or measurements, but one can show that the sum of interior angles of any triangle has 180 degrees by means of the proof like the one above” (*Tokyo-shoseki 8^{th} grade*, p. 98). The figures used in proving task also advocate this function. They are not specific ones whose dimensions (length and angle measure) are fixed, but general ones. On the other hand, in the French textbook, the generality is not often emphasised, and even a figure with a fixed dimension is used for proving task. For example, in Chapter 9 “Right triangle and Pythagorean Theorem” of *Triangle 4e*, an exemplar proof is given for the exercise of “Prove [démontrer] that the lines (AI) and (AB) are perpendicular” (p. 164) in which the length of three sides of the triangle ABI are 32, 24, and 40. The solution given to this exercise is called “mathematical proof” in French textbook, while it is rather an explanation in the Japanese textbook (similar exercise and solution can be found in Ch. 6 “Pythagorean Theorem” of *Tokyo-shoseki 9th grade*, p. 155).

CONCLUDING REMARKS

In this paper, I report a part of results obtained in the comparative analysis of French and Japanese textbooks. While some details could not be reported due to the restriction of pages, I expect that the reader can find some differences on the nature of proof in two countries. The proof in the textbook is a proof *to be taught*. Its differences would imply different consequences in the teaching and learning of proof in the classroom of different countries. The nature of *taught* proof is a further question, in addition to the question on the system of conditions and constraints that forms the nature of proof.

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AN ANALYTICAL TOOL FOR THE CHARACTERISATION OF WHOLE-GROUP DISCUSSIONS INVOLVING DYNAMIC GEOMETRY SOFTWARE

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In this report we present an analytical tool to characterise whole-group discussions in mathematics classrooms with Dynamic Geometry Software (DGS). Following an exploratory study with 8th grade students, our interest lies in identifying and characterising interactional episodes and certain actions that support them. The developed analytical tool should help to detect potentially rich situations that may enhance students' mathematical learning. From a methodological perspective, the tool appears to operate in our orchestrated DGS environment.

INTRODUCTION

Whole-group discussion is an important resource for the teaching and learning of mathematics (Yackel, 2001). Much emphasis has been placed in existing research on presenting pedagogical models to specify key practices that teachers can learn in order to be more effective in discussions and in determining their didactical implications (Stein and Smith, 2011). However, developing systematic research on whole-group discussions is a relatively new endeavour. It would seem, therefore, that further investigations are needed to carry out an in-depth study of the nature and configurations of whole-group discussions from an analytical perspective.

The main goal of the on-going research is to create a systematic analytical tool to: (a) identify and characterise interactional episodes of whole-group discussion involving Dynamic Geometry Software (DGS) work, (b) identify and codify specific actions by the participants in these episodes and (c) determine rich situations in terms of their potential (and de facto) contributions to the progress of students' mathematical learning supported by the interaction between the actions. This new analytical tool enhances the quality of research in classroom practice and contributes to gaining a better understanding of the specific nature of whole-group discussions.

THEORETICAL FRAMEWORK

The final purpose of this study is to determine how interactions between *actions*, which are the main components of an *episode*, support *rich situations*. In this section we establish the nature of these three constructs.

Identifying and characterising episodes

Using a particular whole-group discussion in order to identify episodes, we focus on those most likely to influence the students' mathematical learning by promoting both conceptual and procedural progress. We see students' progress as a situated notion of our research: progress is based on qualitative shifts in the mathematical thinking over the course of participation in whole-group discussion, linked with "the travel of ideas" in classroom communities (Saxe et al., 2009).

The characterisation of the selected episodes of the whole-group discussion orchestrated by the teacher is based on two elements. First, the types of instrumental orchestration which Drijvers and colleagues (2010) called *Technical-demo*, which refers to the demonstration of tool techniques by the teacher; *Explain-the-screen*, which refers to whole-class explanation by the teacher, guided by what happens on the computer screen; *Link-screen-board*, where the teacher emphasises the relationship between what happens in the technological environment and how this is represented in conventional mathematics of paper, book and blackboard; *Discuss the-screen*, which involves a whole-class discussion about what happens on the computer screen; *Spot-and-show*, where student reasoning is brought to the fore through the identification of interesting student work during preparation of the lesson, and its deliberate use in a classroom discussion, and *Sherpa-at-work*, where a so-called Sherpa-student uses the technology to present their work, or to carry out actions requested by the teacher. We use this typology to classify the nature of the episodes according to the didactical performance of the discussion.

The other element on which the analytical tool is based is the sequence of mathematical concepts and procedures that have been used or mentioned during the whole-group discussion. We use the term problem steps to refer to the succession of mathematical concepts and procedures that have been studied during a specific whole-group discussion. This construct is the trajectory that the class (involving all participants) has followed to reach what they co-assumed to be the solution of the problem. When the class takes place, the topics are determined in a linear sequence of steps, and for each situation the problem steps will vary. We use this typology to classify the nature of the episodes according to the content of the discussion.

Identifying and codifying significant actions from the episodes to support rich situations

We consider significant actions to include the participants' interventions and the instrumented acts related to DGS. This consideration is related to the instrumental approach introduced by Rabardel (1995), because it deals with the mutual relationships and interaction between humans and technologies. In this context, we also agree that knowledge is constructed socially by subjects involving the media because the participants collaborate to re-organise thinking with a different role than that assumed by written or oral language (Arzarello and Robutti, 2010).

The relation between different significant actions in an episode may provide rich situations, which create opportunities to greatly enhance students' progress. We adopted the idea of rich situations according to which cognitive constructs may develop in parallel with the transformation of actions during the discussions (Hershkowitz and Schwarz, 1999). Thus, we classify the significant actions as follows:

- students' actions considered as thinking-math interventions (TMI);
- teacher's actions, considered as didactical interventions (DI) and
- instrumented actions performed by the participants centred on DGS, considered as DGS uses (DU).

METHODOLOGY

For this study, ten class sessions of an experienced grade 8 teacher were chosen. A sequence of five inquiry problems was designed to study isometries in a collaborative way using DGS. After the students had solved a problem in pairs, the teacher led a 50-minute whole-group discussion. DGS was involved in both the students working in pairs and the whole-group discussions.

During the whole-group discussions, the teacher and the students were observed and video-taped with three video cameras and additional voice recorders to capture all teacher and student actions in detail. The attention was focused on the interactions between the participants and the software as well as on all different series of significant actions produced by both the teacher and the students.

After data collection, the qualitative data analysis software Atlas.ti© was used to organise and codify the whole-group discussion videos. Three researchers worked together on carrying out the codification. The purpose of this work was to create a well-established analytical tool which determines how interactions between actions, which are the main components of an episode, support rich situations.

THE RESULTING ANALYTICAL TOOL

The resulting analytical tool organises the discussions in characterised episodes. Within each episode, the tool also helps to characterise the actions involved in it. After the characterisations, a representation of the interaction between actions provides the expected rich situations. In this section, we present a detailed explanation of the newly-developed tool and then, an example of its effectiveness.

The nature of the episodes

The episodes are identified and characterised by analysing whole-group discussions. The nature of the episodes is characterised by two elements: the type of orchestration and the specific problem step. The first element is based on the type of orchestration presented by Drijvers et al. (2010). The second is the problem step dealt with during the specific selected segment of the whole-group discussion. In the example presented, the problem steps were: Correct and argued solution of the generic case, Double solution taken into account, Understanding of the properties of the homologous, Detection of invariants, Consideration of particular cases, Generation of parallel cases, and Generation of coincident cases.

Figure 1 illustrates the structure of the specific whole-group discussion exemplified in this report in terms of orchestration and mathematical content. All the episodes identified in a whole-group discussion can be represented in a two-dimensional matrix: type of orchestration and problem step.

Whole-group discussion (Problem 3)	Correct and argued solution of the generic case	Double solution taken into account	Understanding of the properties of the homologous	Detection of invariants	Consideration of particular cases	Generation of parallel cases	Generation of coincident cases	*(Steps of the problem)*
Technical-demo								
Explain-the-screen							(e_{10})	
Link-screen-board		(e_3)						
Discuss-the-screen			(e_5)				(e_9)	
Spot-and-show					(e_7)			
Sherpa-at-work	(e_1)	(e_2), (e_4)		(e_6)		(e_8)		
(Types of orchestrations)								

Figure 1: Representation of the episodes (e_i) involved in a whole-group discussion.

(Subscript *i* help to follow the episodes chronologically).

Once the episodes have been defined and the whole-group discussion has been structured, an in-depth study of the actions involved in each episode is required.

The nature of the actions that support rich situations

The actions involved in the episodes are defined only by one element. As presented in the theoretical framework, the nature of this element differs depending on the action's agent (i.e. students' actions considered as thinking-math interventions; series of teacher's actions, considered as didactical interventions and series of instrumented actions performed by the participants centred on DGS uses, considered as DGS uses).

To start the analysis of the actions, firstly, each action is characterised as one of the three categories presented. Secondly, each action is codified to describe its nature in more detail with an emergent codification. Apart from the chronological succession of actions, the relations between them also have to be taken into account. Thus, we add oriented segments to connect the actions that are influenced by others. These oriented segments are called influence connectors. Finally, a structure is created to summarise all this information: a) the nature of all the actions, b) the participants who are performing each action, c) the time sequence when the actions occur and d) the influence connectors that relate the different actions (see, e.g. Figure 2). This representation of the episodes can reveal a first approximation of possible progress in the students' learning if we focus on the influence connectors.

An example of a characterised episode

We present an example of the use of the developed tool to show its effectiveness. A whole-group discussion extract of the 3rd problem of the sequence is analysed. The problem asked to find the centre of a rotation mapping two line segments given in the plane. After observing the didactical performance of the whole-group discussion, a list of the problem steps dealt with in the discussion was created. Using the types of orchestrations and the list of the problem steps, the episodes were defined and represented in a two-dimensional matrix (see Figure 1).

This representation provides an overall view of the 50-minute discussion in terms of its characterised episodes. To continue with the analysis, an in-depth study of each episode was performed. In this report, we present the analysis of the first episode identified in the whole-group discussion. The episode presented (e_1) was characterised as “Sherpa-at-work orchestration”, because two students were using the technology to present their work and to carry out actions the teacher requested and “Correct and argued solution of the generic case”, because they were presenting this part of the problem.

In the transcription, we can observe the participants involved and the different nature of their actions. Each action was codified and characterised, through the codification system created by the researchers, as one of the three categories presented above.

Student 1: We decided to do a perpendicular bisector between a point and its homologous. Then we tried this intersection and we realised that it coincided and that it was the right rotation centre.

(TMI): Explain a procedure to reach a solution

Student 2: [While Student 1 was explaining their solution, Student 2 was constructing the situation with GeoGebra]

(DU): Complete an explanation with visual construction

Teacher: Now that you know that this point is right, could you argue why?

(DI): Ask for a mathematical argument

Student 2: If two points, when we rotate a piece, coincide, it means that they are at the same distance from that point. So if they are at the same distance, for example, the perpendicular bisector is the “locus point” of all equidistant points between the two original ones.

(TMI): Elaborate on a deductive justification

Teacher: The locus.

(DI): Reformulate technical vocabulary

Student 2: Ah, yes! The locus.

(TMI): Correction of technical vocabulary

Student 2: [While Student 2 was expressing the justification, he was using visual GGB figures which were on the screen for his explanation]

(DU): Draw on visual GGB figures

After having identified and codified the actions involved in the episode, the situation was represented in a visual diagram (see Figure 2).

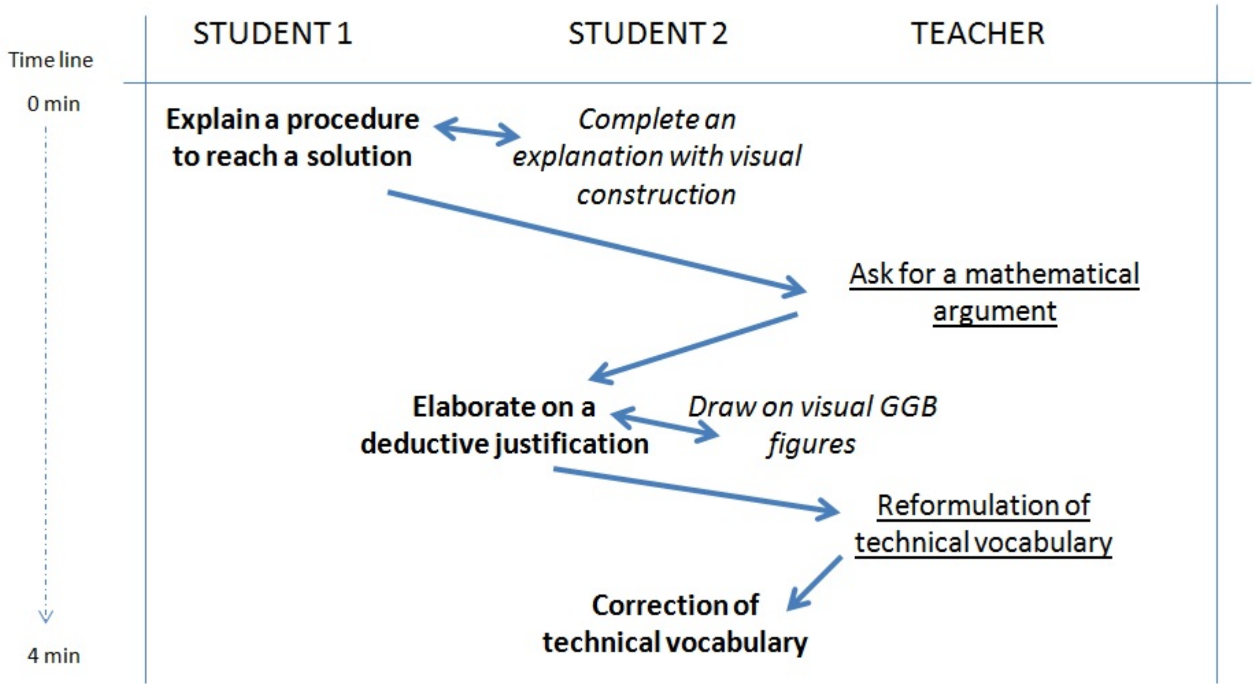

Figure 2: Visual representation of the episode 1 (e_1). (Legend: **TM-I**, <u>D-I</u> and *DGS-U*)

Actually, our focus is on the connections between all the actions of the episode because the potentially rich situations may emerge from these connectors of influence.

From a descriptive point of view, we can observe that the participants involved in this 4-minute episode are two students and the teacher. If we focus on the nature of the actions, we observe that the students' thinking-math interventions are central in the episode. We also observe that the didactical interventions by the teacher are also important. In his first intervention, he asks for an argument and in the second one, he makes a vocabulary correction of the expression "locus". The students' use of the software is also crucial in this episode. In the transcription, we can observe that first they use the software to make a construction and then, they point to the screen and show the completed construction.

From an interpretative point of view, the last use of the analytical tool is to identify rich situations that can influence the students' learning process in a middle-term perspective. The focus is now on the connectors of influence between different categories of actions, as represented by oriented segments in Figure 2. Afterwards, in this episode, three different situations that can enhance the students' mathematical learning are interpreted. The first two rich situations identified are derived from relations between thinking-maths and didactical interventions and the third rich situation is derived from relations between thinking-maths interventions and DGS-uses.

- *Importance of dual explanations with communicative and technological skills.*

The fact that the student uses software every time he wants to explain or show something to the class, as occurs twice in this episode, is a crucial relation between the thinking-maths interventions and the uses of GGB. The students complement their explanations by drawing on visual GGB figures, as we can observe, while Student 1 explains the procedure to reach the solution and while Student 2 develops a deductive justification. We consider that this situation gives importance to the dual explanation: a combination of communicative and technological skills. In further research, we will have to assess the possibility of being influenced by this episode when a student uses the DGS to complement a written or oral solution.

- *Importance of reasoning and learning the specific justification.*

The fact that the student is asked for a mathematical argument after the incomplete solution is presented by Student 1 points out the importance of arguing the solutions. Moreover, in this specific situation, as after the statement, the correct justification is given by Student 2. This also enhances the situation, because it provides specific knowledge (the correct argumentation) that could also influence the other participants.

- *Importance of correct use of the mathematical language.*

The fact that the teacher corrects the mathematical expression "locus point" after its inappropriate use by Student 2 during his argumentation also creates a rich situation in which the importance of using correct vocabulary is demonstrated. After the correction by the teacher, we can observe that Student 2 reacts correcting his expression. So, the positive influence of the intervention is evident, and we can suggest that more students could be positively influenced by this situation.

Overall, in this episode we have identified three potentially rich situations that can enhance the progress of the students. With this example, we show that analysing the episodes with the analytical tool presented, gives us an overall view of how the actions of an episode are interacting. Therefore, this overview facilitates the detection of potential situations emerging as a convergence of different agents involved in the episode.

DISCUSSION

A newly-developed analytical tool has been presented. The main objective of this tool is to identify rich situations that may enhance the students' mathematical progress during whole-group discussions involving DGS. Its effectiveness was tested by applying it to a transcript of a whole-group discussion.

This tool analyses whole-group discussions from multiple perspectives. On the one hand, the framework of instrumental orchestration (Drijvers et al., 2010) provides a crucial episode-structure of the discussions. On the other hand, the findings about the actions involved in the episodes further support the humans-with-media theoretical approach that takes both the subject and the tool involved in a mathematical activity into account (Arzarello and Robutti, 2010). These results are consistent with Hershkowitz and Schwarz's (1999) findings that students' progress is not caused by verbal interactions only, but also by computer manipulations as well as communicative nonverbal actions.

This analytical tool has certain limitations, which constrain the generalisation of the results. The tool has emerged from the analysis of certain whole-group discussions with particular teacher, students and problems. Despite these limitations, we conjecture that the tool could also be applicable, maybe after being adapted, outside the scope of this study. Therefore, it would be worthwhile to investigate the effectiveness of the tool in different contexts such as discussions involving different teachers' and students' profiles or other kinds of problems. Further research would help to refine the tool and carry out a well-established analysis of its weaknesses.

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STUDYING CHANGES IN SCHOOL MATHEMATICS OVER TIME THROUGH THE LENS OF EXAMINATIONS: THE CASE OF STUDENT POSITIONING

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This paper reports part of a study of the changing nature of school mathematics. The study uses the question papers of high stakes examinations as a window on the discourse of school mathematics. An analytic framework, using tools drawn from social semiotics (Morgan, 2006) and from Sfard (2008), has been developed to characterise this discourse. Here part of the framework is presented, addressing the ways in which students are positioned in relation to mathematical activity. Analysis of question papers from two different years focuses on the presence of agency.

Over the last decades there has been public and academic concern in many countries about the nature and standards of school mathematics. This concern has frequently driven revisions of curriculum and examinations, yet there are contradictory opinions about the effects of reforms. The project "*The Evolution of the Discourse of School Mathematics through the Lens of GCSE examinations*"[1] aims to investigate changes in school mathematics in England over the last three decades by analysing the discourse in which students are expected to engage in order to be judged to be successful in mathematics. We use national examinations taken by the vast majority of students in England at the end of compulsory schooling (age 16), the General Certificate of Secondary Education (GCSE) as our window onto the nature of school mathematics. The GCSE is widely used as a qualification for entry into further education and employment and its results are used to rank schools. It thus has high significance for students, teachers and school administrators. The existence of an intimate relationship between high stakes examinations and curriculum and pedagogy has been well established (e.g., Broadfoot, 1996) and has been an explicit focus of debate about the design of assessment tasks for school mathematics (e.g., Bell, Burkhardt, & Swan, 1992). Therefore, although the discourse of examinations has distinct characteristics, we see changes in examinations as a good index of changes in school mathematics. High-stakes examinations such as the GCSE play an important role in the mathematics that students experience, influencing the content of teaching, the ways tasks are defined and the kinds of student responses that are valued.

Rather than comparing syllabi or teaching methods, we seek to probe deeply into the nature of the mathematical activity construed by examination texts and expected of

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students by developing and applying a discourse analytic approach, drawing on social semiotics (Halliday, 1978; Morgan, 2006) and Sfard's theory of mathematical thinking as communicating (Sfard, 2008). Studying discourse in this way allows a subtle characterisation of the nature of mathematics and of student mathematical activity construed through the forms of language used. We argue that the analysis of change produced by this approach will provide insight into how changes in curriculum and assessment may affect students' mathematical learning. In this paper, we present one small part of our developing analytic framework, focussing on the way in which agency in mathematical activity is construed and how this affects students' positioning within the practice of school mathematics.

LANGUAGE, MATHEMATICS AND ASSESSMENT

Our theoretical perspective on the relationship between language and mathematics sees difference in the form of language to be associated with different construal of the nature of mathematics and mathematical activity (Morgan, 2006; Schleppegrell, 2007). Indeed, doing mathematics can be considered as participating in forms of communication endorsed by a mathematical community (Sfard, 2008). Within the literature on assessment, concern has generally been with the effects of language on the difficulty of tasks. For example, studies of examination questions have identified factors such as the structure of the question, use made of diagrams and technical notation and language (e.g. Fisher-Hoch, Hughes, & Bramley, 1997) to affect the difficulty of questions. Shorrocks-Taylor and Hargreaves (1999) summarise the findings of research into the syntactic aspects of mathematical text that may make reading more difficult. The findings of such studies have been used by the English examination boards in recent years to inform their design of examination questions, with the aim to reduce "unnecessary" difficulty that might be due to language issues. Our interest, however, is not simply with difficulty but with more complex questions about the ways that the mathematical activity itself is altered by changing the language used to present it.

STUDENT POSITIONING

From a social semiotic perspective, language not only construes our experience of the world but also construes our identities and relationships to each other and to our experiences (Halliday, 1978; Morgan, 2006). Powerful texts such as textbooks and examination papers provide specific positions for students, that is, ways in which students may interact with the text and act within the practice of school mathematics. Of course, it is possible for individuals to resist such positioning but the text provides a 'natural' way of reading (Hodge & Kress, 1993). Previous studies of school mathematics texts have identified differences in the ways that students may be positioned in relation to mathematics, to their teacher or more generally in relation to social roles. Dowling (1998) characterised the ways that a textbook series for high attainers *apprenticed* students as potential mathematicians, while the texts for low attaining students constructed them as *dependent* and projected their futures as manual workers. In analyses of textbook presentations of definitions of trigonometric functions, Morgan (2005a, 2005b) identified some texts which, by making explicit the reasoning and decision making involved in forming a definition, made it possible for students to be positioned as decision makers themselves, while other texts presented definitions as absolute and unquestionable facts, obscuring any possibilities for creative human engagement in shaping mathematics. Studies of classroom interactions have also identified differences in the possibilities made available for students to engage in mathematical forms of discourse as well as in the kinds of relationships established between teachers and students (Atweh, Bleicher, & Cooper, 1998; O'Halloran, 2004). We are interested in the kinds of positions made available for students in the examination papers we are studying and the ways in which these positions may affect the kind of mathematical activity students engage in. In the next section we outline the analytic tools we use to investigate positioning.

ANALYTIC FRAMEWORK

For the project as a whole, we are developing an analytic framework that allows us to describe the nature of the mathematics and mathematical activity expected of students as they read and respond to the examination paper. In this paper, we focus on aspects of the framework that provide insight into the positioning of the students in relation to mathematical activity, addressing the following questions:

1. To what extent is mathematics construed as a human activity in which claims are based on the outcomes of investigations or decision-making rather than on agentless facts? We code the nature of the processes evident in the text (finite and non-finite verbs and nouns formed from verbs) and whether the agents in these processes are human beings or mathematical objects. Imperatives addressed as instructions to the student are excluded from this analysis but are considered separately (see point 2 below). We note cases where agency in mathematical processes is obscured by, for example, the use of nominalisations (nouns formed from verbs such as *reflection* or *relation*), passive voice or non-finite verb forms. Obscuring agency in such ways plays an important role in the development of mathematical objects and relations between them but also serves to alienate the student from the mathematical activity.

2. What kinds of actions are students expected to carry out? Examinations present tasks for students to do, often expressed using imperative forms. We consider the types of tasks demanded of students to play a role in positioning them in relation to mathematics. Here we distinguish between material and mental processes, drawing on Rotman's (Rotman, 1988) distinction between the *thinker* and *scribbler* roles implied by imperatives. The student is construed as a *thinker*, invited to participate in an intellectual community of mathematicians, if, for example, she is instructed to "Suppose …", "Prove …" or "Explain …". On the other hand, she is construed as a *scribbler*, carrying out a determined material process, if instructed to "Calculate …", "Measure …" or "Write down …".

3. What degree of responsibility or freedom is the student afforded to shape the answer? Here we consider:
    a. the extent and type of decisions the student must make in order to construct an acceptable answer
    b. explicit shaping of the form of the answer (by, for example, the amount of space provided to write the answer, specifying units, specifying a degree of accuracy)

DATA

Our data consist of the examination question papers for the most popular syllabus published by each of two examination boards (from the three operating in England during the period) in each of 8 years between 1980 and 2011. The sampling points were chosen following a review of curriculum documents and interviews with key correspondents to identify points at which changes in curriculum or examination policy or practice occurred. Digitised versions of the question papers have been uploaded into NVivo and coded for each aspect of the analytic framework. This allows us to construct quantitative comparisons across years and to search the question papers for examples of interest.

AGENCY

In this paper we present some examples of our analysis of question papers from one of the examination boards for two of our sampling points, 1995 and 2011. In the space available here we are only able to discuss the first aspect of the framework identified above: the nature of agency evident in the text.

Table 1 summarises the agency in different kinds of processes found in the question papers for each of the two years. We have coded the processes as mathematical (e.g. *estimate*, *measure*, *reflect*), non-mathematical (e.g. *buy*, *share*, *exercise*) or verbal (denoting any form of communication e.g. *show*, *draw*). The same word may be coded differently depending on context. So, for example, the process *pass* in the sentence *John passed the finish line in 12.2 seconds* would be coded as non-mathematical, while in the sentence *The line passes through the point (2,4)* it would be coded as mathematical because such a sentence is clearly part of a mathematical register. The frequency of relational processes (e.g. *be*, *have*) is also noted. These processes do not involve agency but assert identities or assign properties to objects.

Most of the processes coded as non-mathematical occur in questions presented in "real" contexts, for example, a 1995 question involving probability starts with a description of a non-mathematical context:

> Some pupils have thought of a game to use at a school fair.
> A tennis ball is rolled down a slope into one of eight holes. (1995)

In this example the process *thought* is coded as non-mathematical with human agency, while *rolled* is coded as non-mathematical with obscured agency. The overall proportions of non-mathematical processes in the two years are similar. While there are notable differences in the type of agency for these processes between the two years, we are more concerned in this paper with agency in mathematical processes.

Table 1: Agency and processes in examination papers

agent type	process type	1995		2011	
obscured agency	mathematical	26	(19%)	7	(8%)
	non-mathematical	21	(15%)	8	(9%)
	verbal	12	(9%)	4	(4%)
	all	*59*	*(43%)*	*19*	*(21%)*
mathematical object as agent	mathematical	7	(5%)	3	(3%)
	verbal	13	(10%)	13	(14%)
	all	*20*	*(15%)*	*16*	*(17%)*
human agent	mathematical	6	(4%)	0	(0%)
	non-mathematical	23	(17%)	23	(25%)
	verbal	1	(1%)	1	(1%)
	all	*30*	*(22%)*	*24*	*(26%)*
	relational	27	(20%)	33	(36%)
	total	*136*	*(100%)*	*92*	*(100%)*

There is a substantial change between the two years in the extent of the use of processes in which the agency is obscured, mainly by use of the passive voice. In 1995 43% of all the processes in the text were presented without explicit agents, while in 2011 only 21% were without agents. This change appears consistent with the examination board's aim of improving the readability of the text. It is commonly perceived that texts involving extensive use of passive voice are difficult to read, though there are strong arguments for the importance of the rhetorical role played by passive voice and indications that in some cases it may actually increase readability (e.g. Riley, 1991). It is interesting to note, however, that the 2011 reduction in processes in which agency has been obscured does not seem to have been achieved by rewriting passive sentences in the active voice, as there is not a comparable increase of processes with either mathematical or human agents. Rather, the proportion of relational processes has increased. How may we interpret this difference in terms of how students may be positioned in relation to the subject matter?

Obscuring agency through the use of passive voice and nominalisation functions as a form of alienation, distancing the student from the mathematical activity. By hiding the role of the human mathematician, the mathematical processes appear to be beyond human control, possibly making it more difficult for a student to position themselves as potential active participants in relation to these processes. Moreover, a statement such as

> At C, a tangent to the circle has been drawn. (1995)

presents the drawing of the tangent as an unquestionable fact rather than an act that has been carried out purposefully by a human mathematician. Of course, the author of the question in which this statement occurred made a decision to draw this tangent with the intention to prompt the student to display particular forms of mathematical knowledge. However, the absence of the author's explicit agency hides the significance of the decision and the intention behind it. Nevertheless, this statement does involve the material act of drawing to produce a tangent and indicates by the use of the past tense that this act has taken place as part of a sequence of acts. This temporality has potential to include the student's response to the question as part of the sequence, thus positioning the student as a possible actor in a mathematical narrative.

On the other hand, the use of relational processes, as found more frequently in 2011, removes all reference to mathematical actions. For example, the information provided in one question:

> In the diagram,
> ABC is a triangle,
> angle ACB=90°,
> P lies on the line AB,
> CO is perpendicular to AB.
> (2011)

presents the triangle and its attributes as absolute facts with a timeless, completely autonomous and apparently arbitrary existence. This seems to distance the student even further from any possibility of constructing such mathematical objects themselves or questioning the reasons for the properties presented.

We consider the presence of explicit human agency in mathematical processes to play an important role in positioning students as potential members of a mathematical community. As may be seen in Table 1, there are a few instances in 1995. A question about statistics, for example, includes the following:

> Sam took a sample of 80 pupils.
> Explain whether or not he should have sampled equal number of boys and girls in Year 8.
> (1995)

In this example, *a sample* is construed as the product of activity by a human being, who may even be perceived as being a young person with whom the student might identify (*Sam* being a diminutive – and hence informal - given name). Moreover, the action demanded of the student construes mathematical activity as involving decision making and justification. It also positions the student as able to evaluate the action that Sam "should have" done. (There is not space in this paper to consider further the actions expected of students. This will be the subject of future work addressing point 2 of the framework provided above.)

In the 2011 question papers, we identified no instances of human agency in mathematical processes. (As noted above, this analysis omits the processes demanded of students in imperatives, which clearly construe the student-addressee as agent.) Given the small numbers involved and the fact that the 2011 papers contain a lower proportion of mathematical processes overall, we cannot be confident in drawing conclusions from this about differences between the years in the explicit presence of human agency in mathematical processes.

CONCLUSION

In this paper we have presented a preliminary analysis of examination question papers from two year, demonstrating the application of one part of our analytic framework to consider how students may be positioned in relation to mathematics. This analysis suggests that, although some of the linguistic features that contribute to alienation from mathematical activity (passive voice and nominalisation) have been reduced between 1995 and 2011, the increase in the use of atemporal relational statements contributes to a continued, and possibly reduced, absence of human agency in the way that mathematical activity is construed. The size of the sample considered so far is too small to draw definite conclusions about how this may have changed the nature of school mathematics discourse but the analysis presented here indicates issues that we intend to address in our analysis of the full sample of examination papers.

The reduction in the use of passive voice and nominalisations seems likely to be the result of deliberate attempts by the examination boards to reduce sources of reading difficulty such as those identified by Shorrocks-Taylor and Hargreaves (1999). (See also Morgan, Tang & Sfard (2011) for a discussion of reduction of grammatical complexity.) These grammatical features are characteristic of advanced scientific and mathematical discourse (Halliday & Martin, 1993). The question of how such changes to the linguistic form may affect the kinds of mathematical activity in which students engage is a focus of our on-going project.

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THE *SIXNESS* OF SIX: CONTRASTING AND COMPARING PIAGETIAN AND VYGOTSKYAN THEORIES

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The examination of a group task is carried out through Piagetian and Vygotskyan perspectives. Whilst it is recognised that theoretical inconsistencies in neo-Piagetian and neo-Vygotskyan ideas make a complementary view of social and individual learning difficult, the use of both theories to contrast and compare is seen to shed light on children's learning in arithmetic.

INTRODUCTION

This paper examines children's learning in arithmetic by comparing and contrasting Piagetian and Vygotskyan based theories. An example of children's collaborative group task is used for illustration. The children are working with the associativity property to find equivalent solutions to six. This can be related to the neo-Piagetian notion of duality as the children work with both the process of counting and the abstract object of the number, in this regard it is an illustration of Gray and Tall's (1994) *threeness* of three. The children's *sixness* of six is also examined from a social Vygotskyan perspective and semiotic function where children are involved in relations of significance (Walkerdine, 1999).

It is recognised that these two theories have inconsistencies that make a complementary viewpoint difficult. The aim here is to use these theories comparatively to shed further light onto children's learning in arithmetic.

THEORETICAL PERSPECTIVES

Research in mathematics education has for some time been informed by the perspective of psychology. Since the 1980s research in mathematics education has had an increased interest in the social aspects of learning mathematics (Bishop, 1988). It is now accepted that teaching and learning in mathematics can be seen as both social and psychological products. The submergence of social theory into mathematics education has produced some debate between sociocultural theories and individualistic ones and a complementary use of these two perspectives is not straightforward (Lerman, 1994; Ernest, 1999).

The intention of this paper is not to further this debate but to realise the main tenant in relation to interpretations of radical and social constructivism (Ernest, 1999). Piagetian frameworks, based on the idea of a set of universal culturally invariant structures, have led to the notion of radical constructivism as an individual idiosyncratic construction of meaning (Ernest, 1999). In this way systematic errors or misconceptions are seen as part of individual cognitive schemes. Ernest, (1994; 1999), amongst others, argued for

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a view of constructivism in a more thoroughly social Vygotskyan sense which takes the centrality of language to knowledge and thought. Ernest proposed that a thoroughly social constructivism reflects a metaphor of conversation where 'mind is viewed as social and conversational'.

CHILDREN'S LEARNING IN ARITHMETIC

Neo-Piagtian theory: Process/object duality

Gray and Tall (1994) proposed that number is an elementary *procept*, an amalgamation of a process, an object and a symbol. As such *threeness* of three (or in this paper the *sixness* of six) is an abstract concept where six is seen as both a process and the product of a process. The notion of a procept refers to a process-object duality and Piagetian notions of assimilation and encapsulation (Tall et al., 2000). The move from the physical act of counting to the use of number in arithmetic is achieved through 'compression' of the process of counting. In this way the word six is not just a counting word, it is also 'compressed' into the concept of six as an 'economical unit' that can be held both as a focus of attention and as an access to the process of counting. Hence a symbol such as six evokes both the counting process and the number six itself. Children who are able to work with both the concept of six as a unit and to access the process of counting are said to have a *proceptual* view of number.

Within the Piagetian notion of reflective abstraction understanding is derived from interiorised actions where children's sensory experiences are represented at an abstract level. There is a clear distinction between reflective abstraction and empirical abstraction. Whereas empirical abstraction draws from perception and sensory-motor experiences, reflective abstraction does not draw on information from empirical abstraction (Piaget, 2001) so sensory motor or perceptual knowledge is not seen as a source of new constructs. In a child's understanding of number and arithmetic, abstractions are not based on perceptual information received from experience with the world. Understanding is drawn from apprehending the properties that are presented by an object but where the properties were introduced by previous actions. The focus is on the actions of the objects and the properties of those actions. Understanding equivalence through associativity is realised through reflective abstraction that does not draw its information from the sensory perceptual world of a child but from the coordination of objects.

In developing the *sixness* of six from a Piagetian perspective communication is seen as key to developing a proceptual view but language and signs as a social persp**e**ctive are not part of this framework. The symbol is integrated as mathematical meaning, not as a social phenomenon, but as access to the structure of number.

Neo-Vygotskyan: semiotic perspective

From a social perspective attention is turned to the individual concept construction of social knowledge and how an individual relates to and gives meaning to signs. By signs we mean symbols, words or gestures. Ernest (1997) stated that mathematics can be seen as a study of abstract sign systems. In this way learning mathematics is seen as mastering these systems. So as the children are engaged in the *sixness* of six they are engaged in giving meaning to a cultural system and this meaning is mediated semiotically.

Within Vygotskyan theory (1986) a sign or word embodies a generalisation, a concept. Words or concepts already have a meaning in the adult world and a child is seen to negotiate this meaning and instantiate the concepts. Vygotsky also differentiated between lower order, spontaneous concepts and higher order, scientific concepts. Scientific concepts are the factual pre-existing social knowledge associated with classroom instruction whereas spontaneous concepts are grounded in everyday experience and arise through perception. Unlike Piaget, Vygotsky reconciled the two orders of concepts in a dialogic and dynamic relationship. A child's ability to comprehend scientific concepts is connected to development of spontaneous concepts.

> Spontaneous concepts, in working their way 'upward' toward greater abstractedness, clear a path for scientific concepts in their 'downward' development toward their concreteness (Vygotsky, 1986, p.xxxiv).

Within Vygotskyan theory scientific concepts do not displace spontaneous concepts, they are both part of the system. Concept formation is dialogic and dynamic and is represented by a system where perception is mediated by language and other signs/gestures. The use of signs and gestures leads to the semiotic function of signifier and signified (Ernest, 1994). With reference to Saussure, Walkerdine (1999) further explained the unifying nature of this function. Signified is the concept, meaning or thing (what the signifier is indicating), signifier is the word, the pointing finger, the sound image (the indicator). The relationship is arbitrary, a convention, so cognition and context are not two separate phenomena.

From a Vygotskyan perspective the dialogic and dynamic movement upwards and downwards between scientific and spontaneous concepts is a movement between abstract social knowledge and concrete reality where perception, association and imagery are intertwined with the formal and social through language and signs. Language and signs direct the children's attention to what is important to focus on.

THE CHILDREN'S TASK: WORKING WITH THE *SIXNESS* OF SIX

The group task given here as illustration comes from the 'Talking Counts' project funded by Esmee Fairbairn and carried out at the University of Exeter with colleagues Rupert Wegerif and Ros Fisher. The project investigated the development of group collaboration based on the notions of 'Exploratory Talk' (Mercer et al., 1999) with young children ages 6-7 years old who were seen as low attainers in mathematics.

The group of three children (John, Claire, Sally) were completing a form of the Magic Square using the numbers one, two and three on a 3x3 grid.

1	2	3
2	3	1
3	1	2

Figure 1: The Magic Square with numbers one, two and three

Transcript excerpt1:

John: No, we need to make them equal

Claire puts numbers on the grid.

Sally: What are you doing?

John: It equals six; three, two, one (shows on fingers)

Sally: You’ve got to make six

John: Or you could do it another way round

Claire: One and a two and a three, that’s six

All children count on their fingers and agree.

John: Now we could do it the other way. We could do two, three ... No... We could do three, two, one again if we wanted

Claire: Still equals six

John positions numbers three, two and one.

John: Three, two and one...yeah that’s six, six, six, six

It would seem that John has seen the equivalence from the start. When John confirms the answer, “Yeah, that’s six, six, six, six”, he seems to be making a firm pronouncement that they are equivalent. From this Claire seems to see the equivalence and states, “Still equals six.” Sally seems more puzzled as she says, “You’ve got to make six.” Her statement seems more related to totalling the numbers than the idea of equivalence.

Transcript excerpt 2:

Sally: Now we are making four

Claire: You need to have the one there

John: Right I’ll put three

Sally counts on her fingers.

Sally: One, two, three....six. Can’t make just six, that’s six

Claire: One, two, three; three, one, two; two, one, three; one, two, three

Sally: That's six. You can't just make six all the time

Sally continues to suggest some puzzlement at the task as she is looking to make different totals, "Now we are making four." Claire points out that other numbers are needed and she and John rearrange them to make six. Sally checks this on her fingers and seems surprised that they arrive at the same total, "Can't just make six, that's six." Claire then points to the different ways they have made six and reads them out. Sally's statement, "That's six. You can't just be making six all the time," suggests that she is beginning to realise the equivalence.

As the children were talking they often used their fingers to help count out the total. John and Claire abandon the use of fingers and start to focus on the different arrangements of one, two and three. Provided these three numbers are present they begin to see that the total must be six. Sally continues to use her fingers to check the total and the teacher decides to intervene to encourage her in seeing the equivalence without counting.

Transcript excerpt 3:

Teacher: Sally, I want you to see if you can do this without counting on your fingers. What can you tell me about two, one and three? What do they equal? You've got a clue (points to the grid).

Sally: (pause...) Six

Teacher: So if two, one and three equals six, what can you tell me about three, two and one? They're going to equal...?

Sally: Six

Teacher: What about one and three and two?

Sally: Six

Teacher: Do you need to keep checking on your fingers?

Sally shakes her head

Teacher: Now you know that two, one and three; three, two and one and one, three and two always equals...

Sally: Six

DISCUSSION

It seems that the task supports the children in moving their focus from 'how to do the task' (counting) to 'how to use the solutions' (the idea of equivalence). If the children focus on the actions on the objects from a concrete or perceptual way they will have difficulty in seeing the equivalence. For example the fingers and actions used to total 1+2+3 is not the same as 3+2+1. Different fingers are raised and counted at different times. If looking at the numbers as they are placed on the square the arrangement '1 2 3' and '3 2 1' do not look the same.

From a Piagetian duality perspective we see the children reflecting on the coordination of the actions and in this regard could claim that John and Claire were developing a proceptual view of number, or a *sixness* of six. They were able to see six both as a process and a product of a process. One could say that the coordination of the actions enabled them to move through layers of reflective abstraction. The repetition of these actions enabled them to abstract the notion of equivalence as a universal structure. Whilst this explanation might go some way to explain John and Claire's understanding, it disregards the discourse and the context of the task. It is also difficult to explain why Sally was less able to see the equivalence.

Another way of looking at the learning in the task is through a chain of signification (Walkerdine, 1999). The children used routine exchanges initially with fingers for counting. The fingers could be said to be signifiers. The signifiers (the fingers) become the signified which in turn become united with new signifiers, in this case the numbers on the square. Repetition of the actions supports the "move out of the relations of practice and into the internal relations of the number system" (p.120). The fingers are used as iconic signifiers that "hold the practice left behind" (p. 121). As the children construct the chain of signification through their fingers to the numbers, they are also transitioning into mathematical discourse and, hence, relating discourse and practice (p. 123). As the children act on the fingers and the numbers they use terms to explain what they are thinking and so 'read out' the actions. These actions are repeated using different arrangements with the same statements, hence providing a discursive unity.

When the teacher intervenes she appears to use a similar chain of signification to the teacher in Walkerdine (1999, p.123). The teacher reinforces the actions as she points to the Magic Square but with different discursive statements. She also leaves statements unfinished. In this way she is seen to unite the signified, now the Magic Square, with a new signifier, her statements.

The discursive statements along with the actions are focusing attention on equivalence as formal abstract cultural knowledge. Perception of the different arrangements has linked with the actions on the fingers which has linked to the numbers and then to the Magic Square. This chain has enabled the children to move upwards and downwards between the abstract and the concrete in a way that was semiotically mediated.

CONCLUDING REMARKS

Children's lack of attainment is often seen as a need for 'gap filling' resulting in an explicit teaching of strategies and techniques but it is questionable how successful this can be. The examination of learning from the two perspectives begins to make it clear why this might not always be successful. From a Piagetian perspective systematic errors or misconceptions are seen as part of individual cognitive schemes, hence as 'gaps'. From this examination of the procept, there is a sense that the child is pathologised as unable to work in the dual nature. This does not account for the forms of behaviour and activity of an individual child and their identity in learning mathematics. Bishop (2002) has proposed that "there is only so much that individual learners can do about their situation" (p.193). If we see being mathematical as being able to think/speak mathematics then this may give a different perspective.

> Thus the child is not expected to arrive at the objective reality of the structures of mathematics by herself or himself, pulling herself or himself up by the bootstraps of reflective abstraction and being pathologised if she or he cannot manage to arrive at those structures (Lerman, 2001, p.107).

If mathematics is seen as a social engagement then it may be worth engaging children, and particularly lower attaining children, in meaningful mathematical tasks that can enable the integration between spontaneous and scientific concepts in discursive situations.

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CLASSROOM INTERACTION IN GRADE 5 AND 6 MATHEMATICS CLASSROOMS IN TWO BASIC SCHOOLS IN ZAMBIA

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This article reports on a study of classroom interaction aiming at a student-centred approach in two basic schools in Zambia. The study analysed about 40 lessons in two schools. The results show that these lessons included open questions by teachers and conversations among students, both of which indicate a student-centred approach. On the other hand, there was the fact that the teacher failed to handle students' responses when they were stuck and that students were not able to response as the teacher intended for deeper mathematical thinking. The study, therefore, showed the possibility of a student-centred approach in Zambia and the challenges that remain in practice.

Problem statement and objective

In the context of mathematics education in developing countries, descriptions of process have not been as sufficient as research on educational inputs and outputs (cf: UNESCO, 2004). Processes include teaching and learning in class. In fact, only a small amount of research on teaching and learning in mathematics class has been conducted in developing countries, especially Sub-Saharan African regions. Sawamura (2008) used the metaphor of a 'black box' to express the unveiled realities of classrooms in Sub-Saharan Africa.

Zambia is one Sub-Saharan African country that has been tackling various issues to improve the quality of education since the 1990s. However, students' mathematics performance has been seriously low: the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) showed that Zambian Grade 6 pupils were ranked at the lowest position among 15 regions in 1997 (Hungi, et al., 2010). The national assessment in Zambia (The Examination Council of Zambia, 2008) has also reported that the majority of students do not reach the desirable level of learning outcomes stated in the national syllabus.

There are many factors behind this low level of achievement, including, for instance, choice of medium of instruction, lack of teaching and learning materials, poor facilities and management in schools and so on. One reason could be the teacher-centred approach, where the teacher allows students only to listen to a long lecture, chorus responses of 'Yes' and 'No', and repeat and write long notes written on the board. Some researchers have pointed out the negative aspects of this approach in Sub-Saharan African countries (e.g. Moloi et al., 2008; Ackers and Hardman, 2010). Therefore, the aims of this study were to change the type of teaching and learning from teacher-centred to student-centred, as emphasized in the syllabus in Zambia (Curriculum Development Centre, 2003), and to improve mathematics teaching and

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learning. Focusing on the first aim, the paper particularly sheds light on interaction in the classroom, because active interaction must occur in student-centred mathematics classroom, and would also be a measurable indicator of quality lessons.

Previous studies on mathematics lesson analysis in developing countries

There has been little research on lesson analysis in mathematics in developing countries (e.g. the case of Bangladesh mentioned in Baba and Nakamura, 2005; the case of Nigeria in Hardman et al., 2008; and the case of Zambia in Ikeya, 2009). The quantitative analysis by Hardman et al. (2008) revealed the teacher's one sided explanations, rote learning, repetitions and chorus in the classroom with little interaction. Other writers have also pointed out the same features of African lessons (see Claghorn et al., 1989; Bunyi, 1997; and O-saki and Agu, 2002).

Ikeya (2009) quantitatively analysed 8 mathematics lessons for grade 8 by 4 local teachers in Zambia. He classified different categories of interaction of teacher and students, applying categories used by Baba and Nakamura (2005).

Tables 1 and 2 show the results in Ikeya (2009). Table 1 shows that the highest percentage of teachers' interaction was 'closed question (CQ)', while 'open question (OQ)' was 0.0 % in every lesson. The code at the highest percentage of students' responses in Table 2 was 'answer to teacher's question (Ans-T)'. Moreover, he mentioned that students answered with one word or number toward the teacher's closed question, which was the most frequent responses in all the lessons (Ikeya, 2009).

	Teacher A		Teacher B		Teacher C		Teacher D	
S/N of lessons	1	2	3	4	5	6	7	8
Explanation(Xpl)	4.8	3.5	6.0	13.5	3.6	10.6	9.3	25.3
Closed question(CQ)	36.0	50.3	50.0	21.4	65.6	36.6	40.0	33.8
Open question(OQ)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Agreement(Agr)	1.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Confirmation(Cmf)	23.2	17.9	15.8	25.4	16.8	30.5	30.6	19.7
Instruction(Inst)	12.9	12.9	14.6	13.5	11.4	13.7	6.6	9.8
Justification(Jst)	3.2	0.0	2.4	2.7	0.0	0.7	0.0	0.0
Others(Oth)	17.6	15.1	10.9	22.7	2.4	7.6	13.3	11.2

Table 1: Percentage of teacher's interactions in each lesson (%)

	Teacher A		Teacher B		Teacher C		Teacher D	
S/N of lessons	1	2	3	4	5	6	7	8
Answer to student(Ans-S)	0.0	0.0	1.3	0.0	7.5	0.0	0.0	0.0
Anser to teacher(Ans-T)	97.0	89.7	88.8	100	77.3	89.6	89.0	83.7
No answer/confusion(Na)	1.9	10.2	5.5	0.0	6.3	9.2	11.0	16.2
Question(Qst)	0.9	0.0	1.3	0.0	8.7	1.0	0.0	0.0
Opinion(Op)	0.0	0.0	1.3	0.0	0.0	0.0	0.0	0.0
Others(Oth)	0.0	0.0	1.3	0.0	0.0	0.0	0.0	0.0

(Both were referred from Ikeya, 2009)

Table 2: Percentage of student's interactions in each lesson (%)

This paper employs the above categories of interaction in order to compare the results of lesson analysis. Ikeya's result seems to reveal the typical characteristics of Zambian mathematics classrooms, which can be expressed as 'chalk and talk'; therefore, the comparison between Ikeya's data and the data in this study will show some differences if our activities in class made any changes from the traditional approach of teaching.

Methodology and data analysis

Twenty three-time and twenty four-time mathematics lessons[1] were conducted respectively in two basic schools in Lusaka and Central Province from January 2009 to June 2009. A grade 6 class in L basic school in Lusaka and a grade 5 class in M basic school in Kabwe were selected in consultation with educational stakeholders. Both schools were governmental and accommodated more than 1,500 students from grade 1 to grade 9. M basic school's learning performance was better than L basic school according to the passing rate of the national examination, but not with a huge gap.

Two teachers, introduced by the district educational officers and head teachers, were selected to conduct the lessons. Both of them had more than five-years' teaching experience. Plan-Do-See-Improvement/Reflect cycles were repeated in the whole process of conducting lessons. In planning, the teacher and the author discussed the objective and content of the lessons. In implementation ('Do' and 'See'), while teachers taught in class, the author observed the lesson behind, recording class. In Improving/reflecting sessions, teachers reflected his/her teaching and learning in class, saying successful part and challenges. We, then, discussed the further improvement and some modifications of contents according to students' difficulties, which was connected to the next planning session.

The contents of the lessons were addition, subtraction, multiplication, and division, which must be taught from Grade 1 to Grade 7, together with mathematical higher-order thinking skills, which were mentioned in the syllabus, but are difficult to achieve in the classroom.

The objective of the lessons was that students should be able to calculate correctly and to communicate, discover and reason in given activities. We made use of the same materials, supported by the theory of Substantial Learning Environment (Wittmann, 1995; 2005) (Mentioned in detail in Nakawa, 2010). Considering the different grades and abilities in the two classes, however, we employed different number sizes from 2 digits to 4 digits. Fig. 1 shows an example of questions used in the lessons.

Calculate number bricks and discuss some patterns that you can observe.

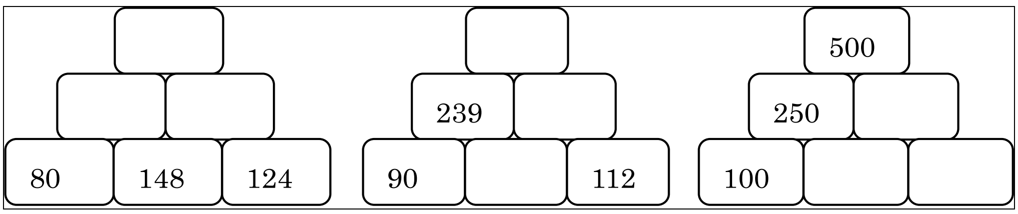

Fig 1: A Task given for addition and subtraction with number pattern in the 6th lesson in Kabwe

In the data analysis, we first applied Ikeya's interaction categories to each conversation or behavior in the lesson protocols that we produced after all the lessons were video-recorded. We added more categories more suitable for the varieties of the conversations we saw in the protocols. We set up new categories for non-verbal activities, such as students' gestures and writing in answering the teacher's questions. Moreover, we added to each category the direction of the interaction, which Ikeya did not classify, in order to identify teacher and students' interactions more analytically. As a result, Table 2 shows our modified categories and codes. The transcript below is an example of the coding in finding number patterns after calculation for Fig 1 in the 6th lesson.

354	T:	CQ Po Inst	OK, who has seen a number pattern here the way we are doing here? Who has seen a number pattern here? Number 1, 2, 3. Who has seen a number pattern here? Nothing? Look at the number as on top in the middle, at the bottom. Who has seen a pattern? Olivia? (Giving chalk to Olivia) Who has seen another one? Time is not with us. Let us what. Ah, write there. The number pattern is…Just write the pattern there.
355	S1:	Wri-T	(Writing on the board)
356	T:	CQ Po	Oh, finish. Ah? What pattern have you seen? 80, 90, what is next? Who can help? Hemfley?
357	S2:	Num-T	100
358	S1:	Wri-T	(Writing 100)

In the analysis, we placed an appropriate code for each interaction and non-verbal activities except for content unrelated to mathematics; for instance, students' counseling and guidance or jokes, which mainly appeared at the beginning and the end of lessons. Also, in the exercise time, which was usually provided for the last 5 to 10 minutes in a lesson, interaction of the teacher's individual guidance was not well captured since the voice level was so low in the video. We therefore decided to remove them from the coding.

Teacher			
Category	**Direction**	**Code**	**Example**
Explanation		Xpl	'What we are going to do is...', 'This rule means'
Closed question		CQ	'What is the answer?', 'What is 7 times 2?'
Open question		OQ	'How did you do this?', 'Why do you think so?', 'Explain what you have done'
Question for agreement		Agr	'Are we together?'
Pointing to student		Po	-
Confirmation		Cmf	'Is s/he right?', 'Is this answer okay?', 'Any question?'
Instruction		Inst	'Write…', 'Tell me the answer', 'Speak', 'Come' 'Find the answer'
Compliment and encouragement		Enc	'Come on, come on', 'You can do it'
Disagreement and justification toward student's answer		Jst	'You have tired', 'This is wrong'
Clapping or singing a song		Cl	'Clap for him/her' 'Let us sing'
Impossible to catch		Imp	-
Others		Oth	-
Student			
Category	**Direction**	**Code**	**Example**
Simple response 1	T	Yn-T	'Yes/No'
	S	Yn-S	
Simple response 2	T	Num-T	'2 x 9', '18', 'Brick', 'Correct'
	S	Num-S	
Question	T	Qst-T	'What are you doing?', 'What is your answer?'
	S	Qst-S	
Opinion	T	Op-T	'That is a wrong answer', 'The number pattern is….' 'The rule is…'
	S	Op-S	
Imcomplete answer		Inc	'Subtr…'
Repetition or reading		Rd	Reading number or sentences
Silence and confusion	T	Na-T	-
	S	Na-S	
Pointing to student		Po	-
Clapping		Cl	-
Writing or gesture without speaking	T	Wri-T	Writing numbers, words or sentences or pointing numbers or places on the board
	S	Wri-S	
Impossible to catch		Imp	-
Others		Oth	-

Table 2: Code of classifications applied in lesson analysis and examples of interaction

Result and discussions

Tables 3, 4 and 5 show the average percentages of teacher's codes and students' codes per lesson obtained in the analysis.

Code	Agr	Cl	Cmf	CQ	Enc	Imp	Inst	Jst	OQ	Oth	Po	Xpl	Total
Kabwe	4.72	4.69	8.00	23.52	0.97	0.09	25.20	6.09	3.80	3.35	12.69	6.88	100
Lusaka	6.93	1.80	15.86	22.88	3.45	0.35	16.19	3.42	5.93	3.10	5.40	14.71	100

Table 3: Average percentage of teacher's interaction per lesson(%)

Code	Cl	Imp	Inc	Na-S	Na-T	Num-S	Num-T	Op-S	Op-T	Oth
Kabwe	5.11	0.86	1.91	0.03	2.08	12.90	26.13	0.33	2.67	1.35
Lusaka	2.15	0.81	0.44	0.04	2.44	6.59	35.53	0.56	4.29	1.33

Table 4: Average percentage of students' interaction per lesson (1) (%)

Code	PO	Qst-S	Qst-T	Rd	Wri-T	Writ-S	Yn-S	Yn-T	Total
Kabwe	2.24	12.31	0.10	6.99	14.68	0.00	0.00	10.29	100
Lusaka	0.04	5.55	0.33	2.59	4.63	0.00	0.00	32.68	100

Table 5: Average percentage of students' interaction per lesson (2) (%)

In comparison with Ikeya (2009)'s results, the findings show a similar tendency: the percentages of teachers' 'closed question (CQ)' (23.52% in Lusaka and 22.88% in Kabwe) and of 'students answering in a word or number to teacher (Num-T)' (26.13% in Lusaka and 35.53% in Kabwe) were higher than the other categories. Also, followed by CQ, 'instruction (Inst)' (25.20% in Lusaka and 16.19% in Kabwe), 'explanation (Xpl)' (14.71% in Kabwe) and 'confirmation (Cmf)' (15.86% in Kabwe) were more frequent categories in both studies. That was to say, mathematical interaction of asking a closed question and of answering with a simplest form were the typical style of lessons in the two classes. This was the same as the findings seen in other Sub-Saharan regions (Pontefract and Hardman, 2005; Hardman et al., 2008; Moloi et al., 2008)

Characteristics

'Open question (OQ)' in the two schools occurred at 5.93% in Lusaka, and 3.80% in Kabwe, while 'open question' did not exist at all in Ikeya's data. Thus, the teacher asked open questions 9.57 times per lesson in Lusaka, and 5.21 times in Kabwe, when we convert the percentage into frequency. 'OQ' has the possibility of promoting students' mathematical thinking. In fact, the two teachers asked questions which deepened the students' thinking, for instance saying 'What is the pattern here?', 'What do you think?', 'Why do you think so?', and 'Tell us the reason why?' Thus, teachers showed some efforts to pose questions with more than one possible answer which stimulated students' mathematical thinking. These questions are rarely utilised in Zambian mathematics classrooms.

There was also little interaction among students in Tables 1 and 2 from Ikeya's study, while it was observed in the two classes in this study, for example, 'Num-S' (12.9% in Lusaka, 6.59% in Kabwe) shown in Table 4. Table 2 shows 0.0% on 'Ans-S' except for 2 lessons. Therefore, we could say that the pattern of interaction in class was different from Ikeya (2009)'s results.

Analysis of characteristics

We then investigated (i) students' responses to open questions and teachers' instructions following students' interactions; and (ii) conversations among students. As for (i), there were three patterns of students' responses: (1) Students did not answer at all and kept silent; (2) They answered with numbers, or in a word, which we included in ‘Num-T’; and (3) They answered in sentence form, saying ‘Because…’ which was shown in ‘Op-T’ (2.67% in Lusaka and 4.29% in Kabwe). Among these three responses, the first and second ones were commonly observed, while the third one, which corresponded appropriately with the teacher’s question form, was observed much less frequently. For example, a student explained the ways of addition for a question, saying ‘Here, 50 and 40, and then 90...90, and 80, 170’. Toward the first and second students’ responses, the two teachers just repeated the same questions or changed the open questions to closed questions, thus they did not change the type of questions.

As for (ii), it was typical that students just asked one simple calculation question standing in front of the board and another individual student answered in a simple way, such that one asked ‘15+7=?, Paul’, and Paul responded ‘22’. Thus, the pattern of communication among students was mimicry of teacher and students’ interaction.

Consequently, the findings indicate that interaction which related to a student-centred approach did exist in the protocol. It means that teachers attempted to create the culture of creating students’ mathematical discussions in class, and that students were learning with others. However, even in these codes, interaction was not inclined to a student-centred approach, because the teachers failed to treat students’ answers as s/he intended, and also students did not answer teacher’s open questions, or they mimicked the chalk and talk approach.

Conclusion

The patterns of interaction of teachers and students were, to some extent, different from the previous research, which was the very first step to take for the improvement of mathematics lessons. On the other hand, qualitative aspects of the interaction, related to some codes such as OQ, Num-S and Qust-S, were still far from a student-centred approach. For the next step, it would be crucial for teachers to consider how quality in each interaction can be deepened in teaching. This paper deliberately mentioned neither the assessment of students’ learning nor the process of lessons; they are mentioned in a different paper (cf: Nakawa, 2010).

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CRAFT KNOWLEDGE IN MATHEMATICS TEACHING

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This paper focuses attention on approaches to teaching which impede teachers' development of craft knowledge through their engagement with students. Using video recordings of mathematics lessons and follow-up conversation with the teachers, two episodes are considered. In the first episode the teacher used an approach which guided the student tightly, overlooking the theories she had learned the previous year, and in the second episode she appears to close down opportunities for discussion of the unanticipated situations that arose. It is argued that both approaches denied the teachers opportunities to learn from the situation and thus to develop craft knowledge.

INTRODUCTION AND BACKGROUND

The study from which this paper reports aims to expose knowledge mathematics teachers deploy while teaching the subject at the level of lower secondary school, particularly focused on the knowledge teachers develop through practice. It is not expected that teacher education courses or teacher training as part of that education can prepare prospective teachers for all complex situations that will probably arise in their teaching practice. An assumption in this paper is that teachers, in the course of their entire career, need to develop new knowledge about, and insight into, the teaching of mathematics. Thus, over years teachers develop what Shulman (1987) describes as the "wisdom of practice", also referred to as craft knowledge (Leinhardt, 1990; Ruthven, 2002).

The purpose of this paper is to draw attention to teachers' opportunities to develop craft knowledge by considering two approaches to mathematics teaching; one episode where the teacher deliberately sets aside scholarly advice, and the second where she exercises tight control over the situation and neglects students' unanticipated responses. However, by overlooking the students' responses she denies herself the opportunity to deal with challenges from which she might have learned. Thus, whereas there is a growing body of evidence of teachers learning through practice (e.g. Leikin & Zazkis, 2010), this paper sets out to demonstrate how learning might be hindered.

This report emerges from my current PhD-work. I engage in this research process with thirty years of experience as a mathematics teacher with the purpose of exposing the "abundance of craft knowledge" (Barth, 2004, p. 59) teachers accrue during a lifelong career while engaging with students. Researchers claim this craft knowledge to be important for informing teaching, teacher education and the scholarly area of mathematics education (Barth, 2004; Leinhardt, 1990; Ruthven, 2002; Shulman, 1987). I want to contribute to the research agenda on this particular theme which

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is ”developed as a product of professional action” and “establishes itself through work and performance in the profession” (Bromme & Tillema, 1995, p. 262).

FOUNDATION FOR THE CONCEPTUAL FRAMEWORK

The international interest in teachers’ knowledge for teaching mathematics has accumulated over several decades. My study is based on a conceptual framework founded on theoretical constructs developed by four scholars/groups that have engaged in research within this area; Shulman (e.g. 1987), Ball, Thames and Phelps (e.g. 2008), Jaworski (1994), and Rowland, Huckstep and Thwaites (e.g. 2005).

Shulman (1987) synthesized knowledge for teaching into a knowledge base that includes as a minimum seven categories of knowledge underlying “teacher understanding needed to promote comprehension among students” (ibid., p. 8); *content knowledge, pedagogical content knowledge, pedagogical knowledge, curriculum knowledge, knowledge of learners, knowledge of educational contexts*, and *knowledge of educational ends*. Of particular interest for the present study are content knowledge and pedagogical content knowledge. The latter is an amalgam of content and pedagogy, a special form of professional understanding that is unique to teaching. Ball and her colleagues (2008) identified mathematical knowledge for teaching through investigating the nature of professionally oriented subject matter knowledge in mathematics. This resulted in refinement of Shulman’s content knowledge and pedagogical content knowledge. Content knowledge is dissected into *common* and *specialized content knowledge*, whereas pedagogical content knowledge is dissected into *knowledge of content and students*, *knowledge of content and teaching*, and *horizon knowledge*. Specialized content knowledge is mathematical knowledge and skill unique to teaching, while common content knowledge is mathematical knowledge and skill also used in other settings than teaching. Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum. By researching into teachers’ engagement with students working on open-ended and problem-solving tasks, Jaworski (1994) identified general characteristics of investigative mathematics teaching from which she deduced the *teaching triad* . The teaching triad is a theoretical construct which links the generalized characteristics to three “domains” of activity in which teachers engage; *management of learning*, *sensitivity to students,* and *mathematical challenge.* Management of learning describes teachers’ role in the constitution of the learning environment. The construct sensitivity to students is about teachers’ knowledge of students and attention to their needs, and is divided into an affective and a cognitive part. The challenges teachers offer to students to engender learning Jaworski refers to as mathematical challenge. Jaworski uses this construct as an attempt to provide a framework for capturing essential elements of the complexity of teaching. Rowland and colleagues’ (2005) knowledge quartet, which to some extent is also derived from Shulman’s (1987) knowledge base for teaching, includes the dimensions *foundation, transformation, connection*, and *contingency*. Foundation is the theoretical background teachers bring into the classroom while transformation is about capacity to transform this knowledge into forms that facilitate students' learning. Connection is about the coherence of planning or teaching displayed across an episode, lesson or series of lessons, and contingency concerns teachers' readiness to respond to students' unanticipated inputs. These categories evolved from locating ways in which pre-service teachers drew on their knowledge of mathematics and mathematics pedagogy in their teaching.

The frameworks briefly outlined above are chosen and synthesized into a conceptual framework for the purpose of enabling me to adopt different perspectives as I focus on the complexity of teaching mathematics. Theories developed by Shulman (1987) and Ball and colleagues (2008) are included so that I can consider the mathematics and how it is communicated to the students. Jaworski's (1994) teaching triad enables me to focus on the didactical relationships between teacher, subject content and the students. Finally Rowland's (2005) knowledge quartet is about categories of actions taken by teachers in the classroom, and thus completes the picture on which I intend to focus.

Research question

The focus in this report is on how teachers enable or impede for opportunities to develop craft knowledge in teaching mathematics, because, I assume, this will enable me to expose one possible way in which teachers learn and develop craft knowledge within and from their practice. Craft knowledge, or in other words "the wisdom of practice itself, the maxims that guide (or provide reflective rationalization for) the practices of able teachers" (Shulman, 1987, p. 11) is a contextualized knowledge encompassing "the wealth of teaching information that very skilled practitioners have about their own practice" (Leinhardt, 1990, p. 18). I claim that one way in which teachers develop craft knowledge in their classroom practice is through reflecting on current experience in the context of existing knowledge, which is an amalgam of scholarly knowledge, experiences and accumulated craft knowledge. The resulting competencies and skills are the substance of the teacher knowledge deployed by teachers as they simultaneously manage the student, the class, and the subject to achieve the optimum outcome. The research question focused here is:

What actions do teachers take in their engagement with students that enable or impede opportunities for their development of craft knowledge in teaching mathematics?

METHOD

The main goal of the study is to expose the knowledge which informs lower secondary mathematics teachers' practice. The data for the study has been gathered by observing experienced teachers in their classrooms. To stand any chance of exposing the knowledge utilized by experienced mathematics teachers, who will often apply their knowledge in routine situations as, it seems, automatic reactions that are below the level of conscious reflection, it is necessary to observe them in regular classroom situations. Following the lessons, teachers were engaged in reflective conversation. In these conversations the teachers were requested to recall events and explain their actions. It follows that it was necessary to use methods that recorded teachers' actions in the classroom in such a way that recall of the knowledge informing those actions could be elicited in the later conversation. I believe that teachers' recall is most likely to be stimulated by video recordings of events that took place in their lessons, which *capture the moment*, thus all lessons were videotaped. The teacher and researcher then viewed the recordings together and talked shortly after the lesson had finished, that is while the lesson was still fresh in the teacher's mind. At any time when either teacher or researcher observed an event that provoked comments, the video was stopped to allow a discussion of the episode. All conversations were also videotaped.

For this report, data are chosen from the classroom of one teacher 'T'. This included lessons of mathematics in a grade ten group of low achieving students (age 15 - 16), and the teacher's subsequent comments on that teaching.

To get the opportunity to observe teaching is a privilege for a researcher. Being given the chance to have the teacher's subsequent comments is an even greater privilege as it requires the teacher to give time from a busy schedule. It thus requires humility and respect on the part of the researcher for the participating teachers, the classes, the data, and how the data are reported. The two episodes have been chosen with the purpose of drawing attention to teaching approaches that impede opportunities to develop craft knowledge, thus experiencing how unproductive approaches occur. It is not the intention here, or in my research in general, to evaluate teachers or teaching. The purpose is to develop categories of teaching approaches that may be productive or unproductive of the development of craft knowledge.

The unit of analysis for the study is the didactical situation, i.e. the interaction between teacher, students and the mathematical challenges the teacher provides the students, focusing on the teacher's knowledge and actions. The basic data collection is about looking at the work of several teachers in their classrooms. This presentation is based upon episodes from one of the observed teachers; T. T is an experienced teacher as she had been teaching primary grades for eight years at the time of the observation. The year before the data collection she took an upgrading course in mathematics, and was in her first year as teacher in lower secondary school at the time of the observation, i.e. she was a novice in teaching mathematics at lower secondary level.

IMPEDING OPPORTUNITIES FOR DEVELOPING CRAFT KNOWLEDGE IN TEACHING MATHEMATICS

Two approaches are illustrated, each by one example. The first approach is about the teacher (T) who guides her student by almost doing all the mathematical work for her student despite the fact that she had learned about the importance of students inquiring into mathematics for themselves. In the second event T appears to ignore students' incorrect answers by repeating the same question until she elicits the correct answer, and thus impedes her opportunity to learn from the situation.

The following event comes from a lesson that focuses on directed number. At the beginning of the lesson, the students were provided with a number line within the range of negative 20 to positive 20 and two dice to be used in a game for engaging them within a realistic and meaningful context that modelled directed number. The session started by T's thorough introduction of the game; the students were to move a counter along the line where the displacements were decided by a regular die (number die) showing the length of the displacement and an operation die (three faces showing plus and three showing minus) telling whether to add to or subtract from the existing number.

After playing this game for about 20 minutes, T introduced operation on negative numbers in the game. She then talked about operations with both positive and negative numbers for about nine minutes before she introduced a sign die (similar to the operation die) which was to determine the sign of the number, and a second number die. T also simulated two turns of the game before she had the students play in turn. In the first round the students were to place their counter on the board, and in the episode described below T engaged with the first student in finding where to put her counter:

T: Just throw that die first (indicates the first number die)

S1: (Throws the number die, it shows 4)

T: Great, you have four, and then you are going to find out whether you are going to add or to subtract

S1: (Throws the operation die which shows +)

T: You are going to add, and then you have to find the sign of the next number

S1: (Throws the sign die, shows -)

T: And that is minus, and then we have (give the second number die to S1)

S1 (Throws the number die which shows five)

T: Yes, then we simply have the task four plus minus five

S1: Minus one

T: Four minus five is minus one, as you said

T was telling S1 to throw (lines: 1, 4, 6), to add (6), and to find the sign (6). She also told the student what the task was (9, 10), thus leaving only the throwing and the answer to S1. This routine was repeated for all the students. In the follow-up conversation T explained that she did this due to her belief about the students' lack of content knowledge and their subsequent reaction; they would start to "guess wildly" and "fling numbers in all directions" if she allowed them to explore by themselves:

T: What I see is that I do not offer them much space for exploration, because last year I learned a lot about doing that, but it is, I feel that they to some extent lack the qualification to (T interrupts herself), and then they start to guess wildly, and it happens very fast that they start to fling numbers in all directions, that is, they are not even able to think

The previous year T had learned about the importance of having students explore themselves (15). But in her teaching she encountered a situation where there was a contradiction between what she learned in teacher education and pupils' response in the class. To achieve what Brown and McIntyre (1993) refers to as a "normal desirable state of student activity" she decided not to use what she had recently learned. Her belief system made her refrain from implementing the theories, thus also denying herself the challenge to learn from the situation.

In the second event, the students were to place heights and depths of several Norwegian mountains and fjords respectively. The students were provided a second number line, that one raging from negative 1000 to positive 1800. When they were to place the mountain Galdhøpiggen, which is 2469 meters above sea level, they realised that they had to extend the number line. T showed the line, and asked the students:

T: What is the end of this? It stops at eighteen hundred and here we have seventeen hundred, didn't we? And then I continue drawing an equally long bit, and then I come to
S2: Nineteen hundred
T: And then another bit at the same length?
S2: Twenty thousand
S3: Twenty hundred
T: You can say that, but what number is it? One thousand nine hundred plus hundred
S2: Twenty
S3: Two thousand
T: Two thousand, yes. If I put on another bit about equally long, how far have I reached now?
S2: Three hundred
S3: Twenty hundred
T: How far have I reached now? Two thousand plus hundred
S2: Twenty one
S3: Twenty one hundred
T: Yes, but we can say it another way
S1: Two thousand one hundred
T: Yes, two thousand one hundred

The excerpt shows a teacher in control of the mathematics. Concerning students' answers, however, she responded only to correct (12, 21) or partly correct suggestions (8, 19). Avoiding commenting on unanticipated suggestions (6, 10, 14, 15, and 17) could be a kind of exploring strategy; she maybe wanted the students rethink. And repeating the question until the correct answer is achieved by teachers is not uncommon. However, it is possible that such a strategy encourages the very guessing the teacher wants to avoid (see previous episode), and thus provides a contradiction to her strategy of discouraging guessing. But S2 and S3 seemed confused about the values and numbering when going from 1900 to 2000 and further, and meeting their uncertainty by discussing numbers and the way they are spoken could have resolved their problem. This might be challenging, but by ignoring them T missed opportunities for learning from the situation. In the follow-up conversation T expressed that she was very content about the way she solved the challenge about extending the line; “it was actually not planned, but it was really a boost for the entire program”. She did not at that time reflect upon what could have been done to resolve the students’ apparent confusion about reading these numbers. However, some days after our conversation T came to see me telling that she had been reflecting on this approach and what problems students could encounter when reading the result of adding one hundred to nineteen hundred. She had not earlier thought of this as a possible problem, but from now on, being aware of it, she claimed that she would bring her new achieved knowledge to future lessons.

CONCLUDING REMARKS

It might be argued that the closed teaching of T was orchestrated by her knowledge of these students as low achievers. I am provided with anecdotal evidence that low achievers might be taught in such a tightly closed approach. However, this presentation is not about how low achieving students are taught. The episodes are selected only to be used as illustrations of situations that impede the opportunity for the teacher to learn from the situation, and could well have occurred in a regular class.

T is an experienced teacher; this is reflected in her planning of the lessons for this particular group, exercising management of learning and sensitivity to students (Jaworski, 1994) by her choice of approach to the topic. Her thorough and thoughtful preparation in advance, her use of practical recourses, and her preparation of the students before starting the game also indicate knowledge of content and students as well as content and teaching (Ball et al., 2008). Her reaction when she realized that there were missing numbers on the number line also demonstrate her capacity to perform contingent actions (Rowland et al., 2005) when necessary. Added up, these actions and reactions show a broad pedagogical repertoire. However, during the lesson she chose to keep close control over the activity in the classroom. The first episode shows T guiding S1 in one round of the game, keeping tight control of S1’s operations. While looking at the video in the follow-up conversation she observed that she did not let the students explore for themselves which she had learned that they should. Her knowledge of the students made her decide to avoid the theoretical recommendations she got from the course. And when unanticipated mathematical situations arose in the second episode, she chose to ignore the situation and stay within a “safe zone”, thus denying herself the opportunity to develop craft knowledge for extending her “kit of tools” (Arzarello & Bartolini Bussi, 1998 in Ruthven, 2002, p. 13).

I am seeking to expose the "abundance of craft knowledge" (Barth, 2004, p. 19) teachers develop through their career in the classroom. The examples presented in this paper may demonstrate that it is difficult for a novice in mathematics teaching to use unanticipated and contingent situations to develop craft knowledge. The knowledge T built on when deciding not to implement recommendations about having students explore for themselves is craft knowledge. However, by connecting the knowledge she has about the students and the theories she learned the previous year, she encountered a double bind without having information about her possibilities to resolve it. Her own sense of agency, inquiry and critical engagement denied her to develop her craft knowledge. Being given more time, however, and probably by having had the opportunity to look at, and reflect on, her own teaching, she apparently developed the kind of craft knowledge that we seek to inform future teachers.

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LEARNING TO SOLVE COMPLEX PERCENTAGE CHANGE TASKS

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The worked example effect was examined from a cognitive load perspective. This study hypothesized that the unitary-pictorial approach would consume less mental effort than either the unitary or equation approach. Fifty one year 9 students were randomly assigned to one of these approaches in a regular classroom setting. Each group was exposed to identical acquisition problems but structured in the respective worked examples. The unitary-pictorial approach outperformed the equation approach, but it had marginal advantage over the unitary method. The differential learning effects arises from varying degree of element interactivity across the approaches.

INTRODUCTION

The use of worked examples to enhance learning is not new (Atkinson, Derry, Renkl, & Wortham, 2000; Rittle-Johnson, Star, & Durkin, 2009; Retnowati, Ayres, & Sweller, 2010). Early research on worked examples focuses on a split-attention effect in which the integration of multiple sources of information is essential to reduce cognitive load (Sweller, Chandler, Tierney, & Cooper, 1990; Chandler & Sweller, 1992); the redundancy effect where the inclusion of unnecessary information may overload working memory (Chandler & Sweller, 1991; Sweller & Chandler, 1994); the modality effect that promotes the combination of the auditory and visual mode to optimize the use of cognitive resources (Mousavi, Low, & Sweller, 1995; Tindall-Ford, Chandler, & Sweller, 1997). Recent research begins to explore an expertise reversal effect (Kalyuga, Ayres, Chandler, & Sweller, 2003; Kalyuga, 2007; Kalyuga & Renkl, 2010) where the instructional impact will decrease as novices gradually gain expertise in a domain. While there is a consensus among researchers concerning the use of worked examples for the novices to acquire schema; however, they are less certain about what constitute effective worked examples.

COGNITIVE LOAD THEORY

Schema acquisition is central to cognitive load theory (Sweller & van Merrienboer, 1998; van Merrienboer & Sweller, 2005). An extraneous load arises from inappropriate instruction that overloads working memory and thus will not be helpful for schema acquisition. The amount of intrinsic load associated with a task varies according to the degree of element interactivity within the task. A possible way to address intrinsic load is to break down the learning task in a hierarchical order to enable especially novices to acquire skills (Gerjets, Scheiter, & Catrambone, 2004). Germane load is helpful for schema acquisition because it encourages the learners to invest cognitive resources to learn relevant aspects of the instruction. Hence, an

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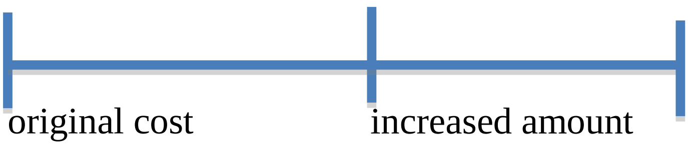
instructional designer should endeavour to design instruction, which minimizes both the extraneous and intrinsic loads but promotes germane load.

CURRENT STUDY

We compared three types of worked examples (unitary, unitary-pictorial, equation) to facilitate learning how to solve complex percentage change tasks. The unitary method is the most popular approach found in mathematics textbooks (e.g., Kalra & Stamell, 2005). Both the unitary-pictorial approach and equation approach are neither found in the mathematics textbooks nor being used in mathematics classroom in Australia. A complex percentage increase task such as, *'After a 12% markup, the shoes now cost $34. How much did they originally cost?'* (Parker & Leinhardt, 1995, p.448), poses a great challenge to the learners. Students tend to draw on their prior knowledge of percentage quantity to calculate $34 x 12% (which is incorrect), and they rely on intuition to subtract $34 (cost after 12% markup) from $34 x 12% as they perceive that the original cost should be less than $34. Clearly, there is a difficulty in decoding what is required. Students' error can be a source of excellent learning experiences if sufficient scaffolding is provided by the teacher to help students visualize the task through diagrams, models and concrete aids. The next section will describe how to solve this task via equation, unitary and unitary-equation approaches.

Equation approach



Let x be the original cost

$\$34 = x + x * 12\%$

$\$34 = x(1+12\%)$

$x = \$34 \div (1+12\%)$

Note: students can use a calculator to find $34 ÷ (1+12%)

The horizontal line is expected to facilitate the construction of a mental representation based on learners' addition intuition – original cost + increased amount. Step 1 has one element and it deals with a variable (x). Step 2 has two elements such as x and 12% and it deals with: (1) the product of x and 12% is equivalent to the increased amount, (2) the sum of x and x * 12% is equivalent to the marked up price of $34. Step 3 has two elements, x and (1+12%), and the learners need know that the multiplicative relation between x and (1+12%) is the result of factorizing step 2. Step 4 has two elements, $34 and (1+12%). The transformation of step 3 to step 4 (i.e., $34 is divided by (1+12%)) involves knowledge of solving an equation with a fraction. Thus, processing multiple elements within and across the steps simultaneously in working memory would give rise to high element interactivity. The interaction between intrinsic load and a learner's expertise (Kalyuga et al., 2003) necessitates the learners to have appropriate prior knowledge (e.g., algebraic skills, factorization, equation with fractions) before they can benefit from the equation approach. It was predicted that participants in this study lacked necessary algebra skills. This coupled with the high element interactivity within the equation approach would result in a high intrinsic load impairing schema acquisition.

Unitary method

Step 1:	markup of 12%	100% + 12% = 112%
Step 2:	calculate 1% (markup price)	\$34 ÷112 = \$0.3035
Step 3:	calculate 100% (markup price)	\$0.3035 x 100 = \$30.35

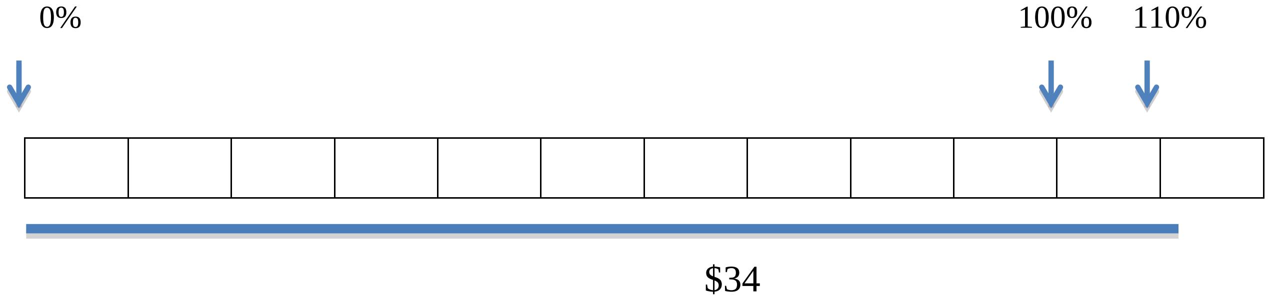
Answer: The original cost is \$30.35

Note: students can use a calculator to verify the answer

This unitary approach calculates 1% of the given quantity and then a multiple of this to solve the task. Step 1 has two elements, 100% and 12%, and the learner needs to learn: (1) 12% is an increased amount, (2) that 112% represents more than the original amount. Step 2 has two elements such as \$34 and 112; and the learner needs to learn: (1) \$34 is the price after a 12% markup, (2) 112 refers to 100 percentage plus a markup of 12 percentage, (3) the sub-goal is 1% of the marked up price. Step 3 has two elements, \$0.3035 and 100, and the learner needs to learn the multiplicative relation between \$0.3035 and 100, which is a multiple of the 1% sub-goal in step 2. The element interactivity not only arises from within but also across the solution steps. For example, an understanding of 100% in step 1 is required to make sense of 100% in step 3. Besides, the solutions steps do not specify the relation between \$34 and 112% so as to provide a point of reference for subsequent interpretation of the 1% sub-goal. Therefore, processing the solution procedure may prove to be too cognitively taxing to enable schema acquisition to occur.

Unitary-pictorial approach

The diagram represents the situatio



From the diagram, 112% represents $34

Therefore, 1% of markup price $34 ÷112 = $0.3035

100% of markup price $0.3035 x 100 = $30.35

Answer: The original cost is $30.35

Note: students can use a calculator to verify the answer

The alignment of percentage and quantity in the diagram provides a clue in regard to the relation between these two, and their close proximity also eliminates a split-attention effect (Yeung, Jin, & Sweller, 1998). Moreover, the integration of quantity and percentage acts as a powerful tool to foster a mental representation of the problem situation where an increase of 12% (112%) is connected with the markup price ($34) – step 1. Since the number and type of elements in steps 2 and 3 are the same as those in the unitary approach, we expected the degree of element interactivity between these steps will be the same too. Essentially, the element interactivity in the unitary-pictorial approach would be reduced by the diagram and thus is expected to facilitate schema acquisition better than the unitary approach.

HYPTHESIS

As argued above, we hypothesized that the learning effect upon the complex percentage change tasks would be in the order: unitary-pictorial approach > unitary approach> equation approach.

METHOD

Participants

Fifty one year 9 students (25 girls and 26 boys, mean age = 15 years) were recruited from a private secondary college in Australia to participate in the study. Students had previous experience in using unitary approach for simple percentage change tasks. Ethic clearance was obtained from relevant authorities prior to data collection.

Materials and Procedure

The procedure was consistent with many of the cognitive load studies (e.g., Kalyuga et al., 2003). Group testing was administered to students in a single 40-minute session in three separate classrooms with each group supervised by a researcher and a class teacher. Students were randomly assigned to three groups: 17 in unitary approach, 18 in unitary-pictorial approach, and 16 in the equation approach. Students were told about the tasks involved before the experiment began. This was followed by all students undertook a pre-test (10 minutes). Then, each group proceeded to the acquisition phase in which they studied an instruction sheet (5 minutes) comprising two worked examples. Having done that, each group completed six worked examples pairs - they studied a worked example and solved a problem for each pair (15 minutes). Throughout the acquisition phase, students were allowed to refer to the instruction sheet, and they could seek help. Lastly, all students completed a post-test which had identical content to the pre-test (10 minutes). In short, the three groups were match with the same materials (except structured in three different formats of worked examples) and time to complete a pre-test, an acquisition phase and a post-test.

Coding

We excluded the data of three (two scored 8 or above in pre-test and one did not complete all three phases) out of 51 students in the analysis. One mark was allocated for a correct solution in the test or practice problems. A computational mistake was ignored. The Cronbach's alpha for the pre-test and post-test were .91 and .94 respectively. Two researchers scored 50% of the test problems and the test scores were found to be correlated at above .9.

Results

For the acquisition phase, the mean correction solution on pre-test was 3.93, 4.33, 2.60 for the unitary, unitary-pictorial and equation approaches respectively. The three groups did not differ on correct solutions, $F(2,45) = 1.98$, $MSE = 13.07$, $p = .15$

The mean correction solution on pre-test was 0.07, 0.11, 0.40 for the unitary, unitary-pictorial and equation approaches respectively. One way ANOVA on pre-test indicated no significant difference between the three groups prior to the intervention, $F(2, 45) = 1.57$, $MSE = .50$, $p = .22$. The mean correction solution on post-test was 3.40, 6.61, 2.33 for the unitary, unitary-pictorial and equation approaches respectively.

A 3 (approach: equation, unitary, unitary-pictorial) x 2 (test: pre-test, post-test) ANOVA with approach as a between group factor and test as a within subjects repeated measure showed a significant test effect, $F(1,45) = 43.94$, $MSE = 8.34$, $p = 0.00$, $\eta 2= 0.49$, observed power = 1, indicating improvement from pre-test to post-test across the three groups, a significant group effect, $F(2,45) = 3.97$, $MSE = 9.56$, $p = 0.03$, $\eta 2= 0.15$, observed power = 0.6, and a significant approach and test interaction effect, $F(2,45) = 5.48$, $MSE = 8.34$, $p = .01$, $\eta 2= .20$, observed power = .83, indicating the improvement from pre-test to post-test varied across the three groups. A follow-up Multiple Comparison based on a Tukey Test revealed a significant mean difference between the unitary-pictorial and equation approaches, mean difference = 1.99, SE = .76, $p = .03$. No other significant mean differences were found between the three groups.

The results partially support the hypothesis. The unitary-pictorial group outperformed the equation group, but the difference between unitary-pictorial group and unitary group was not significant though it was in the predicted direction. Figure 1 further affirms the findings. The unitary-pictorial group experienced greater improvement from pre-test to post-test then either the unitary or equation group.

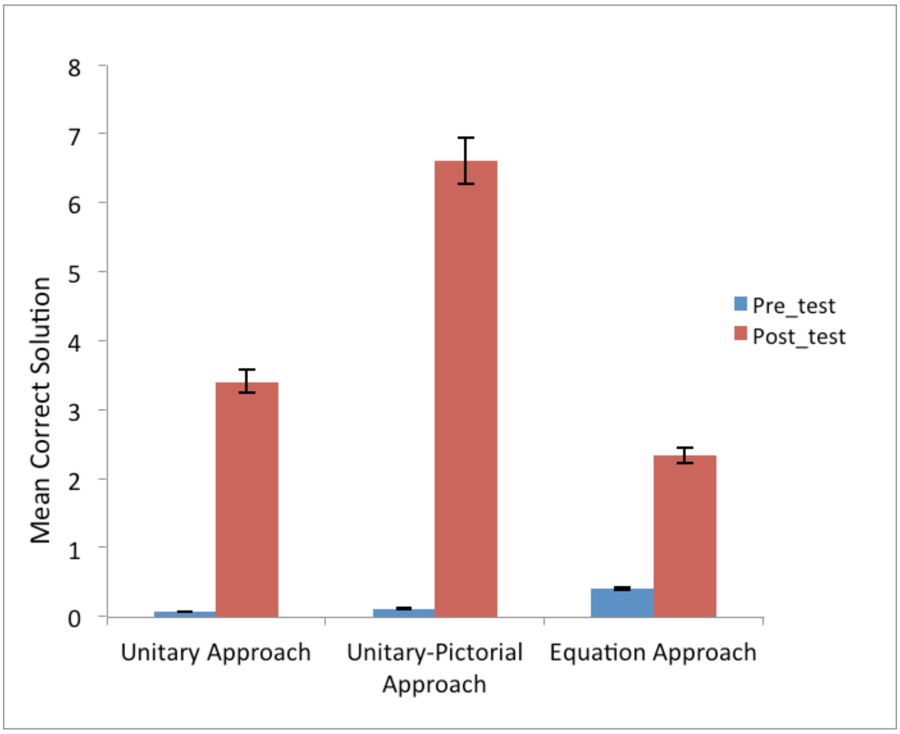


Figure 1. Pre-test and post-test for the unitary, unitary-pictorial and equation approaches

DISCUSSION

The unitary-pictorial group outperformed the equation group, though the former only experienced limited advantage over the unitary method. The poor performance of the equation group as compared to either the unitary-pictorial or unitary approach was evident in the post-test.

Having a diagram in the unitary-pictorial approach presumably helped students to overcome the hurdle of interpreting the problem situation. Students did not need to engage in unprofitable cognitive activities to figure out the connection between percentage and its corresponding quantity. Aligning the quantity and percentage also prevent a split-attention effect.

For the unitary approach, the element interactivity within and across solution steps without the scaffold of percentage and its quantity imposed a high cognitive load. In particular, the interpretation of percentage symbol, % in the context of percentage tasks.

As noted earlier, the equation approach will constitute high intrinsic load for those learners who have limited algebraic skills. It appears that students in the equation approach lacked adequate algebraic skills. As a result, the use of the cognitive resources to deal with high element interactivity within and across solution steps had overwhelmed the students resulting in poor performance in the post-test.

The formats of worked examples impact upon the extent to which students could acquire schema for complex percentage change tasks. The differential element interactivity across the three types of worked examples affects the effectiveness of these worked examples as instructional strategies. Nonetheless, as discussed earlier, the learners' prior knowledge in the domain could potentially reduce the degree of element interactivity. For example, having prior knowledge of a range of algebra skills would likely enable the learners to treat these as a single unit.

The unitary-pictorial approach has provided an important clue to improve the design of the unitary approach though future research is needed to ascertain its merit over the unitary approach. Perhaps the practice of a greater number of acquisition problems may uncover the superiority of the unitary-pictorial approach over the unitary approach. Nevertheless, it is important for mathematics educators to explore the use of the unitary-pictorial approach apart from the traditional unitary approach found in mathematics textbooks.

Test results indicated that the equation approach was harmful for students who have low prior knowledge of algebraic skills. Nonetheless, an instructional designer could focus on strengthening students' algebraic skills prior to introducing the equation approach. Thus future research should investigate the use of equation approach with a sample of students who exhibit varying level of algebraic skills.

To conclude, the formats of the worked examples affect how students' learn complex percentage change tasks. Hence, there is a need to evaluate the suitability of worked examples especially those found in popular mathematics textbooks in the light of the cognitive load theory and instructional design. This should pave the way for reform in junior mathematics education to occur.

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GIVING THE VOICE TO STUDENTS – A CASE STUDY

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Abstract. The contribution focuses on different forms and roles of the teacher's and students' voices during mathematical activities. It develops the notion of an a-didactical situation (Brousseau, 1997) by focusing on the initiator of such situation and the active communication of all actors. The roles of the teacher and students in exploring the situation are compared and their mutual influences are explained.

INTRODUCTION

Giving students the space to actively participate in introduction of new knowledge through their own independent discovery is one of the demands of pedagogical theory and curricular documents. Prerequisite to such approach is providing the space in which the pupil may apply also informal knowledge. Informal knowledge is often subconscious, chaotically connected and unclearly formulated (Hošpesová & Novotná, to appear). If it is to be used, the teachers must be able to listen to their students' voices and make it the basis for construction of a knowledge network (Kaur, 2009). It seems that this is more difficult in mathematics than in other subjects, as mathematical knowledge has a rigorous structure. Our case study demonstrates that a competent teacher who believes in the appropriateness of this approach may use it to activate and motivate her students.

THEORETICAL BACKGROUND

We came out from Brousseau's Theory of didactical situations (TDS); namely the concept *a-didactical situation* and the role of students in it. The organization of an *a-didactical situation* as such (Brousseau, 1997) involves listening to students' voices. This can be observed in the whole a-didactical situation, in the situation of action, but much more distinctly in the situations of formulation and validation. Students not only (for themselves) draw some conclusions from the activities they are involved in, they also share them with their classmates and the teacher. It is the organization of the situation that makes them formulate their ideas, not explicit summons by the teacher.

A-didactical situation and its phases

Brousseau (1997) formulated the concept of *a didactical situation* as a system in which the teacher, student(s), milieu and restrictions necessary for creation of a piece of mathematical knowledge interact "to teach somebody something". The educator "organizes a plan of action which illuminates his/her intention to modify some knowledge or bring about its creation in another actor, a student, for example, and which permits him/her to express himself/herself in actions" (Brousseau & Sarrazy, 2002). In a special case, a-didactical situation, the educator enables the student(s) to

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acquire new knowledge in the learning processes without any explicit intervention from him/her. It is possible to distinguish three phases of an a-didactical situation: *Situation of action* – its result is an anticipated (implicit) model, strategy, initial tactic; *Situation of formulation* – its result is a clear formulation of conditions under which the situation will function; *Situation of validation* – its result is verification of functionality (or non-functionality) of the model.

In our data analysis we focused on the different roles played by the teacher and the students in the different phases of an a-didactical situation. Our work led us to ask several questions, which we want to focus on in this text:

- How is an a-didactical situation initiated? Is it always planned in advance? Do sometimes students bring it about?
- What is the role of teachers and students in exploring the situation?

DATA AND METHOD

The research was a qualitative exploratory case study. We analysed lesson transcripts from 10 consecutive lessons of mathematics (labelled L01 – L10) in the 8th grade of school attendance (students mostly aged 14). Firstly we identified a-didactical situations. Then we explored the role of teacher and her students in it. We confronted our finding with the teacher's and students' reflections from post-lesson interviews.

The teacher was an experienced educator with thirty years of teaching practice. The lessons were given in a middle sized school in Mnichovo Hradiste in January 2010. The data format is based on the LPS design (Clark, 2006). The lessons were video recorded using three cameras. One of cameras was focusing on the teacher; the second camera recorded the whole class. The third camera monitored a selected pair of students different in every lesson. In the course of the 10 recorded lessons almost all pupils became members of the monitored pair. In addition to recording of the lessons, post-lesson interviews (based on the video recording) with the teacher and the selected pair of students were carried out immediately after each lesson.

RESULTS

The teacher as the initiator of the *a-didactical situation*

In our set of data the effort to create an a-didactical situation was evident in all lessons. The incentive was almost in all cases on the teacher's side. Her statements in the lessons and in the post lesson interviews clearly show that she had prepared the situation deliberately. For example she stated at the beginning of the second lesson [L02, 00:03.27]: "Today we will continue ... solving the task from the end of the last lesson. And let's see what will happen. What we'll discover. If we will manage to figure it out or solve something. So that we won't have to guess the solution any more, as we did yesterday."

The a-didactical situation started in analysed lessons by students' independent activities. They worked individually, in pairs, or in groups. These activities were stimulated by the teacher by assigning suitable problems. The students were able to solve the problems, but without any previously learnt and practiced algorithms. The solution of the problems was based on the students' real life experience or on application of previously acquired knowledge or experience. Let us now look at several examples.

The sequence of the lessons was designed as one whole based on one unifying concept to which the teacher kept referring. She decided to start from the solution of word problems using the trial and error strategy. She posed several word problems which led to a linear equation with two unknowns (quoted from L01 and L02):

> Divide 3 l of water into cups sized 0.5 l and 0.2 l so that the cups are full to the mark. You must use all the water and cups of both sizes. Once you have a solution, you can use the cups and water over there to check correctness of your solution;

> When you were on the skiing course in Janov, Veronika and Lucka went to the shop to buy some goods for themselves and for others. When counting and distributing chocolate bars and packets of nuts they found out that the shop assistant only gave them the total cost of two bars of chocolate and three packets of nuts, which was 49 CZK. Find out the price of a bar of chocolate and a packet of nuts.

The knowledge of the context allowed the students to solve the problem without actually knowing the mathematical procedure. In the next step the teacher used this non-mathematical context to introduce systems of equations and different solving methods. The students were asked to solve the problems on their own. Then they showed the different solutions on the blackboard. In most cases the teacher supported the discussion by questions asking for reasons, justification and opinions. Her original idea was that the students would use their everyday life experience for solution of this problem.

However, the progress does not necessarily have to be smooth. Sometimes a student's voice brought in an inappropriate answer, sometimes a student did not answer at all despite the teacher's expectations. At that point the teacher needs much self-control to give students the chance to be heard.

Illustration

The teacher's intention was to bring students to construction and solution of equations with one unknown (the two equations express the same unknown) and to the comparison of the "right sides". She wrote on the blackboard: $x = 3 + 2y$, $x = 9 - 3y$. The explanation went on as follows [L04, 00:32:32]:

Teacher: Can you construct a valid equation for one unknown? ... Let's think about it together. Can anybody see it? We have two equations: *x* equals 3 plus 2 upsilon, and the second: *x* equals something diferent, 9 minus 3 upsilon. What must hold for equalities? If the left sides equal, what does it mean for the right sides of the equations? Any ideas? Peter?

Peter: 3 plus 2 *y* equals 9 minus 3 upsilon.

Teacher: What do you say, Thomas? Could we write it like this? .... Yes? No? ...

Thomas: I don't think so.

Teacher: Why?

Thomas: If I substitute 2, so in one (equation) I get 7 and in the other 3.

Teacher: Hm. When we substitute 2 for *y*, are both equalities right? If we substitute 2 for *y*, do we get here the same *x* as here? [She points at the original equation on the blackboard.]

Students: Yes.

Teacher: So this is not what satisfies both equations. See? So 2 was not well chosen. Veronika?

Veronika: If it should have the same solution it must be equal.

Teacher: Exactly. If both equations must have the same solution, the same number for *x* in the first and the second equation, so they must be equal and the second *x* must therefore be equal to its counterpart. Solve one equation for the unknown *y*.

Different roles of the teacher and students in situations of formulation of conclusions of student individual work

This section focuses on the situation of formulation when the relevant information is transmitted from one student who knows it to other students in a group. The analyses concern its forms and quality, as well as other students' reactions in situations when conclusions are transmitted by students. It is compared to similar situations when the information is transmitted by the teacher.

In this section, the following terminology is used: The person who formulates the conclusions and explains them to the others is called the *transmitter*, those who get the information are called *receivers*. Students have both roles.

Illustration

This extract comes from the 7th lesson. In the final part of the 6th lesson, students were divided into groups of four. Each group was given 4 problems A, B, C, and D. Each member of the group was responsible for one of those problems. Then students left their "home groups" and met in four "expert groups" – in each group one of the four problems was solved collectively. The "expert groups" were given two tasks: to solve the assigned problem correctly and to learn how to explain the correct solution to all members of their "home group". The activity of explaining in "home groups" was planned for the beginning of the 7th lesson.

The following extract is a recording of the work in one "home group". The students are labelled S1, S2, S3 and S4. The problem discussed is B (transmitter S1). In this problem the same system of linear equations as in problem A ($3x – y = -3$, $2x + y = -2$) was to be solved but this time by substitution (solving one equation for one of the unknowns and substituting its value into the other equation). This episode follows presentation of the solution to Problem A (transmitter S2, the same system of equations but solved by comparison, i.e. by eliminating the same unknown from both equations, setting the two expressions equal to each other and then solving this equation). In the beginning S2's explanation was understood without difficulties by the group. The difficulty came when they got to the equation $0 = -5y$. They remembered that there was a problem with division by 0 and did not know what to do with it. They failed to solve the problem before the teacher gave them a hint.

The group continued to S1's solution to problem B.

S1-1: Look how clear my solution is. Copy it and it will all be solved.

S3-1: Could you explain this? [S3 points at the equation where x is substituted by $(-3 + y)/3$]

S1-2: Oh, I forgot how I did it. Wait I'll remember. Substitution method, it means that … Yes, clear. Look, this x is this [S1 circles the expression $x = (-3 + y)/3$] and you put this x here, then in fact you have it three times.

S3 does not understand.

S1-3: If x was for example 2, then you … I am explaining it to you.

S4-1: Don't explain, don't explain.

S1-4: You won't understand it. No, I will explain it to you when you don't understand. This here is x. This here is x. So in fact 3 times this x here. We only substitute in this equation.

S2-1: And what is this?

S1-5: As you have this, you know, you will only write down this. Do you understand?

S2-4: No.

S1-6: You calculate …

S4-2: And why do you have it three times?

S1-7: Well, because here is the 3. Look. If you had 2, then you would have $3 \times 2 - y = -3$. Only x is not 2 but all this. Therefore you write there all this.

Could you tell me why you don't understand it? To begin with you simply calculate how much x is. [Towards S2 who presented the solution of Problem A.] As you did it here [she points out the method of comparison].

S2-2: You said that there could as well be 2.

S1-8: No, I didn't say that. Look, you know how to find what x is from this equation, what x equals. But this x equals $(-3 + y)/3$. So our x equals this and I substitute this in that equation. Therefore the 3 is in fact this and I put there this x. So this is three times this. I substitute it in the equation, calculate it and here is the result. [All the time when talking, she is pointing in the right places in her notation.]

S2, 3, 4: It is clear now.

The episode illustrates the following properties of the situation of formulation.

1. Active role of the transmitter and the receivers

The student who is in the position of the transmitter is very active in the whole episode. Although she has a clear idea what the correct procedure is and understands why it is correct, the transfer to his/her classmates is far from smooth. As the receivers are also active and do not simply passively accept what the transmitter presents, the discussion is very fierce and all its participants are heavily involved.

If we compare this to the situation when the teacher is the transmitter, obviously the difference is mainly on the receivers' side. In case of transmission from the teacher, the students are much less active in trying to express their doubts than when the transmitter is one of the students. In the above transcribed episode, the transmitter had to answer questions 7 times. In a similar episode when the correct solution was presented by the teacher, only two questions were posed by students.

2. Formulations and reformulations; eliminating obstacles

When the first description of the procedure was not grasped by the other students, the transmitter tried to proceed in a way that is used by the teacher in similar situations – she tries to find reformulation of what was presented. Similarly to the teacher she tries to show an analogy to the situation with a concrete number. Although this procedure works when used by the teacher-transmitter, here it looks to be less productive, sometimes even counter-productive (see e.g. S2-2).

There are two reasons for it. One is the lower level of the language used by the student-transmitter. Her explanation is mostly based on what had been written in the model solution in the "expert group", she does not rewrite the calculation step by step, accompanying this rewriting by an accurate description of what she is doing in each step. In consequence, the transmitter's discourse is not clear enough to the receivers. When compared with the teacher's behaviour, it cannot surprise that student-receivers grasp the teacher's accurate explanation much faster and more smoothly.

The other reason for this can be identified as a part of didactical contract valid in the classroom. The students trust the teacher that his/her explanation is correct, which may not necessarily hold for a student-transmitter. They expect the teacher to give them clear and reliable information.

3. Originality of student-transmitter's techniques of explanation

In the analysed episodes, student-transmitters tried to apply the techniques that the teacher was using in mathematics lessons. This can be explained by the quality of the teacher's interventions during mathematics lessons. The students are well aware of the utility and good results of the teacher's techniques and therefore try to use them whenever they face the need of intervention.

4. Motivational potential of discussions in groups without the teacher's direct intervention

In the experiment, the use of students as transmitters was assessed by students as very useful. This is clearly illustrated by the following extracts from post-lesson interviews with two students (S3 and S1) after the 7th lesson.

Interviewer:	What was interesting on group work?
S3:	Well, everybody can express his/her ideas. Everybody calculates in a different way, so.
Interviewer:	But you can do it also in the whole class discussion, can't you?
S3:	Yes, that's true. But when it's in groups, it's more. I don't know, I think we're discussing it more. … In one case none of us knew how to calculate it, we found it strange. But later we grasped it.
Interviewer:	And do you think that it helped you that you could discuss it together?
S3:	Yes, here definitely yes.
Interviewer:	What do you personally find good on group work?
S1:	Well, that the lesson is somehow livelier and we aren't just sitting and looking, but we can at least discuss with the others.
Interviewer:	O.K., livelier, I understand, but is it also important from the perspective that you for example discover something when you're discussing?
S1:	Yes, we have more ideas about it.
Interviewer:	And it helps to find the solution to the problem.

5. Facing failures

In group discussion, students listen to other students'. They also learn that sometimes it happens that their effort to solve a problem may not be successful, that they may fail in the activity. This is a situation they will be facing repeatedly in their life and they must know that it is not a disaster but an impulse to look for other solving strategies, using the lesson they have learned from the unsuccessful attempt. Of course this can also happen when the transmitter is the teacher. But natural school hierarchy influences how students see their failures face to face with the teacher. Although the didactical contract may have some effect on this hierarchy, it is still true that students feel more at ease if they fail working with their classmates than when the teacher is involved. The advantage of the activity based on discussion among students is that after a failure they usually do not cease trying to find another way leading to the correct solution.

DISCUSSION AND SOME CONCLUSIONS

Illustrations of situations which were used in this text clearly show that it is impossible to study students' and the teacher's voices separately. This assertion is justified.The situation may be compared to the situation of an orchestra with a conductor and musicians. The roles both of the conductor and the individual musicians are clearly indispensable. The role of the teacher strongly resembles the role of the conductor. And even when the situation in the class looks like a concert without a conductor, it is never really so.

The teacher’s role is crucial even if it is not always explicit. Even when it is the student’s activity which is in the central position, the student must not be let down. The teacher makes the decisions on how the problem will be presented to the students, what forms of representation will be used, how much space the students will be offered for discussion of the problem, which student strategies will be supported, in other words whether he/she will prepare substantial learning environment for the students and how stimulating it will be.

The teacher from our experiment is exceptionally sensitive to students’ voices in all their possible forms. Not only does she work with students’ suggestions on how to solve a given problem, but also reacts without hesitation to the unforeseen situations arising in consequence to other influences than mathematics. Her reactions do not merely reflect experience of a teacher of mathematics; they are also motivated by her deep knowledge of her students and behaviour of the class. This was transparent in all the observed lessons and the post lesson interviews. The teacher reacts to her students’ voices not only verbally but if necessary also by changes in the intended lesson plan.

To conclude we may say that a student individual discovery (a-didactic situations in school practice) makes strenuous demands on the teacher’s competences, especially in the area of psychology, pedagogy, content knowledge but also in the area of class management.

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WHAT HAS NOT CHANGED IN TRANSFORMING TO EFFECTIVE MATHEMATICS INSTRUCTION?

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Recently student-centered teaching practices seem to be commonly recommended and implemented not only in Western countries but also in East Asia. However, it is a challenging task to transform to student-centered instruction specifically from traditionally well-established teacher-centered pedagogy. Given this challenge, many studies focus on teacher changes. This paper also deals with teacher changes in Korean context but, more importantly, focuses mainly on what has not changed in order to reveal culturally specific values and expectations of mathematics instruction. As such, this paper is expected to foster our understanding of quality mathematics instruction across different education systems.

INTRODUCTION

Educational leaders have sought for effective mathematics instruction to foster students' meaningful learning. Effective teaching is expected to foster not merely conceptual understanding of important mathematical ideas but also meaningful engagement in doing mathematics (NCTM, 2000). Many student-centered teaching methods have been emphasized over teacher-centered instruction wherein the teacher's explanations and ideas are the mainstream of the lesson. This trend seems common across countries (Cai et al., 2009; Li & Kaiser, 2011).

However, incorporating a student-centered approach originating from Western culture into a steady teacher-centered pedagogy in East Asia is a challenging task. This paper explores changes of teaching practice in implementing a student-centered approach in the Korean context. This will lead us understand better what constitutes the process of implementing new ideals into an actual classroom context.

Given that they are only very few attempts exploring teacher changes in Korean contexts, while there are many studies in Western countries, this paper can provide valuable information for international research community in mathematics education. Beyond the merit from a cultural context, more importantly, this paper is expected to make a unique contribution of teacher changes by focusing mainly on what has not changed in the process of transforming a teacher-centered approach into student-centered one. This may be related to culturally-specific characteristics and values in mathematics instruction.

THEORETICAL BACKGROUND

Korean mathematics instruction has rarely been studied in international contexts. An exceptional study, conducted by Grow-Maienza, Hahn, and Joo (1999) reported that

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Korean elementary teachers organized teacher-centered instructions more systematically, coherently, and progressively and Korean students were more deeply engaged in such lessons and more enthusiastically provided choral responses than U.S. counterparts. Clarke, Xu, and Wan (2009) reported that students' choral responses were evident in Korean secondary mathematics classrooms, but only the teacher employed key mathematical terms and students had few opportunities to use them either publicly or privately. The teacher-centered teaching practice has contributed to Korean students' consistent superior mathematics achievement not only in mathematical skills but also in problem solving in international comparisons (e.g., Mullis et al, 2000; OECD, 2004, 2007). However, such a teaching approach has been recognized as problematic partly because Korean students developed the most negative perceptions of mathematics in spite of their success in the subject.

Countering the teacher-centered pedagogy in mathematics, recent revisions of the national mathematics curriculum in Korea have consistently suggested many characteristics of student-centered teaching methods (e.g., Ministry of Education, Science, and Technology [MEST], 2011). This called for significant changes in terms of learning objectives, instructional strategies, mathematical discourse, and the learning environment. As for learning objectives, a teacher is expected to help students develop not only conceptual understanding and problem-solving but also mathematical reasoning, communication, and positive disposition. As for instructional strategies, a teacher is expected to consider not merely the content to be taught but individual students' differences and effective learning materials. As for mathematical discourse, a teacher is expected not to dominate classroom talk but to encourage students to develop and present their own ideas with open-ended questions and timely feedback. As for the learning environment, a teacher is expected to employ small-group or individual activities beyond the prevalent whole-class organization and to establish a permissive learning atmosphere in which all students are actively engaged in classroom discussion.

Creating a recommended teaching practice is not an easy task to accomplish. Huang and Li (2009) reported that the expert teachers greatly emphasized student-centered teaching practice but claimed to keep a balance between innovative features from Western culture and desirable Chinese traditions. Pang (2009) also observed that student-centered practices are rooted well in culturally specific values and nuances in Korea. Given this background, it would be interesting to track down how a reform-oriented Korean teacher may move through the path of implementing a new recommended teaching approach. Specifically, what aspects of typical teaching approaches would remain the same in the process of change and what would not?

METHOD

The data used in this paper are from a one-year research project of transforming teaching practices in Korean elementary mathematics classrooms. The teacher Ms. Y out of 5 participant teachers was selected for this paper because she demonstrated substantial changes in her instructional approaches. The transformation was noticeable not only by the researcher but by the other participating teachers as well. Ms. Y also reflected that her teaching approaches had been significantly changed throughout the project period.

Three kinds of data were collected. First, throughout the year two mathematics lessons per month in Ms. Y's classes were videotaped and transcribed. Second, Ms. Y was interviewed three times to trace her construction of teaching approaches, which were audio-taped and transcribed. The third kind of data were from videotaped inquiry group meetings in which the participant teachers met once per month and had lots of opportunities to analyze their own teaching practices as well as others by watching and discussing the videotaped lessons together. For this paper, Ms. Y's videotaped lessons were mostly used.

An analytic framework was developed to track down the process of incorporating recommended approaches into ordinary Korean teaching practice. Five main dimensions (i.e., overall characteristics, learning objectives, instructional strategies, mathematical discourse, and learning environment) with a total of 24 sub-dimensions were drawn from the literature review of typical and recommended teaching practices in Korea (see Table 1 for an example).

Dimensions	Criteria
1. Overall characteristics	1.1 Is the main topic presented consistently throughout the lesson?
	1.2 Is the main topic presented progressively (from easy/concrete to difficult/abstract)?
	1.3 Is the overall instructional flow systematic? (learning motivation → learning objectives → main activities → practice → evaluation/summary)

Table 1: Part of analytic framework to identify teacher changes

Five lessons of Ms. Y's teaching practice were selected for this paper to trace her instructional changes on a timely basis. Lesson 1 was the first videotaped lesson which could demonstrate Ms. Y's initial teaching practice. While Lessons 2 and 3 were from Ms. Y's sixth grade classroom during the first half of the project period, Lessons 4 and 5 were from her third grade classroom during the second half of the period. Generally speaking, about two or three months difference existed between Lessons.

After reaching satisfactory inter-reliability on the basis of practice sessions, five raters including the researcher coded individually each lesson according to the analytic framework in terms of performance scales from one to five. While a score of five means strongly agree, one means strongly disagree. Bigger scores mostly mean more alignment with student-centeredness. Exceptions occur for three dimensions such as choral and simple responses in which bigger scores mean less alignment with student-centeredness.

Each rater marked scores with regard to each criterion while watching Ms. Y's videotaped lessons one by one with written transcripts. The average score among five raters was calculated and used to identify changes in Ms. Y's teaching practices. A change is counted when the compared score in Lessons 2 to 5 is significantly different from its original score in Lesson 1, indicating that the score should be between 4 and 5 for a positive criterion or between 1 and 2 for a negative one. A description of Ms. Y's teaching practice is added to this paper as needed.

RESUTS

Ms. Y's initial teaching practice

At first, Ms. Y faithfully followed the sequence of activities in the textbook. The following summarizes how she taught a lesson about division of decimal fractions. Ms. Y first asked students why they would need to learn division of decimal fractions and presented the word problem corresponding to 2.5 divided by 0.5 through a projection TV. Ms. Y then asked students to guess what they would learn at day and students could predict the topic. Ms. Y presented three activities to solve the given problem: (a) repeatedly subtract 0.5 from 2.5, (b) figure out 2.5 (cm) divided by 0.5 (cm) by converting 'cm' into 'mm', and (c) convert the given decimals into fractions. She then led students to complete each activity by raising step by step questions, and students easily answered. Students solved 7 exercises in the textbook (e.g., 76.8 divided by 3.2). Ms. Y then presented an additional similar word problem. Ms. Y summarized the lesson by emphasizing that division of decimal fractions can be changed into that of natural numbers by moving their decimal points.

With Ms. Y's detailed guidance, the students in her classroom participated in the classroom activities and solved the given problems easily. The teacher supported students' contributions by providing praise and encouragement. However, students' answers were limited to short or rather fixed responses given the structure and flow of the lesson. In the interview, Ms. Y assessed her initial teaching method as close to teacher-centered pedagogy, even though she had preferred more student-centered instruction.

Overall change of Ms. Y's teaching practice

Ms. Y's teaching practices have been substantially changed with regard to 16 sub-dimensions. Some changes happened dramatically even in the early stage of teacher change. For instance, Ms. Y skillfully used instructional materials for students' manipulative activities and exploration from Lesson 2 in this analysis. She employed small-group or individual activities and students had an opportunity to present their own ideas.

Other changes happened less dramatically but considerably in the middle stage of teacher change (from Lesson 3 in this analysis). For instance, Ms. Y solicited students' diverse ideas and then used them for the lesson. Since students had to explain how they had solved the problem in front of the class, their presentations were no longer limited to choral or simple responses as those in the initial teaching practice.

Still other changes occurred rather gradually over a longer period of time. The satisfactory changes happened only in Lesson 4 or 5 in this analysis. For example, the teacher's dominance of question/answer or demonstration came to be gradually decreased, while students had an opportunity to communicate their ideas directly with peers.

Unchanged aspects over student-centeredness

Table 2 shows an average score of performance scales of Ms. Y's teaching practice across five lessons with regard to six dimensions remained the same throughout the year.

		Lessons				
Dimensions		1	2	3	4	5
1. Overall characteristics	1.1 Consistent	4.8	4.8	5.0	5.0	5.0
	1.2 Progressive	4.2	4.8	4.8	4.6	5.0
	1.3 Systematic	4.6	5.0	5.0	4.4	4.6
2. Learning objectives	2.1 Conceptual understanding	4.8	5.0	5.0	4.6	4.8
	2.2.1 Problem- solving	4.4	4.8	4.8	4.6	5.0
3. Instructional strategies	3.1 Considering content	4.0	4.8	4.8	4.6	4.2

Table 2: Some average score of performance scales of Ms. Y's teaching practice

While Ms. Y employed many new recommended teaching approaches, the overall characteristics of her lessons were still consistent, progressive, and systematic. The lessons focused on students' understanding of a mathematical concept, principle, and law while encouraging them to solve given problems. In the same vein, Ms. Y was consistently skillful in using instructional strategies tailored to the content to be taught. Notice that all these aspects have been regarded as typical Korean teaching practices.

A remarkable constant was Ms. Y's emphasis on important mathematics content across many student-centered approaches. For instance, Ms. Y encouraged students to use manipulative materials but she was cautious to connect such concrete activity with the conceptual structure behind it. In Lesson 5 of "(two digit number) x (one digit number)", for instance, students solved the following 'pencil' problem individually using base-ten blocks: "You can put 13 pencils per pencil box. How many pencils can you put in two pencil boxes?" When Ms. Y initiated the whole class discussion by asking one student to put the blocks in an overhead projector in front to show the multiplication problem 13x2, the student put 13 blocks and then 2 blocks respectively on the projector. The teacher questioned the meaning of 13x2, and the students answered "adding 13 twice". [Note that 13x2 means "2 groups of 13" in Korea.] By paying attention to this meaning, the student could correctly arrange the base-ten blocks: 1 tens-block, 3 ones-blocks, 1 tens-block, and 3 ones-blocks in order. The teacher then related students' manipulative activity to the principle of multiplication. In writing a multiplication expression, students could easily write equations for ones-blocks (3x2=6) but not for tens-blocks. They confused 1x2=2 with 10x2=20. In fact, 2 groups of tens-blocks could be written as 10x2. However, Ms. Y directed students' attention to the exact number of tens blocks by pointing out that there are not 20 but 2 tens-blocks to connect it with the corresponding multiplication expression.

Ms. Y's emphasis on important mathematics content was also salient when she focused on mathematically significant ideas after soliciting students' multiple ideas. In the process of changing her teaching approaches, Ms. Y provided students with an opportunity to come up with their own ideas. However, when students did not come up with a specific idea, the teacher simply introduced it or orchestrated the path of classroom discourse toward exploring it. Meanwhile, a specific solution method was preferred. For instance, in Lesson 5 students used addition even when the teacher clearly asked for multiplication (e.g., as for 42x2, "40 plus 40 is 80. 2 and 2 is 4. So the answer is 84."). Ms. Y at first accepted this method based on her concern that students should understand the basic meaning of multiplication as repeated addition. However, as students were exposed to many different problems, Ms. Y came to emphasize multiplication over addition. She accepted the 'addition method' as mathematically correct but at the same time gave the 'multiplication method' credit for convenience. Moreover, Ms. Y provided detailed feedback only for the multiplication method. In this way, students were able to recognize that multiplication was called for, and tended to write a multiplication expression even after they wrote an addition expression in solving subsequent problems.

Expected but less attained changes

Two dimensions showed some positive changes in Ms. Y's teaching practices throughout the year but did not reach to the full expectation as recommended. Compared to her initial teaching practice, Ms. Y tended to consider students' differences in later lessons but her consideration was not significant enough. The scores were at most 3.2, 2.8, and 3.0 from Lesson 3 to 5 respectively. In a similar vein, Ms. Y was less successful in emphasizing the importance of mathematical communication compared to her other roles. Specifically, the scores of emphasizing mathematical communication were 3.6, 3.8, and 3.8 from Lesson 3 to 5 respectively, whereas those of soliciting and using students' ideas were 4.8, 4.4, and 4.0 correspondingly.

DISCUSSION

Although Ms. Y showed many changes, there are also some dimensions that were not changed throughout the year. Such dimensions can be summarized into three aspects. First, the overall characteristics of Ms. Y's teaching practices were constant in that they were consistent, progressive, and systematic. This characteristic is not surprising because all lessons were based on one textbook series used in common throughout Korea. The textbook presents each topic consistently and progressively from easy and concrete contexts to advanced and abstract forms. This organization of textbooks may engender systematic lessons which include motivating students, introducing learning objectives, doing two or three main activities, practicing, and evaluating or summarizing students' learning.

The second aspect that has remained the same is the focus on important mathematical content. To be clear, the strategies emphasizing conceptual understanding were changed. At first, Ms. Y taught the content by providing step-by-step guidance with direct explanations. In the process of changing her teaching approach, the teacher provided students with an opportunity to explore mathematically important ideas based on concrete activities with manipulative materials. Nevertheless, the teacher was cautious in connecting students' classroom activities to the mathematics content behind them. She also played an important role in directing students' attention to the mathematics content being taught. Despite the changed form of instructional strategies, it is clear that the teaching of conceptual understanding was consistent in Ms. Y's teaching practice throughout the year. This is supported by Korean elementary school teachers' conception of effective mathematics instruction wherein teaching of fundamental concepts in mathematics and their connections was mostly valued (Kwon & Pang, 2009).

The final aspect that was not changed in Ms. Y's teaching practice is her consideration of individual students' differences. This aspect is different from the first two aspects that have remained constant in that Ms. Y showed some improvement in using instructional strategies tailored to students' individual differences. Nevertheless, such improvement was not significant in comparison with changes in other dimensions. This is different from Korean elementary school teachers' conception of effective mathematics instruction wherein they strongly agreed both selection of content by considering students' individual differences and individualized teaching according to their different abilities and needs (Kwon & Pang, 2009). A future study is needed to explain this discrepancy.

The boundary between traditional teacher-centered and student-centered instruction seems blurred (Huang & Li, 2009; Pang, 2009). However, given that teaching is a cultural activity (Stigler and Hiebert, 1999), a close examination of reform-oriented classrooms in East Asia may reveal subtle but significant diversity and variability that have been existed across different education systems. In Ms. Y's case, new aspects of student-centered teaching practices were implemented in conjunction with her well-structured teaching approach. Changing teaching practices seems to be conducted within cultural values and expectations that have been long maintained in a specific education system. Much still remains to be examined about the nature of recent reformed mathematics instruction across different education systems, especially with regard to different nuance of student-centeredness.

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KEY EPISTEMIC FEATURES OF MATHEMATICAL KNOWLEDGE FOR TEACHING THE DERIVATIVE

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In recent years there has been a growing interest in studying the knowledge that mathematics teachers require in order for their teaching to be effective. However, only a few studies have focused on the design and application of instruments that are capable of exploring different aspects of teachers' didactic-mathematical knowledge about specific topics. The present paper reports the results obtained following the application of a questionnaire designed specifically to explore certain key features of prospective, higher secondary-education teachers' knowledge of the derivative. The first part of the paper describes the design of this questionnaire.

BACKGROUND

The mathematical and didactic training of prospective teachers is an area of research that merits the attention not only of researchers in the field of mathematics education but also of educational authorities. Above all, this is because the development of pupils' mathematical thinking and competences is inherently dependent on their teachers' abilities.

One of the questions that have generated the most interest concerns how to determine the didactic-mathematical knowledge that is required to teach mathematics. In this context, the reflections and recommendations of Shulman (1986) and the studies by Ball (2000), Ball, Lubienski and Mewborn (2001) and Hill, Ball and Schilling (2008) have all helped to further our understanding of the different knowledge components that teachers need to acquire in order to teach effectively and foster their pupils' learning. However, a more detailed understanding of the knowledge required in order to teach mathematics needs to focus on specific topics, for example, the knowledge which a secondary teacher needs in order to teach the derivative (Badillo, Azcárate & Font, 2011). This paper reports some of the results obtained following the application of a questionnaire which, based on the model proposed by Godino (2009) for assessing and developing of the didactic-mathematical knowledge, was designed in order to explore key features of prospective secondary teachers' didactic-mathematical knowledge of the derivative.

METHOD

The research is an exploratory study and uses a mixed methods approach; it involves the observation of both quantitative (level of accuracy of items: correct, partially correct, and incorrect answers) and qualitative variables (type of solution or cognitive configurations proposed by the prospective teachers). The latter, qualitative variable is

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closely related to the type of didactic-mathematical knowledge which prospective, higher secondary-education teachers have about the derivative.

Subjects and context

The questionnaire was administered to a sample of 53 students enrolled in the final modules (sixth and eighth semester) of the degree in mathematics teaching offered by the Universidad Autónoma de Yucatán (UADY) in Mexico. This is four-year degree (8 semesters). The Faculty of Mathematics of the UADY is responsible for training teachers to work at higher secondary or university level in the state of Yucatan (Mexico). The 53 students who responded to the questionnaire had studied differential calculus in the first semester of their degree course, and they had subsequently completed other modules related to mathematical analysis (integral calculus, vector calculus, differential equations, etc.). They had also studied subjects related to the teaching of mathematics.

The questionnaire

The questionnaire, called the *Questionnaire regarding didactic-mathematical knowledge about the derivative* (*DMK-Derivative Questionnaire*), comprises seven tasks and was designed in accordance with the model proposed by Godino (2009) for assessing and developing the didactic-mathematical knowledge. This model provides guidelines for categorizing and analysing teachers' didactic-mathematical knowledge in accordance to the onto-semiotic approach to knowledge and mathematics education (OSA) (Godino, Batanero & Font, 2007). The purpose of the questionnaire is to assess certain epistemic features of the didactic-mathematical knowledge (DMK) of prospective secondary teachers on the derivative. According to Ball and colleagues model (Ball, Lubienski & Mewborn, 2001; Hill, Ball & Schilling, 2008) this epistemic facet comprises three types of knowledge: *common content knowledge, specialized content knowledge* and *extended content knowledge*.

When designing the questionnaire, three criteria were considered in order to select the tasks that would be included in it. The first criterion was that the tasks should provide information about the extent to which a prospective teacher's personal understanding of the derivative was consistent with the global or holistic view of this mathematical object (Pino-Fan, Godino & Font, 2011). This was achieved by including items that activate different meanings of the derivative: slope of the tangent line, instantaneous rate of change and instantaneous rate of variation. In this work we distinguish "instantaneous rate of change" that refers specifically to the "quotient" between two quantities of magnitudes, meanwhile "instantaneous rate of variation" refers to the "quotient" of real numbers with no reference to magnitudes. The "instantaneous rate of variation" is commonly known as the limit of the incremental quotient.

The second criterion was that the items selected had to reflect the different types of representations activated in the three sub-processes which, according to Font (2000), are involved in calculating the derivative function: 1) translations and conversions between the different ways of representing $f(x)$; 2) the step from a representation of

$f(x)$ to a representation in the form $f'(x)$; and 3) translations and conversions between the different ways of representing $f'(x)$. Consequently, the tasks included in the questionnaire bring into play the different types of representations that are involved in these three sub-processes, namely verbal description, graphical description, symbolic and tabulation (for both the function and its derivative).

The third criterion, which refers to the didactic-mathematical knowledge held by prospective teachers, considers the inclusion of three types of task: (1) those that require teachers to use their common content knowledge (solving a mathematical problem that would be set at the higher secondary level); (2) those that require specialized content knowledge (using different representations, different partial meanings of a mathematical object, solving the problem by means of various procedures, giving a range of valid arguments, identifying the knowledge that is brought into play when solving a mathematical problem, etc.); and (3) those that require extended content knowledge (generalizing tasks involving common or specialized knowledge, or making links to more advanced mathematical objects that appear in the curriculum). The next section presents an analysis of the aspects that are evaluated by the tasks included in the DMK-Derivative Questionnaire.

CONTENT ANALYSIS OF THE TASKS INCLUDED IN THE *DMK-DERIVATIVE QUESTIONNAIRE*

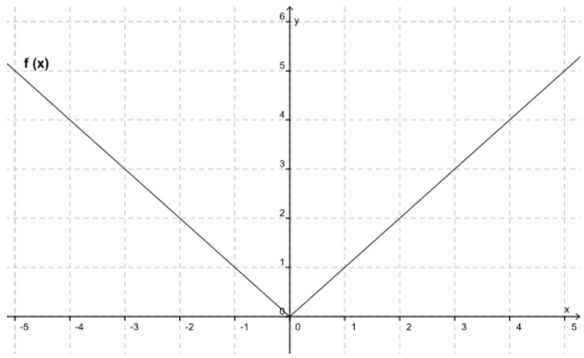
Due to space constraints this section only provides a detailed analysis of the knowledge assessed by five of the seven tasks included in the *DMK-Derivative Questionnaire.* However, both the ‘results and discussion’ and ‘final reflections’ sections include some discussion of all seven tasks in the questionnaire.

Task 1 is a classical question that has been used in a number of studies (Habre & Abboud, 2006; Bingolbali & Monaghan, 2008) to explore the meanings known by the students concerning the derivative. The question is: “What does the derivative mean to you?” As this is a ‘global’ question, prospective teachers are expected to provide a ‘list’ of possible meanings of the derivative. This task therefore explores the prospective teachers’ common knowledge regarding the meanings of the derivative.

Task 2 (Figure 1), which has been the object of several studies (Tsamir, Rasslan & Dreyfus, 2006), explores three types of knowledge that comprise the epistemic facet of didactic-mathematical knowledge about the derivative: 1) common content knowledge (item a), such that the prospective teacher should be able to solve the problem without needing to use various representations or arguments; 2) specialized content knowledge (items b and c), where in addition to solving the problem the teacher is required to use representations (graphs, symbols and verbal descriptions) and valid arguments that justify the procedures; and 3) extended content knowledge (item d), which entails generalization of the initial task about the derivability of the absolute value function at $x=0$, on the basis of valid justifications for the proposition “the graph of a derivable function cannot have peaks” by defining the derivative as the instantaneous rate of variation (limit of the increment quotient). The interpretations of the derivative as the slope of the tangent line and the instantaneous rate of variation are associated with this task.

Task 2

Consider the function $f(x) = |x|$ and its graph.



a) For what values of x is $f(x)$ derivable?

b) If it is possible, calculate $f'(2)$ and draw a graph of your solution. If it is not possible, explain why.

c) If it is possible, calculate $f'(0)$ and draw a graph of your solution. If it is not possible, explain why.

d) Based on the definition of the derivative, justify why the graph of a derivable function cannot have 'peaks' (corners, angles)

Figure 1: Task 2 from the *DMK-Derivative Questionnaire*

Task 4, which is taken from Viholainen (2008), explores the specialized content knowledge of prospective teachers, as it requires the use of various representations (graph, verbal description, symbolic) and a range of justifications for the proposition "the derivative of a constant function is always equal to zero", in which different interpretations of the derivative may be employed: slope of the tangent line, instantaneous rate of change and instantaneous rate of variation.

Task 5 (Figure 2) appears to be the sort of exercise usually found in differential calculus books that are used at the higher secondary level, its solution being obtained by applying certain theorems or propositions about the derivative. Therefore, both item a) and item b) evaluate aspects of common content knowledge related to the derivative, where the latter is understood as the slope of the tangent line or the instantaneous rate of change, respectively. However, the main objective of Task 5 is to explore the associations that prospective teachers make between the different meanings of the derivative, and as such the task evaluates aspects of specialized content knowledge.

Task 5

Given the function $y = x^3 - \frac{x^2}{2} - 2x + 3$

a) Find the points on the graph of the function for which the tangent is horizontal.

b) At what points is the instantaneous rate of change of y with respect to x equal to zero?

Figure 2: Task 5 from the *DMK-Derivative Questionnaire*

Finally, Task 7 (Figure 3), which has been adapted from the paper by Çetin (2009), provides information about the teachers' extended content knowledge, since it involves an approximation to the derivative of a function, described by values in the table, at point *t=0.4* by means of numerical values of the function. Task 7 is not the typical sort of problem that would be encountered at the higher secondary level, and it requires an understanding of the derivative as the instantaneous rate of change, and specifically as instantaneous velocity. This problem can be solved by various methods, for example, Lagrange's interpolating polynomial; this supports the categorization of this task as evaluator of the expanded content knowledge.

Task 7

A ball is thrown into the air from a bridge 11 meters high. $f(t)$ denotes the distance that the ball is from the ground at time *t*. Some values of $f(t)$ are shown in the table below:

t (sec.)	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(t)(m.)$	11	12.4	13.8	15.1	16.3	17.4	18.4

Based on the table, at what speed will the ball be travelling when it reaches a height at $t = 0.4$ seconds? Justify your chosen answer.

a) 11.5 m/s b) 1.23 m/s c) 14.91 m/s d) 16.3 m/s e) Another

Figure 3: Task 7 from the *DMK-Derivative Questionnaire*

RESULTS AND DISCUSSION

In analysing the data obtained through administration of the questionnaire, we considered two variables: *the type of cognitive configuration* (i.e. type of solution proposed by the prospective teachers) and *level of task's accuracy* (i.e. correct, partially correct or incorrect). The analytic technique used with the first variable (type of cognitive configuration) was *semiotic analysis* (Godino, 2002), which provides a systematic description of both the mathematical activity carried out by the prospective teachers in solving the problems, and the mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) that were involved in their practice (Godino, Batanero & Font, 2007). The type of *didactic-mathematical knowledge* is closely related to the variable *type of cognitive configuration* associated to students' answers because the epistemic facet of didactic-mathematical knowledge depends on the presence or absence of the mathematical objects, their meanings and relations among them. These cognitive configurations have a didactic-mathematical nature due to the displayed tasks which have the same nature and therefore the prospective teachers should handle the didactic and mathematical knowledge.

Concerning the variable "level of accuracy" punctuations 2, 1 or 0 were assigned if the answers were correct, partially correct or incorrect correspondingly. Thus, the maximum possible score was 26. Twenty-four of the prospective teachers (45.3%) obtained a score higher than 13, but of these 24 only nine (17%) responded correctly to more than 67% of the questionnaire. The above information reveals that more than

50% of the students showed difficulties for solving questionnaire tasks. The mean score (12.4) obtained by the 53 prospective teachers and the distribution of their scores are shown in Figure 4.

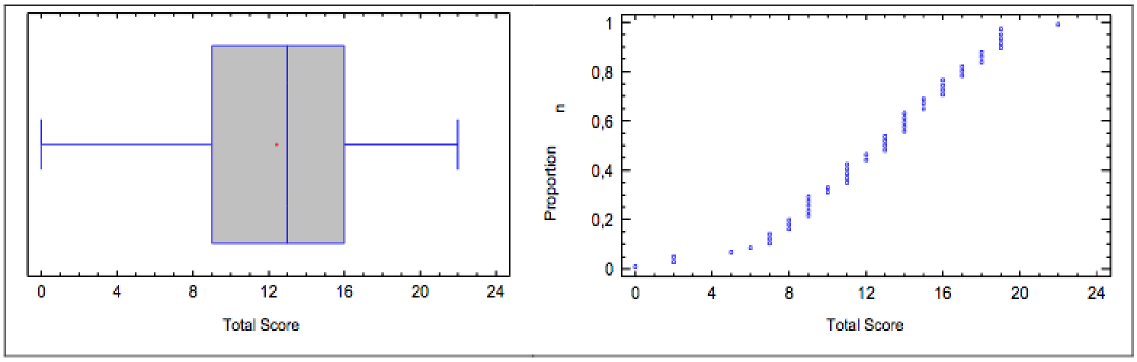


Figure 4: Boxplot and distribution of scores obtained on the *DMK-Derivative Questionnaire*

In general, the *DMK-Derivative Questionnaire* had an intermediate level of difficulty for the prospective teachers (Figure 5). The items they found most difficult were 2-d (Figure 1) and Task 7 as a whole (Figure 3). Task 1 and items 2-a, 3-a and 4-a were the easiest for them to solve.

	Item	Difficulty Index	%
1	I-1	0.8679	86.79
2	I-2a	0.7547	75.47
3	I-2b	0.6038	60.38
4	I-2c	0.6415	64.15
5	I-2d	0.1321	13.21
6	I-3a	0.8491	84.91
7	I-3b	0.5660	56.60
8	I-4a	0.7547	75.47
9	I-4b	0.5849	58.49
10	I-5a	0.5660	56.60
11	I-5b	0.4528	45.28
12	I-6a	0.4528	45.28
13	I-7	0.1132	11.32
Mean: 0.56			

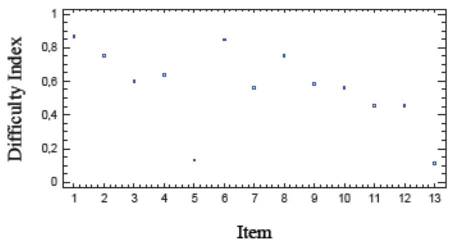


Figure 5: Difficulty index for the items on the *DMK-Derivative Questionnaire*

FINAL REFLECTIONS

The analysis of the responses given by prospective teachers to the tasks included in the *DMK-Derivative Questionnaire* indicates that they had certain difficulties in solving the tasks related to common, specialized and extended knowledge about the derivative. The results obtained with respect to Task 4 show that these teachers performed better when solving tasks in which the derivative is understood as the slope of the tangent line. The results regarding Task 3 highlight the need to improve the advanced knowledge of prospective teachers, as this would help them to solve tasks such as this one. Due to the relationship between Task 4 and the type of knowledge it evaluates, it is clear that the prospective teachers lack certain aspects not only of specialized knowledge (use of different representations, use of different meanings of the derivative, solving the problem through various procedures, giving a range of valid arguments to justify these procedures, etc.) but also of the common knowledge required to solve the task. Indeed, 56.6% of the teachers had problems (on Task 4) demonstrating, by means of the formal definition of the derivative, the proposition “the derivative of a constant function is always zero”. This suggests that they had yet to master the practice of demonstration when this involves using the derivative as the limit of mean rates of change. The results obtained in relation to Tasks 6 and 7 illustrate the difficulties which the prospective teachers experienced when they had to use the derivative as the instantaneous rate of change in a relatively complex situation. Here the *DMK-Derivative Questionnaire* revealed how common content knowledge is in itself not enough to deal with the kind of tasks that will emerge in the teaching context, for which teachers will also need a certain degree of both specialized and extended content knowledge. Indeed, the results show that not only did the prospective teachers lack certain aspects of specialized and extended knowledge, but also that there was a disconnection between the different meanings of the derivative (Tasks 1 and 5).

Both the design of the questionnaire and the responses of these prospective teachers reveal the complex set of mathematical practices, objects and processes that are brought into play when solving tasks related to the derivative. Teachers need to become aware of this complexity during their training so that they will be able to develop and assess the mathematical competence of their future pupils. In this regard, the aspects of specialized and extended knowledge that were lacking in these prospective teachers could hinder their ability to manage appropriately their future pupils’ mathematical knowledge about the derivative. The latter is supported by research showing that the mathematical knowledge of teachers has an effect on the achievements of their pupils (Ball, 1990; Wilson, Shulman & Richert, 1987). The lack of certain knowledge that was revealed in the present study highlights the need for specific training strategies to help prospective teachers develop the epistemic facet of their mathematical and didactic knowledge. The development of these training strategies should take into account the complexity of the global meaning of the derivative (Pino-Fan, Godino & Font, 2011).

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ELEMENTARY SCHOOL CHILDREN SOLVE SPATIAL TASKS A VARIETY OF STRATEGIES

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The study reports about the solving strategies of children in spatial abilities tasks. Four different tasks were designed in reference to the theoretical model of Maier (1999). In guideline-based interviews 57 fourth grade children solved the tasks. The results of the qualitative analyses show a variety of strategies which were structured and characterized in strategy groups. The data show furthermore, that children use much more strategies as the theoretical characterization of the tasks intended.

INTRODUCTION

Psychometrical research about spatial abilities often assumes a one to one matching between the task and the strategy necessary to solve it as well as it traditionally uses two-dimensional paper-and-pencil-tests (s. Hosenfeld et al., 1997). The study presented here describes interviews with fourth-graders. They were asked to solve spatial tasks, which were given in two different presentations: three-dimensional buildings of cubes and two-dimensional photos of such buildings. Since an interview-study with pre-school-children already showed different solving strategies for nearly every task (Lüthje, 2010), our study raises the following questions:

- Which strategies do children use – successfully – to solve different spatial tasks?
- How does the modality of the task presentation influence the strategy choice and the success of the children?
- How do children with high and low mathematic abilities differ in solving spatial tasks?

This article focusses only on the first question and presents qualitative results, which show a great variety of strategies used by the children. These strategies are described and structured into different strategy groups which allow us to contrast them to the strategies intended.

THEORETICAL FRAMEWORK

Rost (1977) describes spatial abilities as the competence of operating mentally with two- or three-dimensional objects whereas Collom et al. (2001) argue much more broadly that spatial abilities imply “the generation, retention, retrieval and transformation of visuo-spatial information” (p. 903). Although a lot of research exists about spatial abilities, there is no consensus about a unitary definition (e.g. Eliot & Smith, 1983; Linn & Petersen, 1985; Hellmich, 2001). As a consequence, it is sometimes difficult to extract the underlying concept of spatial abilities the research is based on. On the other hand it is accepted that spatial abilities should be seen as a complex concept which is a fundamental part of the concept of intelligence and can be divided into different components (e.g. Thurstone, 1938, 1950; Rost, 1977; McGee, 1979; Linn & Petersen, 1985; Carpenter & Just, 1986).

Components of spatial ability

The most common research was done by Thurstone (e.g. 1938, 1950). In the context of his studies about the human intelligence he could differentiate three sub factors within the factor ‘space’: *spatial relations*, *visualization* und *spatial orientation* (Thurstone, 1950; Maier, 1999). Nevertheless, the number of factors was discussed a lot (Linn & Petersen, 1985; Hosenfeld et al., 1997). Up to now, the determination of at least three factors was affirmed very often. Therefore one-factor- and two-factor-theories (Spearman, 1904; Michael et al., 1950; McGee, 1979) will not be discussed here. By a meta-analysis, Linn and Petersen (1985) examined gender differences in spatial abilities. Their analyses confirmed three factors, which were slightly different to those of Thurstone*: spatial visualization, mental rotation* and *spatial perception*.

With references to those studies Maier (1999) generated a summary model of five essential components of spatial abilities: *spatial relations, visualization, mental rotation, spatial orientation* and *spatial perception*. These components can be analysed with regard to two dimensions Maier identified (Maier, 1999; Plath, 2012):

I. The mental solving process can be described as *dynamic* if it contains moving aspects. If there are no such aspects, the mental solving process is categorized as *static*.

II. The situation of the person him- or herself can be seen as being a part of the mental task situation, which is categorized as *inside*, or as looking at the whole situation from an inner distance, which is characterized as *outside*.

According to that differentiation *visualization* implies dynamic processes from an outside point of view. *Mental rotation* is also described as a component with dynamic processes and an outside view but in contrast to visualization the objects remain as a whole. *Spatial orientation* is considered as a component which implies dynamic processes combined with an ‘inside position’ of the person.

Static processes combined with an outside point of view are ascribed to the component *spatial relation*. The component *spatial perception* also shows static processes but these are combined with an ‘inside position’ of the person. Due to the fact that this component is defined very closely and – as already can be seen in the label ‘perception’– a closer analysis shows that it describes much more a visual precondition of spatial abilities, it will be excluded as a component of spatial abilities in the further paper. So, mainly four components are ascribed to the construct of spatial abilities for our investigation.

Solution strategies for spatial ability tasks

With reference to Barrat (1953) strategies can be described as mental process which are used for problem solving and which can be communicated. Classical test procedures as paper-and-pencil-tests are based on the fact that a special strategy is raised by the task and that the ability to use or adopt this strategy differs from subject to subject (Hosenfeld et al., 1997; Pinkernell, 2003). In this perspective, strategies are seen much more as task-oriented than as person-oriented (Plath, 2011).

But other studies show a person-orientation in using strategies as well: Different people show use of different strategies for solving the same task (Barrat, 1953; Kyllonen et al., 1984; Lüthje, 2010). Also one person may even use different strategies to solve similar problems (Kyllonen et al., 1984; Carroll, 1993; Lüthje 2010). There may be intended strategies but these strategies are not a necessity for solving spatial tasks mentally. So, strategies seem to be task- and person-oriented as well.

In the literature a clear difference between analytic and holistic strategies is postulated (Barrat, 1953; Cooper, 1976; Burin et al., 2000). With regard to the model of Maier the analytic strategies comply with static thinking processes because no mental movement of objects is implied. The holistic strategies can much more be assigned to dynamic processes, because mental movements of objects or of oneself are part of these strategies.

An often found differentiation of strategies is that of *key features strategies*, *move object strategies* and *move self strategies* (Barrat, 1953; Carpenter & Just, 1986; Schulz, 1991). *Key features* means to focus on special elements or essential properties of an object. *Move object strategies* stand for the mental movements of objects and *move self strategies* for the mental change of the own position inside the situational context. This differentiation can be matched with the second dimension of Maier's model. Tasks combined to the components *visualization* and *mental rotation* are typically solved with the holistic strategy of *moving objects* while the person mentally stands outside of the situation. Compared with this 'viewpoint outside' tasks of the component *spatial orientation* are also solved by holistic strategies but from an inside viewpoint with *move self strategies*. In contrast to the situations mentioned before, problems assigned to the component *spatial relations* are typically seen as static and are therefore solved by analytic strategies such as the *key features strategies*. These analytical strategies are combined with a perspective from outside the situation.

DATA AND METHOD

Subjects and context

Participants of the investigation were 57 fourth-graders (age 9 to 11) from five classes of three different schools in Germany. Every child was asked to solve 38 tasks mentally in guideline interviews without a time limit. After each solution the children were asked to explain their solution processes. The interviews took place in a separate room in the schools, lasted about 20-30 minutes and were videotaped.

The tasks

For this study different tasks for spatial abilities were developed. Based on the model of Maier (1999), the tasks were analysed and classified in the theoretical framework. The task evaluation by a pilot study resulted in a reduction and revision of the following four tasks for the components *visualization, spatial orientation, mental rotation* and *spatial relations.* All tasks consisted of different constructions with cubes such as the well-known objects in Shepard and Metzler (1971). Altogether four tasks with 38 subtasks were constructed and solved by each child. Due to the theoretical classification specific strategies were expected in the solving process of every task.

T1: Buildings out of Soma-parts

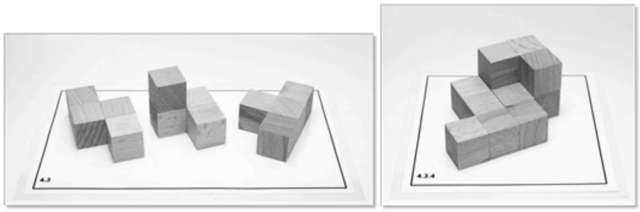

Figure 1: T1 – Buildings out of Soma-parts

The children were shown a complex building and two or three Soma-parts (s. figure 1). They had to decide if the right construction can be built out of the three Soma-parts given on the left. The mental movement of parts, i.e. the *move object strategy*, is typical for the component *visualization.* A dynamic process is expected while the person's viewpoint can be described as outside. Task 1 consisted of three sets with four buildings made of these parts.

T2: Who sees it?

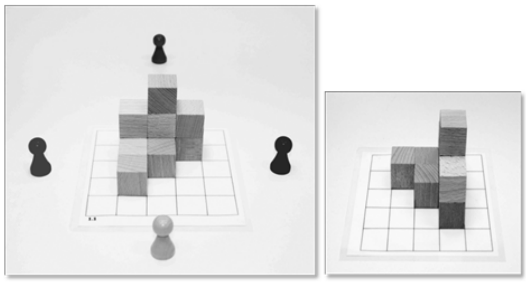

Figure 2: T2 – Who sees it?

Task 2 is a typical perspective change task. The children were shown a building and several photos from different views, including wrong photos (s. figure 2). They had to decide if one of the figures sees the configuration like it is given on the photo. The *move self strategy* is typically expected as a dynamic strategy, in which the person is mentally a part of the task situation. Due to these expectations the task can be classified as a *spatial orientation* task. Task 2 consists of two different buildings with seven photos made from different views.

T3: Is there contact?

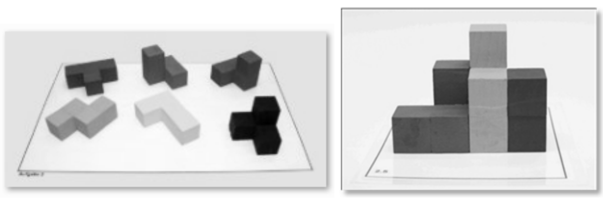

Figure 3: T3 – Is there contact?

Task 3 demands the analysis of the relative positions of Soma-parts in complex buildings (s. figure 3). The children had to decide if the two parts on the left and the right of the building touch each other with a whole face. This task is typical for the component *spatial* *relations*. The person mentally views from outside and is expected to use *key features strategies*, which are seen as static. Task 3 consists of seven different buildings.

T4: Match the cube-snakes

The children were shown two snakes out of six cubes and asked, if they are the same or different (s. figure 4). This task is a typical *mental rotation* task. The dynamic strategy *move object* is expected and the person has to take an internal distance view from outside. Given the snake on the left, five 'matching-snakes' had to be judged.

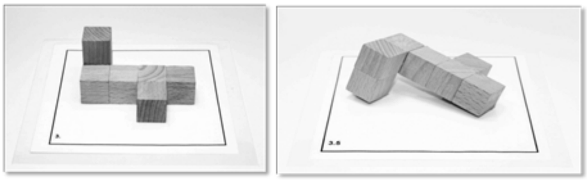

Figure 4: T4 – Match the cube-snakes

Procedure of analysis

The analysis of the strategies was done on different levels. First, verbal expressions, such as *"I put these two together, so I would put this one here"*, and gestures, such as "*shows a rotation of parts with her hands*", of each child and each sub-task were written down. They were marked as successful or not-successful. Secondly, these explanations were labelled with shortcuts of strategies (s. as an example T1 in figure 5). Similar explanations got the same label. Thirdly, the strategies were characterized and structured in strategy groups with different sub strategies (s. figure 5).

***Buildings out of Soma-parts*:**

BA: Reference to the numbers BF: Reference to the form BE: Reference to single elements	→ A: Analytic
B: Mental moving of objects R: Gesture of rotation	→ H: Holistic
ZU1: Building up 1 ZU2: Building up 2	→ (H) ZU: (Holistic) Building up
ZE1: Deconstruction 1 ZE2: Deconstruction 2	→ (H) ZE: (Holistic) Deconstruction

Figure 5: Strategy groups for *"Building out of Soma-parts"*

RESULTS

T1: Buildings out of Soma-parts

To solve this task, the children showed 100 times an analytic strategy and 507 times a holistic strategy. We categorized a solution as **analytic key feature strategy** if the children referred in their explanations to the number of cubes, to the form of the parts or to single elements of the construction. Three different groups of **move object strategies** could be differentiated as **holistic** strategies: Some children showed a *general strategy to move the parts mentally* and/or to support this by a *rotation gesture*. The main group (459 times) consisted in two forms of a mentally *building up* the construction out of the parts, whereas the third group showed different forms of *deconstruction* into the parts.

The data show furthermore, that holistic strategies lead with 70,1% to the correct answer, analytic strategies were with 34,0% successful. These results confirm the theoretical analysis of the task. For *visualization* tasks the *move object strategy* is a typical dynamic strategy with a view from outside.

T2: Who sees it?

Again analytic (495 times) and holistic (189 times) strategies were used. As **analytic key feature strategies** the children showed some kind of *structure analysis*. They verbalized how special parts of the construction helped them or they identified the wrong pictures by referring to the symmetry of the construction. The second analytic group tried to refer to *nonessential aspects*, such as the colour of the cubes. Two major groups of **holistic** strategies could be observed, one *moving mentally the cube-buildings,* which can be seen as a **move object strategy**, the other trying to *change the own perspective,* which is typical for **move self strategies**.

The high number of analytical strategies as well as the better results, the children got by using them (78,6% correctness vs. 67,7% by using holistic strategies) contradict the expected behaviour. The task is a typical perspective change task for *spatial orientation*, in which *move self strategies* are to be expected. Although nearly all holistic strategies show this behaviour (185 times), analytic strategies were used three times as often.

T3: Is there contact?

This task does not show such differentiated results. Only one analytic (310 times) and one holistic (13 times) strategy group could be observed. As **analytic key feature strategies** the children *analysed the structures* by referring to characteristic features or by referring to the positions of parts in the building. The **holistic** strategies which could be identified are typical **move object strategies**, which already were described for task T1: They showed a *general strategy to move the parts mentally* and sometimes supported this by a *rotation gesture*.

The theoretical task analysis and the data correlate. The task, as a *spatial relation* task, implies mainly analytic strategies (*key features)* combined with a viewpoint from outside and no dynamic processes. 67,4% of the analytic strategies succeeded, but also nearly all holistic strategies resulted in the correct answer. This holistic solving process seems to be rare but effective.

T4: Match the cube-snakes

As in every task analytic (95 times) and holistic (172 times) strategies were used. We categorized a solution as **analytic key feature strategy** if the children referred to special parts of the snake, to the number of cubes or to the symmetry of a wrong snake. Two groups of holistic strategies could be observed: 166 times the children told to *move the snakes mentally,* which can be seen as a **move object strategy**, very few tried to *change the own perspective,* which is typical for **move self strategies**.

Both groups of strategies were nearly similar effective: 83,7% succeeded with a holistic strategy whereas 75,8% got the correct answer by an analytic strategy. These results are consistent with the theoretical expectations. The snakes as a *mental rotation* task were thought to evoke dynamic procedures such as the *move object* strategy.

CONCLUSION

The data confirm the hypothesis that solution strategies are not only task-oriented but person-oriented as well. Nearly every subtask showed holistic and analytic strategies although in most cases the strategy expected predominated. But you can always find some children who do it very successfully in an unexpected way. To examine the key issues which influence the choice of a strategy, further analysis of the data is necessary. It may be that some children have their special preferences while others are more leaded by the tasks. Last but not least our other two questions have to be investigated as well: How do the form of the presentation and the mathematics ability influence the choice of a strategy and the success of solving spatial tasks?

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ENTHUSIASM TOWARDS MATHEMATICAL STUDIES IN ENGINEERING

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There are more and more students in engineering struggling in their mathematical studies. The main idea of the study was to recognize those freshmen in engineering who are at risk in their mathematical studies, i.e. they drop out courses or do not pass them (critical students). If these students were recognized at the beginning of their studies, teachers could support their learning from the day one and, hopefully, they would graduate in time. This particular study was focused on students' motivation and self-regulation as they are essential for learning in the universities. Students were asked to fill in two questionnaires with statements about studying mathematical courses. Some critical students could be identified based on their motivation and self-regulation but extra information may also be needed.

INTRODUCTION

Mathematics and physics are the basis for engineering studies. Without mathematical knowledge, it is impossible to work as a professional in the field although programs and computers do most of the calculations. Mathematics also helps in logical thinking that is essential in the field of technology. In Finnish universities of applied sciences, there are more and more students who have difficulties in passing mathematics courses despite the good results of Finnish students in PISA.

In the previous study (Porras 2011), it is shown that freshmen's previous education does not give enough information for progressing in mathematical studies. Proficiency test at the beginning of their studies gives a better hint but much more information is needed. Neither motivation nor self-regulation alone helped much more. In this study, it is analysed whether they together could help in profiling students. The main concern is that factors found should be easily checked by teachers and, also, easy to recognize by freshmen.

THEORETICAL FRAMEWORK

Motivation

Motivation is the base for learning. If there is no earthly motivation, there is no learning either. According to Waitley (1990), motivation cannot be pumped in with incentives, pep talks or diatribes. People can be encouraged and inspired with those but only if they want it by themselves.

Kauppila (2003) separates learning motivation into five groups: *avoidance motivation, diverged motivation, escape motivation, achievement* (or *performance*) *motivation* and *intrinsic motivation*.

A student has avoidance motivation if they are disregard or reluctant toward the subject and they have a weak self-efficacy in learning achievements. There are subconscious and, maybe, also conscious processes which prevent learning. For example, a student with dyslexia may have gotten incorrect results in mathematical problems because of miswriting. These failures may induce avoidance motivation (Förster et al. 2001). It may also be that mathematics is irrelevant to a student as they cannot relate to it personally (Guthrie et.al 2009). Ryan and Deci (2000) define amotivation as “the state of lacking an intention to act”.

Diverged motivated students have lots of other things in their mind such as working, hobbies or acting in a student union. A student is active but there is not enough time to study. These students could be excellent in their studies if they were concentrating on studying.

In escape motivation, a student just tries to pass courses with learning by heart. The motive is to escape failures. Ryan and Deci (2000) define escape motivation as external regulation. These students do everything only for external reward, such as extra points for passing the course, or for external demand, like “it must be done to pass the course”. According to Ryan and Deci, these students experience behaviour as alienated or controlled. Bong (2009) states that these students have performance-avoidance goals. These kinds of goals are usually demonstrated by negative associations with self-efficacy and academic performance. Achievements for these students vary very much and studying is surface level learning according to Kauppila (2003).

Achievement motivated students want to do courses well. They may do this to avoid quilt, anxiety or shame, for effect or to appear to be better than the others. Grades are important to these students. Ryan and Deci (2000) call this as *introjected regulation*. These people act with the feeling of pressure. According to Bong (2009), they have performance-approach goals that often demonstrate positive association with self-efficacy and academic performance.

Ryan and Deci (2000) define intrinsic motivation as “the doing the activity for its inherent satisfaction rather than for some separable consequence”. Intrinsicly motivated students are interested in the subject and want to get deeper understanding: they use a lot of time for studying. They are motivated by the subject and grades are not so important to them. Ryan & Deci (2000) and Middleton & Spanias (1999) also state that all kinds of rewards, directives, deadlines etc. diminish intrinsic motivation. Thus, autonomy seems to be important in intrinsic motivation. Intrinsic motivation can also be problematic in academic studies if a student does not graduate because of being fond of studying (Lonka et al. 2004).

According to Waitley (1990), motivation is an emotion. It is dominated by fear or desire. Fear can sometimes even prevent to reach one’s goals: it reminds of failures. A student may not even have expectations for success (avoidance motivation, escape motivation). These students use lots of phrases like *I cannot*, *I have to* or *I must try*. If motivation roots from desire, it encourages reaching one's goals. These students have lots of expectations and may not even have thought about failure. They use phrases like *I want, I can* or *I do* (Waitley 1990).

Self-Regulation

Self-regulation or self-regulated learning (SRL) may be seen as the self-directive process where learning is viewed as an activity of learning in a proactive way. Zimmerman (2000, 2002), Shunk and Ertmer (2000) define self-regulation as self-generated thoughts, feelings and behaviour that are oriented to attaining goals. Self-regulated learning is learnt by experience and reflection. It could be said to be an academic skill, which matures along the studies (Pintrich 1995).

Four *general assumptions* in SRL (Pintrich 2000, 2004) are that

- students are active participants,
- they are able to monitor, control and regulate certain aspects of their cognition,
- they can reflect their learning to some criteria, goal or standard and
- self-regulation is a compromise between personal, contextual characteristics and actual performance.

All these assumptions are also present in the phases of self-regulation by Zimmerman (2002). Zimmerman sees self-regulation in three phases, whereas Pintrich (2004) sees it in four phases. Differences in processes are not significant, as both consist of the situation before studies, during the studies and after studying.

The first phase describes a student's forethoughts, planning and activation before studying (Zimmerman 2000, 2002). Zimmerman calls them task analysis and self-motivation beliefs. At this phase, a student sets and plans steps for achieving a goal according to perceptions of task difficulty, value and interest. When considering the task difficulty, the student's beliefs about personal capability for doing the task, e.g. self-efficacy, are also perceived. If the student has low outcome expectations and no intrinsic interest or value for a topic, very much cannot be expected either for self-motivation or strategic planning. Students have motivation to learn in a self-regulated fashion, if they are interested in the task and value the learning for its own merits.

The second phase in Zimmerman's self-regulation process consists of self-observation and self-control. Pintrich (2004) separates them into two phases but the basic idea is the same: deployment the methods selected and controlling one's effort. At this phase, a student becomes conscious, for example, of time use and need for help (self-observing). When self-controlling, the student increases or decreases the effort, e.g. finds some tutoring for help. Students may have higher assumptions of actual time spent on studying than it turns out to be if self-recording the time. One part of self-controlling is focusing one's attention. Pintrich (2004) also points out that self-observation and self-control are not distinguished in terms of people's experiences.

The third phase called self-reflection is about reactions and reflections. Self-judgment has two kinds of forms: self-evaluation and causal attribution. In self-evaluation, a student is comparing one's performance against some standard, for example, another person's performance. In causal attribution, a student reasons unsatisfactory results. Causal attribution is very important phase for self-motivation. If a student is attributing unsatisfactory results to factors over which they have only a little control, e.g. limited ability, luck, easy task, (Shunk and Ertmer 2000), it implies that the student cannot improve their results and it may decrease their motivation. On the other hand, if poor results are attributed to a wrong strategy, it may sustain motivation.

The self-reaction involves feelings (self-satisfaction) and reactions (adaptive/ defensive). Self-satisfaction level affect the motivation level: high self-satisfaction may enhance motivation whereas low satisfaction may question the worth of studying. In adaptive reaction, a student increases the effectiveness of learning. The student may modify an ineffective learning strategy or totally discard it. In defensive reaction, a student protects their self-image. It may happen by dropping a course or by being absent from a test. The student is avoiding the opportunities to learn and perform better.

In many models of self-regulation, although Zimmerman presents the phases to be cyclic, the last two phases may occur simultaneously and dynamically (Pintrich 2004). During the study process, a student may change their goals and plans on the basis of the feedback from the monitoring, control and reaction processes. A student with low self-efficacy may after positive reactions change their plans in the hope of, for example, better grades.

QUESTIONNAIRES

In this study, the students were asked to complete motivation and self-regulation questionnaires. Answering was not done anonymously so that grades of mathematical courses could be merged with answers. No identification is included to the final data.

The motivation questionnaire consists of 15 questions with four alternatives in each question. Those 60 alternatives altogether handle all motivation groups presented by Kauppila (2003). Students were asked to give 1, 2 or 3 points for alternatives in each question. The highest points should be given to the alternative best describing the student, the second highest to the next one etc. They did not need to use all points but at least one point must be given. Inside the question, the points could not be given twice. At least one alternative was left empty in every question. This kind of ordering for alternatives was used to know which of the alternatives motivated students most. For example, the question 14 is as follows:

How much do you think to devote to mathematics courses?

- I will put off till the end of studies.
- I work to pass courses.
- I work to get high grades.

- I work because mathematics is interesting/useful.

For example in the question 14, the alternative ‘a’ refers to a lack of motivation, the alternative ‘b’ refers to extrinsic motivation, the alternative ‘c’ to performance motivation and the alternative ‘d’ for intrinsic motivation. Furthermore, the question 16 includes 17 alternatives, how teachers could increase students’ motivation. The student could tag at most five of them but no ordering was asked. These questions were also included in analyses although they were originally meant for giving ideas for further studies. In fact, they give a lot of extra information about enthusiasm.

In the questionnaire of self-regulation, there are 12 questions with four alternatives in each question. Contrary to the motivation questionnaire, students were asked to use five-level Likert scale in self-regulation questionnaire. One point referred that *it does not describe me at all* or *I hardly ever feel like this*. Five points referred *it describes me* or *I feel like this almost all the time.*

MOTIVATIONAL AND SELF-REGULATIONAL FACTORS EXPLAINING THE GRADES

The data is analyzed with IBM SPSS Decision Tree®. The data consists of all those Finnish students who started their studies in the academic year 2010 – 2011. The number of data is 138 but it can change in different cases if all information did not exist. Not all students answered the proficiency test but they answered the questionnaires. These students were also included in the data as questionnaires are more important to this study.

The first factor found when classifying grades by the results in the proficiency test was that 28 of 29 failed students got bad results in the proficiency test. It is worth of bearing in mind as the decision tree does not remind about this information at every stage. Unfortunately, the data does not have the information whether the one failing student with at least average result in the proficiency test dropped out of university or for some other reason was not attending the examination. All grades used in the data are from the first examination. Any following attempts to pass or to raise the grade are not included.

When the grades of the first course in mathematics were classified with background knowledge and questions in motivation and self-regulation questionnaires, the cornerstone seems to be the student’s own beliefs in easiness of mathematics. All students strongly feeling that they probably do not need guidance in mathematics as they have always learnt mathematics easily seem to get higher grades in their first mathematics course. None of these students failed the course and the most grades were either 4 or 5. Grades 4 and 5 were gotten by the students who were using so much time for studying mathematics that they can solve all given problems or did not feel studying mathematics frustrating at all.

If the student gave at most 1 point to the need of guidance and did not give any points to the alternative *I work to pass the courses*, they were more likely to get high grades. These students neither gave points to the alternative *I will put off my mathematics till the end of studies* suggesting they have a good self-esteem. The more points were given to these statements the lower grades students got from their course in general.

When students were categorized according to the results in the proficiency test ('bad', 'average' or 'good'), all those students, who got bad results and marked that they would not probably even notice suddenly cancelled lectures, were failing the course (17 of 19 students). These students seem to neglect the lectures so failing is not a surprise. This study does not tell whether these students drop out of university.

It was also studied whether there can be seen any common factors among those students who failed the course or just squeak through it with the grade 1 or 2. If students got the bad results in the proficiency test and gave any points to *I need guidance as my knowledge in mathematics is not sufficient*, they were getting low grades (42 students of 57 students). It was also found that the less points given to the statement *I work as mathematics is interesting/useful* the lower grades they got, although students did not feel their knowledge in mathematics would be insufficient. In overall, if students gave any points to the statement *I need guidance as my knowledge in mathematics is not sufficient* but did not give high points to *When I am studying mathematical courses I often use more time than I scheduled*, they were marked as critical students.

After these findings, students were categorized to *critical*, *weak*, *average* and *excellent* ones. At the beginning of categorizing, all students were assumed to be average students. Students were categorized differently only if they matched the criteria mentioned earlier. When this factor called *student type* was included to the decision tree, more information could be found. Students, who were categorized to be weak in mathematics and did not feel they should do given problems for learning and, furthermore, did not adjust their learning to meet the course requirements, could be marked as average. These students were probably not studying just to please teachers.

When previous knowledge was included in the student type, it was noticed that students, who were not working for to get high grades but were hoping for well-paid job, were failing the course more often than other weak students. The grades of weaker students were higher, if they were at least slightly working for high grades and wanted to have time to read up about the topic. When this new knowledge was included to the student type all those average students, who were interested in learning as much as possible, could be marked as excellent. The decision tree showed after this that those students marked as excellent at this point got the grade 5 if they did not mind how the grades were determined or solved all the given problems before the next lecture.

It was also found that weaker students seem to miss applied examples from their professional field in mathematical courses. The intuition among mathematics teachers was that applied examples are asked by qualified students. It seems that these applied examples from students' professional fields give motives for weaker students; they know why they should study the topic.

These results are only from students, who started their studies in the academic year 2010 and only from the first mathematic course. The filling data from the next academic years will either confirm the results or supplement them. The first mathematic course was selected because students' motivation to study mathematics after the first course is crucial for advanced courses. If students do not see the importance of the mathematics during the first course or they are getting stale with too easy topics, they are losing their motivation to study.

CONCLUSION

Several factors revealing progress in mathematical studies could be identified. Motivation seems to play a major role compared to self-regulation. According to Miller (2010), motivated students seem to be better in self-regulating their studies. It is not clear whether it is because they are knowingly regulating or because they are just interested in the topic and/or they focus in achieving their goals. Ryan and Deci (2000) define amotivation as "the state of lacking an intention to act". Is it even realistic to assume any self-regulating from weakly motivated students?

The next step is profiling the students. The factors of motivation and self-regulation with the background information (e.g. previous studies, results in the proficiency test) should form a base for profiles. Writing profiles into a form, from which freshmen would recognize themselves, may be hard. Although written profiles might be good enough, the lack of freshmen's self-knowledge may cause misinterpretations. We have to remember that freshmen may not have earlier studies in any university, so they may not know how they act in an academic freedom.

However, results presented here shoe that most of the critical students and, as well as, the excellent students could be recognized. When critical students are recognized, their studies can be supported from the beginning. It gives confidence for students and saves resources. On the other hand, excellent students could have different forms to study the courses to maintain or even to increase their motivation.

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INTERPLAY BETWEEN SEMIOTIC MEANS OF OBJECTIFICATION USED IN PROBABILITY EXPERIMENTS

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Probability has almost vanished from curricula internationally. This article seeks to address the reasons for probability vanishing from curricula by investigating 9th grade students' emerging probabilistic thinking while working with a ThinkerPlots combinatorial problem. In this article attention is paid to the interplay of the semiotic means of objectification involved in probability experiment. Results from this research study with grade 9 students suggest that the interaction between semiotic means of objectification help students to modify, or refine their probabilistic thinking about combinatorial problems.

INTRODUCTION AND THEORETICAL FRAMEWORK

The concept of probability received "different interpretations according to the metaphysical component of people's relationships with reality" (Batanero et al. 2005, p. 16). Probability is a difficult concept to pin down and there are ongoing controversies over the interpretation of the concept; perhaps evidenced by the late formal development of probability as a defined piece of mathematics. The formation of probability within a mathematical theory has been associated with a large number of paradoxes which indicate the disparity between intuitions and formal approaches within the conceptual development of probability.

Probability establishes a different kind of thinking compared to other domains of Mathematics: It does not adopt a causal or a logical approach to quantify randomness (Kapadia and Borovcnik 1991). Subsequently, counterintuitive ideas concerning the estimation of probabilities are evidenced at the very heart of the subject, even when dealing with relatively simple applications at very elementary levels. On the contrary, in other domains of mathematics, counterintuitive ideas occur when working at a high degree of mathematical abstraction. This central feature of probability gives rise to learning difficulties and a set of misconceptions that are so deeply ingrained in a student's underlying knowledge base that these misconceptions still remain at high school level (Shaughnessy, 1992). The literature is abundant with psychological research on how even adults, who are trained in probability and statistics, use various, often misleading, heuristics to make judgments of chance (for example, Kahneman et al. 1982). A weakness, however, of this misconceptions research is that it arguably fails to provide an account of productive ideas that might be used as resources for conceptual growth within a complex system of learning probability. Instead of dismissing or eradicating misconceptions, we should considered them as starting points which provide a pedagogic challenge of how to build on learners' impoverished view of probability and help students develop effective secondary (scientifically learned or taught) intuitions of probability (Fischbein, 1975). Fischbein's work (1975) on probabilistic intuitions offers a more positive outlook promoting the notion that novel pedagogies might provide support for the development of "better" probabilistic intuitions. Since then, research has provided experimental evidence of students' probabilistic thinking and reasoning to support the teaching of probability and the role of technology in fostering students' understanding of the domain. However, a recent investigation among researchers of stochastics (probability and statistics) revealed that probability has almost vanished from curricula internationally (Borovcnik, 2011), because "probability is (a) orientated too much towards mathematics; (b) too tightly connected to games of chance and possibly amoral; and (c) only required to justify the methods for inferential statistics." (p. 78). Pratt (2011), in turn, argued that it is necessary to stress in curricula an alternative meaning for probability, one that is closer perhaps to how probability is used by statisticians in problem solving" (p. 1).

This article seeks to address the first reason for probability vanishing from curricula by using simulation when teaching probability. It does not exclude connections with games of chance because probability concepts originate from such games. It also introduces an alternative meaning for probability –probability as a modelling tool– which is explored through new technological developments. The aforementioned research question is embedded in the theory of *knowledge objectification* (Radford, 2009), a theoretical perspective on teaching and learning, in which learning is taking place as a social process through which students become progressively conversant with cultural forms of reflection (Radford, 2009). Radford advocates that a distinctive feature of this theory is that "thinking cannot be reduced to impalpable mental ideas; it also made up of speech and our actual actions with objects and all types of signs. Thinking, hence, does not occur solely in the head, but *in* and *through* language, body and tools" (Radford, 2009, p. 113).Within the theory of objectification, the distinctive sensuous and artefact-mediated nature of thinking emphasizes the semiotic means of objectification through which knowledge is objectified. The semiotic means of objectification include kinaesthetic actions, gestures, signs (e.g., mathematical symbols, graphs, inscriptions, written and spoken language), and artefacts (e.g. rulers, calculators). Thus, in what follows, in the practical investigation of 9th grade students' probabilistic thinking, attention will be paid to the interplay of semiotic means of objectification in students' probabilistic thinking.

METHODOLOGY

Data collection

The data come from an ongoing research study conducted in a rural secondary school. The data have been collected during regular mathematics lessons designed by the teacher in collaboration with the researcher. In these lessons the students spent extensive time working in pairs. The researcher, who was also acting as a teacher, interacted continuously with different pairs of students during the pair work phase in order to probe the reasons that might explain their thinking. The data collected included audio recordings of each pair's voices and video recordings of the screen output on the computer activity using Camtasia software. When students' body language or facial expression appeared to be indicative of their conceptual evolution, notes were kept. The researcher (Re) prompted students to use the mouse systematically to point to objects on the screen when they reasoned about computer-based phenomena in their attempt to explain their thinking. Plain paper was also available for students' use in case students needed to explain their thinking in a written form. This article focuses on one pair of students, Rafael (Ra) and George (G).

Data Analysis

In the first lesson, students watched an instructional movie that shows how to use *TinkerPlots2* features to build a simulation of rolling two dice. They then built a simulation of rolling four dice and ran the simulation many times.

In the second lesson, students watched a *TinkerPlots2* movie. *ThinkerPlots 2* provides a sampler that is essentially a non-conventional form of probability distribution. The *ThinkerPlots* movie showed how to use two counters to generate a sample space of rolling two dice. Each counter had numbers one through six. As the simulation ran, the right counter circled though the numbers one through six, but the left counter stayed on one. Then the left counter advanced to two and the right counter circled through one to six again until all the possible outcomes were produced. The students observed the systematic listing of all possible outcomes of rolling two dice and the creation of a graph that shows the sum of two dice and the use of the sample space to calculate theoretical probabilities. The students were asked to use counters to build a sample space of rolling four dice and then, after answering some questions, to come up with general rules applied.

In tune with our theoretical framework, to investigate students' probabilistic thinking a *multi−semiotic* data analysis was conducted. At the first stage, the audio recordings were fully transcribed and screenshots were incorporated as necessary to make sense of the transcription. The most salient episodes of the activities were selected. A slow-motion and frame-by-frame analysis of video was also contacted. Focusing on the selected episodes with the support of the transcript, along with a detailed account of significant actions, gestures, and artefacts used; I study the role of spoken language with the gestures including students' actions when pointing to objects on the computer screen and artefacts.

RESULTS

The students spent the first day watching an instructional movie that showed how to use *TinkerPlots2* features to build a simulation of rolling two dice. Then they built a simulation of rolling four dice and made a graph showing the sum of four dice. They ran the simulation and observed the possible outcomes of rolling four dice. The researcher spent the rest of the first day discussing with the students the graph that showed the sum of four dice. They run the simulation 80 times.

1 Ra: We see 14 is the most likely sum. (Gesture–uses the mouse to point to the 14th column of the graph on the screen)

2 G: Run the simulation again.

3 Ra: This time it's 15. (G– uses the mouse to point to the 15th column of the graph on the screen)

4 G: Run it again, 80 times.

5 Ra: Okay, it's16 now. (G– uses the mouse to point to the 16th column of the graph on the screen)

6 G: It's a bit inaccurate.

7 Ra: It's in that general area. 13, 14, 15, 16 but they're not all high values. Like see, 14, 15 taken a drop here (G-uses the thumb and the index finger of his right hand to point to the interval from 13th to 16th column and then uses the mouse to point to the 14th and 15th columns of the graph individually).

Rafael changed from using the mouse to point to individual columns of the graph, to emphasize an interval on the graph by using two pointing fingers (the thumb and the index) of his right hand and then again he used the mouse to point to individual columns of the graph. Gestures dominated Rafael's respond to the researcher's question about the most likely 4-dice sum or a range of 4-dice sums. The coordinated use of spoken language and the use of gestures serve as semiotic means of objectification for Rafael. The boys continued their discussion about the most likely 4-dice sum.

9 G: Um, probably 14 because the first time it was the highest. (G– uses the mouse to point to the 16th column of the graph on the screen)

10 Ra: It's remained constant.

11 G: Second time wasn't yeah. It's remained constantly fairly high.

12 Ra: It's um, generally it's usually the same area most likely so that would seem the most likely one that would come up. 13, 14, 15, 16 yeah, but particularly 14's always stayed constant. Like even though it's not the highest here, it's always stayed in the range of the top highest.

15 G: Yeah, it's in the top 3.

The boys reflected on previously observed graphs. Rafael became consciously aware of a region in which the most likely outcomes were included. This region was placed around the 14th column of the graph. Although the 14th column was not always the highest, it was according to the boys, constantly included in that region. The boys continued to observe features of the graph while running the simulation several times (100 rolls each time).

16 Ra: Yeah, but 13, 14, 15, 16 they always get high. (G-uses the thumb and the index finger of his right hand to point to the interval from 13th to 16th column).

17 G: When 15 was our highest, 15 was the most likely outcome by overall.

18 Ra: If you're viewing in consistency standards, it would be 14. It seems to be the most consistently high, or the most consistently high or most consistent and reliably one to stay around that level or that range.

19 Re: Are you talking about the distribution of the outcomes.

20 Ra: Well it's generally seems to stay between um… 12, 13'ish to about 17. (G-uses the thumb and the index finger of his right hand to point to the interval from 12th to 17th column).

The boys ran the simulation several times increasing the number of rolls from 200 to 250. They both concluded that 14, 15, 16 have the highest consistency. George added:

21 G: That seems to be what's happened. That there the most likely you're going to get.

The next day (2rd lesson) the researcher drew students' attention to permutations without using any explicit mathematical terminology.

22 Re: Which 4-dice sum can be made up by adding different ordered numbers?

23 Ra: It should be …

24 Ra: There is only 4 chances out of the possible outcomes. Like if I was to get 24 I would have to get 4 six's

25 G: That's very unlikely. We didn't get it there (referring to the graph they observed during the previous lesson).

26 Ra: The reason, the range of those numbers (14 to 15 range)…. were so high because there's so many dice. You don't need to get much of each dice. Those numbers there allow you to have a wider variety of combinations like um if I… 13, 14, 15 being the highest ones.

The boys were able to appreciate the connection between the outcomes from the previously observed graphs, with the different faces of 4 dices that summed up to those outcomes. While they attended to the shape of the graph, the permutations were not perceptually distinguishable as actual experimental outcomes. When the researcher asked students to list (on paper) those permutations that added up to 15, the boys began to randomly list the possible outcomes for rolling four dice.

27 Ra: If I write it down once, do you want me to write it down every other way I could write it down? I mean if I have, I have 6 – 4 – 3 – 2 do you want me to also write 3 – 4 – 6 – 2? Is it important?

28 Re: Yes. The order of the inscriptions is important. Why do you think that the order is important?

29 Ra: I'm thinking, I guess, I'm trying to like answer but I don't know if it's right but it's like it's, they vary…

30 Ra: … it's just umm, like the fact that it's variabled. Like it's varied umm, I think is important because like there is a better chance of getting that because the amount if variables that you get with the equation that equals up to 15….If that sort of makes sense, I don't know, I'm just thinking.

As the previous dialogue shows, Rafael seemed unable to understand the uniqueness of permutations (i.e., that [3,5,6,1] and [5,3,1,6] were distinct). When the researcher mentioned that the order of the inscriptions is important, Rafael had difficulties with articulating his thought. The researcher instead of impressing upon the students the importance of the ordered outcomes in a combinatorial experiment, asked students to watch a *ThinkerPlots* movie which showed how to use two counters to generate a sample space of rolling two dice. The students observed the systematic listing of all possible outcomes of rolling four dice and made the graph that showed the sum of four dice.

31 Ra: That's how I would have worked out the four but I would have never. it would take someone too long to do by hand anyway. And I suppose they also do this with locks and stuff. Yeah, when you get your bike lock. The 4 combination 1 to 6. It says how many combinations. I honestly never thought there would be that many. Pretty, amazing!

When Rafael was preciously asked to list all the possible outcomes of the sample space, he had difficulties with constructing combinatorial type outcomes, because he either did not exhaust the sample space or duplicated possibilities. When Rafael watched how *TinkerPlots* features systematically listed all the possible outcomes of rolling four dice using the odometer strategy, he was surprised by the number of possible outcomes. The animated generation of a sample space made it possible for the students to see the use of various representations in solving the combinatorial problem situation including the use of animations, systematic listings, table holding all possible outcomes of rolling 4-dice; These are semiotic means of objectification. When the students attempted to talk about the graph that showed the sum of four dice:

32 Ra: Look at this peak, being 13, 14, 15 high (G-uses the thumb and the index finger of his right hand to point to the interval from 13th to 16th column). We noticed that before (refers to the graph they observed during the first lesson).

33 G: The difference between these twos is that one (the graph created using all the outcomes of the sample space) is a lot more accurate as it's going through every single stage compared to the other one.

34 Ra: This is a lot more thorough as it goes through every single possibility you could have … It would probably be really helpful if you knew this, if you were gambling or placing bets… It shows you the possible outcomes if you're playing a game of Chance. If you want to know the possible outcomes of the ace coming up. Like you want to know if there is more chance than it won't come up, than it will not. You could do that also, at horses.

The gesture (uses the thumb and the index finger of his right hand to point to the interval from 13th to16th column) reminded boys of the previously observed graph, so they began comparing the two graphs before the researcher would ask them (the researcher intended to ask such a question). There is an interplay between three means of objectification: a gesture and a graph within a computer based environment, which is another means of objectification. After this interplay, we observe that although R and G had never been taught at school about the classicist and frequentist approaches to probability, the means of objectification helped them to develop understandings about the two approaches to probability. The students expressed a preference for using the classicist approach to conceptualise the probabilistic experiment. They viewed the classicist approach as a "more accurate" process, because it is based on all the possible outcomes of the sample space. The sample space appeared to the students as an essential property that can regulate the results of the graph, which shows the sums of 4-Dice theoretical frequencies. After the students had seen the systematic construction of the event space via combinatorial analysis performed by features of *TinkerPlots*, both Rafael and George made sense of the role of the event space in relation to the experiment it is said to model. They saw connections between the classicist approach and games of chance and suggested how to apply such a probabilistic approach to the solution of new problems.

DISCUSSION

This preliminary study provides insights into students' thinking about the kind of semiotic means of objectification involved in probabilistic thinking. Framed by the theory of objectification, it was suggested that thinking is a dynamic system consisting of multiple material-ideational components of thinking (e.g., inner and outer speech). Thinking progresses when the *semiotic means of objectification* interact with each other to form structuring relationships. Those relationships are organised and re-organised, until the appearance of new structuring relationships (Radford, 2009).

In this article, I have tried to capture students' use of semiotic means of objectification and the co-ordination between kinaesthetic actions, gestures, graphs, inscriptions, written and spoken language, and artefacts involved in students' probabilistic thinking. Our data offer an initial glimpse of the development of probabilistic thinking about combinatorial problems. It shows how in Grade 9, students with minimal probabilistic knowledge worked towards a solution. Students' "spontaneous" perception was successfully transformed thought the interaction of students with the *ThinkerPlots* combinatorial problem. This interaction might be conceptualize as occurring in the zone of proximal development out of which the students construct new understandings about probability and new probabilistic functions. The researcher questioned students as they explored the combinatorial problem, to promote students' combinatorial understanding. The interaction of spoken language with gestures (using fingers to point to the graph, using the mouse to point to the graph on the screen), signs (graphs, written inscriptions, tables) and artefacts (computer animation) lead students to modify, or refine some of their original structuring relationships between the *semiotic means of objectification*, or to consolidate new structuring relationships and understandings of probabilistic concepts

Future research will investigate more systematically the s*emiotic means of objectification* involved in probabilistic thinking and the impact of the interaction between those *semiotic means of objectification* on students' probabilistic thinking.

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LEARNING FROM ERRORS: EFFECTS OF A TEACHER TRAINING ON STUDENTS' ATTITUDES TOWARDS AND THEIR INDIVIDUAL USE OF ERRORS

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A constructive handling with errors is considered as an important factor for learning processes. Accordingly, instruction should enable learners to develop positive attitudes towards error situations and to use errors as learning opportunities. In a quasi-experimental study with students from 31 classrooms (grades 6 to 9), we investigate effects on students' attitudes towards errors as learning opportunities in two conditions: (1) an error-tolerant classroom culture and (2) condition (1) with an additional teaching of strategies for analyzing errors. Our findings show positive effects of the error-tolerant classroom culture on the affective level, whereas students are not influenced by the cognitive support. Moreover, there is no evidence for differential effects for student groups with different attitudes towards errors.

INTRODUCTION

"Mistakes are often the best teachers.", "Aus Fehlern wird man klug.", "Erreur n'est pas crime." - In many languages all over the world, we find proverbs about errors. Interestingly, many of these proverbs attribute a positive function to errors. This indicates the existence of a cumulative human experience in which errors can have positive effects. However, many people associate negative feelings with errors, which probably arise from the fact that errors are one of the most important criteria to assess the performance of individual actions. Traditionally, mathematics education research has analyzed patterns underlying students' errors related to different mathematical concepts (e.g. Radatz, 1979; Tatsuoka, 1984). We want to stress that our perspective on errors differs from these specific diagnostic research perspective. Our goal is not to analyze why a learner makes an error and which individual misconceptions or problems are responsible for this. Instead, we focus on the error-handling activities that teachers and students perform in mathematics lessons. The main questions are how students experience the activities of their teachers in error situations, how students individually use their own errors as learning opportunities and which aspects of mathematics instruction are beneficial for motivating and supporting students' learning processes when dealing with individual errors in mathematics.

THEORETICAL BACKGROUND

The concept "error" can be defined as a process or a fact that does not match a given norm (Oser & Spychiger, 2005). As expressed by the proverbs quoted above, it seems to be a consensus in educational science that learning from errors is principally possible. One explanation for this assumption follows from the theory of negative knowledge because an understanding of errors is considered to be necessary for distinguishing between correct processes or facts and the incorrect surroundings.

Theory of Negative Knowledge and the Role of Errors

The theory of negative knowledge postulates that individuals accumulate two complementary types of knowledge: positive knowledge about correct facts and procedures and negative knowledge about incorrect facts and procedures (e.g., Minsky, 1994). Negative knowledge is necessary to identify the boundaries of correct facts and processes and therefore, to distinguish between correct and incorrect facts and processes. Since individuals are usually not taught about incorrect facts or processes, individual experiences in error situations are considered necessary to acquire this knowledge (cf. Oser & Spychiger, 2005). Nevertheless, it is questionable whether all error situations are fruitful learning opportunities and how the acquisition of negative knowledge for a future error prevention takes place.

Based on prior research (Garuti, Boero, & Chiappini, 1999; Heinze & Reiss, 2007; Oser & Spychiger, 2005), we propose a process model describing two different ways of dealing with errors (figure 1). We distinguish a pragmatic, outcome-oriented and an analytic, process-oriented path of action. While the former proceeds directly from error detection to error correction, the latter path includes a closer analysis of the error and the generation of error prevention strategies. With respect to generation of negative knowledge, the latter approach can be expected to be more beneficial.

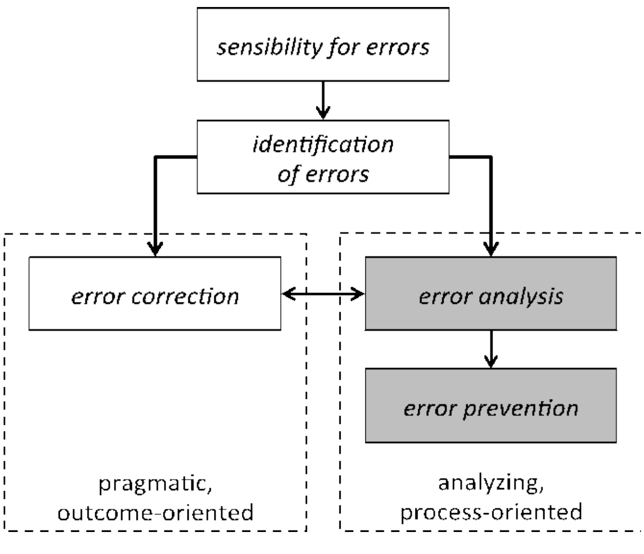

Figure 1: Process model for learning in error situations.

There is strong empirical evidence for this assumption from research on error management trainings in different domains of working life (Keith & Frese, 2008). Nevertheless, the choice of a productive, analyzing approach towards error situations does also rely on affective characteristics of the learner. Errors are often experienced as adverse events, and thus fear can impede their potentially positive effects. On the other hand, dealing with errors in learning showed to be more effective than avoiding them, if there is clear feedback and an error-tolerant culture (Keith & Frese, 2008; Nordstrom, Wendland, & Williams, 1998).

Error-handling Activities in the Mathematics Class

Findings from video-based investigations from Switzerland the USA, Italy and Germany (for a short survey see Heinze & Reiss, 2007) show similar tendencies how teacher deal with errors in mathematics class. Firstly, the number of errors in the public teacher-students interaction is comparatively low (about 3-5 errors per lesson). Secondly, about 90% of the public handling of errors in mathematics lessons are clearly teacher-directed. Oser and Spychiger (2005) explain the low number of errors per mathematics lesson with error avoidance behavior of teachers and students. In particular, teachers try to avoid interruptions in the ongoing instruction process and do not want to expose students who make errors. Accordingly, they pose their questions in such a way that students rarely give erroneous answers.

From the students' perspective on handling errors in mathematics class, there are hardly any empirical studies with a clear focus on this topic. Results from the Swiss group of Oser and colleagues based on questionnaire data from 295 4^{th}-9^{th} grade students indicate that students have a rather positive attitude towards dealing with errors during mathematics class as well as to the role the teachers play (Spychiger, Kuster, & Oser, 2006). The students perceive their teachers' error-handling as friendly and supportive and they rarely experience anxiety due to errors. However, they report only a moderate level for their individual use of errors as learning opportunities. Heinze, Ufer, Rach, and Reiss (2011) analyzed questionnaire data from 1674 students (grade 6-9). They showed that the perceived affective support of teachers correlates with lower anxiety and that the perceived cognitive support of teachers correlates with a more intensive use of errors as learning opportunities.

Heinze and Reiss (2007) investigate the effects of an in-service teacher training in 29 classes. Here, teachers of the experimental group received a combined training in error-handling and in teaching reasoning and proof, whereas the teachers of the control group only participate in a training on teaching reasoning and proof. The students were asked to evaluate how the teachers handled their errors. It turned out that only teachers of the experimental group classrooms improved their error-handling behavior significantly. Moreover, students' achievment in geometry proof increased significantly stronger in the experimental group than in the control group.

Borasi (1996) reports about several teaching experiments she conducted in a series of case studies. She developed a strategy of capitalizing on errors as springboards for inquiry in mathematics classrooms which is integrated in a specific teaching approach. In Borasi (1996) she summarizes that her case studies provide "anecdotal evidence" that learners can benefit from a specific teaching approach focusing the use of errors.

Summarizing these empirical studies, errors can be considered as an important factor of learning processes. However, students do often not use them as learning opportunities and it is unclear which aspects of mathematics instruction are relevant to change this. In particular, it is open if an error-tolerant classroom culture is sufficient to support students (affective support), or if in addition specific meta-cognitive strategies for a beneficial error-handling should be taught (cognitive support). Moreover, it is open if there are different groups of learners with specific perceptions of error situations so that differential effects of classroom interventions can be expected.

RESEARCH QUESTIONS

The aim of our research was to evaluate the effects of an error-tolerant classroom culture as well as specific interventions addressing strategies to use errors as learning opportunities. Moreover, we explored profiles of students' perceptions of error situations and the differential effects of the interventions on students with different profiles. Our study was guided by the following research questions:

- What are the effects of an error-tolerant classroom culture and interventions addressing strategies for learning from errors on students' attitudes towards error-handling?
- Is it possible to identify different types of students showing characteristic profiles with respect to their attitudes towards error-handling?
- If characteristic types of students can be identified, do the interventions affect students of different types in different ways?

DESIGN OF THE STUDY

We conducted a quasi-experimental intervention study with 6^{th} – 9^{th} grade students (12-15 years old) from 31 classrooms in different school types in Germany (comprehensive schools and schools from the academic focused school track Gymnasium). We applied a pre-post-questionnaire design with two experimental groups and a small control group. After cleaning the data from outliers, we obtained a sample of $N = 698$ students for the pre-questionnaire. Because of drop-out only $N = 571$ complete data sets were available for the pre- and post-questionnaire (cf. table 1).

Teachers of the *error-tolerant culture group* took part in a professional training program lead by the researchers. In this training, teachers were informed about the potential use of errors for learning and some of the empirical results described above. They were encouraged to consider errors as learning opportunities instead of useless interruptions in classroom. In particular, aspects of an error-tolerant and error-positive culture for mathematics classroom were discussed. Teachers of the *error-tolerant culture and strategies group* classrooms participated in the same training program and, in addition, they got materials dealing with strategies to learn from errors. These materials encourage learners to reflect on their errors made in homework or exercises and are based on the process model for learning from errors (right part in figure 1). In contrast to the first experimental group, these teachers got concrete ideas (and material) how they can support their students on a cognitive level. For an intervention check, teachers were asked to fill in a questionnaire how they used these materials. Teachers of the *control group* classrooms gave their regular lesson without any training and without using any additional material. Overall, the intervention took five months. The teachers in the second experimental group were asked to implement the material in their mathematics lessons regularly.

Intervention group	Classrooms	n_{pre}	$n_{pre\ \&\ post}$
Control group	4	87	73
Error-tolerant culture	13	267	218
Error-tolerant culture and strategy instruction	14	344	280

Table 1: Structure of the sample.

A slightly adapted version of the Swiss questionnaire on error handling in the mathematics classroom with 22 items (Spychiger et al., 2006) was used to assess students' attitudes towards error-handling before and after the intervention. As described in Heinze et al. (2011), four scales could be extracted (table 2).

Factor	Item example	Cronbach's α
Affective teacher support in error situations (TS_{aff})	*Sometimes our math teacher looks distressed when a student makes an error. [reversed item]*	.88/.91 7 items
Cognitive teacher support in error situations (TS_{cog})	*If I make an error in maths lessons my teacher handles the situation in a way that I can benefit from.*	.77/.83 4 items
No fear of making errors in mathematics lessons (NF)	*I become scared when I make an error in mathematics. [reversed item]*	.69/.73 3 items
Individual use of errors for the learning process (IU)	*In mathematics I explore my errors and try to understand them.*	.78/.81 8 items
Likert scale: 0 = strongly disagree, 1 = disagree, 2 = agree, 3 = strongly agree		

Table 2: Student questionnaire: Item examples and reliabilities (pre/post).

RESULTS OF THE STUDY

To evaluate the effects of the interventions, we conducted ANCOVAs for each of the four scales with the pre-questionnaire results as covariate and the group as a factor. There was a general regression towards disagreeing ratings from pre- to post-test, that has to be taken into account when interpreting the questionnaire data. There was no effect of the intervention on individual use of errors (IU, $F(567,2) = 1.338$, $p > .05$) and on cognitive teacher support (TS_{cog}, $F(567,2) = 0.495$, $p > .05$). The intervention had significant positive effects on affective teacher support (TS_{aff}, $F(567,2) = 9.476$, $p < .001$, $\eta^2 = .032$) and reduced the fear of making errors (NF, $F(567,2) = 3.765$, $p < .05$, $\eta^2 = .013$). There are no significant differences between the two experimental groups in these variables, but students from the experimental groups reported less fear and more affective support than the control group students (table 3).

M (SD)	Control group		Error-tolerant culture		Error-tolerant culture and strategy instruction	
	Pre	Post	Pre	Post	Pre	Post
TS_{aff}	1.71 (0.84)	1.47 (0.96)	2.03 (0.77)	2.02 (0.83)	2.12 (0.69)	2.12 (0.77)
TS_{cog}	1.91 (0.62)	1.74 (0.73)	1.84 (0.77)	1.79 (0.83)	1.91 (0.68)	1.81 (0.78)
NF	1.98 (0.77)	1.93 (0.87)	2.01 (0.73)	2.18 (0.71)	2.12 (0.74)	2.21 (0.75)
IU	1.78 (0.45)	1.57 (0.56)	1.71 (0.49)	1.64 (0.48)	1.68 (0.55)	1.60 (0.57)

Likert scale: 0 = strongly disagree, 1 = disagree, 2 = agree, 3 = strongly agree

Table 3: Development of students' perceptions of error situation.

To describe students' profiles, we carried out a cluster analysis using the four scales of the pre-questionnaire (Ward method). We could identify three types of learners with respect to their attitudes towards error-handling (figure 2), two types showing relatively high resp. low ratings on all scales and one type reporting some teacher support, little fear of errors, but also little use of errors as learning opportunities.

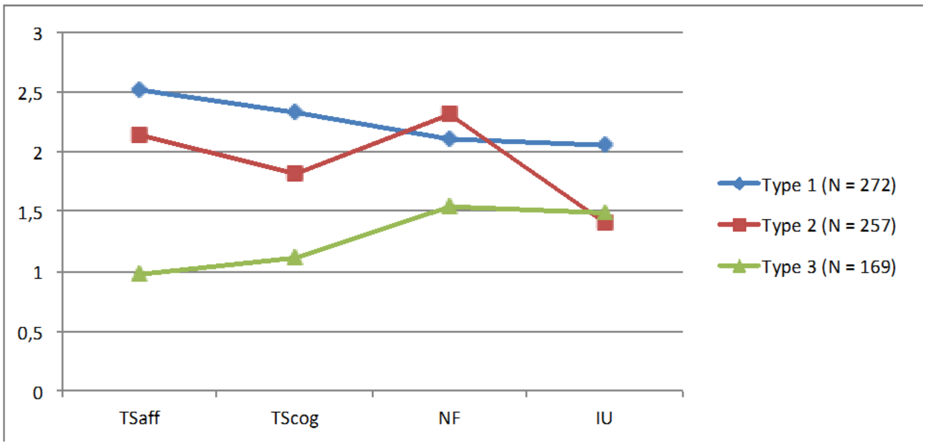

Figure 2: Profiles of the student types identified in the pre-questionnaire.

A MANCOVA with the four scales of the post-questionnaire as dependent variables, the corresponding scales in the pre-questionnaire as covariates, the intervention group and the student types as factors shows a significant effect of the intervention group ($F(1112,8) = 5.059$, $p < .001$, $\eta^2 = .035$) but no effect of student type

($F(1112,8) = 0.674$, $p > .05$) and no interaction effect between the two ($F(2232,16) = 1.220$, $p > .05$).

DISCUSSION

For teaching and learning in school and, in particular, in the mathematics classroom, errors are often considered as an important part of the learning process. In the presented study, teachers of the experimental group classes took part in a training about the role of errors and the importance of an error-tolerant classroom culture. In addition, some of these teachers implemented learning material in their lessons that encourage students to analyze their errors so that they can develop error prevention strategies (right part of figure 1). So in both intervention groups students got an affective support and in the second intervention group an additional cognitive support was provided.

Concerning research question 1, our findings show that - in comparison to a control group - there is a positive effect of both interventions with respect to students' fear of making mistakes and students' perception of affective teacher support. This indicates that the teachers of the experimental groups were able to implement an error-tolerant classroom culture and that this change had positive effects for their students on an affective level. However, a comparison of the two intervention groups does not give evidence for the influence of the additional systematic cognitive support. In particular, students' perception of teachers' cognitive support and students' reports on their individual use of errors do not change. We observe the same result when comparing the intervention groups with the control group. It seems that an affective support based on an error-tolerant classroom culture is not sufficient for inducing a change of students' perception of and handling in error situations with respect to the cognitive level.

The surpising result is that students of the second intervention group got a cognitive support but they did not notice this. One reason might be the unfamiliar demands for students when working with the implemeted material on analyzing errors. In Germany, students are not familiar with reflections about their own learning processes and, in particular, with reflections about their own errors. A possible second reason can be the quality of the intervention. Since we have only questionnaire-based self reports of the teachers, we do not know if they used the learning material in an adequate way.

Concerning research questions 2 and 3, we were able to identify three types of learners which report different perceptions of and handling in error situations during mathematics learning. However, the results of the MANCOVA did not reveal differential effects of the intervention on these three types of learners. Accordingly, for all three types we observed similar positive effects on the affective level and no effects on the cognitive level. Based on these findings, we assume that there is no need for a student type-specific affective support by an error-tolerant classroom culture. Concerning the cognitive support in error situations, the investigation of student type-specific instruction strategies might be a promising idea for further research.

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REFRAMING RESEARCH ON GENDER AND MATHEMATICS EDUCATION: CONSIDERATIONS FROM TRANSGENDER STUDIES

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While gender equity has been a focus of mathematics education research since the 1970s, scholarship to date has been predicated on a dichotomous view of gender that excludes the perspectives and experiences of transgender and gender nonconforming people. This dichotomous view of gender is not only exclusionary, but also limits the research questions posed in relation to gender and mathematics education. This paper aims to introduce a gender complex framework for mathematics education research as well as generate lines of research and pose possible research questions based on the framework.

Gender equity has been a focus of mathematics education research since the 1970s (e.g. Fox, Fennema, & Sherman, 1977; Fennema & Sherman, 1979; Sells, 1974). In fact, based on a survey of 510,241 education articles published between 1982 and 1998, Lubienski and Bowen (2000) found that about half of the articles which focused on both mathematics education and equity addressed gender, while the percentages addressing ethnicity, class, and disability were lower. The authors concluded that "gender research in mathematics education seems to have earned a sustained attention and respect, particularly when contrasted with the situation for ethnicity and class" (p. 631). However, scholarship addressing gender and mathematics education to date has used a simplistic model of gender predicated on a dichotomous view in which each student falls into exactly one of two gender categories: boy or girl. This is evident in the way in which Lubienski and Bowen's (2000) framed the topic of gender and mathematics education: "The data indicate that the mathematics education community has been responsive to the underrepresentation of women in mathematics" (p. 631). While the underrepresentation of women in mathematics is certainly a crucial research focus, the dichotomous view of gender overlooks the complexity of gender as addressed in recent gender theory addressing the existence, experiences, and perspectives of transgender and gender nonconforming people. Beemyn and Rankin (2011) surveyed 3,474 transgender and gender nonconforming adults in the United States; in addition to the categories "female," "male," and "transgender[1]," the participants identified their genders using over 100 different descriptors. Some of the

[1] While some people who have transitioned (Male to Female or Female to Male) identify as male and transgender or female and transgender, others no longer identify as transgender and identify simply as male or female. "Transgender" should not simply be considered adding a third discrete gender to the list.

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 337-344. Taipei, Taiwan: PME.

most common categories included "cross dresser," "male to female," "female to male," "genderqueer," "androgenous/androgyne," "bigender," and "both female and male." Given the diversity in ways people experience gender, a dichotomous view of gender excludes many people and perspectives from research in mathematics education. Such a simplistic view also limits the research questions that are posed. For example, a major focus in the history of research related to gender and mathematics education has been "gender differences." Despite a variety of perspectives and an acknowledgment that the topic of "gender differences in mathematics" is complex, this research continues to be predicated on a dichotomous view of gender. For example, in their edited book *Gender Differences in Mathematics*, Gallagher and Kaufman (2005) summarize the research as follows:

> The true nature of the relationship between gender and mathematics is much more complex than most people have been led to believe. Differences are found in relatively few aspects of mathematics performance. . .and when they are found, their causes are varied and often elusive. Indeed, individual differences in ability and achievement *within* gender are probably much larger than the differences *between* genders. . . .Yet there persists a monolithic stereotype that girls don't like math and aren't as good at it as boys. (p. 316)

Here, although it is acknowledged that differences "within" gender (that is among boys or among girls) are likely to be much larger than the differences "between" genders (that is, boys and girls), the underlying assumption is that gender is a dichotomous construct consisting of two mutually exclusive and exhaustive categories.

To my knowledge, only one other mathematics education scholar, Esmonde (2010), has considered gender in a way that moves beyond the dichotomous view that takes into consideration the existence of transgender people. In responding to the Toronto District School Board's plan for promoting "boy-friendly" education, Esmonde (2010) identified "the need to adopt contemporary views of gender that go beyond binary thinking" in mathematics education (p. 1). Esmonde (2010) noted that research on achievement gaps between boys and girls are "all based on binary conceptions of gender in that boys and girls are separated into two discrete categories and averages are reported as if they apply to the entire group" (p. 14). Esmonde (2010) suggested several lines of research based on a non-binary view of gender which might better serve boys' mathematics educational needs than do the current efforts in Toronto including a focus on "repertoires of practice" (Villalobos, 2009), extending scholarship on sexism to a focus on heterosexism, and a consideration of pedagogical approaches such as Complex Instruction (Cohen, Lotan, Scarloss, & Arellano, 1999) which are designed to intentionally create "equitable classroom spaces in which privileged groups do not dominate" (Esmonde, 2010).

PURPOSES AND METHODOLOGY

This paper responds to "the need to adopt contemporary views of gender that go beyond binary thinking" in mathematics education identified by Esmonde (2010, p. 1).

The purposes of this paper are twofold. The first purpose is to introduce recent developments in gender theory which take into consideration transgender people and perspectives in order to reframe the topic of "gender and mathematics education" in a way that moves beyond the entrenched dichotomous view that has shaped mathematics education research to date. The second purpose is to generate additional lines of research and possible research questions for mathematics education based on this gender complex framework.

In order to generate possible lines of research and research questions, I examined the research questions investigated in Forgasz, Becker, Lee, and Steinthorsdottir's (2010) edited volume *International Perspectives on Gender and Mathematics Education*. This book was chosen as a resource for generating research ideas because of its recent publication date, its focus on both gender and mathematics education, and its international emphasis. With the complex view of gender in mind (see below), I created a chart listing the main research questions for each of the chapters included in the book. In a separate column, I generated possible research questions by considering the original research questions in conjunction with the gender complex framework. After generating possible research questions, I inductively created the six broader categories which could constitute lines of research.

A COMPLEX VIEW OF GENDER

Recent scholarship in gender and transgender studies has provided a more complex lexicon and framework for considering gender. Previously, I (Rands, 2009) have developed a framework for gender complex education in relation to teacher education. In this section, I syntehsize ideas from this previous conceptualization with theoretical ideas from Mendick (2006) and Wall (2010) to provide a framework relevant to mathematics education.

Transgender studies has introduced terminology including gender identity, gender expression or presentation, gender roles, gender attribution, and gender assignment[2]. Gender normativity assumes that all of these aspects of gender will align in particular ways and can therefore be collapsed into a simplistic dichotomous concept of gender. However, the experiences of transgender and gender nonconforming people have demonstrated that these aspects of gender do not necessarily align in presumed ways and that gender is a complex phenomenon. *Gender identity* is the way people define their own gender. According to Bornstein (1994), gender identity "answers the question, 'who am I?' Am I a man or a woman" or an alternative gender? (p. 24). *Gender expression or presentation* refers to the "manifestation of an individual's

[2] Ursini, Ramírez, Rodríguez, Trigueros, and Losano's (2010) chapter in Forgasz et al. (2010) cites Lamas' (1997) definitions of gender assignment, gender identity, and gender roles in their synthesis of studies in México on gender and mathematics, but do not address the perceptions or experiences of transgender or gender nonconforming people. They also point out that "in the Spanish language this term has several distinct and additional referents [including] classes, species, or categories in which people, animals, or things are classified" (p. 148).

fundamental sense of being masculine or feminine through clothing, behaviour, grooming, etc." (Wilchins, 2004, p. 8). Gender presentation is how one presents one's gender to the world. A related concept, *gender attribution*, refers to the process by which others interpret a person's gender presentation. It is the process "whereby we look at somebody and say, 'that's a man,' or 'that's a woman'" (Bornstein, 1994, p. 26). *Gender roles* are "social expectations of proper behaviour and activities for a member of a particular gender" (Stryker, 2008, p. 12). Finally, *gender assignment* is society's official designation of a person's gender. In the United States and many other nations, gender assignment is based on a doctor's perception of a baby's genitalia at birth, which is then recorded on a birth certificate. This gender assignment is reiterated on many official documents such as school enrolment forms, driver's licenses, and passports. Taken together, these gender terms provide a much richer lexicon for disucssing gender than is usually used in mathematics research, especially with the insight from transgender studies that one's relation to the set of terms does not necessarily align in gender normative ways.

Yet, understanding the ways in which gender functions in particular situations also requires an understanding of the ways in which gender relates to power. Wilchins (2004) defines the concept of gender as "a language, a system of meanings and symbols, along with the rules, privileges, and punishments pertaining to their use" (p. 35). These meanings, rules, privileges, and punishments, Wilchins argues are constituted through a process of exclusion: "With gender, we create the meaning of *woman* by excluding everything that is non-Woman, and vice versa for man" (p. 36). In other words, "to understand gender difference we need to see it as *relational*" (Mendick, 2006, p. 11). Mendick (2006) examined how relationships between pairs of terms—social/biological, gender/sex, individual/social, masculine/feminine—come to shape conceptualizations of gender in ways that close off possibilities for mathematics students. Similarly, Wall (2010) introduced the concept of "mathematical genderfication" to describe the "ways in which *male* and *female* are produced as sexuate occupational identities through the learning of mathematics" (p. 87). The meanings of these paired terms (male/female, masculine/feminine, social biological, etc.) tend to be viewed as oppositional and mutually exclusive; however, Mendick (2006) argues, "blurring these distinctions can help generate new understandings of the messiness of everyday life in general and the relationship between mathematics and masculinities in particular" (p. 12). It is also important to note that in such pairs of terms, one term tends to be privileged over the other, as "man" is over "woman." According to Johnson (1997), "privilege exists when one group has something of value that is denied to others simply because of the groups [to which] they belong" (p. 23). Furthermore, whenever one social category is privileged, at least one other social category is oppressed. Oppression, in Johnson's (1997) definition, is "a social phenomenon that happens between different groups in society; it is a system of social inequality through which one group is positioned to dominate and benefit from the exploitation and subordination of another" (p. 136). Feminism has sought to challenge the privileged/oppressed relationship between the categories and material conditions of "men" and "women." This privilege/oppressed relation has traditionally been labelled "sexism," though, as I have argued elsewhere (Rands, 2009), "gender category oppression" is a more apt term. Unfortunately, much of the feminist theorizing of sexism is based on the assumption that "there are two, and only two, genders" (Bornstein, 1994, p. 46), an assumption which overlooks the experiences of those who do not identify within the dichotomous categories of men/boys and women/girls. From a gender complex perspective, the concept of "sexism" or gender category oppression, is insufficient by itself in conceptualizing privilege and oppression related to gender. Those who conform to gender expectations for the alignment of gender assignment-gender identity-gender roles-gender presentation are also privileged over those who cross gender lines in their gender identity or presentation or who reject the binary categories altogether. I have previously called this form of gender oppression "gender transgression oppression" (Rands, 2009). The concept of a "gender oppression matrix" combining these two forms of privilege/oppression relations provides a powerful conceptual framework for explaining the complex forms of gender privilege and oppression that people experience. For example, a girl who presents her gender in traditionally feminine ways may face gender category oppression such as less attention from teachers and the attribution of her mathematical successes to "hard work" instead of "talent;" at the same time, she also experiences privileges that gender nonconforming people may not, such being able to count on others using her preferred pronoun and access to restrooms without being questioned. On the other hand, a person could also face oppression based on gender conformity/transgression, yet also experience privilege based on gender category. For example, a child who identifies as boy may find that other students listen to his explanations and suggestions during math group work during, but is refused membership to a boy scouts group or boys' sports team based on the gender assignment of female on his birth certificate. Expanding the conceptualization of gender oppression to contain both gender category oppression and gender transgression oppression constitutes a move from *feminism* to *transfeminism*. Transfeminists (Koyama, 2001; Crabtree, 2002) have argued that transgender issues are feminist issues, and have called for coalitions challenging both forms of gender oppression. With a more complex view of gender in mind, the next section turns to generating new possibilities for lines of research and research questions that take into consideration insights from transgender studies.

LINES OF RESEARCH AND POSSIBLE RESEARCH QUESTIONS

In this section, I build on existing research based on a dichotomous view of gender to generate new lines of research and possible research questions.[3] This process resulted in six lines of research and numerous possible research questions.

[3] All of the chapters cited in this section can be found in Forgasz et al. (2010).

Perceptions and Experiences of Transgender and Gender Nonconforming People in Learning Mathematics

Many of the chapters (and previous research related to gender and mathematics education) has focused on the achievement, attitudes, perceptions, and experiences of women and girls in mathematics education. A new line of research might examine the achievement, attitudes, perceptions, and experiences of transgender and gender nonconforming people in the context of mathematics education. How do transgender and gender nonconforming people of various ages interpret their experiences of learning mathematics? Building on Lambertus, Bracken, and Berenson's (2010) examination of high achieving young women's perceptions of mathematics over time, a new research question might ask, what are the perceptions of transgender and gender nonconforming students of mathematics over time. To what extent do transgender and gender nonconforming people use gender as a concept through which to relate to mathematics? Building on Wall's (2010) concept of mathematical genderfication and her research on the experiences of boys and girls, how is "mathematical genderfication" implicated in the ways in which transgender and gender nonconforming students engage with mathematics, whether they choose to continue studying mathematics, and the kinds of occupations they take up after schooling?

Gender Complex Mathematical Genderfication as an Interpretive Framework for Research

In addition to focusing on the experiences and perceptions of transgender and gender nonconforming people in mathematics education, a gender complex view of gender might serve as an interpretive framework for research involving people in different gender categories and different degrees of gender conformity. For example, Olson, Olson, Okazaki, and La (2010) examined the use of cognitively demanding language among four types of family dyads: mother-daughter, mother-son, father-daughter, and father-son. How might considering the various aspects of gender including gender identity, gender presentation, gender attribution, gender roles, and gender assignment lead to different ways of conceptualizing family member dyads, as well as the interpretation of results? What might the concepts of "gender category privilege/oppression" and "gender conformity privilege/gender transgression oppression" offer to the interpretation findings?

Family, Community, Institutions: Challenges and Support

In addition to Olson et al.'s (2010) examination of family member dyads, other chapters examined girls' and women's perceptions of the ways in which families provided support or contributed to challenges in their mathematical learning. What type of family, community, and institutional supports and challenges do transgender and gender nonconforming people experience related to mathematical learning? For example, to what extent does participation in Gay-Straight Alliances, math study groups, and other social and school groups provide support and facilitate mathematical learning for transgender and gender nonconforming students?

Strategies for Creating Trans-Friendly Spaces

A case can and has been made for "single-sex" schooling and programs for girls (and for boys). Wiest (2010) examined out-of-school-time (OST) mathematics programs for females. How might a math program consciously and intentionally be designed to create spaces for gender complexity and challenge both gender category oppression and gender transgression oppression? How can teachers create trans-friendly spaces in mathematics classrooms?

Boundary Construction

While gender category oppression of girls and women serve as part of the rationale for "single-sex" schooling and programs, such programs also draw dichotomous gender lines that may serve to exclude transgender and gender nonconforming students. Boundary lines based on various aspects of gender may also be constructed in programs that are not labelled "single-sex." How are boundaries of inclusion and exclusion related to various aspects of gender constituted in mathematics classrooms and programs? In what ways do gender category privilege/oppression and gender conformity privilege/gender transgression oppression shape this process? How might teachers and students address this process of boundary construction?

Teacher Education

The last line of research addresses teacher education. Blömke and Kaiser (2010) examined the connection between gender category and pre-service teachers' selection of programs with different levels of mathematical content load. From a gender complex perspective, a question to consider is how patterns of self-selectivity among teachers relate to multiple aspects of gender beyond gender assignment. Another question related to teacher education to ask is, what are transgender and gender nonconforming mathematics teachers' perceptions and experiences of mathematical genderfication? How does this shape the way in which they teach math? Another important question is the following: How does learning about gender complex vocabulary, gender complex education, and the gender privilege/oppression matrix shape pre-service and in-service teachers' thinking, attitudes, and affiliation with mathematics and their approaches to teaching mathematics?

Reconceptualizing research on gender and mathematics in more complex ways is crucial not only to address the exclusion of transgender and gender nonconforming students from the discussion to date, but also because a gender complex framework contributes powerful a powerful interpretive framework for interpreting the intersection of gender identification and mathematical identification of all students.

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DIFFERENT COLLABORATIVE LEARNING SETTINGS TO FOSTER MATHEMATICAL ARGUMENTATION SKILLS

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Since mathematical argumentation skills comprise both, skills genuine to mathematics of differing complexity, and general abilities in argumentation, domain-general as well as domain-specific interventions, can be expected to have a positive effect on its acquisition. In a collaborative learning setting, we combined both interventions, heuristic worked examples vs. problem solving with collaboration script vs. no script support. Results of this experimental study with 119 teacher students indicate that, in collaborative settings, heuristic worked examples are more effective for the acquisition of low-level argumentation skills, whereas solving corresponding problems is more effective for high-level argumentation skills. Structuring the collaboration by a script did not affect the acquisition of domain-specific argumentation skills significantly.

INTRODUCTION

To inquire mathematical conjectures, convince oneself and also the mathematical community about the truth of a conjecture determines the work of mathematicians (Heintz, 2000). Thereby not only deductions by rules of logic play an important role, but also empirical explorations. During the last two decades national and international curricula also focussed on such complex processes of mathematical work (e.g. CCSSI, 2010). As mathematical argumentation tasks require diverse skills and abilities, mathematical argumentation can be considered as an example of a complex skill (Ufer, Heinze, & Reiss, 2008). Several studies showed, that not only students have problems in this field (Reiss, Heinze, Kessler, Rudolph-Albert, & Renkl, 2007), but also prospective and in-service teachers (e.g. Barkai, Tsamir, Tirosh, & Dreyfus, 2002).

Mathematical argumentation skills are understood here as the ability to find and evaluate a mathematical conjecture, generate adequate arguments for or against this conjecture and finally combine these arguments to a proof in an individual or social discursive context. A closer look on this definition shows, that this skill comprises one component which is genuine to mathematics and another component which refers to more general argumentation skills (Kollar, Fischer, & Slotta, 2007). It is an open question, to what extent it is feasible to foster mathematical argumentation skills by using domain-general interventions compared to interventions that aim at domain-specific knowledge and strategies. Hence this contribution investigates the effects of mathematics-specific interventions – heuristic worked example vs. problem solving – together with domain-general interventions – collaboration script vs. no script support – on individual mathematical argumentation skills in collaborative learning settings.

Moreover, under a domain-specific view mathematical argumentation skills comprise facets of diverse complexity. There are low-level demands like schematic argumentation skills based on a routine application of simple rules, proof tasks that require complex problem-solving processes in building up a coherent line of deductive arguments (Reiss et al., 2007), and even more complex facets like the ability to evaluate and prove or disprove mathematical conjectures (conjecturing).

Domain-general interventions: Collaboration script vs. No Collaboration script

According to Kuhn and Udell (2003) learning in collaborative settings can have positive effects on the acquisition of general argumentation skills. But research has also shown that such collaboration is not always effective, especially when no external structure for the collaboration is provided (Mullins, Rummel, & Spada, 2011). One solution is to provide learners with a computer supported collaboration script (Kollar, Fischer, & Hesse, 2006). It assigns the learners of a small group to specific roles or activities in a defined sequence (e.g. A: give an argument, B: give a counterargument, A&B: try a synthesis). A number of authors have studied if and how collaboration scripts facilitate collaborative argumentation. Indeed, these scripts have positive effects on the general quality of constructed arguments and – less frequently – also on domain specific learning outcomes (Weinberger, Ertl, Fischer, & Mandl, 2005).

Domain-specific interventions: Heuristic worked examples vs. Problem solving

Studying heuristic worked examples as well as solving authentic problems are considered as effective means to foster complex skills, like mathematical argumentation skills. But it remains still an open question, whether one of these two learning modes is superior with respect to different facets of argumentation skills.

To distinguish the two learning modes, we use a categorization of instructional information by Schworm and Renkl (2007) originally developed for worked examples. Instructional settings for complex tasks can differ according to the availability of information on three levels: Structural aspects, which are relevant for the solution of the problem and should be learned, belong to the *learning domain level* (e.g. principles of mathematical proof and argumentation). The *exemplifying domain level* contains information about surface features of a task and especially about the context in which the contents of the learning domain are embedded (e.g. a specific number theory argumentation task). Finally the *strategy level* refers to the meta-cognitive aspects of the task, like the choice of heuristic strategies.

Reiss and Renkl (2002) developed the idea of heuristic worked examples, which provide information on all three content levels. These examples do not only explicate the problem formulation and the solution (as a usual worked example would), but also the solution process, heuristic strategies to approach a problem and a process model of the corresponding skills of a more advanced learner or an expert. When studying a heuristic worked example, the learner follows the solution procedure of a fictitious peer, i.e. the solution process is not perfect and can also contain explorative and misleading approaches. Positive effects of those heuristic worked examples can be explained with Cognitive Load Theory (Kalyuga, Ayres, & Sweller, 2011). With an adapted process model of an expert for proving by Boero (1999), heuristic worked examples have been shown to be more effective than typical school lessons for fostering proving skills (Reiss et al., 2007).

Another promising instructional mode for the learning of such complex skills is solving authentic problems. According to Halmos (1980) mathematics and problem solving belong together. Working on mathematical argumentation tasks means also, solving a problem in the sense of Funke and Frensch (2007). This goes beyond the application of well-known rules or algorithmic steps to an unknown solution method for the learner. According to Funke and Frensch (2007) problem solving can be learned by making various experiences in solving problems. A meta-analysis of 43 studies by Dochy, Segers, van den Bossche, and Gijbels (2003) showed that problem solving has positive effects on the acquisition of problem solving skills but not on the acquisition of domain knowledge.

There is ample research for studying traditional worked examples and also problem solving, restricted to individual learning settings (Kalyuga et al., 2011). What remains mostly open so far is the effectiveness of the two learning modes in collaborative learning settings (Kirschner, Paas, Kirschner, & Janssen, 2011). In a first study, Kirschner et al. (2011) compared problem solving with usual worked examples in an unstructured collaborative setting and found problem solving to be superior in this case. Nevertheless, also in line with the results of Dochy et al. (2003), it is an open question how *heuristic* worked examples in collaborative settings influence the acquisition of facets of mathematical argumentation skills with different complexity.

RESEARCH QUESTIONS AND DESIGN OF THE STUDY

The present study is guided by the following questions:

- Is there a positive effect of the availability of instructional support on all three content levels on differing facets of mathematical argumentation skills? Here we compare problem solving to studying heuristic worked examples in *collaborative settings*.
- What impact do collaboration scripts have on the acquisition of these facets of mathematical argumentation skills?
- Are there differential effects of collaboration scripts on mathematical argumentation skills when combined with two different domain-specific interventions (heuristic worked examples vs. problem solving)?

Sample and Design

119 pre-service mathematics teacher students from two German universities took part in our experimental study with pre- and post-test. The different instructional settings

were implemented during a voluntary two-week preparatory course for university mathematics. The participants were assigned randomly to one of four intervention groups, controlling for high vs. low final school qualification grade (see Table 1).

		Collaboration script	
		Without	With
Learning mode	Problem Solving	N=29	N=29
	Heuristic worked example	N=29	N=32

Table 1: Experimental design

On three days, students worked for 45 minutes on one mathematical argumentation task from elementary number theory (e.g. “Choose an odd amount of consecutive numbers, e.g. 3, 5 or 7 consecutive numbers. Sum up these consecutive numbers. Do you notice anything special? Find a conjecture and prove it.”). The dyads were homogenous with respect to their final high-school qualification grade and were changed every day. The students worked face to face in a computer supported learning environment, each equipped with a laptop, a graphic tablet and a mouse.

Materials and Instruments

On the right side of the screen (see Fig. 1) the two students working together had a shared work space, which functioned like a (graphical) chat window. At the top right side the students got varying instructions depending on the intervention group. In the condition with collaboration script the students had additionally a range of script Buttons at the bottom of the right side.

The left side of the screen contained the domain-specific instruction. In the *heuristic worked example condition*, illustrated texts, describing how a fictitious peer solved the problem following a 6-phase process model adapted from Boero (1999) were shown. To prevent superficial processing of the heuristic worked example, a self-explanation prompt addressing the strategy level (Schworm & Renkl, 2007) was presented in each phase: After the students were asked to think individually about that question, they were prompted to discuss their thoughts with their partner. The heuristic worked example of the two learning partners differed on the strategy level in every second phase. In the *problem solving condition* students were given the problem formulation and asked to find a solution. They were first prompted to think individually about a possible solution step and afterwards discuss their ideas with their partner.

Thus, individual learning phases and collaborative discussions were systematically alternated. Collaborative discussions were structured in three phases according to the cycle of argumentative discourse of Leitão (2000) in the conditions *with collaboration script*: (1) argument, (2) counterargument and (3) synthesis. Additional support was provided in each step based on Toulmin’s (1958) argumentation model.

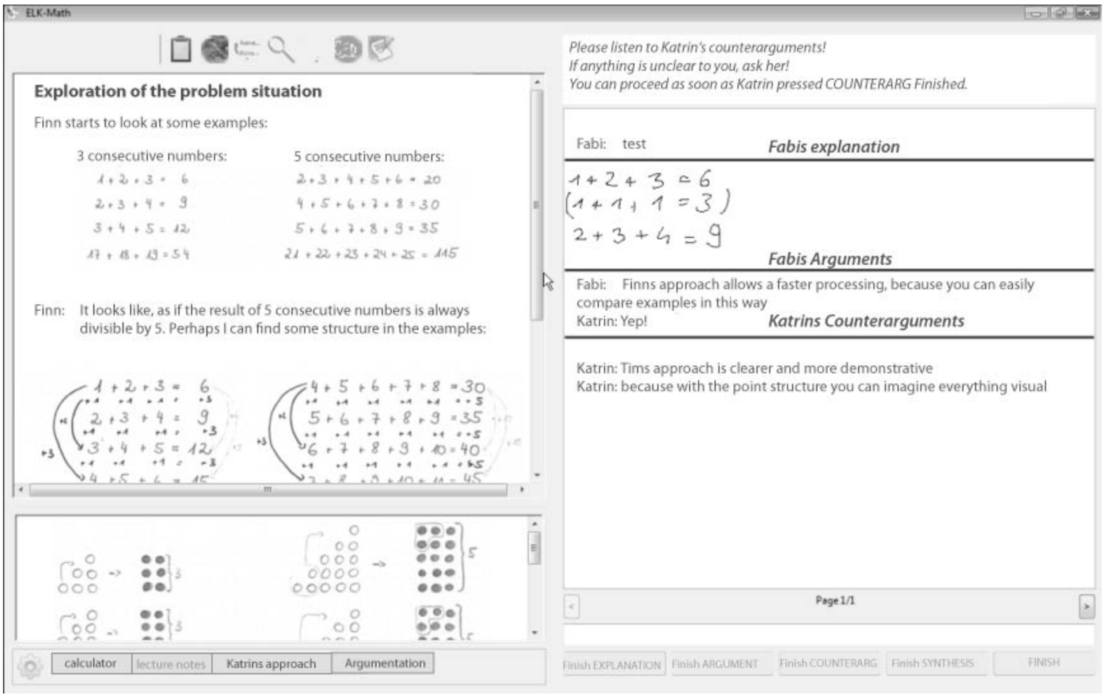

Fig. 1: Screenshot of the computer supported learning environment (heuristic worked example with collaboration script; translated by the authors)

To assess students' progress, parallel pre- and post-tests were designed covering different facets of mathematical argumentation skills. The first part required *schematic argumentation* through divisibility rules (e.g., "Show that for natural numbers, a and b the following statement is true: If 5 divides (a+2b) then 5 divides (4a+3b).") (5 items, Cronbach's α =.66/.67). Students *proof skills* in elementary number theory were measured in the second part (e.g. "Prove the following statement: The sum of a natural number, its square plus one is odd.") (6 items, Cronbach's α =.74/.73) and in the third part, the students had to solve open ended *conjecturing* problems (e.g. "Prove or refute the following statement: The sum of six consecutive numbers is divisible by 6.") (6 items, Cronbach's α =.57/.57). A three-level coding (see also Reiss et al., 2007) was applied to score students' answers. No and irrelevant trials were scored with zero points. For partially correct solutions the students got one point and for a correct solution two points were given. A fourth part of the post-test contained a question on *heuristic strategies* for the mathematical argumentation process ("You should formulate and proof a mathematical conjecture. How would you proceed?"). The students were given one point for each strategy corresponding to the 6-phase model underlying the heuristic worked examples (max. 6 points). All the items were coded by two independent raters and interrater reliability for each part of the pre- and post-test was found to be good (Mean of ICC_{unjust}=.86, SD =.12).

RESULTS

All four learning conditions were appropriate for the learning of mathematical argumentation skills. To get a deeper insight, four ANCOVAs, one for each test part as dependent variable, *learning mode* and *collaboration script* (with/without) as independent variables and pre-test scores as a covariate were conducted.

	Problem solving		Heuristic worked example	
Coll. script	Without	With	Without	With
schematic arg.	.52 (.04)	.58 (.04)	.61 (.04)	.67 (.04)
proof	.50 (.04)	.50 (.03)	.43 (.04)	.46 (.03)
conjecturing	.59 (.03)	.61 (.03)	.54 (.03)	.50 (.03)
heuristic strat.	.30 (.05)	.24 (.05)	.43 (.05)	.35 (.05)

Table 2: Adjusted means (standard deviations indicated in brackets) for mathematical argumentation skills in the four experimental conditions.

For post-test performance, the ANCOVA results show a significant main effect of the learning mode in the first ($F(1,114)=5.93$; $p<.05$; $\eta^2=.05$), third ($F(1,114)=6.63$; $p<.05$; $\eta^2=.06$) and fourth ($F(1,114)=7.66$; $p<.05$; $\eta^2=.06$) part of the test. For *schematic argumentation* (part 1) and *heuristic strategies* (part 4), students who learned in the heuristic worked example condition outperformed those who studied in the problem solving condition (see Table 2). The opposite effect was found for *conjecturing skills* (part 3). Students from the problem solving condition did significantly better when working on open ended conjecturing problems. The main effect for collaboration script and also the interaction effect between collaboration script and learning mode did not reach statistical significance ($F(1,114)<3$, *n.s.*).

DISCUSSION

This study examined the effects of two different domain-specific interventions within a structured resp. unstructured collaborative learning setting on different facets of mathematical argumentation skills.

The main effect of collaboration scripts did not reach statistical significance, but the descriptive results in Table 2 indicate that for most facets of mathematical argumentation skills, students profited from the script. In further analyses, students' behavior in an unstructured collaborative argumentation situation after the intervention will be analysed and we expect to find clearer effects there. We found no interaction effect of the learning mode and the script on the acquisition of mathematical argumentation skills. This indicates that one intervention did not affect the other negatively. Also no synergy effect of combining both interventions was observed.

Regarding the comparison of the two different content-related instructions, our results are at least partly contrary to the results of Kirschner et al. (2011) who found problem solving to be superior to (usual) worked examples in unstructured collaborative settings. Collaborative learning from worked examples led to significantly higher performance in low-level facets of mathematical argumentation skills: Regarding *schematic argumentation skills*, students were required to do transformations of the algebraic expressions, find adequate divisibility rules or use the definition of divisibility to prove the statement. Similarly, knowledge of the *strategies* taught in the heuristic worked example condition can be considered low-level. The reverse relationship was found for *conjecturing skills* and was also indicated for *proof skills*, but failed to reach statistical significance. The *proof skills* test required finding multiple proof steps and for most items a formalization of a verbal statement was conducive. A further demand in the *conjecturing test* was to evaluate the conjecture as true or false. Students' performance on false statements (M=4.18, SD=1.76) was better than on true statements (M=2.52, SD=1.41), since false statements only required counterexamples and no deductive argumentations. This explains higher mean values in the *conjecturing test* compared to the *proof test* (see Table 2). Altogether, the items in the *proof* and *conjecturing tests*, especially the true conjecturing items, can be considered as complex, high-level argumentation tasks.

Our results indicate a first answer on how the availability of solution steps on all content levels influences the acquisition of mathematical argumentation skills. Solution steps on all content levels (heuristic worked examples) proved to be superior for the acquisition of low-level argumentation skills. Problem solving, with no solution steps available, was more effective for the acquisition of high-level argumentation skills in our collaborative setting. Worked examples have repeatedly proved to be effective interventions (Schworm & Renkl, 2007) in individual learning settings. It seems necessary to take the general learning setting as well as the complexity of target skills carefully into account when judging the effectiveness of domain-specific interventions. A noticeable result is that the learning mode influences facets of mathematical argumentation skills differently in our collaborative learning setting. Further research is necessary, modifying these learning modes regarding the availability of solution steps on different levels. Also students' cognitive abilities should be considered (see e.g. Kalyuga et al., 2011)

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AUTHOR INDEX

RESEARCH REPORTS

VOLUME 3

PME 36

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2012